

On Shellability of 3-Cut Complexes of Hexagonal Grid Graphs

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Abstract

The k -cut complex was recently introduced by Bayer et al. as a generalization of earlier work of Fröberg (1990) and Eagon and Reiner (1998), and was shown to be shellable for several classes of graphs. In this article, we prove that the 3-cut complex of the hexagonal grid graph $H_{1 \times m \times n}$ is shellable for all $m, n \geq 1$, by constructing an explicit shelling order using reverse lexicographic ordering. From this shelling, we determine the number of spanning facets, denoted by $\psi_{m,n}$, and deduce that the complex is homotopy equivalent to a wedge of $\psi_{m,n}$ spheres of dimension $(2m + 2n + 2mn - 4)$, where

$$\psi_{m,n} = \binom{2m + 2n + 2mn - 1}{2} - [(6m + 2)n + (2m - 4)].$$

While these topological properties can be obtained from general results of Bayer et al., we provide an explicit combinatorial construction of a shelling order, yielding a direct counting formula for the number of spheres in the wedge sum decomposition.

1 Introduction

Simplicial complexes arising from graph-based properties are well-studied objects in topological combinatorics. The books by Jonsson [14] and Kozlov [15] are excellent works in this regard. Simplicial complexes naturally occur at the intersection of various areas of mathematics. Among these, the interaction with commutative algebra is especially prominent, where these complexes appear via Stanley–Reisner theory (see, for instance, [6, 7, 12, 13, 16, 17]).

For instance, this connection has been explored in detail in recent work of Bayer et al. (see [3, 1, 2]). In 1990, Fröberg proved his celebrated theorem [9, Theorem 1] concerning ideals generated by quadratic monomials. Subsequently, Eagon and Reiner [8, Proposition 8] generalized Fröberg’s result using the Alexander dual of the clique complex $\Delta(G)$, denoted by $\Delta_2(G)$, whose facets correspond to complements of independent sets of size 2. Motivated by this line of work, Bayer et al. recently introduced two classes of graph complexes: the *total k -cut complexes* and the *k -cut complexes*. In this article, we focus on k -cut complexes (see Definition 2.2).

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In their work, Bayer et al. established the shellability of k -cut complexes for a wide range of graph families. This includes trees, threshold graphs, cycles (for $k \geq 3$), complete multipartite graphs, squared paths, and grid graphs (specifically for the case $k = 3$). Additionally, they conjectured that this shellability extends to grid graphs for general $k \geq 1$, as well as to squared cycles when the number of vertices is sufficiently large relative to $k \geq 2$. More recently, Chouhan et al. [5] studied the 3-cut complexes of squared cycle graphs and analyzed their shellability. Also, Chandrakar et al. in [4], showed that the k -cut complex of $(2 \times n)$ -grid graphs are shellable when $n \geq 3$ and $3 \leq k \leq 2n - 2$.

In this article, we study the shellability of the 3-cut complex of hexagonal grid graphs (see Figure 1) and determine its homotopy type. We note that these results can be derived from [1, Proposition 5.4] and [3, Proposition 5.5], where shellability is established using the notion of *minimal forbidden subgraphs*, and from [3, Proposition 2.18], where the homotopy type is computed via the *Möbius function of the face lattice* of the 3-cut complex of a graph.

While the general results of Bayer et al. yield both shellability and the homotopy type of this complex through the use of additional structural notions, we provide an independent proof using only the definition of shellability. Our approach is based solely on constructing an explicit shelling order and describing the associated spanning facets. It is worth noting that determining whether a simplicial complex is shellable is an NP-complete problem [10]. Consequently, finding an explicit shelling order, even for a complex already known to be shellable, is computationally difficult and often requires a detailed understanding of its combinatorial structure.

In particular, we prove the following results,

Theorem 1.1 (Theorem 3.3). *For $m, n \geq 1$, the simplicial complex $\Delta_3(H_{1 \times m \times n})$ is shellable.*

The existence of a shelling order allows us to determine the homotopy type of the complex. This leads to the following theorem.

Theorem 1.2 (Theorem 4.1). *The 3-cut complex of $H_{1 \times m \times n}$ has the homotopy type of wedge of $\psi_{m,n}$ many $(2m + 2n + 2mn - 4)$ -dimensional spheres, where,*

$$\psi_{m,n} = \binom{2m + 2n + 2mn - 1}{2} - [(6m + 2)n + (2m - 4)].$$

In other words,

$$\Delta_3(H_{1 \times m \times n}) \simeq \bigvee_{\psi_{m,n}} \mathbb{S}^{(2m+2n+2mn-4)}.$$

Organization of the article:

In Section 2, we define the hexagonal grid graphs $H_{1 \times m \times n}$ for $m, n \geq 1$ and recall the necessary background on simplicial complexes, including the notions of shellability and spanning facets that will be used throughout the paper. In Section 3, we introduce an explicit shelling order on the facets of the 3-cut complex of $H_{1 \times m \times n}$, defined via reverse lexicographic ordering, and prove that it indeed yields a shelling. Finally, in Section 4, we characterize all spanning facets associated with this shelling order and use this description to determine the homotopy type of $\Delta_3(H_{1 \times m \times n})$.

2 Preliminaries

In this section, we introduce the key concepts and results necessary for discussing the article. For $k_1, k_2 \in \mathbb{N}$, such that $1 \leq k_1 < k_2$, we denote

$$[k_1] = \{1, 2, 3, \dots, k_1\}; \\ [k_1, k_2] = \{k_1, k_1 + 1, k_1 + 2, \dots, k_2\}.$$

2.1 Basics of graph theory

A **graph** G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and

$$E(G) \subseteq \{\{u, v\} \mid u, v \in V(G), u \neq v\}$$

is the set of edges. We use the notation uv to denote edge $\{u, v\}$. We define the **open neighborhood** of the vertex v in G , denoted by $N_G(v)$ as,

$$N_G(v) = \{u \in V(G) \mid u \neq v \text{ and } \{u, v\} \in E(G)\}.$$

Similarly, the **closed neighborhood** of the vertex v in G , denoted by $N_G[v]$ is defined as,

$$N_G[v] = N_G(v) \sqcup \{v\}.$$

When the underlying graph is clear from the context, we omit the subscript and write $N(v)$ and $N[v]$ to denote the open and closed neighborhoods of v , respectively.

Let S be a subset of $V(G)$. The **induced subgraph** of G on S , denoted by $G[S]$, is the graph whose vertex set is S and whose edge set consists of all edges of G that have both vertices in S . We denote the induced subgraph $G[V(G) \setminus H]$ by $G \setminus H$, where $H \subset V(G)$. We now proceed to formally define the hexagonal grid graphs.

2.2 Labelling of vertices in the general hexagonal grid graph.

The hexagonal grid graphs are generally defined using 3 variables $r, s, t \geq 1$ and are denoted by $H_{r \times s \times t}$. For our discussion, we fix $r = 1$, that is, we will deal with $H_{1 \times s \times t}$. Here, our focus is primarily on the labelling of vertices, as it will play a crucial role in defining the shelling order.

Definition 2.1 (Hexagonal grid graphs). For $m, n \geq 1$, we define the hexagonal grid graph, $H_{1 \times m \times n}$, as a graph with the following vertices and edges,

$$V(H_{1 \times m \times n}) = [2m + 2n + 2mn]; \\ E(H_{1 \times m \times n}) = \mathcal{E}_1 \sqcup \mathcal{E}_2 \sqcup \mathcal{E}_3 \sqcup \mathcal{E}_4.$$

where, for $1 \leq i \leq 4$,

$$\begin{aligned} \mathcal{E}_1 &= \{(i)(i + m + n + mn) \mid i \in [m] \cup [1 + n + nm, m + n + nm]\}; \\ \mathcal{E}_2 &= \{(i)(i + n + nm) \mid i \in [m + 1, m + n + nm]\}; \\ \mathcal{E}_3 &= \{(i)(i + m + n + mn + 1) \mid i \in [m, n + nm, m + n + nm - 1]\}; \\ \mathcal{E}_4 &= \{(i)(i + m + n + mn + 2) \mid i \in [m + 1, n + nm - 2]\}. \end{aligned}$$

We will use $[2m + 2n + 2mn]$ and $V(H_{1 \times m \times n})$ interchangeably. For example, if $m = 4$ and $n = 6$, then the graph $H_{1 \times 4 \times 6}$ will look like Figure 1. Since the argument in the main result of the proof is rather intricate, we will refer to this example frequently for better understanding.

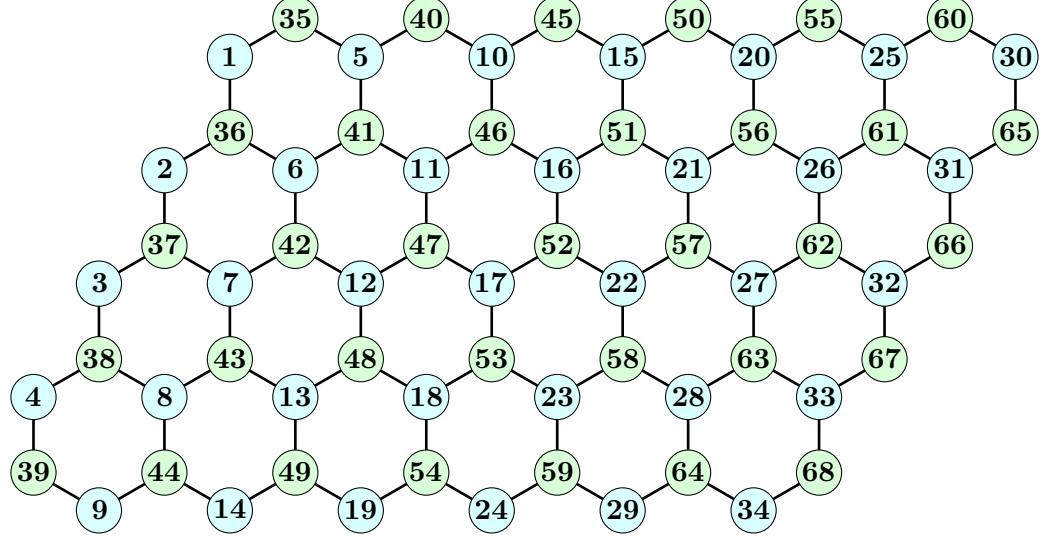


Figure 1: The graph $H_{1 \times 4 \times 6}$.

Before proceeding further, we fix some notations and carefully list some important observations that arise directly from the construction of $H_{1 \times m \times n}$. For $m, n \geq 1$, let $V(H_{1 \times m \times n}) = V_1 \sqcup V_2$, where,

$$\begin{aligned} V_1 &= [m + n + mn] ; \\ V_2 &= [m + n + mn + 1, 2m + 2n + 2mn] . \end{aligned}$$

For instance, in Figure 1, the blue coloured vertices belong to V_1 and the green coloured vertices belong to V_2 .

We say that a subset $A \subseteq V(H_{1 \times m \times n})$ is *connected* if the induced subgraph $H_{1 \times m \times n}[A]$ is connected. It is easy to see that a connected subset of size 3 in $V(H_{1 \times m \times n})$ is always a P_3 (path graph on three vertices). That is, if $\{a, b, c\} \subseteq V(H_{1 \times m \times n})$ is connected, then a and c are the *endpoints* (end vertices) of P_3 , each adjacent to b , the *mid-point* (vertex in between the endpoints). Also, we will say that $\{a, b, c\}$ induces a P_3 in $H_{1 \times m \times n}$.

Observation 2.1. *Let $\{a, b, c\}$ induces a P_3 in $H_{1 \times m \times n}$ with endpoints a and c , then exactly one of the following hold:*

- i. $a, c \in V_1$ and $b \in V_2$ implying $a < c < b$; or,
- ii. $a, c \in V_2$ and $b \in V_1$ implying $b < a < c$.

This means both a and c will either be in V_1 or V_2 , but not one in each, and as a result, b will be in V_2 or V_1 , respectively.

Observation 2.2. If $\{a, b, c\}$ induces P_3 in $H_{1 \times m \times n}$ with endpoints a and c , then these endpoints are unique; that is, no other induced P_3 in $H_{1 \times m \times n}$ has a and c as its endpoints.

Observation 2.3. If $v \in V_1$ then $N(v) \subset V_2$. Similarly, if $v \in V_2$ then $N(v) \subset V_1$. Recall that $N(v)$ is the open neighborhood of the vertex v .

2.3 Topological preliminaries

We follow Hatcher's *Algebraic Topology* [11] and Kozlov's *Combinatorial Algebraic Topology* [15] as our primary references for standard definitions and terminology.

A *simplicial complex* \mathcal{K} is a non-empty family of subsets of a finite set V , referred to as the vertex set, with the property that it is closed under taking subsets, that is, for every $\tau \in \mathcal{K}$ and $\sigma \subset \tau$, we have $\sigma \in \mathcal{K}$. We call an element $\sigma \in \mathcal{K}$ a *face* of \mathcal{K} . A face consisting of $k+1$ vertices is called a k -dimensional face, and the maximum dimension among all faces is called the dimension of the simplicial complex. The maximal faces of a simplicial complex, that is, those faces not properly contained in any other face, are called *facets*.

Since simplicial complexes are closed under taking subsets, it suffices to define their facets, because all other faces are obtained by taking subsets of these. Thus, if a simplicial complex \mathcal{K} has facets F_1, F_2, \dots, F_t , we say that \mathcal{K} is generated by these facets, and we write

$$\mathcal{K} = \langle F_1, F_2, \dots, F_t \rangle.$$

We now recall the definition of the k -cut complex.

Definition 2.2 ([1], Definition 2.7). Let $G = (V, E)$ be a graph and let $k \geq 1$. The *k -cut complex* of G , denoted by $\Delta_k(G)$, is the simplicial complex whose facets are the complements of subsets of $V(G)$ of size k whose induced subgraphs are disconnected. Equivalently, a subset $\sigma \subseteq V(G)$ is a face of $\Delta_k(G)$ if and only if its complement $V(G) \setminus \sigma$ induces a disconnected subgraph on k vertices.

$$\Delta_k(G) = \langle \sigma \in V(G) \mid G[V(G) \setminus \sigma] \text{ is disconnected and } |V(G) \setminus \sigma| = k \rangle.$$

2.3.1 Shellable simplicial complexes and spanning facets

A crucial combinatorial concept that reveals important topological information about a simplicial complex is shellability. We define this as follows,

Definition 2.3 ([15], Defintion 12.1). A simplicial complex \mathcal{K} is said to be *shellable* if the facets of \mathcal{K} can be put together in a linear order F_1, F_2, \dots, F_t such that the subcomplex

$$\left(\bigcup_{s=1}^{j-1} (F_s) \right) \cap (F_j)$$

is pure and of dimension $(\dim(F_j) - 1)$ for each $2 \leq j \leq t$. Such an ordering on \mathcal{K} is called a *shelling order*.

The definition of shellability mentioned above is commonly used. We now discuss an alternative method for determining a shelling order, which will be useful in many cases. A simplicial complex

Δ has a shelling order F_1, F_2, \dots, F_t of its facets if and only if for any i, j satisfying $1 \leq i < j \leq t$, there exists $1 \leq r < j$ such that

$$F_r \cap F_j = F_j \setminus \{\lambda\}, \quad \text{for some } \lambda \in F_j \setminus F_i. \quad (*)$$

A facet F_k ($1 < k \leq t$) is called *spanning* with respect to the given shelling order if the boundary of F_k is contained in union of facets coming before F_k in shelling order, that is,

$$\partial(F_k) \subseteq \bigcup_{i=1}^{k-1} F_i.$$

Here, $\partial(F_k)$ denotes the boundary of F_k . To fix notation, recall that a facet f of a simplicial complex is a d -simplex, where $|f|=d+1$. We now define the boundary of a simplex.

Definition 2.4 (Boundary of a simplex). Let $\sigma = \{v_0, v_1, \dots, v_n\}$ be an n -simplex. The *boundary* of σ , denoted by $\partial\sigma$, is the simplicial complex consisting of all proper faces of σ , that is,

$$\partial\sigma = \{\tau \subsetneq \sigma\}.$$

It is easy to check that F_k is a spanning facet if for each $\lambda \in F_k$, there exists $r < k$ such that

$$F_r \cap F_k = F_k \setminus \{\lambda\}. \quad (\#)$$

Theorem 2.1 ([15, Theorem 12.3]). *Let Δ be a pure shellable simplicial complex of dimension d . Then Δ has the homotopy type of a wedge of β spheres of dimension d , where β is the number of spanning facets in a given shelling order. Hence,*

$$\Delta \simeq \bigvee_{\beta} S^d.$$

3 Shellability of 3-cut complex of hexagonal grid graph

In this section, we prove that the 3-cut complex of a hexagonal tiling $H_{1 \times m \times n}$, that is, $\Delta_3(H_{1 \times m \times n})$, is a shellable simplicial complex. For this, our objective is to define an order on the facets of $\Delta_3(H_{1 \times m \times n})$ and show that it is a shelling order. Before that, we need to specify the ordering on the vertices, and for this graph, we will follow the lexicographic ordering of the vertices of $H_{1 \times m \times n}$.

Before explicitly defining a shelling order on the facets of $\Delta_3(H_{1 \times m \times n})$, let us find out the number of facets.

3.1 Number of facets in $\Delta_3(H_{1 \times m \times n})$

Let $\eta_{m,n}$ be the number of facets of $\Delta_3(H_{1 \times m \times n})$. Precisely speaking, $\eta_{m,n}$ is the number of size-3 subsets of the set $[2m+2n+2mn]$ excluding those that induce P_3 . Note that there are $\binom{2m+2n+2mn}{3}$ subsets of size 3 in $[2m+2n+2mn]$. We state the following lemma to find the number of induced P_3 in $H_{1 \times m \times n}$.

Lemma 3.1. *For $m, n \geq 1$, let $\delta_{m,n}$ be the number of induced P_3 in $H_{1 \times m \times n}$. Then,*

$$\delta_{m,n} = 6mn + 2m + 2n - 4.$$

Proof. Observe that there are exactly $2mn - 2$ vertices of degree three (the vertices in the interior), each contributing to three induced P_3 , and exactly $2m + 2n$ vertices of degree two (the vertices on the boundary), contributing to $2m + 2n + 2$ induced P_3 's. Thus, the total number of induced P_3 in $H_{1 \times m \times n}$ is

$$3(2mn - 2) + (2m + 2n + 2) = 6mn + 2m + 2n - 4.$$

This completes the proof of Theorem 3.1. \square

Hence,

$$\eta_{m,n} = \binom{2m + 2n + 2mn}{3} - (6mn + 2m + 2n - 4).$$

3.2 Shelling order for the facets of $\Delta_3(H_{1 \times m \times n})$

We now give the idea for constructing an ordering to the facets of $\Delta_3(H_{1 \times m \times n})$, which we will prove is indeed a shelling ordering. First, arrange the facets of $\Delta_3(H_{1 \times m \times n})$ in *reverse lexicographic order*, then remove certain facets during the process and append them to the end of the ordering in the order in which they were removed.

Let “ \ll ” and “ \succcurlyeq ” denote the reverse and general lexicographic ordering, respectively. Let $F'_1, F'_2, \dots, F'_{\eta_{m,n}}$ be the facets of $\Delta_3(H_{1 \times m \times n})$ where,

$$F'_1 \ll F'_2 \ll \dots \ll F'_{\eta_{m,n}}. \quad (1)$$

For $m, n \geq 1$, we define T_i , for some $1 \leq i \leq \beta_{m,n}$, where

$$\beta_{m,n} = \begin{cases} mn - 2, & m \geq 2 \\ n - 1, & m = 1 \end{cases},$$

as follows.

1. When $m = 1$ and $n \geq 1$. For $1 \leq i \leq n - 1$, we define T_i 's as

$$T_i = \{2i + 2n, 2i + 2n + 2, 2i + 2n + 3\}^c.$$

2. When $m \geq 2$ and $n \geq 1$.

- (a) For each $k = 1, 2, \dots, n - 1$, and for $(k - 1)m + 1 \leq i \leq km$, define

$$T_i = \left\{ \begin{array}{l} i + m + n + mn + (k - 1), i + 2m + n + mn + k, \\ i + 2m + n + mn + (k + 1) \end{array} \right\}^c.$$

- (b) For $(n - 1)m + 1 \leq i \leq nm - 2$, define

$$T_i = \{i + m + 2n + mn, i + 2m + 2n + mn, i + 2m + 2n + mn + 1\}^c.$$

Let t_i be the index of T_i in Equation (1), i.e., $T_i = F'_{t_i}$. We remove the facets $T_1, T_2, \dots, T_{\beta_{m,n}}$ one by one from Equation (1) and place them at the end in the order in which they were removed.

Definition 3.1. We define the following ordering on the facets of $\Delta_3(H_{1 \times m \times n})$:

$$F_1, F_2, \dots, F_{\eta_{m,n} - \beta_{m,n}}, T_1, T_2, \dots, T_{\beta_{m,n}}. \quad (2)$$

where the facets F_j using Equation (1) is given by:

1. For $1 \leq j \leq t_1 - 1$,

$$F_j = F'_j.$$

2. For $2 \leq i \leq \beta_{m,n} - 1$ and $t_{i-1} + 1 \leq j \leq t_i - 1$,

$$F_{j-i+1} = F'_j.$$

3. For $t_{\beta_{m,n}} + 1 \leq j \leq \eta_{m,n}$,

$$F_{j-\beta_{m,n}} = F'_j.$$

Observation 3.1. Note that T_i^c defined above are the neighborhoods of certain vertices, that is,

$$T_i^c = \begin{cases} \mathcal{N}(i+m), & 1 \leq i \leq m; \\ \mathcal{N}(i+m+1), & m+1 \leq i \leq 2m; \\ \mathcal{N}(i+m+2), & 2m+1 \leq i \leq 3m; \\ \vdots \\ \mathcal{N}(i+m+(k-1)), & (k-1)m+1 \leq i \leq km; \\ \vdots \\ \mathcal{N}(i+m+(n-2)), & (n-2)m+1 \leq i \leq (n-1)m; \\ \mathcal{N}(i+m+n), & (n-1)m+1 \leq i \leq mn-2. \end{cases} \quad (3)$$

The central idea of this construction is to begin with the reverse lexicographic order on facets and then relocate a carefully chosen family of exceptional facets corresponding to neighborhoods of certain vertices. These exceptional facets obstruct shellability if left in place and must be appended at the end.

We call the vertex c_i for which $T_i^c = \mathcal{N}(c_i)$ as the *centre* of T_i , and observe that,

$$T_1 \ll T_2 \ll \dots \ll T_{\beta_{m,n}}.$$

Also, we have some immediate observations from these explicit constructions of T_i .

Observation 3.2. $T_i^c \subset V_2$, for all $1 \leq i \leq \beta_{m,n}$.

Observation 3.3. For all $1 \leq i \leq \beta_{m,n}$, T_i^c is an independent set.

Observation 3.4. If $T_i^c = \{x_1, x_2, x_3\}$, with $x_1 < x_2 < x_3$, then $x_3 = x_2 + 1$, that is, x_2 and x_3 are consecutive integers. Consequently,

$$T_i = \{x_1, x_2, x_2 + 1\}.$$

Before proving our main result, we prove the following lemma, which will play a significant role in proving the shellability of $\Delta_3(H_{1 \times m \times n})$.

Lemma 3.2. *Let $T_1, T_2, \dots, T_{\beta_{m,n}}$ denote the facets that were removed from the reverse lexicographic ordering of the facets of $\Delta_3(H_{1 \times m \times n})$, as defined previously. If a facet T_i were kept at its original position, say j_i (that is, $F_{j_i} = T_i$), in the reverse lexicographic order, then the shellability condition would fail for each such T_i ; that is, there exists an index $i' < j_i$ such that there is no index $r < j_i$ satisfying*

$$F_r \cap F_{j_i} = F_{j_i} \setminus \{\lambda\},$$

where $\lambda \in F_{j_i} \setminus F_{i'}$.

Proof. Fix a facet T_i . By construction, T_i is the open neighborhood of a vertex x_i , as defined in Equation (3). For T_i to satisfy the shellability condition (*), there must exist, for every $i' < j_i$, a facet F_r with $r < j_i$ satisfying

$$F_r \cap F_{j_i} = F_{j_i} \setminus \{\lambda\},$$

for some $\lambda \in F_{j_i} \setminus F_{i'}$. We claim this is impossible because an index $i' < j_i$ exists for which no such F_r exists. Define:

$$F_{i'} := (\{x_i, 2m + 2n + 2mn - 1, 2m + 2n + 2mn\})^c.$$

Observe that

$$T_i \setminus F_{i'} = \begin{cases} F_{i'}^c, & \text{for } 1 \leq i < \beta_{m,n}; \\ \{x_{\beta_{m,n}}, 2m + 2n + 2mn\}, & \text{for } i = \beta_{m,n}. \end{cases}$$

To construct F_r , we must remove a vertex from T_i^c and add a vertex from $T_i \setminus F_{i'}$. However, for any $\lambda \in T_i \setminus F_{i'}$, the following occurs:

- i. If $\lambda = x_i$, then F_r^c induces a P_3 , contradicting the definition of a facet in $\Delta_3(H_{1 \times m \times n})$.
- ii. If $\lambda \neq x_i$, then $T_i^c \succcurlyeq F_r^c$, or equivalently, $T_i \prec F_r$, which contradicts $r < j_i$ under reverse lexicographic ordering.

In both cases, no such F_r satisfying the shellability condition exists, and thus the condition fails for T_i at position j_i . This completes the proof of Theorem 3.2. \square

Remark. There are other possible choices for the facet $F_{i'}$ which also fail to work for each T_i . In each such case, the corresponding F_r constructed from that choice leads to the same issues described in the proof above, and the argument for the failure remains the same.

3.3 Shellability of $\Delta_3(H_{1 \times m \times n})$

We now state the main result of this section.

Theorem 3.3. *For $m, n \geq 1$, the simplicial complex $\Delta_3(H_{1 \times m \times n})$ is shellable.*

Proof. We claim that the ordering given in Definition 3.1 is the required shelling order for the facets of $\Delta_3(H_{1 \times m \times n})$. That is, we aim to show the following:

- A. $F_1, F_2, \dots, F_{\eta_{m,n} - \beta_{m,n}}$ are in shelling order.
- B. If the T_i 's are placed among the F_i 's according to the reverse lexicographic ordering (that is, Equation (1)), the resulting order is not a shelling order.

C. For all $1 \leq i \leq \beta_{m,n}$, T_i satisfies the shellability condition with every F_j , where $1 \leq j \leq \eta_{m,n} - \beta_{m,n}$ and with every $T_{i'}$, where $i' < i$.

D. $T_1, T_2, \dots, T_{\beta_{m,n}}$ are in shelling order.

B follows from Theorem 3.2. In its proof, the failure of T_i to satisfy the shellability condition when kept in reverse lexicographic ordering arose because the required F_r (with $r < j_i$) appeared after T_i . Placing all T_i at the end resolves this issue. Thus, T_i satisfies the shellability condition with F_j . Consequently, D is proved. It remains only to show in D that T_i satisfies the shellability condition with $T_{i'}$ with $i' < i$. The argument for this is analogous to that used in proving A. We therefore proceed to prove A.

To prove A, we need to prove that the ordering $F_1, F_2, \dots, F_{\eta_{m,n} - \beta_{m,n}}$ satisfies the condition Equation (*). In other words, we must construct a suitable F_r for any given choice of F_i and F_j . Note here that, $i < j$ if and only if $F_i \prec F_j$ (or equivalently, $F_i^c \succ F_j^c$). We will deal with the complements of the facets to prove our result. Also, observe that for any $1 \leq i < j \leq (\eta_{m,n} - \beta_{m,n})$, we have the following three scenarios:

1. $|F_i^c \cap F_j^c| = 0$;
2. $|F_i^c \cap F_j^c| = 1$;
3. $|F_i^c \cap F_j^c| = 2$.

We deal with all of these cases one by one.

Case 1: $|F_i^c \cap F_j^c| = 0$.

Let $F_i^c = \{i_1, i_2, i_3\}$ and $F_j^c = \{j_1, j_2, j_3\}$. Without loss of generality (WLOG), let $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$.

Since $F_i^c \succ F_j^c$, we must have $i_1 < j_1$. Note that $F_i^c \subset F_j$ (since $F_i^c \cap F_j^c = \emptyset$), which implies,

$$F_j \setminus F_i = F_i^c.$$

We examine all possible configurations of F_i^c and F_j^c under the given condition, and construct F_r^c accordingly. In this case, F_r is explicitly defined as,

$$F_r = (F_j \cup \{\alpha\}) \setminus \{\lambda\},$$

where, $\lambda \in F_i^c$ and $\alpha \in F_j^c$. Equivalently,

$$F_r^c = (F_j^c \cup \{\lambda\}) \setminus \{\alpha\}.$$

We call the pair (λ, α) an *optimal pair* if it yields such an F_r^c that satisfies (*). Based on the relative ordering of i_2, i_3, j_1, j_2 and j_3 , we have the several sub-cases. The objective here is to look at the worst-case scenarios for each sub-case, that is, if F_r^c is some T_i^c or induces P_3 .

1.1. $i_1 < i_2 < i_3 < j_1 < j_2 < j_3$.

To find an optimal pair, we start by choosing $(\lambda, \alpha) = (i_1, j_2)$. This gives

$$F_r^c = \{i_1 < j_1 < j_3\},$$

such that $F_r^c \succcurlyeq F_j^c$.

At first, suppose that $F_r^c = T_k^c$, for some $1 \leq k \leq \beta_{m,n}$. This will imply that $j_3 = j_1 + 1$, that is, j_1 and j_3 are consecutive integers. However, this is not possible since $j_1 < j_2 < j_3$. Therefore, $F_r^c \neq T_k^c$, for all $1 \leq k \leq \beta_{m,n}$.

Now, if F_r^c induces P_3 , then by Observation 2.1, one of the following two configurations must occur:

- (a) If $i_1, j_1 \in V_1$ and $j_3 \in V_2$, then i_1 and j_1 are the endpoints, each adjacent to j_3 , the midpoint.
- (b) If $i_1 \in V_1$ and $j_1, j_3 \in V_2$, then j_1 and j_3 are the endpoints, each adjacent to i_1 , the midpoint.

It turns out that both of these cases are possible. For instance, in the $\Delta_3(H_{1 \times 4 \times 6})$ (see Figure 1), if we take $F_i^c = \{11, 12, 16\}$ and $F_j^c = \{17, 46, 47\}$, then for the given choice of λ and α we obtain $F_r^c = \{11, 17, 47\}$, which indeed forms an induced P_3 in accordance with condition in (a). Similarly, choosing $F_i^c = \{17, 18, 46\}$ and $F_j^c = \{47, 48, 52\}$, yields $F_r^c = \{17, 47, 52\}$, which again gives an induced P_3 as required by condition in (b). Consequently, the choice $(\lambda, \alpha) = (i_1, j_2)$ does not always yield an optimal pair, as demonstrated above.

Since the previous choice of (λ, α) is not optimal, we retain the same positioning of i_1, i_2, i_3, j_1, j_2 , and j_3 but replace $\alpha = j_2$ by $\alpha = j_3$, while keeping $\lambda = i_1$. Thus, we have

$$F_r^c = \{i_1 < j_1 < j_2\},$$

such that $F_r^c \succcurlyeq F_j^c$. We now establish that this selection is indeed optimal for both conditions given in (a) and (b). In particular, for each part, we will verify both possible worst-case scenarios, namely when F_r^c coincides with some T_k^c and when it induces P_3 .

For part (a), F_r^c cannot coincide with any T_k^c . Indeed, if this were the case, then F_r^c would have j_3 as its center, which is impossible since $j_3 \in V_2$. By Observation 2.3 and Observation 3.2, the center of any T_k^c must lie in V_1 . On the other hand, if F_r^c induces P_3 , then this can only occur if j_2 is in the position of j_3 , that is, $j_2 = j_3$, which is not the case here. Hence, the pair $(\lambda, \alpha) = (i_1, j_3)$ is optimal.

For part (b), F_r^c cannot be a T_k^c because i_1 and j_1 are adjacent, whereas by Observation 3.3, all T_k^c are independent. If instead F_r^c induces a P_3 , then j_2 must be adjacent to either i_1 or j_1 . It cannot be adjacent to i_1 , for in that case $F_j^c = \mathcal{N}(i_1)$, which is precisely of the form of some T_k^c . However, this is a contradiction to F_j^c not being any T_i^c .

At this point, we explicitly mention a possible exception to this argument. Let $z = 2m + 2n + 2mn$ and take $F_j^c = \{z - m - 1, z - 1, z\}$. Clearly, $F_j^c = \mathcal{N}(m + n + mn - 1)$ and it does not coincide with any T_k^c , as is clear from the construction of T_k^c . In this particular case, both previous choices of (λ, α) will satisfy the condition of all possible worst-case scenarios, including part (b). However, this can be easily resolved by replacing the choice of $\lambda = i_1$ with $\lambda = i_2$ (or even i_3), while keeping $\alpha = j_2$. Thus, we will also obtain an optimal pair for this exception.

Now, if j_2 is adjacent to j_1 , then $j_2 \in V_1$, since $j_1 \in V_2$ (by Observation 2.3, which forces $j_2 < j_1$, which is not possible here. Therefore, the pair $(\lambda, \alpha) = (i_1, j_3)$ (or (i_2, j_3) for the exceptional case) forms an optimal pair.

1.2. $i_1 < i_2 < j_1 < i_3 < j_2 < j_3$.

Observe that $(\lambda, \alpha) = (i_3, j_1)$ cannot be an optimal pair, since it will imply $F_j^c \succcurlyeq F_r^c$, or equivalently $F_r \ll F_j$, contrary to our requirement. This subcase then follows using a similar argument given in subcase 1.1.

1.3. $i_1 < i_2 < j_1 < j_2 < i_3 < j_3$.

Observe that (λ, α) cannot be (i_3, j_1) or (i_3, j_2) as a choice for the optimal pair, since this forces $F_j^c \succcurlyeq F_r^c$. The same optimal pairs that are defined in subcase 1.1, and the arguments thereafter, follow here too, with only a modification concerning the exceptional configuration. Earlier, the exception arises when $F_j^c = \{z - m - 1, z - 1, z\}$, with $j_2 = z - 1$ and $j_3 = z$, that is, there is no integer between j_2 and j_3 . However, in this subcase, we have $j_2 < i_3 < j_3$, so this situation cannot arise.

1.4. $i_1 < i_2 < j_1 < j_2 < j_3 < i_3$.

Observe that (λ, α) cannot be any of (i_3, j_1) , (i_3, j_2) , or (i_3, j_3) as a choice for the optimal pair, since this forces $F_j^c \succcurlyeq F_r^c$. The same optimal pairs that are defined in subcase 1.1, and the arguments thereafter, follow here too, with some changes in the exceptional configuration. In subcase 1.1, the exception arises when $F_j^c = \{z - m - 1, z - 1, z\}$ with $j_3 = z = 2m + 2n + 2mn$. In the present subcase, however, this situation does not arise because we require $j_3 < i_3$, which leaves no possible choice for i_3 in $[2m + 2n + 2mn]$.

1.5. $i_1 < j_1 < i_2 < i_3 < j_2 < j_3$.

Observe that (λ, α) cannot be (i_3, j_1) or (i_3, j_2) as a choice for the optimal pair, since this forces $F_j^c \succcurlyeq F_r^c$. The same optimal pairs that are defined in subcase 1.1, and the arguments thereafter, follow here too, with some changes in the exceptional configuration. If $m \geq 3$, then we can proceed with the argument given for the exception in subcase 1.1. If $m = 2$, then $F_j^c = \{z - 3, z - 1, z\}$ implying there is only vertex possible in between $j_1 = z - 3$ and $j_2 = z - 1$ which is $z - 2$. However, in this subcase, we have $j_1 < i_2 < i_3 < j_2$, implying this situation cannot arise here.

1.6. $i_1 < j_1 < i_2 < j_2 < i_3 < j_3$.

Observe that (λ, α) cannot be any of (i_2, j_1) , (i_3, j_1) , or (i_3, j_2) as a choice for the optimal pair, since this forces $F_j^c \succcurlyeq F_r^c$. This subcase then follows using a similar argument given in subcase 1.1 and 1.3.

1.7. $i_1 < j_1 < i_2 < j_2 < j_3 < i_3$.

Observe that (λ, α) cannot be any of (i_2, j_1) , (i_3, j_1) , (i_3, j_2) , or (i_3, j_3) as a choice for the optimal pair, since this forces $F_j^c \succcurlyeq F_r^c$. This subcase then follows using a similar argument given in subcase 1.1 and 1.4.

1.8. $i_1 < j_1 < j_2 < i_2 < i_3 < j_3$.

Observe that (λ, α) cannot be any of (i_2, j_1) , (i_2, j_2) , (i_3, j_1) , or (i_3, j_2) as a choice for the optimal pair, since this forces $F_j^c \succcurlyeq F_r^c$. This subcase then follows using a similar argument given in subcase 1.1 and 1.3.

1.9. $i_1 < j_1 < j_2 < i_2 < j_3 < i_3$.

Observe that (λ, α) cannot be any of (i_2, j_1) , (i_2, j_2) , (i_3, j_1) , (i_3, j_2) , or (i_3, j_3) as a choice for the optimal pair, since this forces $F_j^c \succcurlyeq F_r^c$. This subcase then follows using a similar argument given in subcase 1.1 and 1.4.

1.10. $i_1 < j_1 < j_2 < j_3 < i_2 < i_3$.

Observe that (λ, α) is an optimal pair only when $\lambda = i_1$ for some $\alpha \in F_j^c$; otherwise, we have $F_j^c \succcurlyeq F_r^c$. This subcase then follows using a similar argument given in subcase 1.1 and 1.4.

Case 2: $|F_i^c \cap F_j^c| = 1$.

Let $F_i^c = \{i_1, i_2, x\}$ and $F_j^c = \{j_1, j_2, x\}$. WLOG, let $i_1 < i_2$ and $j_1 < j_2$.

For the ordering $F_i^c \succcurlyeq F_j^c$ to hold, it is necessary that $i_1 < j_1$. This will be easily verified by a case-by-case analysis of all possible orderings of x, i_1, i_2, j_1, j_2 , which we will present in the subsequent discussion. Moreover, observe here that $\{i_1, i_2\} \subset F_j$, implying,

$$F_j \setminus F_i = \{i_1, i_2\}.$$

Thus, for this case, we define F_r explicitly as,

$$F_r = (F_j \cup \{\alpha\}) \setminus \{\lambda\},$$

or equivalently,

$$F_r^c = (F_j^c \cup \{\lambda\}) \setminus \{\alpha\},$$

where, $\lambda \in \{i_1, i_2\}$ and $\alpha \in F_j^c$.

If we fix the position of x in F_i^c and accordingly look at the possibilities of F_j^c , then we will have the following subcases:

2.1. $F_i^c = \{x < i_1 < i_2\}$ and $F_j^c = \{x < j_1 < j_2\}$.

It is clear from the above configuration of F_i^c and F_j^c that in order to have $F_i^c \succcurlyeq F_j^c$, we must have $i_1 < j_1$. Observe that (λ, α) cannot be (i_1, x) or (i_2, x) as a choice for the optimal pair, since it forces $F_j^c \succcurlyeq F_r^c$.

Moreover, it follows that F_j^c can only assume the form $\{x < j_1 < j_2\}$ for the given F_i^c . Indeed, if F_j^c is of the form $\{j_1 < x < j_2\}$ or $\{j_1 < j_2 < x\}$, then we will always have $F_j^c \succcurlyeq F_i^c$, contrary to our requirement. So, it remains to analyze the possible orderings of i_2, j_1 , and j_2 , leading to the following scenarios.

2.1.1. $i_2 < j_1 < j_2$ (that is, $x < i_1 < i_2 < j_1 < j_2$).

We proceed by choosing $(\lambda, \alpha) = (i_1, j_1)$ as our initial choice for the optimal pair. The F_r^c thus obtained will not assume the form of T_k^c for any $1 \leq k \leq \beta_{m,n}$, and the argument follows similar steps to that given in subcase 1.1.

If F_r^c induces P_3 , then, as in subcase 1.1, we obtain two possible scenarios analogous to parts (a) and (b). Both of these can occur in the present subcase. We will not provide a concrete example, since it can be easily constructed by analyzing F_i^c and F_j^c . For completeness, we will explicitly state parts (a) and (b) in this case as well, since doing so will make the later arguments easier to understand.

- (a) If $x, i_1 \in V_1$ and $j_2 \in V_2$, then x and i_1 are the endpoints, each adjacent to j_2 , the midpoint.
- (b) If $x \in V_1$ and $i_1, j_2 \in V_2$, then i_1 and j_2 are the endpoints, each adjacent to x , the midpoint.

Since the initial choice $(\lambda, \alpha) = (i_1, j_1)$ was not optimal, we now replace $\alpha = j_1$ with $\alpha = j_2$, keeping $\lambda = i_1$ and the positioning of x, i_1, i_2, j_1 , and j_2 unchanged in both part (a) and (b). This will give

$$F_r^c = \{x < i_1 < j_1\}.$$

For parts (a) and (b), we verify that the worst-case scenarios, namely, when F_r^c coincides with some T_k^c and when it induces a P_3 , cannot occur. The argument for part (a) here is analogous to that in subcase 1.1. We modify the argument for part (b) here.

For part (b), F_r^c cannot be a T_k^c because x and i_1 are adjacent, whereas by Observation 3.3, all T_k^c are independent. If instead F_r^c induces a P_3 , then j_1 must be adjacent to either x or i_1 . It cannot be adjacent to x , since that would make F_j^c induce a P_3 , contradicting the disconnectedness of F_j^c . If j_1 is adjacent to i_1 , then $j_1 \in V_1$, since $i_1 \in V_2$ (by Observation 2.3), which forces $j_1 < i_1$, which is not possible here. Also, we do not get any exceptions here as we do in subcase 1.1. Therefore, the pair (λ, α) is optimal.

2.1.2. $j_1 < i_2 < j_2$ (that is, $x < i_1 < j_1 < i_2 < j_2$).

Observe that the choice $(\lambda, \alpha) = (i_2, j_1)$ will never be an optimal pair, since it forces $F_j^c \succcurlyeq F_r^c$. This subcase follows from the previous subcase.

2.1.3. $j_1 < j_2 < i_2$ (that is, $x < i_1 < j_1 < j_2 < i_2$).

Observe that (λ, α) cannot be (i_2, j_1) or (i_2, j_2) as a choice for the optimal pair, since it forces $F_j^c \succcurlyeq F_r^c$. The argument then proceeds exactly as in the subcase 2.1.1.

2.2. $F_i^c = \{i_1 < x < i_2\}$ and $F_j^c = \{x < j_1 < j_2\}$.

Clearly, $i_1 < x < j_1$ will always hold and thus $F_i^c \succcurlyeq F_j^c$. Also, observe that (λ, α) cannot be (i_2, x) as a choice for the optimal pair, since it forces $F_j^c \succcurlyeq F_r^c$. We only have to look at the different ordering of i_2, j_1 , and j_2 . This can be done in the following ways:

2.2.1. $i_2 < j_1 < j_2$ (that is, $i_1 < x < i_2 < j_1 < j_2$).

This case is analogous to subcase 1.1, including the exception we get there, and unlike subcase 2.1.1, where we do not.

2.2.2. $j_1 < i_2 < j_2$ (that is, $i_1 < x < j_1 < i_2 < j_2$).

Observe that (λ, α) cannot be (i_2, j_1) a choice for the optimal pair, since it forces $F_j^c \succcurlyeq F_r^c$. This subcase proceeds analogously to subcase 1.1, but without any exception, and the reason for this follows from subcase 1.3.

2.2.3. $j_1 < j_2 < i_2$ (that is, $i_1 < x < j_1 < j_2 < i_2$).

Observe that (λ, α) cannot be (i_2, j_1) or (i_2, j_2) as a choice for the optimal pair, since it forces $F_j^c \succcurlyeq F_r^c$. This subcase proceeds analogously to subcase 1.1, but without any exception, and the reason for this follows from subcase 1.4.

2.3. $F_i^c = \{i_1 < x < i_2\}$ and $F_j^c = \{j_1 < x < j_2\}$.

In order to have $F_i^c \succ F_j^c$, we must $i_1 < j_1 < x$ here. Similar to the subcase 2.1, (λ, α) cannot be (i_2, x) as a choice for the optimal pair, since it forces $F_j^c \succ F_r^c$. Therefore, we need to examine the orderings of i_2 and j_2 . This can be done in the following ways,

2.3.1. $i_2 < j_2$ (that is, $i_1 < j_1 < x < i_2 < j_2$).

Observe that (λ, α) cannot be (i_2, j_1) as a choice for the optimal pair, since it forces $F_j^c \succ F_r^c$. This subcase proceeds analogously to subcase 1.1, but without any exception, and the reason for this follows from subcase 1.3.

2.3.2. $j_2 < i_2$ (that is, $i_1 < j_1 < x < j_2 < i_2$).

Observe that (λ, α) cannot be (i_1, x) or (i_2, x) as a choice for the optimal pair, since it forces $F_j^c \succ F_r^c$. Again, this subcase proceeds analogously to subcase 1.1, but without any exception, and the reason for this follows from subcase 1.4. The only modification is in the choice of the pair (λ, α) , where the initial choice is (i_1, x) , and if this fails, we then choose (i_1, j_2) .

2.4. $F_i^c = \{i_1 < x < i_2\}$ and $F_j^c = \{j_1 < j_2 < x\}$.

Clearly, $i_1 < j_1 < j_2 < x < i_2$ is the only possibility here. Also, (λ, α) cannot be any of (i_2, j_1) , (i_2, j_2) , or (i_1, x) , as a choice for the optimal pair, since this forces $F_j^c \succ F_r^c$. This subcase proceeds analogously to subcase 1.1, but without any exception, and the reason for this follows from subcase 1.4. The only modification is in the choice of the pair (λ, α) , where the initial choice is (i_1, j_2) , and if this fails, we then choose (i_1, x) .

2.5. $F_i^c = \{i_1 < i_2 < x\}$ and $F_j^c = \{x < j_1 < j_2\}$.

Clearly, $i_1 < i_2 < x < j_1 < j_2$ is the only possibility here. This subcase follows analogously from the subcase 1.1, including the exception we get there.

2.6. $F_i^c = \{i_1 < i_2 < x\}$ and $F_j^c = \{j_1 < x < j_2\}$.

In order to have $F_i^c \succ F_j^c$, we must have $i_1 < j_1$, and $i_2 < x < j_2$ will always hold. So, we only need to see how i_2 and j_1 are placed. This can be done in the following ways:

2.6.1. $i_2 < j_1$ (that is, $i_1 < i_2 < j_1 < x < j_2$).

This subcase follows analogously to the subcase 1.1, including the exception we get there.

The only modification is in the choice of the pair (λ, α) , where the initial choice is (i_1, x) , and if it fails, we then choose (i_1, j_2) .

2.6.2. $j_1 < i_2$ (that is, $i_1 < j_1 < i_2 < x < j_2$).

This case follows the exact same argument given for the previous subcase 2.6.1.

2.7. $F_i^c = \{i_1 < i_2 < x\}$ and $F_j^c = \{j_1 < j_2 < x\}$.

In order to have $F_i^c \succ F_j^c$, we must have $i_1 < j_1$ here. Also, the position of x is fixed, so we only need to see how i_2 , j_1 , and j_3 are arranged. We have the following possible scenarios,

2.7.1. $i_2 < j_1 < j_2$ (that is, $i_1 < i_2 < j_1 < j_2 < x$).

This subcase follows analogously to the subcase 1.1, including the exception we get there.

The only modification is in the choice of the pair (λ, α) , where the initial choice is (i_1, j_2) , and if it fails, we then choose (i_1, x) .

2.7.2. $j_1 < i_2 < j_2$ (that is, $i_1 < j_1 < i_2 < j_2 < x$).

This case follows the exact same argument given for the previous subcase 2.7.2.

2.7.3. $j_1 < j_2 < i_2$ (that is, $i_1 < j_1 < j_2 < i_2 < x$).

This subcase follows analogously to the subcase 1.1, but without any exception, and the reason for this follows from subcase 1.3. The only modification is in the choice of the pair (λ, α) , where the initial choice is (i_1, j_2) , and if it fails, we then choose (i_1, x)

$$\text{Case 3: } \left| (F'_i)^c \cap (F'_j)^c \right| = 2.$$

Let $(F'_i)^c = \{i_1, x, y\}$ and $(F'_j)^c = \{j_1, x, y\}$. In this case, we choose $\alpha = j_1$ and $\lambda = i_1$, that is, we choose $F'_i = F'_r$. The conclusion then follows immediately.

This completes the proof of Theorem 3.3, showing $\Delta_3(H_{1 \times m \times n})$ is a shellable simplicial complex. \square

4 Spanning facets and the homotopy type of the 3-cut complex of hexagonal tiling

Let \mathcal{S} denote the set of spanning facets of $\Delta_3(H_{1 \times m \times n})$ in the shelling order given in Definition 3.1. Recall, the equivalent condition given in (#), states that F_k is a spanning facet if for each $\lambda \in F_k$, there exists $r < k$, such that

$$F_r \cap F_k = F_k \setminus \{\lambda\}.$$

We state the main theorem of this section before building the necessary argument to prove it.

Theorem 4.1. *The 3-cut complex of $H_{1 \times m \times n}$ has the homotopy type of wedge of $\psi_{m,n}$ many $(2m + 2n + 2mn - 4)$ -dimensional spheres, where,*

$$\psi_{m,n} = \binom{2m + 2n + 2mn - 1}{2} - [(6m + 2)n + (2m - 4)].$$

In other words,

$$\Delta_3(H_{1 \times m \times n}) \simeq \bigvee_{\psi_{m,n}} \mathbb{S}^{(2m+2n+2mn-4)},$$

where $\psi_{m,n}$ is the number of spanning facets for the shelling order.

The proof of Theorem 4.1 is an immediate consequence of Theorem 4.3 and Theorem 4.4. We now proceed to prove these lemmas.

Before stating and proving these results, we must explicitly describe the elements of \mathcal{S} . For this, we first give the following important lemma.

Before stating and proving these results, we must explicitly describe the elements of \mathcal{S} . Unlike approaches based on Möbius inversion of the face lattice, our method explicitly identifies spanning facets, allowing the homotopy type to be read off directly from the shelling. For this, we first give the following important lemma.

Lemma 4.2. For all $F \in \mathcal{S}$, the vertex $2m + 2n + 2mn \notin F$, that is, no spanning facet of $\Delta_3(H_{1 \times m \times n})$ contains $2m + 2n + 2mn$.

Proof. Let $\lambda = 2m + 2n + 2mn$ and assume that there exists an $F_j \in \mathcal{S}$, for some $1 \leq j \leq \eta_{m,n} - \beta_{m,n}$, in the Equation (2) such that $\lambda \in F_j$. The choice of the upper bound $\eta_{m,n} - \beta_{m,n}$ for j is explained in the remark following the proof. Using (#), there exists $k < j$ (in Equation (2)) such that

$$F_k \cap F_j = F_j \setminus \{\lambda\}.$$

This implies that $F_k = (F_j \setminus \{\lambda\}) \cup \{\alpha\}$, for some $\alpha \in F_j^c$, and equivalently,

$$F_k^c = (F_j^c \setminus \{\alpha\}) \cup \{\lambda\}.$$

Note that, $\lambda \notin F_j^c$. Let $F_j^c = \{j_1, j_2, j_3\}$ with $j_1 < j_2 < j_3$. Since $\lambda > j_3$, replacing any $\alpha \in F_j^c$ will give $F_j^c \succcurlyeq F_k^c$ or equivalently $F_j \ll F_k$, which is a contradiction to $k < j$. \square

Remark. Using the above lemma, it is clear that none of the T_i removed from the reverse lexicographic ordering and appended at the end in the order of their removal will be a spanning facet.

Let $z = 2m + 2n + 2mn$. By the Theorem 4.2, it follows that z must always lie in the complement of any spanning facet. That is, for any $F \in \mathcal{S}$, we have

$$F^c = \{x, y, z\}, \quad \text{where } x, y \in [2m + 2n + 2mn - 1].$$

Therefore, to determine F completely, we must uniquely determine the pair (x, y) . Note that there are exactly

$$\binom{2m + 2n + 2mn - 1}{2}$$

such pairs, which gives the binomial term appearing in $\psi_{m,n}$. However, not all of these pairs correspond to spanning facets. Some lead to what we call *non-spanning pairs*.

We will explicitly describe these non-spanning pairs and prove that there are exactly

$$(6m + 2)n + (2m - 4)$$

of them. This gives the quantity subtracted from the binomial term in $\psi_{m,n}$.

For a better understanding of these non-spanning facets, we make some arrangements. Let $C_1, C_2, \dots, C_{2m+2n+2mn-2}$ denote the sets containing all possible pairs, that is, for $1 \leq i \leq 2m + 2n + 2mn - 2$, define

$$C_i = \{(i, j) \mid j \in [i + 1, 2m + 2n + 2mn - 1]\}$$

We claim that the list of non-spanning pairs in C_i 's is as follows:

1. For $1 \leq i \leq m - 1$, the non-spanning pairs are $(i, i + 1)$, $(i, i + m)$, $(i, i + m + 1)$ and $(i, i + m + n + mn)$. So, there are a total of $4(m - 1)$ such non-spanning pairs.
2. For $i = m$, the non-spanning pairs are $(i, i + m)$, $(i, i + m + 1)$ and $(i, i + m + n + mn)$. So, there are a total of 3 such non-spanning pairs.

3. Fix $k_1 \in [n - 1]$. For $k_1m + k_1 \leq i \leq k_1m + k_1 + m - 1$, the non-spanning pairs are $(i, i + 1)$, $(i, i + m + 1)$, $(i, i + m + 2)$, $(i, i + (m + 1)n)$ and $(i, i + (m + n + mn) + 1)$. So, there are a total of $5m(n - 1)$ such non-spanning pairs.
4. For $i = k_2(m + 1) - 1$, where $k_2 \in [2, n]$, the non-spanning pairs are $(i, i + (m + 1))$ and $(i, i + (m + 1)n)$. So, there are a total of $2(n - 1)$ such non-spanning pairs.
5. For $n + mn + 1 \leq i \leq m + n + mn - 1$, the non-spanning pairs are $(i, i + 1)$, $(i, i + (m + 1)n)$, and $(i, i + m + n + mn)$. So, there are a total of $3(m - 1)$ such non-spanning pairs.
6. For $i = (m + 1)n$, the non-spanning pairs are $(i, i + 1)$, and $(i, i + (m + 1)n)$. So, there are a total of 2 such non-spanning pairs.
7. For $i = m + n + mn$, the non-spanning pair is $(i, i + (m + 1)n)$. So, there is 1 such non-spanning pair.
8. For all

$$i \in [m + n + mn + 1, 2m + 2n + 2mn - m - 1] \\ \setminus (\{2n + 2mn\} \cup \{(m + n + mn) + t(m + 1) \mid t \in [n - 1]\}),$$

the non-spanning pair is $(i, i + m + 1)$. So, there are a total of

$$(m + n + mn - m - 1) - (n) = mn - 1$$

such non-spanning pairs.

For all other i 's which are not listed above, have no non-spanning pairs, that is, if

$$i = \{2n + 2mn\},$$

or,

$$i \in \{(m + n + mn) + t(m + 1) \mid t \in \{1, 2, \dots, n\}\} \\ \cup [m + 2n + 2mn + 1, 2m + 2n + 2mn - 2],$$

then C_i do not contain any non-spanning pair. The next lemma proves our claim that the above-listed pairs are indeed non-spanning pairs.

Lemma 4.3. *The pairs listed in C_i , for $1 \leq i \leq 2m + 2n + 2mn - 2$ are non-spanning pairs, with a total $6mn + 2m + 2n - 6$ of them.*

Proof. Recall that, using the condition (#), we know that if F is a spanning facet for a given shelling order, then for each $\lambda \in F$, there exists an k such that $F_k \prec F$ and

$$F_k \cap F = F \setminus \{\lambda\}.$$

In other words,

$$F_k = (F \cup \{\alpha\}) \setminus \{\lambda\},$$

where, $\alpha \in F^c$.

Recall that $z = 2m + 2n + 2mn$. If the complement of any of the above pairs, along with the z , does not form a spanning facet, then the condition in (#) must fail. Thus, to prove that the above pairs are non-spanning, it suffices to find a vertex in the facets formed by a pair defined above that fails (#). Let this vertex be $\lambda_{x,y}$ for the pair (x,y) . Also, let $S_{x,y}$ denote the facets formed by the pair (x,y) (that is, $S_{x,y}^c = \{x,y,z\}$), with $j_{x,y}$ being the position of $S_{x,y}$ in the Equation (2). In other words, $S_{x,y} = F_{j_{x,y}}$, where $1 \leq j_{x,y} \leq \eta_{m,n} - \beta_{m,n}$.

We group the non-spanning pairs defined above into the following types and, for each case, explicitly define a vertex $\lambda_{x,y}$ for which the spanning condition fails for $S_{x,y}$.

Type 1. The pairs of the form $(i, i+1)$.

We define $\lambda_{i,i+1}$ for these pairs as follows, based on the indices where i varies,

- (a) For $1 \leq i \leq m-1$ or $(m+1)n \leq i \leq m+n+mn-1$, we have,

$$\lambda_{i,i+1} = i + m + n + mn + 1.$$

- (b) For some fixed $k_1 \in [n-1]$, and $k_1m + k_1 \leq i \leq k_1m + k_1 + m - 1$, we have

$$\lambda_{i,i+1} = i + m + n + mn + 2.$$

Type 2. The pairs of the form $(i, i+m)$.

For $1 \leq i \leq m$, we define $\lambda_{i,i+m}$ for these pairs as,

$$\lambda_{i,i+m} = (m+1)n + i.$$

Type 3. The pairs of the form $(i, i+m+1)$.

We define $\lambda_{i,i+m+1}$ for these pairs as follows, based on the indices where i varies,

- (a) For $1 \leq i \leq m$ or $i = k_2(m+1) - 1$, where $k_2 \in [2, n]$, we have,

$$\lambda_{i,i+m+1} = i + m + n + mn + 1.$$

- (b) For some fixed $k_1 \in [n-1]$, and $k_1m + k_1 \leq i \leq k_1m + k_1 + m - 1$, we have

$$\lambda_{i,i+m+1} = i + m + n + mn + 2.$$

- (c) For all,

$$i \in [m+n+mn+1, 2m+2n+2mn-m-1] \\ \setminus (\{2n+2mn\} \cup \{(m+n+mn)+t(m+1) \mid t \in \{1, 2, \dots, n\}\}),$$

we have,

$$\lambda_{i,i+m+1} = i + m + 2.$$

Type 4. The pairs of the form $(i, i+m+2)$.

Fix $k_1 \in [n-1]$. For $k_1m + k_1 \leq i \leq k_1m + k_1 + m - 1$, we define $\lambda_{i,i+m+2}$ for these pairs as,

$$\lambda_{i,i+m+2} = i + m + n + mn + 2.$$

Type 5. The pairs of the form $(i, i + (m + 1)n)$.

We define $\lambda_{i,i+(m+1)n}$ for these pairs as follows, based on the indices where i varies,

- (a) Fix $k_1 \in [n - 1]$. For $k_1m + k_1 \leq i \leq k_1m + k_1 + m - 1$, we have the following two choices for $\lambda_{i,i+(m+1)n}$,

$$\lambda_{i,i+(m+1)n} = i + m + n + mn + 1 \quad \text{or} \quad \lambda_i = i + m + n + mn + 2.$$

- (b) For $i = k_2(m + 1) - 1$, where $k_2 \in [2, n]$, or $i = (m + 1)$, we have,

$$\lambda_{i,i+(m+1)n} = i + m + n + mn + 1.$$

- (c) For $n + mn + 1 \leq i \leq m + n + mn - 1$, we have,

$$\lambda_{i,i+(m+1)n} = i + m + n + mn \quad \text{or} \quad \lambda_{i,i+(m+1)n} = i + m + n + mn + 1.$$

- (d) For $i = m + n + mn$, we define $\lambda_{i,i+(m+1)n}$ as follows,

$$\lambda_{i,i+(m+1)n} = i + m + n + mn.$$

Type 6. The pairs of the form $(i, i + m + n + mn)$.

For $1 \leq i \leq m$, or $n + mn + 1 \leq i \leq m + n + mn - 1$, we define $\lambda_{i,i+m+n+mn}$ for these pairs as,

$$\lambda_{i,i+m+n+mn} = i + m + n + mn + 1.$$

Type 7. The pairs of the form $(i, i + m + n + mn + 1)$.

Fix $k_1 \in [n - 1]$. For $k_1m + k_1 \leq i \leq k_1m + k_1 + m - 1$, we define $\lambda_{i,i+m+n+mn+1}$ for these pairs as,

$$\lambda_{i,i+m+n+mn+1} = i + m + n + mn + 2.$$

We now analyze the possible α for all the pairs listed above. For all the $\lambda_{x,y}$ defined above for the pairs (x, y) , observe that $\lambda_{x,y} > x$ and $\lambda_{x,y} > y$. So, if we choose α to be either x or y , then $S_{x,y} \not\ll F_k$, contradicting $k < j_{x,y}$.

Now, if $\alpha = z$, then we first deal with Type 3(c) and then the rest. For Type 3(c),

$$F_k^c = \{i, i + m + 1, \lambda_{x,y}\} = \{i, i + m + 1, i + m + 2\},$$

where the range of i is specified in Type 3(c). In this case, $F_k = \mathcal{N}(i - (m + 1)n)$, which is one of the T_i , implying $S_{x,y} \not\ll F_k$ since T_i are placed at the end. For all other types, $S_{x,y}^c$ is connected (that is, it induces a path P_3), contradicting F_k^c being disconnected.

Hence, all the pairs defined above are non-spanning pairs. Moreover, summing up all of these non-spanning pairs gives a total of $6mn + 2m + 2n - 6$. \square

Observe that the pairs that were not listed above as non-spanning are indeed spanning. This follows from two facts. First, the analysis shown in the previous proof explicitly lists out all those pairs for which the resulting facet $S_{x,y}$ will fail (#) because

- i. any constructed F_k is forced to appear after $S_{x,y}$,

- ii. any constructed F_k^c is connected (induces a P_3), or
- iii. any constructed F_k coincides with one of the removed facets T_i .

Second, for any remaining pair $\{x, y\}$, one can construct a valid F_k by choosing a suitable $\alpha \in S_{x,y}^c$ that will avoid all three obstructions listed above for some $\lambda \in S_{x,y}$. The next result proves that the pairs given in types 1 to 7 in the proof of Theorem 4.3 are the only non-spanning pairs.

Lemma 4.4. *All pairs that are not listed in C_i , for $1 \leq i \leq 2m + 2n + 2mn - 2$ are spanning pairs.*

Proof. Let (a, b) be a pair that does not fit into any of the types 1 through 7 described in the proof of Theorem 4.3. To show that $S_{a,b}$ is a spanning facet, it suffices to verify that for each $\lambda \in S_{a,b}$ there exists a facet F_r with $r < j_{a,b}$ (equivalently, $F_r \not\ll S_{a,b}$) satisfying $(\#)$.

Observe that if the pair (a, b) does not belong to any of the types 1 through 7, then exactly one of the following three cases must occur.

Case 1: The vertices a and b are adjacent, that is, $\{a, b\} \in E(H_{1 \times m \times n})$.

It follows from the description of non-spanning pairs given in types 1 through 7 that this case can occur only when the pair (a, b) admits one of the following forms:

- i. $(a, b) = (i, i + m + n + mn + 1)$, when $i \in [m] \sqcup [n + mn, m + n + mn - 2]$; or
- ii. $(a, b) = (i, i + m + n + mn + 2)$, when $i \in [k_1 m + k_1, (k_1 + 1)m + k_1 - 1]$, for all $k_1 \in [n-1]$.

Note that, for any $\lambda \in S_{a,b}$ we choose to swap with $\alpha \in S_{a,b}^c$, the resulting F_r^c will never coincide with any T_i^c , for $1 \leq i \leq \beta_{m,n}$. This is because a and b are adjacent to each other, whereas by Observation 3.3, each T_i^c is an independent set.

Let $\lambda \in S_{a,b}$. First, suppose that λ is adjacent to neither a nor b . In this subcase, we choose $\alpha = z$, which yields

$$F_r^c = \{a, b, \lambda\}.$$

Since every element of $S_{a,b}$ is smaller than z , we have $F_r^c \not\gg S_{a,b}^c$. Moreover, the F_r^c obtained here does not induce a P_3 , as λ is adjacent to neither a nor b .

Now suppose that λ is adjacent to either a or b . In this case, choosing $\alpha = z$ is not possible, since the resulting F_r^c would induce a P_3 . Before choosing α , we observe that in all the possible forms of (a, b) listed above, we have $a \in V_1$ and $b \in V_2$, and hence $a < b$. In fact, b is the largest element in the neighborhood of a , that is, $b = \max\{N(a)\}$.

For this subcase, we choose $\alpha = b$. It suffices to show that $\lambda < b$, which guarantees $F_r^c \not\gg S_{a,b}^c$. Indeed, if λ is adjacent to a , then by the above observation $\lambda < b$; and if λ is adjacent to b , then by Observation 2.3, we have $\lambda \in V_1$, again implying $\lambda < b$. Finally, the resulting F_r^c does not induce a P_3 , since in all the cases listed above the vertex b is at a distance of at least 2 from z .

Case 2: The vertices a and b are at a distance of two from each other, that is, a and b are endpoints of an induced P_3 .

This case is only possible when (a, b) admits the following forms:

- i. $(a, b) = (i, i + 1)$, when

$$i \in [(k_2 - 1)(m + 1) + m + n + mn + 1, k_2(m + 1) + m + n + mn - 1],$$

for each $k_2 \in [n]$, or when

$$i \in [m + 2n + 2mn, 2m + 2n + 2mn - 2].$$

ii. $(a, b) = (i, i + m + 1)$, when

$$i = m + n + mn + k_3(m + 1),$$

for all $k_3 \in [n - 1]$, or when,

$$i \in [2n + 2mn, m + 2n + 2mn - 2].$$

iii. $(a, b) = (i, i + m + 2)$, when

$$i \in [(k_3 - 1)(m + 1) + m + n + mn + 1, k_3(m + 1) + m + n + mn - 1],$$

for each $k_3 \in [n - 1]$.

Before proceeding, we record some observations. In all the above cases, both a and b belong to V_2 . Hence, by Observation 2.3, every vertex adjacent to either a or b must lie in V_1 .

Let $\lambda \in S_{a,b}$. We consider the following two subcases.

1. λ is adjacent to both a and b .

In this situation, choosing $\alpha = z$ is not possible, since the resulting F_r^c will induce a P_3 . Instead, we choose $\alpha = b$. Since λ is adjacent to b , we have $\lambda < b$, and hence $F_r^c \succcurlyeq S_{a,b}^c$.

Moreover, F_r^c cannot coincide with any T_i^c , as λ is adjacent to a and each T_i^c is an independent set by Observation 3.3. Finally, F_r^c does not induce P_3 , since a and z are always at a distance greater than two. The only exception when F_r^c induces P_3 occurs when $\lambda = m + n + mn - 1$ and

$$(a, b) = (m + 2n + 2mn - 1, m + 2n + 2mn).$$

This exceptional case can be handled by choosing $\alpha = a$ instead of b , while keeping all other arguments unchanged.

2. λ is at a distance of at least two from at least one of a or b .

First, suppose that $\{a, b, \lambda\}$ coincides with some T_i^c . In this case, choosing $\alpha = z$ is not possible, as it would yield $F_r^c = T_i^c$. We therefore choose $\alpha = b$. As in the previous subcase, this choice guarantees $F_r^c \succcurlyeq S_{a,b}^c$.

Moreover, the resulting F_r^c cannot coincide with any T_i^c , since it contains z and no T_i^c contains z . Also, F_r^c does not induce a P_3 , as a is adjacent to neither λ nor z .

If the above situation does not arise, then we may safely choose $\alpha = z$. This again yields $F_r^c \succcurlyeq S_{a,b}^c$, and the resulting F_r^c neither coincides with any T_i^c nor induces a P_3 , by arguments analogous to those above.

Case 3: The vertices a and b are at distance greater than two from each other.

We do not list explicit types in this case, since it consists of all pairs (a, b) that were not covered earlier. Here, we choose $\alpha = z$. Then the resulting facet F_r^c satisfies $F_r^c \succcurlyeq S_{a,b}^c$, as $\lambda < z$ for every $\lambda \in S_{a,b}$.

As in the previous cases, the facet F_r^c obtained here neither coincides with any T_i^c nor induces a P_3 . Both assertions follow from arguments analogous to those used above.

This completes the proof of Theorem 4.4 \square

5 Future directions

Motivated by our results on the shellability of the 3-cut complex of hexagonal grid graphs, we investigated similar results for higher values of k . Computational experiments using `SageMath` suggest that similar behavior persists for $k = 4, 5$, and 6 , especially for hexagonal line tilings. This leads us to propose the following conjecture.

Conjecture 5.1. *For $k = 4, 5$, and 6 , the k -cut complex of the hexagonal line tiling is shellable.*

Building on this conjecture, it is natural to ask whether a similar shellability result holds for the k -cut complex of the general hexagonal grid graphs $H_{r \times s \times t}$.

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