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Correcting K-correction: dark energy is based on a math error from 1930

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ABSTRACT

We rederive the K-correction equation and discover that the version of the equation that is commonly used to calculate magnitudes contains three errors. The oldest of these errors traces back nine decades; in 1930 the equation did not correct for spectral bandwidth stretching, followed by a rederivation in 1968 that accounted for spectral bandwidth stretching but did not correct for time dilation. Two new errors were introduced in 1996, one of which causes systematic distance biases by inverting the zero-point correction term. We conclude that properly correcting apparent magnitudes for redshift resolves two of the preeminent mysteries in cosmology: first, dark energy is not supported by observations of Type Ia supernovae and second, the Hubble tension is due to the errors in the K-correction equation.

Keywords: Type Ia supernovae(1728) — Apparent magnitude(59)

1. INTRODUCTION

The estimate of Type Ia supernovae (SnIa) distances is one of the primary sources of data for cosmological models. Performing these distance estimates involves taking a measurement of redshifted light in our observation frame and then calculating the rest frame magnitude as if no redshift had occurred. It is standard to report rest frame magnitudes in the B filter, which is sensitive to blue light, regardless of the filter used to take measurements in the observation frame. The equation that converts from an observation frame magnitude to a rest frame magnitude in the B filter is called K-correction.

Tolman (1930) published the first formal derivation of K-correction, but he did not account for spectral bandwidth stretching, one of the dimming effects of redshift. An alternative K-correction was correctly derived by de Sitter (1934). However, in Hubble & Tolman (1935), the discrepancy was noted yet dismissed. Oke & Sandage (1968) rederived the K-correction equation where they accounted for spectral bandwidth stretching, but they did not account for time dilation, one of the other dimming effects of redshift. The paper by Kim et al. (1996), which is the basis for modern implementations of K-correction, extended the work of Oke & Sandage (1968) to handle additional cross-filter comparisons and added a term that was intended to address zero-point corrections. This history is explored in more depth in Section 2.

The effect of using an incorrect magnitude for SnIa is that we think objects are farther away than they really are, and the effect compounds for greater distances. Up until Riess et al. (1998), we did not have distant enough observations for this error to matter much for cosmological models, but by the late 1990s, it became clear that our measurements for distance and redshift were not linear. This realization led to a model that utilized a cosmological constant Λ and dark energy in order to explain the non-linear distance-redshift graph. We will show in Section **??** that correcting the flaws with K-correction leads to a linear graph that does not need to rely on a cosmological constant.

Planck-Collaboration et al. (2016) used an alternative technique to compute the Hubble constant H_0 that is based on measurements of the cosmic microwave background (CMB). CMB-based computations of H_0 have made it evident that something is missing with our understanding of cosmology because this technique produces a significantly different value than when the constant is derived using SnIa. As we will discuss in Section **??**, fixing the errors with K-correction produces a measurement of the Hubble constant that is compatible with models based on the CMB.

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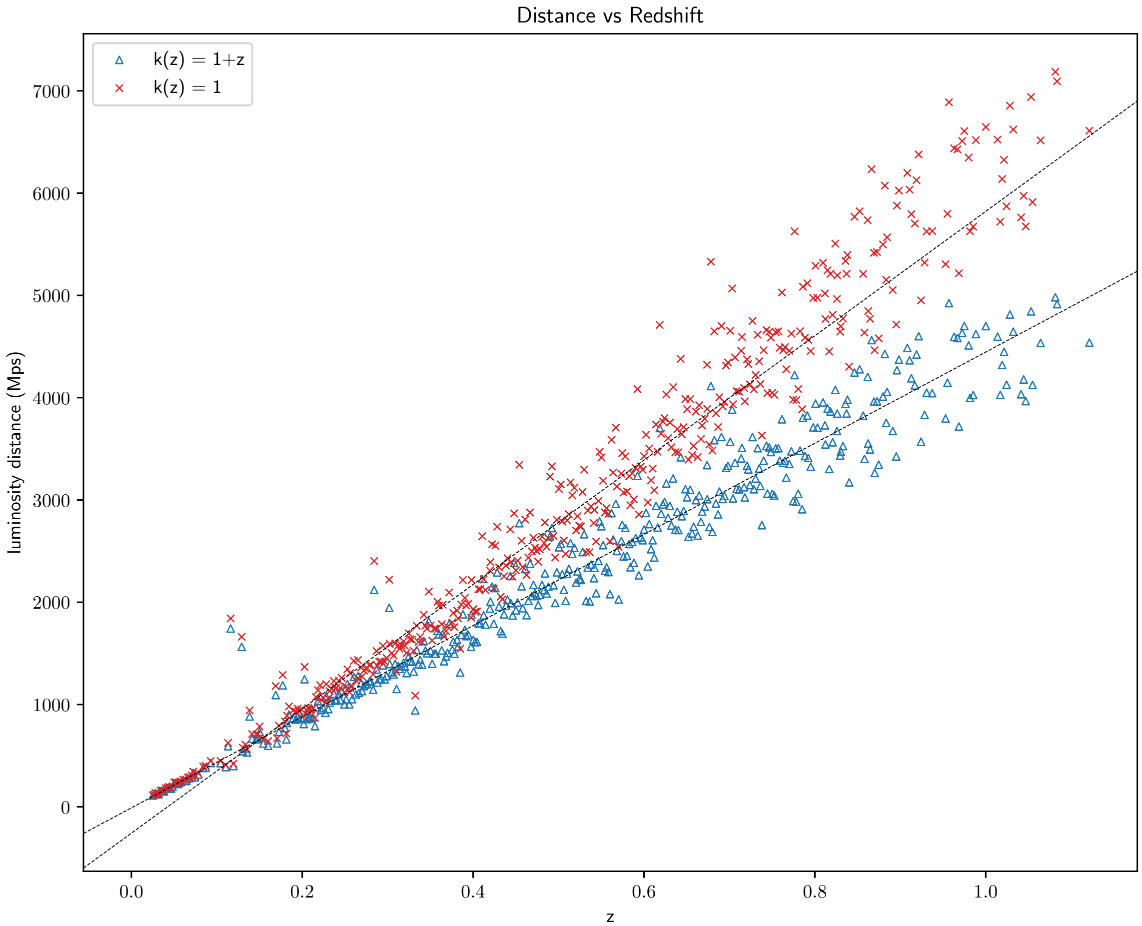

Figure 1. The relationship between luminosity distance and redshift for two treatments of magnitude data. The displayed points are roughly a third of the values in the full SnIa dataset published in DES-Collaboration et al. (2024), selected evenly to aid visibility. The red $k(z) = 1$ treatment, which represents the uncorrected values, is clearly non-linear while the blue $k(z) = 1 + z$ treatment, which fixes the K-correction redshift error, appears to be linear.

2. A BRIEF HISTORY OF K-CORRECTION

Observational data from SnIa are essential for the calibration of models that use redshift to estimate distance. The approximately linear relationship between redshift and luminosity distance, often called the Hubble-Lemaître law, describes how quickly the universe is expanding. However, Riess et al. (1998) and Perlmutter et al. (1999) presented evidence that there isn't a linear relationship between redshift and distance, but instead, distant objects are farther away than their redshift would predict (see Figure 1). This phenomenon implies that the acceleration of the universe is faster today than it was for old observations. Previously the cause of this phenomenon was unknown and was referred to as dark energy.

SnIa are used to explore the relationship between distance and redshift because these events always happen in similar ways, which keeps the absolute brightness roughly constant. An analogy is to imagine someone walking in the dark and lighting matches. If we know how brightly a match burns at a known distance, we can estimate the distance to any match by measuring the apparent brightness before applying some geometry.

However, redshifting dims the luminosity of an observation in three ways:

- The energy of a wave is inversely proportional to its wavelength; thus a redshift of z means the amount of energy per photon is reduced by a factor of $1/(1 + z)$.

- The bandwidth stretches. If the rest frame observes photons in the bandwidth from 400nm and 401nm, then a redshift of $z = 1$ stretches the wavelengths to the bandwidth from 800nm to 802nm. The number of photons per bandwidth is reduced by a factor of $1/(1+z)$.
- Cosmological time dilation reduces the rate at which photons arrive. The number of photons that arive per second is reduced by a factor of $1/(1+z)$.

To complicate matters, we use the B band magnitude to calculate luminosity distance. We want to know how bright a supernova appears for wavelengths around 445nm as if no redshift had occurred. However, we typically need to observe the supernova with a filter that is sensitive to longer wavelengths, such as the i band filter which is sensitive to wavelengths from 700nm-850nm (Flaugher et al. 2015). The K-correction formula allows us to take a magnitude measured in an observation filter y and compute the magnitude in the target filter x.

The first mathematical treatment of K-correction was performed by Tolman (1930). However, when Tolman made his derivation, he did not consider the effects of a spectrum that is stretched due to redshift.

A few years later, de Sitter (1934), discussed all three issues that reduce the observed magnitude of a distant observation. The correction for each of these issues is identical: take a measurement for luminosity and multiply it by the factor $1 + z$.

A year later, Hubble & Tolman (1935) published a similar set of calculations where they noted:

> It should be specially noted that this expression differs from the correction to m proposed by de Sitter, which contains the term $(1 + z)^3$ instead of $(1 + z)^2$. Expression (28), however, would seem to give the proper correction to use in connection with our equation (21), since it has been derived in such a way as to make appropriate allowance, first, for the double effect of nebular recession in reducing both the individual energy and the rate of arrival of photons, and then for the further circumstance that a change in spectral distribution of the energy that does arrive will lead to changes in its photographic effectiveness.

However, even though they considered all three dimming effects of redshift, they started their derivation by copying the incorrect equation from 1930. The incorrect correction term has been used ever since.

In a paper published by Oke & Sandage (1968), the two factors of $(1 + z)$ were attributed to the change in energy and to the spectral bandwidth elongation, which leaves time dilation as the factor that was omitted.

The modern treatment of K-correction is based on the work of Kim et al. (1996). This work extended the calculations of K-correction to extend to filters beyond the blue B and visible V bands. It also introduced a term that deals with the zero-point for the actual filters. Historically, bolometric devices would measure the energy flux, but modern charge-coupled device (CCD) cameras effectively measure the photon flux.

Quoting Kim et al. (1996):

> Therefore, the correct K correction calculation to be used with measured photometric magnitudes is the integral photon counts:

$$K_{xy} = -2.5\log\left(\frac{\int \lambda \mathcal{Z}(\lambda)S_x(\lambda)d\lambda}{\int \lambda \mathcal{Z}(\lambda)S_y(\lambda)d\lambda}\right) + 2.5\log(1+z) + 2.5\log\left(\frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int \lambda F(\lambda/(1+z))S_y(\lambda))d\lambda}\right). \tag{1}$$

This equation has three errors that we will explore in Section 4.

3. DERIVATION OF K-CORRECTION

With modern CCD cameras, a telescope observation consists of a single value $\mathcal{F}_y$ erg/s, which represents the energy collected per second in filter y. A summary of how this works is provided by Lesser (2015). Photogenerated electrons are collected in potential wells, which means that the measured energy is proportional to the number of photons. However, we need to calculate the expected number of photons collected by using a spectral energy density function F. The value $F(\lambda)$ gives the amount of energy collected by a bolometric device for the wavelength λ.

To convert the spectral energy density F to the photon density F', we need to use the Plank relation $E = hc/\lambda$ where E is the energy, h is Planck's constant, and c is the speed of light. This gives us

$$\begin{aligned} F(\lambda) &= F'(\lambda) \times \frac{hc}{\lambda} \\ F'(\lambda) &= \frac{\lambda F(\lambda)}{hc}. \end{aligned} \tag{2}$$

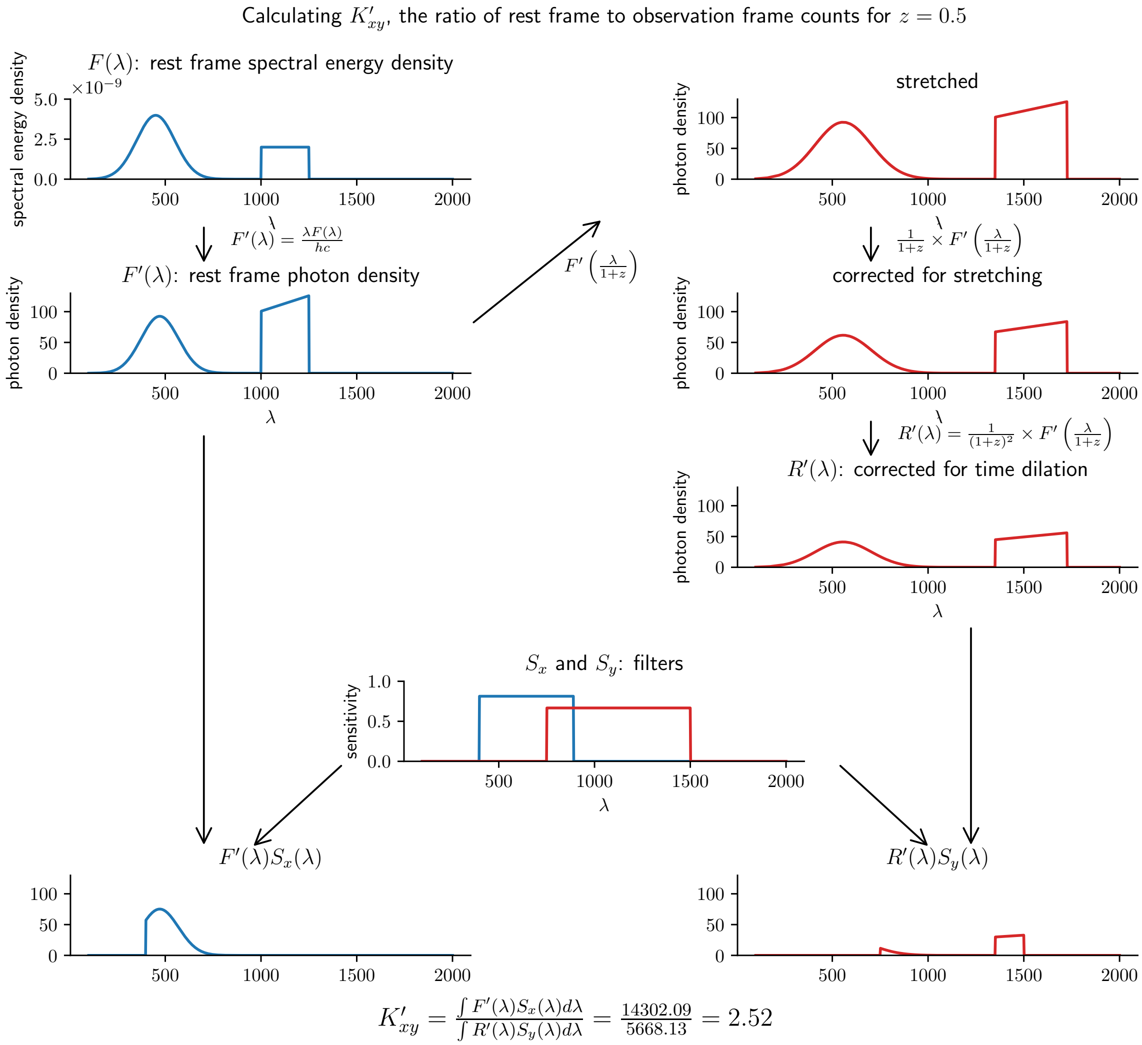

Figure 2. The ratio K'_{xy} is the crux of the K-correction algorithm. We are able to observe a SnIa in the redshifted observation frame, but we want to compute the magnitude in the rest frame as if no redshift has occurred. This computation starts with $F(\lambda)$, the spectral energy density, which we then convert to $F'(\lambda)$, the photon density for the rest frame. We show the process of computing $R'(\lambda)$, the photon density in the observation frame, in three phases starting at the top right of the figure. First we stretch the photon density, but this step inflates counts. In the second step, we correct for the inflated counts by multiplying by $1/(1+z)$; at this point, the total count equals the total count of $F'(\lambda)$. Finally, in the third step we finish calculating the observation frame photon density $R'(\lambda)$ by accounting for time dilation, which also reduces photon counts by a factor of $1/(1+z)$. We can compute the expected ratio of photons by totalling the area under the curves in the bottom two panels. This example does not show the process of converting photon flux to magnitude, nor does it address the zero-point correction.

It is important to note that for the blueshifted wavelength $\lambda/(1+z)$, this equation produces

$$F'\left(\frac{\lambda}{1+z}\right) = \frac{\lambda}{(1+z)hc} F\left(\frac{\lambda}{1+z}\right). \tag{3}$$

However, this equation is misleading and error prone. We will want to use it to help calculate the amount of flux in an observation filter at the redshifted wavelength $\lambda \times (1+z)$. In other words, we want to produce the photon density function R' that is the redshifted version of F'. Redshifting the photon density does two things:

- All wavelengths are increased by a factor of $1+z$. When we integrate R' from wavelength λ_a to wavelength λ_b, the values correspond to the wavelengths $\lambda_a/(1+z)$ to $\lambda_b/(1+z)$ in F'. We will integrate over a width of $\lambda_b - \lambda_a$, but $F'(\lambda/(1+z))$ will refer to values in a width of $(\lambda_b - \lambda_a)/(1+z)$.

In order to account for this spectral bandwidth stretching effect, we will need to multiply F' by a second factor of $1/(1+z)$.

- The stretching of space increases the distance between photons while they are traveling. This phenomenon appears to an observer like time dilation, although cosmological time dilation is due to a different mechanism than relativistic time dilation. This effect reduces the photon arrival rate by a factor of $1/(1+z)$.

  In order to account for cosmological time dilation, we will need to multiply F' by $1/(1+z)$.

Combining these two phenomena, we can calculate the redshifted spectral energy density R using

$$\begin{aligned} R'(\lambda) &= F'\left(\frac{\lambda}{1+z}\right) \times \frac{1}{(1+z)^2} \\ R(\lambda) &= \frac{\lambda}{(1+z)hc} \times F\left(\frac{\lambda}{1+z}\right) \times \frac{1}{(1+z)^2}. \end{aligned} \tag{4}$$

We deliberately do not combine all of the factors of $1+z$ together because this form is more natural to implement in code.

In order to calculate the amount of flux $\mathcal{F}_x$ measured in filter x, we need to compute the photon density $F'(\lambda)$ and multiply it by sensitivity $S_x(\lambda)$, which represents the proportion of photons filter x will measure at wavelength λ. We then need to sum over all wavelengths, which is expressed with the equation

$$\begin{aligned} \mathcal{F}_x &= \int F'(\lambda) S_x(\lambda) d\lambda \\ &= \int \lambda F(\lambda) S_x(\lambda) d\lambda. \end{aligned} \tag{5}$$

The limits of integration are technically from 0 to ∞, but these are usually not written because the sensitivity $S(\lambda)$ is 0 for wavelengths outside of a filter's bandpass.

We will also use the energy flux $\mathcal{F}$ to magnitude m formula:

$$\begin{aligned} m_x &= -2.5\log(\mathcal{F}_x) + P_x \\ -2.5\log(\mathcal{F}_x) &= m_x - P_x. \end{aligned} \tag{6}$$

P_x represents the zero-point for the filter x on some particular telescope. In order to use consistent magnitude values across telescopes that have different light gathering abilities, we take the measured magnitude and multiply it by the ratio of the standard flux rate to the flux rate for this particular telescope and filter. For convenience, we use $P_x = -2.5\log(P'_x)$ so that we can work with flux instead of with magnitude.

Now that we have the identities in Equations 5 and 6, we will change directions and look at the definition of K-correction K_{xy}. This value allows us to make an observation in filter y and report what the magnitude would have been in filter x if no redshift occurred:

$$\begin{aligned} m_y &= M_x + \mu + K_{xy} \\ &= M_x + m_x - M_x + K_{xy} \\ &= m_x + K_{xy} \\ m_x &= m_y - K_{xy}. \end{aligned} \tag{7}$$

The second line of Equation 7 expands the distance modulus μ using $\mu = m - M$ where m is the observed magnitude and M is the absolute magnitude.

Since the K-correction is a magnitude value and we wish to work on flux values, it is convenient to define the following substitution:

$$K_{xy} = 2.5\log(K'_{xy}). \tag{8}$$

Note that this substitution omits the minus $(-)$ sign that we used on a similar substitution for P_x.

Starting with Equation 6 and then recombining the flux term $\mathcal{F}_y$ with m_y from Equation 7, we have

$$
\begin{aligned}
-2.5\log(\mathcal{F}_x) &= m_x - P_x \\
&= m_y - K_{xy} - P_x \\
&= -2.5\log(\mathcal{F}_y) + P_y - K_{xy} - P_x \\
&= -2.5\log(\mathcal{F}_y) - 2.5\log(P'_y) - 2.5\log(K'_{xy}) + 2.5\log(P_x) \\
&= -2.5\left(\log(\mathcal{F}_y) + \log(K'_{xy}) + \log(P'_y) - \log(P'_x)\right) \\
&= -2.5\log\left(\mathcal{F}_y \times K'_{xy} \times \frac{P'_y}{P'_x}\right) \\
\mathcal{F}_x &= \mathcal{F}_y \times K'_{xy} \times \frac{P'_y}{P'_x}.
\end{aligned}
\tag{9}
$$

It is useful to consider whether the term P'_y/P'_x is correct or if it should be inverted, but if we rewrite Equation 9 as

$$\mathcal{F}_x \times P'_x = \mathcal{F}_y \times P'_y \times K'_{xy} \tag{10}$$

then the meaning becomes more clear. The left side, $\mathcal{F}_x \times P'_x$, is the rate of photon collection for an idealized telescope in filter x while $\mathcal{F}_y \times P'_y$ is the rate of photon collection for an idealized telescope in filter y. We observe the value $\mathcal{F}_y \times P'_y$ and want to use the fudge factor K'_{xy} to produce the value $\mathcal{F}_x \times P'_x$.

We can isolate K'_{xy} and then use Equation 5 to expand

$$
\begin{aligned}
\mathcal{F}_x &= \mathcal{F}_y \times K'_{xy} \times \frac{P'_y}{P'_x} \\
K'_{xy} &= \frac{P'_x}{P'_y} \times \frac{\mathcal{F}_x}{\mathcal{F}_y}.
\end{aligned}
\tag{11}
$$

We now use Equations 4 and 5 to calculate the fluxes $\mathcal{F}_x$ and $\mathcal{F}_y$ in terms of the spectral energy density function F. Note that $\mathcal{F}_x$ uses the rest frame spectral energy density F, while $\mathcal{F}_y$ uses the redshifted spectral energy density $R(\lambda)$:

$$
\begin{aligned}
K'_{xy} &= \frac{P'_x}{P'_y} \times \frac{\mathcal{F}_x}{\mathcal{F}_y} \\
&= \frac{P'_x}{P'_y} \times \frac{\int F'(\lambda)S_x(\lambda)d\lambda}{\int R'(\lambda)S_y(\lambda)d\lambda} \\
&= \frac{P'_x}{P'_y} \times \frac{\int F'(\lambda)S_x(\lambda)d\lambda}{\int F'(\lambda/(1+z)) \times (1/(1+z)^2)S_y(\lambda)d\lambda} \\
&= \frac{P'_x}{P'_y} \times (1+z)^2 \times \frac{\int(\lambda/hc)F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/((1+z)hc))F(\lambda/(1+z))S_y(\lambda)d\lambda} \\
&= \frac{P'_x}{P'_y} \times (1+z)^2 \times \frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z)F(\lambda/(1+z))S_y(\lambda)d\lambda}.
\end{aligned}
\tag{12}
$$

Finally, we can use Equation 8 to convert K'_{xy} back into the magnitude K_{xy}:

$$
\begin{aligned}
K_{xy} &= 2.5\log(K'_{xy}) \\
&= 2.5\log\left(\frac{P'_x}{P'_y} \times (1+z)^2 \times \frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z))F(\lambda/(1+z))S_y(\lambda)d\lambda}\right) \\
&= 2.5\left(\log\left(\frac{P'_x}{P'_y}\right) + \log((1+z)^2) + \log\left(\frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z))F((\lambda)/(1+z))S_y(\lambda)d\lambda}\right)\right) \\
&= 2.5\log\left(\frac{P'_x}{P'_y}\right) + 5\log(1+z) + 2.5\log\left(\frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z))F(\lambda/(1+z))S_y(\lambda)d\lambda}\right) \\
&= 5\log(1+z) + 2.5\log\left(\frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z))F(\lambda/(1+z))S_y(\lambda)d\lambda}\right) - P_x + P_y.
\end{aligned}
\tag{13}
$$

4. CONSEQUENCES

In order to fully compare our derivation against Equation 1, we need to use the identity

$$P = -2.5\log\left(\int \frac{\lambda}{hc}\mathcal{Z}(\lambda)S(\lambda)d\lambda\right) \tag{14}$$

where, according to Kim et al. (1996), "$\mathcal{Z}(\lambda)$ is an idealized spectral energy distribution at $z = 0$ for which $U = B = V = R = I = 0$ in the photometric system being used." Combining this with Equation 13, we have

$$\begin{aligned} K_{xy} &= 5\log(1+z) + 2.5\log\left(\frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z))F(\lambda/(1+z))S_y(\lambda)d\lambda}\right) - P_x + P_y \\ &= 5\log(1+z) + 2.5\log\left(\frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z))F(\lambda/(1+z))S_y(\lambda)d\lambda}\right) \\ &\quad + 2.5\log\left(\int(\lambda/hc)\mathcal{Z}(\lambda)S_x(\lambda)d\lambda\right) - 2.5\log\left(\int(\lambda/hc)\mathcal{Z}(\lambda)S_y(\lambda)d\lambda\right) \\ &= 2.5\log\left(\frac{\int\lambda\mathcal{Z}(\lambda)S_x(\lambda)d\lambda}{\int\lambda\mathcal{Z}(\lambda)S_y(\lambda)d\lambda}\right) + 5\log(1+z) + 2.5\log\left(\frac{\int \lambda F(\lambda)S_x(\lambda)d\lambda}{\int(\lambda/(1+z))F(\lambda/(1+z))S_y(\lambda)d\lambda}\right). \end{aligned} \tag{15}$$

Three differences with Equation 1 stand out:

- The sign is different for the first term. This will have a subtle but frustrating effect for real measurements. The zero-point values are usually, but not always, close to zero. As we observe more distant supernovae, it is useful to switch our observational filter to a filter that is sensitive to longer wavelengths. Each observation filter will have its own small bias.

  This effect by itself can alter estimates of H_0 by several percent. While we present an estimate of the Hubble constant in Figure 4, we are not terribly confident in the numbers because we were unable to correct for the zero-point errors, or verify that these errors may not exist.

- The second term is multiplied by 2.5 in Kim et al. (1996), but is multiplied by 5 here. This is the manifestation of the error in Tolman (1930).

- In our derivation, the third term has an extra $1 + z$ expression. Based on an inspection of the SN(oo)py software package presented by Burns (2024), this error is ignored and software package authors implement it as intended, not precisely as written.

5. CONCLUSIONS

As shown in Figure 1, when reported magnitudes are corrected by adding the magnitude $-2.5\log(1 + z)$, there is a linear relationship between redshift and luminosity distance. This is in accordance with the Hubble-Lemaître law. As opposed to the uncorrected values, there is no visual acceleration. Since the corrected values are approximately linear, we can use them to estimate the Hubble constant H_0. This is demonstrated in Figure 3. Furthermore, we performed a bootstrap calculation to estimate the distribution of H_0 estimates in Figure 4.

The value estimated here, $H_0 = 65.94 \pm 1.3$ is highly suspicious because of the flipped zero-point corrections, but it is consistent with estimates of H_0 that are based on the CMB. Planck-Collaboration et al. (2020) published the CMB value of $H_0 = 67.27 \pm 0.6$, and the one sigma error bars overlap. It is interesting to note that restricting the Dark Energy Survey observations of SnIa (Vincenzi et al. 2024) to only those with $z < 0.1$, which presumably is a small enough redshift that all observations will have been made with the same filter, gives the estimate $H_0 = 67.8 \pm 1.9$.

Since the K-correction error has corrupted all astronomical measurements that depend on both magnitude and redshift, a lot of data needs to be reanalysed. Cosmological parameters, such as those used by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric or the Λ-CDM model, can only be recalculated once the data is cleaned.

The reasons that the K-correction errors went undiscovered for so long should also be explored. Many hints existed that something was wrong, yet the problems persisted.

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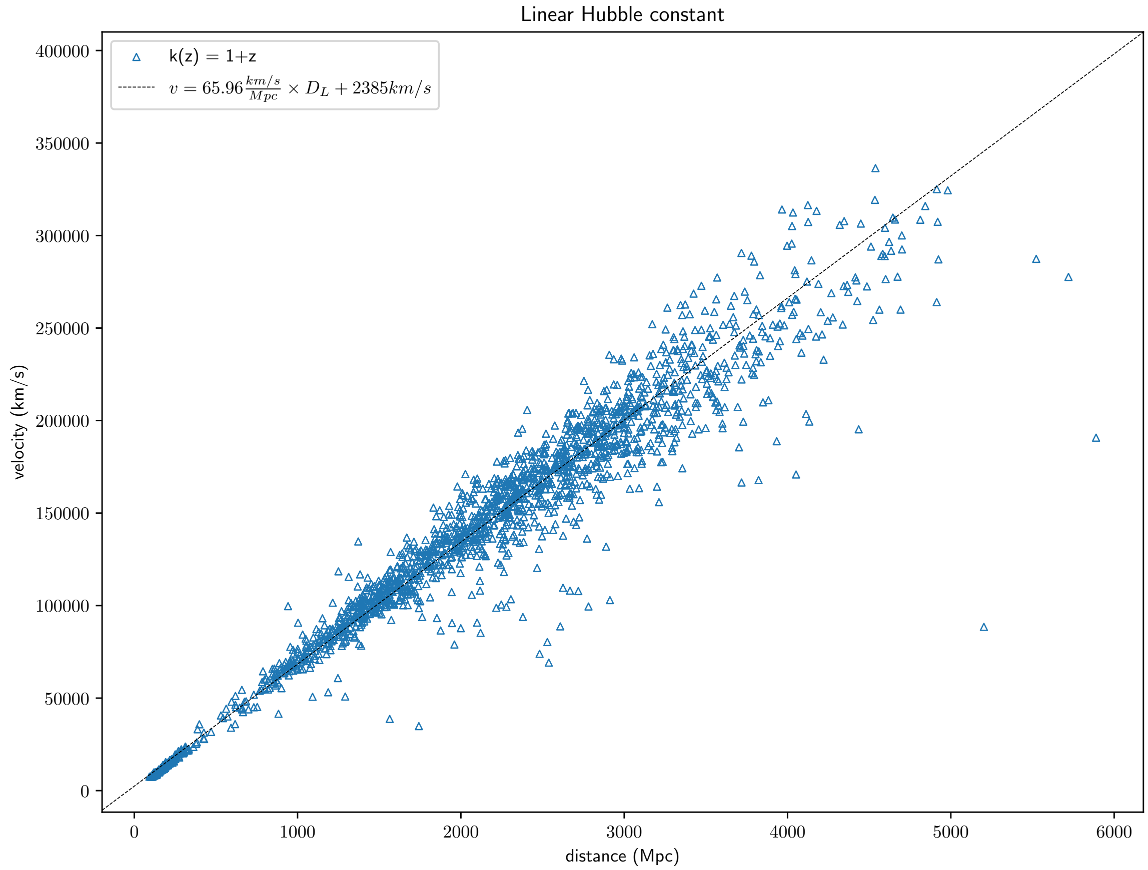

Figure 3. The relationship between expansion velocity and distance. The slope of this graph demonstrates the Hubble constant. The data for these SnIa observations comes from the Dark Energy Spectroscopic Instrument (Vincenzi et al. 2024), but the magnitudes are corrected to account for time dilation by adding $-2.5log(1+z)$.

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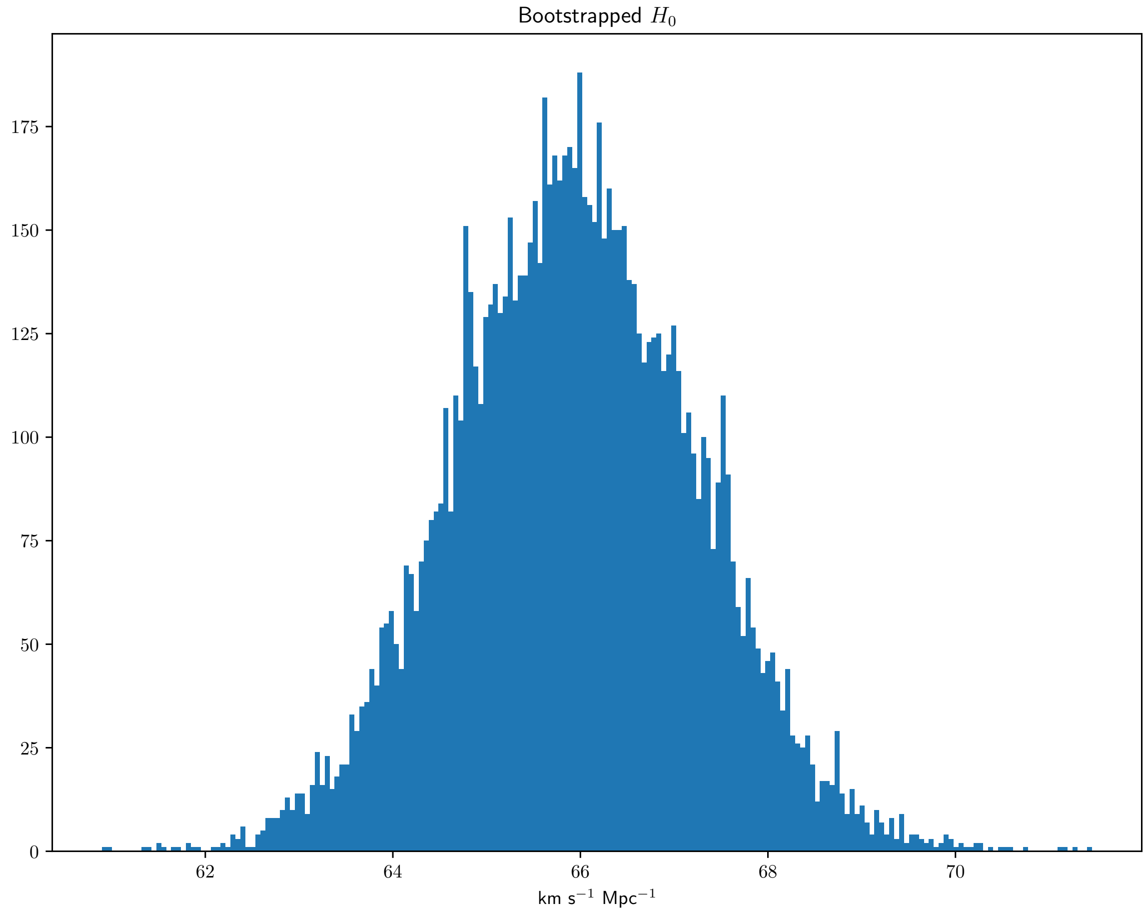

Figure 4. A histogram of 10,000 bootstrap trials measuring the Hubble constant H_0. Each trial samples an absolute magnitude $M \sim Norm(-19.2334, 0.0404)$ for SnIa based on data published by Camarena & Marra (2020). It then samples, with replacement, a population of supernovae from the dataset published by DES-Collaboration et al. (2024). Finally, it uses the non-parametric linear regression technique described by Siegel (1982). The result of the bootstrap is $H_0 \sim Norm(65.94, 1.3^2)$.