

Topological Quantum Gravity through Harmonic S^2 Maps

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(Dated: December 30, 2025)

By virtue of harmonic maps on two-dimensional spheres (S^2), a topological quantization in space-time is proposed. The discrete character of all physical quantities follows naturally. A Schwarzschild black hole, non-black hole and wormhole based geometries are considered in which a quantum hair becomes effective. A thermometer or curvature-detecting device can record the macroscopic quantumness of spacetime.

This paper is dedicated to Prof. Metin Gürses on the occasion of his completion of sixty fruitful years of contributions to the field of General Relativity.

Introduction: Harmonic mapping (HM) between two Riemannian manifolds \mathcal{M} and \mathcal{M}' , described by the line elements $ds^2 = g_{ab}dx^a dx^b$ ($a, b = 1, 2, \dots, m$) and $ds'^2 = g_{AB}dy^A dy^B$ ($A, B = 1, 2, \dots, n$), respectively, is defined by the energy functional [1]

$$E[y^A] = \int g_{AB} \frac{\partial y^A}{\partial x^a} \frac{\partial y^B}{\partial x^b} g^{ab} \sqrt{|g|} d^m x, \quad (1)$$

in which $|g| = \det g_{ab}$. The extremal condition $\delta E[y^A] = 0$ yields

$$\nabla_a \nabla_b y^A + \Gamma_{BC}^A \frac{\partial y^B}{\partial x^a} \frac{\partial y^C}{\partial x^b} g^{ab} = 0, \quad (2)$$

where ∇_a is the gradient on \mathcal{M} and Γ_{BC}^A are the Christoffel symbols on \mathcal{M}' . For appropriate choices of \mathcal{M} and \mathcal{M}' , these equations are equivalent to the Einstein field equations [2, 3]. The isometries on \mathcal{M}' can be used to generate new solutions from known ones in general relativity [4]. The technique of HMs has aided in obtaining important solutions in the past [5]. In this Letter, instead of arbitrary manifolds, we restrict ourselves to HMs between two-spheres (S^2). Our technique of topological quantization is built on any classical well-known spacetime such that for the quantum parameter $k = 1$, we recover our classical spacetime. To expose the power of HM, symmetry and the natural discreteness that it creates on classical geometry we study three different classes of metrics:

1. The Schwarzschild black hole (SBH) in which a minimum quantum length (ℓ_h) is greater than the event horizon radius (r_h). Such a choice is more applicable to microscopic BHs.

2. The zero mass case of a BH, which becomes a non-BH spacetime consisting of pure topological quantum sources and Weyl tetrad scalar as representative of gravitational field. The minimum length can be identified as the Planck length $\ell_p \sim 10^{-35} m$.

3. In this class of spacetimes, the minimum length is identified with the throat radius of a wormhole which provides a natural lower bound $\ell_h = \ell_0 > 0$. This choice makes our quantum spacetime nonsingular and the entire spacetime can be built on a wormhole with an exotic matter.

1. All spherically symmetric, static (3+1)-dimensional BHs possess a 2-dimensional spherical (S^2) sector. A harmonic mapping of an S^2 into another S^2 [1] yields a discrete transformation that defines a topological quantum number [6]. This is the integer $k \in \mathbb{N}$ representing the number of wrappings of the base S^2 manifold. Given this discrete symmetry, the vacuum SBH can be re-expressed as ($G = c = 1 = \hbar$)

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 f_k^2(\theta) d\Omega^2, \quad (3)$$

where

$$f_k(\theta) = \frac{2k(\sin \theta)^{k-1}}{(1 - \cos \theta)^k + (1 + \cos \theta)^k}, \quad (4)$$

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and

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \quad (5)$$

Here $k \in \mathbb{N}$ measures the degree of the harmonic map on S^2 [1]. The energy of the map is $E_k = 4\pi|k|$, and in what follows we restrict ourselves to $0 < k < \infty$. The physical interpretation of E_k as a quantum energy requires multiplication by a length in geometrical units. Our formalism involves the strong gravity regime of a black hole. In the absence of a black hole, a quantum theory for free gravitational waves analogous to quantum electrodynamics might be developed. Our strategy differs as we quantize geometry itself, assuming that such quantization manifests in physical quantities and gravity.

The quantized Schwarzschild black hole: To elevate the classical picture to a quantized geometry, we employ the integer $k > 1$ as a quantum number and modify the Schwarzschild metric as

$$ds^2 = -g(r, k)dt^2 + \frac{dr^2}{g(r, k)} + r^2 f_k^2(\theta)d\Omega^2, \quad (6)$$

where

$$g(r, k) = 1 - \frac{2m}{r} + \frac{Q_k^2}{r^2}, \quad (7)$$

and $Q_k^2 = (1 - \frac{1}{k})\ell_h^2$. Here ℓ_h is a fundamental length and Q_k can be interpreted as a quantum charge. For the extremal black hole, ℓ_h lies in the interval $m \leq \ell_h \leq \sqrt{2}m$, where m is the geometrical mass related to the physical mass M by $m = GM/c^2$. For a non-extremal Schwarzschild black hole, the event horizon is

$$r_h = m \left(1 + \sqrt{1 - \left(\frac{Q_k}{m} \right)^2} \right). \quad (8)$$

To have a horizon distinct from the extremal one, we choose $Q_k < m$, provided by $Q_k = \alpha_k m$ with $0 < \alpha_k < 1$. This ensures $r_h < \ell_h$, so that the energy integral does not cross the horizon. For example, $\frac{2\sqrt{2}}{3} < \alpha_2 < 1$, $\frac{2\sqrt{6}}{5} < \alpha_3 < 1$, and $\frac{4\sqrt{3}}{7} < \alpha_4 < 1$, and so on. For each $2 \leq k < \infty$, one finds an α_k ensuring $r_h < \ell_h$, and α_k approaches unity as k increases.

The quantized energy-momentum tensor $T_a^b = \text{diag}\{-\rho, p_r, p_\theta, p_\varphi\}$ takes the form

$$T_a^b = \frac{Q_k^2}{r^4} \text{diag}\{-1, -1, 1, 1\}, \quad (9)$$

with energy density ρ and directional pressures p_i . The integrated energy is

$$E_k = \int_{l_h}^{\infty} \rho \sqrt{g} d^3x = 4\pi(k-1)\ell_h, \quad (10)$$

implying $k = 1$ corresponds to the vacuum case. Our quantum description therefore applies outside a black hole, similar to the Bohr model which covers orbital electrons but not nuclear structure. The quantized area, entropy, and Hawking temperature are given by [6]

$$A_k = 4\pi m^2 k (1 + \chi)^2, \quad (11)$$

$$S_k = \frac{1}{4} A_k, \quad (12)$$

and

$$T_H = \frac{\chi}{2\pi m(1 + \chi)^2}, \quad (13)$$

respectively, where

$$\chi = \sqrt{1 - \left(\frac{Q_k}{m} \right)^2}, \quad (14)$$

and the horizon radius

$$r_h = m(1 + \chi). \quad (15)$$

The quantum hair: To examine the role of quantum hair (i.e., dependence on k), we compute in a Newman-Penrose (NP) tetrad [7] the nonzero curvature quantities Ψ_2 (Weyl component) and Φ_{11} (Ricci part):

$$\Psi_2(r, k) = -\frac{m}{r^3} + \frac{Q_k^2}{r^4}, \quad (16)$$

$$\Phi_{11}(r, k) = \frac{1}{2} \frac{Q_k^2}{r^4}. \quad (17)$$

The Weyl curvature Ψ_2 contains classical and quantum parts, with the former dominating asymptotically. The Ricci term Φ_{11} is purely quantum, decaying as r^{-4} . The quantum hair becomes significant near the outer horizon. For microscopic black holes, as $r \rightarrow 0$, the quantum hair dominates. For a black hole with solar mass $M = 1.989 \times 10^{30}$ kg, Eq. (10) yields

$$E_k = 4\pi\sqrt{k(k-1)} \frac{GM}{c^4} \alpha_k. \quad (18)$$

We obtain $E_2 = 1.72$ MeV, $E_3 = 3.09$ MeV, and $E_4 = 4.36$ MeV. As the quantum number grows ($k \gg 1$, $k < \infty$),

$$E_k \simeq 4\pi k \frac{GM}{c^2}. \quad (19)$$

Unlike the Bohr model, the quantum energy levels of the SBH increases linearly with k , yet remain bounded.

2. For $m = 0$, no black hole remains, but a discrete geometrical structure persists. The fundamental length ℓ_h then corresponds to the Planck length, and the quantized line element becomes

$$d\Omega^2 = -\left(1 + \frac{Q_h^2}{r^2}\right) dt^2 + \frac{dr^2}{1 + \frac{Q_h^2}{r^2}} + r^2 f_k^2(\theta) d\Omega^2. \quad (20)$$

This can be interpreted as a pure quantum metric with quantized curvature $\Psi_2(r, k)$ and source $\Phi_{11}(r, k)$. The $r = 0$ is a spacetime singularity which can be avoided by choosing the minimum length to be the Planck length so that $\ell_p \leq r < \infty$. According to this model each point r is the center of a "spacetime atom" around which orbital levels with curvature $\Psi_2(r, k)$ and quantized energy levels $\Phi_{11}(r, k)$ are established. In series these are given exactly by

$$\Psi_2 = \frac{\ell_p^2}{r^4} \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1} \right), \quad (21)$$

and

$$\Phi_{11} = \frac{1}{2} \Psi_2. \quad (22)$$

It is desired that each k level represents a ripple of gravitation and between any two successive diffusions k 's we have void, as in the H-atom model of quantum mechanics. We note also that the quantum metric gives rise to repulsive gravity and may contribute to the dark energy.

3. Our next application of the topological quantization with a wormhole base is given by the line element

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + (r^2 + \ell_0^2)f_k^2(\theta)d\Omega^2 \quad (23)$$

where $g(r) = 1 - \frac{2m}{\sqrt{r^2 + \ell_0^2}} + \frac{Q_k^2}{r^2 + \ell_0^2}$, with the constant $\ell_0 > 0$. For $m = 0 = Q_k$ corresponding to $k = 1$, this reduces to the Ellis wormhole in which ℓ_0 =throat radius. For $m \neq 0 \neq Q_k$, we have a spacetime that interpolates between a black hole and wormhole which extends the metric of Simpson and Visser [9]. Letting $m = 0$, with $\ell_0 \neq 0$, gives us a regular spacetime with exotic quantum source ($\rho < 0$) which fails to satisfy some of the null energy conditions

(NEC), i.e., $\rho + p_i \geq 0$. For $m = \ell_0 = 0 \neq Q_k$ our line element satisfies the weak energy conditions (WEC), i.e., $\rho > 0$, $\rho + p_i \geq 0$ which is nothing but the singular case considered above in part 2. Such singularity can be cured by introducing the Plank length as a minimal radius so that a particle can't reach $r = 0$.

Conclusion: By virtue of the harmonic map symmetry of two spheres all Schwarzschild-type black holes admit ripple of gravity in Weyl curvature $\Psi_2(r)$ as a quantum hair i.e. the index k . A fundamental length scale ℓ_h , of the order of horizon or throat radius in case of a wormhole is introduced. For a non-black hole case ℓ_h can be identified with the Planck scale ℓ_p . In our model each spacetime point can be considered as an 'atom' of spacetime, encircled by Bohr-like orbits. Our approach is analogous to the quantum mechanical particle theory and as a paradigm shift, is distinct from the perturbative mean graviton field theoretic approach [10]. We admit that our topological quantum method doesn't rid gravity from all exotic sources. This non-perturbative, exact topological method can also be applied to string and loop quantum gravity metrics.

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