

Linear varieties and matroids with applications to the Cullis' determinant

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Abstract

Let V be a vector space of rectangular $n \times k$ matrices annihilating the Cullis' determinant. We show that $\dim(V) \leq (n-1)k$, extending Dieudonné's result on the dimension of vector spaces of square matrices annihilating the ordinary determinant.

Furthermore, for certain values of n and k , we explicitly describe such vector spaces of maximal dimension. Namely, we establish that if k is odd, $n \geq k+2$ and $\dim(V) = (n-1)k$, then V is equal to the space of all $n \times k$ matrices X such that alternating row sum of X is equal to zero.

Our proofs rely on the following observations from the matroid theory that have an independent interest. First, we provide a notion of matroid corresponding to a given linear variety. Second, we prove that if the linear variety is transformed by projections and restrictions, then the behaviour of the corresponding matroid is expressed in the terms of matroid contraction and restriction. Third, we establish that if M is a matroid, I^* its coindependent set $M|S$ and its restriction on a set S , then the union of $I^* \setminus S$ with every cobase of $M|S$ is coindependent set of M .

Keywords: linear varieties, matroids, rectangular matrices, Cullis' determinant

1 Introduction

The determinant of a matrix is used in many fields of mathematics and its applications. Its study has a long and rich history, with contributions from mathematicians across the world since ancient times.

One of the subjects in the investigations of the determinant is describing the structure of spaces of matrices annihilating it, i.e., vector spaces $V \subseteq \mathcal{M}_n(\mathbb{F})$ such that $\det(X) = 0$ for all $X \in V$. Dieudonné in [4] obtained the sharp bound for the dimension of matrix space having this property and established that every such space of maximal possible dimension should be a left or right maximal ideal in the ring of all matrices.

Theorem 1.1 ([4, Théorème 1]). Assume that $n \in \mathbb{N}$ and \mathbb{F} is a field such that $(|\mathbb{F}|, n) \neq (2, 2)$. Let $V \subseteq \mathcal{M}_n(\mathbb{F})$ be a vector space, $A \in \mathcal{M}_{n,k}(\mathbb{F})$ and $\mathsf{K} = \{A\} + V$. Then the following statements hold:

1. If $\det(X) = 0$ for all $X \in \mathsf{K}$, then $\dim(V) \leq n^2 - n$.
2. If $\det(X) = 0$ for all $X \in \mathsf{K}$ and $\dim(V) = n^2 - n$, then either the nullspaces of the matrices from K have a common nonzero vector, or the same is true for K^t .

This allowed Dieudonné to provide a description of linear maps preserving the matrix determinant without any restrictions on ground field [4, Théorème 3].

Since the ordinary determinant is defined only for the square matrices, there were made several attempts to extend this notion to the rectangular matrices. One of such extension is due to Cullis who introduced the concept of determinant (he called it *determinoid*) of a rectangular matrix in his monograph [2] which can be expressed as an alternating sum of its maximal minors (see Corollary 3.6) and is denoted by $\det_{n,k}$. Several properties known for the classical determinant were studied and shown to be valid for the Cullis' determinant in [2, §5, §27, §32].

In 1966 Radić [17] independently proposed an equivalent definition of the Cullis determinant. He also provided in [18, 19] its geometrical applications to the calculating areas of polygons on the plane and volumes of certain polyhedrons in space. According to that, it is also sometimes called *Radić's determinant* [1] or *Cullis-Radić determinant* [12, 11] in some papers.

Nakagami and Yanai in [14] formulate the definition of the Cullis' determinant in the terms of vectors in a Grassmann algebra and provide a definition of the Cullis' determinant through several characteristic properties. For this reason they consider $n \times k$ matrices only with $n \geq k$, whereas the original definition covers rectangular matrices of any size. But according to it, the Cullis determinant of $n \times k$ matrix with $n < k$ is equal to the Cullis determinant of its transpose and consequently it is sufficient to consider only the case $n \geq k$.

Theorems 1.1 suggest a natural direction for studying the Cullis determinant: describing the maximal vector spaces annihilating the Cullis determinant.

In this paper the authors make the first steps in the second direction of research consisting in studying vector spaces of matrices annihilating the Cullis' determinant. In particular, we obtain a generalization of the Dieudonné's theorem for the Cullis' determinant to the case where $n \geq k + 2$ and k is odd. The main result of this paper is formulated as follows.

Theorem 1.2 (Cf. Theorem 7.11). Let $n \geq k + 2$ and $V \subseteq \mathcal{M}_{n,k}(\mathbb{F})$ be a vector space $A \in \mathcal{M}_{n,k}(\mathbb{F})$ and $\mathsf{K} = \{A\} + V$. Then $\det_{n,k}(X) = 0$ for all $X \in \mathsf{K}$ and $\text{codim}(\mathsf{K}) = k$ if and only if k is odd and alternating row sum of every $X \in \mathsf{K}$ is equal to zero.

Thus, this theorem implies that if $n \geq k + 2$ and k is odd, there is a **unique** vector subspace of $\mathcal{M}_{n,k}(\mathbb{F})$ of maximal dimension annihilating $\det_{n,k}$.

The argumentation presented in this paper is similar to that used by Dieudonné to prove Theorem 1.1. The main difference is that the authors rely on the following considerations from the matroid theory. First, we provide a notion of matroid corresponding to a given linear variety (Definition 6.3). Second, we prove that if the linear variety is transformed by projections and restrictions, then the behaviour of the corresponding matroid is expressed in the terms of matroid contraction and restriction (Lemma 6.7 and Lemma 6.16). Third, we

establish that if M is a matroid, I^* its coindependent set $M|S$ and its restriction on a set S , then the union of $I^* \setminus S$ with every cobase of $M|S$ is coindependent set of M (Lemma 5.14).

Note that these observations may be useful for solving other similar problems. It is also worth to mention that the considerations from the matroid theory were already used to solve the problems of such kind. For example, Meshulam in [6] applies the Lovász Matroid Parity Theorem to find the maximal dimension of vector space V consisting of skew-symmetric matrices such that every element of V is a singular matrix.

Similarly to Theorem 1.1, Theorem 1.2 finds its applications in studying linear maps that preserve the Cullis' determinant. Using this theorem, it is possible to obtain a description of linear maps preserving the Cullis' determinant of $n \times k$ rectangular matrices for the case where $k = 3$, see [10].

In addition to finding a precise description of vector spaces of matrices of size $n \times k$ annihilating the Cullis' determinant for k odd we also provide the following upper bound of the dimension of such spaces for arbitrary k .

Theorem 1.3 (Cf. Theorem 7.6). *Let $V \subseteq \mathcal{M}_{nk}(\mathbb{F})$ be a vector space, $A \in \mathcal{M}_{nk}(\mathbb{F})$ and $\mathsf{K} = \{A\} + V$ be such that $\det_{nk}(X) = 0$ for every $X \in \mathsf{K}$. Then $\text{codim}(V) \geq k$.*

Here $\text{codim}(V)$ denotes a *codimension* of vector space $V \subseteq \mathcal{M}_{nk}(\mathbb{F})$, which is defined by $\text{codim}(V) = \dim(\mathcal{M}_{nk}(\mathbb{F})) - \dim(V) = n \cdot k - \dim(V)$.

Theorem 1.2 and Theorem 1.3 imply in particular that if k is even and $n \geq k+2$, then the codimension of any vector space of matrices of size $n \times k$ annihilating the Cullis' determinant is strictly greater than k . This is the sharpest bound known by authors at present, and the question of its exact value remains open.

In accordance with the text above, the two cases ($n = k$ and $n = k + 1$) remain undisussed. The case if $n = k$ is completely covered by Theorem 1.1 because the Cullis' determinant of square matrix is equal to the ordinary determinant. The case $n = k + 1$, in turn, will be considered in the separate paper because it is carried out using different methods.

It is worth mentioning that Theorem 1.1 admits other generalizations. For example, Pazzis in [3] provides a description of vectors spaces of matrices of size $n \times k$ (where $n \geq k$) of maximal dimension which do not contain a matrix of rank k . Unfortunately, it is not possible to apply his results directly to the spaces of matrices annihilating the Cullis' determinant because there exist matrices of rank k belonging to $\mathcal{M}_{nk}(\mathbb{F})$ such that their Cullis' determinant is zero. For example, this is true for the matrix $E_{11} + \dots + E_{kk} + E_{(k+1)k}$.

This paper is organized as follows. In Section 2 we provide the necessary notations; Sections 3, 5 contain the preliminary facts from the theory of the Cullis' determinant and the matroid theory, correspondingly; in Section 4 we provide an explanation of all necessary facts from the theory of linear varieties; in Section 6 we introduce the matroid corresponding to a given linear variety, study the behaviour of this matroid while the corresponding linear variety is transformed using different operations and establish the relationship between matroids corresponding to linear varieties and vector matroids corresponding to matrices; Section 7 is devoted to investigations of linear varieties of matrices annihilating the Cullis' determinant and contains the proofs of main results of this paper; in Section 8 we discuss the possible further work in this direction.

2 Notation and basic definitions

By \mathbb{F} we denote a field without any restrictions on its characteristic.

We denote by $\mathcal{M}_{nk}(\mathbb{F})$ the set of all $n \times k$ matrices with the entries from a certain field \mathbb{F} . In addition, if A and B are finite sets, then $\mathcal{M}_{AB}(\mathbb{F})$ denotes the set of all $|A| \times |B|$ matrices with the entries from \mathbb{F} whose rows and columns are indexed by the elements of A and B , respectively. $O_{nk} \in \mathcal{M}_{nk}(\mathbb{F})$ denotes the matrix with all the entries equal to zero. $I_{nn} = I_n \in \mathcal{M}_{nn}(\mathbb{F})$ denotes an identity matrix. Let us denote by $E_{ij} \in \mathcal{M}_{nk}(\mathbb{F})$ a matrix, whose entries are all equal to zero besides the entry on the intersection of the i -th row and the j -th column, which is equal to one. By x_{ij} we denote the element of a matrix X lying on the intersection of its i -th row and j -th column. For a set S and $i, j \in S$ by δ_{ij} we denote a Kronecker delta of i and j , which is equal to 1 if $i = j$ and is equal to 0 otherwise. If $A \in \mathcal{M}_{nk}(\mathbb{F})$, then by $\mathsf{R}(A) \subseteq \mathbb{F}^k$ and $\mathsf{C}(A) \subseteq \mathbb{F}^n$ we denote a row space of A and a column space of A , correspondingly.

For $A \in \mathcal{M}_{nk_1}(\mathbb{F})$ and $B \in \mathcal{M}_{nk_2}(\mathbb{F})$ by $A|B \in \mathcal{M}_{nk_1+k_2}(\mathbb{F})$ we denote a block matrix defined by $A|B = \begin{pmatrix} A & B \end{pmatrix}$.

For $A \in \mathcal{M}_{nk}(\mathbb{F})$ by $A^t \in \mathcal{M}_{kn}(\mathbb{F})$ we denote a transpose of the matrix A , i.e. $A_{ij}^t = A_{ji}$ for all $1 \leq i \leq k, 1 \leq j \leq n$.

We use the notation for submatrices following [13] and [16]. That is, by $A[J_1|J_2]$ we denote the $|J_1| \times |J_2|$ submatrix of A lying on the intersection of rows with the indices from J_1 and the columns with the indices from J_2 . By $A(J_1|J_2)$ we denote a submatrix of A derived from it by striking out from it the rows with indices belonging to J_1 and the columns with the indices belonging to J_2 . If the set is absent, then it means empty set, i.e. $A(J_1|)$ denotes a matrix derived from A by striking out from it the rows with indices belonging to J_1 . We may skip curly brackets, i.e. $A[1, 2|3, 4] = A[\{1, 2\}|\{3, 4\}]$. The notation with mixed brackets is also used, i.e. $A(|1)$ denotes the first column of the matrix A . This notation is also used for vectors as well. In this case vectors are considered as $n \times 1$ or $1 \times n$ matrices.

We use the bold font to denote vectors and lower indices to denote their coordinates. In the case if we need the series of vectors, we use the upper indices placed in braces. For example, if $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\mathbf{v}^t = (1 \ 0)$ and $\mathbf{v}_1 = 1$. If $\mathbf{u}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{u}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\mathbf{u}_1^{(1)} = 1$ and $\mathbf{u}_1^{(2)} = 0$.

We introduce the definitions below following [14].

Definition 2.1. By $[n]$ we denote the set $\{1, \dots, n\}$.

Definition 2.2. By \mathcal{C}_X^k we denote the set of injections from $[k]$ to X .

Definition 2.3. By $\binom{X}{k}$ we denote the set of the images of injections from $[k]$ to X , i.e. the set of all subsets of X of cardinality k .

Definition 2.4. Suppose that $c \in \binom{[n]}{k}$ equal to $\{i_1, \dots, i_k\}$, where $i_1 < i_2 < \dots < i_k$, and $1 \leq \alpha \leq k$ is a natural number. Then $c(\alpha)$ is defined by

$$c(\alpha) = i_\alpha.$$

Definition 2.5. Given a set $c \in \binom{[n]}{k}$ we denote by $\text{sgn}(c) = \text{sgn}_{[n]}(c)$ the number

$$(-1)^{\sum_{\alpha=1}^k (c(\alpha)-\alpha)}.$$

Note 2.6. $\text{sgn}_{[n]}(c)$ depends only on c and does not depend on n .

Definition 2.7. Given an injection $\sigma \in \mathcal{C}_{[n]}^k$ we denote by $\text{sgn}_{nk}(\sigma)$ the product $\text{sgn}(\pi_\sigma) \cdot \text{sgn}_{[n]}(c)$, where $\text{sgn}(\pi)$ is the sign of the permutation

$$\pi_\sigma = \begin{pmatrix} i_1 & \dots & i_k \\ \sigma(1) & \dots & \sigma(k) \end{pmatrix},$$

where $\{i_1, \dots, i_k\} = \sigma([k])$ and $i_1 < i_2 < \dots < i_k$.

We omit the subscripts for $\text{sgn}_{[n]}$ and sgn_{nk} if this cannot lead to a misunderstanding.

Definition 2.8 ([14], Theorem 13). Let $n \geq k$, $X \in \mathcal{M}_{nk}(\mathbb{F})$. Then Cullis' determinant $\det_{nk}(X)$ of X is defined to be the function:

$$\det_{nk}(X) = \sum_{\sigma \in \mathcal{C}_k^{[n]}} \text{sgn}_{nk}(\sigma) X_{\sigma(1)1} X_{\sigma(2)2} \dots X_{\sigma(k)k}.$$

We also denote $\det_{nk}(X)$ as follows

$$\det_{nk}(X) = \left| \begin{array}{ccc} X_{11} & \dots & X_{1k} \\ \vdots & \dots & \vdots \\ X_{n1} & \dots & X_{nk} \end{array} \right|_{nk}.$$

In the case if $n = k$, then \det_{nk} is also denoted as \det_k and is clearly equal to a classical determinant of a square matrix.

3 Preliminaries from the theory of the Cullis' determinant

Now we list the properties of \det_{nk} which are similar to corresponding properties of the ordinary determinant (see [2, §5, §27, §32] or [14] for detailed proofs).

- Theorem 3.1** ([14, Theorem 13, Theorem 16]).
1. For $X \in \mathcal{M}_n(\mathbb{F})$, $\det_{nn}(X) = \det(X)$.
 2. For $X \in \mathcal{M}_{nk}(\mathbb{F})$, $\det_{nk}(X)$ is a linear function of columns of X .
 3. If a matrix $X \in \mathcal{M}_{nk}(\mathbb{F})$ has two identical columns or one of its columns is a linear combination of other columns, then $\det_{nk}(X)$ is equal to zero.
 4. For $X \in \mathcal{M}_{nk}(\mathbb{F})$, interchanging any two columns of X changes the sign of $\det_{nk}(X)$.
 5. Adding a linear combination of columns of X to another column of X does not change $\det_{nk}(X)$.
 6. For $X \in \mathcal{M}_{nk}(\mathbb{F})$, $\det_{nk}(X)$ can be calculated using the Laplace expansion along a column of X (see Lemma 3.8 for precise formulation).

Corollary 3.2. Let $n \geq k$, $A, B \in \mathcal{M}_{n,k}(\mathbb{F})$. Then

$$\begin{aligned} & \det_{n,k}(A + \lambda B) \\ &= \sum_{d=0}^k \lambda^d \left(\sum_{1 \leq i_1 < \dots < i_d \leq k} \det_{n,k} \left(A(|1] \mid \dots \mid B(|i_1] \mid \dots \mid B(|i_d] \mid \dots \mid A(|k]) \right) \right), \quad (3.1) \end{aligned}$$

where both sides of the equality are considered as formal polynomials in λ , i.e. as elements of $\mathbb{F}[\lambda]$.

Proof. This is a direct consequence from the multilinearity of $\det_{n,k}$ with respect to the columns of a matrix. \square

Corollary 3.3. If $A, B \in \mathcal{M}_{n,k}(\mathbb{F})$, then $\deg_\lambda (\det_{n,k}(A + \lambda B)) \leq k$.

Definition 3.4. Suppose that $c \in \binom{[n]}{k}$ and $c = \{i_1, \dots, i_k\}$. By $P_c \in \mathcal{M}_{n,n}(\mathbb{F})$ we denote a matrix defined by

$$P_c = E_{i_1 i_1} + \dots + E_{i_k i_k}.$$

Lemma 3.5 (Cf. [14, Lemma 8]). Let $X \in \mathcal{M}_{n,k}(\mathbb{F})$. Then

$$\operatorname{sgn}_{[n]}(c) \det_{n,k}(P_c X) = \det_k(X[c|])$$

and

$$\det_{n,k}(X) = \sum_{c \in \binom{[n]}{k}} \det_{n,k}(P_c X).$$

Corollary 3.6. The Cullis' determinant of a matrix $X \in \mathcal{M}_{n,k}(\mathbb{F})$ is an alternating sum of basic minors of X . That is,

$$\det_{n,k}(X) = \sum_{c \in \binom{[n]}{k}} \operatorname{sgn}_{[n]}(c) \det_k(X[c|]). \quad (3.2)$$

Lemma 3.7 (Cf. [14, Lemma 20]). Assume that $n > k \geq 1$. Let $X \in \mathcal{M}_{n,k}(\mathbb{F})$ and $Y \in \mathcal{M}_{n,(k+1)}$ is defined by $Y = X | \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Then

$$\det_{n,(k+1)}(Y) = \begin{cases} \det_{n,k}(X), & n+k \text{ is odd}, \\ 0, & n+k \text{ is even}. \end{cases}$$

Lemma 3.8 (Cf. [14, Theorem 16]). Let $1 < k \leq n$. For any $n \times k$ matrix $X = (x_{ij})$ the expansion of $\det_{n,k}(X)$ along the j -th column is given by

$$\det_{n,k}(X) = \sum_{i=1}^n (-1)^{i+j} x_{ij} \det_{n-1,k-1}(X(i|j)),$$

where $X(i|j)$ denotes the $(n-1) \times (k-1)$ matrix obtained from Y by deleting the i -th row and the j -th column.

Lemma 3.9 (Invariance of $\det_{n,k}$ under cyclic shifts, Cf. [1, Theorem 3.5]). *If $k \leq n$, and $k+n$ is odd, then for all $X = (x_{ij}) \in \mathcal{M}_{n,k}(\mathbb{F})$ and $i \in \{1, \dots, n\}$*

$$(-1)^{(i+1)k} \begin{vmatrix} x_{i1} & \dots & x_{ik} \\ \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{nk} \\ x_{11} & \dots & x_{1k} \\ \vdots & \vdots & \vdots \\ x_{(i-1)1} & \dots & x_{(i-1)k} \end{vmatrix}_{n,k} = \begin{vmatrix} x_{11} & \dots & x_{1k} \\ \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{nk} \end{vmatrix}_{n,k}.$$

Here the matrix on the left-hand side of the equality is obtained from X by performing the row cyclical shift sending i -th row of X to the first row of the result.

Lemma 3.10 (Invariance of $\det_{n,k}$ under semi-cyclic shifts, Cf. [1, Theorem 3.6]). *If $k \leq n$, and $k+n$ is even, then for all $X = (x_{ij}) \in \mathcal{M}_{n,k}(\mathbb{F})$ and $i \in \{1, \dots, n\}$*

$$(-1)^{(n-i)k} \begin{vmatrix} x_{i1} & \dots & x_{ik} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nk} \\ -x_{11} & \dots & -x_{1k} \\ \vdots & \ddots & \vdots \\ -x_{(i-1)1} & \dots & -x_{(i-1)k} \end{vmatrix}_{n,k} = \begin{vmatrix} x_{11} & \dots & x_{1k} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nk} \end{vmatrix}_{n,k}.$$

Here the matrix on the left-hand side of the equality is obtained from X by performing the following sequence of operations: the row cyclical shift sending i -th row of X to the first row of the result; multiplying the bottom $i-1$ rows by -1 .

In the following lemma we introduce an invertible linear map satisfying certain properties which will be used in the proof of Lemma 7.11.

Lemma 3.11. *Let k be an odd integer, $n \geq k \geq 1$ and $1 \leq i^\circ \leq n$. Then there exists an invertible linear map $\mathbf{S}_{i^\circ}: \mathcal{M}_{n,k}(\mathbb{F}) \rightarrow \mathcal{M}_{n,k}(\mathbb{F})$ having the following properties:*

(S1) if $X \in \mathcal{M}_{n,k}(\mathbb{F})$ and $\mathbf{z} \in \mathbb{F}^n$ are such that $\mathbf{z}^t X = 0$, then $\mathbf{z}^{t^*} \mathbf{S}_{i^\circ}(X) = 0$, where

$$\mathbf{z}^\circ = (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, (-1)^{n+k+1} \mathbf{z}_1 \dots, (-1)^{n+k+1} \mathbf{z}_{i^\circ-1})^t; \quad (3.3)$$

(S2) $\det_{n,k}(X) = 0 \Rightarrow \det_{n,k}(\mathbf{S}_{i^\circ}(X)) = 0$;

(S3) If k is odd, then $\sum_{i=1}^n (-1)^i (\mathbf{S}_{i^\circ}(X)) [i] = 0 \Rightarrow \sum_{i=1}^n (-1)^i X[i] = 0$ for all $X \in \mathcal{M}_{n,k}$.

Proof. The proof of this lemma relies on Lemma 3.9 and Lemma 3.10, which are applicable for the case of $n+k$ odd and $n+k$ is even, respectively. For this reason, each of these two cases will be considered separately.

The case if $n + k$ is odd. Let \mathbf{S}_{i° be defined by

$$\mathbf{S}_{i^\circ}(X) = \begin{pmatrix} x_{i^{\text{circ}} 1} & \dots & x_{i^\circ k} \\ \vdots & \vdots & \vdots \\ x_{n 1} & \dots & x_{n k} \\ \vdots & \vdots & \vdots \\ x_{(i^\circ - 1) 1} & \dots & x_{(i^\circ - 1) k} \end{pmatrix} \quad \text{for all } X \in \mathcal{M}_{n k}(\mathbb{F}). \quad (3.4)$$

This map is clearly linear and invertible. Now we verify that \mathbf{S}_{i° indeed satisfies the properties (S1)–(S3).

Property (S1). Let $X \in \mathcal{M}_{n k}(\mathbb{F})$ and $\mathbf{z} \in \mathbb{F}^n$ be such that $\mathbf{z}^t X = 0$. It follows from (3.4) that $\mathbf{z}^{\text{odd}} \mathbf{S}_{i^\circ}(X) = 0$, where \mathbf{z}^{odd} is defined by $\mathbf{z}^{\text{odd}} = (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, \mathbf{z}_1, \dots, \mathbf{z}_{i^\circ - 1})^t$. Since $n + k$ is odd, then

$$\mathbf{z}^{\text{odd}} = (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, \mathbf{z}_1, \dots, \mathbf{z}_{i^\circ - 1})^t = (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, (-1)^{n+k+1} \mathbf{z}_1, \dots, (-1)^{n+k+1} \mathbf{z}_{i^\circ - 1})^t = \mathbf{z}^\circ,$$

where \mathbf{z}° be as defined in (3.3).

Property (S2). Let $X \in \mathcal{M}_{n k}(\mathbb{F})$ be such that $\det_{n k}(X) = 0$. Then by Lemma 3.9 we have

$$\det_{n k}(\mathbf{S}_{i^\circ}(X)) = (-1)^{(i^\circ + 1)k} \det_{n k}(X) = 0.$$

Property (S3). Assume that k is odd. Let $X \in \mathcal{M}_{n k}$ be such that

$$\sum_{i=1}^n (-1)^i X^\circ[i] = 0, \quad (3.5)$$

where $X^\circ = \mathbf{S}_{i^\circ}(X)$. Rewrite the sum $\sum_{i=1}^n (-1)^i X[i]$ as follows

$$\sum_{i=1}^n (-1)^i X[i] = \sum_{i=1}^{i^\circ - 1} (-1)^i X[i] + \sum_{i=i^\circ}^n (-1)^i X[i].$$

Since n is even in this case,

$$\sum_{i=1}^{i^\circ - 1} (-1)^i X[i] = \sum_{i=n-i^\circ+2}^n (-1)^{i-n+i^\circ-1} X^\circ[i] \quad \text{and} \quad \sum_{i=i^\circ}^n (-1)^i X[i] = \sum_{i=1}^{n-i^\circ+1} (-1)^{i+i^\circ-1} X^\circ[i],$$

then

$$\begin{aligned} \sum_{i=1}^n (-1)^i X[i] &= \sum_{i=1}^{i^\circ - 1} (-1)^i X[i] + \sum_{i=i^\circ}^n (-1)^i X[i] \\ &= \sum_{i=n-i^\circ+2}^n (-1)^{i-n+i^\circ-1} X^\circ[i] + \sum_{i=1}^{n-i^\circ+1} (-1)^{i+i^\circ-1} X^\circ[i] \\ &= \sum_{i=n-i^\circ+2}^n (-1)^{i+i^\circ-1} X^\circ[i] + \sum_{i=1}^{n-i^\circ+1} (-1)^{i+i^\circ-1} X^\circ[i] = (-1)^{i^\circ - 1} \sum_{i=1}^n (-1)^i X^\circ[i]. \end{aligned}$$

Thus, using the equality (3.5) we conclude that

$$\sum_{i=1}^n (-1)^i X[i] = (-1)^{i^\circ - 1} \sum_{i=1}^n (-1)^i X^\circ[i] = 0.$$

The case if $n + k$ is even. Let us define $\mathbf{S}_{i^\circ} : \mathcal{M}_{n,k}(\mathbb{F}) \rightarrow \mathcal{M}_{n,k}(\mathbb{F})$ by

$$\mathbf{S}_{i^\circ}(X) = \begin{pmatrix} x_{i^\circ 1} & \dots & x_{i^\circ k} \\ \vdots & \vdots & \vdots \\ x_{n 1} & \dots & x_{n k} \\ -x_{1 1} & \dots & -x_{1 k} \\ \vdots & \vdots & \vdots \\ -x_{(i^\circ-1) 1} & \dots & -x_{(i^\circ-1) k} \end{pmatrix} \text{ for all } X \in \mathcal{M}_{n,k}(\mathbb{F}).$$

This map is clearly linear and invertible. Now we verify that \mathbf{S}_{i° indeed satisfies the properties (S1)–(S3).

Property (S1). Let $X \in \mathcal{M}_{n,k}(\mathbb{F})$ and $\mathbf{z} \in \mathbb{F}^n$ be such that $\mathbf{z}^t X = 0$. It follows from (3.4) that $\mathbf{z}^{\text{odd}} \mathbf{S}_{i^\circ}(X) = 0$, where \mathbf{z}^{even} is defined by $\mathbf{z}^{\text{even}} = (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, -\mathbf{z}_1, \dots, -\mathbf{z}_{i^\circ-1})^t$. Since $n + k$ is even, then

$$\begin{aligned} \mathbf{z}^{\text{even}} &= (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, -\mathbf{z}_1, \dots, -\mathbf{z}_{i^\circ-1})^t \\ &= (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, (-1)^{n+k+1} \mathbf{z}_1, \dots, (-1)^{n+k+1} \mathbf{z}_{i^\circ-1})^t = \mathbf{z}^\circ, \end{aligned}$$

where \mathbf{z}° be as defined in (3.3).

Property (S2). Let $X \in \mathcal{M}_{n,k}(\mathbb{F})$ be such that $\det_{n,k}(X) = 0$. Then by Lemma 3.10 we have

$$\det_{n,k}(\mathbf{S}_{i^\circ}(X)) = (-1)^{(n-i^\circ)k} \det_{n,k}(X) = 0.$$

Property (S3). Assume that k is odd. Let $X \in \mathcal{M}_{n,k}$ be such that

$$\sum_{i=1}^n (-1)^i X^\circ[i] = 0, \quad (3.6)$$

where $X^\circ = \mathbf{S}_{i^\circ}(X)$. Rewrite the sum $\sum_{i=1}^n (-1)^i X[i]$ as follows

$$\sum_{i=1}^n (-1)^i X[i] = \sum_{i=1}^{i^\circ-1} (-1)^i X[i] + \sum_{i=i^\circ}^n (-1)^i X[i].$$

Since n is odd in this case,

$$\sum_{i=1}^{i^\circ-1} (-1)^i X[i] = - \left(\sum_{i=n-i^\circ+2}^n (-1)^{i-n+i^\circ-1} X^\circ[i] \right)$$

and

$$\sum_{i=i^\circ}^n (-1)^i X[i] = \sum_{i=1}^{n-i^\circ+1} (-1)^{i+i^\circ-1} X^\circ[i],$$

then

$$\begin{aligned}
\sum_{i=1}^n (-1)^i X[i] &= \sum_{i=1}^{i^\circ-1} (-1)^i X[i] + \sum_{i=i^\circ}^n (-1)^i X[i] \\
&= - \left(\sum_{i=n-i^\circ+2}^n (-1)^{i-n+i^\circ-1} X^\circ[i] \right) + \sum_{i=1}^{n-i^\circ+1} (-1)^{i+i^\circ-1} X^\circ[i] \\
&= \sum_{i=n-i^\circ+2}^n (-1)^{i+i^\circ-1} X^\circ[i] + \sum_{i=1}^{n-i^\circ+1} (-1)^{i+i^\circ-1} X^\circ[i] = (-1)^{i^\circ-1} \sum_{i=1}^n (-1)^i X^\circ[i].
\end{aligned}$$

Thus, using the equality (3.6) we conclude that

$$\sum_{i=1}^n (-1)^i X[i] = (-1)^{i^\circ-1} \sum_{i=1}^n (-1)^i X^\circ[i] = 0.$$

□

4 Preliminaries from the theory of linear varieties

In order to use the matroid theory we provide a consistent explanation of the theory of linear varieties (Definition 4.1) and the relationship between their equational representation (Definition 4.17) following [5], [7] and [4].

Definition 4.1 ([20, Observation 3 on p. 8]). Let $V \subseteq W$ be vector spaces. Then a set

$$K = \{\mathbf{s}\} + V = \{\mathbf{s} + \mathbf{v} : \mathbf{v} \in V\}$$

is called *a linear variety*. In particular, every vector subspace of W is a linear variety. Linear varieties are also called *translated subspaces* (e.g. [7, §2.1, Definition of p. 15]).

We extensively use the properties of linear varieties in the next sections and therefore revise them in some detail. The main reason why we cannot use only the theory of vector spaces is the induction step in the proof of Lemma 7.8, where we yield one linear variety from another by equating some coordinates of its elements to nonzero elements of \mathbb{F} .

Lemma 4.2 (Cf. [7, §2.1, Lemma 1 on p. 15]). *The following statements are equivalent:*

- (1) $\mathbf{s} + V = \mathbf{s}' + V$;
- (2) $\mathbf{s}' \in \mathbf{s} + V$.

Lemma 4.3 (Cf. [7, §2.1, Lemma 2 on p. 16]). *If $\mathbf{s} + V = \mathbf{s}' + V'$, then $V = V'$.*

Definition 4.4 (Cf. [7, §2.1, p. 16]). Let $V \subseteq W$ be vector spaces, $K \subset W$ be a linear variety. The space V is called *a subspace belonging to K* if $K = \mathbf{s} + V$ for some $\mathbf{s} \in W$. In [20] the space V is called *a vector space associated with K*.

Lemma 4.3 implies that Definition 4.4 is unambiguous.

Lemma 4.5 (Cf. [7, §2.1, Proposition 1 on p. 16]). *A non-empty intersection of any family of linear varieties is a linear variety.*

Lemma 4.6 (Cf. [7, §2.1, proof of Proposition 1 on p. 16]). *Let $(K_i \mid i \in I)$ be a family of linear varieties having non-empty intersection, and V_i be a vector space belonging to K_i . Then $\bigcap_{i \in I} V_i$ belongs to $\bigcap_{i \in I} K_i$.*

Definition 4.7 (Cf. [7, §2.1, Definition on p. 16]). Let K be a linear variety and $K = s + V$. Then $\dim(V)$ is called a *dimension* of K and denoted by $\dim(K)$.

Definition 4.8. Let K be a linear variety and $K = s + V$. Then $n - \dim(V)$ is called a *codimension* of K and denoted by $\text{codim}(K)$.

Lemma 4.9 (Cf. [7, §2.2, Theorem 1 on p. 19]). *Let $K = s + V$ and $K' = s + V'$ are linear varieties and $K \subseteq K'$. Then*

- (a) $\text{codim}(K) \geq \text{codim}(K')$;
- (b) if $\text{codim}(K) = \text{codim}(K')$, then $K = K'$.

Definition 4.10. For $A \in \mathcal{M}_{mn}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^n$ we denote by $\mathcal{S}(A, \mathbf{b})$ the solution set of a system of linear equations $A\mathbf{x} = \mathbf{b}$.

Lemma 4.11 (Cf. [5, Sec. 3.3, Theorem 3.8]). *Let $A \in \mathcal{M}_{mn}(\mathbb{F})$ and $A\mathbf{x} = 0$ be a homogeneous system of m linear equations in n unknowns over a field \mathbb{F} . Then $\mathcal{S}(A, \mathbf{0})$ is a vector subspace of \mathbb{F}^n of dimension $n - \text{rk}(A)$.*

Lemma 4.12 (See [5, Sec. 3.3, Theorem 3.9]). *Let $A \in \mathcal{M}_{mn}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^n$. Then for any solution \mathbf{s} to $A\mathbf{x} = \mathbf{b}$*

$$\mathcal{S}(A, \mathbf{b}) = \{\mathbf{s}\} + \mathcal{S}(A, \mathbf{0}) = \{\mathbf{s} + \mathbf{k} : \mathbf{k} \in \mathcal{S}(A, \mathbf{0})\}.$$

Corollary 4.13. *Let $A \in \mathcal{M}_{mn}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^n$. If a system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent, then its solution set is a linear variety and $\mathcal{S}(A, \mathbf{0})$ is a vector space belonging to it.*

Lemma 4.14 (See [5, Sec. 3.3, Theorem 3.11]). *Let $A \in \mathcal{M}_{mn}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^n$ and $A\mathbf{x} = \mathbf{b}$ be a system of linear equations. Then the system is consistent if and only if $\text{rk}(A) = \text{rk}(A|\mathbf{b})$.*

Definition 4.15 (See [5, Sec 3.4, Definition on p. 182]). Two systems of linear equations are called *equivalent* if they have the same solution set.

Lemma 4.16 (See [5, Sec. 3.4, Theorem 3.13]). *Let $A\mathbf{x} = \mathbf{b}$ be a system of m linear equations in n unknowns, and let C be an invertible $m \times m$ matrix. Then the system $(CA)\mathbf{x} = C\mathbf{b}$ is equivalent to $A\mathbf{x} = \mathbf{b}$.*

Definition 4.17. Let $A \in \mathcal{M}_{mn}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^n$ and $K \subseteq \mathbb{F}^n$ be a linear variety. If $K = \mathcal{S}(A, \mathbf{b})$, then the pair (A, \mathbf{b}) is called an *equational representation* for K .

Lemma 4.18 (Cf. [7, §3.3, Proposition 2]). *Every linear variety has an equational representation.*

Lemma 4.19. Let (A, \mathbf{b}) be an equational representation of a linear variety $\mathsf{K} \subseteq \mathbb{F}^n$. Then $\text{rk}(A) = \text{codim}(\mathsf{K})$.

Proof. Let $\mathbf{s} \in \mathsf{K}$. Lemma 4.12 implies that $\mathsf{K} = \mathbf{s} + \mathcal{S}(A, \mathbf{0})$. Then

$$\text{rk}(A) = n - \dim(\mathcal{S}(A, \mathbf{0})) = \text{codim}(\mathsf{K})$$

by Lemma 4.11 because $\mathcal{S}(A, \mathbf{0})$ is a vector space which also follows from Lemma 4.11. \square

Lemma 4.20. Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^m$, $\mathsf{K} = \mathcal{S}(A, \mathbf{b}) \subsetneq \mathbb{F}^n$ be a linear variety. Then there exists $R \subseteq [m]$ such that $\mathsf{K} = \mathcal{S}(A[R|], \mathbf{b}[R|])$ and $A[R|]$ has full rank.

Proof. Let K_H be the solution set of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. Lemma 4.12 implies that $\mathsf{K}_H \neq \mathbb{F}^n$. According to Lemma 4.11, the linear variety K_H is a vector subspace of \mathbb{F}^n and $\text{rk}(A) = r = n - \dim(\mathsf{K}_H)$. Since $\mathsf{K}_H \neq \mathbb{F}^n$, then $\dim(\mathsf{K}_H) < n$ and consequently $r > 0$.

Let $R = \{i_1, \dots, i_r\}$ be any set of indices of linearly independent rows in A . Since $\text{rk}(A) = r$, then every equation of system $A\mathbf{x} = \mathbf{b}$ is a linear combination of equations with indices belonging to R . Therefore, $\mathsf{K} = \mathcal{S}(A[R|], \mathbf{b}[R|])$ for this R . \square

Lemma 4.21. Suppose that $\emptyset \neq \mathcal{S}(A, \mathbf{b}) = \mathcal{S}(A', \mathbf{b}') \subseteq \mathbb{F}^n$. Then

- (a) $\mathcal{S}(A, \mathbf{0}) = \mathcal{S}(A', \mathbf{0})$;
- (b) $\text{rk}(A) = \text{rk}(A')$;
- (c) if A and A' have full rank, then there exists an invertible matrix $C \in \mathcal{M}_{\text{rk}(A) \times \text{rk}(A)}(\mathbb{F})$ such that $CA = A'$ and $C\mathbf{b} = \mathbf{b}'$.

Proof. (a) Let $\mathbf{s} \in \mathcal{S}(A, \mathbf{b})$. Since $\mathcal{S}(A, \mathbf{b}) = \mathcal{S}(A', \mathbf{b}')$, then $\mathbf{s} \in \mathcal{S}(A', \mathbf{b}')$. It follows from Lemma 4.12 that

$$\{\mathbf{s}\} + \mathcal{S}(A, \mathbf{0}) = \mathcal{S}(A, \mathbf{b}) \quad \text{and} \quad \{\mathbf{s}\} + \mathcal{S}(A', \mathbf{0}) = \mathcal{S}(A', \mathbf{b}').$$

Then the assumption of the lemma implies that

$$\{\mathbf{s}\} + \mathcal{S}(A, \mathbf{0}) = \{\mathbf{s}\} + \mathcal{S}(A', \mathbf{0}).$$

Therefore,

$$\mathcal{S}(A, \mathbf{0}) = \mathcal{S}(A', \mathbf{0}) \tag{4.1}$$

by Lemma 4.3.

(b) Lemma 4.11 implies that $\mathcal{S}(A, \mathbf{0})$ and $\mathcal{S}(A', \mathbf{0})$ are both vector spaces. Then the equality (4.1) implies that $\dim(\mathcal{S}(A, \mathbf{0})) = \dim(\mathcal{S}(A', \mathbf{0}))$. Therefore,

$$\text{rk}(A) = n - \dim(\mathcal{S}(A, \mathbf{0})) = n - \dim(\mathcal{S}(A', \mathbf{0})) = \text{rk}(A')$$

by Lemma 4.11.

(c) Let $A'' = \begin{pmatrix} A \\ A' \end{pmatrix}$ and $\mathbf{b}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{b}' \end{pmatrix}$. Since $\mathcal{S}(A, \mathbf{b}) = \mathcal{S}(A', \mathbf{b}')$, then

$$\mathcal{S}(A'', \mathbf{b}'') = \mathcal{S}(A, \mathbf{b}) \cap \mathcal{S}(A', \mathbf{b}') = \mathcal{S}(A, \mathbf{b}).$$

Hence, $\text{rk}(A'') = \text{rk}(A)$ by the part (b) of the lemma. Therefore, every row of A'' corresponding to A' is a linear combination of rows corresponding to A , which means that there exists a matrix $C \in \mathcal{M}_{\text{rk}(A) \times \text{rk}(A)}(\mathbb{F})$ such that

$$CA = A'. \quad (4.2)$$

Since $\text{rk}(A) = \text{rk}(A')$, this matrix is invertible.

Let \mathbf{s} be any solution of the system of linear equations $Ax = \mathbf{b}$. It exists because we suppose that $\mathcal{S}(A, \mathbf{b}) \neq \emptyset$. Thus, $A\mathbf{s} = \mathbf{b}$. This equality together with the equality (4.2) imply that

$$C\mathbf{b} = CA\mathbf{s} = A'\mathbf{s} = \mathbf{b}'$$

because $\mathcal{S}(A, \mathbf{b}) = \mathcal{S}(A', \mathbf{b}')$ and consequently $\mathbf{s} \in \mathcal{S}(A', \mathbf{b}')$ as well. \square

Corollary 4.22. Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^n$ and $\mathsf{K} = \mathcal{S}(A, \mathbf{b}) \subsetneq \mathbb{F}^n$ be a linear variety. If $R \subseteq [m]$ is a nonempty set such that $A[R]$ has full rank, then $\mathsf{K} = \mathcal{S}(A[R]), \mathbf{b}[R])$.

Lemma 4.23. Let $\mathsf{K} \subseteq \mathbb{F}^n$ be a linear variety, $V \subseteq \mathbb{F}^n$ be a vector space belonging to K and $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ be a linear map. Then $T(\mathsf{K})$ is a linear variety and $T(V)$ is a vector space belonging to $T(\mathsf{K})$.

Proof. Indeed, if $\mathsf{K} = \mathbf{s} + V$ for some $\mathbf{s} \in \mathbb{F}^n$ and a vector subspace $V \subseteq \mathbb{F}^n$, then $T(\mathsf{K}) = T(\mathbf{s}) + T(V)$, where $T(\mathbf{s}) \in \mathbb{F}^m$ and $T(V) \subseteq \mathbb{F}^m$ is a vector subspace of \mathbb{F}^m . \square

Lemma 4.24. Let $m, n \in \mathbb{N}$, $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $\mathbf{b} \in \mathbb{F}^m$, $\mathsf{K} = \mathcal{S}(A, \mathbf{b})$ be a linear variety, $C \in \mathcal{M}_{n,n}(\mathbb{F})$ be an invertible matrix and $\mathsf{K}_C = \{Cx \mid x \in \mathsf{K}\}$. Then

$$(a) \quad \mathsf{K}_C = \mathcal{S}(AC^{-1}, \mathbf{b});$$

$$(b) \quad \text{codim}(\mathsf{K}_C) = \text{codim}(\mathsf{K}).$$

Proof. (a) Indeed, the conditions $AC^{-1}(Cx) = \mathbf{b}$ and $Ax = \mathbf{b}$ are equivalent for all $x \in \mathbb{F}^n$.

(b) Assume that $\mathsf{K} = \mathbf{s} + V$ for some $\mathbf{s} \in \mathbb{F}^n$ and vector subspace $V \subseteq \mathbb{F}^n$. Lemma 4.23 implies that $V_C = \{Cx \mid x \in \mathsf{K}\}$ is a vector space belonging to K_C . Since C is invertible, then $\dim(V_C) = \dim(V)$, which implies the required equality. \square

We also need the definition for a vector space over \mathbb{F} whose coordinates are indexed by an arbitrary finite set.

Definition 4.25. Let E be a finite set. Then by \mathbb{F}^E we denote the set of all functions from E to \mathbb{F} endowed with the standard structure of vector space over \mathbb{F} ; that is,

$$(f + \lambda g)(x) = f(x) + \lambda g(x)$$

for all $f, g \in \mathbb{F}^E$, $\lambda \in \mathbb{F}$, $x \in E$. We call this space a *coordinate space indexed by S*.

For every $i \in E$ let us also denote by \mathbf{e}_i the function $E \rightarrow \mathbb{F}$ which is equal to 1 if its argument is equal to s , otherwise it is equal to 0. Since E is finite, then the set $\{\mathbf{e}_i \mid i \in E\}$ forms a basis of \mathbb{F}^E .

For every $\mathbf{v} \in \mathbb{F}^E$ and $i \in E$ we denote by \mathbf{v}_i the value of the function identified with \mathbf{v} at the point i .

It is clear that \mathbb{F}^n could also be considered as $\mathbb{F}^{[n]}$ and for every finite set S the space \mathbb{F}^S could be considered as $\mathbb{F}^{|S|}$. Therefore, all the theory developed above could be applied to \mathbb{F}^S with S finite. We also identify $\mathcal{M}_{nk}(\mathbb{F})$ with $\mathbb{F}^{[n] \times [k]}$ by the correspondence $E_{ij} \leftrightarrow \mathbf{e}_{(i,j)}$.

Definition 4.26. If $E' \subset E$ are finite sets, then by $\pi_E^{E'}: \mathbb{F}^E \rightarrow \mathbb{F}^{E'}$ we denote a *standard projection* from \mathbb{F}^E on $\mathbb{F}^{E'}$. That is, if $\mathbf{x} \in \mathbb{F}^E$, then $\pi_E^{E'}(\mathbf{x})$ is defined by

$$\pi_E^{E'}(\mathbf{x})_i = \mathbf{x}_i \text{ for all } i \in E'.$$

Definition 4.27. Let E_1 and E_2 be two finite sets and $f: E_2 \rightarrow E_1$ be an injective function. Denote by $\pi_f: \mathbb{F}^{E_1} \rightarrow \mathbb{F}^{E_2}$ a linear map defined by

$$\pi_f(\mathbf{v})_a = \mathbf{v}_{f(a)} \text{ for all } \mathbf{v} \in \mathbb{F}^{E_1} \text{ and } a \in E_2.$$

We call this map *f-projection*. Since f is injective, then π_f is surjective.

Definition 4.28. If E is a finite set, $i \in E$ its element, then by $x_i^E: \mathbb{F}^E \rightarrow \mathbb{F}$ we denote i -coordinate function. That is,

$$x_i^E(\mathbf{x}) = \mathbf{x}_i \text{ for all } \mathbf{x} \in \mathbb{F}^E.$$

Note that the set $\{x_i \mid i \in E\}$ is a basis of a dual space \mathbb{F}^{E^*} .

5 Preliminaries from the matroid theory

Let us recall the necessary facts from the matroid theory following [15].

Definition 5.1 ([15, Definition on p.7]). A *matroid* M is an ordered pair (E, \mathcal{I}) consisting of a finite set E and a collection \mathcal{I} of subsets of E having the following three properties:

- (I1) $\emptyset \in \mathcal{I}$.
- (I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- (I3) If I_1 and I_2 are in \mathcal{I} and $|I_1| < |I_2|$, then there is an element e of $I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

If M is the matroid (E, \mathcal{I}) , then M is called a *matroid on* E . The members of \mathcal{I} are the *independent sets* of M , and E is the *ground set* of M .

A subset of E that is not in \mathcal{I} is called *dependent*.

Definition 5.2 (Cf. [15, Proposition 1.1.1]). Let $A \in \mathcal{M}_{mn}(\mathbb{F})$. By $\mathbf{M}[A]$ we denote a matroid with E being the set of column labels of A and \mathcal{I} being the set of subsets X of E for which the multiset of columns labelled by X is a set and is linearly independent in the vector space \mathbb{F}^m . This matroid is called the *vector matroid* of A .

It is indeed a matroid by [15, Proposition 1.1.1].

Definition 5.3 (Cf. [15, Definition on p. 20]). Let M be the matroid (E, \mathcal{I}) and suppose that $X \subseteq E$. Let $I|X$ be $\{I \subseteq X: I \in \mathcal{I}\}$. Then it is easy to see that the pair $(X, I|X)$ is a matroid. We call this matroid the *restriction of M to X* or the *deletion of $E \setminus X$ from M*. It is denoted by $M|X$ or $M \setminus (E \setminus X)$.

Definition 5.4. Let M be the matroid (E, \mathcal{I}) and $X \subseteq E$. We define the *rank* $r(X)$ of X to be the cardinality of a basis B of $M|X$ and call such a set B a *basis of X* .

We often write r as r_M . In addition, we usually write $r(M)$ for $r(E(M))$.

Lemma 5.5 (Cf. [15, Lemma 1.3.1 and text on the top of p. 21]). *The rank function r of a matroid M on a set E has the following properties:*

(R1) *If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.*

(R2) *If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.*

(R3) *If X and Y are subsets of E , then*

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

Lemma 5.6 (See [15, Theorem 1.3.2]). *Let E be a set and r be a function that maps 2^E into the set of non-negative integers and satisfies (R1)–(R3). Let I be the collection of subsets X of E for which $r(X) = |X|$. Then (E, I) is a matroid having rank function r .*

Definition 5.7 ([15, Definition on p. 65]). Let M be a matroid. The matroid, whose ground set is $E(M)$ and whose set of bases is $\{E(M) \setminus B : B \text{ is the basis of } M\}$, is called the *dual* of M and is denoted by M^* .

M^* is indeed a matroid by [15, Theorem 2.1.1].

Definition 5.8 (Cf. [15, Definitions on p. 65 and p. 67]). The bases of M^* are called *cobases* of M . A similar convention applies to other distinguished subsets of $E(M^*)$. Hence, for example, the independent sets and spanning sets of M^* are called coindependent sets and cospanning sets of M , and r^* denotes r_{M^*} and is called a *corank* function of M .

Lemma 5.9 (Cf. [15, Proposition 2.1.9]). *For all subsets X of the ground set E of a matroid M ,*

$$r^*(X) + r(E) = r(E \setminus X) + |X|.$$

Lemma 5.10 (See [15, Lemma 2.1.10]). *Let I and I^* be disjoint subsets of $E(M)$ such that I is independent and I^* is coindependent. Then M has a basis B and a cobasis B^* such that B and B^* are disjoint, $I \subseteq B$, and $I^* \subseteq B^*$.*

Definition 5.11 (Cf. [15, Definition on p. 100]). Let M be a matroid on E , and T be a subset of E . Let M/T , the *contraction of T from M* , be given by

$$M/T = (M^* \setminus T)^*.$$

Lemma 5.12 (C.f. [15, Proposition 3.1.6]). *If $T \subseteq E$, then, for all $X \subseteq E \setminus T$,*

$$r_{M/T}(X) = r_M(X \cup T) - r_M(T).$$

Corollary 5.13. *Let M be a matroid on E and T be an independent set of M . Then $X \supseteq T$ is an independent set of M if and only if $X \setminus T$ is an independent set of M/T .*

For purpose of the proof of the main theorem of this article we need to provide the correspondence between certain coindependent sets of matroid \mathbf{M} and contraction matroid $\mathbf{M}|S$ and their coranks.

Lemma 5.14. Let \mathbf{M} be a matroid on E , $S \subseteq E$ be a set and $I^* \subseteq E$ be a coindependent set of \mathbf{M} . Then

- (a) If B'^* is a cobasis of $\mathbf{M}|S$, then $(I^* \setminus S) \cup B'^*$ is a coindependent set of \mathbf{M} .
- (b) $r^*(\mathbf{M}|S) \leq r^*(\mathbf{M}) - |I^* \setminus S|$.

Proof. (a) Since B'^* is a cobasis of $\mathbf{M}|S$, then $B' = (S \setminus B'^*) \subseteq S$ is an independent set of $\mathbf{M}|S$. Hence, B' is an independent set of \mathbf{M} by the definition of $\mathbf{M}|S$.

I^* is a coindependent set of \mathbf{M} . Therefore, $I_S^* = I^* \setminus S$ is a coindependent set of \mathbf{M} . In addition, $I_S^* \cap S = \emptyset$.

Thus, I_S^* and B' are correspondingly coindependent and independent subsets of \mathbf{M} such that $I_S^* \cap B' = \emptyset$. Hence, from Lemma 5.10 we conclude that there exist a basis B_0 and a cobasis B_0^* of \mathbf{M} such that $B_0 \supseteq B'$, $B_0^* \supseteq I_S^*$ and $B_0 \cap B_0^* = \emptyset$.

The inclusion $B_0 \supseteq B'$ implies the inclusion

$$B_0 \cap S \supseteq B' \cap S = B'. \quad (5.1)$$

The definition of $\mathbf{M}|S$ implies that $B_0 \cap S$ is an independent set of $\mathbf{M}|S$. Thus, having the inclusion (5.1) and the fact that B' is base of $\mathbf{M}|S$, we obtain that

$$B_0 \cap S = B'. \quad (5.2)$$

Since $E = B_0 \sqcup B_0^*$ and $S \subseteq E$, then it follows from (5.2) that $B_0^* \cap S = B'^*$. Thus, $B_0^* \supseteq B'^*$ and consequently

$$B_0^* \supseteq I_S^* \cup B'^* = (I^* \setminus S) \cup B'^*.$$

Therefore, $(I^* \setminus S) \cup B'^*$ is a coindependent set of \mathbf{M} .

(b) Let $B'^* \subseteq S$ be any cobasis of $\mathbf{M}|S$. Then $r^*(\mathbf{M}|S) = |B'^*|$. From the part (a) of the lemma we conclude that

$$r^*(\mathbf{M}) \geq |(I^* \setminus S) \cup B'^*|.$$

Let us consider the right hand side of this inequality. Since $(I^* \setminus S) \cap B'^* = \emptyset$, then

$$|(I^* \setminus S) \cup B'^*| = |I^* \setminus S| + |B'^*| = |I^* \setminus S| + r^*(\mathbf{M}|S).$$

Thus,

$$r^*(\mathbf{M}) \geq |(I^* \setminus S)| + r^*(\mathbf{M}|S).$$

which is equivalent to the required inequality. \square

The following corollary is a direct consequence of Lemma 5.14 applied to the contraction matroid \mathbf{M}/T instead of \mathbf{M} and the definition of contraction matroid. It will be used in sequel.

Corollary 5.15. Let \mathbf{M} be a matroid on E , $T \subseteq E$, $S \subseteq E \setminus T$ be two sets and $I^* \subseteq E$ be a coindependent set of \mathbf{M} . Then

- (a) If B'^* is a cobasis of $\mathbf{M}/T|S$, then $((I^* \setminus T) \setminus S) \cup B'^*$ is a coindependent set of \mathbf{M} .
- (b) $r^*(\mathbf{M}/T|S) \leq r^*(\mathbf{M}) - |(I^* \setminus T) \setminus S|$.

6 Matroids corresponding to vector spaces and linear varieties

In this section we introduce a matroid $\mathbf{M}(\mathsf{K})$ corresponding to a given linear variety K (Definition 6.3). After that, we study the behaviour of $\mathbf{M}(\mathsf{K})$ while K is transformed using the projection map or intersected with a slice of \mathbb{F}^S (Definition 4.27 and Definition 6.11). The corresponding results are stated and proved in Lemma 6.7 and Lemma 6.16. We also prove that a matroid linear variety and vector matroid of a matrix in its equational representation are dual to each other (Lemma 6.17).

Definition 6.1. Let $V \subseteq \mathbb{F}^E$ be a vector space. Then by $\mathbf{M}(V)$ we denote a matroid determined by rank function $r(X) = \dim(\text{span}(\mathbf{x}_i|_V \mid i \in X))$.

Since r satisfies the properties (R1)–(R3), then $\mathbf{M}(V)$ is indeed a matroid by Lemma 5.6.

Remark 6.2. Every matroid of a subspace $V \subseteq F$ defined above can be regarded as an example of a matroid *representable over \mathbb{F}* (see [21, Chap. 9.1, definition on p. 136]).

Definition 6.3. Let $\mathsf{K} \subseteq \mathbb{F}^E$ be a linear variety. Then by $\mathbf{M}(\mathsf{K})$ we denote a matroid of the linear variety K which is defined by $\mathbf{M}(\mathsf{K}) = \mathbf{M}(V)$, where $V \subseteq \mathbb{F}^E$ is the vector space belonging to K .

Corollary 6.4. If K is a linear variety, then $r(\mathbf{M}(\mathsf{K})) = \dim(\mathsf{K})$ and $r^*(\mathbf{M}(\mathsf{K})) = \text{codim}(\mathsf{K})$.

Lemma 6.5. Let $V \subseteq \mathbb{F}^E$ be a vector space, $X \subseteq E$. Then a linear map $\pi_E^X|_V$ is a surjection if and only if X is an independent set of $\mathbf{M}(V)$.

Proof. Let $\phi: (\mathbb{F}^X)^* \rightarrow V^*$ be defined by $\phi(f) = \sum_{i \in X} f(\mathbf{e}_i)\mathbf{x}_i|_V$. Note that

$$\phi = (\pi_E^X|_V)^*. \quad (6.1)$$

Indeed, since $\mathbf{x}_i^X \circ \pi_E^X|_V = \mathbf{x}_i^E|_V$ for all $i \in X$, then $(\pi_E^X|_V)^*(\mathbf{x}_i^X) = \phi(\mathbf{x}_i^X)$ for all $i \in X$. Therefore, the equality 6.1 holds since it holds on the basis $\{\mathbf{x}_i^X \mid i \in X\}$ of $(\mathbb{F}^X)^*$.

The basic properties of dual vector spaces (see [5, Sec. 2.7, Exercise 20 on p. 127]) imply that in this case $\pi_E^X|_V$ is a surjection if and only if ϕ is an injection. The last condition is equivalent to the condition that X is an independent set of $\mathbf{M}(V)$. Indeed, the definition of $\mathbf{M}(V)$ implies that $X \in \mathbf{M}(V)$ if and only if $\dim(\text{span}(\mathbf{x}_i|_V \mid i \in X)) = |X|$ which means that the set $\{\mathbf{x}_i|_V \mid i \in X\}$ is linearly independent. Thus, the required equivalence is established. \square

Corollary 6.6. Let $\mathsf{K} \subseteq \mathbb{F}^E$ be a linear variety, $X \subseteq E$. Then X is an independent set of $\mathbf{M}(\mathsf{K})$ if and only if for every $\mathbf{c} \in \mathbb{F}^I$ there exists $\mathbf{x} \in \mathsf{K}$ such that $\mathbf{x}_i = \mathbf{c}_i$ for all $i \in I$

Let us study behaviour of $\mathbf{M}(\mathsf{K})$ under projections and restrictions. The next lemma provide a correspondence between a matroid of linear variety and a matroid of its f -projection.

Lemma 6.7. Let E_1 and E_2 be two finite sets, $f: E_2 \rightarrow E_1$ be an injective function, $\mathsf{K} \subseteq \mathbb{F}^{E_1}$ be a linear variety. Then

- (a) $r_{\mathbf{M}(\pi_f(\mathsf{K}))}(X) = r_{\mathbf{M}(\mathsf{K})}(f(X))$ for all $X \subseteq E_2$;

(b) f is an isomorphism of matroids $\mathbf{M}(\pi_f(\mathsf{K}))$ and $\mathbf{M}(\mathsf{K})|f(E_2)$ (restriction matroid);

Proof. (a) Let $V \subseteq \mathbb{F}^{E_1}$ be vector space belonging to K . Then Lemma 4.23 implies that $\pi_f(V)$ is a vector space belonging to $\pi_f(\mathsf{K})$. Let us consider $\pi_f|_V$ as a linear map on its image, that is, we assume that $\pi_f|_V$ sends V on $\pi_f(V)$. Note that this together with the definition of π_f implies that

$$(\pi_f|_V)^*(\mathbf{x}_i^{E_2}|_{\pi_f(V)}) = \mathbf{x}_{f(i)}^{E_1}|_V \text{ for all } i \in E_2. \quad (6.2)$$

Since $\pi_f|_V$ is a surjection, then $(\pi_f|_V)^*$ is an injection. Therefore,

$$\dim(\text{span}(\mathbf{x}_i^{E_2}|_{\pi_f(V)} \mid i \in X)) = \dim(\text{span}(\mathbf{x}_{f(i)}^{E_1}|_V \mid i \in f(X)))$$

and consequently

$$\begin{aligned} r_{\mathbf{M}(\pi_f(\mathsf{K}))}(X) &= \dim(\text{span}(\mathbf{x}_i^{E_2}|_{\pi_f(V)} \mid i \in X)) \\ &= \dim(\text{span}(\mathbf{x}_{f(i)}^{E_1}|_V \mid i \in f(X))) = r_{\mathbf{M}(\mathsf{K})}(f(X)) \text{ for all } X \subseteq E_2. \end{aligned} \quad (6.3)$$

(b) It follows directly from the part (a) of the lemma and the definition of restriction matroid. \square

Definition 6.8. Let E_1 and E_2 be two finite sets such that $E_2 \subseteq E_1$ and $V \subseteq \mathbb{F}^{E_1}$ be a vector space. Then by $V_{E_2} \subseteq \mathbb{F}^{E_1}$ we denote a vector space defined by

$$V_{E_2} = \{\mathbf{x} \in V \mid \mathbf{x}_i = 0 \text{ for all } i \in E_2\}.$$

Lemma 6.9. Let E_1 and E_2 be two finite sets such that $E_2 \subseteq E_1$, $V \subseteq \mathbb{F}^{E_1}$ be a vector space. Then

(a)

$$r_{\mathbf{M}(V_{E_2})}(X) = r_{\mathbf{M}(V)}(X \cup E_2) - r_{\mathbf{M}(V)}(E_2) \text{ for all } X \subseteq (E_1 \setminus E_2) \quad (6.4)$$

(b) $\mathbf{M}(V_{E_2}) \setminus E_2 = \mathbf{M}(V)/E_2$.

Proof. (a) Let $X \subseteq (E_1 \setminus E_2)$. The definition of V_{E_2} implies that

$$\mathbf{x}_i^{E_1}|_{V_{E_2}} = 0 \text{ for all } i \in E_2 \quad (6.5)$$

Let $\phi: \text{span}(\mathbf{x}_i^{E_1}|_V \mid i \in (X \cup E_2)) \rightarrow \text{span}(\mathbf{x}_i^{E_1}|_{V_{E_2}} \mid i \in X)$ be a linear map defined by

$$\phi\left(\sum_{i \in (X \cup E_2)} \lambda_i \mathbf{x}_i^{E_1}|_V\right) = \sum_{i \in X} \lambda_i \mathbf{x}_i^{E_1}|_{V_{E_2}}.$$

The equality (6.5) implies that ϕ is defined properly, i.e. does not depend of the representation of an element of $\text{span}(\mathbf{x}_i^{E_1}|_V \mid i \in (X \cup E_2))$ as a linear combination $\sum_{i \in (X \cup E_2)} \lambda_i \mathbf{x}_i^{E_1}|_V$. Therefore, the following sequence is exact

$$\begin{aligned} 0 \rightarrow \text{span}(\mathbf{x}_i^{E_1}|_V \mid i \in E_2) &\hookrightarrow \text{span}(\mathbf{x}_i^{E_1}|_V \mid i \in (X \cup E_2)) \\ &\xrightarrow{\phi} \text{span}(\mathbf{x}_i^{E_1}|_{V_{E_2}} \mid i \in X) \rightarrow 0. \end{aligned} \quad (6.6)$$

Hence,

$$\begin{aligned} \dim(\text{span}(\mathbf{x}_i|_V \mid i \in E_2)) + \dim(\text{span}(\mathbf{x}_i^{E_1}|_{V_{E_2}} \mid i \in X)) \\ = \dim(\text{span}(\mathbf{x}_i|_V \mid i \in (X \cup E_2))). \end{aligned}$$

Since

$$\begin{aligned} \dim(\text{span}(\mathbf{x}_i|_V \mid i \in E_2)) &= r_{\mathbf{M}(V)}(E_2), \\ \dim(\text{span}(\mathbf{x}_i^{E_1}|_{V_{E_2}} \mid i \in X)) &= r_{\mathbf{M}(V_{E_2})}(X) \quad \text{and} \\ \dim(\text{span}(\mathbf{x}_i|_V \mid i \in (X \cup E_2))) &= r_{\mathbf{M}(V)}(X \cup E_2) \end{aligned}$$

by the corresponding definitions, then we conclude that

$$r_{\mathbf{M}(V)}(E_2) + r_{\mathbf{M}(V_{E_2})}(X) = r_{\mathbf{M}(V)}(X \cup E_2),$$

which is equivalent to the equality (6.4).

(b) This follows directly from Lemma 5.12, the part (a) of the lemma and the definitions of the corresponding matroids. \square

Corollary 6.10. *Let E be a finite set, $X \subseteq E$ and $V \subseteq \mathbb{F}^E$ be a vector space. Then*

$$\dim(V_{E \setminus X}) = r(\mathbf{M}(V)) - r_{\mathbf{M}(V)}(E \setminus X).$$

In order to generalize Lemma 6.9 to linear varieties we need to introduce a notion of slice.

Definition 6.11. For $\{e_1, \dots, e_r\} = E' \subseteq E$ and $\mathbf{c}' \in \mathbb{F}^{E'}$ we call a *slice* $U(x_{e_1} = \mathbf{c}_{e_1}, \dots, x_{e_r} = \mathbf{c}_{e_r})$ a linear variety of \mathbb{F}^E defined by

$$U(x_{e_1} = \mathbf{c}_{e_1}, \dots, x_{e_r} = \mathbf{c}_{e_r}) = \mathbf{c} + \text{span}(\mathbf{e}_i \mid i \in E \setminus E'),$$

where $\mathbf{c}_i = \mathbf{c}'_i$ for $i \in E'$ and is equal to zero otherwise. Equivalently,

$$U(x_{e_1} = \mathbf{c}_{e_1}, \dots, x_{e_r} = \mathbf{c}_{e_r}) = \{\mathbf{s} \in \mathbb{F}^E \mid \mathbf{s}_e = \mathbf{c}_e \forall e \in E'\}.$$

We will also use a shorter notation $U(x_e = \mathbf{c}_e \mid e \in E')$.

Corollary 6.12. *If $E' \subseteq E$ and $\mathbf{c}' \in \mathbb{F}^{E'}$, then $U(x_e = 0 \mid e \in E')$ is a vector space belonging to $U(x_e = \mathbf{c}_e \mid e \in E')$.*

Lemma 6.13. *Let $\mathsf{K} \subseteq \mathbb{F}^E$ be a linear variety, V be a subspace belonging to K , I be an independent set of $\mathbf{M}(\mathsf{K})$, $\mathbf{c} \in \mathbb{F}^I$ and $\mathsf{K}_I = \mathsf{K} \cap U(x_e = \mathbf{c}_e \mid e \in I)$. Then K_I is a linear variety and V_I is a subspace belonging to K_I .*

Proof. Lemma 4.5 implies that in order to prove that K_I is a linear variety it is sufficient to show that K and $U(x_e = \mathbf{c}'_e \mid e \in I)$ have non-empty intersection. This follows directly from Corollary 6.6.

Next, since $U(x_e = 0 \mid e \in I)$ is a subspace belonging to $U(x_e = \mathbf{c}_e \mid e \in I)$ by Corollary 6.12, then $V \cap U(x_e = 0 \mid e \in I)$ is a subspace belonging to K_I by Lemma 4.6. In addition, $V \cap U(x_e = 0 \mid e \in I) = V_I$ by the definition of V_I . \square

Remark 6.14. If I is not an independent set of $\mathbf{M}(\mathsf{K})$, then the intersection $\mathsf{K} \cap U(x_e = \mathbf{c}_e \mid e \in I)$ could be empty.

In the next lemma we provide a correspondence between a matroid of linear variety K and a matroid of its intersection with slice $U(x_e = \mathbf{c}_e \mid e \in I)$ assuming that I is an independent set of K which follows directly from the correspondence established in Lemma 6.13. This lemma together with its generalized version (Lemma 6.16) will be used in the next section.

Lemma 6.15. Let E be a finite set, $\mathsf{K} \subseteq \mathbb{F}^E$ be a linear variety, I be an independent set of $\mathbf{M}(\mathsf{K})$, $\mathbf{c} \in \mathbb{F}^I$. Then $\mathsf{K}_I = \mathsf{K} \cap U(x_e = \mathbf{c}_e \mid e \in I)$ is a linear variety and $\mathbf{M}(\mathsf{K}_I) \setminus I = \mathbf{M}(\mathsf{K})/I$.

Proof. Lemma 6.13 implies that $\mathsf{K}_I \neq \emptyset$ is indeed a linear variety and V_I is a vector space belonging to K_I , where V is a vector space belonging to K . Therefore, by Lemma 6.9(b) applied to I, E and V , we obtain the statement of the lemma. \square

Lemma 6.16. Let E_1, E_2 be finite sets, $\mathsf{K} \subseteq \mathbb{F}^{E_1}$ be a linear variety, I be an independent set of $\mathbf{M}(\mathsf{K})$, $\mathbf{c} \in \mathbb{F}^I$, $f: E_2 \rightarrow E_1$ be an injective function. Then f provides an isomorphism between the matroids $\mathbf{M}(\pi_f(\mathsf{K}_I)) \setminus f^{-1}(I)$ and $(\mathbf{M}(\mathsf{K})/I)|_{(f(E_2) \setminus I)}$, where K_I is defined by $\mathsf{K}_I = \mathsf{K} \cap U(x_e = \mathbf{c}_e \mid e \in I)$

Proof. Lemma 6.15 applied to E_1, K and I implies that

$$\mathbf{M}(\mathsf{K}_I) \setminus I = \mathbf{M}(\mathsf{K})/I.$$

Hence,

$$(\mathbf{M}(\mathsf{K}_I) \setminus I)|(f(E_2) \setminus I) = (\mathbf{M}(\mathsf{K})/I)|(f(E_2) \setminus I).$$

In addition, since $(E_1 \setminus I) \supseteq (f(E_2) \setminus I)$,

$$(\mathbf{M}(\mathsf{K}_I) \setminus I)|(f(E_2) \setminus I) = \mathbf{M}(\mathsf{K}_I)|(f(E_2) \setminus I).$$

Therefore,

$$\mathbf{M}(\mathsf{K}_I)|(f(E_2) \setminus I) = (\mathbf{M}(\mathsf{K})/I)|(f(E_2) \setminus I). \quad (6.7)$$

By Lemma 6.7(b), f is an isomorphism between $\mathbf{M}(\pi_f(\mathsf{K}_I))$ and $\mathbf{M}(\mathsf{K}_I|f(E_2))$. Hence, $f|_{(E_2 \setminus f^{-1}(I))}$ is an isomorphism between $\mathbf{M}(\pi_f(\mathsf{K}_I)) \setminus f^{-1}(I)$ and $\mathbf{M}(\mathsf{K}_I)|(f(E_2) \setminus I)$. This fact together with (6.7) implies that f provides an isomorphism between $\mathbf{M}(\pi_f(\mathsf{K}_I)) \setminus f^{-1}(I)$ and $(\mathbf{M}(\mathsf{K})/I)|(f(E_2) \setminus I)$. \square

The following lemma provides a simple correspondence between a matroid of linear variety K and a vector matroid of matrix in equational representation of K which will be used in sequel and has an independent importance.

Lemma 6.17. Let $\mathsf{K} = \mathcal{S}(A, \mathbf{b}) \subseteq \mathbb{F}^E$ be a linear variety for some $A \in \mathcal{M}_{mE}(\mathbb{F}), \mathbf{b} \in \mathbb{F}^n$. Then

(a)

$$r_{\mathbf{M}[A]}(X) + r(\mathbf{M}(\mathsf{K})) = |X| + r_{\mathbf{M}(\mathsf{K})}(E \setminus X) \quad \text{for all } X \subseteq E. \quad (6.8)$$

(b) $\mathbf{M}[A] = \mathbf{M}^*(\mathsf{K})$.

Proof. (a) Let $V \subseteq \mathbb{F}^E$ be a vector space belonging to K . It follows from Corollary 4.13 that

$$V = \mathcal{S}(A, \mathbf{0}). \quad (6.9)$$

Let $X \subseteq E$. Consider the following sequence of vector spaces and linear maps

$$0 \rightarrow V_{E \setminus X} \xrightarrow{\pi_E^X} \mathbb{F}^X \xrightarrow{\mathbf{x} \mapsto \sum_{i \in X} \mathbf{x}_i A(|i|)} \mathsf{C}(A(|X|)) \rightarrow 0, \quad (6.10)$$

where $\mathsf{C}(A(|X|))$ is a column space of a matrix $A(|X|)$. The definition of $V_{E \setminus X}$, π_E^X and the equality (6.9) imply that this sequence is exact. Hence,

$$\dim(\mathbb{F}^X) = \dim(V_{E \setminus X}) + \dim(\mathsf{C}(A(|X|))). \quad (6.11)$$

From Corollary 6.10 we conclude that

$$\dim(V_{E \setminus X}) = r(\mathbf{M}(V)) - r_{\mathbf{M}(V)}(E \setminus X),$$

which is equal to $r_{\mathbf{M}(\mathsf{K})}(E) - r_{\mathbf{M}(\mathsf{K})}(E \setminus X)$ by the definition of $\mathbf{M}(\mathsf{K})$.

The definition of $\mathbf{M}[A]$ implies that

$$\dim(\mathsf{C}(A(|X|))) = r_{\mathbf{M}[A]}(X).$$

Thus, (6.11) is rewritten as follows

$$\dim(\mathbb{F}^X) = r(\mathbf{M}(V)) - r_{\mathbf{M}(\mathsf{K})}(E \setminus X) + r_{\mathbf{M}[A]}(X).$$

Since $\dim(\mathbb{F}^X) = |X|$, this equality is equivalent to the equality (6.8).

(b) This follows directly from the part (a) of the lemma and Lemma 5.9. \square

Corollary 6.18. *Let $\mathsf{K} = \mathcal{S}(A, \mathbf{b}) \subseteq \mathbb{F}^E$ be a linear variety. Then $\mathbf{M}(\mathsf{K}) = \mathbf{M}^*[A]$.*

Corollary 6.19. *If $\emptyset \neq \mathcal{S}(A, \mathbf{b}) = \mathcal{S}(A', \mathbf{b}')$, then $\mathbf{M}[A] = \mathbf{M}[A']$.*

The following definition and two lemmas are used in the proof of Lemma 7.8 in the next section.

Definition 6.20. Let $\mathsf{K} = \mathcal{S}(A, \mathbf{b})$ be a linear variety and $B^* \subseteq E$ be a cobasis of $\mathbf{M}[A]$. Then A is called *reduced with respect to B^** if A has full rank and $A(|B^*|)$ is a permutation matrix.

Lemma 6.21. *Let $\mathsf{K} \subsetneq \mathbb{F}^E$ be a linear variety and $B^* \subseteq E$ be a cobasis of $\mathbf{M}(\mathsf{K})$. Then there exists an equational representation (A, \mathbf{b}) of K such that A is reduced with respect to B^* .*

Proof. Let (A, \mathbf{b}) be any equational representation of K . Then by Lemma 4.20 there exists $R \subseteq [m]$ such that $(A', \mathbf{b}') = (A[R], \mathbf{b}[R])$ is an equational representation of K and A' has full rank.

Since B^* is a cobasis of $\mathbf{M}(\mathsf{K})$, then it is a basis of $\mathbf{M}[A']$ by the definition of $\mathbf{M}(\mathsf{K})$. Hence,

$$\text{rk}\left(A'(|B^*|)\right) = |B^*| = \text{rk}(A').$$

Thus, $A'(|B^*|)$ is a square matrix having full rank, which implies that it is an invertible matrix. Let us denote its inverse by C . Then by Lemma 4.15

$$\mathcal{S}(CA', C\mathbf{b}') = \mathcal{S}(A', \mathbf{b}')$$

and $(CA')(|B^*|) = C(A'(|B^*|))$ is equal to $I_{\text{codim}(\mathsf{K})}$ which is a permutation matrix. \square

Lemma 6.22. Let $\mathsf{K} \subseteq \mathbb{F}^E$ be a linear variety, $B^* \subseteq E$ be a cobasis of $\mathbf{M}(\mathsf{K})$, (A, \mathbf{b}) be an equational representation of K such that A is reduced with respect to B^* , $1 \leq i \leq \text{codim}(\mathsf{K})$, $e_{\text{old}} \in B^*$ be a unique element such that $A_{ie_{\text{old}}} \neq 0$. If $A_{ie_{\text{new}}} \neq 0$ for some e_{new} , then $B^* \Delta \{e_{\text{old}}, e_{\text{new}}\}$ is a cobasis of $\mathbf{M}(\mathsf{K})$.

Proof. Indeed, in this case $A(|B^* \Delta \{e_{\text{old}}, e_{\text{new}}\}|)$ has the form

$$\begin{pmatrix} A(i|B^* \setminus \{e_{\text{old}}\}) & * \\ 0 & A_{ie_{\text{new}}} \end{pmatrix}$$

after an appropriate rearrangement of rows and columns. It follows from the definition of e_{old} that $A(i|B^* \setminus \{e_{\text{old}}\})$ is a permutation matrix and hence it has full rank. Therefore, $A(|B^* \Delta \{e_{\text{old}}, e_{\text{new}}\}|)$ is a square matrix having full rank. It implies that

$$\text{rk}\left(A(|B^* \Delta \{e_{\text{old}}, e_{\text{new}}\}|)\right) = |B^* \Delta \{e_{\text{old}}, e_{\text{new}}\}| = |B^*| = \text{rk}(A).$$

Therefore, $B^* \Delta \{e_{\text{old}}, e_{\text{new}}\}$ is a basis of $\mathbf{M}[A]$. \square

7 Linear varieties of matrices annihilating $\det_{n,k}$

In this section we prove our main result regarding linear varieties in matrix spaces annihilating $\det_{n,k}$ (Theorem 7.11).

Note that our argumentation in the proofs of Lemma 7.8 and Theorem 7.11 is motivated by the argumentation used in the proof of [4, Theorem 2] and similar to it, but for convenience we use the language of matroid theory.

First we define the projection map which formalizes the concept of operation of striking out one row and one column of matrix. For example, this operation is performed when we use the Laplace expansion formula.

Definition 7.1. Let $n \geq k > 1$ and $1 \leq i \leq n, 1 \leq j \leq k$. We denote by $\text{ins}_{n,k}^{(i,j)} : [n-1] \times [k-1] \rightarrow [n] \times [k] \setminus (\{i\} \times [k] \cup [n] \times \{j\})$ a map defined by

$$\text{ins}_{n,k}^{(i,j)}((a, b)) = \begin{cases} (a+1, b+1), & a \geq i, b \geq j \\ (a+1, b), & a \geq i, b < j \\ (a, b+1), & a < i, b \geq j \\ (a, b), & \text{otherwise} \end{cases}$$

for all $(a, b) \in [n-1] \times [k-1]$.

Lemma 7.2. Suppose that $(a, b), (a', b') \subseteq [n-1] \times [k-1]$ belong to the same row (column). Then $\text{ins}_{n,k}^{(i,j)}((a, b)), \text{ins}_{n,k}^{(i,j)}((a', b')) \subseteq [n] \times [k]$ belong to the same row (column) for all $1 \leq i \leq n, 1 \leq j \leq k$.

Proof. It follows from the definition of $\text{ins}_{n,k}^{(i,j)}$ that if $a = a'$ then the first elements of tuples $\text{ins}_{n,k}^{(i,j)}((a, b))$ and $\text{ins}_{n,k}^{(i,j)}((a', b'))$ are equal.

Similarly, if $b = b'$, then the second elements of tuples $\text{ins}_{n,k}^{(i,j)}((a, b))$ and $\text{ins}_{n,k}^{(i,j)}((a', b'))$ are equal. \square

In the following lemma we provide an algorithm to obtain a linear variety in $\mathcal{M}_{(n-1)(k-1)}(\mathbb{F})$ annihilating $\det_{(n-1)(k-1)}$ from a given linear variety in $\mathcal{M}_{nk}(\mathbb{F})$ annihilating \det_{nk} and a relationship between their codimensions and cobases.

Lemma 7.3. *Let $k > 1$, $\mathsf{K} \subseteq \mathcal{M}_{nk}(\mathbb{F})$ be a linear variety, $B^* \subseteq [n] \times [k]$ be a cobasis of $\mathbf{M}(\mathsf{K})$, $1 \leq j' \leq k$ such that $B^* \cap ([n] \times \{j'\}) = \emptyset$, $1 \leq i' \leq n$ and $c_1, \dots, c_n \in \mathbb{F}$. Then*

(a) $\mathsf{K}' = \pi_{\text{ins}_{nk}^{(i',j')}}(\mathsf{K} \cap U(x_{(i,j')} = c_i \mid 1 \leq i \leq n))$ is a linear variety.

(b)

$$\text{codim}(\mathsf{K}') \leq \text{codim}(\mathsf{K}) - |B^* \cap (\{i'\} \times [k])|. \quad (7.1)$$

(c) If B'^* is a cobasis of $\mathbf{M}(\mathsf{K}')$, then $(B^* \cap (\{i'\} \times [k])) \cup \text{ins}_{nk}^{(i',j')}(B'^*)$ is a coindependent set of $\mathbf{M}(\mathsf{K})$.

(d) If $\det_{nk}(X) = 0$ for all $X \in \mathsf{K}$ and $c_i = \delta_{i,i'}$ for all $1 \leq i \leq n$, then $\det_{n-1,k-1}(X') = 0$ for all $X' \in \mathsf{K}'$.

Proof. (a) From Lemma 6.15 we conclude that $\mathsf{K} \cap U(x_{(i,j')} = c_i \mid 1 \leq i \leq n)$ is a linear variety. Hence, K' is a linear variety by Lemma 4.23 because $\pi_{\text{ins}_{nk}^{(i',j')}}$ is a linear map.

(b) and (c) Since B^* is a cobasis of $\mathbf{M}^*(\mathsf{K})$, then $I = [n] \times \{j'\}$ is an independent set of $\mathbf{M}(\mathsf{K})$. By Lemma 6.16 applied to K , $I = [n] \times \{j'\}$ and $f = \text{ins}_{nk}^{(i',j')}$ we obtain that

$\text{ins}_{nk}^{(i',j')}$ is an isomorphism between $\mathbf{M}(\mathsf{K}')$ and \mathbf{M}' ,

$$\text{where } \mathbf{M}' = \left(\mathbf{M}(\mathsf{K}) / ([n] \times \{j'\}) \right) \mid \left(\text{ins}_{nk}^{(i',j')}([n-1] \times [k-1]) \right) \quad (7.2)$$

because $f^{-1}(I) = \left(\text{ins}_{nk}^{(i',j')} \right)^{-1}([n] \times \{j'\}) = \emptyset$. This claim will be used in the proofs of the parts (b) and (c) of the lemma.

Let us prove the part (b) of the lemma. By the claim (7.2) we have

$$\text{codim}(\mathsf{K}') = r^*(\mathbf{M}(\mathsf{K}')) = r^*(\mathbf{M}'). \quad (7.3)$$

Corollary 5.15(b) applied to $\mathbf{M} = \mathbf{M}(\mathsf{K})$, $E = [n] \times [k]$, $T = [n] \times \{j'\}$, $S = \text{ins}_{nk}^{(i',j')}([n-1] \times [k-1])$ and $I^* = B^*$ implies that

$$r^*(\mathbf{M}') \leq r^*(\mathbf{M}(\mathsf{K})) - \left| \left(B^* \setminus ([n] \times \{j'\}) \right) \setminus \left(\text{ins}_{nk}^{(i',j')}([n-1] \times [k-1]) \right) \right| \quad (7.4)$$

Let us simplify the left-hand side of (7.4). First,

$$r^*(\mathbf{M}(\mathsf{K})) = \text{codim}(\mathsf{K}) \quad (7.5)$$

by Corollary 6.4. Second, the equality

$$B^* \cap ([n] \times \{j'\}) = \emptyset \quad (7.6)$$

implies that

$$\left(B^* \setminus ([n] \times \{j'\}) \right) \setminus \left(\text{ins}_{nk}^{(i',j')} ([n-1] \times [k-1]) \right) = B^* \setminus \left(\text{ins}_{nk}^{(i',j')} ([n-1] \times [k-1]) \right). \quad (7.7)$$

Since $\text{ins}_{nk}^{(i',j')} ([n-1] \times [k-1]) = ([n] \times [k]) \setminus \left(([n] \times \{j'\}) \cup (\{i'\} \times [k]) \right)$ and $B^* \subseteq ([n] \times [k])$, then

$$B^* \setminus \left(\text{ins}_{nk}^{(i',j')} ([n-1] \times [k-1]) \right) = B^* \cap \left(([n] \times \{j'\}) \cup (\{i'\} \times [k]) \right). \quad (7.8)$$

By the equality (7.6) we have

$$B^* \cap \left(([n] \times \{j'\}) \cup (\{i'\} \times [k]) \right) = B^* \cap (\{i'\} \times [k]). \quad (7.9)$$

Hence, by aligning the equalities (7.7)–(7.9) we conclude that

$$\left| \left(B^* \setminus ([n] \times \{j'\}) \right) \setminus \left(\text{ins}_{nk}^{(i',j')} ([n-1] \times [k-1]) \right) \right| = \left| B^* \cap (\{i'\} \times [k]) \right|. \quad (7.10)$$

Therefore, by substituting (7.3), (7.5) and (7.10) into the inequality (7.4) we obtain the inequality (7.1). This establishes the part (b) of the lemma.

Let us prove the part (c) of the lemma. Let B'^* be a cobasis of $\mathbf{M}(K')$. Then $\text{ins}_{nk}^{(i',j')} (B'^*)$ is a cobasis of \mathbf{M}' as it follows from (7.2). Thus,

$$\left(\left(B^* \setminus ([n] \times \{j'\}) \right) \setminus \left(\text{ins}_{nk}^{(i',j')} ([n-1] \times [k-1]) \right) \right) \cup \text{ins}_{nk}^{(i',j')} (B'^*) \quad (7.11)$$

is a coindependent set of $\mathbf{M}(K)$ by Corollary 5.15(a) applied to $\mathbf{M} = \mathbf{M}(K)$, $E = [n] \times [k]$, $T = [n] \times \{j'\}$, $S = \text{ins}_{nk}^{(i',j')} ([n-1] \times [k-1])$, $I^* = B^*$ and $\text{ins}_{nk}^{(i',j')} (B'^*)$.

By substituting (7.10) into (7.11) we obtain that $\left(B^* \cap (\{i'\} \times [k]) \right) \cup \text{ins}_{nk}^{(i',j')} (B'^*)$ is a coindependent set of $\mathbf{M}(K)$ as it is claimed in the part (c) of the lemma.

(d) Let $X' = (x'_{ij}) \in A'$. It means that there exists $X = (x_{ij}) \in A \cap U(x_{(i,j')} = \delta_{ii'} \mid 1 \leq i \leq n)$ such that $X' = \pi_{\text{ins}_{nk}^{(i',j')}}(X) = X(i'|j')$. This implies that X has the form

$$\begin{pmatrix} x'_{11} & \cdots & x'_{1j'-1} & 0 & x'_{1j'} & \cdots & x'_{1(k-1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x'_{i'-11} & \cdots & x'_{i'-1j'-1} & 0 & x'_{i'-1j'} & \cdots & x'_{(i'-1)(k-1)} \\ * & \cdots & * & 1 & * & \cdots & * \\ x'_{i'1} & \cdots & x'_{i'j'-1} & 0 & x'_{i'j'} & \cdots & x'_{i'(k-1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x'_{n-11} & \cdots & x'_{n-1j'-1} & 0 & x'_{n-1j'} & \cdots & x'_{(n-1)(k-1)} \end{pmatrix}. \quad (7.12)$$

By the Laplace expansion along the j' -th column (Lemma 3.8), we have an equality

$$\det_{nk}(X) = \sum_{i=1}^n (-1)^{i+j'} x_{ij'} \det_{(n-1)(k-1)}(X(i|j)),$$

where $X(i|j)$ denotes a matrix obtained from X by striking out the i -th row and the j -th column. Since $x_{i'j'} = 1$ is the only nonzero value in the sequence $x_{1j'}, \dots, x_{nj'}$, then

$$\sum_{i=1}^n (-1)^{i+j'} x_{ij'} \det_{(n-1)(k-1)}(X(i|j)) = (-1)^{i'+j'} \det_{(n-1)(k-1)}(X(i'|j')).$$

Note that $X(i'|j') = X'$ because X has the form (7.12). Therefore,

$$\det_{(n-1)(k-1)}(X') = \det_{(n-1)(k-1)}(X(i'|j')) = (-1)^{i'+j'} \det_{nk}(X) = 0.$$

□

Corollary 7.4. *Let $k > 1$, $\mathsf{K} \subseteq \mathcal{M}_{nk}(\mathbb{F})$ be a linear variety such that $\det_{nk}(X) = 0$ for all $X \in A$, $B^* \subseteq [n] \times [k]$ be a cobasis of $\mathbf{M}(\mathsf{K})$. If there exist $1 \leq i' \leq n$ and $1 \leq j' \leq k$ such that $|B^* \cap (\{i'\} \times [k])| = l$ and $B^* \cap ([n] \times \{j'\}) = \emptyset$, then there exists a linear variety $\mathsf{K}' \subseteq \mathcal{M}_{(n-1)(k-1)}(\mathbb{F})$ with $\text{codim}(\mathsf{K}') \leq \text{codim}(\mathsf{K}) - l$ such that $\det_{(n-1)(k-1)}(X') = 0$ for all $X' \in \mathsf{K}'$.*

Corollary 7.5. *Let $k > 1$, $\mathsf{K} \subseteq \mathcal{M}_{nk}(\mathbb{F})$ be a linear variety such that $\det_{nk}(X) = 0$ for all $X \in A$, $B^* \subseteq [n] \times [k]$ be a cobasis of $\mathbf{M}(\mathsf{K})$. If there exists $1 \leq j' \leq k$ such that $B^* \cap ([n] \times \{j'\}) = \emptyset$, then there exists a linear variety $\mathsf{K}' \subseteq \mathcal{M}_{(n-1)(k-1)}(\mathbb{F})$ with $\text{codim}(\mathsf{K}') \leq \text{codim}(\mathsf{K}) - 1$ such that $\det_{(n-1)(k-1)}(X') = 0$ for all $X' \in \mathsf{K}'$.*

Corollary 7.5 allows us to provide a lower bound the codimension of linear variety consisting of matrices annihilating \det_{nk} .

Theorem 7.6. *Let $\mathsf{K} \subseteq \mathcal{M}_{nk}(\mathbb{F})$ be a linear variety such that $\det_{nk}(X) = 0$ for every $X \in \mathsf{K}$. Then $\text{codim}(\mathsf{K}) \geq k$.*

Proof. We prove it by induction on k . If $k = 1$ and K is such that $\det_{n1}(X) = 0$ for every $X \in \mathsf{K}$, then this condition implies that $\sum_{i=1}^n (-1)^{i-1} X_{i1} = 0$. Hence,

$$\mathsf{K} \subseteq \mathcal{S}((1, -1, 1, \dots, (-1)^{n-1}), 0)$$

and consequently

$$\text{codim}(\mathsf{K}) \geq \text{codim}(\mathcal{S}((1, -1, 1, \dots, (-1)^{n-1}), 0)) = 1 = k.$$

Assume now that $k \geq 2$, the statement of the lemma holds for $k - 1$ and $\mathsf{K} \subseteq \mathcal{M}_{nk}(\mathbb{F})$ is a linear variety such that $\det_{nk}(X) = 0$ for every $X \in \mathsf{K}$. We will prove that $\text{codim}(\mathsf{K}) \geq k$ by contradiction. Suppose that $\text{codim}(\mathsf{K}) < k$. Let B^* be any cobasis of $\mathbf{M}(\mathsf{K})$. Since $|B^*| = \text{codim}(\mathsf{K}) < k$, then there exists $1 \leq j' \leq k$ such that $B^* \cap ([n] \times \{j'\}) = \emptyset$. Hence, B^* , K and j' satisfy the conditions of Corollary 7.5 and there exists a linear variety $\mathsf{K}' \subseteq \mathcal{M}_{n-1k-1}(\mathbb{F})$ with

$$\text{codim}(\mathsf{K}') \leq \text{codim}(\mathsf{K}) - 1 < k - 1$$

which leads to a contradiction. Therefore, $\text{codim}(\mathsf{K}) \geq k$. □

We apply Corollary 7.4 in the next lemma establishing that every set of independent coordinates of linear variety annihilating \det_{nk} of minimal possible codimension cannot contain two elements in any column of $[n] \times [k]$.

Lemma 7.7. Assume that $n \geq k > 1$. Let $\mathsf{K} \subseteq \mathcal{M}_{n,k}(\mathbb{F})$ be a linear variety such that $\text{codim}(\mathsf{K}) = k$ and $\det_{n,k}(X) = 0$ for all $X \in \mathsf{K}$, $B^* \subseteq [n] \times [k]$ is a cobasis of K . If there is $1 \leq j' \leq k$ such that $B^* \cap ([n] \times \{j'\}) = \emptyset$, then

$$\left| B^* \cap (\{i\} \times [k]) \right| \leq 1 \text{ for all } 1 \leq i \leq n.$$

Proof. By contradiction. If we suppose that K satisfies the conditions of the lemma, then by Corollary 7.4 there exists a linear variety $\mathsf{K}' \subseteq \mathcal{M}_{(n-1)(k-1)}(\mathbb{F})$ with

$$\text{codim}(\mathsf{K}') \leq \text{codim}(\mathsf{K}) - l = k - l < k - 1$$

such that $\det_{(n-1)(k-1)}(X') = 0$ for all $X' \in \mathsf{K}'$ which contradicts Theorem 7.6. \square

In the following lemma we prove that every linear variety annihilating $\det_{n,k}$ of minimal possible codimension is actually a vector space such that rows of its elements satisfy a common linear relation.

Lemma 7.8. Assume that $n \geq k+2$. Let $\mathsf{K} \subseteq \mathcal{M}_{n,k}(\mathbb{F})$ be a linear variety with $\text{codim}(\mathsf{K}) = k$ such that $\det_{n,k}(X) = 0$ for every $X \in \mathsf{K}$. Then the following is true:

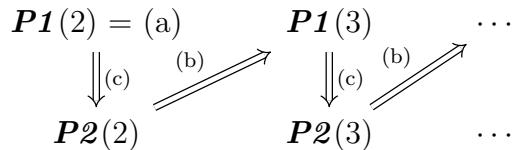
P1. if B^* is a cobasis of $\mathbf{M}(\mathsf{K})$, then $B^* \cap ([n] \times \{j\}) \neq \emptyset$ for all $1 \leq j \leq k$;

P2. there exists $\mathbf{z} \in \mathbb{F}^n$ such that $\mathbf{z}^t X = 0$ for every $X \in \mathsf{K}$.

Proof. The statement of the theorem is clearly true for $k = 1$. To prove it for $k > 1$ we prove separately the following claims

- (a) **P1** holds for $k = 2$;
- (b) if $k > 2$, then **P2** for $k - 1$ implies **P1** for k ;
- (c) if **P1** holds for k , then **P2** holds for k .

The corresponding diagram of implications obtained from (a), (b) and (c) could be represented as follows



Thus, the principle of mathematical induction will imply the statement of the lemma for $k > 1$ if we prove the claims (a), (b) and (c).

(a) By contradiction. Let $n \geq k + 2 = 4$ and $\mathsf{K} \subseteq \mathcal{M}_{n,k}(\mathbb{F})$ be a linear variety with $\text{codim}(\mathsf{K}) = k = 2$ such that $\det_{n,2}(X) = 0$ for every $X \in \mathsf{K}$. Suppose that there exists a cobasis B^* of $\mathbf{M}(\mathsf{K})$ and $1 \leq j' \leq 2$ such that $B^* \cap ([n] \times \{j'\}) = \emptyset$. Without loss of generality let us assume that $j' = 1$.

First, we show that then $\{(1, 2), (2, 2)\}$ is a cobasis of $\mathbf{M}(\mathsf{K})$. If $B^* = \{(1, 2), (2, 2)\}$, then the claim follows immediately. Now consider the case if $B^* \neq \{(1, 2), (2, 2)\}$. Assume without loss of generality that $(1, 2) \notin B^*$. Then there exists $(i', 2) \in B^*$ such that $i' \neq 1$.

Let K' be a linear variety defined by

$$K' = \text{ins}_{n2}^{(i',1)} (\mathsf{K} \cap U(x_{(1)i} = \delta_{i,i'} \mid 1 \leq i \leq n)).$$

By Corollary 7.5 $\det_{n-11}(X') = 0$ for all $X' \in K'$ and $\text{codim}(K') \leq \text{codim}(\mathsf{K}) - 1 = 1$. Hence, $\text{codim}(K') = 1$ by Lemma 7.7.

Since $\det_{n-11}(X') = 0$ for all $X' \in A'$, then

$$K' \subseteq \mathcal{S}(A', \mathbf{0}),$$

where $A' \in \mathcal{M}_{1[n-1] \times \{1\}}(\mathbb{F})$ defined by $A'_{1(i,1)} = (-1)^{i-1}$ for all $1 \leq i \leq n-1$. From this we conclude that

$$K' = \mathcal{S}(A', \mathbf{0})$$

by Lemma 4.9 because $\text{codim}(\mathcal{S}(A', \mathbf{0})) = 1$.

Since $A'_{1(i,1)} \neq 0$, then $\text{rk}(A'([i,1])) = 1$. Therefore, $B^{*'} = \{(1,1)\}$ is a cobasis of K' . By Lemma 7.3 the set I^* defined by

$$I^* = \left(B^* \cap (\{i'\} \times [k]) \right) \cup \text{ins}_{n2}^{(i',1)}(B^{*'})$$

is contained in a set of dependent coordinates of K . Since

$$\left(B^* \cap (\{i'\} \times [k]) \right) = \{(i',2)\}$$

and

$$\text{ins}_{n2}^{(i',1)}(B^{*'}) = \{(1,2)\},$$

then we obtain that $B_{\text{new}}^* = \{(1,2), (i',2)\}$ is a cobasis of $\mathbf{M}(\mathsf{K})$. By applying the same arguments to B_{new}^* and $(i',2) = (1,2)$ we obtain that $B^{C*} = \{(1,2), (2,2)\}$ is a cobasis of K .

Thus, since B^{C*} is a cobasis of $\mathbf{M}(\mathsf{K})$, then $B^C = ([n] \times [2]) \setminus B^{C*}$ is a basis of $\mathbf{M}(\mathsf{K})$. Let $I^C = \{(1,1), \dots, (n,1), (5,2), \dots, (n,2)\} \subseteq B^C$ be an independent set of K , $\mathbf{c}^C \in \mathbb{F}^{I^C}$ be defined by

$$\mathbf{c}_e^C = \begin{cases} 1, & e = (1,1) \\ 1, & e = (2,1) \\ 0, & e = (3,1) \quad \text{for all } e \in I^C \\ -1, & e = (4,1) \\ 0, & \text{otherwise} \end{cases}$$

and $\mathsf{K}^C \subseteq \mathsf{K}$ be defined by

$$\mathsf{K}^C = \mathsf{K} \cap U(x_e = \mathbf{c}_e^C \mid e \in I^C).$$

On the one hand, Lemma 6.15 applied to K , I_C and \mathbf{c}^C implies that K^C is a linear variety and

$$\mathbf{M}(\mathsf{K}^C) \setminus I^C = \mathbf{M}(\mathsf{K})/I^C. \tag{7.13}$$

Let $I^{(3,2)} = \{(3,2)\} \cup I^C$. Since $I^{(3,2)} \cap B^{C*} = \emptyset$, then $I^{(3,2)}$ is an independent set of $\mathbf{M}(\mathsf{K})$. Hence, $\{(3,2)\} = I^{(3,2)} \setminus I^C$ is an independent set of $\mathbf{M}(\mathsf{K})/I^C$ by Corollary 5.13. The equality (7.13) implies that $\{(3,2)\}$ is an independent set of $\mathbf{M}(\mathsf{K}^C) \setminus I^C$. By the definition

of restriction matroid (Definition 5.3), $\{(3, 2)\}$ is an independent set of $\mathbf{M}(\mathsf{K}^C)$. From this and Corollary 6.6 we conclude that there exists $X^C = (x_{ij}^C) \in \mathsf{K}^C$ such that

$$x_{32}^C = 1 \quad (7.14)$$

On the other hand, since $\mathsf{K}^C \subseteq \mathsf{K}$, then $\det_{n,2}(X) = 0$ for all $X \in \mathsf{K}^C$ by the initial assumption on K . Let us express $\det_{n,2}(X)$ for $X = (x_{ij}) \in \mathsf{K}^C$ in terms of x_{ij} using Laplace expansion along the first column. By Lemma 3.8 and the definition of K^C we have

$$\begin{aligned} \det_{n,k}(X) &= \sum_{i=1}^n (-1)^{i+1} x_{ij} \det_{n-1,k-1}(X(i|j)) \\ &= 1 \cdot \det_{n,1}(X(1|1)) - 1 \cdot \det_{n,1}(X(2|1)) + 0 \cdot \det_{n,1}(X(3|1)) \\ &\quad - (-1) \cdot \det_{n,1}(X(4|1)) + \sum_{i=5}^n (-1)^{i+1} 0 \cdot \det_{(n-1)(k-1)}(X(i|j)) \\ &= (x_{22} - x_{32} + x_{42} + 0 + \dots + 0) - (x_{12} - x_{32} + x_{42} + 0 + \dots + 0) \\ &\quad + (x_{12} - x_{22} + x_{32} + 0 + \dots + 0) = x_{32} \end{aligned}$$

for all $X = (x_{ij}) \in \mathsf{K}^C$. Hence, $x_{32} = 0$ for all $X \in \mathsf{K}^C$ which contradicts with (7.14).

(b) Suppose that $k > 2$ and **P2** holds for $k - 1$. Let us prove **P1** for k by contradiction. Let $\mathsf{K} \subseteq \mathcal{M}_{n,k}(\mathbb{F})$ be a linear variety with $\text{codim}(\mathsf{K}) = k$ such that $\det_{n,k}(X) = 0$ for all $X \in \mathsf{K}$, and B^* be cobasis of K . Suppose that $1 \leq j' \leq k$ is such that $B^* \cap [n] \times \{j'\} = \emptyset$. First note that $|B^* \cap (\{i\} \times [k])| \leq 1$ for all $1 \leq i \leq n$ by Lemma 7.7.

Since $|B^*| = k > 0$, there exists $1 \leq i' \leq n$ such that $|B^* \cap (\{i'\} \times [k])| = 1$. Let $\mathsf{K}' = \pi_{\text{ins}_{n,k}^{(i',j')}}(\mathsf{K} \cap U((i,j') = \delta_{ii'} \mid 1 \leq i \leq n))$. Then $\det_{n-1,k-1}(X') = 0$ for all $X' \in \mathsf{K}'$ by Lemma 7.3(d) applied to K , B^* , j' , i' and $c_i = \delta_{ii'}$ for all $1 \leq i \leq n$. Since we assume that **P2** holds for $k - 1$, then by applying **P2** to K' we conclude that there exists $0 \neq \mathbf{z}' \in \mathbb{F}^{n-1}$ such that

$$\mathbf{z}'^t X' = 0 \quad (7.15)$$

for all $X' \in \mathsf{K}'$. Let us assume without loss of generality that $\mathbf{z}'_{i_0} = 1$ for some $1 \leq i_0 \leq n - 1$.

Let $A' \in \mathcal{M}_{k-1}[n-1] \times [k-1](\mathbb{F})$ be defined by

$$A'_{p(i,j)} = \begin{cases} \mathbf{z}'_i, & j = p \\ 0, & \text{otherwise} \end{cases}$$

for all $1 \leq p \leq k - 1$ and $(i,j) \in [n - 1] \times [k - 1]$. Then $\text{rk}(A') = k - 1$ because $A'([i_0, 1], \dots, [i_0, k - 1])$ is a permutation matrix. Therefore, a linear variety $\mathcal{S}(A', 0)$ of codimension $k - 1$ is properly defined and

$$B'^* = \{(i_0, 1), \dots, (i_0, k - 1)\} \quad (7.16)$$

is a cobasis of $\mathbf{M}(\mathcal{S}(A', 0))$ by Lemma 6.17.

The equality (7.15) means that $\mathsf{K}' \subseteq \mathcal{S}(A', 0)$. Hence, $\mathsf{K}' = \mathcal{S}(A', 0)$ by Lemma 4.9 because $\text{codim}(\mathsf{K}') = k - 1 = \text{codim}(\mathcal{S}(A', 0))$. Therefore, B'^* is a cobasis of $\mathbf{M}(\mathsf{K}')$.

By Lemma 7.3(c), $I^* = \left(B^* \cap (\{i'\} \times [k]) \right) \cup \text{ins}_{nk}^{(i',j')}(B'^*)$ is a coindependent set of $\mathbf{M}(\mathsf{K})$. Hence, I^* is a cobasis of K because

$$\begin{aligned} & \left| \left(B^* \cap (\{i'\} \times [k]) \right) \cup \text{ins}_{nk}^{(i',j')}(B'^*) \right| \\ &= \left| B^* \cap (\{i'\} \times [k]) \right| + \left| \text{ins}_{nk}^{(i',j')}(B'^*) \right| = 1 + (k - 1) = k = \text{codim}(\mathsf{K}). \end{aligned}$$

Thus, I^* contains $k - 1 \geq 2$ elements of the set $\text{ins}_{nk}^{(i',j')}(B'^*)$. The equality (7.16) implies that all elements of $\text{ins}_{nk}^{(i',j')}(B'^*)$ belong to the same row by Lemma 7.2. This leads to a contradiction because by Lemma 7.7 every cobase of K no more than one element in every row of $[n] \times [k]$.

(c) Let $\mathsf{K} \subseteq \mathcal{M}_{nk}(\mathbb{F})$ be a linear variety with $\text{codim}(\mathsf{K}) = k$ such that $\det_{nk}(X) = 0$ for all $X \in \mathsf{K}$, B^* be a cobasis of $\mathbf{M}(\mathsf{K})$. Since **P1** holds for k and $|B^*| = \text{codim}(\mathsf{K}) = k$, then B^* has exactly one element in each column of $[n] \times [k]$. By Lemma 4.18 there exist $m \geq 1$, $A \in \mathcal{M}_{m([n] \times [k])}$ and $b \in \mathbb{F}^m$ such that $\mathsf{K} = \mathcal{S}(A, b)$. In addition, Lemma 6.21 allows us to assume that is A reduced with respect to B^* and consequently $m = k$. For every $1 \leq p \leq k$ let us denote by a_p a unique element of B^* such that

$$A_{p a_p} = 1. \quad (7.17)$$

Since the permutation of rows of system of linear equations does not affect on the set of its solutions, then we also assume that a_p belongs to $[n] \times \{p\}$ for every $1 \leq p \leq k$.

Note that if $A_{p a} \neq 0$, then a and a_p belong to the same column. Indeed, if $A_{p a} \neq 0$ and the elements a , a_p do not belong to the same column, then the set $B^* \Delta \{a, a_p\}$ is a cobasis of $\mathbf{M}(\mathsf{K})$ by Lemma 6.22. This contradicts **P1** for k because $B^* \Delta \{a, a_p\}$ does not have an element belonging to the column containing a_p (because a belongs to other column).

Thus, A has the form

$$\begin{pmatrix} A_{1(1,1)} & \cdots & A_{1(n,1)} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & A_{2(1,2)} & \cdots & A_{2(n,2)} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & A_{k(1,k)} & \cdots & A_{k(n,k)} \end{pmatrix}$$

For every $1 \leq p \leq k$ let $\mathbf{w}^{(p)} \in \mathbb{F}^n$ be defined by

$$\mathbf{w}^{(p)} = (A_{p(1,p)}, \dots, A_{p(n,p)}).$$

All these vectors are nonzero because

$$\mathbf{w}_{a_p}^{(p)} = 1 \text{ for all } 1 \leq p \leq k \quad (7.18)$$

by the equality (7.17).

Let us prove that $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(k)}$ are scalar multiples of each other by contradiction. Suppose that $\mathbf{w}^{(p_2)}$ is not a scalar multiple of $\mathbf{w}^{(p_1)}$ for some $1 \leq p_1 < p_2 \leq k$. Consider a linear map $T: \mathcal{M}_{nk}(\mathbb{F}) \rightarrow \mathcal{M}_{nk}(\mathbb{F})$ sending $X \in \mathcal{M}_{nk}(\mathbb{F})$ to X with p_1 -th column subtracted from its p_2 -th column. This map is invertible and preserves \det_{nk} by the property 5 in

Theorem 3.1. Let $D \in \mathcal{M}_{[n] \times [k]}([n] \times [k])(\mathbb{F})$ be a matrix of T in the standard basis of $\mathcal{M}_{[n] \times [k]}$ identified with $\mathbb{F}^{[n] \times [k]}$. Then by direct calculations we obtain from the definition of T that

$$D^{-1} = I_{[n] \times [k]} + \sum_{i=1}^n E_{(i,p_2)(i,p_1)}.$$

Thus, $\mathsf{K}_T = T(\mathsf{K})$ is a linear variety of codimension k such that $\det_{n,k}(X) = 0$ for all $X \in \mathsf{K}_T$. In addition, $\mathsf{K} = \mathcal{S}(AD^{-1}, \mathbf{b})$ by Lemma 4.24.

The matrix AD^{-1} could be represented as follows

$$\begin{pmatrix} A_{p_1(1,1)} & \cdots & A_{p_1(n,1)} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ A_{p_2(1,1)} & \cdots & A_{p_2(n,1)} & A_{p_2(1,2)} & \cdots & A_{p_2(n,2)} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & A_{k(1,2)} & \cdots & A_{k(n,2)} \end{pmatrix},$$

where we put p_1 -th and p_2 -th row of AD^{-1} first for convenience. If we subtract the p_1 -th row of AD^{-1} with the coefficient $\alpha = \frac{\mathbf{w}_{\alpha p_2}^{(p_2)}}{\mathbf{w}_{\alpha p_1}^{(p_1)}}$ from its p_2 -th row, we obtain the matrix A'' which has the form

$$\begin{pmatrix} A_{p_1(1,p_1)} & \cdots & A_{p_1(n,p_1)} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ A_{p_2(1,p_2)} - \alpha A_{p_1(1,p_1)} & \cdots & 0 & A_{p_2(n,p_2)} - \alpha A_{p_1(n,p_1)} & A_{p_2(1,p_2)} & \cdots & A_{p_2(n,p_2)} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & A_{k(1,k)} & \cdots & A_{k(n,k)} \end{pmatrix}.$$

Therefore, A'' is reduced with respect to $B^* = \{a_{p_1}, \dots, a_{p_k}\}$. Hence, B^* is a cobasis of $\mathbf{M}(\mathsf{K}_T)$.

Since $\mathbf{w}^{(p_2)}$ is not scalar multiple of $\mathbf{w}^{(p_1)}$, $A''_{p_2 a}$ is nonzero for some $a \in [n] \times \{p_1\}$. Therefore, by Lemma 6.22 $B^* \triangle \{a_{p_2}, a\}$ is a cobasis of $\mathbf{M}(\mathsf{K}_T)$ which does not contain an element belonging to $[n] \times \{p_2\}$. This lead us to a contradiction because we assume that **P1** holds for k .

Thus, $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(k)}$ are proportional. In addition, all $w^{(p)}$ are nonzero by (7.18). Hence, there exists $0 \neq \mathbf{z} \in \mathbb{F}^n$ such that $\mathbf{w}^{(p)} = \alpha_p \mathbf{z}$ for some $0 \neq \alpha_1, \dots, \alpha_p \in \mathbb{F}$ and all $1 \leq p \leq k$. Since the multiplication of the row of system of linear equation by a nonzero scalar does not change the set of its solutions, then

$$\mathsf{K} = \mathcal{S}(\text{diag}(\alpha_1^{-1}, \dots, \alpha_k^{-1})A, \text{diag}(\alpha_1^{-1}, \dots, \alpha_k^{-1})\mathbf{b}).$$

This implies that

$$\mathsf{K} = \{X \in \mathcal{M}_{n,k}(\mathbb{F}) \mid \mathbf{z}^t X = \mathbf{b}'\}, \quad (7.19)$$

where $\mathbf{b}' = \text{diag}(\alpha_1^{-1}, \dots, \alpha_k^{-1})\mathbf{b}$.

Let us prove that $\mathbf{b}' = 0$ by contradiction. Suppose that $\mathbf{b}' \neq 0$ and assume without loss of generality that $\mathbf{b}'_1 \neq 0$. Let $1 \leq i_0 \leq n$ be such that $\mathbf{z}_{i_0} \neq 0$. Then by the equality (7.19) the set $B^* = \{i_0\} \times [k]$ is a cobasis of $\mathbf{M}(\mathsf{K})$. Hence, $B = ([n] \times [k]) \setminus (\{i_0\} \times [k])$ is a basis of $\mathbf{M}(\mathsf{K})$. By Corollary 6.6 there exists $X^n = (x_{i,j}^n) \in \mathsf{K}$ such that

$$x_{i,1}^n = 0 \text{ for all } 1 \leq i \neq i_0 \leq n \text{ and } \det_{(n-1)(k-1)}(X^n(i_0|1)) \neq 0 \quad (7.20)$$

because the matrix $X^n(i_0|1)$ could be arbitrary and $\det_{(n-1)(k-1)}$ is clearly a nonzero function. In addition,

$$x_{i_0,1}^n \neq 0 \quad (7.21)$$

because all other elements of the first column of X^n are zero and $\sum_{i=1}^n \mathbf{z}_i x_{i1}^n = \mathbf{b}'_1 \neq 0$ by our assumption.

Let us show that $\det_{nk}(X^n) \neq 0$. Using the Laplace expansion of $\det_{nk}(X^n)$ along the first row we obtain

$$\det_{nk}(X^n) = \sum_{i=1}^n (-1)^{i+1} x_{i1}^n \cdot \det_{(n-1)(k-1)}(X^n(i|1)) = x_{i_01}^n \cdot \det_{(n-1)(k-1)}(X^n(i|1)) \neq 0$$

because $x_{i_01}^n$ and $\det_{(n-1)(k-1)}(X^n(i|1))$ are nonzero as it is stated in (7.20) and (7.21). This leads to a contradiction with initial assumption on X^n as an element of K which annihilates \det_{nk} . Therefore, $\mathbf{b}' = 0$. \square

Remark 7.9. If $k > 1$, then the statement of the lemma is not true for $n = k$ and $n = k + 1$. Indeed, if $n = k$, then $\det_{nk}(X) = \det_k(X) = 0$ for all $X \in \mathsf{K}'$, where

$$\mathsf{K}' = \left\{ \begin{pmatrix} 0 & x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_{n,1} & \cdots & x_{n,n-1} \end{pmatrix} \mid x_{11}, \dots, x_{nn-1} \in \mathbb{F} \right\}.$$

If $n = k + 1$, then $\det_{nk}(X) = \det_k(X) = 0$ for all $X \in \mathsf{K}''$, where

$$\mathsf{K}'' = \left\{ \begin{pmatrix} x & x_{1,1} & \cdots & x_{1,n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x & x_{n,1} & \cdots & x_{n,n-2} \end{pmatrix} \mid x, x_{11}, \dots, x_{nn-2} \in \mathbb{F} \right\}$$

which follows from multilinearity of \det_{nk} with respect to columns and Lemma 3.7 because $n + k - 1$ is even in this case.

Lemma 7.10. Let $\mathbf{z} \in \mathbb{F}^n$ be such that $\mathbf{z}_1 = -1$ and $\mathsf{K} \subseteq \mathcal{M}_{nk}(\mathbb{F})$ be a linear variety defined by $\mathsf{K} = \{X \in \mathcal{M}_{nk}(\mathbb{F}) \mid \mathbf{z}^t X = 0\}$. Then $\det_{nk}(X) = 0$ for all $X \in \mathsf{K}$ if and only if

$$1 + \sum_{\alpha=1}^k \mathbf{z}_{c(\alpha)} (-1)^{\alpha - c(\alpha)} = 0 \quad \text{for all } c \in \binom{[n]}{k} \text{ such that } 1 \notin c. \quad (7.22)$$

Proof. Since $\mathbf{z}_1 = -1$, then the condition $\mathbf{z}^t X = 0$ for all $X \in \mathsf{K}$ means that the first row of every $X \in \mathsf{K}$ could be expressed as a linear combination of other rows as follows

$$X[1] = \mathbf{z}_2 X[2] + \dots + \mathbf{z}_n X[n]. \quad (7.23)$$

Let $X \in \mathsf{K}$. By Corollary 3.6

$$\det_{nk}(X) = \sum_{c \in \binom{[n]}{k}} \operatorname{sgn}_{[n]}(c) \det_k(X[c]).$$

The right-hand side of this equality is divided into two summands as follows.

$$\sum_{c \in \binom{[n]}{k}} \operatorname{sgn}_{[n]}(c) \det_k(X[c]) = \sum_{\substack{c \in \binom{[n]}{k} \\ 1 \in c}} \operatorname{sgn}_{[n]}(c) \det_k(X[c]) + \sum_{\substack{c \in \binom{[n]}{k} \\ 1 \notin c}} \operatorname{sgn}_{[n]}(c) \det_k(X[c]). \quad (7.24)$$

Consider the first summand of the right-hand side of (7.24). Let $c' \in \binom{[n]}{k}$ be such that $1 \in c'$. It follows from (7.22) that

$$\det_k(X[c']) = \begin{vmatrix} X[1]) \\ X[c'(2)]) \\ \vdots \\ X[c'(k)]) \end{vmatrix} = \begin{vmatrix} \mathbf{z}_2 X[2]) + \dots + \mathbf{z}_n X[n]) \\ X[c'(2)]) \\ \vdots \\ X[c'(k)]) \end{vmatrix} = \sum_{2 \leq i \leq n} \mathbf{z}_i \begin{vmatrix} X[i]) \\ X[c'(2)]) \\ \vdots \\ X[c'(k)]) \end{vmatrix} \quad (7.25)$$

Note that

$$\begin{vmatrix} X[i]) \\ X[c'(2)]) \\ \vdots \\ X[c'(k)]) \end{vmatrix} = 0 \text{ for all } i \in c', 2 \leq i \leq n \quad (7.26)$$

because it becomes a determinant of a matrix with two rows equal to each other.

Since the ordinary determinant is an antisymmetric function of the rows of matrix, then for $i \notin c', 2 \leq i \leq n$ we have

$$\begin{vmatrix} X[i]) \\ X[c'(2)]) \\ \vdots \\ X[c'(k)]) \end{vmatrix} = \operatorname{sgn}_{n,k}(\sigma_{i,c'}) \det_k(X[c' \triangle \{1, i\}]), \quad (7.27)$$

where $\sigma_{i,c'} \in \mathcal{C}_{[n]}^k$ is defined by

$$\sigma_{i,c'}(k) = \begin{cases} i, & k = 1 \\ c'(k), & \text{otherwise} \end{cases}.$$

By substituting (7.26) and (7.27) into (7.25) we obtain that

$$\sum_{\substack{c' \in \binom{[n]}{k} \\ 1 \in c'}} \operatorname{sgn}_{[n]}(c') \det_k(X[c']) = \sum_{\substack{c' \in \binom{[n]}{k} \\ 1 \in c'}} \operatorname{sgn}_{[n]}(c') \sum_{\substack{2 \leq i \leq n \\ i \notin c'}} \mathbf{z}_i \operatorname{sgn}_{n,k}(\sigma_{i,c'}) \det_k(X[c' \triangle \{1, i\}]).$$

Let us rearrange the sum in the right-hand side of this equality. Note that if $2 \leq i \leq n$ and $c' \triangle \{1, i\} = c$, then $i = c(\alpha)$ for some unique $1 \leq \alpha \leq k$. Hence, $\operatorname{sgn}_{[n]}(c') = (-1)^{1-i} \operatorname{sgn}_{[n]}(c) = (-1)^{1-c(\alpha)} \operatorname{sgn}_{[n]}(c)$ by the definition of $\operatorname{sgn}_{[n]}$. In addition, $\operatorname{sgn}_{n,k}(\sigma_{i,c'}) = (-1)^{\alpha-1}$ because it is equal to the sign of permutation

$$\begin{pmatrix} c(1) & c(2) & \cdots & c(\alpha) & c(\alpha+1) & \cdots & c(k) \\ c(\alpha) & c(1) & \cdots & c(\alpha-1) & c(\alpha+1) & \cdots & c(k) \end{pmatrix}$$

by the definition.

Observe also that for every pair (c, α) such that $c \in \binom{[n]}{k}$, $1 \notin c$ and $1 \leq \alpha \leq k$ there exist a unique pair (c', i) , where $c' \in \binom{[n]}{k}$ and $2 \leq i \leq n$ such that $i = c(\alpha)$, $c' = c \triangle \{1, i\}$ and consequently $1 \in c'$ and $i \notin c'$.

Thus,

$$\begin{aligned}
& \sum_{\substack{c' \in \binom{[n]}{k} \\ 1 \in c'}} \operatorname{sgn}_{[n]}(c') \sum_{\substack{2 \leq i \leq n \\ i \notin c'}} \mathbf{z}_i \operatorname{sgn}_{nk}(\sigma_{ic'}) \det_k \left(X[c' \Delta \{1, i\}] \right) \\
&= \sum_{\substack{c' \in \binom{[n]}{k}, 2 \leq i \leq n \\ 1 \in c', i \notin c'}} \mathbf{z}_i \operatorname{sgn}_{[n]}(c') \operatorname{sgn}_{nk}(\sigma_{ic'}) \det_k \left(X[c' \Delta \{1, i\}] \right) \\
&= \sum_{\substack{c \in \binom{[n]}{k}, 1 \leq \alpha \leq k \\ 1 \notin c}} \mathbf{z}_{c(\alpha)}(-1)^{1-c(\alpha)} \operatorname{sgn}_{[n]}(c) (-1)^{\alpha-1} \det_k \left(X[c] \right) \\
&= \sum_{\substack{c \in \binom{[n]}{k} \\ 1 \notin c}} \left(\sum_{\alpha=1}^k \mathbf{z}_{c(\alpha)}(-1)^{\alpha-c(\alpha)} \right) \operatorname{sgn}_{[n]}(c) \det_k \left(X[c] \right)
\end{aligned}$$

By substituting this instead of the first summand into (7.24) we obtain that

$$\begin{aligned}
\det_{nk}(X) &= \sum_{c \in \binom{[n]}{k}} \operatorname{sgn}_{[n]}(c) \det_k \left(X[c] \right) \\
&= \sum_{\substack{c \in \binom{[n]}{k} \\ 1 \notin c}} \left(\sum_{\alpha=1}^k \mathbf{z}_{c(\alpha)}(-1)^{\alpha-c(\alpha)} \right) \operatorname{sgn}_{[n]}(c) \det_k \left(X[c] \right) + \sum_{\substack{c \in \binom{[n]}{k} \\ 1 \in c}} \operatorname{sgn}_{[n]}(c) \det_k \left(X[c] \right) \\
&= \sum_{\substack{c \in \binom{[n]}{k} \\ 1 \notin c}} \left(1 + \sum_{\alpha=1}^k \mathbf{z}_{c(\alpha)}(-1)^{\alpha-c(\alpha)} \right) \operatorname{sgn}_{[n]}(c) \det_k \left(X[c] \right) \quad (7.28)
\end{aligned}$$

for all $X \in \mathbb{K}$.

Now let $c_0 \in \binom{[n]}{k}$ be such that $1 \notin c_0$. Consider X_{c_0} defined by

$$X_{c_0} = \sum_{\alpha=1}^k (E_{c_0(\alpha)\alpha} + \mathbf{z}_{c_0(\alpha)} E_{1\alpha}).$$

X_{c_0} clearly satisfies the equation $\mathbf{z}^t X_{c_0} = 0$ and consequently $X_{c_0} \in \mathbb{K}$. In addition, for every $c \in \binom{[n]}{k}$ such that $1 \notin c$

$$\det_k(X_{c_0}[c]) = \begin{cases} 1, & c = c_0 \\ 0, & \text{otherwise} \end{cases}.$$

On the one hand, the equality (7.28) implies that

$$\det_{nk}(X_{c_0}) = \left(1 + \sum_{\alpha=1}^k \mathbf{z}_{c_0(\alpha)}(-1)^{\alpha-c_0(\alpha)} \right) \operatorname{sgn}_{[n]}(c_0).$$

On the other hand, since $X_{c_0} \in \mathbb{K}$, then $\det_{nk}(X_{c_0}) = 0$. Since $\operatorname{sgn}_{[n]}(c_0) \neq 0$, then we obtain the equality

$$1 + \sum_{\alpha=1}^k \mathbf{z}_{c_0(\alpha)}(-1)^{\alpha-c_0(\alpha)} = 0,$$

which holds for every $c_0 \in \binom{[n]}{k}$ such that $1 \notin c_0$. \square

Theorem 7.11. *Let $n \geq k + 2$ and $\mathsf{K} \subseteq \mathcal{M}_{n,k}(\mathbb{F})$ be a linear variety. Then $\det_{n,k}(X) = 0$ for all $X \in \mathsf{K}$ and $\text{codim}(\mathsf{K}) = k$ if and only if k is odd and alternating row sum of every $X \in \mathsf{K}$ is equal to zero.*

Proof. First we prove the necessity and second we prove the sufficiency.

Necessity. Let K be a linear variety satisfying the conditions of the lemma. Lemma 7.8 implies that there exists $\mathbf{z} \in \mathbb{F}^n$ such that

$$\mathbf{z}^t X = 0 \text{ for all } X \in \mathsf{K}. \quad (7.29)$$

The proof is built up from the following three claims proven independently for all n, k and K satisfying the conditions of the lemma:

(i) If $\mathbf{z}_1 = -1$, then

$$\mathbf{z}_i = -\mathbf{z}_{i+1} \text{ for all } 1 < i < n. \quad (7.30)$$

(ii) The equality (7.30) implies the statement of this part the lemma.

(iii) If the claims (i) and (ii) hold, then this part of the lemma holds (we mean the **Necessity** part).

Thus, this part of the lemma follows from the claims (i)–(iii).

Claim (i). To establish the equality (7.30), consider any subsequence m_1, \dots, m_{k+1} of the sequence $2, \dots, n$ containing both i and $i+1$ (such a sequence exists because $n \geq k+2$). Suppose that $i = m_l$. Then consider $c_1 = \{m_1, \dots, \widehat{m_{l+1}}, \dots, m_{k+1}\} \in \binom{[n]}{k}$ and $c_2 = \{m_1, \dots, \widehat{m_l}, \dots, m_{k+1}\} \in \binom{[n]}{k}$. Since $1 \notin c_1, c_2$, then the equalities

$$1 + \sum_{1 \leq \alpha \leq k} (-1)^{\alpha - c_1(\alpha)} \mathbf{z}_{c_1(\alpha)} = 0 \quad \text{and} \quad 1 + \sum_{1 \leq \alpha \leq k} (-1)^{\alpha - c_2(\alpha)} \mathbf{z}_{c_2(\alpha)} = 0 \quad (7.31)$$

hold by Lemma 7.10. Since $c_1(\alpha) = c_2(\alpha)$ for all $\alpha \neq l$, then the difference of the left-hand sides of equalities in (7.31) is equal to

$$(-1)^{l - c_1(l)} \mathbf{z}_{c_1(l)} - (-1)^{l - c_2(l)} \mathbf{z}_{c_2(l)} = (-1)^l ((-1)^i \mathbf{z}_i - (-1)^{i+1} \mathbf{z}_{i+1}).$$

Hence,

$$\mathbf{z}_i + \mathbf{z}_{i+1} = 0.$$

This implies the equality (7.30).

Claim (ii). From (i) we conclude that there exists $z \in \mathbb{F}$ such that

$$\mathbf{z}_i = (-1)^i z \text{ for all } i \geq 2. \quad (7.32)$$

Lemma 7.10 applied to $c = \{2, \dots, k+1\}$ implies that $1 + \sum_{1 \leq \alpha \leq k} (-1)^{\alpha+c(\alpha)} \mathbf{z}_{c(\alpha)} = 0$ and consequently

$$\sum_{1 \leq \alpha \leq k} (-1)^{\alpha+c(\alpha)} \mathbf{z}_{c(\alpha)} = -1. \quad (7.33)$$

Consider the left-hand side of (7.33). By the equality (7.32) we obtain that

$$\sum_{1 \leq \alpha \leq k} (-1)^{\alpha+c(\alpha)} \mathbf{z}_{c(\alpha)} = \sum_{1 \leq \alpha \leq k} (-1)^{\alpha-\alpha-1} (-1)^{i+1} z = \sum_{1 \leq \alpha \leq k} (-1)^i z. \quad (7.34)$$

Substituting the right-hand side of (7.34) into (7.33) yields

$$\sum_{1 \leq \alpha \leq k} (-1)^i z = -1. \quad (7.35)$$

If k is even, we obtain a contradiction because the left part of (7.35) is zero while the right part is nonzero.

If k is odd, then the left part of (7.35) is equal to $-z$, which implies that $z = 1$. Thus, $\mathbf{z}_i = (-1)^i$.

Claim (iii). Since $\mathbf{z} \neq 0$, then there exists $1 \leq i^\circ \leq n$ such that $\mathbf{z}_{i^\circ} \neq 0$. The equality (7.29) implies that

$$\mathbf{z}'^t X = 0 \text{ for all } X \in \mathcal{K}, \quad (7.36)$$

where \mathbf{z}' is defined by $\mathbf{z}'_i = -\frac{\mathbf{z}_i}{\mathbf{z}_{i^\circ}}$ for all $1 \leq i \leq n$. Additionally,

$$\mathbf{z}'_{i^\circ} = -1 \quad (7.37)$$

by the definition of \mathbf{z}' .

By Lemma 3.11, there exists an invertible linear map $\mathbf{S}_{i^\circ}: \mathcal{M}_{n,k}(\mathbb{F}) \rightarrow \mathcal{M}_{n,k}(\mathbb{F})$ having the following properties:

(S1) if $X \in \mathcal{M}_{n,k}(\mathbb{F})$ and $\mathbf{z} \in \mathbb{F}^n$ are such that $\mathbf{z}^t X = 0$, then $\mathbf{z}^{\circ t} \mathbf{S}_{i^\circ}(X) = 0$, where

$$\mathbf{z}^\circ = (\mathbf{z}_{i^\circ}, \dots, \mathbf{z}_n, (-1)^{n+k+1} \mathbf{z}_1, \dots, (-1)^{n+k+1} \mathbf{z}_{i^\circ-1})^t;$$

(S2) $\det_{n,k}(X) = 0 \Rightarrow \det_{n,k}(\mathbf{S}_{i^\circ}(X)) = 0$;

(S3) If k is odd, then $\sum_{i=1}^n (-1)^i (\mathbf{S}_{i^\circ}(X)) [i] = 0 \Rightarrow \sum_{i=1}^n (-1)^i X[i] = 0$ for all $X \in \mathcal{M}_{n,k}$.

Let K° be defined by $K^\circ = \mathbf{S}_{i^\circ}(\mathcal{K})$. Since \mathbf{S}_{i° is an invertible linear map, then K° is a linear variety with $\text{codim}(K^\circ) = \text{codim}(\mathcal{K}) = k$. By the property (S2) of \mathbf{S}_{i° we have $\det_{n,k}(X^\circ) = 0$ for all $X^\circ \in K^\circ$.

It follows from the property (S1) of \mathbf{S}_{i° and (7.36) that $\mathbf{z}^\circ X^\circ = 0$ for all $X^\circ \in K^\circ$, where \mathbf{z}° is defined by

$$\mathbf{z}^\circ = (\mathbf{z}'_{i^\circ}, \dots, \mathbf{z}'_n, (-1)^{n+k+1} \mathbf{z}'_1, \dots, (-1)^{n+k+1} \mathbf{z}'_{i^\circ-1})^t. \quad (7.38)$$

The equalities (7.37) and (7.38) imply that $\mathbf{z}_1^\circ = -1$. Hence, we can apply the claims (i) and (ii) to K° and conclude that k is odd and

$$\sum_{i=1}^n (-1)^i X^\circ[i] = 0 \text{ for all } X^\circ \in \mathsf{K}^\circ. \quad (7.39)$$

Let $X \in \mathsf{K}$. Then $\sum_{i=1}^n (-1)^i (\mathbf{S}_{i^\circ}(X)) [i] = 0$ by (7.39). Hence, $\sum_{i=1}^n (-1)^i X[i] = 0$ by the property (S3) of \mathbf{S}_{i° . That is, the alternating row sum of every element in K is equal to zero. This establishes the claim (iii).

Sufficiency. Assume that k is odd and $\mathsf{K} \subseteq \mathcal{M}_{n,k}(\mathbb{F})$ is a vector space consisting of all the matrices such that alternating sum of their rows is zero. It is clear that $\text{codim}(\mathsf{K}) = k$.

Assume that $X \in \mathsf{K}$. Then $\mathbf{z}^t X = 0$ for $\mathbf{z} = (-1, 1, \dots, (-1)^n)^t$ by the definition of K . Lemma 7.10 implies that in order to show that $\det_{n,k}(X) = 0$ it is sufficient to check the condition (7.22) for every $c \in \binom{[n]}{k}$ such that $1 \notin c$. For every such c we have

$$1 + \sum_{\alpha=1}^k \mathbf{z}_{c(\alpha)}(-1)^{\alpha-c(\alpha)} = 1 + \sum_{\alpha=1}^k (-1)^{c(\alpha)}(-1)^{\alpha-c(\alpha)} = 1 + \sum_{\alpha=1}^k (-1)^\alpha = 1 - 1 = 0.$$

Therefore, the condition (7.22) is satisfied for every $c \in \binom{[n]}{k}$ such that $1 \notin c$. This implies that $\det_{n,k}(X) = 0$ for all $X \in \mathsf{K}$. \square

8 Further work

First, a natural continuation of this work is to complete the investigation by determining the vector spaces of $n \times k$ matrices of maximal dimension that annihilate $\det_{n,k}$ when k is even. As it is mentioned in the introduction, the only available information is that the codimension of such spaces is strictly larger than k . There is not only no hypothesis about the description of such spaces for even k of maximal dimension, but also no strict lower bound of codimension of such spaces. Any progress in this direction will be interesting.

Second, the found complete solution to the problem of maximal spaces annihilating the Cullis' determinant could help to solve linear preserver problem for the Cullis' determinant in the full generality, without any restriction on the ground field, in contrast to how it is done in [8] and [9].

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