Martingale Markets are Inefficient

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Abstract

We present the G-score for a portfolio and develop a trading strategy for which this value monotonically increases. We show that the expected value of this algorithm has a higher expected value than a buy-and-hold strategy, which is a contradiction to the Efficient Market Hypothesis.

1 Introduction

The Efficient Market Hypothesis (EMH) has been a useful way to think about markets for over half a century. According to Fama [1970], "A market in which prices always "fully reflect" all available information is called 'efficient'."

2 Wave Arbitrage

Consider a scenario where a trader is fully invested in two equities and at time t must decide the proportion of their assets invested in each company. Let s_i represent the number of shares a portfolio holds of security i. The value of the portfolio, using s_i as the basis of measurement, is $S_i = s_i + \frac{s_j p_j}{p_i}$. However, since both S_i and S_j are both representations for the value of a portfolio, we take the geometric mean of the two and obtain the G-score:

$$G = \sqrt{\left(s_i + \frac{s_j p_j}{p_i}\right)\left(s_j + \frac{s_i p_i}{p_j}\right)} = \frac{s_i p_i + s_j p_j}{\sqrt{p_i p_j}}. \tag{1}$$

Let α_i represent the proportion of the portfolio invested in security i. At time t, the prices p_i are constant, so G has the property that for any α_i, G is a constant value. More plainly, if a trader can ignore exchange fees, then the value of G is the same both before and after any trade.

Outside of rebalance events, G is a function of the prices p_i and p_j. To find the minimum, we take the derivative with respect to p_i:

$$\begin{aligned} G &= \frac{s_i p_i + s_j p_j}{\sqrt{p_i p_j}} \\ \frac{dG}{dp_i} &= \frac{p_j(s_i p_i - s_j p_j)}{2(p_i p_j)^{\frac{3}{2}}} \end{aligned} \tag{2}$$

Assuming prices are always greater than zero, the only root is at

$$s_i p_i = s_j p_j. \tag{3}$$

This represents the global minimum. This leads to

Lemma 1 *The sequence $\{G\}$ will monotonically increase if the only actions performed is to rebalance the portfolio so that $\alpha_i = \frac{1}{2}$.*

We refer to this algorithm as "wave arbitrage" because it works by harvesting energy from waves in stock prices.

3 Disproof of the EMH

Fama [1970] describes a test that can show that a market is inefficient. The total excess market value at $t+1$ is

$$V_{t+1} = \sum_{j=1}^{n} \alpha_j(\Phi_t)[r_{j,t+1} - E(\tilde{r}_{j,t+1}|\Phi_t)]. \tag{4}$$

The value $\alpha_j(\Phi_t)$ represents the proportion of assets invested in security j, $r_{j,t+1}$ is the actual return at $t+1$, and $E(\tilde{r}_{j,t+1}|\phi_t)$ is the expected equilibrium returns given the available information Φ_t. Using the "fair game" property, meaning that price movements follow a martingale, he concludes

$$E(\tilde{V}_{t+1}|\Phi_t) = \sum_{j=1}^{n} \alpha_j(\Phi_t) E(\tilde{z}_{j,t+1}|\Phi_t) = 0. \tag{5}$$

In other words, no strategy will outperform any other strategy. Since the buy-and-hold strategy avoids all trading costs, it can be used as a benchmark.

Since Lemma 1 implies that $\{G\}$ monotonically increases under wave arbitrage, the goal is to determine whether the expected value of the portfolio also increases.

Inspecting the expected value of wave arbitrage, we let $S_i = s_i + \frac{s_j p_j}{p_i}$ represent the potential number of shares of security i that a portfolio can produce if all shares of security j are sold and the proceeds are used to buy shares of security i. Since G is the geometric mean of two quantities, we can use the inequality of arithmetic and geometric means to write

$$G = \sqrt{S_i S_j} \leq \frac{S_i + S_j}{2}. \tag{6}$$

When using wave arbitrage, since the sequence $\{G\}$ grows monotonically and diverges due to Lemma 1, the arithmetic mean of S_i and S_j will also diverge. From this we conclude that the expected number of potential shares increases:

$$E(S_{i,t+1}) > E(S_{i,t}). \tag{7}$$

Under the buy-and-hold strategy, the number of shares held does not change, but S_i is a function of the prices and can change. However, the expected returns from security i is equal to the expected returns from security j; if that were not the case, an investor could allocate all of their assets to one or the other of the securities and have an abnormally high expected return. This implies that the expected price change δ is equal for the two assets. However, if $E(p_{i,t+1}) = \delta p_{i,t}$ and $E(p_{j,t+1}) = \delta p_{j,t}$, then

$$E(S_{i,t+1}) = s_i + \frac{s_j(\delta p_{j,t})}{\delta p_{i,t}} = E(S_i). \tag{8}$$

Therefore, the expected return rate for wave arbitrage is higher than the expected return rate for buy-and-hold and Equation 5 is false.

4 Conclusion

Martingale markets are inefficient.

References

Eugene F Fama. Efficient capital markets: A review of theory and empirical work. *The journal of Finance*, 25(2):383–417, 1970.