Some Patterns of Duplications in the outputs of Mersenne Twister Pseudorandom Number Generator MT19937

Alain Schumacher[a], Takuji Nishimura[b], Makoto Matusmoto[c]

[a]*SICAP R&D, L-1272 Luxembourg-Beggen 68, rue de Bourgogne, Luxembourg, , Luxembourg*
[b]*Yamagata Ujniversity, 1-4-12 Kojirakawa, Yamagata, 990-8560, Japan*
[c]*AMAGAERU Institute of Free Mathematics, 2-37-6, Narita-Higashi,
Suginami-ku, Tokyo, 166-0015, Japan*

Abstract

The Mersenne Twister MT19937 pseudorandom number generator, introduced by the last two authors in 1998, is still widely used. It passes all existing statistical tests, except for the linear complexity test, which measures the ratio of the even-odd of the number of 1's among specific bits (and hence should not be important for most applications). Harase reported that MT19937 is rejected by some birthday-spacing tests, which are rather artificially designed.

In this paper, we report that MT19937 fails in a natural test based on the distribution of run-lengths on which we found an identical value in the output 32-bit integers. The number of observations of the run-length 623 is some 40 times larger than the expectation (and than the numbers of the observations of 622 and 624, etc.), which implies that the corresponding p-value is almost 0.

We mathematically analyze the phenomena, and obtain a theorem which explains these failures. It seems not to be a serious defect of MT19937, because finding the defect requires astronomical efforts. Still, the phenomena should be reported to the academic society relating to pseudorandom number generation.

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1. Introduction

A Mersenne Twister MT19937 pseudo random number generator [1] is still widely used. As far as the authors know, no reasonable statistical tests reject MT19937, except for the linear complexity test. The test measures whether the number of 1's in the bit-presentation of some of the output integers is even or not, and hence it matters seldomly. The first failure of MT19937 in other statistical tests than the linear complexity test is reported by Harase [2]. It is found that MT19937 has linear relations among three non-successive outputs and is rejected by a birthday-spacing test on them. However, the test is rather artificial, concentrating on outputs of fixed non-successive outputs, such as y_i, y_{i+396} and y_{i+623} $(i = 0, 1, \ldots)$.

In this paper, we report that a modified version of the repetition test introduced by Gil, Gonnet, and Petersen [3] naturally detects some flaws of MT19937. The flaws are shown in Section 2, which are on positive correlations between repetitions of identical 32-bit integers. In Section 3, we prove that these flaws are due to the sparseness of a matrix C appeared in the recursion of MT19937 (this is not the case for MT19937-64 [4]). We explain how the C yields some clear patterns on the repetitions. This section is mathematically rather complicated, and one may skip it

and go to Section 4, where we explain how these flaws are found through experiments on the repetition tests, i.e., by observing the distribution of the run-lengths to find a repetition.

2. Flaws of Mersenne Twister

Let $\mathbb{N}$ denote the set of non-negative integers. We report flaws of Mersenne Twister MT19937.

Fact 2.1. Put $n = 624$ and $m = 397$. Let $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots$ be the 32-bit integer outputs of MT19937. Let i be an arbitrary integer. Then, the following are strongly positively correlated.

(1) $\mathbf{y}_{i+(m-1)} = \mathbf{y}_{i+(n-1)}$.
(2) $\mathbf{y}_{i+2(m-1)} = \mathbf{y}_{i+2(n-1)}$.
(3) $\mathbf{y}_{i+4(m-1)} = \mathbf{y}_{i+4(n-1)}$.
(4) $\mathbf{y}_{i+8(m-1)} = \mathbf{y}_{i+8(n-1)}$.
(5) $\mathbf{y}_{i+16(m-1)} = \mathbf{y}_{i+16(n-1)}$.
(6) $\mathbf{y}_{i+32(m-1)} = \mathbf{y}_{i+32(n-1)}$.

To be more precise, under the condition (i) above $(i = 1, 2, 3, 4, 5)$, $(i+1)$ occurs with the probability *exactly* $1/2, 1/4, 1/16, 1/256, 1/65536$, respectively. These numbers are too large, since if the outputs were truly random, each probability should be 2^{-32}. One observes that, for example, a triplet consisting from three of the six conditions appears too often. E.g., the probability that (2), (3) and (4) occur at the same time is $1/64 \cdot 2^{-32}$, which is far too large than the expectation $2^{-3\times32}$ for truely random sequences.

See Example 3.11 below, and for the general formula Theorem 3.10. These flaws are discovered by the first author experimentally, when a large variant of the repetition test introduced by Gil, Gonnet and Petersen [3] is executed, see Section 4.

3. Theorem and Proof

Notation 3.1. Let $\mathbb{F}_2$ denote the two element field $\{0, 1\}$ with addition is specified by the exor $1 + 1 = 0$. We identify the set of 32-bit integers with the set of horizontal vectors $W = \mathbb{F}_2^{32}$ (W for words). Let w be 32. Consider the set of sequences of W indexed with $\mathbb{Z}$, namely,

$$W^{\mathbb{Z}} := \{(\ldots, \mathbf{y}_2, \mathbf{y}_1, \mathbf{y}_0, \mathbf{y}_{-1}, \mathbf{y}_{-2}, \ldots) \mid \mathbf{y}_i \in W\}.$$

Let $\mathcal{Y} = (\mathbf{y}_i)_{i \in \mathbb{Z}}$ be a sequence in $W^{\mathbb{Z}}$. We define a delay operator

$$D : W^{\mathbb{Z}} \to W^{\mathbb{Z}}$$

by mapping

$$(\mathbf{y}_i \mid i \in \mathbb{Z}) \mapsto (\mathbf{y}_{i+1} \mid i \in \mathbb{Z}).$$

We denote this action from the right:

$$\mathcal{Y} \mapsto \mathcal{Y}D.$$

This means that D shifts the components by one to the right. Cleary, D^{-1} exists, which shifts the components in the other direction. A $(w \times w)$ matrix M acts on $W^{\mathbb{Z}}$ diagonally from right, i.e.,

$$\mathcal{Y}M = (\mathbf{y}_i M \mid i \in \mathbb{Z}).$$

This action is denoted by the same letter, thus

$$M : W^{\mathbb{Z}} \to W^{\mathbb{Z}}.$$

It is easy to check that D and M commute. We define evaluation at $j \in \mathbb{Z}$ by

$$\mathrm{ev}_j : W^{\mathbb{Z}} \to W, \quad \mathcal{Y} \to \mathbf{y}_j.$$

We have

$$\mathrm{ev}_j(D^k(\mathcal{Y})) = \mathbf{y}_{j+k}.$$

Since the output of MT19937 is purely periodic, we may consider them as a sequence indexed by $\mathbb{Z}$. Also, since a tempering is a (linear) bijection, which preserves the identity relation, we may consider the sequences generated by the recursion [1, (2.1)] (i.e., before tempering) as the output sequence of MT19937, as far as we consider only the repetition of outputs. The next lemma gives an equivalent condition to the recursion formula in [1, Section 2.1].

Lemma 3.2. *The sequences which MT19937 produces (before tempering) are characterized as a kernel of the operator*

$$D^n - D^m + DB + C : W^{\mathbb{Z}} \to W^{\mathbb{Z}}, \tag{3.1}$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_{w-1} & a_{w-2} & a_{w-3} & a_{w-4} & \ddots & a_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \cdots 0 \\ 0 & \\ \vdots & F \\ 0 & \end{pmatrix},$$

where F is a companion matrix, and

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.2}$$

Proof. Let $\mathcal{X} = (\mathbf{x}_i)_{i \in \mathbb{Z}}$ be a sequence in $W^{\mathbb{Z}}$. We compute

$$\begin{aligned}
\mathrm{ev}_k((\mathcal{X})(D^n - D^m - DB - C)) &= \mathrm{ev}_k((\mathcal{X})D^n - \mathcal{X}D^m - \mathcal{X}DB - \mathcal{X}C) \\
&= \mathbf{x}_{k+n} - \mathbf{x}_{k+m} - \mathbf{x}_{k+1}B - \mathbf{x}_k C.
\end{aligned}$$

This is zero for every k, if and only if the recursion [1, (2.1)] is satisfied. $\square$

Note that C has only one non-zero component. This sparseness of C comes from the smallness of the remainder: $19937 \div 32 = 623$ with remainder 1 (the case $w - r = 1$ in [1, Section 3.1]). For example, such a phenomenon is not observed for MT19937-64 [4].

Since D is invertible, we have the following

Corollary 3.3. *The space of the outputs (before tempering) of MT19937 is the kernel of the operator*

$$D^{n-1} - D^{m-1} + B + D^{-1}C : W^{\mathbb{Z}} \to W^{Z}. \tag{3.3}$$

Lemma 3.4. *As an operator on $W^{\mathbb{Z}}$, for any non-negative integer s,*

$$(D^{n-1} - D^{m-1} + B + D^{-1}C)^{2^s} = D^{2^s(n-1)} - D^{2^s(m-1)} + (B + D^{-1}C)^{2^s}.$$

Proof. In the case $s = 1$, paying attention to $1 + 1 = 0$ and that D commutes with any other operators, we have

$$
\begin{aligned}
& (D^{n-1} - D^{m-1} + B + D^{-1}C)^2 \\
=\ & (D^{n-1} - D^{m-1})^2 + 2((D^{n-1} - D^{m-1}))(B + D^{-1}C) + (B + D^{-1}C)^2 \\
=\ & D^{2(n-1)} - D^{2(m-1)} + (B + D^{-1}C)^2.
\end{aligned}
$$

The cases for general s follow by induction. $\square$

Corollary 3.5. *Let $\mathcal{X} = (\mathbf{x}_i \mid i \in \mathbb{Z})$ be a sequence generated by MT19937. Let s be a non-negative integer. Then,*

$$\mathbf{x}_{i+2^s(m-1)} = \mathbf{x}_{i+2^s(n-1)}$$

holds if and only if

$$\mathrm{ev}_i(\mathcal{X}(B + D^{-1}C)^{2^s}) = 0.$$

Proof. Since $\mathcal{X}$ is generated by MT19937,

$$\mathcal{X}(D^{n-1} - D^{m-1} + B + D^{-1}C)^{2^s} = 0.$$

It follows that

$$\mathcal{X}(D^{2^s(n-1)} - D^{2^s(m-1)} + (B + D^{-1}C)^{2^s}) = 0.$$

By ev_i, this yields the desired statement. $\square$

We compute the above for some small s. It is easy to check that $C^2 = 0$, and

$$
CB = \begin{pmatrix}
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

and

$$
BC = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

both having only one non-zero component.

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Lemma 3.6. *For $s < w$, we have*

$$B^s = \begin{pmatrix} \begin{array}{c|ccc} 0 & 0 \cdots 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & F^s & \\ & & & \\ \vdots & & & \\ * & & & \end{array} \end{pmatrix}$$

where in the first column, 1 is at the s-th row from the bottom (i.e. the $(w-s+1)$-st row from the top). We have

$$CB^sC = 0$$

for $s \leq w - 2$.

Proof. The first statement follows from the induction on s and the property of a companion matrix F. For the last statement, CXC has only one possibly non-zero component, whose value is the $(2,1)$-component of X, which is zero for B^s with $s \leq w - 2$. $\square$

Corollary 3.7. *For $0 \leq t \leq w - 2$, CB^t is a matrix whose components are all zero, except the one at the $(1, t+2)$-component. For $1 \leq s \leq w - 2$, B^sCB^t is a matrix whose columns are zero, except the $(t+2)$-nd column, which has $w - s$ zeroes at the top, and the $(w - s + 1)$-st component is one.*

Proof. For a horizontal vector $\mathbf{y} = (y_1, \ldots, y_w)$ with $y_w = 0$,

$$\mathbf{y}B = (0, 0, y_2, y_3, \ldots, y_{w-1}),$$

namely, obtained from $\mathbf{y}$ by replacing y_1 by 0 and shift right. Because each row of C has $w - 2$ zeroes at the right, CB^t is obtained by this shifting for $t \leq w - 2$, which proves the first statement. By Lemma 3.6, we know the first column of B^s, and B^sC has a unique nonzero column at the second row, which is identical with the first column of B^s. The form of B^sCB^t follows. $\square$

Lemma 3.8. *For $k \leq w - 2$, we have*

$$(B + D^{-1}C)^k = B^k + D^{-1}\sum_{i=0}^{k-1} B^iCB^{k-i-1}. \tag{3.4}$$

We define

$$Q_k := \sum_{i=0}^{k-1} B^iCB^{k-i-1}.$$

Then, for $2^s \leq w - 2$,

$$Q_{2^s} = Q_{2^{s-1}}B^{2^{s-1}} + B^{2^{s-1}}Q_{2^{s-1}} \tag{3.5}$$

holds.

Proof. The left hand side of (3.4) is the sum of all the possible 2^k monomials consisting of k of B or $D^{-1}C$. By Lemma 3.6, the terms with two C's are zero. Thus (3.4) follows. By the case division of the place of C, (3.5) follows. $\square$

Lemma 3.9. *For any $s \geq 0$, the rank of $\begin{pmatrix} B^s \\ Q_s \end{pmatrix}$ is w.*

Proof. We remark that $B + C$ is a companion matrix, and since $a_{w-1} = 1$, it is invertible. Thus, $(B + C)^s = B^s + Q_s$ is invertible and has rank w. Hence $\begin{pmatrix} B^s \\ Q_s \end{pmatrix}$ has rank no less than w, and being $2w \times w$ matrix, it has rank w. $\qquad\square$

Theorem 3.10. *Assume that the initialization of MT19937 is done uniformly (including zero). Let $0 \leq s < t$ be integers with $2^t \leq w - 2$. The probability that*

$$\mathbf{x}_{i+2^k(m-1)} = \mathbf{x}_{i+2^k(n-1)} \tag{3.6}$$

holds for all k, $s \leq k \leq t$, is

$$2^{-w} \cdot 2^{-(2^t - 2^s)}.$$

(This is much higher than $2^{-w(t-s+1)}$ for a true random sequence, since $2^t \leq w - 2$.)

For example, we choose $s = 0$ and $t = 1$. Then, the probability that

$$\mathbf{x}_{i+(m-1)} = \mathbf{x}_{i+(n-1)} \text{ and } \mathbf{x}_{i+2(m-1)} = \mathbf{x}_{i+2(n-1)}$$

occur is $1/2 \cdot 2^{-w}$, while for true random numbers, the probability is 2^{-2w}.

Proof. By Corollary 3.5, (3.6) is equivalent to

$$\mathrm{ev}_i(\mathcal{X}(B + D^{-1}C)^{2^k}) = 0,$$

and by Lemma 3.8 equivalent to

$$\mathrm{ev}_i(\mathcal{X}(B^{2^k} + D^{-1}Q_{2^k})) = 0,$$

which is

$$\mathbf{x}_i B^{2^k} + \mathbf{x}_{i-1} Q_{2^k} = 0. \tag{3.7}$$

The conditions for all k satisfying $s \leq k \leq t$ can be described by the product of a vector and a matrix

$$(\mathbf{x}_i, \mathbf{x}_{i-1}) \begin{pmatrix} B^{2^s} & B^{2^{s+1}} & \cdots & B^{2^{t-1}} & B^{2^t} \\ Q_{2^s} & Q_{2^{s+1}} & \cdots & Q_{2^{t-1}} & Q_{2^t} \end{pmatrix} = 0. \tag{3.8}$$

Because of the random choice of the initial seed, $(\mathbf{x}_i, \mathbf{x}_{i-1})$ is uniformly random (note that this property is called the 2-dimensional equidistribution, while MT19937 is known to be 623-dimensionally equidistributed), and the probability that this equality holds is 2^{-r}, where r is the rank of the matrix in (3.8).

By multiplying $B^{2^{t-1}}$ from the right to the second (from the right end) row and subtracting it from the right most row, we have a matrix with the same rank

$$\begin{pmatrix} B^{2^s} & B^{2^{s+1}} & \cdots & B^{2^{t-1}} & 0 \\ Q_{2^s} & Q_{2^{s+1}} & \cdots & Q_{2^{t-1}} & B^{2^{t-1}}Q_{2^{t-1}} \end{pmatrix},$$

where we use (3.5) for the right-bottom corner. Then, we multiply $B^{2^{t-2}}$ to the third row from the right, and subtract it from the second row from the right. By iteration, we have a matrix with the same rank

$$\begin{pmatrix} B^{2^s} & 0 & \cdots & 0 & 0 \\ Q_{2^s} & B^{2^s}Q_{2^s} & \cdots & B^{2^{t-2}}Q_{2^{t-2}} & B^{2^{t-1}}Q_{2^{t-1}} \end{pmatrix}. \tag{3.9}$$

Here we have

$$B^{2^k}Q_{2^k} = B^{2^k}\sum_{i=0}^{2^k-1}B^iCB^{2^k-i-1} = \sum_{i=0}^{2^k-1}B^{2^k+i}CB^{2^k-i-1}.$$

By Corollary 3.7, $B^{2^k+i}CB^{2^k-i-1}$ has a unique nonzero column as the $(2^k-i-1+2)$-nd column with top $w-(2^k+i)$ components being zeroes and the $(w-(2^k+i)+1)$-st component is one. The range of i is $0 \le i \le 2^k-1$. This means that all these columns for k, $s \le k \le t$, and $0 \le i \le 2^k-1$ are linearly independent. Their number is

$$2^s + 2^{s+1} + \cdots + 2^{t-1} = 2^t - 2^s.$$

This means that we have a $2w \times (w+2^t-2^s)$-matrix with the same rank as (3.9)

$$\begin{pmatrix} B^{2^s} & 0 \\ Q_{2^s} & G \end{pmatrix}, \tag{3.10}$$

where G consists of the above $2^t - 2^s$ columns. We show that the columns of (3.10) are independent. Let $b_1, \ldots, b_w, c_1, \ldots, c_{2^t-2^s}$ be elements of $\mathbb{F}_2$, and the linear combination of the columns with these coefficients is zero. Then, since G has independent columns, it follows that c_i for $i = 1, \ldots, 2^t - 2^s$ are zeroes. Then, by Lemma 3.9, b_i for $i = 1, \ldots, w$ are zeroes, hence the columns of (3.10) are linearly independent. Thus the matrix (3.10) has rank $w+2^t-2^s$. Thus, the probability that all equalities in (3.7) for $s \le k \le t$ hold is $2^{-w-(2^t-2^s)}$, which proves the theorem. $\square$

Example 3.11. Put $n = 624$ and $m = 397$. Let $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots$ be the 32-bit integer outputs of MT19937. Let i be an arbitrary integer. Then, the following are strongly positively correlated.

(1) $\mathbf{y}_{i+(m-1)} = \mathbf{y}_{i+(n-1)}$.
(2) $\mathbf{y}_{i+2(m-1)} = \mathbf{y}_{i+2(n-1)}$.
(3) $\mathbf{y}_{i+4(m-1)} = \mathbf{y}_{i+4(n-1)}$.
(4) $\mathbf{y}_{i+8(m-1)} = \mathbf{y}_{i+8(n-1)}$.
(5) $\mathbf{y}_{i+16(m-1)} = \mathbf{y}_{i+16(n-1)}$.
(6) $\mathbf{y}_{i+32(m-1)} = \mathbf{y}_{i+32(n-1)}$.

For example, the probability that (1) and (2) occur is the case $s = 0$ and $t = 1$, hence $2^{-w}2^{-(2^t-2^s)} = 1/2 \cdot 2^{-w}$, while the truly random sequence has the probability $2^{-w} \cdot 2^{-w}$.

The probability that (4) and (5) occur is the case $s = 3$ and $t = 4$, hence $2^{-w}2^{-(2^4-2^3)} = 1/256 \cdot 2^{-w}$.

The probability that (5) and (6) occur is $2^{-w}2^{-(2^5-2^4)} = 1/65536 \cdot 2^{-w}$.

For triples, for example, the probability that (2), (3) and (4) occur is the case $s = 1$ and $t = 3$, hence is $2^{-w}2^{-(8-2)} = 1/64 \cdot 2^{-w}$, while the probability for a truly random sequence is 2^{-3w}.

4. Modified repetition tests found the patterns

The original repetition test counts the run-length for observing the identical 32-bit occurring at two distinct places. Namely, starting from the output $\mathbf{y}_0$ of MT19937, we memorize consecutive outputs, until we found r such that $\mathbf{y}_{r-d} = \mathbf{y}_r$. This r is called the run-length for the repetition.

After finding such an r, we do not initialize MT19937, just continue to find the next repetition. In the original test, one set is to iterate this 100 times, and the average of r's is taken and compared with its theoretical distribution. (Three sets are repeated.)

The first author iterated this 100 billion times, and instead of taking the average, he observed the number of occurrences of r, for $r = 2, 3, \ldots, 2100000$. The probability that $r > 2100000$ occurs is negligibly small ($0.000 \cdots$ with 222 zeros). Then, the number of the occurrences of the case $r = 623$ is extremely high (more than 40 times larger than the expectation, as stated above). Analogous phenomena are observed at $r = 1246, 2492, \ldots$.

Then, the second and the third authors analyzed the phenomena, and found Theorem 3.10, which explains these phenomena. Probability that $r = 623$ occurs is very small for a true random 32-bit integer sequence. For the case of MT19937, after observing the first repetition, the continuous search for the next repetition is affected by the strong correlation among (1) and (2) in Theorem 3.10. Namely, it is often the case that (2) is preceded by (1), and (1) is detected in the previous run. A part of Theorem implies that (2) is very often after (1) (i.e., with probability $1/2$, whereas it should be 2^{-32} if the sequence is truly random), which means that we should observe the run-length $n - 1 = 623$ quite often (after finding (1), we continue to search for the next, which implies that we observe (2) very often, where the run-length is $n - 1 = 623$).

The first author noticed that the cases (2) after (1), (3) after (2), and (3) after (2) after (1), etc., are extremely frequent. The main theorem is proved to explain these phenomena. The exact figures of the cases encountered are available for download at: [5].

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