# Quaternions

$$\begin{align\*}
q = a+\bold{i}b+\bold{j}c+\bold{k}d,\quad a,b,c,d\in \mathbb{R}
\end{align\*}$$

## Abstract Algebra

### group

a set G with an operation which satisfies the axioms:

1. associative
2. element *zero*

$$\forall a \in G,\exist e,s.t.a\*e=e\*a$$

1. element *inverse*

$$\forall a, \exist a^{-1},s.t. a\*a^{-1}=a^{-1}\*a=e$$

### ring

a set A with operations called addition and multiplication which satisfy the following axioms:

1. A with addition alone is an abelian group
2. Multiplication is associative
3. Multiplication is distributive over addition

#### some terms

**commutative ring** multiplication is also commutative.

**filed** a commutative ring with unity in which every nonzero element is invertible.

**unity** neutral element for multiplication.

**division ring** non-commutative ring

# Optimization

least square optimization

$$\min\_{\bold x \in \chi} \sum\_{i=1}^{N} r^{2}(\bold y\_{i},\bold x) \label{eq:op\_1}$$

where is the variable to estimate, e.g. the pose in ICP problems,

is the domain of , e.g. in ICP problems,

$\bold y\_{i}$ is the measurement,

the function $r(\bold y\_{i},\bold x)$ is the residual for the measurement.

Generally $\eqref{eq:op\_1}$ is difficult to solve globally, due to the nonlinearity of the residual function and non-convexity of . Further more, in the presence of outliers, $\eqref{eq:op\_1}$ gives false estimation. This call for a robust cost

$$\min\_{\bold x \in \chi} \sum\_{i=1}^{N} \rho (r(\bold y\_{i},\bold x)) \label{eq:op\_2}$$

## Graduated non-convexity (GNC)

### Black-Rangarajan duality

Given a robust cost function , define . If satisfies , , and , then the robust estimation problem $\eqref{eq:op\_2}$ is equivalent to

$$\min\_{\bold x \in \chi,w\_i\in [0,1]} \sum\_{i=1}^{N} [w\_i r^2(\bold y\_{i},\bold x)+\Phi\_\rho (w\_i)] \label{eq:op\_3}$$

where are weight associated to measurement , and the function defines a penalty on the weight , which depends on the choice of robust cost function .

### GNC

GNC is a popular approach for the optimization of a generic non-convex cost function .

The basic idea is to introduce a surrogate cost , governed by a control parameter , which adjust the non-convexity.

#### Example 1 (Geman McClure and GNC)

[GM](#GM) as following, blue is original residual, red is residual reduced. Smaller c indicate larger penalty for outliers. Results in slower converge but more robust.

the surrogate function satisfies (i) becomes convex for large . (ii) recovers the original form [image](#gnc)

GNC

#### Example 2 (TLS)

[TLS](#TLS) is defined as

GNC surrogate is

$$\rho\_\mu(r)=\begin{cases}
r^2 & if \, r^2\in [0,\frac{\mu}{\mu+1}\bar c^2] \\
2\bar c \abs{r} \sqrt{\mu(\mu+1)}-\mu(\bar c^2+r^2) & if \, r^2 \in [\frac{\mu}{\mu+1}\bar c^2,\frac{\mu+1}{\mu}\bar c^2]\\
\bar c^2 & if \,r^2 \in [\bar c^2,+ \infty)
\end{cases}$$

### Implementation

#### variable update

For inner loop, first fix and minimize $\eqref{eq:op\_3}$ with respect to $\bold x$

$$\DeclareMathOperator\*{\argmax}{arg\,max}
\DeclareMathOperator\*{\argmin}{arg\,min}
\bold x^{(t)} = \argmin\_{\bold x \in \chi} \sum\_{i=1}^{N} w\_i^{(t-1)} r^2(\bold y\_{i},\bold x) \label{eq:op\_4}$$

which can be solved globally.

#### weight update

Then fix and minimize $\eqref{eq:op\_3}$ with respect to .

$$\bold w^{(t)}=\argmin\_{\bold x \in \chi,w\_i\in [0,1]} \sum\_{i=1}^{N} [w\_i r^2(\bold y\_{i},\bold x^{(t)})+\Phi\_\rho (w\_i)] \label{eq:op\_6}$$

we will now try to optimize in closed form.

#### update

#### GM-GNC

given

then $\eqref{eq:op\_6}$ can be solved in closed form

where $\hat{r}=r(\bold y\_i,\bold x^{(t)})$

#### TLS-GNC

closed form as following:

## Residuals

### notation

$$\bold b\_i\in B \\
\bold a\_i\in A$$

## generalized distance function

$$r=d\_{B}(\bold T\*\bold a\_i)$$

denote the minimum distance to B.

### generalized form

$$\begin{align}
d\_B(\bold x) & =\min\_{\bold b\_i\in B}\norm{\bold x-\bold b\_i}\_{\bold C}\\
& =(\bold x - \bold b\_i)^T\*\bold C\*(\bold x - \bold b\_i)
\end{align}$$

#### point to point

$$d\_B(\bold x)=\min\_{\bold b\_i\in B}\norm{\bold x-\bold b\_i}\_{\bold I}$$

#### point to line

$$d\_B(\bold x)=\min\_{\bold b\_i\in B}\norm{\bold x-\bold b\_i}\_{(\bold I-\bold v\bold v^T)}$$

where is the unit direction vector for a line.

#### point to plane

$$d\_B(\bold x)=\min\_{\bold b\_i\in B}\norm{\bold x-\bold b\_i}\_{(\bold n\bold n^T)}$$

where is the unit normal vector for a plane.

## Quadratic formulation

$$d\_B^2(\bold T\* \bold a\_i)=(\bold T\*\bold a\_i - \bold b\_i)^T\*\bold C\*(\bold T\*\bold a\_i - \bold b\_i)$$

$\bold T\* \bold a\_i$ is in fact linear in the elements of $\bold T$

$$\bold T\* \bold a\_i=\bold R\bold a\_i+\bold t=\underbrace{(\bold{\tilde a}\_i\,\otimes\,\bold I\_3)}\_{\bold A\_i}vec(\bold T)$$

where $\tilde{\bold a}\_i=[\bold a\_i^T,1]^T$, $vec(T)=\begin{bmatrix}vec(\bold R)\\\bold t\end{bmatrix}$.

we name $\tau=vec(\bold T)$, the generalized distance is a quadratic function of

$$d\_B^2(\bold T\* \bold a\_i)=\tilde \tau^T\underbrace{\bold N\_i^T\bold C\_i\bold N\_i}\_{\tilde {\bold M}\_i}\tilde \tau$$

with $\bold N\_i=[\tilde {\bold a}\_i\,\otimes\,\bold I\_3|-\bold b\_i]$ and $\tilde \tau=\begin{bmatrix} vec(\bold T\\1\end{bmatrix}^T$.

compression for the whole point cloud

$$f(\bold T)=\sum^m\_{i=1}d^2\_{B\_i}(\bold T\,\otimes\,\bold a\_i)=\tilde{\bold \tau}^T\underbrace{\Bigg( \sum^m\_{i=1}\tilde{\bold M}\_i\Bigg)}\_{\tilde{\bold M}}\tilde {\bold{\tau}}$$

$\bold t$ can be derived in terms of $\bold R$

$$\bold t(\bold R)=-\tilde{\bold M}\_{\bold t,\bold t}^{-1}\tilde{\bold M}\_{\bold t,\bold !t}\tilde{\bold r},\quad \tilde{\bold r}=\begin{bmatrix} vec(\bold R)\\1\end{bmatrix}$$

the marginalized optimization problem is then

$$f=\min\_{\bold R\in SO(3)}\underbrace{\tilde{\bold r}^T\tilde{\bold Q}\tilde{\bold r}}\_{q(\tilde{\bold r})},\quad \tilde{\bold r}=\begin{bmatrix} vec(\bold R)\\1\end{bmatrix}\label{eq:lg\_01}$$

where $\tilde{\bold Q}=\tilde{\bold M}/\tilde{\bold M}\_{\bold t,\bold t}$.

### SO(3) constraints

in $\refeq{eq:lg\_01}$, the constraints are as follows

$$SO(3)=\{\bold R\in \R^{3\times3}:\,\bold R^T\bold R=\bold I\_3,det(\bold R)=+1\}$$

the orthonormality is quadratic, but the determinant constraint is cubic.

### TIMs

translation invariant measurement where $\bar{\bold b}\_{ij}=\bold b\_i-\bold b\_j$, $\bar{\bold a}\_{ij}=\bold a\_i-\bold a\_j$. (Construct with complete graph, can be simplified with max clique)

$$\bar{\bold b}\_{ij}=R\bar {\bold a}\_{ij}+\bold o\_{ij}+\bold{\epsilon}\_{ij}$$

in $\eqref{eq:op\_1}$, set $\bold x$ as $\bold R$, $\bold y\_i$ as $\bar{\bold b}\_{ij},\bar {\bold a}\_{ij}$.

$$r(\bar{\bold b}\_{ij},\bar {\bold a}\_{ij},\bold R)=\bar{\bold b}\_{ij}-\bold R\*\bar {\bold a}\_{ij}$$

### Li's method

### symmetric workpieces