Splitting Functions

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1 Charged Scalar Splitting functions

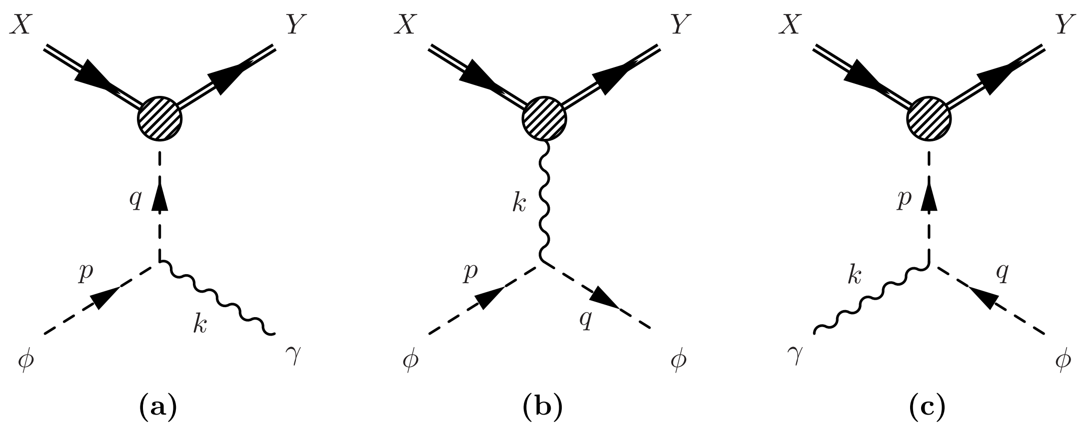


Figure 1

In this section, we will derive the splitting function for a charged scalar particle. We wish to consider the diagrams shown in Fig. (1). In particular, we wish to investigate the cross sections for these diagrams in the limit in which the final state particle (on the bottom of the diagram) is collinear to the initial state particle (again, on the bottom of the diagram.) We will do this individually for each diagram. We will consider the mass of the charged scalar to be negligible. We will take the interaction of the charged scalar and the photon to be a gauge-invariant one:

$$\mathcal{L}_{\text{int}} \supset -ieA_{\mu}(\phi^{\dagger}\partial_{\mu}\phi - \partial_{\mu}\phi^{\dagger}\phi) \tag{1.1}$$

In the collinear limit, the differential cross sections go like:

$$\frac{d\sigma}{dz} = \frac{\alpha}{\pi}P(z)\log\left(\frac{s}{m^2}\right) \tag{1.2}$$

where z is a fraction of energy carried by the final state particle and $P(z)$ is a splitting function. We will show that the splitting functions are:

$$P_{\phi\to\gamma\phi} = \frac{2\tilde{z}}{z} \tag{1.3}$$
$$P_{\gamma\to\phi\phi^\dagger} = z\tilde{z} \tag{1.4}$$

where $\tilde{z} = 1 - z$.

1.1 Diagram (a)

It isn't hard to see that we can write the cross section for this diagram as:

$$\sigma(X\phi \to Y\gamma) = \frac{1}{2E_X 2E_\phi v_{\text{rel}}} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} d\Pi_Y \frac{1}{q^4} |\mathcal{M}_{\phi\to\phi\gamma}|^2 |\mathcal{M}_{X\phi\to Y}| \tag{1.5}$$

where $\mathcal{M}_{\phi\to\phi\gamma}$ is the matrix element for the charged scalar to split into a charged scalar and photon, and $\mathcal{M}_{X\phi\to Y}$ is the amplitude for the state X to annihilate with $\phi^\dagger$ into the state Y. In order to compute the first amplitude, we will need to understand the kinematics. We will take the initial charged scalar's momentum to be oriented along the z axis:

$$p^\mu = (p, 0, 0, p) \tag{1.6}$$

We can take the momenta q, k to be:

$$k^\mu = (zp, p_\perp, 0, zp + x) \tag{1.7}$$
$$q^\mu = ((1-z)p, -p_\perp, 0, (1-z)p - x) \tag{1.8}$$

Here we have taken the transverse part of the momentum to be along the x direction and we have set the magnitude of the transverse momentum to be $p_\perp$. The factor of x will need to be determined in order to ensure that the photon is on-shell: $k^2 = 0$. If we set this requirement, we find that:

$$x = -\frac{p_\perp}{2pz} + \mathcal{O}\big(p_\perp^4\big) \tag{1.9}$$

Now that we have the kinematics worked out, we can write down the amplitude:

$$i\mathcal{M}_{\phi\to\phi\gamma} = -ie(p^\mu + q^\mu)\epsilon^*_\mu(k) \tag{1.10}$$

We will sum over the polarizations of the final state photon using:

$$\epsilon^{\mu *}_{L,R}(k) = \frac{1}{\sqrt{2}}\left(0, 1, \pm i, -\frac{p_\perp}{zp}\right) \tag{1.11}$$

where the factor of $p_\perp/(zp)$ is equal to $\sin\theta$, the sine of the angle the photon makes w.r.t. the z-axis. If we sum over these polarizations, we find that:

$$i\mathcal{M}_{\phi\to\phi\gamma} = -\frac{ie}{\sqrt{2}} \frac{p_\perp(4p^2 z + p_\perp^2)}{p^2 z^2} \tag{1.12}$$

Squaring the amplitude and dividing by q^4 yields the following simple result:

$$\frac{|\mathcal{M}_{\phi\to\phi\gamma}|^2}{q^4} = \frac{32\pi\alpha}{p_\perp^2} \tag{1.13}$$

Therefore, the cross section is

$$\sigma(X\phi \to Y\gamma) = \frac{1}{2E_X 2E_\phi v_{\mathrm{rel}}} \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} d\Pi_Y \frac{32\pi\alpha}{p_\perp^2} |\mathcal{M}_{X\phi\to Y}| \tag{1.14}$$

Note that we can write the integration measure d^3k as:

$$\mathrm{d}^3k = 2\pi p_\perp \,\mathrm{d}p_\perp \, p \,\mathrm{d}z = p\pi \,\mathrm{d}p_\perp^2 \,\mathrm{d}z \tag{1.15}$$

Using $k_0 = zp$, we find:

$$\sigma(X\phi \to Y\gamma) = \frac{2\alpha}{\pi} \int \frac{\mathrm{d}p_\perp^2}{p_\perp^2} \frac{\mathrm{d}z}{z} \frac{1}{2E_X 2E_\phi v_{\mathrm{rel}}} \int \mathrm{d}\Pi_y \, |\mathcal{M}_{X\phi\to Y}| \tag{1.16}$$

If we replace E_ϕ with q_0 in the pre-factor of the integral over $\mathrm{d}\Pi_y$, then the last part of the expression will simply be the cross-section $\sigma(X\phi \to Y)$. Using:

$$\frac{1}{E_\phi} = \frac{p(1-z)}{p} \frac{1}{q_0} \tag{1.17}$$

(where we used $E_\phi = p$), we find that

$$\sigma(X\phi \to Y\gamma) = \frac{2\alpha}{\pi} \int \frac{\mathrm{d}p_\perp^2}{p_\perp^2} \frac{\mathrm{d}z\,(1-z)}{z} \sigma(X\phi \to Y) \tag{1.18}$$

Integrating over $p_\perp$, we find that:

$$\frac{\mathrm{d}\sigma(X\phi \to Y\gamma)}{\mathrm{d}z} = \frac{2\alpha}{\pi} \log\left(\frac{s}{m^2}\right) \frac{(1-z)}{z} \sigma(X\phi \to Y) \tag{1.19}$$

1.2 Diagram (b)

The cross section for diagram two is:

$$\sigma(X\phi \to Y\phi) = \frac{1}{2E_X 2E_\phi v_{\mathrm{rel}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3 2q_0} \,\mathrm{d}\Pi_Y \, \frac{1}{k^4} |\mathcal{M}_{\phi\to\gamma\phi}|^2 |\mathcal{M}_{X\gamma\to Y}|^2 \tag{1.20}$$

The kinematics are slightly different for this case. Here we have that $q^2 = 0$ and:

$$q^\mu = (zp, p_\perp, 0, zp - \frac{p_\perp^2}{2pz}) \tag{1.21}$$

$$k^\mu = ((1-z)p, -p_\perp, 0, (1-z)p + \frac{p_\perp^2}{2pz}) \tag{1.22}$$

In this case, we find that the squared amplitude divided by k^2 is given by:

$$\frac{|\mathcal{M}_{\phi\to\gamma\phi}|^2}{k^4} = \frac{32\pi\alpha(1-z)^2}{p_\perp^2 z^2} \tag{1.23}$$

Thus, the cross section is again given by:

$$\frac{\mathrm{d}\sigma(X\phi \to \phi Y)}{\mathrm{d}z} = \frac{2\alpha}{\pi} \log\left(\frac{s}{m^2}\right) \frac{1-z}{z} \sigma(X\phi \to Y) \tag{1.24}$$

1.3 Diagram (c)

The cross section is:

$$\sigma(X\gamma \to Y\phi^\dagger) = \frac{1}{2E_X 2E_\gamma v_{\rm rel}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p_0}\, \mathrm{d}\Pi_Y\, \frac{1}{p^4} \big|\mathcal{M}_{\gamma\to\phi\phi^\dagger}\big|^2 \big|\mathcal{M}_{X\gamma\to Y}\big|^2 \tag{1.25}$$

The kinematics are:

$$k^\mu = (p, 0, 0, p) \tag{1.26}$$

$$p^\mu = (zp, p_\perp, 0, zp + \frac{p_\perp^2}{2p(1-z)}) \tag{1.27}$$

$$q^\mu = ((1-z)p, -p_\perp, 0, (1-z)p - \frac{p_\perp^2}{2p(1-z)}) \tag{1.28}$$

$$\epsilon^{\mu *}_{L,R} = (0, -1, \pm i, 0) \tag{1.29}$$

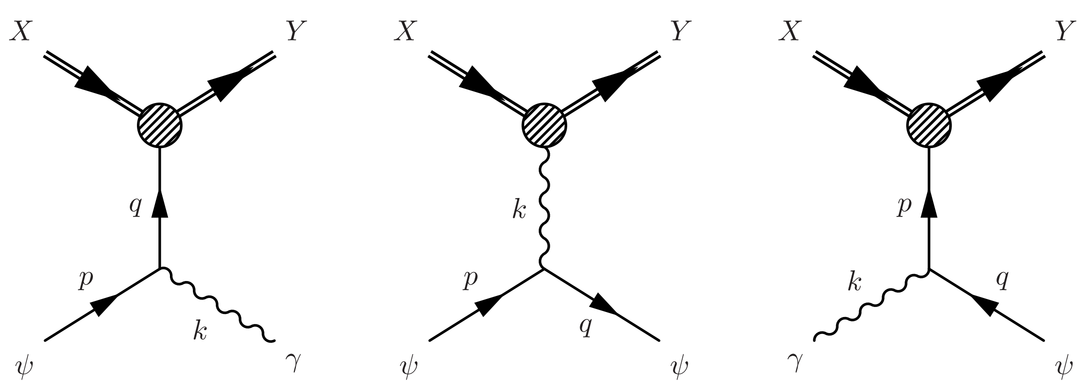
The amplitude over the propagator squared is:

$$\frac{\big|\mathcal{M}_{\gamma\to\phi\phi^\dagger}\big|^2}{p^4} = \frac{16\pi\alpha(1-z)^2}{p_\perp^2} \tag{1.30}$$

(note the extra factor of 1/2 due to the averaging over polarizations of the photon.) The differential cross section is:

$$\frac{\mathrm{d}\sigma(X\gamma\to\phi^\dagger Y)}{\mathrm{d}z} = \frac{\alpha}{\pi}\log\Big(\frac{s}{m^2}\Big) z(1+z)\sigma(X\phi\to Y) \tag{1.31}$$

2 Charged Fermion Splitting Functions



(a) (b) (c)

Figure 2

$$\mathcal{L}_{\rm int} \supset -ieA_\mu \bar{\psi}\gamma^\mu\psi \tag{2.1}$$