

Multi-agent Adaptive Mechanism Design

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We study a sequential mechanism design problem in which a principal seeks to elicit truthful reports from multiple rational agents while starting with no prior knowledge of agents' beliefs. We introduce Distributionally Robust Adaptive Mechanism (DRAM), a general framework combining insights from both mechanism design and online learning to jointly address truthfulness and cost-optimality. Throughout the sequential game, the mechanism would estimate agents' beliefs, then iteratively updates a distributionally robust linear program with shrinking ambiguity sets to reduce payments while preserving truthfulness. Our mechanism guarantees truthful reporting with high probability while achieving $\tilde{O}(\sqrt{T})$ cumulative regret, and we establish a matching lower bound showing that no truthful adaptive mechanism can asymptotically do better. The framework generalizes to plug-in estimators (DRAM+), supporting structured priors and delayed feedback. To our knowledge, this is the first adaptive mechanism under the general settings that maintains truthfulness and achieves optimal regret when incentive constraints are unknown and must be learned.

1 Introduction

The theory of mechanism design studies rules and institutions in various disciplines, ranging from auctions and online advertisements to business contracts and trading rules. The formulation often involves a central principal (system) and one or many rational agents (players), where the principal designs a mechanism to achieve a given objective subject to agents' incentives. A typical component in such formulations is the *common knowledge assumption*: certain information about agents is presumed known to the principal and can be exploited to design analytically tractable, often optimal mechanisms. For example, the design of revenue-optimal auctions requires knowledge of bidders' value distributions over the item [Myerson, 1981]. However, the availability of such knowledge is difficult to justify in practice. This observation, originally due to [Wilson, 1985], is now known as Wilson's critique. It proposes that some information is too private to be common knowledge, and such assumptions should be weakened to approximate reality.

In parallel, the theory of online learning studies algorithms that learn and make decisions in unfamiliar environments, aiming to approach the performance of oracles that have full knowledge from the start. The principal typically begins with no knowledge of the environment, and information is acquired through repeated data collection and carefully designed statistical methods. A common assumption is that the environment is unknown but *stationary*. For example, in the classical multi-armed bandit model [Lattimore and Szepesvári, 2020], each arm's reward is a stochastic distribution, and the best arm can be discovered via repeated sampling. An alternative is to assume the worst-case scenario from the environment, i.e., fully adversarial feedback. These algorithms have wide applications in recommendation, pricing, scheduling, and more [Lattimore

*Author ordering alphabetical. All authors made valuable contributions to the writing, editing, and overall management of the project. **Renfei Tan** led the project, and is the first to propose the idea of achieving cost-efficient adaptive mechanisms via sequentially accurate distributionally robust mechanisms. He is the main developer of the modeling, algorithm, and corresponding theorems, as well as the main writer of the paper. **Zishuo Zhao** contributed on discussions, proofreading, and comprehensive review. He proposed the main idea of distributionally robust mechanisms with insights on the synergy between it and online learning. He also helped formulating the examples and wrote the literature review part of peer prediction. **Qiushi Han** proposed the initial idea of the two-phased (warm-start and adaptive) approach to relax the common knowledge assumptions and contributed to the development of the main algorithm. He led the design and conduct of the numerical experiments in this work. **David Simchi-Levi** supervised the research and assisted in writing the paper.

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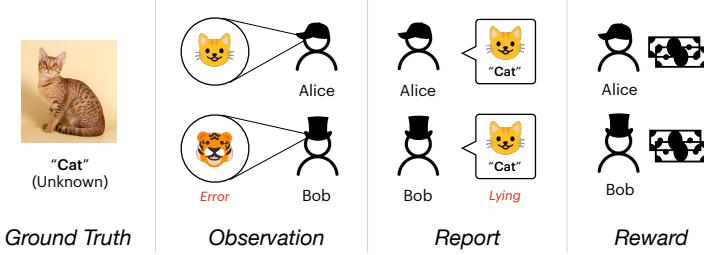


Fig. 1. An image labeling example. Nature samples an unlabeled image with an unknown ground truth, which is then independently observed by multiple agents. Each agent’s observation (type) is private to herself. The agents then report to the principal and receive rewards in the end. Lying or lazy behavior is possible, since the principal does not know the ground truth or the agents’ observations. One objective is to incentivize truthful behavior via reward mechanisms based on only agents’ reports.

and Szepesvári, 2020]. In application, however, they often interact with humans, who are neither stationary nor fully adversarial. In fact, a foundational assumption in economics is that humans are *rational* [Von Neumann and Morgenstern, 2007].

Therefore, the strength and weaknesses from both fields seems to be complementary to each one. Mechanism design can incentivize nice behavior for proper learning guarantees from rational agents, while online learning can provide the necessary knowledge for efficient mechanisms. For this reason, the combination of mechanism design and online learning has received increasing attention, most notably in settings such as online contract design [Ho et al., 2014, Zhu et al., 2022] and online auctions [Blum et al., 2004, Cesa-Bianchi et al., 2014]. However, the design of general multi-agent adaptive mechanisms remains an under-explored problem.

In this work, we consider a general sequential mechanism design problem in which the principal starts with *no knowledge* of the players. As a motivating example, consider the image labeling task where the principal assigns unlabeled images to multiple agents. In each round, each agent makes an independent observation of the image as her type, then reports her observation (type), and finally receives a payment from the principal. The principal’s objective is three-fold: data quality, truthfulness, and cost-optimality. The principal wants to design a reward mechanism that can obtain the highest-quality data from the crowdsourced task, while incentivizing truthful report from players, and do so in a cost-minimal way.

As the principal cannot directly and cheaply observe ground truth at scale, the design of such a reward mechanism faces several challenges. First, the true label is unavailable or expensive to obtain for most tasks (otherwise the principal would prefer that route), making it hard to control report quality or infer agent skills. Second, each agent’s observation (type), or whatever actions she does before reporting, is unknown to the principal. Agents are rational and pursue utility, so they may lie or become lazy (skip observation and report random labels) during this phase. Finally, classical mechanism designs that rely on common knowledge are inapplicable: even mechanisms that only aim to elicit truthful reports often assume accurate knowledge of posteriors or correlation structure [Miller et al., 2005]. In addition, we cannot exploit the special structures in specific mechanism design problems (such as in auctions, a second-price auction incentivizes truthfulness even *without* common knowledge). For a general mechanism design problem, the principal must learn this knowledge while simultaneously incentivizing effort and honesty even when such knowledge is inaccurate.

1.1 Our Contribution

Our model can be viewed as a generalization of the optimal mechanism design problem in the mechanism design field and prediction with expert advice problem in the online learning field. From a mechanism design perspective, we relaxed the common knowledge assumption. The principal would rely on online learning techniques to gain knowledge. From an online learning perspective, we relaxed the setting from always-honest agents to the more realistic rational agents. To incentivize honest work, the principal would utilize proper mechanism design.

The necessity of truthfulness. We show that in our sequential setting, mechanisms that aim to maximize downstream decision quality while reducing payments over time must, up to label permutations, induce truthful reports under mild regularity conditions. This result clarifies the target of design: once reports are (approximately) truthful, the principal can consistently learn the distributions required by the payment rule and drive down payments. The statement is structural rather than the main focus, and it guides the robust and adaptive designs that follow.

Distributionally robust mechanisms. We formulate the single-round mechanism design problem as a linear program that minimizes expected payment subject to (i) individual rationality, (ii) truthfulness, and (iii) a no-free-lunch constraint for uninformative strategies. Because the principal initially misspecifies the required knowledge, we introduce distributional robustness: incentive constraints are enforced against all posteriors within a total-variation ambiguity set around current estimates. The resulting mechanism preserves truthfulness under bounded misspecification, at a quantifiable payment premium that scales with the ambiguity radius. As data accumulates, the radius shrinks and payments decrease accordingly.

Optimal adaptive mechanism design. Finally, in the sequential mechanism design problem, we show that the fundamental bottleneck lies in learning players' beliefs. This learning problem is unavoidable: without sufficiently accurate knowledge of players' beliefs, no mechanism can simultaneously guarantee truthfulness and low payment. Motivated by this observation, we design an algorithm named *Distributionally Robust Adaptive Mechanism* (DRAM) that carefully balances robustness and learning. Our algorithm achieves an $\tilde{O}(N\sqrt{T})$ regret guarantee (up to logarithmic factors) while preserving truthfulness with high probability at every round, and we complement this result with a matching lower bound showing that no truthful adaptive mechanism can do better in the worst case. The theoretical results are validated by numerical simulations. This framework extends to any plug-in estimators (e.g., structured or regularized estimators for discrete distributions) and is compatible with delayed or batched feedback. To our knowledge, this is the first general adaptive mechanism that maintains truthfulness and achieves optimal regret when incentive constraints depend on unknown and learned information.

1.2 Related Work

Our work draws insights from both the mechanism design and online learning literature, including adaptive mechanism design and online learning.

1.2.1 Online and adaptive Mechanism Design. The combination of mechanism design and online learning is a fast-growing direction in algorithmic game theory [Roughgarden, 2010]. Prior work spans a variety of participation structures and problem settings. Some works study *synchronous* settings, where the same players have multiple encounters with each other, as in repeated games [Hart and Mas-Colell, 2000, Papadimitriou et al., 2022, Satchidanandan and Dahleh, 2023]. Others consider *asynchronous* settings, where new agents may arrive and depart over time, a structure particularly relevant in online auctions and advertising platforms [Cesa-Bianchi et al., 2007, Choi et al., 2020, Hajiaghayi et al., 2004, Milgrom, 2019, Wang et al., 2017]. Within both settings, two domains have received the most attention: online contract design [Ho et al., 2014, Zhu et al., 2022]

and online auctions [Blum et al., 2004, Cesa-Bianchi et al., 2014]. These works typically focus on learning an optimal mechanism, such as an optimal contract or an optimal reserve price, using tools from bandit learning.

A key observation is that preserving truthfulness is not a substantive difficulty in these existing models. In contract design, agents' choices naturally reflect their private information, and no truthful reporting constraint is involved. In online auctions, structural properties ensure incentive compatibility: for instance, in second-price auctions, truthful bidding remains a dominant strategy even when the reserve price is inaccurate. Even without any knowledge, the second-price auction guarantees players' truthfulness. As a result, works such as [Cesa-Bianchi et al., 2014] can safely explore suboptimal mechanisms during learning without risking incentive distortion or data contamination. The focus is solely on learning an optimal mechanism, which makes these problems are more or less reducible to a bandit problem [Cesa-Bianchi et al., 2014, Zhu et al., 2022].

In contrast, in a general mechanism design problem, preserving truthfulness becomes a significant difficulty. When the principal begins with limited or mis-specified knowledge, an improperly constructed mechanism can immediately encourage agents to lie or exert low effort, thereby corrupting the collected data undermining subsequent learning processes. Thus, unlike prior literature, maintaining truthfulness throughout the entire learning trajectory is not merely desirable but essential, and this requirement is one of the central challenge we must address.

1.2.2 Prediction with Expert Advice. We note that our setup is a stochastic, label-efficient variant of the classical *prediction with expert advice* problem. The prediction with expert advice problem is fundamental in online learning [Cesa-Bianchi and Lugosi, 2006]. In the standard framework, the players (often called “experts”) can sequentially provide arbitrary and even adversarial signals, and the principal’s objective is to implement an *aggregation* algorithm that achieves sublinear decisional regret. A simplification is to assume players behave stochastically (report signals according to a probability law), under which the aggregation regret can be significantly improved [Cesa-Bianchi et al., 2004]. The stochastic variant also has connections with other online learning problems such as online optimization [Agarwal et al., 2017, Cesa-Bianchi et al., 2007, Gaillard et al., 2014], with extensions in contextual or non-stationary settings [Besbes et al., 2016].

In the practical setting, acquiring a true label might be expensive. It may be only feasible to query the true label for a small portion of rounds. This is the label-efficient setting of prediction with expert advice [Cesa-Bianchi et al., 2005, Helmbold and Panizza, 1997]. Roughly speaking, the aggregation regret decreases as the inverse square root of the number of queries. There also exist adaptive algorithms that achieve the same regret with far fewer queries in benign cases [Castro et al., 2023, Mitra and Gopalan, 2020].

Compared to the standard framework, our setup deals with *rational* experts who need proper incentives for nice behaviors, which (on the difficulty of response aggregation) lies between the adversarial and the stochastic setting. The assumption on rationality brings additional considerations on the design of incentives, which is the main concern of our work. Also, different from the standard or the label-efficient settings, in our model, the true signal is never revealed (or only revealed for a constant number of rounds), adding difficulty to distinguish poorly-performed experts.

1.2.3 Information elicitation and peer prediction. The field of *information elicitation* studies the mechanism design task to incentivize honest feedback from untrusted but rational participants, generally via designing *scoring rules* [Li et al., 2022] as rewards or penalties for participants. Particularly, *peer prediction* [Miller et al., 2005] studies the scenarios in which ground truth is unavailable for direct verification of collected reports, with applications in dataset acquisition and evaluation [Chen et al., 2020, Zheng et al., 2024], crowdsourcing [Dasgupta and Ghosh, 2013], and recent blockchain-based decentralized ecosystems [Wang et al., 2023, Zhao et al., 2024]. The general

paradigm of peer prediction mechanisms is to ask multiple participants the same question and reward them according to the comparison among their reports. While peer prediction mechanisms provide elegant results on truthful Nash equilibria without requirement of ground-truth information, most of existing mechanism rely on strong unrealistic assumptions on know prior and observation matrices, forming a gap to practical usage in real-world systems.

For practical usage, researcher develop a series of works with relaxed assumptions or stronger incentive guarantees. Particularly, [Kong, 2024] develops a prior-free multi-task peer prediction mechanism with dominant-strategy incentive compatibility, with a lack of permutation-proof property that is impossible for any prior-free peer prediction mechanisms [Kong and Schoenebeck, 2019]. Besides, [Shnayder et al., 2016] provides an *informed truthful* mechanism ensuring that the truthful equilibrium achieves highest utilities among all Nash equilibria, and [Zhang et al., 2025] develops a mechanism with a *stochastic dominance* property ensuring incentive compatibility even under non-linear utilities.

In our setting, we address the gap between existing prior-dependent designs and reality via acquiring the prior distribution by *online learning*, with a multi-round adaptive mechanism that learns the distributional information during the process. Besides, we also explicitly consider the robustness property that ensures incentive guarantees under inaccurate knowledge, thus making the peer prediction framework applicable in realistic applications.

2 Problem Formulation

We consider the sequential mechanism design problem where a principal seeks to elicit truthful reports from rational agents. The principal sequentially assigns T prediction tasks to a group of N rational players. Each task has a true label $Y_t \in \mathcal{Y}$, i.i.d. sampled from an unknown and stationary distribution $p_Y(\cdot)$. Unless stated otherwise, this true label is not revealed to anyone, either the principal or the players. We let \mathcal{Y} be finite to avoid mathematical complications, and assume each true label y appears with uniformly bounded probability $\underline{p} \leq p_Y(y) \leq \bar{p}$. In each round, each player $i = 1, \dots, n$ independently studies the task, acquiring her own observation $X_{it} \in \mathcal{Y}$ with a constant cost c . X_{it} is generated according to the player's skill $p_i(x | y)$, a stationary conditional probability law. We assume p_i is non-degenerate, i.e., there exists two labels y, y' where $p_i(\cdot | y) \neq p_i(\cdot | y')$. In other words, observation should bring new information by stochastically distinguishing at least two of the labels. Each player might know her own skill distribution p_i , but has no information about anyone else's, and the principal initially knows none of them. Aside from observation, players also has an outside option of lazily reporting a random label without observing the label (does not incur cost c as well).

After studying the task, players independently produce their *public reports* $Z_{it} \in \mathcal{Y}$ to the principal. We assume players are *risk-neutral* and *myopic*. Being risk-neutral means players aim to maximize their expected reward, conditional on the public and private information they have. Being myopic means players only care about immediate reward in the current round but not future rewards. Under these settings, we have rational players who do not necessarily report their observations. Instead, they would *lie* (report $Z_{it} \neq X_{it}$) or be *lazy* (report Z_{it} without observation X_{it}) when they expect an advantage in doing so. We denote the observation and report profile of all players in a round by X_t and Z_t .

Collecting the reports, the principal rewards each player i with an reward mechanism $R_{it}(Z_1, \dots, Z_t)$. The reports can then be used for downstream decision-making tasks, such as aggregation. Note that the reward mechanism is non-anticipating, meaning the mechanism can only decide on past and current but not future report profiles.

The principal aims to design the online reward mechanism $\mathbf{R} = (R_{it})_{i \in [N], t \in [T]}$ with three objectives:

- **Truthfulness** (aka *incentive-compatibility*): given all other players act honestly, a player would maximize her own expected utility when she works, obtains observations, and then reports honestly ($Z_{it} = X_{it}$).
- **Data quality**: the reward mechanism should incentivize the highest-quality reports, such that downstream decision-making tasks may achieve the optimal objective.
- **Cost-optimality**: maintaining truthfulness and data-quality, the principal minimizes its total expected payment to players.

We now compare between our setup and typical modeling assumptions in the online learning and mechanism design literature. Online learning mainly targets statistical efficiency—minimizing cumulative prediction error—while treating all reports as truthful. Mechanism design, by contrast, centers on strategic incentives, but usually presumes players' type distributions are known or even common knowledge. These assumptions ease analysis, yet rarely hold in practice. Our model pursues both goals at once and relaxes the assumptions from both fields.

Remark 1. The proposed model can be further generalized to match the classical model in the mechanism design literature. Here, players' observations are their own *types* of the round. Assume in each round players' types X are sampled from a stationary joint distribution. Players then report their types (not necessarily truthful) Z to the principal and receive rewards. All of our analysis and algorithms applies to this generalized setting. In fact, our analysis does not require how the types are generated and make no use of p_Y and p_i and focuses exclusively on the joint law p_X .

Remark 2. We note that each agent's utility is linear in only her own reward ($u_i = r_i$). In the most general setting of mechanism design, an agent's utility is a function of all agents' types and the resource allocation from principal: $u_i : \mathcal{Y}^N \times \mathcal{R} \rightarrow \mathbb{R}$, where \mathcal{R} is the space of the principal's resource allocation decisions. For example, in contracts, utility depends on the agent's own type x_i and the principal's payment r_i ($u_i = f(x_i, r_i)$). In auctions, utility depends on whether or not the agent gets the item (with probability p), her valuation of the item (type x_i), and her requested payment r_i ($u_i = p \cdot x_i - r_i$). Our analysis could be potentially generalized to the case when such utility function is exactly known by the principal. Nonetheless, we believe substantial hurdle exists when such utility functions are not known and need to be learned.

We conclude this section by revealing the importance of truthfulness. After all, the principal's top objectives in outsourcing tasks are to improve data quality and lower costs. Truthfulness, as a mechanism design objective, might not be of interest if the mechanism that reaches the highest quality or the lowest cost promotes dishonest behaviors. From the revelation principle [Myerson, 1979], we know that truthfulness is “free”, in the sense that we don't lose anything by focusing only on mechanisms with their incentive-compatible Nash equilibria. For the same reason, it suffices to consider the setting where the true label Y , observation X , and report Z all belong to the same space \mathcal{Y} . The following proposition actually proves a stronger result, showing that truthfulness is not only free, but in fact almost *necessary* for maximal quality.

PROPOSITION 2.1 (TRUTHFULNESS IS NECESSARY FOR MAXIMAL QUALITY). *Consider a general non-anticipating decision-making task $A_t(Z_1, \dots, Z_t)$ with objective $\text{Obj}(A_t, Y_t)$. Assume players' skills p_i are non-degenerate. For any round t , the highest possible performance is attainable if and only if every player first observes the task, and then either*

- *reports truthfully, i.e. $Z_{it} = X_{it}$ for all i ; or*
- *reports a fixed permutation of her observation, i.e. for each player i there exists a bijection $\pi_i : \mathcal{Y} \rightarrow \mathcal{Y}$ such that $Z_{it} = \pi_i(X_{it})$.*

Proposition 2.1 is derived from Blackwell's informativeness theorem [Blackwell, 1953]. Each round, the true label, observations, reports, and decisions form a Markov chain: $Y_t \rightarrow X_t \rightarrow Z_t \rightarrow A_t$. An intuition is that optimal decision-making requires maximal *information* from upstream. Due to the data processing inequality [Cover, 1999], information never increases going downstream; therefore, the best approach is to preserve as much information as possible at each link. Truthful reporting preserves full information at the link $X_t \rightarrow Z_t$, which allows for optimal subsequent decisions. Any lies from players erode information. Lazy behavior also produces less information than observing. Aside from truthfulness, an alternative case that preserves full information is when players use a permutation reporting strategy. However, such a case is unrealistic in practical settings, as the principal would need to know each player's permutation rule to reverse the encoding and uncover the true observation. Therefore, this proposition essentially shows that eliciting truthfulness is the practical way to achieve maximal performance.

Truthfulness is necessary not only for data quality but also cost-optimality. Our adaptive mechanism relies on the idea of gradually learning players' skills across tasks to lower costs in later rounds. Truthful reporting provides faithful data that allows accurate estimates of skills.

3 Mechanism Design without Common Knowledge

In our work, a central relaxation of modeling is that we don't assume prior distribution of labels p_Y or players' skills p_i are known by players or the principal. The principal's central focus is to maintain truthfulness with unknown or inaccurate estimation of that knowledge. The absence of the so-called *common knowledge assumption* is a major distinction of our model from the classical mechanism design literature. We would follow Wilson's critique, which appeals for relaxing the assumption of that player behaviors are common knowledge. In this section, we focus on (distributionally) robust mechanisms, which aim to incentivize truthful behavior under knowledge ambiguity. The notion of (distributionally) robust mechanisms is not new, and has been studied in [Bergemann and Morris, 2005, Koçyiğit et al., 2020] in general settings. Nonetheless, the mechanism in our section is specifically tailored to the estimation process of unknown player information, with the purpose of being applied in each stage of adaptive mechanisms.

3.1 Optimal Single-round Mechanism Design

We begin with the analysis of optimal mechanism design with *known* p_Y and p_i within a single round. When there are no true labels available, we apply the principles of peer prediction, which is to use other players' report to verify a focal player's report. The delicacy lies in the careful design of the reward mechanism to ensure that truthfulness is a Nash Equilibrium.

We start with the two-player mechanism. With a focal player i and a reference player j , the optimal 2-player mechanism design problem could be formulated as a linear programming problem. The objective is to minimize expected reward to players, and the constraints are the desired properties of the mechanism, including truthfulness. (we hide subscript t for simplicity.)

$$\begin{aligned} \min_{R_i} \quad & \mathbb{E}[R_i(X_i, X_j)] \\ \text{s.t.} \quad & \mathbb{E}[R_i(X_i, X_j) | X_i] \geq c \\ & \mathbb{E}[R_i(Z_i, X_j) | X_i] \leq c, \quad \forall Z_i \neq X_i \\ & \mathbb{E}[R_i(Z_i, X_j)] \leq 0, \quad \forall Z_i \perp\!\!\!\perp X_i, X_j \end{aligned} \tag{1}$$

The expectation is taken under the joint probability law p_X induced by p_Y , p_i , and p_j . The first constraint is the *individual rationality* property, meaning when the player obtains observations and then reports honestly, she would get a non-negative reward. The second one states the *truthfulness* property, where the player receives a non-positive reward when she lies. The final constraint

implements the *no-free-lunch* property, meaning the player cannot get a positive utility when she is lazy. We introduce the individual rationality and the no-free-lunch constraint to prevent arbitrary decrease of the objective function by applying an affine transformation to the reward mechanisms. (For risk-neutral players, affine transformations on reward do not affect utility ordering and strategic behavior.)

Example 3.1 (Image Labeling). Suppose there are two types of images $\mathcal{Y} = \{\text{Cat}, \text{Tiger}\}$, abbreviated with C and T respectively. We further assume that the prior distribution of the image types is balanced, i.e., $p_Y(C) = p_Y(T) = 0.5$. For each image with an unknown true label $Y \in \mathcal{Y}$, the principal would like to let two players 1, 2 individually observe it, and truthfully report their observations X_1, X_2 to label that image. Assume that both players are 90% accurate: $p_i(C | C) = p_i(T | T) = 0.9$, and $p_i(T | C) = p_i(C | T) = 0.1$.

The principal designs the reward mechanism as follows: both players receive 1 reward if their reports Z_1, Z_2 agree with each other, and receive -1 otherwise, i.e. $R_{\{1,2\}}(Z_1, Z_2) = 2 \cdot \mathbf{1}_{[Z_1=Z_2]} - 1$, and assume that observation incurs cost $c = 0.1$. Now we assume that player 2 observes and report honestly, and analyze the incentive of player 1.

Suppose player 1 observes Cat, Bayes' formula gives that $\mathbb{P}(X_2 = C | X_1 = C) = 0.82$ and $\mathbb{P}(X_2 = C | X_1 = T) = 0.18$. On the other hand, if player 1 does not pay the effort to toss the coin, then her Bayesian belief on player 2's observation (and report) is $P(X_2 = C) = P(X_2 = T) = 0.5$. She can then work out expected reward under truthful, lying, and lazy strategies: truthful (0.54) > lazy (0) > lying (-0.74). Hence truthful behavior is desired. In fact, this simple mechanism is a feasible solution to Eq.(1). Intuitively, after the focal player's observation, her posterior probability on the other player's observing the same label is higher than observing a different label. Therefore, it is preferable to report whatever you observe in the first place. Such a mechanism is called *peer prediction* [Miller et al., 2005], originated the fact that rational players always tries to predict their peers' observations before action.

Define the belief matrix \mathbf{B} where $B_{xx'} = \mathbb{P}(X_j = x' | X_i = x)$, and reward matrix \mathbf{R} where $R_{xx'} = R_i(x, x')$. We also let \mathbf{d} a column vector of the prior distribution of j 's observation: $\mathbf{d}_x = \mathbb{P}(X_j = x)$. Then we can reformulate (1) into the following equivalent problem:

$$\begin{aligned} \min_{\mathbf{R}} \quad & \sum_{x,x'} \mathbb{P}(X_i = x) \mathbf{B}_{xx'} \mathbf{R}_{xx'} \\ \text{s.t.} \quad & (\mathbf{B} \mathbf{R}^\top)_{xx} \geq c, \quad \forall x \in \mathcal{Y} \\ & (\mathbf{B} \mathbf{R}^\top)_{xy} \leq c, \quad \forall x \neq y \in \mathcal{Y} \\ & \mathbf{R} \mathbf{d} \leq \mathbf{0} \end{aligned} \tag{2}$$

Note that the second constraint only enforces *pure* lying strategies incur non-positive reward, nevertheless, it is sufficient since any mixed strategy is a convex combination of pure strategies and its corresponding reward is also a convex combination with the same weights. The final constraint assumes all entries of $\mathbf{R} \mathbf{d}$ are negative, thus making sure any report strategy without observing incurs a non-positive reward.

THEOREM 3.2 (OPTIMAL COST OF A TWO-PLAYER PEER-PREDICTION MECHANISM). *Suppose the belief matrix \mathbf{B} is invertible, and there does not exist $x \in \mathcal{Y}$ such that $\mathbb{P}(X_i = x) = 1$. Then the linear program (1) (equivalently, its matrix form (2)) is feasible. Moreover, the minimum achievable expected payment equals the individual rationality threshold c ; that is,*

$$\min_{R_i \text{ satisfying (1)}} \mathbb{E}[R_i(X_i, X_j)] = c.$$

In addition, at optimality, the first constraint in (1) is binding.

The tight result on the objective function is in the spirit of the classical Crémer-McLean mechanism [Crémer and McLean, 1988], which can extract full surplus from the players when type distributions are common knowledge. The conditions are satisfied for “almost all” \mathbf{B} and \mathbf{d} .

Example 3.3 (Optimal Mechanism in Image Labeling). Continuing the image labeling example from Example 3.1, we show that the optimal mechanism pays both players the observation cost $c = 0.1$ in expectation, as a demonstration of Theorem 3.2. The optimal mechanism can be acquired by solving Eq.(2). Here, we first compute the belief matrix and player 2’s observation distribution.

$$\mathbf{B} = \begin{bmatrix} 0.82 & 0.18 \\ 0.18 & 0.82 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Solving the linear program would give the following mechanism: both players receive $5/32$ reward if their reports agree, and receive $-5/32$ otherwise.

We now show that the mechanism satisfies all constraints and is cost-optimal. First, no-free-lunch is satisfied since when a player reports without observation, no matter what strategy she follows, expected reward is always $0.5 \times 5/32 - 0.5 \times 5/32 = 0$ (since the other player observes both labels with equal probability). When she lies, expected reward is $(\mathbf{BR}^\top)_{xy} = -0.1$. When she is truthful, she gets $\sum_{x,x'} \mathbb{P}(X_i = x) \mathbf{B}_{xx'} \mathbf{R}_{xx'} = 0.1$ in reward, exactly equal to her observation cost. The optimal mechanism extracts full surplus from players.

Remark 3. From Lemma 1 of [Radanovic and Faltings, 2013], it is known that for any mechanism with more than two players with a truthful Bayesian Nash Equilibrium, it is possible to construct a 2-player mechanism (with one focal and one reference player) that achieves a truthful Bayesian Nash Equilibrium with the same expected payment. This lemma narrows our attention to reward mechanisms with only two players. In application, the reference player can be randomly picked to avoid collusion.

3.2 Distributionally Robust Mechanism Design with Inaccurate Knowledge

Now we move on to the scenario where players and the principal only have *inaccurate* knowledge of the true p_Y and p_i . Suppose they only know that the true distributions players’ observations X belong to some ambiguity sets $p \in \mathcal{P}_X$. From the principal’s perspective, the challenge would be to design a *distributionally robust* mechanism, such that for any possible realizations within the ambiguity set, the truthfulness constraint would still be met. Its objective now becomes minimizing the expected payment in the worst case.

The notion of distributionally robust mechanisms has been studied in [Koçyiğit et al., 2020], which primarily focus on auctions. We focus on the general mechanism design problem, and try to obtain mechanisms that achieve certain guarantees under the worst case, but we don’t pursue exact worst-case optimality. This greatly reduces computational complexity, and turns out to be sufficient for subsequent adaptive mechanism design. From (2) we know the mechanism is evaluated under prior and posterior distribution of the reference player’s observation. With ambiguity set \mathcal{P}_X , the mechanism design problem can now be formulated as the following distributionally robust optimization problem:

$$\begin{aligned}
\min_{R_i} \quad & \sup_{p \in \mathcal{P}} \mathbb{E}_p [R_i(X_i, X_j)] \\
\text{s.t.} \quad & \inf_{p \in \mathcal{P}_X} \mathbb{E}_p [R_i(X_i, X_j) | X_i] \geq c \\
& \sup_{p \in \mathcal{P}_X} \mathbb{E}_p [R_i(Z_i, X_j) | X_i] \leq c, \quad \forall Z_i \neq X_i \\
& \sup_{p \in \mathcal{P}_X} \mathbb{E}_p [R_i(Z_i, X_j)] \leq 0, \quad \forall Z_i \perp\!\!\!\perp X_i, X_j
\end{aligned} \tag{3}$$

The constraints are formulated as the three properties are guaranteed even in the worst case. Typically, the analysis revolves around acquiring tight bounds on the objective under different structures of ambiguity set. In the most general case when \mathcal{P} is the set of all distributions on X , we would have a mechanism that achieves dominant strategy incentive compatibility (DSIC, in other words, we have a *strategyproof* mechanism). For our purposes, we don't need to achieve exact worst-case optimal result on the objective function. Instead, we take an agnostic approach where we don't require any structure of ambiguity set. All we need to know is the diameter and location of the ambiguity set.

We begin with a sensitivity analysis of (1) with respect to shifts in probability law. Assume the principal obtain a design by solving (1) according to an erroneous probability law p , but the true law is p^* . According to Theorem 3.2, at optimality, we have a binding constraint in (1). Therefore, any slight deviation from p would lead to the potential violation of constraints. To hedge against violations, following the idea of [Zhao et al., 2024], the principal could add safety *margins* on the constraints. For example, instead of only requiring the expected reward of truthful behaviors to be greater than c , the principal could let it be no less than $c + \delta$, where $\delta > 0$ is the margin width. In this case, even if the expected reward of truthful behaviors might decrease under p' , as long as the decrease is no more than δ , the individual rationality property is still preserved. Under this idea, we look at a variant of the mechanism design problem with margin δ :

$$\begin{aligned}
\min_{R_i} \quad & \kappa \\
\text{s.t.} \quad & |R_i| \leq \kappa \\
& \mathbb{E}_p [R_i(X_i, X_j) | X_i] \geq c + \delta \\
& \mathbb{E}_p [R_i(Z_i, X_j) | X_i] \leq c - \delta, \quad \forall Z_i \neq X_i \\
& \mathbb{E}_p [R_i(Z_i, X_j)] \leq -\delta, \quad \forall Z_i \perp\!\!\!\perp X_i, X_j
\end{aligned} \tag{4}$$

There are two curious features to this problem variant. First, the margin δ added to the constraints protects the principal against inaccurate knowledge at an additional cost of at least δ , since the expected payment under each possible observation is at least $c + \delta$. This is a lower bound on the cost of robustness. Increasing δ means the mechanism from (4) is robust for a higher degree of inaccuracies, but it would also cost more. Pursuing optimality, the principal want to find the lowest δ just enough to guarantee constraints are still satisfied under p^* . It would be crucial to understand the connection between the degree of misspecification and the minimal required margin δ . Second, the objective function alters from minimizing expected payment to minimizing worst-case payment. The reason for this change is to limit the sensitivity of expected payment to worst case probability law deviation from p^* to p . Following the “compactness” criteria discussed in [Zhao et al., 2024], the outcome incurring the highest absolute payment has the highest sensitivity to probability deviation, hence a large δ/κ ratio would ensure a high robustness to such deviations. Therefore, for a fixed δ , we would like to lower κ as much as possible.

THEOREM 3.4 (ROBUSTNESS TO DISTRIBUTIONAL MISSPECIFICATION). *Let p be the distribution used in designing the margin- δ mechanism R_i in (4) and let p^* be the true distribution. Let $p(x_j | x_i)$ be the induced prior or posterior distribution of the reference player's observation x_j , conditional on the focal player's observation $x_i \in \mathcal{Y} \cup \{\emptyset\}$.*

Denote total variation distance by $\text{TV}(p, p^)$. If*

$$\text{TV}(p(\cdot | x_i), p^*(\cdot | x_i)) \leq \frac{\delta}{2\kappa}, \quad \forall x_i \in \mathcal{Y} \cup \{\emptyset\}, \quad (5)$$

where $\kappa = \max |R_i|$. Then the mechanism R_i produced under p remains feasible for the original problem (1) when evaluated under p^ .*

Theorem 3.4 proves the robustness of the margin- δ mechanism under inaccurate distributional knowledge. This suggests it is possible to design a mechanism that guarantees truthfulness for all distributions in an ambiguity set centered at a distribution estimation $p: \mathcal{P} = \{p' | p' \text{ satisfying (5)}\}$. However, notice that the objective κ itself is influenced by the margin δ we choose and the distribution p used. Actually, when we increase δ , the minimal achievable κ would also increase, hence the increment in the provided robustness (i.e. δ/κ) may diminish. Therefore, one cannot infinitely increase δ hoping for unlimited robustness. In the following, we first provide an objective upper bound on (4), then provide the upper and lower bounds on the amount of robustness that can be provided by (4).

THEOREM 3.5 (BOUNDS ON PAYMENTS OF ROBUST MECHANISM). *Suppose the belief matrix \mathbf{B} is invertible, and there does not exist $x \in \mathcal{Y}$ such that $\mathbb{P}(X_i = x) = 1$. Let (κ^*, R_i^*) be the optimal solution of (4) with design distribution p and margin δ . Then we have*

- Worst-case payment:

$$c + \delta \leq \kappa^* \leq \|\mathbf{B}^{-1}\|_2 \cdot \left(c \cdot \frac{\gamma|\mathcal{Y}| + 1}{1 - \gamma} + \delta \cdot \frac{(1 + \gamma)|\mathcal{Y}| + 2}{1 - \gamma} \right). \quad (6)$$

where $\gamma = \max_x \mathbb{P}(X_i = x) < 1$.

- Expected payment: there exists a solution (κ, R_i) that satisfies the bound (6), while ensuring the expected payment of truthful equilibrium under p is $c + \delta$, the lowest possible.

The essential insight from Theorem 3.5 is that (4) is a linear programming problem. Hence, if we consider δ as a perturbation on the constraints, the shift in the objective itself should also linearly relate to the perturbation. In other words, the sensitivity of both worst-case and expected payment to perturbation δ is $O(\delta)$. Rearranging terms in (6) would easily give us the following bounds.

COROLLARY 3.6 (BOUNDS ON ACHIEVABLE ROBUSTNESS). *Let the (possibly inaccurate) design distribution be p . Then for any margin parameter $\delta > 0$, there exists a mechanism R_i feasible for (4) such that*

$$\frac{\delta}{2\kappa} \geq \frac{\delta \cdot (1 - \gamma)}{2\|\mathbf{B}^{-1}\|_2(c \cdot (\gamma|\mathcal{Y}| + 1) + \delta \cdot ((1 + \gamma)|\mathcal{Y}| + 2))}, \quad (7)$$

where $\kappa = \max |R_i|$ and $\gamma = \max_x \mathbb{P}(X_i = x) < 1$.

Moreover, we have $\delta/2\kappa \leq 1/2$ for all δ and corresponding feasible mechanism R_i , meaning no mechanism can achieve robustness more than $1/2$.

Corollary 3.6 primarily provides the minimum robustness achieved by solving (4). Notice that increasing δ provides more robustness, but comes at increased cost. Also, the marginal robustness from increasing δ would diminish: as $\delta \rightarrow \infty$, the lower bound is at most $(1 - \gamma)/\|\mathbf{B}^{-1}\|((1 + \gamma)|\mathcal{Y}| + 2)$. When $\delta \rightarrow 0$, the robustness provided scales linearly with δ . In addition, from the lower bound on the worst case payments, we have an upper bound $1/2$ on the maximum robustness that can

be provided from our scheme. This result does not contradict the impossibility result (Theorem 1) shown in [Radanovic and Faltings, 2013], which proves that no mechanism can guarantee truthfulness for *all* distributions.

Our theorem shows that while an all-round strictly dominantly truthful mechanism does not exist, it is still possible to design a mechanism that covers distributions that are relatively close to a design distribution p . For distributions distanced too far away, the impossibility result still holds. In other words, our scheme is enough to cover inaccuracies that are not too extreme. For that reason, Corollary 3.6 is already sufficient for our purposes of adaptive mechanism design. If the principal starts with a distribution estimation not too off (such that the total variation distance condition is satisfied with the robustness floor), then it is possible to maintain truthfulness and refine estimation at the same time. The principal would first apply a mechanism that provides abundant robustness for its initial ambiguity set. As more data is obtained and estimation becomes more accurate, it shrinks the ambiguity set and selects a smaller δ , eventually converging to $\delta = 0$ and the optimal mechanism design.

THEOREM 3.7 (COST OF ROBUSTNESS). *Suppose we have an ambiguity set*

$$\{p' \in \mathcal{P} \mid \text{TV}(p(\cdot | x_i), p'(\cdot | x_i)) \leq \eta, \forall x_i \in \mathcal{Y} \cup \{\emptyset\}\}$$

with design distribution p and ambiguity parameter η . If $\eta \leq \tilde{\eta} = (1 - \gamma)/2\|\mathbf{B}^{-1}\|_2((1 + \gamma)|\mathcal{Y}| + 2)$, there exists a mechanism R_i such that player i would act truthfully when her belief belongs to this ambiguity set.

Moreover, if the actual distribution p^ also belongs to this set, then the principal's expected payment for guaranteeing such truthful behavior is at most*

$$c + \frac{4\|\mathbf{B}^{-1}\|_2(\gamma|\mathcal{Y}| + 1) \cdot \eta}{(1 - \gamma) - 2\|\mathbf{B}^{-1}\|_2((1 + \gamma)|\mathcal{Y}| + 2) \cdot \eta} \cdot c \quad (8)$$

where the second part is the additional cost of robustness.

Theorem 3.7 is essentially a combination of Theorem 3.4 and 3.5. Notice that the additional cost takes the format of $c \cdot C_1\eta/(1 - C_2\eta)$. This means when $\eta \rightarrow 0$, the additional cost of robustness is roughly $O(\eta)$. Also, this theorem holds only when the ambiguity level is lower than a constant threshold $\tilde{\eta}$. This means there is only so much our robustness scheme can do: if the principal starts with no knowledge and the ambiguity level is too high, it is impossible for our robustness mechanism to guarantee truthfulness. We need certain warm-start procedure (for example, an oracle's opinion) to lower uncertainty under the threshold first.

Example 3.8 (Distributionally Robust Mechanism in Image Labeling). We follow the same example as in Example 3.1 and 3.3. Now we compare the simple mechanism that pays 1 on agreement and the optimal mechanism that pays 5/32. Although the simple mechanism is suboptimal, it is robust to misspecification of players' skills.

For example, suppose that the players' true observation accuracy is 0.8 instead of 0.9. With the same procedure in Example 3.1, one can show that the simple mechanism still guarantees truthfulness (truthful (0.26) > lazy (0) > lying (-0.46)). On the other hand, the previously optimal mechanism breaks down (truthfulness gives 9/160 < $c = 0.1$, so players have no incentives in participation). In fact, one can show that as long as both players' accuracies are the same and stay within the range [0.66, 1], the simple mechanism always guarantees truthfulness. The lower bound $(10 + \sqrt{10})/20 \approx 0.66$ is when the truthful strategy's expected reward falls to 0.1. This property holds true even if players know the actual skill level, while the principal does not have that information. In a word, the additional payment in the simple mechanism serves as insurance against ambiguity.

4 Adaptive Mechanism Design

In this section, we study the problem of adaptive mechanism design where the principal has no knowledge of the nature or the players. Starting from oblivion, the principal's aim is to dynamically design mechanisms that eventually converge to the optimum, while ensuring players' truthfulness along the convergence process. At the same time, the principal would attempt to converge as fast as possible, so as to minimize the regret. Define the principal's (empirical cumulative) *regret* after T rounds as $\sum_{t=1}^T \sum_{i=1}^N (R_{it} - R_{it}^*)$, where R_{it}^* is the optimal payment ($\mathbb{E}[R_{it}^*] = c$ as shown by Theorem 3.2).

We would demonstrate that the bottleneck of the adaptive mechanism design problem is the difficulty to learn the conditional distribution $p^*(x_j | x_i)$. Surrounding this observation, we design a statistically optimal adaptive mechanism that achieves regret guarantee with matching lower bounds. We present our algorithm, "Distributionally Robust Adaptive Mechanism" (DRAM), in Algorithm 1. The algorithm maintains truthfulness and reduces cost by designing a sequence of *distributionally robust mechanisms* with shrinking ambiguity parameter η . The ambiguity parameter tracks the accuracy of the principal's estimation of nature and players' behavior at each round.

In DRAM, the entire T tasks are divided into two phases: *warm-start* phase and *adaptive* phase. As suggested by Theorem 3.7 (see discussions after Theorem 3.7), initially, when the ambiguity level is above a certain threshold $\tilde{\eta}$, no mechanism can ensure truthfulness for all distributions in the ambiguity set. The warm-start phase is designed to reduce ambiguity below that threshold. In this phase, the principal would use the true label Y_t for verification. This phase lasts $O(\log \log T)$ tasks, so cost is controlled even when true label might be expensive to get. Then, the principal moves to the adaptive phase. As the principal obtains more data, it constructs a more accurate estimation. This allows it to design a mechanism with decreasing η and therefore reduce cost of robustness. The ambiguity parameter η shrinks at a proper rate, so as to make sure truthfulness is preserved with high probability for each round.

During the entire algorithm, the principal uses players' past report to estimate $p^*(x_j | x_i)$, which is the posterior distribution of reference player's observation x_j conditional on the focal player x_i . Note that the principal only has players' reports z but not the true observation x . This means truthfulness must be guaranteed at all times to ensure the estimation fidelity. This adds another evidence on the necessity of truthfulness - it is not only necessary for optimal downstream aggregation tasks (Theorem 2.1), but also necessary for cutting costs.

Inputs. Out of the three input variables, the failure tolerance is one that can be decided arbitrarily, and the other two depend on the players we work with. The ambiguity threshold for all players is defined as $\tilde{\eta} = \min_i \tilde{\eta}_i$, where the player-specific threshold $\tilde{\eta}_i$ is computed according to Theorem 3.7:

$$\tilde{\eta}_i = \frac{1 - \gamma(i)}{2\|\mathbf{B}_{(i)}^{-1}\|(1 + \gamma(i))|\mathcal{Y}| + 2}. \quad (9)$$

Warm-start phase. The main objective of the warm-start phase is to reduce the principal's ambiguity below the threshold suggested by Theorem 3.7 so that distributionally robust mechanisms can be applied. There are multiple approaches to reduce ambiguity, and here the principal learns $p^*(x_j | x_i)$ by collecting truthful reports from players. The main challenge is to ensure truthfulness when peer prediction mechanisms do not work. We achieve this by using true label verification. Suppose the principal can now obtain the correct label Y_t from an external expert. This expert verification might require a cost much higher than the total observation cost per round Nc , therefore it is costly to apply it for too many rounds. The principal can now compare players' reports with ground truth and reward according to a fact-checking mechanism $R_{it}(Z_{it}, Y_t)$.

ALGORITHM 1: Distributionally Robust Adaptive Mechanism (DRAM)

Input: ambiguity threshold $\tilde{\eta}$; failure tolerance ε ; lower bound on observation frequency

$$0 < \rho < \min_{i,x \in \mathcal{Y}} \mathbb{P}(X_i = x).$$

Update ambiguity threshold $\tilde{\eta} = \min(\tilde{\eta}, 1/\sqrt{2})$;

Compute the warm-start phase length $\tau = \log((d+1)2^d N \log T / \varepsilon) / 2\rho\tilde{\eta}^2$;

For each player i , assign a corresponding reference player j .

Warm-start phase.

for $t = 1, 2, \dots, \tau$ **do**

Obtain true label Y_t from an external source.

Deploy the fact-checking mechanism $R_i = \mathbf{1}\{Z_{it} = Y_t\}$ for each player i .

end

Adaptive phase.

Define epoch schedule $\tau = \tau_0 < \tau_1 < \tau_2 < \dots$ with $\tau_k - \tau_{k-1} = 2^{k-1}\tau$ (*continue until $\tau_m \geq T$ or anytime*).

for $k = 1, 2, \dots, m$ **do**

Estimate reference distribution with all past reports:

$$\hat{p}_{ik}(x_j | x_i) = \frac{\sum_{t=1}^{\tau_{k-1}} \mathbf{1}\{Z_{it} = x_i, Z_{jt} = x_j\}}{\sum_{t=1}^{\tau_{k-1}} \mathbf{1}\{Z_{it} = x_i\}}.$$

Let ambiguity parameter $\eta_k = \sqrt{\log((d+1)2^d N (\log T) / \varepsilon) / 2\rho\tau_{k-1}}$.

For each player i , set their safety margin

$$\delta_{ik} = \frac{2\|\mathbf{B}_{(i)}^{-1}\|_2(Y_{(i)}|\mathcal{Y}| + 1) \cdot \eta_k}{(1 - \gamma_{(i)}) - 2\|\mathbf{B}_{(i)}^{-1}\|_2((1 + \gamma_{(i)})|\mathcal{Y}| + 2) \cdot \eta_k} \cdot c.$$

Here $\mathbf{B}_{(i)}$ is the matrix representation of $\hat{p}_{ik}(x_j | x_i)$ (see Section 3), and $\gamma_{(i)} = \max_x \mathbb{P}(X_i = x)$.

Compute the mechanism R_{ik} for each player by solving Eq.(4) with parameter \hat{p}_{it} and δ_{ik} .

Deploy the mechanism R_{ik} for rounds $t = \tau_{k-1} + 1, \dots, \tau_k$.

end

LEMMA 4.1 (FACT CHECKING UNDER DIAGONAL DOMINANCE). *Recall the assumption that each true label y appears with uniformly bounded probability $\underline{p} \leq p_Y(y) \leq \bar{p}$. If for all $y \in \mathcal{Y}$ and $x \neq y$, player i 's skill p_i satisfies the diagonal dominance property:*

$$p_i(y | y) \geq \frac{\bar{p}}{\underline{p}} \cdot p_i(x | y), \quad (10)$$

then the simple fact-checking rule $R_{it}(Z_{it}, Y_t) = \mathbf{1}\{Z_{it} = Y_t\}$ guarantees player i 's truthfulness.

The *diagonal dominance* condition essentially assumes that players are more likely to obtain the correct observation than to make a mistake. Therefore, any lying behavior would decrease the probability of the report being correct; thus, the player should be incentivized to tell the truth. We note that it is generally impossible to design a fact-checking rule that guarantees truthfulness for arbitrary skill distribution p_i and p_Y . It is shown in [Lambert, 2011] that if a player's observation has overlaps, i.e., the player can have the same observation under two different labels, then there always exists an adversarial prior under which a fact-checking mechanism fails.

Adaptive phase. After the ambiguity is lower than the threshold, the principal moves into the adaptive phase. This phase divides the entire time horizon into epochs, with each epoch double the size of the previous one. In total, we would have $O(\log T)$ epochs.

At the beginning of each epoch, the principal would call two oracles: i) an offline estimation oracle for the reference distribution $p^*(x_j | x_i)$ (in Algorithm 1 it is the empirical distribution

estimator), and ii) an optimization oracle that computes the distributionally robust mechanisms by solving Eq.(4). Then, the principal would use the same produced mechanism throughout the entire epoch, and no further computation is needed. This indicates DRAM is also *computationally efficient* with $O(N \log T)$ total calls to both oracles.

At the same time, DRAM is also *statistically efficient* for we have the following regret guarantee.

THEOREM 4.2 (REGRET UPPER BOUND OF DRAM). *Consider the sequential mechanism design problem with N players and T rounds. With probability at least $1 - \varepsilon$, Algorithm 1 simultaneously achieves:*

- *truthfulness is guaranteed for all N players in all T rounds.*
- *expected total regret of the algorithm is at most*

$$O\left(N\sqrt{T} \log(N \log T / \varepsilon)\right). \quad (11)$$

Theorem 4.2 recovers the $O(\sqrt{T})$ terms typically seen in bandits and online learning literature [Lattimore and Szepesvári, 2020]. In fact, oracle calls can be further reduced to $O(N \log \log T)$ when T is known. In DRAM, we use the classical *doubling trick* [Cesa-Bianchi and Lugosi, 2006] from the online learning literature. This trick does not require exact knowledge of the number of tasks. (Although we need to know the magnitude $\log(T)$ to compute epochwise ambiguity parameter η_k s.) The epoch schedule $\tau_k - \tau_{k-1} = T^{1-2^{-(k-1)}} \tau$ (similar to [Cesa-Bianchi et al., 2014]) maintains the same regret guarantee, while requiring even fewer oracle calls $O(\log \log T)$.

COROLLARY 4.3 (REGRET UPPER BOUND WITH KNOWN T). *Consider replacing the epoch schedule in Algorithm 1 with $\tau_k - \tau_{k-1} = T^{1-2^{-(k-1)}} \tau$. With probability at least $1 - \varepsilon$, the principal simultaneously guarantees truthfulness across all rounds for all players, and the expected total regret maintains the same upper bound*

$$O\left(N\sqrt{T} \log(N \log T / \varepsilon)\right)$$

with only $O(\log \log T)$ epochs.

4.1 Regret Lower Bound of Adaptive Mechanism Design

We now show that DRAM is *statistically optimal* up to logarithmic factors. In particular, we prove a matching lower bound demonstrating that any policy which guarantees truthfulness with high probability must incur regret of order at least $\Omega(N\sqrt{T})$. Throughout this section, we consider the general setting described in Remark 1, where players' types (observations) are directly sampled from an unknown joint distribution p_X over the type space \mathcal{Y}^N .

THEOREM 4.4. *Consider the sequential mechanism design problem with N players and T rounds. Fix any failure tolerance $\varepsilon \in (0, 1/4)$. For any (possibly randomized) non-anticipating reward policy that guarantees truthfulness across all players and rounds with probability at least $1 - \varepsilon$, there exists a type distribution $p_X \in \Delta(\mathcal{Y}^N)$ under which, with probability at least $1 - \varepsilon$, the total regret is at least*

$$\Omega(N\sqrt{T \log(1/\varepsilon)}).$$

The proof proceeds by constructing a pair of statistically indistinguishable problem instances whose corresponding cost-optimal truthful mechanisms are incompatible. Specifically, any reward mechanism that is both truthful and near-optimal under one instance must either violate truthfulness or incur strictly larger payments under the other. This incompatibility allows us to reduce adaptive mechanism design to a hypothesis testing problem, and we invoke Le Cam's two-point method to derive the lower bound.

The result reveals that the inherent difficulty of adaptive mechanism design lies in learning players' conditional beliefs, namely the posterior distributions $p^*(x_j | x_i)$ that govern incentives.

Because the lower bound is derived via a two-point argument, it does not explicitly depend on the alphabet size $d = |\mathcal{Y}|$. It remains to be studied the optimality of . However, since estimating a discrete distribution over \mathcal{Y} incurs a minimax risk of order $\sqrt{d/T}$ [Han et al., 2015], we conjecture that the regret bound achieved by DRAM is also optimal in its dependence on d , up to logarithmic factors.

4.2 Extension to General Estimator

An important observation of DRAM is that the estimation oracle and the optimization oracle are separate modules. They are connected via the ambiguity parameter η_k , which measures the distance between the estimated *distribution* and the actual one. This means that DRAM is flexible with estimators, so long as its estimation could satisfy the requirement in Eq.(5). Therefore, the empirical estimator could be swapped with any other distribution estimator that may better exploit and reflect the underlying structures of players' skills. Based on this, we propose the algorithm DRAM+, which works with general distribution estimators.

Definition 4.5 (General Discrete Distribution Estimator). Let q be a discrete distribution on space \mathcal{Y} . Given t samples independently and identically drawn from q , the generalized distribution estimator provides an estimation \hat{q}_t such that with probability $1 - \varepsilon$, we have

$$\text{TV}(q, \hat{q}_t) \leq \eta_\varepsilon(t). \quad (12)$$

This estimation guarantees $\eta_\varepsilon(t)$ should monotonically decrease with t , and monotonically increase with failure probability ε . It bounds the gap between \hat{q}_t and q under a dataset size of t . Such a bound is commonly seen in the probably approximately correct (PAC) framework of statistical learning, where a good estimator should achieve a lower gap with higher probability and lower t .

Now we introduce DRAM+ (Algorithm 2), which modifies DRAM to work with general discrete distribution estimators in Definition 4.5. Compared to DRAM, the main difference of DRAM+ lies in the epoch schedule and ambiguity parameters. In DRAM+, we don't restrict how epoch schedules are designed. This may lead to suboptimal schedules. Generally, one should aim for a geometric epoch schedule, as this typically results in the best possible bounds and only $m = O(\log T)$ epochs. The impact of schedule is studied in [Besson and Kaufmann, 2018, Perchet et al., 2016], from the perspective of batched feedback and doubling trick, respectively. Moreover, the ambiguity parameters now follow the guarantee $\eta_\varepsilon(t)$, in order to ensure truthfulness holds with high probability.

THEOREM 4.6 (REGRET UPPER BOUND OF DRAM+). *Consider the sequential mechanism design problem with N players and T rounds. With probability at least $1 - \varepsilon - N(d+1) \cdot \sum_{k=1}^m \exp(-\rho\tau_{k-1}/8)$, Algorithm 2 simultaneously achieves:*

- *truthfulness is guaranteed for all N players in all T rounds.*
- *expected total regret of the algorithm is at most*

$$O\left(N \sum_{k=1}^m \eta_{\varepsilon/Nm(d+1)}(\rho\tau_{k-1}/2) \cdot (\tau_k - \tau_{k-1})\right). \quad (13)$$

Compared to Theorem 4.6, an additional overhead $N(d+1) \cdot \sum_{k=1}^m \exp(-\rho\tau_{k-1}/8)$ appears in the high probability guarantee in the Theorem. This is due to a failed event when a player does not observe a certain label x_i for enough number of times, and there are not enough data to recover $p^*(\cdot|x_i)$. This failed case is inherent and universal for whatever estimator the principal uses. Without a closed-form description of the estimation gap, we cannot absorb this probability into ε , like what we did in Theorem 4.6. (See Appendix A for more details.) Nonetheless, with appropriately chosen schedule (such as $\tau_k - \tau_{k-1} = 2^{k-1}\tau$), this exponential term should be dominated by ε .

ALGORITHM 2: Distributionally Robust Adaptive Mechanism+ (DRAM+)

Input: ambiguity threshold $\tilde{\eta}$; failure tolerance ε ; lower bound on observation frequency $0 < \rho < \min_{i,x \in \mathcal{Y}} \mathbb{P}(X_i = x)$; distribution estimator \mathcal{E} .

Compute the warm-start phase length as the smallest τ such that $\eta_{\varepsilon/Nm(d+1)}(\rho\tau/2) < \tilde{\eta}$.

For each player i , assign a corresponding reference player j .

Warm-start phase.

Follows the same procedure as in Algorithm 1.

Adaptive phase.

Define epoch schedule $\tau = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_m = T$.

for $k = 1, 2, \dots, m$ **do**

Estimate reference distribution with the general distribution estimator for each $x_i \in \mathcal{Y}$:

$$\hat{p}_{ik}(x_j | x_i) \leftarrow \{Z_{it}, Z_{jt}\}_{t \leq \tau_{k-1}}.$$

Let ambiguity parameter $\eta_k = \eta_{\varepsilon/Nm(d+1)}(\rho\tau_{k-1}/2)$.

Compute the safety margin δ_{ik} and deploy the mechanism R_{ik} the same way as in Algorithm 1.

end

The central interpretation of DRAM+ and Theorem 4.6 is that any estimation guarantees for discrete distribution can be immediately translated to mechanism regret guarantees. When the principal starts with assumptions or prior beliefs on the distribution $p^*(x_j | x_i)$, it can apply a different estimator to exploit that knowledge.

We note that the design of DRAM and DRAM+ is conceptually related to the multi-armed bandit algorithm from [Simchi-Levi and Xu, 2022]. In fact, both DRAM and DRAM+ are partly inspired by their insight that an online contextual bandit problem can be reduced to an offline regression problem. Compared to the classical multi-armed bandit, the adaptive mechanism design problem is, in some sense, both simpler and harder. On the one hand, the core challenge in bandit problems lies in the exploration-exploitation trade-off, since the arm-sampling policy in previous rounds affects observed data distributions. From that perspective, the mechanism design problem is simpler than bandits, since the underlying distributions remain unaffected by the principal's mechanism decision as long as players behave truthfully. On the other hand, when participants are rational, incentivizing truthfulness and doing it optimally is nontrivial. Players' incentives and skills are unknown, and any deviation can cause unpredictable dynamics. In contrast, there are no incentives involved in bandits, and each arm always gives truthful feedback. Nonetheless, the connection to [Simchi-Levi and Xu, 2022], together with Theorem 4.6, suggests that a principled reduction from online mechanism design to offline learning may indeed be possible.

4.3 Discussions

We collect some interesting observations from the Algorithm 1 and 2 and their corresponding guarantees (Theorem 4.2 and 4.6). These observations further demonstrate the generality of our results.

Truthfulness is weakly necessary for cost-optimality. Reducing cost of robustness requires accurate knowledge on the posterior distribution of peer's observation $p^*(x_j | x_i)$. Therefore, to ensure estimation is not biased, all players should be incentivized to stay truthful at all times. This provides yet another evidence on the necessity of truthfulness, in addition to Proposition 2.1.

Robustness to fluctuation/non-stationarity of player performance. We assume each player's skill (i.e., law $p_i(x_i | y)$) is consistent throughout the sequential tasks, an assumption not necessarily true in practice. Players may under or over-perform in certain rounds compared to

other rounds, resulting in skill fluctuation and non-stationarity. In DRAM, we apply a distributionally robust mechanism each round. This robustness holds not only for estimation inaccuracy, but also to inaccuracy from other sources. This means as long as the actual reference distribution stays within the ambiguity set defined by \hat{p}_{ik} and η_k , players are still incentivized to stay truthful. Indeed, the principal could even widen up or narrow the ambiguity set by adjusting η_k , looking for more robustness or less cost.

Robustness to adversary. The distributionally robust mechanism would also provide robustness to adversarial behavior from players. When a player intentionally lies in a small portion of rounds, it would only slightly bias the estimation. As long as it does not surpass the ambiguity margin as designed in each epoch, the mechanism would not break down. In addition, the assignment procedure means an adversary would at most disrupt at most $2T$ out of NT player–task interactions in total (being one focal player and one reference player), possibly spread across different players and tasks. Of course, here we only talk about the mechanism’s robustness, and an adversary would also contaminate the collected data, potentially biasing subsequent inference or learning procedures. For defending against adversaries in downstream tasks such as aggregation, there is a line of research that focuses on robust estimators and aggregators [Arieli et al., 2018, Caragiannis et al., 2016, Gibbard, 1973, Moulin, 1980].

Flexibility with reference player assignment procedure. In DRAM, each player is assigned one corresponding reference player j , to which her reports will be compared. Any assignment procedure (deterministic or randomized) could be used for this process, and some could provide robustness to possible adverse players. As an example, suppose we use cyclic matching, where player $i + 1$ is assigned to player i as reference for $i < N$, and player 1 is assigned to player N . Under cyclic matching, any adversary would disrupt at most two players, and the majority is not affected. Furthermore, at the beginning of each epoch, we could rerun the procedure and assign new players. Such replacement generates little extra computational costs, since we need to update the estimation and regenerate mechanism anyway.

Compatibility with delayed/batched feedback. In the practical setting, the feedback to the principal might not be immediately available and may come in batches [Chapelle and Li, 2011, McMahan et al., 2013]. The delayed/batched feedback setting has been studied in multiple online learning and decision-making problems [Gao et al., 2019, Joulani et al., 2013]. In DRAM, the mechanisms are computed at the beginning of each epoch, and stay the same throughout. This means DRAM naturally handles delayed and batched feedback since report data is only required for computation at the beginning of each epoch. Particularly, Corollary 4.3 suggests that $O(\log \log T)$ epochs are already sufficient for the $O(\sqrt{T})$ bound up to logarithmic terms. Nevertheless, such small epoch counts rely on a carefully designed epoch schedule. For example, DRAM uses a geometric epoch schedule, under which it quickly adapts early on, and slows down when sufficient data are gained. Deviation from the $O(\sqrt{T})$ bound may appear when the principal faces a different epoch schedule constraint [Perchet et al., 2016].

5 Experiments

In this section, we perform numerical experiments to verify and demonstrate the effectiveness of our proposed algorithm.

Environment. We consider a sequential labeling game (as in Figure 1 and Example 3.1) with $N = 3$ players and $d = 3$ labels with a uniform prior $p_Y(y) = 1/3$. Each player i has a diagonally-dominant skill distribution $p_i(\cdot | y)$ that is symmetric across labels:

$$p_i(x | y) = \begin{cases} \alpha_i, & x = y, \\ \frac{1-\alpha_i}{d-1}, & x \neq y, \end{cases} \quad \alpha_i = 0.7 + \left(i - \frac{N-1}{2}\right) \cdot 0.02.$$

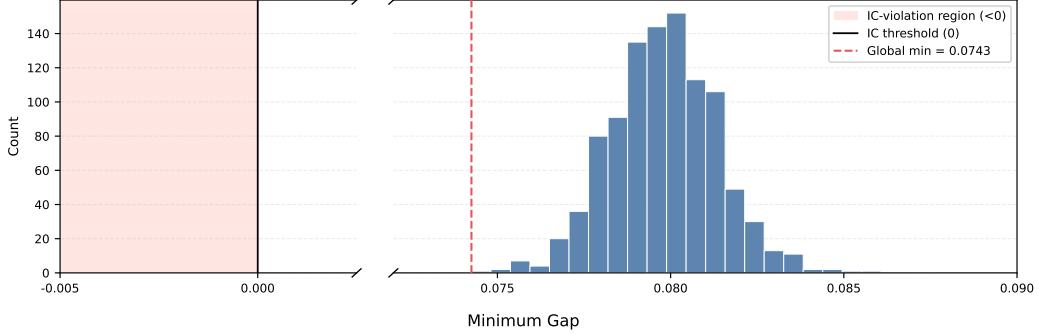


Fig. 2. Minimum reward gap between truthful reporting and other pure strategies across 1000 runs of a sequential labeling game. Negative gap means the constraints are violated. In this simulation, the minimum gap distribution is well separated from 0, meaning that truthful reporting dominates other strategies by a considerable margin, and DRAM guarantees truthfulness even with spare robustness.

Thus for $N = 3$, $\alpha_0 = 0.68$, $\alpha_1 = 0.70$, and $\alpha_2 = 0.72$, with the remaining probability mass spread uniformly over the $d - 1$ incorrect labels. We use horizon $T = 10^6$ and observation cost $c = 0.3$. During warm-start, the principal acquires the ground-truth label Y_t from an external expert at cost $C_{\text{lab}} = 3.0$ per round. We run 1000 independent episodes.

Algorithm setup. We implement the exact DRAM algorithm (Algorithm 1) in this simulation. To match the theoretical parameterization, we compute

$$\rho_{\text{true}} = \min_{i, x \in \mathcal{Y}} \mathbb{P}(X_i = x), \quad \gamma_{(i)} = \max_{x \in \mathcal{Y}} \mathbb{P}(X_i = x),$$

and the player-wise robustness thresholds $\tilde{\eta}_i$ from Theorem 3.7, then set

$$\tilde{\eta}_{\text{true}} = \min_i \tilde{\eta}_i, \quad \tilde{\eta}_{\text{used}} = \min \left(0.9 \tilde{\eta}_{\text{true}}, 1/\sqrt{2} \right), \quad \rho_{\text{used}} = 0.99 \rho_{\text{true}}.$$

Given $\varepsilon = 10^{-3}$, we plug $(\tilde{\eta}_{\text{used}}, \rho_{\text{used}})$ into the warm-start length formula in Algorithm 1 to obtain τ . For this setting, τ is on the order of 10^5 , so the warm-start phase occupies only a moderate fraction of the horizon. In the warm-start phase, we use the simple fact-checking mechanism: reward both player 1 if their report agrees, and 0 if not.

Truthfulness checks. We verify truthfulness via a retrospective approach. We set all participating players to be always truthful. At the beginning of every epoch, we perform a truthfulness check using the true joint distribution $p^*(X_i, X_j)$.

We compute the truthful expected utility

$$U_{ik}^{\text{truth}} = \mathbb{E}_{(X_i, X_j) \sim p^*} [R_{ik}(X_i, X_j)] - c,$$

and compare it against two families of deviations:

- **Lazy strategies:** the best constant report (without observation) $z \in \mathcal{Y}$ with no observation cost,

$$U_{ik}^{\text{lazy}} = \max_{z \in \mathcal{Y}} \mathbb{E}_{X_j \sim p^*} [R_{ik}(z, X_j)].$$

- **Misreporting strategies:** all deterministic mappings from observation to $g : \mathcal{Y} \rightarrow \mathcal{Y}$ excluding the identity,

$$U_{ik}^g = \mathbb{E}_{(X_i, X_j) \sim p^*} [R_{ik}(g(X_i), X_j)] - c.$$

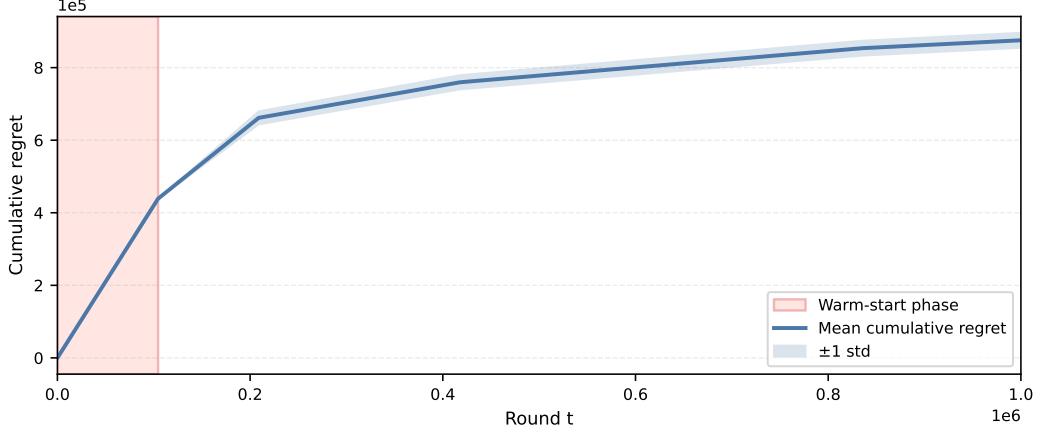


Fig. 3. Average cumulative regret over time across 1000 runs of a sequential labeling game. The first $\sim 10^5$ rounds are warm-start phase, and then comes with doubling epoch lengths. The regret curve is piecewise linear, as the expected round-wise regret within each epoch stays unchanged. The geometric epoch schedule ensures $O(\sqrt{T})$ regret.

We define the IC gap for player i in epoch k as

$$\text{Gap}_{ik} = U_{ik}^{\text{truth}} - \max \left\{ U_{ik}^{\text{lazy}}, \max_{g \neq \text{id}} U_{ik}^g \right\},$$

and, for each episode, record the minimum Gap_{ik} across all players and epochs. If any Gap_{ik} is negative, we count the episode as an IC violation. Figure 2 shows the histogram of per-episode minimum IC gaps.

Regret checks. We collect the cumulative regret at each round within each episode of the game. In addition to the regret notion defined in Section 4, we include the warm-start verification cost in the regret formula. Specifically, we plot the cumulative regret up to time t :

$$\text{Reg}(t) = \sum_{s=1}^t \left(\sum_{i=1}^N R_{is} - Nc + \mathbf{1}\{s \leq \tau\} C_{\text{lab}} \right),$$

and report the mean and standard deviation across 1000 episodes. Figure 3 shows the resulting regret trajectory.

Results. DRAM passes the truthfulness checks as across the 1000 episodes, we observe *no* truthfulness violations. The global minimum gap is approximately $0.0743 > 0$, and the distribution of per-episode minimum gaps is well separated from zero. This indicates that, in a setting that exactly matches our assumptions and with theoretically chosen parameters (τ, δ) , DRAM indeed implements a truthful mechanism in practice.

DRAM also consistently achieves the $O(\sqrt{T})$ regret, as shown in Figure 3. In this simulation we have 5 epochs in total. Within each epoch, the mechanism stays unchanged, therefore the cumulative regret curve is piecewise linear. In summary, this experiment demonstrates the efficiency and robustness of the vanilla DRAM algorithm, and validates the correctness of Theorem 4.2. In fact, the existence of extra IC gap seems to suggest that further refinement are possible, as the current parameters are set for theoretical proofs rather than optimized for practical implementations.

6 Conclusions

In this paper, we designed an adaptive mechanism for the sequential mechanism design problem. The studied problem assumes rational feedback compared to the prediction with expert advice problem from online learning, and relaxes the common knowledge assumption compared to the peer prediction problem from mechanism design. Drawing insights from both fields, our proposed mechanism ensures truthful behaviors with high probability, while achieving optimal payment regret. It also remains robust and adaptable in changing environments.

Looking forward, our work motivates interesting questions. In Section 3, the mechanism design problem is formulated as a linear optimization problem, with truthfulness encoded as constraints. A key idea of our algorithm is to solve a distributionally robust variant of this problem while gradually learning the relevant constraints over time. This principle seems to be broadly applicable: since many decision-making problems can be cast as optimization tasks, the same approach might extend naturally to online, adaptive, or sequential variants of other real-world decision-making problems beyond mechanism design.

Acknowledgments

The authors thank Rui Ai, Jiachun Li, Chonghuan Wang, Yunzong Xu, Yuan Zhou for helpful comments and suggestions.

Author Contributions

Author ordering alphabetical. All authors made valuable contributions to the writing, editing, and overall management of the project.

Renfei Tan led the project, and is the first to propose the idea of achieving cost-efficient adaptive mechanisms via sequentially accurate distributionally robust mechanisms. He is the main developer of the modeling, algorithm, and corresponding theorems, as well as the main writer of the paper.

Zishuo Zhao contributed on discussions, proofreading, and comprehensive review. He proposed the main idea of distributionally robust mechanisms with insights on the synergy between it and online learning. He also helped formulating the examples and wrote the literature review part of peer prediction.

Qiushi Han proposed the initial idea of the two-phased (warm-start and adaptive) approach to relax the common knowledge assumptions and contributed to the development of the main algorithm. He led the design and conduct of the numerical experiments in this work.

David Simchi-Levi supervised the research and assisted in writing the paper.

References

- Naman Agarwal, Brian Bullins, and Elad Hazan. 2017. Second-order stochastic optimization for machine learning in linear time. *Journal of Machine Learning Research* 18, 116 (2017), 1–40.
- Itai Arieli, Yakov Babichenko, and Rann Smorodinsky. 2018. Robust forecast aggregation. *Proceedings of the National Academy of Sciences* 115, 52 (2018), E12135–E12143.
- Dirk Bergemann and Stephen Morris. 2005. Robust mechanism design. *Econometrica* (2005), 1771–1813.
- Omar Besbes, Yonatan Gur, and Assaf Zeevi. 2016. Optimization in online content recommendation services: Beyond click-through rates. *Manufacturing & Service Operations Management* 18, 1 (2016), 15–33.
- Lilian Besson and Emilie Kaufmann. 2018. What doubling tricks can and can't do for multi-armed bandits. *arXiv preprint arXiv:1803.06971* (2018).
- David Blackwell. 1953. Equivalent comparisons of experiments. *The Annals of Mathematical Statistics* (1953), 265–272.
- Avrim Blum, Vijay Kumar, Atri Rudra, and Felix Wu. 2004. Online learning in online auctions. *Theoretical Computer Science* 324, 2-3 (2004), 137–146.
- Ioannis Caragiannis, Ariel Procaccia, and Nisarg Shah. 2016. Truthful univariate estimators. In *International Conference on Machine Learning*. PMLR, 127–135.

- Rui Castro, Fredrik Hellström, and Tim van Erven. 2023. Adaptive selective sampling for online prediction with experts. *Advances in Neural Information Processing Systems* 36 (2023), 134–154.
- Nicolo Cesa-Bianchi, Alex Conconi, and Claudio Gentile. 2004. On the generalization ability of on-line learning algorithms. *IEEE Transactions on Information Theory* 50, 9 (2004), 2050–2057.
- Nicolo Cesa-Bianchi, Claudio Gentile, and Yishay Mansour. 2014. Regret minimization for reserve prices in second-price auctions. *IEEE Transactions on Information Theory* 61, 1 (2014), 549–564.
- Nicolo Cesa-Bianchi and Gábor Lugosi. 2006. *Prediction, learning, and games*. Cambridge university press.
- Nicolo Cesa-Bianchi, Gábor Lugosi, and Gilles Stoltz. 2005. Minimizing regret with label efficient prediction. *IEEE Transactions on Information Theory* 51, 6 (2005), 2152–2162.
- Nicolo Cesa-Bianchi, Yishay Mansour, and Gilles Stoltz. 2007. Improved second-order bounds for prediction with expert advice. *Machine Learning* 66, 2 (2007), 321–352.
- Olivier Chapelle and Lihong Li. 2011. An empirical evaluation of thompson sampling. *Advances in neural information processing systems* 24 (2011).
- Yiling Chen, Yiheng Shen, and Shuran Zheng. 2020. Truthful data acquisition via peer prediction. *Advances in Neural Information Processing Systems* 33 (2020), 18194–18204.
- Hana Choi, Carl F Mela, Santiago R Balseiro, and Adam Leary. 2020. Online display advertising markets: A literature review and future directions. *Information systems research* 31, 2 (2020), 556–575.
- Thomas M Cover. 1999. *Elements of information theory*. John Wiley & Sons.
- Jacques Crémer and Richard P McLean. 1988. Full extraction of the surplus in Bayesian and dominant strategy auctions. *Econometrica: Journal of the Econometric Society* (1988), 1247–1257.
- Anirban Dasgupta and Arpita Ghosh. 2013. Crowdsourced judgement elicitation with endogenous proficiency. In *Proceedings of the 22nd international conference on World Wide Web*. 319–330.
- Pierre Gaillard, Gilles Stoltz, and Tim Van Erven. 2014. A second-order bound with excess losses. In *Conference on Learning Theory*. PMLR, 176–196.
- Zijun Gao, Yanjun Han, Zhimei Ren, and Zhengqing Zhou. 2019. Batched multi-armed bandits problem. *Advances in Neural Information Processing Systems* 32 (2019).
- Allan Gibbard. 1973. Manipulation of voting schemes: a general result. *Econometrica: journal of the Econometric Society* (1973), 587–601.
- Mohammad Taghi Hajiaghayi, Robert Kleinberg, and David C Parkes. 2004. Adaptive limited-supply online auctions. In *Proceedings of the 5th ACM Conference on Electronic Commerce*. 71–80.
- Yanjun Han, Jiantao Jiao, and Tsachy Weissman. 2015. Minimax estimation of discrete distributions. In *2015 IEEE International Symposium on Information Theory (ISIT)*. IEEE, 2291–2295.
- Sergiu Hart and Andreu Mas-Colell. 2000. A simple adaptive procedure leading to correlated equilibrium. *Econometrica* 68, 5 (2000), 1127–1150.
- David Helmbold and Sandra Panizza. 1997. Some label efficient learning results. In *Proceedings of the tenth annual conference on Computational learning theory*. 218–230.
- Chien-Ju Ho, Aleksandrs Slivkins, and Jennifer Wortman Vaughan. 2014. Adaptive contract design for crowdsourcing markets: Bandit algorithms for repeated principal-agent problems. In *Proceedings of the fifteenth ACM conference on Economics and computation*. 359–376.
- Pooria Joulani, Andras Gyorgy, and Csaba Szepesvári. 2013. Online learning under delayed feedback. In *International conference on machine learning*. PMLR, 1453–1461.
- Çağrı Koçyiğit, Garud Iyengar, Daniel Kuhn, and Wolfram Wiesemann. 2020. Distributionally robust mechanism design. *Management Science* 66, 1 (2020), 159–189.
- Yuqing Kong. 2024. Dominantly truthful peer prediction mechanisms with a finite number of tasks. *J. ACM* 71, 2 (2024), 1–49.
- Yuqing Kong and Grant Schoenebeck. 2019. An information theoretic framework for designing information elicitation mechanisms that reward truth-telling. *ACM Transactions on Economics and Computation (TEAC)* 7, 1 (2019), 1–33.
- Nicolas S Lambert. 2011. Elicitation and evaluation of statistical forecasts. *Preprint* (2011).
- Tor Lattimore and Csaba Szepesvári. 2020. *Bandit algorithms*. Cambridge University Press.
- Yingkai Li, Jason D Hartline, Liren Shan, and Yifan Wu. 2022. Optimization of scoring rules. In *Proceedings of the 23rd ACM Conference on Economics and Computation*. 988–989.
- H Brendan McMahan, Gary Holt, David Sculley, Michael Young, Dietmar Ebner, Julian Grady, Lan Nie, Todd Phillips, Eugene Davydov, Daniel Golovin, et al. 2013. Ad click prediction: a view from the trenches. In *Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining*. 1222–1230.
- Paul Milgrom. 2019. Auction market design: Recent innovations. *Annual Review of Economics* 11, 1 (2019), 383–405.
- Nolan Miller, Paul Resnick, and Richard Zeckhauser. 2005. Eliciting informative feedback: The peer-prediction method. *Management Science* 51, 9 (2005), 1359–1373.

- Siddharth Mitra and Aditya Gopalan. 2020. On adaptivity in information-constrained online learning. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 34. 5199–5206.
- Hervé Moulin. 1980. On strategy-proofness and single peakedness. *Public Choice* 35, 4 (1980), 437–455.
- Roger B Myerson. 1979. Incentive compatibility and the bargaining problem. *Econometrica* (1979), 61–73.
- Roger B Myerson. 1981. Optimal auction design. *Mathematics of operations research* 6, 1 (1981), 58–73.
- Christos Papadimitriou, George Pierrakos, Alexandros Psomas, and Aviad Rubinstein. 2022. On the complexity of dynamic mechanism design. *Games and Economic Behavior* 134 (2022), 399–427.
- Vianney Perchet, Philippe Rigollet, Sylvain Chassang, and Eric Snowberg. 2016. Batched Bandit Problems. *The Annals of Statistics* 44, 2 (2016), 660–681.
- Goran Radanovic and Boi Faltings. 2013. A robust bayesian truth serum for non-binary signals. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 27. 833–839.
- Tim Roughgarden. 2010. Algorithmic game theory. *Commun. ACM* 53, 7 (2010), 78–86.
- Bharadwaj Satchidanandan and Munther A Dahleh. 2023. Incentive compatibility in two-stage repeated stochastic games. *IEEE Transactions on Control of Network Systems* 11, 1 (2023), 295–306.
- Victor Shnayder, Arpit Agarwal, Rafael Frongillo, and David C Parkes. 2016. Informed truthfulness in multi-task peer prediction. In *Proceedings of the 2016 ACM Conference on Economics and Computation*. 179–196.
- David Simchi-Levi and Yunzong Xu. 2022. Bypassing the monster: A faster and simpler optimal algorithm for contextual bandits under realizability. *Mathematics of Operations Research* 47, 3 (2022), 1904–1931.
- John Von Neumann and Oskar Morgenstern. 2007. Theory of games and economic behavior: 60th anniversary commemorative edition. In *Theory of games and economic behavior*. Princeton university press.
- Jun Wang, Weinan Zhang, Shuai Yuan, et al. 2017. Display advertising with real-time bidding (RTB) and behavioural targeting. *Foundations and Trends® in Information Retrieval* 11, 4-5 (2017), 297–435.
- Shengling Wang, Xidi Qu, Qin Hu, Xia Wang, and Xiuzhen Cheng. 2023. An uncertainty-and collusion-proof voting consensus mechanism in blockchain. *IEEE/ACM Transactions on Networking* 31, 5 (2023), 2376–2388.
- Tsachy Weissman, Erik Ordentlich, Gadiel Seroussi, Sergio Verdu, and Marcelo J Weinberger. 2003. Inequalities for the l_1 deviation of the empirical distribution. *Hewlett-Packard Labs, Tech. Rep* (2003), 125.
- Robert Wilson. 1985. *Game-Theoretic Analysis of Trading Processes*. Technical Report.
- Yichi Zhang, Shengwei Xu, David Pennock, and Grant Schoenebeck. 2025. Stochastically Dominant Peer Prediction. *arXiv preprint arXiv:2506.02259* (2025).
- Zishuo Zhao, Xi Chen, and Yuan Zhou. 2024. It Takes Two: A Peer-Prediction Solution for Blockchain Verifier’s Dilemma. *arXiv preprint arXiv:2406.01794* (2024).
- Shuran Zheng, Xuan Qi, Rui Ray Chen, Yongchan Kwon, and James Zou. 2024. Proper Dataset Valuation by Pointwise Mutual Information. *arXiv preprint arXiv:2405.18253* (2024).
- Banghua Zhu, Stephen Bates, Zhuoran Yang, Yixin Wang, Jiantao Jiao, and Michael I Jordan. 2022. The sample complexity of online contract design. *arXiv preprint arXiv:2211.05732* (2022).

Appendix

A Deferred Proofs

A.1 Proof of Proposition 2.1

Suppose the past rounds' report history is $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{t-1}$. All probability laws and strategies discussed below are conditional on such history.

In round t , denote player i 's *report strategy* by $s_{it} : \mathcal{Y} \rightarrow \Delta(\mathcal{Y})$, where $s_{it}(z | x)$ is the probability to report z given x . Define player i 's *information structure* σ_{it} as her conditional probability law to report z conditional on true label y . Then we have

$$\sigma_{it}(z | y) = \sum_{x \in \mathcal{Y}} s_{it}(z | x) p_i(x | y).$$

Alternatively, we can write $\sigma = s_{it} \circ p_i$ as the information structure is induced by strategy s_{it} . We let $\sigma = (\sigma_{1t}, \dots, \sigma_{Nt})$ be the information structure profile of all players. Suppose the principal observes report $\mathbf{z}_t = (z_{1t}, \dots, z_{Nt-1})$. The principal's optimal expected accuracy under such report is

$$\begin{aligned} V(\mathbf{z}, \sigma) &= \max_{A_t \in A} \sum_{y \in \mathcal{Y}} \mathbb{P}(A_t = y | Y_t = y, Z_t = \mathbf{z}) \mathbb{P}(Y_t = y | Z_t = \mathbf{z}) \\ &= \max_{A_t \in A} \sum_{y \in \mathcal{Y}} \mathbb{P}(A_t = y | Z_t = \mathbf{z}) \mathbb{P}(Y_t = y | Z_t = \mathbf{z}) \end{aligned}$$

since A_t and Y_t are independent conditional on Z_t . Note that conditional law $\mathbb{P}(Y_t = y | Z_t = \mathbf{z})$ is decided by and only by players' skills p_i and report strategy s_{it} . Then the optimal expected accuracy under σ is

$$W(\sigma) = \sum_{\mathbf{z} \in \mathcal{Y}^N} V(\mathbf{z}, \sigma) \mathbb{P}(Z_t = \mathbf{z}).$$

We now state the Blackwell's informativeness theorem. The theorem states three equivalent conditions, of which we only use the following two.

LEMMA A.1 ([BLACKWELL, 1953]). *Suppose we have two information structure profiles σ and σ' , the following two conditions are equivalent:*

- $W(\sigma) \leq W(\sigma')$. The decision-maker attains a higher expected utility under σ than under σ' ,
- There exists a stochastic map Γ such that $\sigma' = \Gamma \circ \sigma$. That is, σ' is a garbling of σ .

Note that this lemma applies for general σ , even those where player i 's report may depend on other players' observation. In our setting, we are only concerned with the subset of structures where each player's report is independent from others' observation.

Sufficiency. Define a permutation strategy π as a bijection of the set \mathcal{Y} onto itself. (This includes the identity mapping, or the truthful strategy, $\pi(z) = z$.) Any report strategy s is a garbling of π , since $s = (s \circ \pi^{-1}) \circ \pi$. Therefore the corresponding information structure $\sigma'_i = s \circ p_i$ is also a garbling of $\sigma_i = \pi \circ p_i$. This applies for all i , therefore the induced information structure profile σ_s is also a garbling for σ_π . From Lemma A.1, we know for any strategy profile s , we would have $W(\sigma_s) \leq W(\sigma_\pi)$, meaning under permutation strategy we achieve maximal attainable accuracy.

Necessity. Since we have a information structure profile σ that achieves maximal accuracy, from Lemma A.1 we know that for any player i , any information structure induced by a strategy $\sigma'_i = s' \circ p_i$ must be a garbling of $\sigma_i = s \circ p_i$. We now show that such property indicates that s must be a permutation strategy π .

First, we prove that such a strategy cannot have overlapped labels, meaning there does not exist $z \in \mathcal{Y}$, such that both $s(z | x_1)$ and $s(z | x_2)$ are greater than 0 for some x_1, x_2 . Because p_i can be arbitrary, it means any strategy s' is a garbling of s . Consider the truthful strategy $s'(z | x) = \mathbf{1}\{z = x\}$, we would have

$$1 = s'(x | x) = \sum_{z \in \mathcal{Y}} \Gamma(x | z) s(z | x).$$

Since $\sum_{z \in \mathcal{Y}} s(z | x) = 1$, for each z where $s(z | x) > 0$, we must have $\Gamma(x | z) = 1$, otherwise the sum of weighted average would fall short of 1. Now, assume we have such overlapped labels z and the corresponding x_1 and x_2 , it would mean that $\Gamma(x_1 | z) = \Gamma(x_2 | z) = 1$. But $\Gamma(\cdot | z)$ is a single probability distribution, so it cannot assign probability 1 to two different outcomes simultaneously, forming a contradiction and we prove the overlapped signals cannot exist.

Since s cannot have overlapped labels, by counting we know each observation must be mapped to one and only one label, meaning it is a permutation strategy.

Suboptimality of laziness. Finally, we show that the lazy option (directly report according to a prior belief \hat{p}_Y) is strictly worse than observation with permutation strategy regardless of \hat{p}_Y used. The induced information structure is $\sigma_Y(z | y) = \hat{p}_Y(z)$, and the information structure from a permutation strategy is $\sigma_\pi = \pi \circ p_i$. Actually $\Gamma = \sigma_Y \circ \pi^{-1}$ makes σ_Y a garbling of σ_π , since $\sigma_Y \circ \pi^{-1} \circ \pi \circ p_i = \sigma_Y \circ p_i = \sigma_Y$. However, σ_π is not a garbling of σ_Y since the corresponding row-stochastic matrix of σ_Y is rank 1 but because of non-degeneracy σ_π is at least rank 2. Therefore Lemma A.1 suggests lazy option is strictly dominated by observation with permutation strategy.

□

A.2 Proof of Theorem 3.2

Feasibility. Suppose \mathbf{B} is invertible. Then for arbitrary matrix \mathbf{M} , there exists $\mathbf{R} = (\mathbf{B}^{-1}\mathbf{M})^\top$ such that $\mathbf{B}\mathbf{R}^\top = \mathbf{M}$. Notice that the entry \mathbf{M}_{xy} is exactly player 1's expected reward given she observes label x and reports label y . Hence for our purposes, we can construct an \mathbf{M} whose diagonal entries are greater than c , and off-diagonal entries are less than c , then the corresponding \mathbf{R} satisfies the first two constraints. (Actually, if \mathbf{B} is invertible, the first two constraints of (2) can be satisfied for arbitrary right-hand side values.)

Now we consider the third constraint $\mathbf{R}\mathbf{d} \leq \mathbf{0}$. Notice that we have $\mathbf{d}^\top = \sum_x \mathbb{P}(X_i = x) \cdot \mathbf{B}_{x:}$. Therefore letting $\mathbf{B}\mathbf{R}^\top = \mathbf{M}$, we have

$$\begin{aligned} \mathbf{R}\mathbf{d} &= \sum_x \mathbb{P}(X_i = x) \cdot \mathbf{R}(\mathbf{B}_{x:})^\top \\ &= \sum_x \mathbb{P}(X_i = x) \cdot (\mathbf{M}_{x:})^\top. \end{aligned}$$

Therefore, to satisfy all the three constraints, we need to find a matrix \mathbf{M} whose linear combinations of its rows under coefficients $\{\mathbb{P}(X_i = x)\}_{x \in \mathcal{Y}}$ yield a vector with non-positive entries. Let $\gamma = \max_x \mathbb{P}(X_i = x)$, we know that $\gamma < 1$. Then if we let all diagonal values of \mathbf{M} be c , and all off-diagonal values of \mathbf{M} equals $-c\gamma/(1 - \gamma)$, then we would have for all $x' \in \mathcal{Y}$,

$$\begin{aligned} (\mathbf{R}\mathbf{d})_{x'} &= \sum_x \mathbb{P}(X_i = x) \cdot (\mathbf{M}_{xx'})^\top \\ &= \mathbb{P}(X_i = x') \cdot c + (1 - \mathbb{P}(X_i = x')) \cdot (-c\gamma/(1 - \gamma)) \\ &\leq \gamma \cdot c + (1 - \gamma) \cdot (-c\gamma/(1 - \gamma)) \\ &= 0. \end{aligned}$$

Hence such a matrix \mathbf{M} exists and the corresponding \mathbf{R} is a feasible solution.

Optimality. Notice that the objective is essentially $\sum_x \mathbb{P}(X_i = x) \mathbf{M}_{xx}$. Since we constructed \mathbf{M} with all diagonal values being c , the objective value is c . The first constraint is binding. Smaller objective is not possible as it would require $\mathbf{M}_{xx} < c$ for some x , violating the first constraint.

□

A.3 Proof of Theorem 3.4

It is more convenient to use the matrix notation (see (2)). The condition is essentially saying $\|\mathbf{B} - \mathbf{B}^*\|_\infty \leq \delta/\kappa$ and $\|\mathbf{d} - \mathbf{d}^*\|_1 \leq \delta/\kappa$, where $\|\cdot\|_\infty$ is the matrix norm induced by vector ∞ -norm. (It is essentially the maximum absolute row sum of the matrix.)

Therefore, we have

$$\begin{aligned} \max_{x,y} |\mathbf{B}\mathbf{R}^\top - \mathbf{B}^*\mathbf{R}^\top|_{xy} &= \max_{x,y} \sum_z (\mathbf{B} - \mathbf{B}^*)_{xz} \mathbf{R}_{yz} \\ &\leq \max_x \|(\mathbf{B} - \mathbf{B}^*)_{x:}\|_1 \cdot \kappa \\ &\leq 2 \cdot (\delta/2\kappa) \cdot \kappa \\ &= \delta. \end{aligned}$$

Here it is crucial to notice that $\|(\mathbf{B} - \mathbf{B}^*)_{x:}\|_1 = 2 \text{TV}(p(\cdot | x_i), p^*(\cdot | x_i))$. Similarly, we can show that $\|\mathbf{R}\mathbf{d}\|_\infty \leq \delta$.

Therefore, the constraints in (4) shift by at most δ , which means the δ -margin mechanism R_i satisfies (1).

□

A.4 Proof of Theorem 3.5

Worst-case payment. We still use the matrix formulation for the problem (see (2)). Under this notation, the problem becomes

$$\begin{aligned} \min_{\mathbf{R}} \quad & \kappa \\ \text{s.t.} \quad & \|\mathbf{R}\|_{\max} \leq \kappa, \\ & (\mathbf{B}\mathbf{R}^\top)_{xx} \geq c + \delta, \quad \forall x \in \mathcal{Y} \\ & (\mathbf{B}\mathbf{R}^\top)_{xy} \leq c - \delta, \quad \forall x \neq y \in \mathcal{Y} \\ & \mathbf{R}\mathbf{d} \leq -\delta \cdot \mathbf{1}. \end{aligned}$$

We call this problem $\text{LP}(p, c, \delta)$, since it is a linear programming problem with distribution p , cost c and margin δ . Notice that if (κ, \mathbf{R}) is a feasible solution to $\text{LP}(p, c, 0)$, and (κ', \mathbf{R}') is a feasible solution to $\text{LP}(p, 0, 1)$, then $(\kappa + \delta\kappa', \mathbf{R} + \delta\mathbf{R}')$ is a feasible solution to $\text{LP}(p, c, \delta)$. Therefore, we can construct upper bounds of $\text{LP}(p, c, \delta)$ by constructing upper bounds on $\text{LP}(p, c, 0)$ and $\text{LP}(p, 0, 1)$ separately. We apply the same strategy as proof of Theorem 3.2, that is, to consider the intermediate solution $\mathbf{M} = \mathbf{B}\mathbf{R}^\top$. The mechanism can be easily acquired by $\mathbf{R} = (\mathbf{B}^{-1}\mathbf{M})^\top$. With this reformulation (see Section A.2 for details), the problem can be constructed as

$$\begin{aligned} \min_{\mathbf{M}} \quad & \kappa \\ \text{s.t.} \quad & \|\mathbf{B}^{-1}\mathbf{M}\|_{\max} \leq \kappa, \\ & \mathbf{M}_{xx} \geq c + \delta, \quad \forall x \in \mathcal{Y} \\ & \mathbf{M}_{xy} \leq c - \delta, \quad \forall x \neq y \in \mathcal{Y} \\ & \mathbf{M}^\top \mathbf{d}' \leq -\delta \cdot \mathbf{1}, \end{aligned}$$

where $\mathbf{d}'_x = \mathbb{P}(X_i = x)$.

The lower bound is apparent, since when absolute maximal payment goes under $c + \delta$ we violate the constraint $\mathbf{M}_{xx} \geq c + \delta$.

Upper bounds of LP(p, c, 0). Following the same construction as Appendix A.2, we let \mathbf{M} has all diagonal values being c , and all off-diagonals equal $-c\gamma/(1 - \gamma)$, where $\gamma = \max_x \mathbb{P}(X_i = x)$. Then \mathbf{M} satisfies the three constraints on matrix. With this \mathbf{M} , we have

$$\begin{aligned}\|\mathbf{B}^{-1}\mathbf{M}\|_{\max} &\leq \|\mathbf{B}^{-1}\mathbf{M}\|_2 \leq \|\mathbf{B}^{-1}\|_2 \|\mathbf{M}\|_2 \\ &= \|\mathbf{B}^{-1}\|_2 \cdot \frac{c}{1 - \gamma} \max(1, |1 - \gamma \cdot |\mathcal{Y}||) \\ &\leq \|\mathbf{B}^{-1}\|_2 \cdot \frac{c(\gamma|\mathcal{Y}| + 1)}{1 - \gamma}.\end{aligned}$$

Here, all eigenvalues of $\|\mathbf{M}\|_2$ can be easily calculated since \mathbf{M} is a combination of identity matrix \mathbf{I} and all-ones matrix \mathbf{J} , whose eigenvalues are known. In the end we can take $\kappa \leq \|\mathbf{B}^{-1}\|_2 \cdot c(\gamma|\mathcal{Y}| + 1)/(1 - \gamma)$.

Upper bounds of LP(p, 0, 1). Similarly, construct \mathbf{M}' with diagonal 1 and off-diagonal $-(1 + \gamma)/(1 - \gamma)$. This \mathbf{M}' satisfies all three constraints on matrix. A similar argument gives us

$$\kappa' \leq \|\mathbf{B}^{-1}\|_2 \cdot \frac{(1 + \gamma)|\mathcal{Y}| + 2}{1 - \gamma}.$$

Combining the two upper bounds, it means that $(\kappa + \delta\kappa', \mathbf{M} + \delta\mathbf{M}')$ is a feasible solution, and we end up with an upper bound on $\text{LP}(p, c, \delta)$, which is:

$$\kappa \leq \|\mathbf{B}^{-1}\|_2 \cdot \left(c \cdot \frac{\gamma|\mathcal{Y}| + 1}{1 - \gamma} + \delta \cdot \frac{(1 + \gamma)|\mathcal{Y}| + 2}{1 - \gamma} \right).$$

Expected payment. The solution $(\kappa + \delta\kappa', \mathbf{M} + \delta\mathbf{M}')$ ensures that the constraint $\mathbf{M}_{xx} \geq c + \delta$ is binding. Therefore, the expected payment under truthful equilibrium is $\mathbb{E}_p[R_i(X_i, X_j)] = \sum_x \mathbb{P}(X_i = x) \mathbb{E}_p[R_i(X_i, X_j) | X_i] = c + \delta$.

□

A.5 Proof of Theorem 3.7

To ensure player i stays truthful, from Theorem 3.4 we need to find a margin δ such that $\delta/2\kappa^* \geq \eta$. We consider the best case $\delta/2\kappa^* = \eta$. Combining with Theorem 3.5 leads to

$$\begin{aligned}\delta &= 2\kappa^* \eta \\ &\leq 2\|\mathbf{B}^{-1}\|_2 \cdot \left(c \cdot \frac{\gamma|\mathcal{Y}| + 1}{1 - \gamma} + \delta \cdot \frac{(1 + \gamma)|\mathcal{Y}| + 2}{1 - \gamma} \right).\end{aligned}$$

This gives us

$$\delta \leq \frac{2\|\mathbf{B}^{-1}\|_2(\gamma|\mathcal{Y}| + 1) \cdot \eta}{(1 - \gamma) - 2\|\mathbf{B}^{-1}\|_2((1 + \gamma)|\mathcal{Y}| + 2) \cdot \eta} \cdot c$$

And under this margin we have a robust mechanism that guarantees truthfulness.

Note that the second part of Theorem 3.5 tells there exists a mechanism that ensures the above bound holds, while making sure the expected payment of truthful equilibrium under p is $c + \delta$. So if the actual distribution p^* is in the required ambiguity set, it shifts this expected payment by at most an additional δ , making the final expected payment at most $c + 2\delta$. Combining it with the upper bound gives us the final result.

□

A.6 Proof of Lemma 4.1

Suppose player i 's observation $X_{it} = x$. Then from Bayes' rule we have

$$\mathbb{P}(Y_t = x \mid X_{it} = x) = p_i(x \mid x) p_Y(x) / \mathbb{P}(X_{it} = x) \geq \underline{p} \cdot p_i(x \mid x) / \mathbb{P}(X_{it} = x)$$

$$\mathbb{P}(Y_t = y \mid X_{it} = x) = p_i(x \mid y) p_Y(y) / \mathbb{P}(X_{it} = x) \leq \bar{p} \cdot p_i(x \mid x) / \mathbb{P}(X_{it} = x)$$

for any $y \in \mathcal{Y}$. The diagonal dominance then implies that $\mathbb{P}(Y_t = x \mid X_{it} = x) \geq \mathbb{P}(Y_t = y \mid X_{it} = x)$ for any $y \in \mathcal{Y}$. Hence a truthful strategy $Z_{it} = X_{it}$ uniquely maximizes payoff under $\mathbf{1}\{Z_{it} = Y_t\}$.

□

A.7 Proof of Theorem 4.2

Estimation of the reference distribution $p^*(x_j \mid x_i)$. As suggested in Section 3, optimizing the cost of mechanism relies on accurate knowledge over the reference distribution $p^*(x_j \mid x_i)$. Therefore, we first focus on accurate estimations on this distribution using players reports. Throughout this part, players' reports are assumed to be truthful, i.e. $z_i = x_i$, so intuitively speaking principal should faithfully recover p^* if given enough data.

We begin with a lemma on concentration bound on using the empirical estimator for a discrete distribution. Let q be a discrete distribution on sample space \mathcal{Y} , from which we obtain t i.i.d. samples. Let \hat{q} be the empirical probability distribution where $\hat{q}_t(y) = t_y/t$. Here t_y is number of times label y appears in the t samples. We also define $d = |\mathcal{Y}|$.

LEMMA A.2 (CONCENTRATION INEQUALITY OF THE EMPIRICAL DISTRIBUTION [WEISSMAN ET AL., 2003]). *For all $\eta > 0$, we have*

$$\mathbb{P}(\text{TV}(q, \hat{q}_t) \geq \eta) \leq (2^d - 2) \exp(-t\varphi(\pi_q)\eta^2) \leq (2^d - 2) \exp(-2t\eta^2),$$

where $\varphi(x) = \log((1-x)/x)/(1-2x)$ with $\varphi(1/2) = 2$, and $\pi_q = \max_{A \subseteq \mathcal{Y}} \min(\mathbb{P}(A), 1 - \mathbb{P}(A))$.

Lemma A.2 gives an concentration inequality on the empirical distribution. We now apply this lemma to derive a concentration bound on estimating *conditional* distribution using the empirical *conditional* distribution estimator. The empirical *conditional* distribution, defined as $\hat{p}_t(x_j \mid x_i) = t_{x_j|x_i}/t_{x_i}$, is what we ended up using in Algorithm 1.

LEMMA A.3 (CONCENTRATION PROPERTY OF THE EMPIRICAL CONDITIONAL DISTRIBUTION). *Suppose the principal has received T rounds of reports from player i and j . Assuming players are always truthful. Let $\hat{p}(x_j \mid x_i)$ be the empirical conditional distribution defined in Algorithm 1. Define the ambiguity set with ambiguity level η as*

$$S_\eta(\hat{p}) = \{p \in \mathcal{P} \mid \text{TV}(\hat{p}(\cdot \mid x_i), p(\cdot \mid x_i)) \leq \eta, \forall x_i \in \mathcal{Y} \cup \{\emptyset\}\}.$$

If the number of rounds satisfies

$$T \geq \frac{1 + 2\eta^2}{2\rho\eta^2} \log\left(\frac{(d+1)2^d}{\varepsilon}\right),$$

where $\rho = \min_{x \in \mathcal{Y}} \mathbb{P}(X_i = x)$, then with probability at least $1 - \varepsilon$, the true distribution p^* belongs to S_η .

We note the increasing rate of T is on the order of $O(\log(1/\varepsilon)/\eta^2)$ for arbitrary ε and small η . Even when η is large, there is still a threshold $T > O(\log(1/\varepsilon))$ that must be satisfied. This is because there are two possible ways for the event S_η to fail: the first case is when the estimator for a certain conditional distribution is η -away from the true distribution, and the second case is when a certain symbol x_i never appears in i 's report. To ensure the second case does not happen with probability larger than ε , we need T to be large enough.

PROOF. We first consider one label $x_i \in \mathcal{Y}$. Within T rounds, the count of x_i from player i 's report follows a binomial distribution. Let $\rho = \min_{x \in \mathcal{Y}} \mathbb{P}(X_i = x)$.

We then have

$$\begin{aligned} \mathbb{P}(\text{TV}(\hat{p}(\cdot | x_i), p^*(\cdot | x_i)) > \eta) &= \sum_{t=0}^T \mathbb{P}(\text{TV}(\hat{p}(\cdot | x_i), p^*(\cdot | x_i)) > \eta | T_{x_i} = t) \cdot \mathbb{P}(T_{x_i} = t) \\ &\leq \sum_{t=0}^T 2^d \exp(-2t\eta^2) \cdot \binom{T}{t} \rho^t (1-\rho)^{T-t} \\ &= 2^d \binom{T}{t} \sum_{t=0}^T [\rho \exp(-2\eta^2)]^t (1-\rho)^{T-t} \\ &= 2^d [\rho \exp(-2\eta^2) + 1 - \rho]^T. \end{aligned}$$

Utilizing union bound across all $(d+1)$ symbols $x_i \in \mathcal{Y} \cup \{\emptyset\}$ would give us

$$\mathbb{P}(\exists x_i, \text{TV}(\hat{p}(\cdot | x_i), p^*(\cdot | x_i)) > \eta) \leq (d+1)2^d [\rho \exp(-2\eta^2) + 1 - \rho]^T.$$

Inverting this inequality gives that when

$$T \geq \frac{\log((d+1)2^d/\varepsilon)}{-\log(1 - \rho(1 - \exp(-2\eta^2)))},$$

the original event in lemma holds with probability at least $1 - \varepsilon$.

Notice that we have

$$\frac{\log((d+1)2^d/\varepsilon)}{-\log(1 - \rho(1 - \exp(-2\eta^2)))} \leq \frac{\log((d+1)2^d/\varepsilon)}{\rho(1 - \exp(-2\eta^2))} \leq \frac{1+2\eta^2}{2\rho\eta^2} \log\left(\frac{(d+1)2^d}{\varepsilon}\right),$$

where the first inequality holds since $-\log(1-x) \geq x$, and the second holds since $1 - \exp(-x) \geq x/(1+x)$. Thus a sufficient bound is

$$T \geq \frac{1+2\eta^2}{2\rho\eta^2} \log\left(\frac{(d+1)2^d}{\varepsilon}\right).$$

Also, for $\eta < 1/\sqrt{2}$, we have a sufficient bound

$$T \geq \frac{1}{\rho\eta^2} \log\left(\frac{(d+1)2^d}{\varepsilon}\right).$$

□

Warm-starting. The ambiguity threshold $\tilde{\eta}$ is the smallest value across players, on the maximum ambiguity a distributionally robust mechanism can tolerate. For mathematical convenience we update the parameter to let $\tilde{\eta} < 1/\sqrt{2}$, so that we have a cleaner bound in subsequent derivations. The fact-checking mechanism ensures truthfulness because of Lemma 4.1. The length of this phase is $O(\log(N \log T))$, which results in smaller order total regrets even when we have constant regret in each round.

Bounding the regret. Now we focus on the algorithm for a single player i . Consider what happens in epoch k , where $t \in (\tau_{k-1}, \tau_k]$. At the beginning of the epoch, we have $\tau_{k-1} = 2^k \tau$ data points, and we set η_k in a specific way so that

$$\tau_{k-1} \geq \frac{1}{\rho\eta_k^2} \log\left(\frac{(d+1)2^d N \log T}{\varepsilon}\right),$$

thus we know from Lemma A.3 that $p_i^* \in S_{\eta_k}(\hat{p}_{ik})$ with probability at least $1 - \varepsilon/N \log T$.

From Theorem 3.7, we know that the mechanism R_{ik} acquired by solving Eq.(3) guarantees player i 's truthfulness when $p_i^* \in S_{\eta_k}(\hat{p}_{ik})$. Also, in the duration of epoch k , the expected regret for a single round is $c \cdot C_1 \eta_k / (1 - C_2 \eta_k)$ where C_1 and C_2 are constants as defined in Eq.(8). Therefore, the total expect regret for player i across all T rounds is

$$\begin{aligned} \sum_{k=1}^m \frac{C_1 \eta_k}{1 - C_2 \eta_k} c \cdot (\tau_k - \tau_{k-1}) &\leq \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sum_{k=1}^m \eta_k \cdot (\tau_k - \tau_{k-1}) \\ &= \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sqrt{\log((d+1)2^d N(\log T)/\varepsilon)} \cdot \sum_{k=1}^m \frac{1}{\sqrt{\tau_{k-1}}} \cdot (\tau_k - \tau_{k-1}) \\ &= \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sqrt{\log((d+1)2^d N(\log T)/\varepsilon)} \cdot \sum_{k=1}^m \sqrt{2^{k-1}\tau} \\ &\preceq O(\sqrt{T} \log(d \cdot 2^d N \log T/\varepsilon)) \end{aligned}$$

In the last inequality we used the equation on the sum of geometric sequences.

Finally, we know that for one player in one epoch, the scheme ensures that applying union bound across all N players and $\log T$ periods. Hence applying union bound, we know that truthfulness is held with probability $1 - \varepsilon$. Also, note that the regret in the warm start phase is at most $O(\tau)$ and therefore dominated by the regret from the adaptive phase. Thus the total regret is N times the single player's regret and therefore $O(N\sqrt{T} \log(N(\log T)/\varepsilon))$.

A.8 Proof of Corollary 4.3

The updated upper bound under the epoch schedule $\tau_k - \tau_{k-1} = T^{1-2^{-(k-1)}}\tau$. All steps are identical to, except for the final step, where we sum over the regret across epochs:

$$\begin{aligned} \sum_{k=1}^m \frac{C_1 \eta_k}{1 - C_2 \eta_k} c \cdot (\tau_k - \tau_{k-1}) &\leq \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sum_{k=1}^m \eta_k \cdot (\tau_k - \tau_{k-1}) \\ &= \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sqrt{\log((d+1)2^d N(\log \log T)/\varepsilon)} \cdot \sum_{k=1}^m \frac{1}{\sqrt{\tau_{k-1}}} \cdot (\tau_k - \tau_{k-1}) \\ &\leq \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sqrt{\log((d+1)2^d N(\log \log T)/\varepsilon)} \cdot \sum_{k=1}^m \frac{T^{1-2^{-(k-1)}}}{\sqrt{T^{1-2^{-(k-2)}}}} \tau \\ &= \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sqrt{\log((d+1)2^d N(\log \log T)/\varepsilon)} \cdot \sum_{k=1}^m T^{\frac{1}{2}} \tau \\ &= \frac{C_1}{1 - C_2/\sqrt{2}} c \cdot \sqrt{\log((d+1)2^d N(\log \log T)/\varepsilon)} \cdot m \sqrt{T} \\ &\preceq O\left(\sqrt{T} \log((d+1)2^d N \log \log T/\varepsilon)\right). \end{aligned}$$

We note that this epoch schedule is sub-geometric, but it grows faster at the first few steps than the geometric epoch schedule $\tau_k - \tau_{k-1} = 2^{k-1}\tau$, therefore it uses logarithmically smaller $O(\log \log T)$ epoch count to reach T .

A.9 Proof of the Lower Bound (Theorem 4.4)

We start by considering the two-player, two-label case. Notice that as long as agents are perfectly truthful in the first t rounds, on round t the principal essentially has t data points i.i.d. sampled from p_X . In other words, the reward policy in previous rounds will not affect data distribution as long as it satisfies all the constraints. This specialty structure avoids cross-round (bandit-style) discussions on sequential policies. Instead, we can examine optimal policy round-by-round. In the following, we suppose the game is currently at round $T + 1$, and all the previous rounds are truthful, so the principal has T i.i.d. data points from p_X . Without loss of generality we let cost $c = 1$.

We first start by constructing a pair of problem instances $p_0, p_1 \in \Delta(\mathcal{X}^N)$ that simultaneously satisfies the two requirements: i) the two instances are statistically close; ii) the optimal mechanisms differ sharply. Let the focal player be player 1 and the reference player be 2. We now study the reward mechanisms for the focal player 1. Consider the following examples on p_0 and p_1 , whereas from the focal player's perspective, the two instances has:

$$\mathbf{B}_0 = \begin{bmatrix} 0.5 - \delta & 0.5 + \delta \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0.5 + \delta & 0.5 - \delta \\ 1 & 0 \end{bmatrix}$$

$$\mathbb{P}_k(X_1 = x) = 0.5, \quad \forall x \in \mathcal{X}, k \in \{0, 1\}.$$

We call δ the *cheapness* parameter, as it relates to how low the principal's expected payment can be.

LEMMA A.4 (HARD INSTANCES). *For the aforementioned two instances p_0 and p_1 with parameter $\delta \in (0, 1/4)$, we have:*

- *Competition: under instance p_0 , any reward mechanism R for that satisfies the incentive-compatibility (IC) constraints in Eq.(2) and whose expected payment is less than $1 + \delta$ must either violate IC constraints or pays more than $1 + \delta$ under instance p_1 . The statement holds with the roles of p_0 and p_1 reversed.*
- *Similarity: p_0 and p_1 are statistically close, i.e., $\text{KL}(p_0 \| p_1) \leq 8\delta^2$.*

PROOF. *Competition.* We follow a similar procedure as in Section A.2. Let $\mathbf{BR}^\top = \mathbf{M}$, notice that \mathbf{M} and \mathbf{R} has 1-1 correspondence since both \mathbf{B}_k are invertible. We therefore studies in the space of \mathbf{M} . The problem becomes:

$$\begin{aligned} \text{Find } \mathbf{M} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \text{s. t. } \mathbf{M}_{xx} &\geq 1, \quad \forall x \in \mathcal{Y} \\ \mathbf{M}_{xy} &\leq 1, \quad \forall x \neq y \in \mathcal{Y} \\ \mathbf{M}^\top \mathbf{d}' &\leq 0 \\ \sum_x \mathbb{P}(X_1 = x) \mathbf{M}_{xx} &\leq 1 + \delta \end{aligned}$$

where $\mathbf{d}'_x = \mathbb{P}(X_1 = x) = 1/2$. This gives a set of necessary conditions on the entries for feasible \mathbf{M} :

$$1 \leq a \leq 1 + 2\delta$$

$$1 \leq d \leq 1 + 2\delta$$

$$b \leq 1$$

$$c \leq 1$$

$$a + c \leq 0$$

$$b + d \leq 0.$$

(Here the $1 + 2\delta$ is a relaxation on the bound $(a + d)/2 < 1 + \delta$.) Notice that \mathbf{M}_{xy} is exactly player 1's expected reward given she observes label x and reports label y . Therefore, for any \mathbf{M}_0 that is cheap and satisfies the constraints under p_0 , its performance under p_1 is $\mathbf{M}_1 = \mathbf{B}_1 \mathbf{B}_0^{-1} \mathbf{M}_0$. However, we have that:

$$\mathbf{M}_1 = \begin{bmatrix} \frac{1-2\delta}{1+2\delta} & \frac{4\delta}{1+2\delta} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{(1-2\delta)a+4\delta c}{1+2\delta} & \frac{(1-2\delta)b+4\delta d}{1+2\delta} \\ c & d \end{bmatrix}.$$

Therefore, the first entry must have

$$\frac{(1-2\delta)a+4\delta c}{1+2\delta} \leq \frac{(1-2\delta)(1+2\delta) - 4\delta}{1+2\delta} < 1,$$

leading to a violation of the truthfulness constraint. Similar procedure would also prove the statement with p_0 and p_1 reversed. For any cheap and feasible mechanism under p_0 , the resulting first entry of corresponding \mathbf{M}_0 would be

$$\frac{(1+2\delta)a-4\delta c}{1-2\delta} \geq \frac{(1+2\delta)+4\delta}{1-2\delta} > 1+\delta.$$

Hence we prove the competition property in both ways. *Similarity.*

$$\begin{aligned} \text{KL}(p_0 \| p_1) &= \frac{1-2\delta}{4} \log \frac{(1-2\delta)/4}{(1+2\delta)/4} + \frac{1+2\delta}{4} \log \frac{(1+2\delta)/4}{(1-2\delta)/4} + \frac{1}{2} \log \frac{1/2}{1/2} \\ &= \delta \log \left(1 + \frac{4\delta}{1-2\delta} \right) \\ &\leq \frac{4\delta^2}{1-2\delta} \\ &\leq 8\delta^2. \end{aligned}$$

The first inequality holds since $\log(1+x) \leq x$, and the second holds since $\delta \in (0, 1/4)$. \square

Now we consider the mechanism design problem. Suppose we have a policy π that maps the i.i.d. collected data H_t to a reward mechanism R_{t+1} . Consider the good event G_k , $k \in \{0, 1\}$:

$$G_k = \{R_{t+1} \text{ satisfies IC constraints and pays less than } 1 + \delta \text{ in expectation}\}$$

Notice that G_0 and G_1 are disjoint since we have the competition property from Lemma A.4. Give such policy, we can construct a test $\tilde{\phi}$ that distinguishes p_0 and p_1 : the test outputs 0 if $R_{t+1} \in G_0$, outputs 1 if $R_{t+1} \in G_1$, and arbitrarily if neither is satisfied. Under Bretagnolle–Huber inequality, we have that the minimax error rate of any tests on p_0 and p_1 has lower bound as:

$$\begin{aligned} \inf_{\phi} (\mathbb{P}_0(\phi(H_t) = 1) + \mathbb{P}_1(\phi(H_t) = 0)) &\geq \frac{1}{2} \exp\{-\text{KL}(p_0^T \| p_1^T)\} \\ &\geq \frac{1}{2} \exp\{-8T\delta^2\}, \end{aligned}$$

where the second inequality is due to the tensorization property of divergence and the similarity property from Lemma A.4. Therefore, our special test $\tilde{\phi}$ must also follow this condition, and thus any policy π must fail to ensure both IC constraints and cheapness with probability at least $\exp(-8T\delta^2)/4$ in one of the hard instances.

In other words, for possible nice policies that guarantee truthfulness and cheapness with worst-case probability at least $1 - \varepsilon$, we must set the cheapness threshold to be greater than the threshold:

$$\delta \geq \sqrt{\frac{1}{8T} \log \frac{1}{4\varepsilon}},$$

and we must have the one-player, single-round expected regret be greater than the same lower bound (since all mechanism with expected regret smaller than the lower bound would fail the test with probability greater than ε .)

Therefore, summing across all periods would give us:

$$\inf_{\pi} \sup_{p \in \Delta(\mathcal{X}^2)} \text{Reg}(\pi, p) \geq \Omega\left(\sqrt{T \log(1/\varepsilon)}\right).$$

From Lemma 1 of [Radanovic and Faltings, 2013], it is known that for any mechanism with more than two players, its truthful Bayesian Nash Equilibrium can correspond to a 2-player mechanism with the same expected payment at its truthful Bayesian Nash Equilibrium. Therefore, more players would not bring additional benefits than the 2-player case, and hence we can simply apply the same lower bound for the N -player case. Hence the N -player mechanism design problem would have regret lower bound as:

$$\inf_{\pi} \sup_{p \in \Delta(\mathcal{X}^2)} \text{Reg}(\pi, p) \geq \Omega\left(N \sqrt{T \log(1/\varepsilon)}\right).$$

□

A.10 Proof of Theorem 4.6

The proof roughly follows the same procedure as Theorem 4.2, with a few modifications. First, since we don't have an explicit formula for the PAC guarantee, we cannot invert the function $\eta_\varepsilon(T)$ to get a closed-form bound for T under certain η and ε . This may lead to a relatively looser bound for certain estimators. The tightest bound can always be specifically derived following proof of Theorem 4.2. Second, we would use the estimator for conditional distribution $p(\cdot | x_i)$, for each $x_i \in \mathcal{Y}$. There are two scenarios where the estimator could be off:

- (1) Player i does not observe x_i for enough number of times. (T_{x_i} is small).
- (2) The estimation on $p(\cdot | x_i)$ is off.

The first scenario is not decided by whatever the estimator used by principal, since it is a tail events of a multinomial distribution. For the same reason as the first, we cannot directly merge the two probabilities together as it is done in Lemma A.3, resulting in the following lemma.

LEMMA A.5 (CONCENTRATION PROPERTY OF GENERAL DISCRETE DISTRIBUTION ESTIMATOR). *Suppose the principal has received T rounds of reports from player i and j . Assuming players are always truthful. Let $\hat{p}(x_j | x_i)$ be the conditional distribution estimation from the general estimator in Definition 4.5. Let $\rho = \min_{x \in \mathcal{Y}} \mathbb{P}(X_i = x)$. Define the ambiguity set with ambiguity level $\eta_\varepsilon(\rho T/2)$ as*

$$S_{\eta_\varepsilon(\rho T/2)}(\hat{p}) = \{p \in \mathcal{P} \mid \text{TV}(\hat{p}(\cdot | x_i), p(\cdot | x_i)) \leq \eta_\varepsilon(\rho T/2), \forall x_i \in \mathcal{Y} \cup \{\emptyset\}\}.$$

Then, with probability $1 - (d + 1)(\varepsilon + \exp(-\rho T/8))$, the true distribution p^* belongs to $S_{\eta_\varepsilon(\rho T/2)}$.

PROOF. For one label $x_i \in \mathcal{Y}$, we have

$$\begin{aligned} \mathbb{P}(\text{TV}(\hat{p}(\cdot | x_i), p^*(\cdot | x_i)) > \eta) &= \sum_{t=0}^T \mathbb{P}(\text{TV}(\hat{p}(\cdot | x_i), p^*(\cdot | x_i)) > \eta \mid T_{x_i} = t) \cdot \mathbb{P}(T_{x_i} = t) \\ &\leq \mathbb{P}(T_{x_i} \leq \rho T/2) + \mathbb{P}(\text{TV}(\hat{p}(\cdot | x_i), p^*(\cdot | x_i)) > \eta) \\ &\leq \exp(-\rho T/8) + \varepsilon \end{aligned}$$

Notice that the final inequality is Chernoff bound applied to the binomial distribution $\text{Bin}(T, \rho)$.

Applying union bound across all labels would give us

$$\mathbb{P}(\exists x_i, \text{TV}(\hat{p}(\cdot | x_i), p^*(\cdot | x_i)) > \eta) \leq (d+1)(\exp(-\rho T/8) + \varepsilon).$$

□

Now we focus on the regret for a single player i . Consider what happens in epoch k , where $t \in (\tau_{k-1}, \tau_k]$. At the beginning of the epoch, we have τ_{k-1} data points, and we set $\eta_k = \eta_{\varepsilon/Nm(d+1)}(\rho\tau_{k-1}/2)$. Thus we know from Lemma A.5 that $p_i^* \in S_{\eta_k}(\hat{p}_{ik})$ with probability at least $1 - \varepsilon/(Nm) - (d+1)\exp(-\rho\tau_{k-1}/8)$.

From Theorem 3.7, we know that the mechanism R_{ik} acquired by solving Eq.(3) guarantees player i 's truthfulness when $p_i^* \in S_{\eta_k}(\hat{p}_{ik})$. Also, in the duration of epoch k , the expected regret for a single round is $c \cdot C_1 \eta_k / (1 - C_2 \eta_k)$ where C_1 and C_2 are constants as defined in Eq.(8). Therefore, the total expect regret for player i across all T rounds is

$$\begin{aligned} \sum_{k=1}^m \frac{C_1 \eta_k}{1 - C_2 \eta_k} c \cdot (\tau_k - \tau_{k-1}) &\leq C_1 c \cdot \sum_{k=1}^m \eta_k \cdot (\tau_k - \tau_{k-1}) \\ &= C_1 c \cdot \sum_{k=1}^m \eta_{\varepsilon/Nm(d+1)}(\rho\tau_{k-1}/2) \cdot (\tau_k - \tau_{k-1}) \end{aligned}$$

Finally, we know that for one player in one epoch, the scheme ensures that applying union bound across all N players and m periods. Hence applying union bound, we know that truthfulness is held with probability

$$1 - \varepsilon - N(d+1) \cdot \sum_{k=1}^m \exp(-\rho\tau_{k-1}/8).$$

Also, note that the regret in the warm start phase is at most $O(\tau)$ and therefore dominated by the regret from the adaptive phase. Thus the total regret is N times the single player's regret and therefore

$$O\left(N \sum_{k=1}^m \eta_{\varepsilon/Nm(d+1)}(\rho\tau_{k-1}/2) \cdot (\tau_k - \tau_{k-1})\right)$$

□