MATH 217 - Advanced Honors Calculus I: Logan's Notes

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Chapter 1

Topology in $\mathbb{R}^N$

1.1 Justification of $\mathbb{R}$

1.1.1 Definition: Justfication

x with property $P(x)$ is justified $\iff \vdash \exists x\, P(x)$

1.1.2 The Top Down Approach (Axioms of R)

$\exists \mathbb{R} = (R, +, \cdot, <)$ which is formed from

- A set R with at least two elements
- Two functions $+$, $\cdot$ of the form
  $+ : R \times R \to R, \cdot : R \times R \to R$
- A strict order relation $<$ on $\mathbb{R}$

Which satisfy the following

1. **Field Axioms:**
   (a) $a + b = b + a$ (commutativity of $+$)
   (b) $a + (b + c) = (a + b) + c$ (associativity of $+$)
   (c) $\exists 0 : a + 0 = a$ (additive identity)
   (d) $\forall a, \exists - a : a + (-a) = 0$ (additive inverse)
   (e) $a \cdot b = b \cdot a$ (commutativity of $\cdot$)
   (f) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity of $\cdot$)
   (g) $\exists 1 \neq 0 : a \cdot 1 = a$ (multiplicative identity)
   (h) $\forall a \neq 0, \exists a^{-1} : a \cdot a^{-1} = 1$ (multiplicative inverse)
   (i) $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity)
2. **Order Axioms:**
   (a) $a \leq b$ or $b \leq a$ (totality)
   (b) $a \leq b$ and $b \leq c \implies a \leq c$ (transitivity)
   (c) $a \leq b \implies a + c \leq b + c$ (addition compatible)
   (d) $0 \leq a, 0 \leq b \implies 0 \leq a \cdot b$ (multiplication compatible)
3. **Completeness:** Every nonempty subset of $\mathbb{R}$ bounded above has a least upper bound.

1.1.3 The Bottom Up Approach

Definition of $\mathcal{N}$

We define a set $\mathcal{N}$ and a function $S : \mathcal{N} \to \mathcal{N}$ satisfying:

1. $0 \in \mathcal{N}$.
2. $S : \mathcal{N} \to \mathcal{N}$.
3. $\forall n, m \in \mathcal{N},\ S(n) = S(m) \implies n = m$ (injectivity).
4. $\nexists n \in \mathcal{N}$ such that $S(n) = 0$.
5. **(Axiom of Induction).** For any set K, if
   (a) $0 \in K$, and
   (b) $\forall n \in \mathcal{N},\ n \in K \implies S(n) \in K$,

   then $\mathcal{N} \subseteq K$.

Justification of $\mathbb{N}_0$

The set $\mathcal{N}$ can be identified with $\mathbb{N}_0$ through use of the successor function where

1. $0_{\mathbb{N}_0} :=$ the object $0 = 0_{\mathcal{N}}$
2. $1_{\mathbb{N}_0} := S(0_{\mathcal{N}}) = S(0_{\mathbb{N}_0})$
3. $2_{\mathbb{N}_0} := S(S(0_{\mathcal{N}})) = S(1_{\mathbb{N}_0})$ ...

Funcions of $\mathbb{N}_0$ $\exists + : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ $\exists + : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ satisfying

1. Neutral element of addition and multiplication
2. Commutativity over addition and multiplication
3. Associativity over addition and multiplication
4. Distributivity of multiplication over addition

1.1.4 Binary Relation

$\mathbb{N}_0$ can be ordered with a total order