Integral Tables

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1 Tables for $2 \to 2$ Cross Sections

Here we give expressions for integrals that appear in $2 \to 2$ cross sections. $2 \to 2$ cross sections will be functions of the masses and three Mandelstam variables s, t and u. We can always get rid of u using:

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{1.1}$$

We can therefore always get rid of u is favor of masses, s and t. The cross section, in the center-of-mass frame is given by:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{EM}} = \frac{1}{64\pi^2 s}\frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)}|M|^2\theta\big(\sqrt{s} - m_3 - m_4\big) \tag{1.2}$$

where $d\Omega = d\phi d\cos\theta$ and

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \tag{1.3}$$

We can replace the integration over $\cos\theta$ with an integration over t using:

$$t = (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2\cos\theta|\mathbf{p}_1||\mathbf{p}_3|, \qquad \mathrm{d}t = 2|\mathbf{p}_1||\mathbf{p}_3|\,\mathrm{d}\cos\theta \tag{1.4}$$

Using

$$|\mathbf{p}_1| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}, \qquad |\mathbf{p}_3| = \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\sqrt{s}} \tag{1.5}$$

$$E_1^2 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \qquad E_3^2 = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}} \tag{1.6}$$

we find that

$$\sigma = \frac{1}{16\pi\lambda(s, m_1^2, m_2^2)}\int_{t_{\min}}^{t_{\max}}\mathrm{d}t\,|M|^2\theta\big(\sqrt{s} - m_3 - m_4\big) \tag{1.7}$$

$$\tag{1.8}$$

with

$$t_{\min} = m_1^2 + m_3^2 - \frac{\lambda^{1/2}(s, m_1^2, m_2^2)\lambda^{1/2}(s, m_3^2, m_4^2) + (s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2)}{2s} \tag{1.9}$$

$$t_{\max} = m_1^2 + m_3^2 + \frac{\lambda^{1/2}(s, m_1^2, m_2^2)\lambda^{1/2}(s, m_3^2, m_4^2) - (s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2)}{2s} \tag{1.10}$$

We assume that the matrix element squared can be written as:

$$|\mathcal{M}|^2 = \frac{p(t)}{t+a} + \frac{q(t)}{(t+a)^2} + \frac{r(t)}{(t+a)(t+b)} + \frac{v(t)}{at^2 + bt + c} \tag{1.11}$$

where $(t+a)$ and $(t+b)$ are propagator factors and p, q and r are polynomials in t. We can write the polynomials as:

$$p(t) = \sum_{k=0}^{n} p_k t^k \tag{1.12}$$

To compute the integrals of $|\mathcal{M}|^2$, we will need to integrate $t^n/(t+a)$, $t^n/(t+a)^2$ and $t^n/(t+a)(t+b)$. To perform the first of these integrals, we first shift $t \to \tau = t + a$:

$$\int \mathrm{d}t\, \frac{t^n}{t+a} = \int \mathrm{d}\tau\, \frac{(\tau - a)^n}{\tau} \tag{1.13}$$

Then, we use the binomial expansion in the numerator:

$$\frac{(\tau - a)^n}{\tau} = \frac{(-a)^n}{\tau} + \sum_{k=0}^{n-1} \binom{n}{k} (-a)^k \tau^{n-k-1} \tag{1.14}$$

Integrating, we obtain:

$$\int \mathrm{d}\tau\, \frac{(\tau - a)^n}{\tau} = (-a)^n \log(\tau) + \sum_{k=0}^{n-1} \binom{n}{k} (-a)^k \frac{\tau^{n-k}}{n-k} \tag{1.15}$$

Replacing $\tau = t + a$ and adding integration bounds, we find:

$$\int_{t_0}^{t_1} \mathrm{d}t\, \frac{t^n}{t+a} = (-a)^n \log\left(\frac{t_1 + a}{t_0 + a}\right) + \sum_{k=0}^{n-1} \binom{n}{k} (-a)^k \frac{(t_1 + a)^{n-k} - (t_0 + a)^{n-k}}{n-k} \tag{1.16}$$

Performing a similar trick to the second integral we wish to evaluate, we find that:

$$\begin{aligned} \int \mathrm{d}t\, \frac{t^n}{(t+a)^2} &= \int \mathrm{d}\tau\, \frac{(\tau - a)^n}{\tau^2} && (1.17) \\ &= -\frac{(-a)^n}{\tau} + n(-a)^{n-1} \log(\tau) + \sum_{k=0}^{n-2} \binom{n}{k} (-a)^k \frac{\tau^{n-k-1}}{n-k-1} && (1.18) \end{aligned}$$

Plugging in $\tau = t + a$ and integrating with bounds, we obtain:

$$\int_{t_0}^{t_1} \mathrm{d}t\, \frac{t^n}{(t+a)^2} = -(-a)^n \left(\frac{1}{t_1 + a} - \frac{1}{t_0 + a} \right) + n(-a)^{n-1} \log\left(\frac{t_1 + a}{t_0 + a} \right) + \sum_{k=0}^{n-2} \binom{n}{k} (-a)^k \frac{(t_1 + a)^{n-k-1} - (t_0 + a)^{n-k-1}}{n-k-1} \tag{1.19}$$

Lastly, we can integrate over $t^n/(t+a)(t+b)$ using:

$$\frac{1}{(t+a)(t+b)} = \frac{1}{b-a} \left(\frac{1}{t+a} - \frac{1}{t+b} \right) \tag{1.20}$$

to obtain:

$$\int_{t_0}^{t_1} \mathrm{d}t\, \frac{t^n}{(t+a)(t+b)} = \frac{(-1)^n}{b-a} \log\left(\frac{(t+a)^{a^n}}{(t+b)^{b^n}} \right) + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k \frac{a^k (t+a)^{n-k} - b^k (t+b)^{(n-k)}}{(b-a)(n-k)} \tag{1.21}$$

We can also integrate over $t^n/(at^2 + bt + c)$ using:

$$at^2 + bt + c = a\left(t - \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right)\left(t - \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) \tag{1.22}$$