

# ASYMPTOTICS AND INEQUALITIES FOR THE DISTINCT PARTITION FUNCTION

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**ABSTRACT.** In this paper, we give explicit error bounds for the asymptotic expansion of the shifted distinct partition function  $q(n+s)$  for any nonnegative integer  $s$ . Then based on this refined asymptotic formula, we give the exact thresholds of  $n$  for the inequalities derived from the invariants of the quartic binary form, the double Turán inequalities, the Laguerre inequalities and their corresponding companion versions.

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## 1. INTRODUCTION

The objective of this paper is to derive an asymptotic expansion for the distinct partition function and further show that this function satisfies the inequalities derived from the invariants of the quartic binary form, the double Turán inequalities, the Laguerre inequalities and their companion versions.

A sequence  $\{\alpha_n\}_{n \geq 0}$  is said to satisfy the double Turán inequalities if for  $n \geq 2$ ,

$$(\alpha_n^2 - \alpha_{n-1}\alpha_{n+1})^2 \geq (\alpha_{n-1}^2 - \alpha_{n-2}\alpha_n)(\alpha_{n+1}^2 - \alpha_n\alpha_{n+2}).$$

Additionally, the sequence satisfies Laguerre inequalities of order  $m$  if

$$\frac{1}{2} \sum_{k=0}^{2m} (-1)^{k+m} \binom{2m}{k} \alpha_{n+k} \alpha_{2m-k+n} \geq 0.$$

It should be noted that the study of the Turán-type and Laguerre inequalities is closely related to the Laguerre-Pólya class and the Riemann hypothesis, see, for example, [9, 11, 12, 22, 24]. Consequently, this area has received widespread attention. Many scholars have

been investigating the Turán-type and their companion inequalities [6, 10, 14], as well as the Laguerre and corresponding companion inequalities [25, 26, 27] for different kind of sequences.

A comprehensive study on inequalities arising from invariants of a binary form was studied by Chen [5]. For the background on the theory of invariants, see, for example, Hilbert [16], Kung and Rota [19] and Sturmfels [23]. A binary form  $P_d(x, y)$  of degree  $d$  is a homogeneous polynomial of degree  $d$  in two variables  $x$  and  $y$ , defined by

$$P_d(x, y) := \sum_{i=0}^d \binom{d}{i} a_i x^i y^{d-i},$$

where the coefficients  $a_i$  are complex numbers. Restricting  $a_i$  to be real numbers, the binary form  $P_d(x, y)$  is transformed into a new binary form

$$Q_d(\bar{x}, \bar{y}) = \sum_{i=0}^d \binom{d}{i} c_i \bar{x}^i \bar{y}^{d-i}$$

under the action of an invertible complex matrix  $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$  as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.$$

The transformed coefficients  $c_i$  are polynomials in  $a_i$  and  $m_{ij}$ . For nonnegative integer  $k$ , a polynomial  $I(a_0, a_1, \dots, a_d)$  in the coefficients  $(a_i)_{0 \leq i \leq d}$  is called an invariant of index of  $k$  of the binary form  $P_d(x, y)$  if for any invertible matrix  $M$ ,

$$I(\bar{a}_0, \bar{a}_1, \dots, \bar{a}_d) = (\det M)^k I(a_0, a_1, \dots, a_d).$$

We observe that three invariants of the quartic binary form

$$P_4(x, y) = a_4 x^4 + 4a_3 x^3 y + 6a_2 x^2 y^2 + 4a_1 x y^3 + a_0 y^4$$

are of the following form

$$\begin{aligned} A(a_0, a_1, a_2, a_3, a_4) &= a_0 a_4 - 4a_1 a_3 + 3a_2^2, \\ B(a_0, a_1, a_2, a_3, a_4) &= -a_0 a_2 a_4 + a_2^3 + a_0 a_3^2 + a_1^2 a_4 - 2a_1 a_2 a_3, \\ I(a_0, a_1, a_2, a_3, a_4) &= A(a_0, a_1, a_2, a_3, a_4)^3 - 27B(a_0, a_1, a_2, a_3, a_4)^2. \end{aligned}$$

Chen [5] conjectured that both the partition function and the spt-function satisfy the inequalities derived from the invariants of the quartic binary form and the associated companion inequalities. For the definition of spt-function, we refer to Andrews [1]. Chen's conjectures related to  $A(a_0, a_1, a_2, a_3, a_4)$  and  $B(a_0, a_1, a_2, a_3, a_4)$  have recently been proved by Banerjee [3], Jia and Wang [18] and Wang and Yang [26].

Recently, inequalities for the distinct partition function have also received significant attention from many researchers. Recall that a distinct partition of integer  $n$  is a partition of

$n$  into distinct parts. For example, there are eight distinct partitions of 9:

$$(9), (8, 1), (7, 2), (6, 3), (6, 2, 1), (5, 4), (5, 3, 1), (4, 3, 2).$$

Let  $q(n)$  denote the number of distinct partitions of  $n$ . Hagis [15] and Hua [17] established a Hardy-Ramanujan-Rademacher type formula for  $q(n)$  by applying the Hardy-Ramanujan circle method. Applying this formula, Craig and Pun [8] showed that  $q(n)$  satisfies the Turán inequalities of any order for sufficiently large  $n$  by using a general result of Griffin, Ono, Rolen, and Zagier [14]. They also conjectured that  $q(n)$  is log-concave for  $n \geq 33$  and satisfies the third-order Turán inequalities for  $n \geq 121$ . Instead of using the Hardy-Ramanujan-Rademacher type formula for  $q(n)$  due to Hagis [15] and Hua [17], Dong and Ji [13] used Chern's asymptotic formulas for  $\eta$ -quotients [7] to obtain an asymptotic formula for  $q(n)$  with an effective bound on the error term. Based on this asymptotic formula, they proved Craig and Pun's [8] conjectures. But they couldn't prove the inequalities pertaining to the invariants of the quartic binary form for  $q(n)$  and stated them as conjectures [13, Conjecture 6.1]. In this paper, we derive an asymptotic expansion of the shifted distinct partition function  $q(n+s)$  for any nonnegative integer  $s$ . Subsequently, we prove the first two conjectures of Dong and Ji [13] and establish their associated companion inequalities.

To state our asymptotic expansion, we need to introduce some definitions. Here and throughout, we use the notation  $f(n) = O_{\leq C}(g(n))$  to mean  $|f(x)| \leq C \cdot g(x)$  for a positive function  $g$  and the domain for  $x$  will be specified accordingly. For  $(s, N) \in \mathbb{N}_0 \times \mathbb{N}$ , we define

$$n(N, s) := \max \left\{ 206, \frac{\frac{72}{\pi^2} N_0^2 (N+2) - 1}{24}, \left\lceil \frac{2(24s+1)}{3} \right\rceil \right\}, \quad (1.1)$$

$$\begin{aligned} Er_N(s) &:= |a_N(1)| \left( \frac{\pi \cdot 2^{N-1}}{\sqrt{3}} \sqrt{\frac{24s+1}{24}} + A(s) \left( 1 + \frac{2^{N+1}}{3} \right) \right) \left( \frac{24s+1}{24} \right)^{\frac{N+1}{2}} \\ &+ \left( 1 + \frac{\pi}{2\sqrt{3}} \frac{24s+1}{24} + \frac{A(s)}{12} \right) Er_6(N, s) + 2|a_N(1)| Er_4(N, s) + \frac{Er_4(N, s) Er_6(N, s)}{n(N, s)^{\frac{N+1}{2}}}, \end{aligned} \quad (1.2)$$

where  $a_m(v)$ ,  $Er_3(N, s)$ ,  $Er_4(N, s)$ ,  $Er_6(N, s)$ , and  $A(s)$  are as defined in (2.8), (3.11), (3.13), (3.25), and (4.6) respectively.

**Theorem 1.1.** *For all  $(s, N) \in \mathbb{N}_0 \times \mathbb{N}$  and  $n \geq n(N, s)$ , we have*

$$q(n+s) = \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{1/4} n^{3/4}} \left( \sum_{m=0}^N \frac{\widehat{B}_m(s)}{n^{\frac{m}{2}}} + O_{\leq Er_N(s)} \left( n^{-\frac{N+1}{2}} \right) \right),$$

where the coefficient sequence  $(\widehat{B}_m(s))_{m \geq 0}$  is given explicitly in (4.1).

In light of Theorem 1.1, we obtain the following inequalities by taking appropriate values of  $N$ .

**Theorem 1.2.** *For  $n \geq 230$ , we have*

$$A(q(n-1), q(n), q(n+1), q(n+2), q(n+3)) > 0.$$

**Theorem 1.3.** For  $n \geq 279$ , we have

$$4 \left( 1 + \frac{\pi^2}{32n^3} \right) q(n)q(n+2) > q(n-1)q(n+3) + 3q(n+1)^2.$$

**Theorem 1.4.** For  $n \geq 272$ , we have

$$B(q(n-1), q(n), q(n+1), q(n+2), q(n+3)) > 0.$$

**Theorem 1.5.** For  $n \geq 309$ , we have

$$\begin{aligned} \left( 1 + \frac{\pi^3}{288\sqrt{3}n^{\frac{9}{2}}} \right) (2q(n)q(n+1)q(n+2) + q(n-1)q(n+1)q(n+3)) \\ > q(n+1)^3 + q(n-1)q(n+2)^2 + q(n)^2q(n+3). \end{aligned}$$

**Theorem 1.6.** For  $n \geq 273$ ,  $q(n)$  satisfies the double Turán inequalities, i.e.,

$$(q(n)^2 - q(n-1)q(n+1))^2 > (q(n-1)^2 - q(n-2)q(n)) (q(n+1)^2 - q(n)q(n+2)).$$

**Theorem 1.7.** For  $n \geq 346$ ,  $q(n)$  satisfies the companion double Turán inequalities, i.e.,

$$\begin{aligned} (q(n)^2 - q(n-1)q(n+1))^2 \\ < (q(n-1)^2 - q(n-2)q(n)) (q(n+1)^2 - q(n)q(n+2)) \left( 1 + \frac{\pi}{2\sqrt{3}n^{\frac{3}{2}}} \right). \end{aligned}$$

More recently, similar to the general result of Griffin, Ono, Rolen, and Zagier [14] for the Turán inequalities of any order, Wang and Yang [27] gave a criteria for the Laguerre inequalities of any order. It is worth noting that if a sequence satisfies the conditions of Griffin, Ono, Rolen, and Zagier [14], then it also satisfies the conditions of Wang and Yang [27]. Craig and Pun [8] have showed that  $q(n)$  satisfies the criteria of Griffin, Ono, Rolen, and Zagier [14]. This directly implies the following result.

**Corollary 1.8.** For sufficiently large  $n$ ,  $q(n)$  satisfies the Laguerre inequalities of any order.

With the aid of Theorem 1.1, we determine the threshold for the Laguerre inequalities of order 3 and their companion inequalities. In fact, the Laguerre inequalities of order 2 is equivalent to the inequality  $A(a_0, a_1, a_2, a_3, a_4) \geq 0$ . The findings regarding the third-order Laguerre inequalities are presented as follows.

**Theorem 1.9.** For  $n \geq 651$ ,  $q(n)$  satisfies the Laguerre inequalities of order 3, i.e.,

$$10q(n+3)^2 + 6q(n+1)q(n+5) > 15q(n+2)q(n+4) + q(n)q(n+6).$$

**Theorem 1.10.** For  $n \geq 715$ ,  $q(n)$  satisfies the companion Laguerre inequalities of order 3, i.e.,

$$\begin{aligned} 10q(n+3)^2 + 6q(n+1)q(n+5) \\ < (15q(n+2)q(n+4) + q(n)q(n+6)) \left( 1 + \frac{5\pi^3}{256\sqrt{3}n^{\frac{9}{2}}} \right). \end{aligned}$$

The paper is organized as follows. In Section 2, we establish the upper and lower bounds for  $q(n)$ , which in turn furnish an asymptotic expansion of  $q(n+s)$ . In Section 3, we calculate each term of the expansion separately by Taylor's theorem. These enable us to prove Theorem 1.1 in Section 4. Finally, in Section 5, we prove all the inequalities by applying Theorem 1.1.

## 2. SET UP

In this section, by giving an upper bound and a lower bound for  $q(n)$ , and along with the estimations of the first modified Bessel function of the first kind, we shall obtain a preliminary asymptotic expansion of  $q(n+s)$ .

Define

$$N_0(m) := \begin{cases} 1, & \text{if } m = 1, \\ 4m \log m - 3m \log \log m, & \text{if } m \geq 2, \end{cases} \quad (2.1)$$

and

$$M(n) := \frac{\sqrt{2}\pi^2}{12\nu(n)} I_1(\nu(n)), \quad (2.2)$$

where  $I_1(s)$  is the first modified Bessel function of the first kind and

$$\nu(n) := \frac{\pi\sqrt{24n+1}}{6\sqrt{2}}.$$

**Lemma 2.1.** *For  $\nu(n) \geq \max\{26, N_0(m+1)\}$ , we have*

$$M(n) \left(1 - \frac{4}{\nu(n)^m}\right) \leq q(n) \leq M(n) \left(1 + \frac{4}{\nu(n)^m}\right). \quad (2.3)$$

*Proof.* From [13, Theorem 1.2], we know that for  $\nu(n) \geq 21$ ,

$$q(n) = \frac{\sqrt{2}\pi^2}{12\nu(n)} I_1(\nu(n)) + R(n),$$

where

$$|R(n)| \leq \frac{\sqrt{3}\pi^{\frac{3}{2}}}{6\nu^{\frac{1}{2}}} \exp\left(\frac{\nu(n)}{3}\right).$$

As a result, let

$$G(n) := \sqrt{\frac{6\nu(n)}{\pi}} \cdot \frac{\exp\left(\frac{\nu(n)}{3}\right)}{I_1(\nu(n))},$$

we have

$$M(n)(1 - G(n)) \leq q(n) \leq M(n)(1 + G(n)).$$

Hence, to verify (2.3), it suffices to show that for  $\nu(n) \geq \max\{26, N_0(m+1)\}$ ,

$$G(n) \leq \frac{4}{\nu(n)^m}. \quad (2.4)$$

Dong and Ji [13, Eq.(3.16)] gave that for  $\nu(n) \geq 26$ ,

$$G(n) \leq 2\sqrt{3}\nu(n) \left(1 + \frac{1}{\nu(n)}\right) \exp\left(-\frac{2\nu(n)}{3}\right). \quad (2.5)$$

It's easy to check that for  $\nu(n) \geq 7$ ,

$$2\sqrt{3}\nu(n) \left(1 + \frac{1}{\nu(n)}\right) \leq 4\nu(n). \quad (2.6)$$

Analogous to the proof of [21, Lemma 2.1], for  $x \geq N_0(m)$ , we have

$$\exp\left(-\frac{2x}{3}\right) < x^{-m}.$$

It follows that for  $\nu(n) \geq N_0(m+1)$ ,

$$\exp\left(-\frac{2\nu(n)}{3}\right) < \frac{1}{\nu(n)^{m+1}}. \quad (2.7)$$

Plugging (2.6) and (2.7) into (2.5), we get (2.4) and thus complete the proof.  $\square$

In view of [2, Theorem 3.9], for  $N \geq 1$ , we have

$$\left| \frac{\sqrt{2\pi x}}{e^x} I_1(x) - \sum_{m=0}^N \frac{(-1)^m a_m(1)}{x^m} \right| < \frac{Er_{N,1}}{x^{N+1}},$$

where

$$a_m(v) := \frac{\binom{v-\frac{1}{2}}{m} \binom{v+\frac{1}{2}}{m}_m}{2^m}, \quad (2.8)$$

$$Er_{N,1} := \frac{3^{\frac{N+1}{2}}}{\pi^{N+1}} \left( \frac{1 + \frac{9}{\log(N+1)} + \frac{9}{N+2}}{\sqrt{2\pi}} + \frac{\sqrt{2} + (N + \frac{5}{2})^{-\frac{1}{2}}}{\log(N+1)} \right) |a_{N+1}(1)|. \quad (2.9)$$

Combining with Lemma 2.1, for  $(s, N) \in \mathbb{N}_0 \times \mathbb{N}$  and  $\nu(n) \geq \max\{26, N_0(N+2)\}$ , we have

$$\begin{aligned} q(n+s) &= M(n+s) \left( 1 + O_{\leq 4} \left( \frac{3^{\frac{N+1}{2}}}{\pi^{N+1}} n^{-\frac{N+1}{2}} \right) \right) \\ &= \frac{e^{\pi\sqrt{\frac{n}{3}}} e^{\pi\sqrt{\frac{n}{3}} \left( \sqrt{1 + \frac{24s+1}{24n}} - 1 \right)}}{4 \cdot 3^{1/4} n^{3/4} \left( 1 + \frac{24s+1}{24n} \right)^{3/4}} \left( \sum_{m=0}^N \frac{(-1)^m a_m(1)}{(\frac{\pi}{\sqrt{3}})^m n^{\frac{m}{2}}} \left( 1 + \frac{24s+1}{24n} \right)^{-\frac{m}{2}} \right. \\ &\quad \left. + O_{Er_{N,1}} \left( n^{-\frac{N+1}{2}} \right) \right) \left( 1 + O_{\leq 4} \left( \frac{3^{\frac{N+1}{2}}}{\pi^{N+1}} n^{-\frac{N+1}{2}} \right) \right). \end{aligned} \quad (2.10)$$

### 3. PRELIMINARY AND PREPARATORY LEMMAS

In this section, through the application of Taylor's theorem, we separately compute each term of the preliminary asymptotic expansion of  $q(n+s)$  given in (2.10), and these estimations are indispensable for the proof of Theorem 1.1. To this end, we need the following result.

**Lemma 3.1.** [4, Lemma 3.3] *For  $r, m \in \mathbb{N}_0$  with  $r < 2m$ , we have*

$$\sum_{s=0}^r (-1)^s \binom{r}{s} \binom{\frac{s}{2}}{m} = \begin{cases} 1, & \text{if } r = m = 0, \\ (-1)^m \frac{r \cdot 2^r}{m \cdot 2^{2m}} \binom{2m-r-1}{m-r}, & \text{otherwise.} \end{cases}$$

Following the work done in [20], we have the following preparatory Lemmas 3.2 - 3.6. Before that, for  $k \in \mathbb{N}_0$ , define

$$B_{2k}(s) := \begin{cases} 1, & \text{if } k = 0, \\ \frac{\left(\frac{24s+1}{24}\right)^k \left(\frac{1}{2}-k\right)_{k+1}}{k!} \sum_{\ell_1=1}^k \frac{(-k)_{\ell_1}}{(k+\ell_1)!} \frac{\left(\pi\sqrt{\frac{24s+1}{72}}\right)^{2\ell_1}}{(2\ell_1-1)!}, & \text{if } k \geq 1, \end{cases}$$

and

$$B_{2k+1}(s) := \frac{\pi}{\sqrt{3}} \left(\frac{24s+1}{24}\right)^{k+1} \left(\frac{1}{2}-k\right)_{k+1} \sum_{\ell_1=0}^k \frac{(-k)_{\ell_1}}{(\ell_1+k+1)!} \frac{\left(\pi\sqrt{\frac{24s+1}{72}}\right)^{2\ell_1}}{(2\ell_1)!}. \quad (3.1)$$

Moreover, we let

$$Er_2(N, s) := \frac{4}{3} \sqrt{\frac{2\pi}{3}} N^{-\frac{3}{2}} \left(\frac{24s+1}{24}\right)^{\frac{N+2}{2}} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right). \quad (3.2)$$

**Lemma 3.2.** *For  $N \geq 1$  and  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ , we have*

$$e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)} = \sum_{k=0}^N \frac{B_k(s)}{n^{\frac{k}{2}}} + O_{\leq Er_2(N, s)}\left(n^{-\frac{N+1}{2}}\right),$$

*Proof.* By Taylor expansion, we have

$$\begin{aligned} e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)} &= \sum_{r_1=0}^{\infty} \frac{\left(\frac{\pi\sqrt{n}}{3}\right)^{r_1}}{r_1!} \left(\sqrt{1+\frac{24s+1}{24n}} - 1\right)^{r_1} \\ &= \sum_{r_1=0}^{\infty} \frac{\left(\frac{\pi\sqrt{n}}{3}\right)^{r_1}}{r_1!} \sum_{s_1=0}^{r_1} (-1)^{r_1+s_1} \binom{r_1}{s_1} \sum_{m_1=0}^{\infty} \binom{\frac{s_1}{2}}{m_1} \left(\frac{24s+1}{24n}\right)^{m_1} \\ &= \sum_{m_1=0}^{\infty} \sum_{r_1=0}^{2m_1} \sum_{s_1=0}^{r_1} \frac{\left(\frac{\pi}{\sqrt{3}}\right)^{r_1} \left(\frac{24s+1}{24}\right)^{m_1}}{r_1!} (-1)^{r_1+s_1} \binom{r_1}{s_1} \binom{\frac{s_1}{2}}{m_1} n^{-\frac{2m_1-r_1}{2}}. \end{aligned} \quad (3.3)$$

Since  $e^{\frac{\pi}{z\sqrt{3}}\left(\sqrt{1+\frac{24s+1}{24}}z^2-1\right)}$  with  $z := \frac{1}{\sqrt{n}}$  is analytic in the neighbourhood of 0 and thus the Taylor expansion of  $e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)}$  is of the form  $\sum_{\ell=0}^{\infty} \frac{a_{\ell}}{\sqrt{n}^{\ell}}$ , so the range  $0 \leq r_1 \leq \infty$  in the last step (3.3) is truncated to  $0 \leq r_1 \leq 2m_1$ .

Define

$$D' := \{(r_1, s_1, m_1) \in \mathbb{N}_0 : 0 \leq s_1 \leq r_1\}$$

and for  $k \in \mathbb{N}_0$ ,

$$D'_k := \{(r_1, s_1, m_1) \in \mathbb{N}_0 : 2m_1 - r_1 = k\}.$$

For  $t_1 = (r_1, s_1, m_1) \in D'$ , set

$$a'_{t_1} := \frac{\left(\frac{\pi}{\sqrt{3}}\right)^{r_1} \left(\frac{24s+1}{24}\right)^{m_1}}{r_1!} (-1)^{r_1+s_1} \binom{r_1}{s_1} \binom{\frac{s_1}{2}}{m_1}, \text{ and } d_{t_1} := 2m_1 - r_1.$$

Therefore (3.3)) can be rewritten as

$$e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)} = \sum_{t_1 \in (r_1, s_1, m_1) \in D'} \frac{a'_{t_1}}{n^{\frac{d_{t_1}}{2}}} = \sum_{k \geq 0} \sum_{t_1 \in D'_{2k}} \frac{a'_{t_1}}{n^{\frac{k}{2}}} = \sum_{k \geq 0} \sum_{t_1 \in D'_{2k}} \frac{a'_{t_1}}{n^k} + \sum_{k \geq 0} \sum_{t_1 \in D'_{2k+1}} \frac{a'_{t_1}}{n^{k+\frac{1}{2}}}. \quad (3.4)$$

We see that

$$\begin{aligned} D'_{2k} &= \{(r_1, s_1, m_1) \in D' : r_1 - 2m_1 = -2k\} \\ &= \{(r_1, s_1, m_1) \in D' : r_1 \equiv 0 \pmod{2}, r_1 - 2m_1 = -2k\} \\ &= \{(2\ell_1, s_1, \ell_1 + k) \in \mathbb{N}_0^3 : 0 \leq s_1 \leq 2\ell_1\}. \end{aligned}$$

By Lemma 3.1, it follows that,

$$\sum_{k \geq 0}^{\infty} \sum_{t \in D'_{2k}} \frac{a'_{t_1}}{n^k} = 1 + \sum_{k \geq 1}^{\infty} \left( \frac{\left(\frac{24s+1}{24}\right)^k (\frac{1}{2} - k)_{k+1}}{k} \sum_{\ell_1=1}^k \frac{(-k)_{\ell_1}}{(k + \ell_1)!} \frac{\left(\pi\sqrt{\frac{24s+1}{72}}\right)^{2\ell_1}}{(2\ell_1 - 1)!} \right) \frac{1}{n^k} = \sum_{k \geq 0} \frac{B_{2k}(s)}{n^k}. \quad (3.5)$$

Similarly, noting that

$$D'_{2k+1} = \{(2\ell_1 + 1, s_1, \ell_1 + k + 1) \in \mathbb{N}_0^3 : 0 \leq s_1 \leq 2\ell_1 + 1\}$$

and applying Lemma 3.1 followed by (3.1), we obtain

$$\begin{aligned} &\sum_{k \geq 0}^{\infty} \sum_{t \in D'_{2k+1}} \frac{a'_{t_1}}{n^{k+\frac{1}{2}}} \\ &= \frac{\pi}{\sqrt{3}} \sum_{k \geq 0}^{\infty} \left( \left(\frac{24s+1}{24}\right)^{k+1} \left(\frac{1}{2} - k\right)_{k+1} \sum_{\ell_1=0}^k \frac{(-k)_{\ell_1}}{(k + \ell_1 + 1)!} \frac{\left(\pi\sqrt{\frac{24s+1}{72}}\right)^{2\ell_1}}{(2\ell_1)!} \right) \frac{1}{n^{k+\frac{1}{2}}} \\ &= \sum_{k \geq 0} \frac{B_{2k+1}(s)}{n^{k+\frac{1}{2}}}. \end{aligned} \quad (3.6)$$

Applying (3.5) and (3.6) to (3.4), we get

$$e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)} = \sum_{k=0}^N \frac{B_k(s)}{n^{\frac{k}{2}}} + \sum_{k \geq N+1}^{\infty} \frac{B_k(s)}{n^{\frac{k}{2}}}.$$

To estimate  $|B_k(s)|$  for  $k \in \mathbb{N}$ , we estimate  $B_{2k}(s)$  and  $B_{2k+1}(s)$  separately as follows.

$$\begin{aligned} |B_{2k}(s)| &\leq \frac{\left|\left(\frac{24s+1}{24}\right)^k (\frac{1}{2} - k)_{k+1}\right|}{k} \sum_{\ell_1=1}^k \frac{\left(\pi\sqrt{\frac{24s+1}{72}}\right)^{2\ell_1} |(-k)_{\ell_1}|}{(2\ell_1 - 1)!(k + \ell_1)!} \\ &\leq \frac{\left(\frac{24s+1}{24}\right)^k (2k)!}{2k \cdot 4^k} \sum_{\ell_1=1}^k \left| \left( (-1)^{\ell_1} \prod_{j=0}^{\ell_1-1} \frac{k-j}{k+j-1} \right) \right| \frac{\left(\pi\sqrt{\frac{24s+1}{72}}\right)^{2\ell_1}}{(2\ell_1 - 1)!} \end{aligned}$$

$$\leq \sqrt{\frac{\pi}{3}} \frac{\left(\frac{24s+1}{24}\right)^{k+\frac{1}{2}}}{2k^{\frac{3}{2}}} \sum_{\ell_1=1}^k \frac{\left(\pi \sqrt{\frac{24s+1}{72}}\right)^{2\ell_1-1}}{(2\ell_1-1)!} \leq \sqrt{\frac{\pi}{3}} \frac{\left(\frac{24s+1}{24}\right)^{k+\frac{1}{2}}}{2k^{\frac{3}{2}}} \sinh\left(\pi \sqrt{\frac{24s+1}{72}}\right). \quad (3.7)$$

Analogous to (3.7), for  $k \in \mathbb{N}$ , we get

$$|B_{2k+1}(s)| \leq \sqrt{\frac{\pi}{3}} \frac{\left(\frac{24s+1}{24}\right)^{k+1}}{2k^{\frac{3}{2}}} \cosh\left(\pi \sqrt{\frac{24s+1}{72}}\right). \quad (3.8)$$

Combining (3.7) and (3.8), it follows that for  $k \geq 2$ ,

$$|B_k(s)| \leq \sqrt{\frac{\pi}{3}} \frac{\left(\frac{24s+1}{24}\right)^{\frac{k+1}{2}}}{2\left(\frac{k-1}{2}\right)^{\frac{3}{2}}} \cosh\left(\pi \sqrt{\frac{24s+1}{72}}\right), \quad (3.9)$$

and therefore, applying (3.9), we get for all  $N \geq 1$  and  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ ,

$$\begin{aligned} \left| \sum_{k \geq N+1} \frac{B_k(s)}{n^{\frac{k}{2}}} \right| &\leq \sqrt{\frac{2\pi}{3}} \cosh\left(\pi \sqrt{\frac{24s+1}{72}}\right) \sum_{k \geq N+1} \frac{\left(\frac{24s+1}{24}\right)^{\frac{k+1}{2}}}{(k-1)^{\frac{3}{2}} n^{\frac{k}{2}}} \\ &\leq N^{-\frac{3}{2}} \sqrt{\frac{2\pi}{3}} \cdot \sqrt{\frac{24s+1}{24}} \cosh\left(\pi \sqrt{\frac{24s+1}{72}}\right) \sum_{k \geq N+1} \frac{(24s+1)^{\frac{k}{2}}}{(24n)^{\frac{k}{2}}} \\ &\leq \frac{4}{3} N^{-\frac{3}{2}} \sqrt{\frac{2\pi}{3}} \left(\frac{24s+1}{24}\right)^{\frac{N+2}{2}} \cosh\left(\pi \sqrt{\frac{24s+1}{72}}\right) n^{-\frac{N+1}{2}} = Er_2(N, s) n^{-\frac{N+1}{2}}. \end{aligned}$$

Thus, we conclude the proof.  $\square$

Define

$$\bar{A}_\ell\left(\frac{3}{4}, s\right) := \begin{cases} \left(\frac{24s+1}{24}\right)^{\frac{\ell}{2}} \left(-\frac{3}{2}\right)^{\frac{3}{2}}, & \text{if } \ell \equiv 0 \pmod{2}, \\ 0, & \text{otherwise,} \end{cases} \quad (3.10)$$

and

$$Er_3\left(\frac{3}{4}, N, s\right) := \frac{4}{3} \left(\frac{24s+1}{24}\right)^{\frac{N+1}{2}}. \quad (3.11)$$

**Lemma 3.3.** For  $N \geq 1$  and  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ , we have

$$\left(1 + \frac{24s+1}{24n}\right)^{-\frac{3}{4}} = \sum_{\ell=0}^N \frac{\bar{A}_\ell\left(\frac{3}{4}, s\right)}{n^{\frac{\ell}{2}}} + O_{\leq Er_3\left(\frac{3}{4}, N, s\right)}\left(n^{-\frac{N+1}{2}}\right).$$

*Proof.* By Taylor expansion of  $(1 + \frac{24s+1}{2n})^{-\frac{3}{4}}$ , and (3.10), we get

$$\left(1 + \frac{24s+1}{24n}\right)^{-\frac{3}{4}} = \sum_{k_1=0}^{\infty} \binom{-\frac{3}{4}}{k_1} \left(\sqrt{\frac{24s+1}{24n}}\right)^{2k_1} =: \sum_{\ell=0}^N \frac{\bar{A}_\ell\left(\frac{3}{4}, s\right)}{n^{\frac{\ell}{2}}} + \sum_{\ell=N+1}^{\infty} \frac{\bar{A}_\ell\left(\frac{3}{4}, s\right)}{n^{\frac{\ell}{2}}}.$$

It is clear that for all  $\ell \in \mathbb{N}$ ,

$$\left|\bar{A}_\ell\left(\frac{3}{4}, s\right)\right| \leq \left(\frac{24s+1}{24}\right)^{\frac{\ell}{2}} \quad \left(\text{since} \quad \left|\binom{-\frac{3}{4}}{\frac{\ell}{2}}\right| \leq 1\right). \quad (3.12)$$

Thus for  $n \geq \left\lceil \frac{2(24s+1)}{3} \right\rceil$ , it follows that

$$\left| \sum_{\ell=N+1}^{\infty} \frac{\bar{A}_{\ell}(\frac{3}{4}, s)}{n^{\frac{\ell}{2}}} \right| \leq \sum_{\ell=N+1}^{\infty} \left( \frac{24s+1}{24n} \right)^{\frac{\ell}{2}} \leq Er_3\left(\frac{3}{4}, N, s\right) \cdot n^{-\frac{N+1}{2}},$$

and this finishes the proof.  $\square$

Next, we combine Lemmas 3.2 and 3.3 to estimate an error bound after extracting the main terms for the factor  $\frac{e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)}}{\left(1+\frac{24s+1}{24n}\right)^{3/4}}$ . Letting  $Er_2(N, s)$  and  $Er_3\left(\frac{3}{4}, N, s\right)$  be as in (3.2) and (3.11) respectively, we define

$$\begin{aligned} Er_4(N, s) := & \left( \frac{4}{3}N^{\frac{3}{2}} + 1 \right) Er_2(N, s) + \frac{\pi}{2\sqrt{3}} \left( \frac{24s+1}{24} \right)^{\frac{N+2}{2}} + Er_3\left(\frac{3}{4}, N, s\right) \\ & \times \left( 1 + \frac{\pi}{2\sqrt{3}} \frac{24s+1}{24} + \frac{\sqrt{\pi(24s+1)}}{72} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right) \right). \end{aligned} \quad (3.13)$$

**Lemma 3.4.** *Let  $Er_4(N, s)$  be as given in (3.13). For  $N \geq 1$  and  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ , we have*

$$\frac{e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)}}{\left(1+\frac{24s+1}{24n}\right)^{3/4}} = \sum_{k=0}^N \frac{\bar{B}_k(s)}{n^{\frac{k}{2}}} + O_{\leq Er_4(N, s)}\left(n^{-\frac{N+1}{2}}\right),$$

where for  $k \in \mathbb{N}_0$ ,

$$\bar{B}_k(s) := \sum_{\ell=0}^k B_{\ell}(s) \bar{A}_{k-\ell}\left(\frac{3}{4}, s\right). \quad (3.14)$$

*Proof.* Utilizing Lemmas 3.2 and 3.3, we get

$$\begin{aligned} \frac{e^{\pi\sqrt{\frac{n}{3}}\left(\sqrt{1+\frac{24s+1}{24n}}-1\right)}}{\left(1+\frac{24s+1}{24n}\right)^{3/4}} &= \left( \sum_{\ell=0}^N \frac{B_{\ell}(s)}{n^{\frac{k}{2}}} + O_{\leq Er_2(N, s)}\left(n^{-\frac{N+1}{2}}\right) \right) \left( \sum_{\ell=0}^N \frac{\bar{A}_{\ell}(\frac{3}{4}, s)}{n^{\frac{\ell}{2}}} + O_{\leq Er_3(\frac{3}{4}, N, s)}\left(n^{-\frac{N+1}{2}}\right) \right) \\ &= \sum_{\ell=0}^N \frac{B_{\ell}(s)}{n^{\frac{\ell}{2}}} \sum_{\ell=0}^N \frac{\bar{A}_{\ell}(\frac{3}{4}, s)}{n^{\frac{\ell}{2}}} + \sum_{\ell=0}^N \frac{B_{\ell}(s)}{n^{\frac{\ell}{2}}} \cdot O_{\leq Er_3(\frac{3}{4}, N, s)}\left(n^{-\frac{N+1}{2}}\right) \\ &\quad + \left( 1 + \frac{24s+1}{24n} \right)^{-3/4} \cdot O_{\leq Er_2(N, s)}\left(n^{-\frac{N+1}{2}}\right). \end{aligned} \quad (3.15)$$

Clearly, it follows that

$$\begin{aligned} \sum_{\ell=0}^N \frac{B_{\ell}(s)}{n^{\frac{\ell}{2}}} \sum_{\ell=0}^N \frac{\bar{A}_{\ell}(\frac{3}{4}, s)}{n^{\frac{\ell}{2}}} &= \sum_{k=0}^N \frac{1}{n^{\frac{k}{2}}} \sum_{\ell=0}^k B_{\ell}(s) \bar{A}_{k-\ell}\left(\frac{3}{4}, s\right) + n^{-\frac{N+1}{2}} \sum_{k=0}^{N-1} \frac{1}{n^{\frac{k}{2}}} \sum_{\ell=k}^{N-1} B_{\ell+1}(s) \bar{A}_{N+k-\ell}\left(\frac{3}{4}, s\right) \\ &=: \sum_{k=0}^N \frac{\bar{B}_k(s)}{n^{\frac{k}{2}}} + T_1(N, s, n). \end{aligned}$$

Applying (3.9) and (3.12), for  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ , we have

$$\begin{aligned} |T_1(N, s, n)| &\leq n^{-\frac{N+1}{2}} \left( \sum_{k=1}^{N-1} \frac{1}{n^{\frac{k}{2}}} \sum_{\ell=k}^{N-1} \sqrt{\frac{2\pi}{3}} \frac{\left(\frac{24s+1}{24}\right)^{\frac{N+k+2}{2}}}{\ell^{\frac{3}{2}}} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right) \right. \\ &\quad \left. + \frac{\pi}{2\sqrt{3}} \left(\frac{24s+1}{24}\right)^{\frac{N+2}{2}} + \sum_{\ell=1}^{N-1} \sqrt{\frac{2\pi}{3}} \frac{\left(\frac{24s+1}{24}\right)^{\frac{N+2}{2}}}{\ell^{\frac{3}{2}}} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right) \right) \\ &\leq n^{-\frac{N+1}{2}} \left( \frac{N^{\frac{3}{2}}}{2} Er_2(N, s) \left( \sum_{k=1}^{N-1} \left(\frac{24s+1}{24n}\right)^{\frac{k}{2}} \sum_{\ell=k}^{N-1} \ell^{-\frac{3}{2}} + \sum_{\ell=1}^{N-1} \ell^{-\frac{3}{2}} \right) + \frac{\pi}{2\sqrt{3}} \left(\frac{24s+1}{24}\right)^{\frac{N+2}{2}} \right) \\ &\leq n^{-\frac{N+1}{2}} \left( \frac{4}{3} N^{\frac{3}{2}} Er_2(N, s) + \frac{\pi}{2\sqrt{3}} \left(\frac{24s+1}{24}\right)^{\frac{N+2}{2}} \right), \end{aligned}$$

where in the last step, we use that

$$\sum_{\ell=1}^{N-1} \ell^{-\frac{3}{2}} \leq \sum_{\ell=1}^{\infty} \ell^{-\frac{3}{2}} = \zeta\left(\frac{3}{2}\right) \leq 2. \quad (3.16)$$

Then letting

$$Er_{4,1}(N, s) := \frac{4}{3} N^{\frac{3}{2}} Er_2(N, s) + \frac{\pi}{2\sqrt{3}} \left(\frac{24s+1}{24}\right)^{\frac{N+2}{2}},$$

for all  $N \geq 2$  and  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ , we obtain

$$\sum_{\ell=0}^N \frac{B_k(s)}{n^{\frac{\ell}{2}}} \sum_{\ell=0}^N \frac{\overline{A}_\ell(\frac{3}{4}, s)}{n^{\frac{\ell}{2}}} = \sum_{k=0}^N \frac{\overline{B}_k(s)}{n^{\frac{k}{2}}} + O_{\leq Er_{4,1}(N, s)}\left(n^{-\frac{N+1}{2}}\right). \quad (3.17)$$

To finish the proof, it remains to estimate the two terms involving the factor  $n^{-\frac{N+1}{2}}$  in (3.15). It is clear that for all  $n \geq 1$ ,

$$\left(1 + \frac{24s+1}{24n}\right)^{-3/4} \cdot O_{\leq Er_2(N, s)}\left(n^{-\frac{N+1}{2}}\right) = O_{\leq Er_2(N, s)}\left(n^{-\frac{N+1}{2}}\right). \quad (3.18)$$

For all  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ , employing (3.9) gives that

$$\begin{aligned} \sum_{\ell=0}^N \frac{B_\ell(s)}{n^{\frac{\ell}{2}}} &\leq 1 + \frac{\pi}{2\sqrt{3}} \frac{24s+1}{24} + \sum_{\ell=2}^N \sqrt{\frac{2\pi}{3}} \frac{\left(\frac{24s+1}{24}\right)^{\frac{\ell+1}{2}}}{n^{\frac{\ell}{2}} (\ell-1)^{\frac{3}{2}}} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right) \\ &\leq 1 + \frac{\pi}{2\sqrt{3}} \frac{24s+1}{24} + \frac{\sqrt{\pi(24s+1)}}{72} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right). \end{aligned}$$

and therefore,

$$\begin{aligned} \sum_{k=0}^N \frac{B_k(s)}{n^{\frac{k}{2}}} \cdot O_{\leq Er_3(\frac{3}{4}, N, s)}\left(n^{-\frac{N+1}{2}}\right) \\ = O_{\leq \left(1 + \frac{\pi}{2\sqrt{3}} \frac{24s+1}{24} + \frac{\sqrt{\pi(24s+1)}}{72} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right)\right) Er_3(\frac{3}{4}, N, s)}\left(n^{-\frac{N+1}{2}}\right). \quad (3.19) \end{aligned}$$

Finally, combining (3.17)-(3.19) to (3.15), we conclude the proof.  $\square$

For all  $\ell \in \mathbb{N}_0$ , define

$$\begin{aligned}\widehat{C}_{2\ell}(s) &:= \sum_{k=0}^{\ell} \binom{-k}{\ell-k} \frac{a_{2k}(1)}{(\frac{\pi}{\sqrt{3}})^{2k}} \left( \frac{24s+1}{24} \right)^{\ell-k}, \\ \widehat{C}_{2\ell+1}(s) &:= - \sum_{k=0}^{\ell} \binom{-\frac{2k+1}{2}}{\ell-k} \frac{a_{2k+1}(1)}{(\frac{\pi}{\sqrt{3}})^{2k+1}} \left( \frac{24s+1}{24} \right)^{\ell-k}.\end{aligned}\quad (3.20)$$

**Lemma 3.5.** *For  $N \geq 1$  and  $n \geq \max \left\{ \left\lceil \frac{2(24s+1)}{3} \right\rceil, \frac{48}{\pi^2} \right\}$ , we have*

$$\sum_{m=0}^N \frac{(-1)^m a_m(1)}{(\frac{\pi}{\sqrt{3}})^m n^{\frac{m}{2}}} \left( 1 + \frac{24s+1}{24n} \right)^{-\frac{m}{2}} = \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} + O_{\leq Er_5(N,s)} \left( n^{-\frac{N+1}{2}} \right),$$

where

$$Er_5(N,s) := \frac{4 \cdot |a_N(1)|}{3} \left( \frac{24s+1}{24} + \frac{3}{\pi^2} \right)^{\lfloor \frac{N}{2} \rfloor + 1} + \frac{4 \cdot |a_N(1)|}{\sqrt{3}\pi} \left( \sqrt{\frac{24s+1}{24}} + \frac{\sqrt{3}}{\pi} \right)^{2\lfloor \frac{N-1}{2} \rfloor + 2}. \quad (3.21)$$

*Proof.* We set

$$\widehat{b}_k := \frac{(-1)^k a_k(1)}{(\frac{\pi}{\sqrt{3}})^k}, \quad \widehat{c}_{k,\ell}(s) := \left( \frac{24s+1}{24} \right)^{\ell} \binom{-\frac{k}{2}}{\ell}, \quad \text{and } w := n^{-\frac{1}{2}},$$

and consequently, rewrite the sum as

$$\begin{aligned}\sum_{m=0}^N \frac{(-1)^m a_m(1)}{(\frac{\pi}{\sqrt{3}})^m n^{\frac{m}{2}}} \left( 1 + \frac{24s+1}{24n} \right)^{-\frac{m}{2}} &= \sum_{k=0}^N \sum_{\ell=0}^{\infty} \widehat{b}_k \widehat{c}_{k,\ell}(s) w^{2\ell+k} \\ &= \sum_{m=0}^N \widehat{C}_m(s) w^m + \sum_{m=\lfloor \frac{N}{2} \rfloor + 1}^{\infty} \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \widehat{b}_{2k} \widehat{c}_{2k,m-k}(s) w^{2m} \\ &\quad + \sum_{m=\lfloor \frac{N-1}{2} \rfloor + 1}^{\infty} \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \widehat{b}_{2k+1} \widehat{c}_{2k+1,m-k}(s) w^{2m+1} \\ &=: \sum_{m=0}^N \widehat{C}_m(s) w^m + \widehat{E}_1(N,w) + \widehat{E}_2(N,w).\end{aligned}\quad (3.22)$$

On the one hand, we have

$$\begin{aligned}\left| \widehat{E}_1(N,w) \right| &\leq \sum_{m=\lfloor \frac{N}{2} \rfloor + 1}^{\infty} \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} |\widehat{b}_{2k}| |\widehat{c}_{2k,m-k}(s)| w^{2m} \quad (\text{by (3.22)}) \\ &= \sum_{m=\lfloor \frac{N}{2} \rfloor + 1}^{\infty} \frac{1}{n^m} \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \left| \frac{a_{2k}(1)}{(\frac{\pi}{\sqrt{3}})^{2k}} \right| \left| \binom{-k}{m-k} \right| \left( \frac{24s+1}{24} \right)^{m-k}\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=\lfloor \frac{N}{2} \rfloor + 1}^{\infty} \frac{1}{n^m} \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{m-1}{k-1} \left| \frac{a_{2k}(1)}{(\frac{\pi}{\sqrt{3}})^{2k}} \right| \left( \frac{24s+1}{24} \right)^{m-k} \\
&\leq |a_N(1)| \sum_{m=\lfloor \frac{N}{2} \rfloor + 1}^{\infty} \left( \frac{24s+1}{24n} \right)^m \sum_{k=0}^{m-1} \binom{m-1}{k-1} \left( \frac{3}{\pi^2} \cdot \frac{24}{24s+1} \right)^k \\
&\leq |a_N(1)| \sum_{m=\lfloor \frac{N}{2} \rfloor + 1}^{\infty} \left( \frac{24s+1}{24n} + \frac{3}{\pi^2 n} \right)^m \\
&\leq \frac{4 \cdot |a_N(1)|}{3} \left( \frac{24s+1}{24n} + \frac{3}{\pi^2 n} \right)^{\lfloor \frac{N}{2} \rfloor + 1} \quad \left( \forall n \geq \max \left\{ \left\lceil \frac{24s+1}{3} \right\rceil, \frac{24}{\pi^2} \right\} \right) \\
&\leq \frac{4 \cdot |a_N(1)|}{3} \left( \frac{24s+1}{24} + \frac{3}{\pi^2} \right)^{\lfloor \frac{N}{2} \rfloor + 1} n^{-\frac{N+1}{2}}. \tag{3.23}
\end{aligned}$$

On the other hand, we estimate

$$\begin{aligned}
&|\widehat{E}_2(N, w)| \\
&\leq \sum_{m=\lfloor \frac{N-1}{2} \rfloor + 1}^{\infty} \frac{1}{n^{m+\frac{1}{2}}} \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} |\widehat{b}_{2k+1}| |\widehat{c}_{2k+1, m-k}(s)| \quad (\text{by (3.22)}) \\
&= \sum_{m=\lfloor \frac{N-1}{2} \rfloor + 1}^{\infty} \frac{1}{n^{m+\frac{1}{2}}} \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \left| \frac{a_{2k+1}(1)}{(\frac{\pi}{\sqrt{3}})^{2k+1}} \right| \left| \frac{\binom{2m}{2k} \binom{2m-2k}{m-k}}{4^{m-k} \binom{m}{k}} \right| \left( \frac{24s+1}{24} \right)^{m-k} \\
&\leq \frac{\sqrt{3} \cdot |a_N(1)|}{\pi} \sum_{m=\lfloor \frac{N-1}{2} \rfloor + 1}^{\infty} \frac{1}{n^{m+\frac{1}{2}}} \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \binom{2m}{2k} \left( \frac{3}{\pi^2} \cdot \frac{24}{24s+1} \right)^k \left( \frac{24s+1}{24} \right)^m \\
&\leq \frac{\sqrt{3} \cdot |a_N(1)|}{\pi n^{\frac{1}{2}}} \sum_{m=\lfloor \frac{N-1}{2} \rfloor + 1}^{\infty} \left( \frac{24s+1}{24n} \right)^m \sum_{k=0}^{2m} \binom{2m}{2k} \left( \frac{3}{\pi^2} \cdot \frac{24}{24s+1} \right)^k \\
&\leq \frac{\sqrt{3} \cdot |a_N(1)|}{\pi n^{\frac{1}{2}}} \sum_{m=\lfloor \frac{N-1}{2} \rfloor + 1}^{\infty} \left( \sqrt{\frac{24s+1}{24n}} + \frac{\sqrt{3}}{\pi \sqrt{n}} \right)^{2m} \\
&\leq \frac{4 \cdot |a_N(1)|}{\sqrt{3}\pi} \left( \sqrt{\frac{24s+1}{24}} + \frac{\sqrt{3}}{\pi} \right)^{2\lfloor \frac{N-1}{2} \rfloor + 2} n^{-\frac{N+1}{2}} \quad \left( \forall n \geq \max \left\{ \left\lceil \frac{2(24s+1)}{3} \right\rceil, \frac{48}{\pi^2} \right\} \right). \tag{3.24}
\end{aligned}$$

Applying (3.23) and (3.24) to (3.22), we conclude the proof.  $\square$

**Lemma 3.6.** Let  $Er_{N,1}$  and  $Er_5(N,s)$  be as given in (2.9) and (3.21) separately. For  $N \geq 1$  and  $n \geq \max \left\{ \left\lceil \frac{2(24s+1)}{3} \right\rceil, \frac{48}{\pi^2} \right\}$ , we have

$$\begin{aligned} & \left( \sum_{m=0}^N \frac{(-1)^m a_m(1)}{\left(\frac{\pi}{\sqrt{3}}\right)^m n^{\frac{m}{2}}} \left(1 + \frac{24s+1}{24n}\right)^{-\frac{m}{2}} + O_{\leq Er_{N,1}} \left(n^{-\frac{N+1}{2}}\right) \right) \left(1 + O_{\leq 4} \left(\frac{3^{\frac{N+1}{2}}}{\pi^{N+1}} n^{-\frac{N+1}{2}}\right)\right) \\ &= \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} + O_{\leq Er_6(N,s)} \left(n^{-\frac{N+1}{2}}\right), \end{aligned}$$

where

$$Er_6(N,s) := 8 \left( \frac{\sqrt{3}}{\pi} \right)^{N+1} |a_N(1)| + \left( 1 + 4 \left( \frac{3}{\pi \sqrt{2(24s+1)}} \right)^{N+1} \right) (Er_5(N,s) + Er_{N,1}). \quad (3.25)$$

*Proof.* Using Lemma 3.5, for all  $n \geq \max \left\{ \left\lceil \frac{2(24s+1)}{3} \right\rceil, \frac{48}{\pi^2} \right\}$ , we get

$$\begin{aligned} & \sum_{m=0}^N \frac{(-1)^m a_m(1)}{\left(\frac{\pi}{\sqrt{3}}\right)^m n^{\frac{m}{2}}} \left(1 + \frac{24s+1}{24n}\right)^{-\frac{m}{2}} + O_{\leq Er_{N,1}} \left(n^{-\frac{N+1}{2}}\right) \\ &= \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} + O_{\leq (Er_5(N,s)+Er_{N,1})} \left(n^{-\frac{N+1}{2}}\right), \end{aligned}$$

and hence,

$$\begin{aligned} & \left( \sum_{m=0}^N \frac{(-1)^m a_m(1)}{\left(\frac{\pi}{\sqrt{3}}\right)^m n^{\frac{m}{2}}} \left(1 + \frac{24s+1}{24n}\right)^{-\frac{m}{2}} + O_{\leq Er_{N,1}} \left(n^{-\frac{N+1}{2}}\right) \right) \left(1 + O_{\leq 4} \left(\frac{3^{\frac{N+1}{2}}}{\pi^{N+1}} n^{-\frac{N+1}{2}}\right)\right) \\ &= \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} + \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} \cdot O_{\leq 4} \left(\frac{3^{\frac{N+1}{2}}}{\pi^{N+1}} n^{-\frac{N+1}{2}}\right) + O_{\leq Er_{6,1}(N,s)} \left(n^{-\frac{N+1}{2}}\right), \quad (3.26) \end{aligned}$$

where

$$Er_{6,1}(N,s) := \left( 1 + 4 \left( \frac{3}{\pi \sqrt{2(24s+1)}} \right)^{N+1} \right) (Er_5(N,s) + Er_{N,1}).$$

For all  $n \geq \max \left\{ \left\lceil \frac{2(24s+1)}{3} \right\rceil, \frac{48}{\pi^2} \right\}$ , we obtain

$$\begin{aligned} \left| \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} \right| &\leq \sum_{m=0}^{\lfloor \frac{N}{2} \rfloor} \left| \frac{\widehat{C}_{2m}(s)}{n^m} \right| + \sum_{m=0}^{\lfloor \frac{N-1}{2} \rfloor} \left| \frac{\widehat{C}_{2m+1}(s)}{n^{m+\frac{1}{2}}} \right| \\ &\leq \sum_{m=0}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{n^m} \sum_{k=0}^m \binom{m-1}{k-1} \frac{|a_{2k}(1)|}{\left(\frac{\pi}{\sqrt{3}}\right)^{2k}} \left(\frac{24s+1}{24}\right)^{m-k} \\ &+ \sum_{m=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{n^{m+\frac{1}{2}}} \sum_{k=0}^m \left| \binom{-\frac{2k+1}{2}}{m-k} \right| \frac{|a_{2k+1}(1)|}{\left(\frac{\pi}{\sqrt{3}}\right)^{2k+1}} \left(\frac{24s+1}{24}\right)^{m-k} \quad (\text{by Lemma 3.5}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{n^m} \sum_{k=0}^m \binom{m-1}{k-1} \frac{|a_{2k}(1)|}{(\frac{\pi}{\sqrt{3}})^{2k}} \left( \frac{24s+1}{24} \right)^{m-k} \\
&\quad + \sum_{m=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{n^{m+\frac{1}{2}}} \sum_{k=0}^m \frac{\binom{2m}{2k} \binom{2m-2k}{m-k}}{4^{m-k} \binom{m}{k}} \frac{|a_{2k+1}(1)|}{(\frac{\pi}{\sqrt{3}})^{2k+1}} \left( \frac{24s+1}{24} \right)^{m-k} \\
&\leq 2 \cdot |a_N(1)|.
\end{aligned}$$

Therefore,

$$\sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} \cdot O_{\leq 4} \left( \frac{3^{\frac{N+1}{2}}}{\pi^{N+1}} n^{-\frac{N+1}{2}} \right) = O_{\leq 8 \cdot 3^{\frac{N+1}{2}} \cdot \pi^{-N-1} |a_N(1)|} \left( n^{-\frac{N+1}{2}} \right). \quad (3.27)$$

Applying (3.27) to (3.26), we conclude the proof.  $\square$

#### 4. PROOF OF THEOREM 1.1

In this section, by making use of the estimations of each terms in the asymptotic expansion which were presented in Section 3, we prove Theorem 1.1.

*Proof of Theorem 1.1.* Applying Lemmas 3.4 and 3.6 to (2.10), for  $n \geq n(N, s)$  (cf. (1.1)), we have

$$\begin{aligned}
&q(n+s) \\
&= \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{1/4} n^{\frac{3}{4}}} \left( \sum_{m=0}^N \frac{\overline{B}_m(s)}{n^{\frac{m}{2}}} + O_{\leq Er_4(N,s)} \left( n^{-\frac{N+1}{2}} \right) \right) \left( \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} + O_{\leq Er_6(N,s)} \left( n^{-\frac{N+1}{2}} \right) \right).
\end{aligned}$$

Now, expanding the above, it gives

$$\begin{aligned}
&\left( \sum_{m=0}^N \frac{\overline{B}_m(s)}{n^{\frac{m}{2}}} + O_{\leq Er_4(N,s)} \left( n^{-\frac{N+1}{2}} \right) \right) \left( \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} + O_{\leq Er_6(N,s)} \left( n^{-\frac{N+1}{2}} \right) \right) \\
&= \sum_{m=0}^N \sum_{k=0}^m \frac{\overline{B}_k(s) \cdot \widehat{C}_{m-k}(s)}{n^{\frac{m}{2}}} + n^{-\frac{N+1}{2}} \sum_{m=0}^{N-1} \frac{1}{n^{\frac{m}{2}}} \sum_{k=m}^{N-1} \overline{B}_{k+1}(s) \cdot \widehat{C}_{N+m-k}(s) \\
&\quad + \sum_{m=0}^N \frac{\overline{B}_m(s)}{n^{\frac{m}{2}}} \cdot O_{\leq Er_6(N,s)} \left( n^{-\frac{N+1}{2}} \right) + \sum_{m=0}^N \frac{\widehat{C}_m(s)}{n^{\frac{m}{2}}} \cdot O_{\leq Er_4(N,s)} \left( n^{-\frac{N+1}{2}} \right) \\
&\quad + O_{\substack{\leq Er_4(N,s) \cdot Er_6(N,s) \\ n(N,s)}} \left( n^{-\frac{N+1}{2}} \right) \\
&=: \sum_{m=0}^N \frac{\widehat{B}_m(s)}{n^{\frac{m}{2}}} + \overline{Er}_1(n, N, s) + \overline{Er}_2(n, N, s) + \overline{Er}_3(n, N, s) + O_{\substack{\leq Er_4(N,s) \cdot Er_6(N,s) \\ n(N,s)}} \left( n^{-\frac{N+1}{2}} \right),
\end{aligned}$$

where for  $\ell \in \mathbb{N}_0$ ,

$$\widehat{B}_\ell(s) := \sum_{k=0}^{\ell} \overline{B}_k(s) \cdot \widehat{C}_{\ell-k}(s). \quad (4.1)$$

We first estimate  $\overline{B}_m(s)$  and  $\widehat{C}_m(s)$ . For  $m \geq 2$ , Using (3.9) and (3.16), we have

$$\begin{aligned} |\overline{B}_m(s)| &= \left| \sum_{\ell=0}^m B_\ell(s) \overline{A}_{m-\ell}\left(\frac{3}{4}, s\right) \right| \quad (\text{by (3.14)}) \\ &\leq B_0(s) \left| \overline{A}_m\left(\frac{3}{4}, s\right) \right| + |B_1(s)| \left| \overline{A}_{m-1}\left(\frac{3}{4}, s\right) \right| + \sum_{\ell=2}^m |B_\ell(s)| \cdot \left| \overline{A}_{m-\ell}\left(\frac{3}{4}, s\right) \right| \\ &\leq \left( \frac{24s+1}{24} \right)^{\frac{m}{2}} \left( 1 + \frac{\pi}{2\sqrt{3}} \sqrt{\frac{24s+1}{24}} + \sqrt{\frac{2\pi}{3}} \sqrt{\frac{24s+1}{24}} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right) \sum_{\ell=2}^m (l-1)^{-\frac{3}{2}} \right) \\ &\leq \left( \frac{24s+1}{24} \right)^{\frac{m}{2}} \left( 1 + \frac{\pi}{2\sqrt{3}} \sqrt{\frac{24s+1}{24}} + \frac{\sqrt{\pi(24s+1)}}{12} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right) \right). \end{aligned} \quad (4.2)$$

Next, we have for all  $m \in \mathbb{N}$  as

$$\begin{aligned} |\widehat{C}_{2m}(s)| &\leq \sum_{k=0}^m \left| \binom{-k}{m-k} \right| \frac{|a_{2k}(1)|}{(\frac{\pi}{\sqrt{3}})^{2k}} \left( \frac{24s+1}{24} \right)^{m-k} \\ &\leq |a_{2m}(1)| \left( \frac{24s+1}{24} \right)^m \sum_{k=0}^m \binom{m-1}{k-1} \leq |a_{2m}(1)| \cdot \left( \frac{24s+1}{12} \right)^m, \end{aligned} \quad (4.3)$$

and in a similar fashion, we get for  $m \in \mathbb{N}$ ,

$$|\widehat{C}_{2m+1}(s)| \leq |a_{2m+1}(1)| \cdot \left( \frac{24s+1}{6} \right)^m. \quad (4.4)$$

From (4.3), (4.4) and the fact that  $\widehat{C}_0(s) = 1$  (by (3.20)), it follows that

$$|\widehat{C}_m(s)| \leq |a_m(1)| \cdot \left( \frac{24s+1}{6} \right)^{\frac{m}{2}}. \quad (4.5)$$

With the above estimation, now we can estimate  $\overline{Er}_i(n, N, s)$  for all  $i = 1, 2, 3$ . Setting

$$A(s) := 1 + \frac{\pi}{2\sqrt{3}} \sqrt{\frac{24s+1}{24}} + \frac{\sqrt{\pi(24s+1)}}{12} \cosh\left(\pi\sqrt{\frac{24s+1}{72}}\right), \quad (4.6)$$

by (4.2) and (4.5), we see that, for  $n \geq \left\lceil \frac{2(24s+1)}{3} \right\rceil$ ,

$$\begin{aligned} |\overline{Er}_1(n, N, s)| &\leq \left| n^{-\frac{N+1}{2}} \sum_{m=0}^{N-1} \frac{1}{n^{\frac{m}{2}}} \sum_{k=m}^{N-1} \overline{B}_{k+1}(s) \cdot \widehat{C}_{N+m-k}(s) \right| \\ &\leq \left( \frac{\pi|a_N(1)| \cdot 2^{N-1}}{\sqrt{3}} \left( \frac{24s+1}{24} \right)^{\frac{N}{2}+1} + A(s) \sum_{k=1}^{N-1} \left( \frac{24s+1}{24} \right)^{\frac{k+1}{2}} |a_{N-k}(1)| \left( \frac{24s+1}{6} \right)^{\frac{N-k}{2}} \right. \\ &\quad \left. + A(s) \sum_{m=1}^{N-1} \frac{1}{n^{\frac{m}{2}}} \sum_{k=m}^{N-1} \left( \frac{24s+1}{24} \right)^{\frac{k+1}{2}} |a_{N+m-k}(1)| \left( \frac{24s+1}{6} \right)^{\frac{N+m-k}{2}} \right) n^{-\frac{N+1}{2}} \\ &\leq \left( \frac{\pi|a_N(1)| \cdot 2^{N-1}}{\sqrt{3}} \left( \frac{24s+1}{24} \right)^{\frac{N}{2}+1} + A(s)|a_N(1)| \left( \frac{24s+1}{24} \right)^{\frac{N+1}{2}} \right) \end{aligned}$$

$$\begin{aligned}
& + A(s)|a_N(1)| \cdot \left( \frac{24s+1}{6} \right)^{\frac{N}{2}} \left( \frac{24s+1}{24} \right)^{\frac{1}{2}} \sum_{m=1}^{N-1} \frac{1}{2^m} \sum_{k=m}^{N-1} \frac{1}{2^k} \right) n^{-\frac{N+1}{2}} \\
& \leq |a_N(1)| \left( \frac{\pi \cdot 2^{N-1}}{\sqrt{3}} \sqrt{\frac{24s+1}{24}} + A(s) \left( 1 + \frac{2^{N+1}}{3} \right) \right) \left( \frac{24s+1}{24} \right)^{\frac{N+1}{2}} n^{-\frac{N+1}{2}}. \tag{4.7}
\end{aligned}$$

For all  $n \geq \lceil \frac{2(24s+1)}{3} \rceil$ , we get

$$\begin{aligned}
|\overline{Er}_2(n, N, s)| & \leq Er_6(N, s) \cdot n^{-\frac{N+1}{2}} \sum_{m=0}^N \frac{|\overline{B}_m(s)|}{n^{\frac{m}{2}}} \\
& \leq \left( 1 + \frac{\pi}{2\sqrt{3}} \frac{24s+1}{24} + \frac{A(s)}{12} \right) Er_6(N, s) n^{-\frac{N+1}{2}}, \tag{4.8}
\end{aligned}$$

and

$$|\overline{Er}_3(n, N, s)| \leq Er_4(N, s) \cdot n^{-\frac{N+1}{2}} \sum_{m=0}^N \frac{|\widehat{C}_m(s)|}{n^{\frac{m}{2}}} \leq 2|a_N(1)| Er_4(N, s) n^{-\frac{N+1}{2}}. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9) with following (1.2), we finish the proof of Theorem 1.1.  $\square$

## 5. INEQUALITIES FOR $q(n)$

In this section, we prove all the inequalities by applying Theorem 1.1 and choosing appropriate values of  $N$ .

From Theorem 1.1, we have for  $(s, N) \in \mathbb{N}_0 \times \mathbb{N}$  and  $n \geq n(N, s)$ ,

$$\frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{1/4} n^{3/4}} L(n, s, N) \leq q(n+s) \leq \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{1/4} n^{3/4}} U(n, s, N), \tag{5.1}$$

where

$$L(n, s, N) := \sum_{m=0}^N \frac{\widehat{B}_m(s)}{n^{\frac{m}{2}}} - \frac{Er_N(s)}{n^{\frac{N+1}{2}}}, \quad U(n, s, N) := \sum_{m=0}^N \frac{\widehat{B}_m(s)}{n^{\frac{m}{2}}} + \frac{Er_N(s)}{n^{\frac{N+1}{2}}}.$$

*Proof of Theorem 1.2.* Making the shift  $n \rightarrow n+1$ , it is equivalent to show that for  $n \geq 229$ ,

$$A(q(n), q(n+1), q(n+2), q(n+3), q(n+4)) > 0.$$

Following (5.1), we have

$$\begin{aligned}
& A(q(n), q(n+1), q(n+2), q(n+3), q(n+4)) \\
& > \left( \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{1/4} n^{3/4}} \right)^2 (L(n, 0, N)L(n, 4, N) + 3L^2(n, 2, N) - 4U(n, 1, N)U(n, 3, N)). 
\end{aligned}$$

Thus to prove the theorem, we first aim to show that

$$L(n, 0, N)L(n, 4, N) + 3L^2(n, 2, N) - 4U(n, 1, N)U(n, 3, N) > 0. \tag{5.2}$$

Choosing  $N = 14$  and with `Reduce` command in Mathematica, for  $n \geq 2469$ , we have (5.2). Computing

$$\max_{s \in \{0,1,2,3,4\}} \{n(14, s)\} \leq 5019,$$

we finally conclude that for  $n \geq 5019$ ,

$$A(q(n), q(n+1), q(n+2), q(n+3), q(n+4)) > 0.$$

For the remaining cases  $229 \leq n \leq 5018$ , we verified with Mathematica.  $\square$

*Proof of Theorem 1.3.* Making the shift  $n \rightarrow n+1$ , it is equivalent to show that for  $n \geq 278$ ,

$$4 \left(1 + \frac{\pi^2}{32(n+1)^3}\right) q(n+1)q(n+3) > q(n)q(n+4) + 3q(n+2)^2.$$

Since for  $n \geq 1$ ,

$$\frac{\pi^2}{32(n+1)^3} \geq \frac{\pi^2}{32n^3} - \frac{1}{n^{\frac{7}{2}}},$$

thus it is suffices to show

$$4 \left(1 + \frac{\pi^2}{32n^3} - \frac{1}{n^{\frac{7}{2}}}\right) q(n+1)q(n+3) > q(n)q(n+4) + 3q(n+2)^2.$$

Therefore, following (5.1), we need to prove

$$4 \left(1 + \frac{\pi^2}{32n^3} - \frac{1}{n^{\frac{7}{2}}}\right) L(n, 1, N)L(N, 3, N) > U(n, 0, N)U(n, 4, N) + 3U^2(n, 2, N). \quad (5.3)$$

Choosing  $N = 14$  and with `Reduce` command in Mathematica, for  $n \geq 5885$ , we have (5.3). Computing

$$\max_{s \in \{0,1,2,3,4\}} \{n(14, s)\} \leq 5019,$$

we finally conclude that for  $n \geq 5885$ ,

$$4 \left(1 + \frac{\pi^2}{32(n+1)^3}\right) q(n+1)q(n+3) > q(n)q(n+4) + 3q(n+2)^2.$$

For the remaining cases  $278 \leq n \leq 5884$ , we verified with Mathematica.  $\square$

*Proof of Theorem 1.4.* Making the shift  $n \rightarrow n+1$ , it is equivalent to show that for  $n \geq 271$ ,

$$B(q(n), q(n+1), q(n+2), q(n+3), q(n+4)) > 0.$$

Following (5.1), we have

$$\begin{aligned} B(q(n), q(n+1), q(n+2), q(n+3), q(n+4)) \\ > \left( \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{1/4} n^{3/4}} \right)^2 \left( L^3(n, 2, N) + L(n, 0, N)L^2(n, 3, N) + L^2(n, 1, N)L(n, 4, N) \right. \\ \left. - U(n, 0, N)U(n, 2, N)U(n, 4, N) - 2U(n, 1, N)U(n, 2, N)U(n, 3, N) \right). \end{aligned}$$

Thus to prove the theorem, we show that

$$\begin{aligned} L^3(n, 2, N) + L(n, 0, N)L^2(n, 3, N) + L^2(n, 1, N)L(n, 4, N) \\ > U(n, 0, N)U(n, 2, N)U(n, 4, N) + 2U(n, 1, N)U(n, 2, N)U(n, 3, N). \end{aligned} \quad (5.4)$$

Choosing  $N = 24$  and with **Reduce** command in Mathematica, for  $n \geq 9800$ , we have (5.4). Computing

$$\max_{s \in \{0,1,2,3,4\}} \{n(24, s)\} \leq 18502,$$

we finally conclude that for  $n \geq 18502$ ,

$$B(q(n), q(n+1), q(n+2), q(n+3), q(n+4)) > 0.$$

For the remaining cases  $271 \leq n \leq 18501$ , we verified with Mathematica.  $\square$

*Proof of Theorem 1.5.* Making the shift  $n \rightarrow n+1$ , it is equivalent to show that for  $n \geq 308$ ,

$$\begin{aligned} \left(1 + \frac{\pi^3}{288\sqrt{3}(n+1)^{\frac{9}{2}}}\right) (2q(n+1)q(n+2)q(n+3) + q(n)q(n+2)q(n+4)) \\ > q(n+2)^3 + q(n)q(n+3)^2 + q(n+1)^2q(n+4). \end{aligned}$$

Since for  $n \geq 1$ ,

$$\frac{\pi^3}{288\sqrt{3}(n+1)^{\frac{9}{2}}} \geq \frac{\pi^3}{288\sqrt{3}n^{\frac{9}{2}}} - \frac{1}{4n^5},$$

thus it suffices to show

$$\begin{aligned} \left(1 + \frac{\pi^3}{288\sqrt{3}n^{\frac{9}{2}}} - \frac{1}{4n^5}\right) (2q(n+1)q(n+2)q(n+3) + q(n)q(n+2)q(n+4)) \\ > q(n+2)^3 + q(n)q(n+3)^2 + q(n+1)^2q(n+4). \end{aligned}$$

Therefore, following (5.1), we need to prove

$$\begin{aligned} \left(1 + \frac{\pi^3}{288\sqrt{3}n^{\frac{9}{2}}} - \frac{1}{4n^5}\right) (L(n, 0, N)L(n, 2, N)L(n, 4, N) + 2L(n, 1, N)L(n, 2, N)L(n, 3, N)) \\ > U^3(n, 2, N) + U(n, 0, N)U^2(n, 3, N) + U^2(n, 1, N)U(n, 4, N). \end{aligned} \quad (5.5)$$

Choosing  $N = 24$  and with **Reduce** command in Mathematica, for  $n \geq 18225$ , we have (5.5). Computing

$$\max_{s \in \{0,1,2,3,4\}} \{n(24, s)\} \leq 18502,$$

we finally conclude that for  $n \geq 18502$ ,

$$\begin{aligned} \left(1 + \frac{\pi^3}{288\sqrt{3}(n+1)^{\frac{9}{2}}}\right) (2q(n+1)q(n+2)q(n+3) + q(n)q(n+2)q(n+4)) \\ > q(n+2)^3 + q(n)q(n+3)^2 + q(n+1)^2q(n+4). \end{aligned}$$

For the remaining cases  $308 \leq n \leq 18501$ , we verified with Mathematica.  $\square$

*Proof of Theorem 1.6.* Making the shift  $n \rightarrow n+2$ , it is equivalent to show that for  $n \geq 271$ ,

$$(q(n+2)^2 - q(n+1)q(n+3))^2 > (q(n+1)^2 - q(n)q(n+2))(q(n+3)^2 - q(n+2)q(n+4)).$$

Following (5.1), it is enough to show that

$$\begin{aligned} (L^2(n, 2, N) - U(n, 1, N)U(n, 3, N))^2 &> (U^2(n, 1, N) - L(n, 0, N)L(n, 2, N)) \\ &\quad \times (U^2(n, 3, N) - L(n, 2, N)L(n, 4, N)). \end{aligned} \quad (5.6)$$

Choosing  $N = 14$  and with `Reduce` command in Mathematica, for  $n \geq 3153$ , we have (5.6). Computing

$$\max_{s \in \{0, 1, 2, 3, 4\}} \{n(14, s)\} \leq 5019,$$

we finally conclude that for  $n \geq 5019$ ,

$$(q(n+2)^2 - q(n+1)q(n+3))^2 > (q(n+1)^2 - q(n)q(n+2))(q(n+3)^2 - q(n+2)q(n+4)).$$

For the remaining cases  $271 \leq n \leq 5018$ , we verified with Mathematica.  $\square$

*Proof of Theorem 1.7.* Making the shift  $n \rightarrow n+2$ , it is equivalent to show that for  $n \geq 344$ ,

$$\begin{aligned} (q(n+1)^2 - q(n)q(n+3))^2 \\ < (q(n+1)^2 - q(n)q(n+2))(q(n+3)^2 - q(n+2)q(n+4)) \left(1 + \frac{\pi}{2\sqrt{3}(n+1)^{\frac{3}{2}}}\right). \end{aligned}$$

Since for  $n \geq 1$ ,

$$\frac{\pi}{2\sqrt{3}(n+1)^{\frac{3}{2}}} \geq \frac{\pi}{2\sqrt{3}n^{\frac{3}{2}}} - \frac{1}{n^2},$$

thus it is suffices to show

$$\begin{aligned} (q(n+1)^2 - q(n)q(n+3))^2 \\ < (q(n+1)^2 - q(n)q(n+2))(q(n+3)^2 - q(n+2)q(n+4)) \left(1 + \frac{\pi}{2\sqrt{3}n^{\frac{3}{2}}} - \frac{1}{n^2}\right). \end{aligned}$$

Therefore, following (5.1), we need to prove

$$\begin{aligned} \left(1 + \frac{\pi}{2\sqrt{3}n^{\frac{3}{2}}} - \frac{1}{n^2}\right) (L^2(n, 1, N) - U(n, 0, N)U(n, 2, N)) (L^2(n, 3, N) - U(n, 2, N)U(n, 4, N)) \\ > (U^2(n, 2, N) - L(n, 1, N)L(n, 3, N))^2. \end{aligned} \quad (5.7)$$

Choosing  $N = 14$  and with `Reduce` command in Mathematica, for  $n \geq 7056$ , we have (5.7). Computing

$$\max_{s \in \{0, 1, 2, 3, 4\}} \{n(14, s)\} \leq 5019,$$

we finally conclude that for  $n \geq 7056$ ,

$$\begin{aligned} (q(n+1)^2 - q(n)q(n+3))^2 \\ < (q(n+1)^2 - q(n)q(n+2))(q(n+3)^2 - q(n+2)q(n+4)) \left(1 + \frac{\pi}{2\sqrt{3}(n+1)^{\frac{3}{2}}}\right). \end{aligned}$$

For the remaining cases  $344 \leq n \leq 7055$ , we verified with Mathematica.  $\square$

*Proof of Theorem 1.9.* According to (5.1), it is sufficient to show that

$$10L^2(n, 3, N) + 6L(n, 1, N)L(n, 5, N) > 15U(n, 2, N)U(n, 4, N) + U(n, 0, N)U(n, 6, N). \quad (5.8)$$

Choosing  $N = 24$  and with `Reduce` command in Mathematica, for  $n \geq 12884$ , we have (5.8). Computing

$$\max_{s \in \{0,1,2,3,4,5,6\}} \{n(24, s)\} \leq 18502,$$

we finally conclude that for  $n \geq 18502$ ,

$$10q(n+3)^2 + 6q(n+1)q(n+5) > 15q(n+2)q(n+4) + q(n)q(n+6).$$

For the remaining cases  $651 \leq n \leq 18501$ , we verified with Mathematica.  $\square$

*Proof of Theorem 1.10.* Based on (5.1), we need to prove

$$\begin{aligned} \left(1 + \frac{5\pi^3}{256\sqrt{3}n^{\frac{9}{2}}}\right) (15L(n, 2, N)L(n, 4, N) + L(n, 0, N)L(n, 6, N)) \\ > 10U^2(n, 3, N) + 6U(n, 1, N)U(n, 5, N). \end{aligned} \quad (5.9)$$

Choosing  $N = 24$  and with `Reduce` command in Mathematica, for  $n \geq 17880$ , we have (5.9). Computing

$$\max_{s \in \{0,1,2,3,4,5,6\}} \{n(24, s)\} \leq 18502,$$

we finally conclude that for  $n \geq 18502$ ,

$$\begin{aligned} 10q(n+3)^2 + 6q(n+1)q(n+5) \\ < (15q(n+2)q(n+4) + q(n)q(n+6)) \left(1 + \frac{5\pi^3}{256\sqrt{3}n^{\frac{9}{2}}}\right). \end{aligned}$$

For the remaining cases  $715 \leq n \leq 18501$ , we verified with Mathematica.  $\square$

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