

Marginal flows of non-entropic weak Schrödinger bridges*

Camilo HERNÁNDEZ [†] Ludovic TANGPI [‡]

Abstract

This paper introduces a dynamic formulation of divergence-regularized optimal transport with weak targets on the path space. In our formulation, the classical relative entropy penalty is replaced by a general convex divergence, and terminal constraints are imposed in a weak sense. We establish well-posedness and a convex dual formulation of the problem, together with explicit structural characterizations of primal and dual optimizers. Specifically, the optimal path measure is shown to admit an explicit density relative to a reference diffusion, generalizing the classical Schrödinger system. For the pure Schrödinger case, i.e., when the transport cost is zero, we further characterize the flow of time marginals of the optimal bridge, recovering known results in the entropic setting and providing new descriptions for non-entropic divergences including the χ^2 -divergence.

1 Introduction

Optimal transport (OT) provides a powerful framework for comparing probability measures, and has become a cornerstone in modern statistics, data science, machine learning and operations research. Formulated as a convex optimization problem over couplings between two distributions μ_0, μ_T not only compares distributions, but also exhibits geometric properties of the set of probability measures, and plays a fundamental role in partial differential equations. However, a well-known limitation is its poor statistical scalability: When the distributions μ_0, μ_T are replaced by empirical measures based on i.i.d. samples, the convergence of the empirical OT problem to its population counterpart deteriorates exponentially with the dimension, an instance of the curse of dimensionality; see, e.g. S. Chewi [51] for a recent survey. To overcome this limitation, entropic regularized optimal transport introduces an entropic penalty to the OT objective, yielding smoother dual problems and faster statistical convergence, see for instance [8; 9; 14; 47; 43] among many works on the subject. The entropic regularized perspective exhibit great statistical properties. In fact, the empirical optimal costs, optimal couplings, and dual potentials all converge to their population counterparts at the rate $\mathcal{O}(\sqrt{N})$ when the measures μ_0, μ_T are approximated by empirical measures of N i.i.d. random variables Genevay et al. [24]; Mena and Niles-Weed [43].

Despite these advantages, the entropy penalty unfortunately produces full-support couplings (a consequence of Brenier's theorem), a phenomenon known as overspreading, which can lead to undesirable blurring effects in imaging and manifold learning applications Blondel et al. [10]. Moreover, entropic OT becomes numerically unstable when the regularization parameter is small, as the dual variables may attain exponentially large or small values Li et al. [39]. To overcome these shortcomings, an increasingly popular alternative to entropic regularization is quadratically regularized optimal transport, which corresponds to the problem

$$v(\mu_0, \mu_T) := \inf_{\pi \in \Pi(\mu_0, \mu_T)} \left(\int_{\mathbb{R}^m \times \mathbb{R}^m} c(x, y) \pi(dx, dy) + \varepsilon \int_{\mathbb{R}^m \times \mathbb{R}^m} \ell\left(\frac{d\pi}{d\mu_0 \otimes \mu_T}\right) \mu_0 \otimes \mu_T(dx, dy) \right) \quad (1.1)$$

where $\Pi(\mu_0, \mu_T)$ is the set of couplings of the probability measures μ_0, μ_T and $\ell(x) := \frac{1}{2}|x|^2$. This problem has attracted a sustained interest in recent years. While the theoretically of this problem is a lot less well-understood than its entropic counterpart, it has already been showed notably that quadratic OT retains many computational benefits of entropic OT and produces sparse approximations that more faithfully reflect the geometry of unregularized OT, see González-Sanz and Nutz [25]; Wiesel and Xu [59]. We refer for instance to [45; 25; 61; 58; 59; 18] for recent works on sparsity, and González-Sanz et al. [28, 26] on gradient descent and sample approximation. Quadratic regularized OT also enjoys numerical stability even for small regularization parameters and admits fast gradient-based optimization as showed by González-Sanz et al. [28]. While quadratically regularized OT may suffer the curse of dimensionallity, the interesting recent paper of González-Sanz et al. [27] shows that for a sufficiently large class of convex functions ℓ the divergence-regularized OT (1.1) overcomes it and the empirical version of the problem converges at the parametric rate. This further motivates the study of (1.1) beyond the quadratic case. Divergence regularized OT also an emerging research topic, with some recent references including

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[†]University of Southern California, ISE department, USA. camilohe@usc.edu.

[‡]Princeton University, ORFE department, USA. ludovic.tangpi@princeton.edu.

[5; 27; 54; 44; 15]. Research on divergence-regularized OT has been restricted to the static case. The goal of this work is to introduce divergence-regularized OT in the dynamic setting, and to derive the structure of the optimizer.

Divergence-regularized OT and weak constraints. Let us introduce the divergence on the path space. Let $m \in \mathbb{N}$, $T > 0$ and μ_0, μ_T be given Borel probability measures on \mathbb{R}^m . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space supporting a Brownian motion W and such that \mathbb{P} is the unique weak solution of the stochastic differential equation

$$dX_t = b(t, X_t)dt + dW_t, \quad \mathbb{P} \circ X_0^{-1} = \mu_0.$$

The process X belongs to \mathcal{C}_T , the space of continuous functions on $[0, T]$ with values in \mathbb{R}^m . The (dynamic) divergence-regularized OT is

$$\mathcal{V}^\varepsilon(\mu_0, \mu_T) = \inf \left\{ \mathbb{E}^{\mathbb{Q}}[C(X)] + \varepsilon \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}), \mathbb{Q} \circ X_0^{-1} = \mu_0, \mathbb{Q} \circ X_T^{-1} = \mu_T \right\}, \quad (1.2)$$

where $C : \mathcal{C}_T \rightarrow \mathbb{R}$ is a given cost function and $\mathcal{I}_\ell(\cdot|\mathbb{P})$ denotes the divergence operator with respect to the measure \mathbb{P} and the function ℓ i.e.

$$\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}\left[\ell\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ \infty & \text{otherwise.} \end{cases} \quad (1.3)$$

In essence, we consider the divergence-regularization of the celebrated Benamou–Brenier continuous-time transport problem [7]. This viewpoint reveals deep connections between optimal transport, fluid mechanics, and gradient flows, but also enables powerful analytical and numerical tools that are unavailable in the static case. Obviously when $C = 0$, this problem corresponds to a form of (non-entropic) Schrödinger problem [21; 52; 53] in which the discrepancy between probability measures considered with respect to a general divergence. This is in its own right an interesting generalization of Schrödinger problem that, as far as we know, has only been considered by Léonard [37]. From a probabilistic standpoint, divergences arise naturally when one seeks to control deviations between laws without imposing symmetry or additivity properties inherent to relative entropy. Their flexibility makes them well-suited for capturing different aspects of distributional distance, while still retaining structural features such as convexity and monotonicity under Markov kernels. However, they do not, in general, satisfy the so-called data-processing equality

$$\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) = \mathcal{I}_\ell(\mathbb{Q}_{0T}|\mathbb{P}_{0T}) + \int_{(\mathbb{R}^m)^2} \mathcal{I}_\ell(\mathbb{Q}_{xy}|\mathbb{P}_{xy}) \mathbb{P}_{0T}(dx dy), \quad (1.4)$$

where $\mathbb{Q}_{xy}(\cdot) := \mathbb{Q}(\cdot|X_0 = x, X_T = y)$ and $\mathbb{Q}_{0T} = \mathbb{Q} \circ (X_0, X_T)^{-1}$; which is well-known to hold (essentially) only in the case $\ell(x) = x \log(x) - x + 1$, which corresponds to the relative entropy see e.g. Lacker [36, Theorem 5.5 and Remark 5.6]. Consequently, the dynamic problem (1.2) does not necessarily have the same value as its static counterpart (1.1) for arbitrary ℓ (as is known in the entropic case), see for instance the arguments of proof of [38, Proposition 2.3].

A key feature of (1.2) is that the divergence regularization allows mass to diffuse along many possible paths rather than following a single deterministic trajectory, hence the reward accounts for the average trajectories induced by a path distribution \mathbb{Q} , not the precise microscopic path itself. This naturally suggests considering problems where we no longer require X to hit the target distribution exactly, i.e., $\mathbb{Q} \circ X_T^{-1} = \mu_T$, but to match it on average. *Weak optimal transport* problems introduced by Gozlan et al. [30; 31] have emerged as a particularly flexible, yet tractable framework for modeling broader relationships between distributions. See (2.2) for precise definition and Backhoff-Veraguas and Pammer [3] for references on the topic and a sample of applications. Specifically, given a weak cost $c : \mathbb{R}^m \times \mathcal{P}_p(\mathbb{R}^m) \rightarrow \mathbb{R}$ and associated weak optimal transport value $\mathcal{W}_c(\cdot, \cdot)$, we will focus on the more general problem

$$V_c(\mu_0, \mu_T) = \inf \left\{ \mathbb{E}^{\mathbb{Q}}[C(X)] + \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}), \mathbb{Q} \circ X_0^{-1} = \mu_0, \mathcal{W}_c(\mathbb{Q} \circ X_T^{-1}, \mu_T) = 0 \right\} \quad (1.5)$$

where, for simplicity, we let $\varepsilon = 1$. Examples will be discussed in subsection 3.3 where we will see that adequate choices of c allow to recover the constraint $\mathbb{Q} \circ X_T^{-1} = \mu_T$, but also to model cases where the terminal law $\mathbb{Q} \circ X_T^{-1}$ of X is constrained to be in convex order with μ_T or to belong to a ball around it.

Main results. In this work, we derive the dual problem of (1.5) in the sense of convex analysis and, as main contributions of the paper, we recover fundamental structural properties of the primal and dual optimizers of (1.5). Let us give a brief synopsis of the main results of the work here; precise statements and assumptions are

given in Section 3. For a generic probability measure P and ℓ^* being the convex conjugate of ℓ , let us introduce the functions

$$\Phi_P(\xi) := \inf_{r \in \mathbb{R}} (\mathbb{E}^P[\ell^*(\xi - r)] + r), \text{ and } Q_c \varphi(x) := \inf_{\rho \in \mathcal{P}_p(\mathbb{R}^m)} (c(x, \rho) + \langle \varphi, \rho \rangle) \quad x \in \mathbb{R}^m, \xi \in \mathbb{L}^0.$$

Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be the canonical filtration. We show the duality relationship

$$V_c(\mu_0, \mu_T) = \sup_{\varphi \in C_{b,p}(\mathbb{R}^m)} \left(- \int_{\mathbb{R}^m} \Phi_{\mathbb{P}_x}(-Q_c \varphi(X_T)) \mu_0(dx) - \int_{\mathbb{R}^m} \varphi(x) \mu_T(dx) \right) + \mathcal{I}_\ell(\mu_0 | \nu_0)$$

where $(\mathbb{P}_x)_{x \in \mathbb{R}^m}$ is the regular conditional distribution of \mathbb{P} with respect to \mathcal{F}_0 , and $C_{b,p}(\mathbb{R}^m)$ is a set of continuous functions defined below. We show that if φ is a dual optimizer, then the probability measure \mathbb{Q} given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{d\mu_0}{d\nu_0}(X_0) \partial_x \ell^*(-Q_c \varphi(X_T) - C(X) - \psi(X_0)) \quad (1.6)$$

for some measurable function ψ is a primal optimizer; and when $c(x, \rho) = \int_{\mathbb{R}^m} \mathbf{1}_{\{x \neq y\}} \rho(dy)$, if an admissible measure \mathbb{Q} takes the form (1.6) then it is primal optimal and the function φ is dual optimal. Moreover, the functions φ and ψ satisfy an associated Schrödinger system (3.3). Lastly, in the case of Schrödinger problem with divergence cost, i.e. when $C = 0$, we can further characterize the time-marginals of the optimal bridge. In fact, in general the probability density function \mathbb{Q}_t of $\mathbb{Q} \circ X_t^{-1}$ takes the form

$$\partial_x \log(\mathbb{Q}_t(x)) = \mathbb{E}^\mathbb{Q} \left[\alpha_t^* + \overleftarrow{\alpha}_{T-t} \circ \overleftarrow{\mathcal{T}}^{-1} | X_t = x \right] + b(t, x) + b(T-t, x), \quad dt \otimes d\mathbb{Q}_t \text{-a.e.} \quad (1.7)$$

for two predictable processes α^* and $\overleftarrow{\alpha}$ where $\overleftarrow{\mathcal{T}}$ is the time reversal mapping on the path space. The explicit form of this formula can be obtained on a case-by-case basis, leveraging optimal control theory. This is illustrated in (3.6) for the case of the χ^2 -divergence.

Related literature. In the entropic case $\ell(x) = x \log(x) - x + 1$, and with $C = 0$, our results build on the well-established literature on the dynamic Schrödinger's problem. In particular, the characterization of the density of the optimal bridge in terms of potentials, the Schrödinger system, as well as the characterization of the flow of marginal laws (with respect to a reversible reference measure) in terms of Hamilton–Jacobi–Bellman equations are well understood in this setting, see the survey of Léonard [38] and the references therein. In the non-entropic case, a first step towards understanding the infinitesimal behaviour of the quadratically penalized optimal transport was discussed in Garriz–Molina et al. [23].

In the static setting, Schrödinger problems for general divergences were studied in Léonard [37]. Regarding the study of divergence-penalized optimal transport problems, as mentioned above, an emerging body of literature investigates the effects of general penalization on the transport map. In particular, what can be construed as the static version of the characterization of the density of the optimal static bridge appears in the work of Bayraktar et al. [5] and more recently in Nutz [45]; González-Sanz et al. [27]. We remind the reader that the setting of this paper corresponds to the dynamic non-entropic case, which, to the best of our knowledge, remained unexplored. Moreover, the present work seems to be the first Schrödinger problems with weak targets.

Organization of the paper. In the next section, we present the divergence-regularized optimal transport we are interested in, with weak terminal constraints, and state our main results before providing a few examples. Section 4 gathers some preparatory results for the proofs of our main results, which are presented in Section 5. Some standard but frequently used results from convex analysis and variational calculus are gathered in the appendix.

Notation. Throughout this work, $\Omega := \mathcal{C}([0, T], \mathbb{R}^m)$ denotes the canonical space of continuous paths on $[0, T]$ with values in \mathbb{R}^m . X denotes the canonical process, i.e., $X_t(\omega) := \omega_t, (t, \omega) \in [0, T] \times \Omega$, and canonical filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$, $\mathcal{F}_t := \sigma(X_s : s \in [0, t])$. \mathbb{L}^0 denotes the collection of real-valued random variables ξ on (Ω, \mathcal{F}_T) . Given $\mathbb{Q} \in \text{Prob}(\Omega)$, we denote by $(\mathbb{Q}_x)_{x \in \mathbb{R}^m}$ the regular conditional probability distribution of \mathbb{Q} given \mathcal{F}_0 .

For a function $\ell : \mathbb{R} \rightarrow \mathbb{R}$, its convex conjugate is given by $\ell^*(x) := \sup_{y \in \mathbb{R}} (xy - \ell(y))$. We also introduce the sets

$$B_{b,p}(\mathbb{R}^m) := \{\varphi : \mathbb{R}^m \rightarrow \mathbb{R} \text{ Borel measurable s.t. } \exists a, b \in \mathbb{R} : a \leq \varphi(y) \leq b(1 + |y|^p), \forall y \in \mathbb{R}^m\}.$$

and $C_{b,p}(\mathbb{R}^m) := \{\varphi \in B_{b,p}(\mathbb{R}^m) \text{ continuous}\}$.

Given a Polish space E , we denote by $\mathcal{P}(E)$ the set of Borel probability measures on E and $\mathcal{P}_p(E)$ the set of elements of $\mathcal{P}(E)$ with finite p^{th} moments. For a μ -integrable function φ defined on E , we put $\langle \varphi, \mu \rangle := \int_E \varphi(x) \mu(dx)$. For $\mu, \nu \in \mathcal{P}(E)$, $\Pi(\mu, \nu)$ is the set of couplings between μ and ν and the p -Wasserstein distance given by $W_p^p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{E \times E} \|x - y\|^p \pi(dx, dy)$.

2 Problem statement

Let $T > 0$, m a nonnegative integer, and $\nu_0 \in \mathcal{P}(\mathbb{R}^m)$ be fixed. We assume to be given a mapping $b : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, such that there is a unique $\mathbb{P} \in \text{Prob}(\Omega)$ weak solution to the SDE

$$dX_t = b(t, X_t)dt + dW_t, \quad \mathbb{P} \circ X_0^{-1} = \mu_0, \quad (2.1)$$

where W denotes a \mathbb{P} -Brownian motion. This is the case, for instance, if either b is locally Lipschitz or bounded, see [34, Section III.2d].

We are also given coefficients C , ℓ , and c , satisfying the following set of assumptions.

Assumption 2.1. (i) *The function $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ is strictly convex and twice continuously differentiable on $(0, \infty)$, satisfying $\ell(1) = 0$ and $\ell(x)/x \rightarrow \infty$ as $x \rightarrow \infty$. Moreover, its convex conjugate ℓ^* satisfies*

$$\forall r \in \mathbb{R}, \exists \gamma > 0, \exists x_0 \in \mathbb{R} : \forall x \geq x_0, \ell^*(x + r) \leq \gamma \ell^*(x).$$

(ii) *There is $p \geq 1$ such that $c : \mathbb{R}^m \times \mathcal{P}_p(\mathbb{R}^m) \rightarrow \mathbb{R}_+$ is jointly l.s.c. with respect to the product topology of $\mathbb{R}^m \times \mathcal{P}_p(\mathbb{R}^m)$, $\mathcal{P}_p(\mathbb{R}^m)$ being equipped with the topology generated by W_p . There is $L > 0$ such that*

$$c(x, \rho) \leq L \left(1 + \|x\|^p + \int_{\mathbb{R}^m} \|y\|^p \rho(dy) \right), \quad (x, \rho) \in \mathbb{R}^m \times \mathcal{P}_p(\mathbb{R}^m).$$

In addition, the map $\rho \mapsto c(x, \rho)$ is linearly convex for any $x \in \mathbb{R}^m$, i.e.,

$$c(x, \lambda \rho_1 + (1 - \lambda) \rho_2) \leq \lambda c(x, \rho_1) + (1 - \lambda) c(x, \rho_2), \quad \text{for all } \rho_1, \rho_2 \in \mathcal{P}_p(\mathbb{R}^m), \lambda \in [0, 1].$$

(iii) *The function C is Borel-measurable and bounded from below.*

Observe that under the above assumptions, the convex conjugate ℓ^* is continuously differentiable.

Weak targets. In this work, we will consider transportation problems for which the terminal configuration is specified in a broader sense by leveraging the weak optimal transport problem (WOT) first considered by Gozlan, Roberto, Samson, and Tetali [30]. Introduced as an equivalent tool to derive novel concentration inequalities, this generalization of optimal transport enables the construction of couplings between probability measures that possess desired structures or properties. Given $c : \mathbb{R}^m \times \mathcal{P}(\mathbb{R}^m) \rightarrow \mathbb{R}_+$, as above, WOT is given by

$$\mathcal{W}_c(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^m} c(x, \pi_x) \mu(dx). \quad (2.2)$$

We write \mathcal{W}_c to stress the dependence of WOT in c . Notice that when $c(x, \rho) = \int_{\mathbb{R}^m} \|x - y\|^p \rho(dy)$, for $p \geq 1$, $\mathcal{W}_c^{1/p}$ is nothing but the p -Wasserstein distance W_p . Further details, results relevant to our analysis, and examples are deferred to Section 3.3. Recalling that $\mu = \nu \in \mathcal{P}_p(\mathbb{R}^m) \iff W_p(\mu, \nu) = 0$, it seems natural to introduce a binary relation induced by \mathcal{W}_c over the elements of $\mathcal{P}(\mathbb{R}^m)$.

Definition 2.2. *Given any two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$, we write $\mu \stackrel{c}{=} \nu$ whenever $\mathcal{W}_c(\mu, \nu) = 0$.*

The non-entropic optimal transportation problem. Given two probability measures μ_0, μ_T such that $\mathcal{I}(\mu_0 | \nu_0) < \infty$, the goal of this work is to analyze the optimal transport problem

$$V_c(\mu_0, \mu_T) := \inf_{\mathbb{Q} \in \mathcal{P}_c(\mu_0, \mu_T)} \left(\mathbb{E}^{\mathbb{Q}}[C(X)] + \mathcal{I}_{\ell}(\mathbb{Q} | \mathbb{P}) \right), \quad (2.3)$$

with \mathcal{I}_{ℓ} denoting the divergence operator (1.3) and

$$\mathcal{P}_c(\mu_0, \mu_T) := \left\{ \mathbb{Q} \in \mathcal{P}(\mu_0) : \mathbb{Q} \circ X_T^{-1} \stackrel{c}{=} \mu_T \right\}, \quad \text{where, } \mathcal{P}(\mu_0) := \left\{ \mathbb{Q} \in \text{Prob}(\Omega) : \mathbb{Q} \circ X_0^{-1} = \mu_0 \right\}.$$

3 Main results and Examples

We will now rigorously state the main results of this work. In addition to deriving well-posedness and convex duality for the divergence-regularized transport problem, we are chiefly interested in explicit representations of the optimal transport plan in terms of the optimal potential. This property is a crucial building block for deriving the Sinkhorn algorithm, which has proven extremely efficient for the numerical simulation of entropic optimal transport [14; 12].

3.1 Optimal transport plan in divergence-regularized OT

To state the dual representation, for any $P \in \text{Prob}(\Omega)$, we will need the functional Φ_P defined as

$$\Phi_P : \mathbb{L}^0 \longrightarrow \mathbb{R}, \Phi_P(\xi) := \inf_{r \in \mathbb{R}} (\mathbb{E}^P[\ell^*(\xi - r)] + r),$$

To alleviate the notation we will simply write $\Phi := \Phi_{\mathbb{P}}$. This function is sometimes referred to as *optimized certainty equivalent* in the literature. We refer the interested reader for instance to [48; 4; 6] for detailed accounts of properties of this function.

Theorem 3.1. *Let Theorem 2.1 hold.*

- (i) *If $\mathcal{P}_c(\mu_0, \mu_T) \neq \emptyset$, then the problem (2.3) admits a unique optimizer $\mathbb{Q}^* \in \mathcal{P}_c(\mu_0, \mu_T)$.*
- (ii) *The problem (2.3) admits the convex dual representation*

$$V_c(\mu_0, \mu_T) = \sup_{\varphi \in C_{b,p}(\mathbb{R}^m)} (\Psi^\varphi(\mu_0) - \langle \varphi, \mu_T \rangle) \quad (3.1)$$

with the functional Ψ^φ being defined as

$$\Psi^\varphi(\mu) := - \int_{\mathbb{R}^m} \Phi_{\mathbb{P}_x}(-Q_c \varphi(X_T) - C(X)) \mu(dx) + \mathcal{I}_\ell(\mu|\nu_0), \text{ with, } Q_c \varphi(x) := \inf_{\rho \in \mathcal{P}_p(\mathbb{R}^m)} (c(x, \rho) + \langle \varphi, \rho \rangle). \quad (3.2)$$

Remark 3.2. *We remark for future reference that the case of bounded c corresponds to the limiting case $p = \infty$ in $C_{b,p}(\mathbb{R}^m)$. In particular, in the case $c(x, \rho) = \int_{\mathbb{R}^m} \mathbf{1}_{\{x \neq y\}} \rho(dy)$ associated to the terminal constraint $\mathbb{Q} \circ X_T^{-1} = \mu_T$, thanks to Theorem A.3, we have that $Q_c \varphi(x) = \varphi(x)$.*

In the rest of the paper we assume that $\mathcal{P}_c(\mu_0, \mu_T) \neq \emptyset$.

Our next result characterizes the density with respect to \mathbb{P} of the optimal primal path measure \mathbb{Q}^* in terms of the dual optimizers, i.e. of Problem (3.1). Because (1.4) fails for most divergences, we will consider the following notions:

Definition 3.3. *The divergence operator \mathcal{I}_ℓ is superadditive relative to \mathbb{P} if for all $\mathbb{Q} \in \mathcal{P}(\mu_0)$*

$$\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) \geq \mathcal{I}_\ell(\mu_0|\nu_0) + \int_{\mathbb{R}^m} \mathcal{I}_\ell(\mathbb{Q}_x|\mathbb{P}_x) \mu_0(dx),$$

where $\mathbb{Q}_x(d\omega)\mu_0(dx)$ denotes the disintegration of \mathbb{Q} with respect to \mathcal{F}_0 . The divergence operator \mathcal{I}_ℓ is subadditive relative to \mathbb{P} if the inequality “ \geq ” above is replaced by the “ \leq ”.

Remark 3.4. *The notions of super/sub-additive divergence probably first explicitly appeared in the work of Lacker [36]. Interestingly, this author showed that beyond the relative entropic which is additive (i.e. both sub and superadditive), other examples of functions ℓ leading to additive divergence are functions whose Legendre transform are the exponential of a function of at most linear growth, see [36, Remark 5.3].*

When $\mu_0 = \nu_0$, our characterization of optimal plans will hold for essentially all divergence operators; otherwise we will need to assume that ℓ is such that the associated divergence \mathcal{I}_ℓ is superadditive. We collect these two cases in the following assumption:

Assumption 3.5. *One of the following holds: (i) $\mu_0 = \nu_0$, (ii) \mathcal{I}_ℓ is superadditive.*

Theorem 3.6. *Let Assumptions 2.1 and 3.5 hold. Let $\varphi^* \in B_{b,p}(\mathbb{R}^m)$ be optimal for the dual problem (3.1) satisfying $\mathbb{E}^{\mu_0 \otimes \mathbb{P}}[\ell^*((Q_c \varphi^*(X_T) + C(X))^+)] < \infty$. There is a measurable function $\psi^* : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that*

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{d\mu_0}{d\nu_0}(X_0) \partial_x \ell^* \left(-Q_c \varphi^*(X_T) - C(X) - \psi^*(X_0) \right).$$

Remark 3.7. *Let us comment on the above result. First, note that the existence of a dual optimizer is complemented by the integrability condition $\mathbb{E}^{\mu_0 \otimes \mathbb{P}}[\ell^*((Q_c \varphi^*(X_T) + C(X))^+)] < \infty$. In the case $\mu_0 = \nu_0$, the condition reduces to $\mathbb{E}^{\mathbb{P}}[\ell^*((Q_c \varphi^*(X_T) + C(X))^+)] < \infty$ and guarantees a dual representation for the functional $\Phi_{\mathbb{P}_x}$, for μ_0 -a.e. $x \in \mathbb{R}^m$, see Theorem A.1. Second, the reader might find it illustrative to note that in the case $\ell(x) = x \log(x) - x + 1$ and $c(x, \rho) = \int_{\mathbb{R}^m} \mathbf{1}_{\{x \neq y\}} \rho(dy)$, we have that $\ell^*(x) = \exp(x)$ and $Q_c \phi(x) = \phi(x)$. In this case, we recover the well-known characterization of the optimal entropic optimal transport plan. That is,*

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{d\mu_0}{d\nu_0} e^{-C(X)} e^{-\varphi^*(X_T) - \psi^*(X_0)}.$$

We will now state a verification theorem. In the statement below, we use the time reversal mapping $\overleftarrow{\mathcal{T}} : \Omega \longrightarrow \Omega$ given by $\overleftarrow{\mathcal{T}}(\omega) := \overleftarrow{\omega}$ where $\overleftarrow{\omega}_t := \omega_{T-t}$ and the time-reversed measure $\overleftarrow{\mathbb{P}} = \mathbb{P} \circ \overleftarrow{\mathcal{T}}$.

Theorem 3.8. Let Assumptions 2.1 and 3.6 hold, and fix $c(x, \rho) = \int_{\mathbb{R}^m} \mathbf{1}_{\{x \neq y\}} \rho(dy)$.

Let $\hat{\varphi} \in B_{b,p}(\mathbb{R}^m)$ satisfy $\mathbb{E}^{\mu_0 \otimes \mathbb{P}}[\ell^*((\hat{\varphi}(X_T) + C(X))^+)] < \infty$, $\hat{\psi}$ be measurable, and let $\mathbb{Q} \in \mathcal{P}_c(\mu_0, \mu_T)$ be given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{d\mu_0}{d\nu_0}(X_0) \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0)).$$

Then it holds:

(i) The probability measure \mathbb{Q} is primal optimal and the function $\hat{\varphi}$ is dual optimal.

(ii) The functions $\hat{\varphi}$ and $\hat{\psi}$ satisfy the Schrödinger system

$$\begin{cases} 1 = \mathbb{E}^{\mathbb{P}}[\partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0))|X_0 = x], \mu_0\text{-a.e.} \\ 1 = \mathbb{E}^{\hat{\mathbb{P}}}[\partial_x \ell^*(-\hat{\varphi}(\hat{X}_T) - C(\hat{X}) - \hat{\psi}(\hat{X}_0))|\hat{X}_0 = x], \mu_T\text{-a.e.} \end{cases} \quad (3.3)$$

Remark 3.9. When the measure \mathbb{P} is reversible, i.e. when $\hat{\mathbb{P}} = \mathbb{P}$, the system (3.3) becomes

$$\begin{cases} 1 = \mathbb{E}^{\mathbb{P}}[\partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0))|X_0 = x], \mu_0\text{-a.e.} \\ 1 = \mathbb{E}^{\mathbb{P}}[\partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0)|X_T = x], \mu_T\text{-a.e.} \end{cases}$$

To keep the paper self-contained we give the argument after the proof of Theorem 3.8 below. In fact, in the entropic case and when \mathbb{P} is reversible, the system (3.3) reduces to the “classical” Schrödinger system. In fact if $C(X) = \tilde{C}(X_0, X_T)$ for some function \tilde{C} on $\mathbb{R}^m \times \mathbb{R}^m$, the equations become

$$\begin{cases} \hat{\psi}(x) = -\log \mathbb{E}[e^{-\hat{\varphi}(X_T) - \tilde{C}(x, X_T)}|X_0 = x], \mu_0\text{-a.s.} \\ \hat{\varphi}(x) = -\log \mathbb{E}[e^{-\hat{\psi}(X_T) - \tilde{C}(x, X_T)}|X_T = x], \mu_T\text{-a.s.} \end{cases}$$

derived, e.g., by Léonard [38]. This is not the only case of interest. As we will comment in Section 3.3, the case of the χ^2 -divergence (3.3) results in a tractable system of equations as well. In general, the system remains amenable to regression-type methodologies.

3.2 Flows of marginals for non-entropic dynamic Schrödinger bridges

We now turn our attention to describing the flow of marginal measures $(\mathbb{Q}_t^*)_{t \in [0, T]}$, where $\mathbb{Q}_t^* := \mathbb{Q}^* \circ X_t^{-1}$ denotes the marginal law of X , induced by \mathbb{Q}^* , the optimal solution to the non-entropic Schrödinger problem. Our first result consists of a characterization in terms of Markovian projections in the spirit of Föllmer [21].

Theorem 3.10. Let Assumptions 2.1 and 3.5 hold. Assume that $C = 0$ and c is such that $\mathcal{W}_c(\mu, \nu) = 0$ if and only if $\mu = \nu$ and that $b(t, x) = -\partial_x U(x)/2$ for some continuously differentiable function $U : \mathbb{R}^m \rightarrow \mathbb{R}$. Further assume that $V_c(\mu_0, \mu_T)$ and $V_c(\mu_T, \mu_0)$ admit dual optimizers $\varphi^* \in B_{b,p}(\mathbb{R}^m)$ and $\hat{\varphi} \in B_{b,p}(\mathbb{R}^m)$ respectively. Then there are two predictable processes α^* and $\hat{\alpha}$ such that

$$\mathbb{E}^{\mathbb{Q}^*}[\alpha_t^* + \hat{\alpha}_{T-t} \circ \hat{\mathcal{T}}|X_t = x] - \partial_x U(x) = \partial_x \log(\mathbb{Q}_t^*(x)), dt \otimes d\mathbb{Q}_t^*\text{-a.e.} \quad (3.4)$$

where \mathbb{Q}_t^* is the probability density function of $\mathbb{Q}^* \circ X_t^{-1}$, and where the derivative is in the sense of distributions.

The above characterization result can be further specialized, depending on the choice of the function ℓ . We illustrate this with two corollary concerning popular choices of divergences.

Corollary 3.11 (Relative entropy). Let the assumptions of Theorem 3.10 hold and let $\ell(x) = x \log x - x + 1$. Then we have:

(i) The Equation (3.4) reduces to

$$f_t(x) + g_{T-t}(x) - \partial_x U(x) = \partial_x \log(\mathbb{Q}_t^*(x)), dt \otimes d\mathbb{Q}_t^*\text{-a.e.}$$

for two Borel-measurable functions $f, g : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$.

(ii) If in addition the functions $\partial_x U, Q_c \varphi^*$ and $\hat{\varphi}$ are continuously differentiable. Then f and g satisfy $f_t(x) = \partial_x v^*(t, x)$ and $g(t, x) = \partial_x \hat{v}(t, x)$ where v and \hat{v} are solutions of the HJB equation

$$\partial_t v(t, x) - \partial_x v(t, x) \cdot \partial_x U(x) + \frac{1}{2} \partial_{xx} v(t, x) - \frac{1}{2} |\partial_x v(t, x)|^2 = 0$$

with respective terminal conditions $v^*(T, x) = Q_c \varphi^*(x)$ and $\hat{v}(T, x) = \hat{\varphi}(x)$.

Remark 3.12. Some remarks are in order concerning the above characterization of the marginals of the Schrödinger bridge: Based on the fact that the potentials f and g are also gradients, the characterization in Theorem 3.11 becomes the well-known formula

$$\mathbb{Q}_t^*(x) = \exp(v^*(t, x)) \exp(\overleftarrow{v}(T - t, x)) \exp(U(x)).$$

We do not expect this property to extend to other divergence operators beyond the relative entropy. This is because the functions f and g are constructed through solutions of stochastic optimal control problems (given in (5.9)). By verification theorems, such processes are constructed as minimizers of a Hamiltonian. The said Hamiltonian would typically not have closed-form minimizers. Nonetheless, we can obtain pointwise characterizations in some non-entropic cases, see e.g. Corollary 3.13 below.

Moreover, Theorem 3.10 also assumes that \mathcal{W}_c is a distance. This assumption is important to relate the values $V_c(\mu_0, \mu_T)$ and $V_c(\mu_T, \mu_0)$, which a key element of the proof. We believe that this assumption cannot be dropped.

As discussed in the introduction, quadratic regularized optimal transport has been an active area in recent years, notably due to the sparsity property of optimal transport plans, see, e.g., [27; 26; 45; 25; 61; 58; 59]. In the dynamic setting this corresponds to the choice of χ^2 -divergence, the flow of marginals of which the following result characterizes:

Corollary 3.13. (χ^2 -divergence) Let the assumptions of Theorem 3.10 hold and let $\ell(x) = (x - 1)^2/2$ if $x \geq 0$ and ∞ else. Further assume that the functions $\partial_x U$, $Q_c \varphi^*$ and $\overleftarrow{\varphi}$ are twice continuously differentiable with bounded derivatives. Let v and \overleftarrow{v} solve the PDEs

$$\begin{cases} \partial_t v(t, x) - \partial_x v(t, x) \cdot \partial_x U(x) + \frac{1}{2} \partial_{xx} v(t, x) - \frac{1}{2} |\partial_x \tilde{v}(t, x)|^2 = 0 \\ v(T, x) = 0 \\ \partial_t \tilde{v}(t, x) - \partial_x \tilde{v}(t, x) \cdot \partial_x U(x) + \frac{1}{2} \partial_{xx} \tilde{v}(t, x) = 0 \\ v(T, x) = \zeta(x) \end{cases} \quad (3.5)$$

with respective terminal conditions $\zeta(x) = Q_c \varphi^*(x)$ and $\zeta(x) = \overleftarrow{\varphi}(x)$. Let Z be given by

$$dZ_t = -\partial_x v(t, X_t) dW_t, \quad Z_0 = 1.$$

Then the density \mathbb{Q}_t^* of $\mathbb{Q}^* \circ (X_t, Z_t)^{-1}$ exists and satisfies

$$\mathcal{F}(t, x, z) + \overleftarrow{\mathcal{F}}(T - t, x, z) = \Sigma \Sigma^\top(t, x, z) \nabla \log(\mathbb{Q}_t^*) \quad (3.6)$$

where the functions \mathcal{F} , $\overleftarrow{\mathcal{F}}$ and Σ are defined as

$$\mathcal{F}(t, x, z) = \begin{pmatrix} -\frac{\partial_x v(t, x)}{z} - \frac{\partial_x U(x)}{2} \\ \frac{|\partial_x v(t, x)|^2}{z} + \frac{\partial_{xx} v(t, x)}{2} \end{pmatrix}, \quad \overleftarrow{\mathcal{F}}(t, x, z) = \begin{pmatrix} -\frac{\partial_x \overleftarrow{v}(t, x)}{z} - \frac{\partial_x U(x)}{2} \\ \frac{|\partial_x \overleftarrow{v}(t, x)|^2}{z} + \frac{\partial_{xx} \overleftarrow{v}(t, x)}{2} \end{pmatrix}, \quad \text{and, } \Sigma(t, x, z) = \begin{pmatrix} I_m \\ \partial_x v(t, x) \end{pmatrix}$$

where I_m is the identity matrix of $\mathbb{R}^{m \times m}$.

3.3 Examples

Let us conclude this section with some examples of possible choices of divergences and weak costs c .

Divergence. Beyond the classical case of relative entropy $\ell(x) = x \log x - x + 1$ discussed above, one could also consider the following choices of divergence for which we illustrate the form of our results.

(i) χ^2 -divergence. This corresponds to $\ell(x) = (x - 1)^2/2$ if $x \geq 0$ and $+\infty$ else. In this case, the convex conjugate is $\ell^*(x) = x + x^2/2$ for $x \geq -1$ and $\ell^*(x) = -1/2$ else. In this case, the primal optimizer \mathbb{Q}^* is given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{d\mu_0}{d\nu_0}(X_0) \left(1 - \varphi^*(X_T) - C(X) - \psi^*(X_0) \right)^+$$

where φ^* is a dual optimizer and x^+ denotes the positive part function; and if $c(x, \rho) = \int_{\mathbb{R}^m} \mathbb{1}_{\{x \neq y\}} \rho(dy)$, the functions φ^* and ψ^* satisfy the system

$$\begin{cases} 1 = \mathbb{E}^{\mathbb{P}} \left[\left(1 - \varphi^*(X_T) - C(X) - \psi(X_0) \right)^+ | X_0 = x \right], \mu_0\text{-a.e.} \\ 1 = \mathbb{E}^{\overleftarrow{\mathbb{P}}} \left[\left(1 - \varphi^*(\overleftarrow{X}_T) - C(\overleftarrow{X}) - \psi^*(\overleftarrow{X}_0) \right)^+ | \overleftarrow{X}_0 = x \right], \mu_T\text{-a.e.} \end{cases}$$

(ii) *Tsallis entropy.* Given in terms of $\ell(x) = \frac{x^q - 1}{q-1}$, $x \geq 1$ and $+\infty$ else, for $q \neq 1$. This family encompasses a rather large class of f -divergences that fits our setting for $q > 1$. The convex conjugate is $\ell^*(x) = \frac{1}{q-1} + (\frac{q-1}{q}x)^{q/(q-1)}$ for $x \geq 0$ and $\ell^*(x) = \frac{1}{q-1}$ else. In this case, the primal optimizer \mathbb{Q}^* is given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{q-1}{q} \frac{1}{q-1} \frac{d\mu_0}{d\nu_0}(X_0) \left((-\varphi^*(X_T) - C(X) - \psi^*(X_0))^+ \right)^{\frac{1}{q-1}}.$$

(iii) *Squared-Hellinger divergence.* Defined by $\ell(x) = (1 - \sqrt{x})^2$ if $x \geq 0$ and $+\infty$ else. This choice offers a slight variation of the previous example. Though it does not satisfy the growth assumption at infinity, we illustrate the form of our results. The convex conjugate is $\ell^*(x) = x/(1-x)$ for $x < 1$ and $\ell^*(x) = \infty$ else. In this case, the primal optimizer \mathbb{Q}^* is given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{d\mu_0}{d\nu_0}(X_0) \frac{1}{\{(1 - \varphi^*(X_T) - C(X) - \psi^*(X_0))^+\}^2}$$

where φ^* is a dual optimizer

Weak transport costs. Let us first note that the weak OT problem (2.2) admits the probabilistic interpretation

$$\mathcal{W}_c(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[c(X, \mathbb{E}[Y|X])],$$

revealing how, though initially motivated by applications to geometric inequalities, weak optimal transport theory highlights the relevance and implications of imposing structural constraints on feasible plans beyond fixed marginals.

We now detail some instances of costs c and elucidate their associated relation $\mathbb{Q}_T := \mathbb{Q} \circ X_T^{-1} \stackrel{c}{=} \mu_T$ induced by $\mathcal{W}_c(\mathbb{Q}_T, \mu_T) = 0$. We refer the reader to [29; 30] for a detailed account of these instances.

(i) *Total variation.* As mentioned above, when $c(x, \rho) = \int_{\mathbb{R}^m} \mathbf{1}_{\{x \neq y\}} \rho(dy)$, \mathcal{W}_c is the total variation distance and $\mathbb{Q}_T \stackrel{c}{=} \mu_T$ if and only if $\mathbb{Q}_T = \mu_T$ and in this case $Q_c \varphi = \varphi$.

(ii) *Marton's cost.* Takes its name from the work [42] and further studied in [30]. The Marton's cost functional is given by $c(x, \pi_x) = \theta \left(\int_{\mathbb{R}^m} \delta(c(x, y)) \pi_x(dy) \right)$, where $c : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ is a classical cost, $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is l.s.c. and $\theta : \mathbb{R}_+ \rightarrow [0, \infty]$ is convex. It has the probabilistic interpretation

$$\mathcal{W}_c(\mathbb{Q}_T, \mu_T) = \inf_{X_T \sim \mathbb{Q}_T, Y \sim \mu_T} \mathbb{E}[\theta(\mathbb{E}[\delta(c(X_T, Y))|X_T])].$$

The classical transport problem with cost c is recovered with the choices $\theta(x) = \delta(x) = x$. In this case, $Q_c \varphi(x) = \varphi^c(x) := \inf_{y \in \mathbb{R}^m} \{\varphi(y) + \|x - y\|^p\}$, is the c -transform of φ , see [57, Chapter 5]. In particular, if $c(x, y) = \|x - y\|^p$ and $c(x, y) = \mathbf{1}_{\{x \neq y\}}$ then \mathcal{W}_c become the p -Wasserstein and the total variation, respectively. In this context, Brenier's Theorem [7] shows that if $c \equiv \|x - y\|^p, p > 1$ and $\mathbb{Q}_T \ll \text{Leb}$, then $\mathbb{Q}_T \stackrel{c}{=} \mu_T$ implies there exists some convex function ϕ such that $\nabla \phi$ pushes forward \mathbb{Q}_T onto μ_T .

(iii) *Barycentric cost.* The Barycentric cost corresponds to $c(x, \pi_x) = \theta(x - \int_{\mathbb{R}^m} y \pi_x(dy))$, for $\theta : \mathbb{R}^m \rightarrow \mathbb{R}_+$ convex and l.s.c. That is, we have probabilistic interpretation

$$\mathcal{W}_c(\mathbb{Q}_T, \mu_T) = \inf_{X_T \sim \mathbb{Q}_T, Y \sim \mu_T} \mathbb{E}[\theta(X_T - \mathbb{E}[Y|X_T])].$$

This choice captures the existence of *martingale couplings* between two measures. That is, $\mathbb{Q}_T \stackrel{c}{=} \mu_T$ ensures that there exists a martingale coupling between \mathbb{Q}_T and μ_T . In this case, $Q_c \varphi(x) = Q_\theta \varphi_{cvx}(x)$, where φ_{cvx} denotes the greatest convex function $\varphi_{cvx} : \mathbb{R}^m \rightarrow \mathbb{R}$, such that $\varphi_{cvx} \leq \varphi$ and $Q_\theta \varphi := \inf_{y \in \mathbb{R}^m} \{\varphi(y) + \theta(x - y)\}$, see [29; 2] for further details.

(iv) *Distributionally robust costs.* This weak cost builds on the *variance case with a quadratic cost studied* in Alibert et al. [1, Section 5.2], the distributionally robust cost takes the form

$$c(x, \pi_x) = \int_{\mathbb{R}^m} c(x, y) \pi_x(dy) - \lambda^{-1} \text{Var}(\pi_x), \quad \text{where, } \text{Var}(\rho) := \int_{\mathbb{R}^m} \left\| y - \int_{\mathbb{R}^m} y' \rho(dy') \right\|^2 \rho(dy),$$

and $\lambda > 0$, acts as an interpolation parameter between the classical cost c and the variance of π_x , thus favouring the spreading of the measures π_x . Thanks to [1, Lemma 5.5], for any $r > 0$, with $\lambda := \text{Var}(\nu)/r^2$ and $c(x, y) = \|x - y\|^2$ it holds that

$$\mathcal{W}_c(\mu, \nu) + r^2 \geq W_2^2(\mu, \nu).$$

Thus, $\mathbb{Q}_T \stackrel{c}{=} \mu_T$ implies $\mathbb{Q}_T \in B_{W_2}(\mu_T, r)$, where $B_{W_2}(\nu, r)$ is the Wasserstein-2 ball of radius $r > 0$ centered at $\nu \in \mathcal{P}_2(\mathbb{R}^m)$. In this case, $Q_c \varphi(x) = \inf_{y \in \mathbb{R}^m} \{\varphi(y) + \frac{\lambda}{2} \|x - y\|^2\}$ is the Moreau-Yosida transform of φ .

4 Preliminaries

Before presenting the proofs of the main results, we gather a few technical lemmas that will be used in the proofs. We first establish some convexity properties.

Lemma 4.1. *Let Theorem 2.1 hold. The set $\mathcal{P}_c(\mu_0, \mu_T)$ and the mapping $(\mu_0, \mu_T) \mapsto V_c(\mu_0, \mu_T)$ are convex.*

Proof. We begin by convexity of $\mathcal{P}_c(\mu_0, \mu_T)$. Let $\bar{\mathbb{Q}} = \lambda \mathbb{Q}^1 + (1 - \lambda) \mathbb{Q}^2$ for $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{P}_c(\mu_0, \mu_T)$. It is clear that $\bar{\mathbb{Q}} \circ X_0^{-1} = \mu_0$, thus $\bar{\mathbb{Q}} \in \mathcal{P}(\mu_0)$. Recall from [2, Theorem 1.3] the duality formula for \mathcal{W}_c

$$\mathcal{W}_c(\mu, \mu_T) = \sup_{\varphi \in C_{b,p}(\mathbb{R}^d)} (\langle Q_c \varphi, \mu \rangle - \langle \varphi, \mu_T \rangle). \quad (4.1)$$

Note that $\bar{\mathbb{Q}} \circ X_T^{-1} = \lambda \mathbb{Q}^1 \circ X_T^{-1} + (1 - \lambda) \mathbb{Q}^2 \circ X_T^{-1}$, and, since c is nonnegative, it follows that

$$0 \leq \mathcal{W}_c(\bar{\mathbb{Q}} \circ X_T^{-1}, \mu_T) \leq \lambda \mathcal{W}_c(\mathbb{Q}^1 \circ X_T^{-1}, \mu_T) + (1 - \lambda) \mathcal{W}_c(\mathbb{Q}^2 \circ X_T^{-1}, \mu_T) = 0.$$

Hence, $\bar{\mathbb{Q}} \in \mathcal{P}(\mu_0, \mu_T)$.

Regarding convexity of V_c , let $\lambda \in (0, 1)$, $\mu_0^i, \mu_T^i \in \mathcal{P}(\mathbb{R}^d)$, $i = 1, 2$ be given, and $\mathbb{Q}^i \in \mathcal{P}_c(\mu_0^i, \mu_T^i)$. Let $\bar{\mathbb{Q}} = \lambda \mathbb{Q}^1 + (1 - \lambda) \mathbb{Q}^2$ and define $\bar{\mu}_0$ and $\bar{\mu}_T$ similarly. We claim that $\bar{\mathbb{Q}} \in \mathcal{P}_c(\bar{\mu}_0, \bar{\mu}_T)$. In fact, $\bar{\mathbb{Q}} \circ X_0^{-1} = \bar{\mu}_0$ is clear. Furthermore, $\bar{\mathbb{Q}} \circ X_T^{-1} = \lambda \mathbb{Q}^1 \circ X_T^{-1} + (1 - \lambda) \mathbb{Q}^2 \circ X_T^{-1}$, and since c is nonnegative, it follows from (4.1) that

$$0 \leq \mathcal{W}_c(\bar{\mathbb{Q}} \circ X_T^{-1}, \bar{\mu}_T) \leq \lambda \mathcal{W}_c(\mathbb{Q}^1 \circ X_T^{-1}, \mu_T^1) + (1 - \lambda) \mathcal{W}_c(\mathbb{Q}^2 \circ X_T^{-1}, \mu_T^2) = 0.$$

Therefore, by convexity of ℓ , $V_c(\bar{\mu}_0, \bar{\mu}_T) \leq \mathbb{E}^{\bar{\mathbb{Q}}}[C(X)] + \mathcal{I}_\ell(\bar{\mathbb{Q}}|\mathbb{P}) \leq \lambda \{\mathbb{E}^{\mathbb{Q}^1}[C(X)] + \mathcal{I}_\ell(\mathbb{Q}^1|\mathbb{P})\} + (1 - \lambda) \{\mathbb{E}^{\mathbb{Q}^2}[C(X)] + \mathcal{I}_\ell(\mathbb{Q}^2|\mathbb{P})\}$. Since \mathbb{Q}^1 and \mathbb{Q}^2 were taken arbitrary we deduce that $V_c(\bar{\mu}_0, \bar{\mu}_T) \leq \lambda V_c(\mu_0^1, \mu_T^1) + (1 - \lambda) V_c(\mu_0^2, \mu_T^2)$. \square

Proposition 4.2. *Let Theorem 2.1 hold. The function $\mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \ni (\mu, \nu) \mapsto V_c(\mu, \nu) \in \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous for the topology of weak convergence.*

Proof. Let (μ_0^n, μ_T^n) be a sequence of elements of $\mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)$ converging to (μ_0, μ_T) . Without loss of generality, we assume that $\liminf_{n \rightarrow \infty} V_c(\mu_0^n, \mu_T^n) < \infty$. Thus, up to a subsequence, $(V_c(\mu_0^n, \mu_T^n))_{n \in \mathbb{N}}$ is bounded. In particular, since C is bounded from below, there is a constant $K > 0$ such that for each $n \geq 1$ there is $\mathbb{Q}^n \in \mathcal{P}_c(\mu_0^n, \mu_T^n)$ satisfying,

$$K > V_c(\mu_0^n, \mu_T^n) - \mathbb{E}^{\mathbb{Q}^n}[C(X)] \geq \mathbb{E}[\ell(Z^n)] - \frac{1}{n}, \text{ where } Z^n := \frac{d\mathbb{Q}^n}{d\mathbb{P}}. \quad (4.2)$$

Thus, the sequence $(Z^n)_{n \geq 1}$ is uniformly integrable and, consequently, tight.

We claim that any of the weak limits of $(Z^n)_{n \geq 1}$ gives rise to a probability measure $\mathbb{Q} \in \mathcal{P}_c(\mu_0, \mu_T)$ with $\mathbb{Q} \ll \mathbb{P}$. Let Z be a weak limit of $(Z^{n_k})_{k \geq 1}$ and consider the process $M_t := \mathbb{E}[Z|\mathcal{F}_t]$, $t \in [0, T]$. The convexity of ℓ shows that $\sup_{t \in [0, T]} \mathbb{E}[\ell(M_t)] \leq \mathbb{E}[\ell(Z)] < \infty$, and, by the growth assumption on ℓ , M is a uniformly integrable martingale. Since $\mathbb{E}[Z^n] = 1$, $n \geq 1$, we deduce from the convergence in law of $(Z^{n_k})_{k \geq 1}$ that $\mathbb{E}[Z] = \mathbb{E}[M_t] = 1$, $t \in [0, T]$. With this, it follows that $\mathbb{Q}(A) := \mathbb{E}[M_T \cdot \mathbf{1}_A]$ is a well defined probability measure and $\mathbb{Q} \ll \mathbb{P}$.

To prove the claim it remains to show that $\mathbb{Q} \in \mathcal{P}_c(\mu_0, \mu_T)$. Recall that $\mathbb{Q}^n \in \mathcal{P}_c(\mu_0^n, \mu_T^n)$, i.e., $\mathbb{Q}^n \circ X_0^{-1} = \mu_0^n$, $\mathbb{Q}^n \circ X_T^{-1} \stackrel{c}{=} \mu_T^n$ for all $n \geq 1$ and (μ_0^n, μ_T^n) converges in law to (μ_0, μ_T) . Thus, for a bounded continuous function f

$$\mathbb{E}^{\mathbb{Q} \circ X_0^{-1}}[f] = \mathbb{E}^{\mathbb{Q}}[f(X_0)] = \lim_{k \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^{n_k}}[f(X_0)] = \lim_{k \rightarrow \infty} \mathbb{E}^{\mu_0^{n_k}}[f] = \mathbb{E}^{\mu_0}[f],$$

and similarly, the lower semicontinuity of $\mu \mapsto \mathcal{W}_c(\mu, \mu_T)$, gives

$$\mathcal{W}_c(\mathbb{Q} \circ X_T^{-1}, \mu_T) \leq \liminf_{k \rightarrow \infty} \mathcal{W}_c(\mathbb{Q}^{n_k} \circ X_T^{-1}, \mu_T) = \liminf_{k \rightarrow \infty} \mathcal{W}_c(\mu_T^{n_k}, \mu_T) = 0.$$

We now conclude. Back in (4.2) we see that

$$\liminf_{n \rightarrow \infty} V(\mu_0^n, \mu_T^n) \geq \liminf_{n \rightarrow \infty} \mathbb{E}[Z^n C(X)] + \mathbb{E}[\ell(Z^n)] \geq \mathbb{E}[M_T C(X)] + \mathbb{E}[\ell(M_T)] \geq V(\mu_0, \mu_T). \quad \square$$

It will be useful to see the regularized optimal transport problem (2.3) as an optimal stochastic control problem. To this end we show in the following lemma that elements of $\mathcal{P}(\mu_0, \mu_T)$ can be written as stochastic exponential of local martingales. Let $\mathbb{H}_{\text{loc}}^2$ be the space of \mathbb{F} -predictable processes α such that $\mathbb{P}\left(\int_0^T \|\alpha_t\|^2 dt < \infty\right) = 1$.

Lemma 4.3. *Let Theorem 2.1 hold. Let*

$$\mathcal{A}(\mu_0) := \left\{ \mathbb{P}^\alpha \in \text{Prob}(\Omega) : \frac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \mathbf{1}_{\{\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} > 0\}} \frac{d\mu_0}{d\nu_0}(X_0) Z_T, dZ_t = \alpha_t Z_t dW_t, Z_0 = 1, \mathbb{P}\text{-a.s.}, \alpha \in \mathbb{H}_{\text{loc}}^2, \mathbb{P}^\alpha \circ X_0^{-1} = \mu_0 \right\}$$

and $J(\mathbb{P}^\alpha) := \frac{1}{2} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt \right] + \mathcal{I}_\ell(\mu_0|\nu_0)$.¹ Then the following holds:

¹ $\mathcal{A}_c(\mu_0, \mu_T)$ is defined analogous to $\mathcal{P}_c(\mu_0, \mu_T)$.

- (i) $\{\mathbb{P}^\alpha \in \mathcal{A}(\mu_0) : J(\mathbb{P}^\alpha) < \infty\} = \{\mathbb{Q} \in \mathcal{P}(\mu_0) : \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) < \infty\}$ and $\mathcal{I}_\ell(\mathbb{P}^\alpha|\mathbb{P}) = J(\mathbb{P}^\alpha)$;
- (ii) $V_c(\mu_0, \mu_T) = \inf_{\mathbb{P}^\alpha \in \mathcal{A}_c(\mu_0, \mu_T)} (\mathbb{E}^{\mathbb{P}^\alpha}[C(X)] + J(\mathbb{P}^\alpha))$.

Proof. (i) Let $\mathbb{Q} \in \mathcal{P}(\mu_0)$ be such that $\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) < \infty$. Then $\mathbb{Q} \ll \mathbb{P}$, so that the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ exists. Letting $M_t := \mathbb{E}[Z_t | \mathcal{F}_t]$, $Z_T := \frac{d\mathbb{Q}}{d\mathbb{P}}$, the convexity of ℓ shows that $\sup_{t \in [0, T]} \mathbb{E}[\ell(M_t)] \leq \mathbb{E}[\ell(Z_T)] = \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) < \infty$, and thus M is a uniformly integrable martingale. Let us now introduce

$$\tau_n := \inf\{t \geq 0 : M_t \leq 1/n\}, \text{ and } \tau := \inf\{t \geq 0 : M_t = 0\}.$$

Note that, $\tau_n \leq \tau_{n+1} \leq \tau$, \mathbb{P} -a.s. and $\tau_n \rightarrow \tau$ \mathbb{P} -a.s. as $n \rightarrow \infty$. Let us put $M_t^n := M_{t \wedge \tau_n}$ and note that, since $M^n > 0$, it follows from Itô's formula that

$$M_t^n = \exp\left(L_t^n - \frac{1}{2}\langle L^n \rangle_t\right), \quad L_t^n := \log(M_0^n) + \int_0^t \frac{1}{M_s^n} dM_s^n.$$

Passing to the limit we find that, for $L_t := \log(M_0) + \int_0^t \frac{1}{M_s} dM_s$,

$$M_t = \exp\left(L_t - \frac{1}{2}\langle L \rangle_t\right), \text{ on } \{t < \tau\}.$$

Noticing that $\{T < \tau\} = \left\{\frac{d\mathbb{Q}}{d\mathbb{P}} > 0\right\}$ we deduce that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathbb{1}_{\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} > 0\right\}} \exp\left(L_T - \frac{1}{2}\langle L \rangle_T\right).$$

Now recall that \mathbb{P} is, by definition, the unique weak solution of (2.1) and thus, by [34, Theorem III.4.29] the predictable martingale representation property holds for (\mathbb{F}, \mathbb{P}) -local martingales. That is, there exists a unique $\alpha \in \mathbb{H}_{loc}^2$, such that $L_t = L_0 + \int_0^t \alpha_s \cdot dW_s$. Thus,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathbb{1}_{\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} > 0\right\}} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_0\right] Z_T, \text{ where, } dZ_t = Z_t \alpha_t \cdot dW_t, Z_0 = 1, \mathbb{P}\text{-a.s.} \quad (4.3)$$

Furthermore, observe that for any bounded continuous function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ we have

$$\mathbb{E}^\mathbb{P}\left[f(X_0) \frac{d\mu_0}{d\nu_0}(X_0)\right] = \mathbb{E}^{\nu_0}\left[f \frac{d\mu_0}{d\nu_0}\right] = \mathbb{E}^{\mu_0}[f] = \mathbb{E}^{\mathbb{Q} \circ X_0^{-1}}[f] = \mathbb{E}^\mathbb{Q}[f(X_0)] = \mathbb{E}^\mathbb{P}\left[f(X_0) \mathbb{E}^\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_0^X\right]\right].$$

That is,

$$M_0 = \mathbb{E}^\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_0\right] = \frac{d\mu_0}{d\nu_0}(X_0), \mathbb{P}\text{-a.s.}$$

With this, we conclude that $\mathbb{Q} = \mathbb{P}^\alpha \in \mathcal{A}(\mu_0)$.

Let us now note that by Itô's formula, on $\left\{\frac{d\mathbb{Q}}{d\mathbb{P}} > 0\right\}$ we have that

$$\ell(M_T) = \ell(M_0) + \frac{1}{2} \int_0^T \ell''(Z_t) Z_t^2 \|\alpha_t\|^2 dt + \int_0^T \ell'(Z_t) Z_t \alpha_t \cdot \sigma dW_t, \mathbb{P}\text{-a.s.},$$

and by [50, Proposition VIII.1.2], M is strictly positive \mathbb{Q} -a.s. Thus, a localization argument leads to

$$J(\mathbb{P}^\alpha) = \mathcal{I}_\ell(\mu_0|\nu_0) + \frac{1}{2} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt \right] = \mathbb{E}^\mathbb{Q} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{-1} \ell \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] = \mathbb{E} \left[\ell \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] = \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) < \infty.$$

The converse direction of the correspondence follows exchanging the roles of $\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P})$ and $J(\mathbb{P}^\alpha)$ in the above line.

(ii) The proof of the second part is now immediate since by (i) we have that $\{\mathbb{P}^\alpha \in \mathcal{A}(\mu_0, \mu_T) : J(\mathbb{P}^\alpha) < \infty\} = \{\mathbb{Q} \in \mathcal{P}(\mu_0, \mu_T) : \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) < \infty\}$. \square

As a direct corollary, and in the case where the reference and admissible measures' initial condition coincide, we obtain the following result. It can be seen as an instance of the chain rule or tensorization of sorts for the divergence operator. It is well-known in the case of the relative entropy, see, e.g., Lacker [36] or Gozlan and Léonard [32].

Corollary 4.4. *Let Theorem 2.1 hold and suppose $\mu_0 = \nu_0$. Then, for every $\mathbb{Q} \in \mathcal{P}(\mu_0)$ we have*

$$\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) = \int_{\mathbb{R}^m} \mathcal{I}_\ell(\mathbb{Q}_x|\mathbb{P}_x) \mu_0(dx).$$

Proof. We know that for every $\mathbb{Q} \in \mathcal{P}(\mu_0)$, it holds $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathbf{1}_{\{\frac{d\mathbb{Q}}{d\mathbb{P}} > 0\}} Z_T$, $dZ_t = Z_t \alpha_t \cdot dW_t$, \mathbb{P} -a.s. and

$$\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) = \mathcal{I}_\ell(\mu_0|\nu_0) + \frac{1}{2} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt \right] = \frac{1}{2} \int_{\mathbb{R}^m} \mathbb{E}_x^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt \right] \mu_0(dx).$$

Applying Itô's formula and using $\ell(1) = 0$, we have $\frac{1}{2} \mathbb{E}_x^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt \right] = \mathcal{I}_\ell(\mathbb{P}^\alpha|\mathbb{P}_x)$. \square

As corollary of the above result, we show that the functional Ψ^φ introduced in (3.2) is actually the value function of an optimal stochastic control problem.

Corollary 4.5. *Let Theorem 2.1 hold. For every $\varphi \in C_{b,p}(\mathbb{R}^m)$, the functional Ψ^φ admits the representation*

$$\begin{aligned} \Psi^\varphi(\mu) &= \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu)} \frac{1}{2} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + Q_c \varphi(X_T) + C(X) \right] + \mathcal{I}_\ell(\mu|\nu_0) \\ &= \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu)} \left(\mathcal{I}_\ell(\mathbb{P}^\alpha|\mathbb{P}) + \mathbb{E}^{\mathbb{P}^\alpha} [Q_c \varphi(X_T) + C(X)] \right). \end{aligned}$$

Proof. Let $\varphi \in C_{b,p}(\mathbb{R}^m)$. Suppose $\mu(x : \mathbb{E}^{\mathbb{P}_x} [\ell^*((Q_c \varphi(X_T) + C(X))^+)] = \infty) > 0$. Notice that since ℓ^* is increasing, we have that $\mathbb{E}[\ell^*(\zeta)] \leq \mathbb{E}[\ell^*(\zeta^+)]$ for any random variable ζ . It then follows by definition that $\Psi^\varphi(\mu) = -\infty$. Moreover, thanks to [13, Theorem 2.3], the other two expressions in the statement are also equal to $-\infty$.

We now consider the case $\mu(x : \mathbb{E}^{\mathbb{P}_x} [\ell^*((Q_c \varphi(X_T) + C(X))^+)] < \infty) = 1$. By Theorem A.1.(ii), we have that

$$\begin{aligned} \Psi^\varphi(\mu) &= - \int_{\mathbb{R}^m} \sup_{\mathbb{Q} \ll \mathbb{P}_x} \left(\mathbb{E}^{\mathbb{Q}} [-Q_c \varphi(X_T) - C(X)] - \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}_x) \right) \mu(dx) + \mathcal{I}_\ell(\mu|\nu_0) \\ &= \int_{\mathbb{R}^m} \inf_{\mathbb{Q} \ll \mathbb{P}_x} \left(\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}_x) + \mathbb{E}^{\mathbb{Q}} [Q_c \varphi(X_T) - C(X)] \right) \mu(dx) + \mathcal{I}_\ell(\mu|\nu_0). \end{aligned} \quad (4.4)$$

Note that $\mathbb{Q} \ll \mathbb{P}_x$ implies $\mathbb{Q} \circ X_0^{-1} = \delta_x = \mathbb{P}_x \circ X_0^{-1}$, and recalling $\ell(1) = 0$, we have that $\mathcal{I}_\ell(\mathbb{Q} \circ X_0^{-1}|\mathbb{P}_x \circ X_0^{-1}) = 0$. It them follows from Theorem 4.3 that

$$\begin{aligned} \Psi^\varphi(\mu) &= \int_{\mathbb{R}^m} \left(\inf_{\mathbb{P}^\alpha \in \mathcal{A}(\delta_x)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\frac{1}{2} \int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + Q_c \varphi(X_T) + C(X) \right] \right) \mu(dx) + \mathcal{I}_\ell(\mu|\nu_0) \\ &= \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\frac{1}{2} \int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + Q_c \varphi(X_T) + C(X) \right] + \mathcal{I}_\ell(\mu|\nu_0) \\ &= \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu)} \left(\mathcal{I}_\ell(\mathbb{P}^\alpha|\mathbb{P}) + \mathbb{E}^{\mathbb{P}^\alpha} [Q_c \varphi(X_T) + C(X)] \right) \end{aligned}$$

where the second equality follows from Theorem A.4. \square

5 Proofs of the main results

5.1 Existence, uniqueness and convex duality

Proof of Theorem 3.1. (i) Since $\mathcal{P}_c(\mu_0, \mu_T) \neq \emptyset$, for every $n \geq 1$ there is $\mathbb{Q}^n \in \mathcal{P}_c(\mu_0, \mu_T)$ such that

$$V_c(\mu_0, \mu_T) \geq \mathbb{E}^{\mathbb{Q}^n} [C(X)] + \mathcal{I}_\ell(\mathbb{Q}^n|\mathbb{P}) - \frac{1}{n}.$$

As argued in the proof of Theorem 4.2 above, there is $\mathbb{Q}^* \in \mathcal{P}_c(\mu_0, \mu_T)$ such that $\mathbb{E}^{\mathbb{Q}^*} [C(X)] + \mathcal{I}_\ell(\mathbb{Q}^*|\mathbb{P}) \leq \liminf_{n \rightarrow \infty} (\mathbb{E}^{\mathbb{Q}^n} [C(X)] + \mathcal{I}_\ell(\mathbb{Q}^n|\mathbb{P}))$. Thus, \mathbb{Q}^* is optimal for Problem (2.3).

Uniqueness follows since by Theorem 4.1 the set $\mathcal{P}_c(\mu_0, \mu_T)$ is convex, and under Theorem 2.1 the map $\mathbb{Q} \mapsto \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P})$ is strictly convex. Consequently, $V_c(\mu_0, \mu_T)$ admits a unique minimizer $\mathbb{Q}^* \in \mathcal{P}_c(\mu_0, \mu_T)$.

(ii) For any $\mu_0, \mu_T \in \mathcal{P}_p(\mathbb{R}^m)$ let us put

$$D(\mu_0, \mu_T) := \sup_{\varphi \in C_{b,p}(\mathbb{R}^m)} (\Psi^\varphi(\mu_0) + \langle \varphi, \mu_T \rangle)$$

with Ψ^φ defined in (3.2). If $V_c(\mu_0, \mu_T) = +\infty$ for all $\mu_T \in \mathcal{P}_2(\mathbb{R}^m)$, then by Theorem 4.5,

$$D(\mu_0, \mu_T) \geq \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu_0)} \mathcal{I}_\ell(\mathbb{P}^\alpha | \mathbb{P}) + \langle Q_c \varphi, \mathbb{P}^\alpha \circ X_T^{-1} \rangle + \mathbb{E}^{\mathbb{P}^\alpha}[C(X)] - \langle \varphi, \mu_T \rangle$$

for all $\varphi \in C_{b,p}(\mathbb{R}^d)$. This shows, due to the convex dual representations of $\mathcal{W}_c(\mathbb{P} \circ X_T^{-1}, \mu_T)$ that $D(\mu_0, \mu_T) \geq \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu_0)} \mathcal{I}_\ell(\mathbb{P}^\alpha | \mathbb{P}) + \mathbb{E}^{\mathbb{P}^\alpha}[C(X)] + \mathcal{W}_c(\mathbb{P}^\alpha \circ X_T^{-1}, \mu_T) \geq \infty$. In particular, $V_c(\mu_0, \mu_T) = D(\mu_0, \mu_T)$.

Let us now assume that there is $\mu_T \in \mathcal{P}_2(\mathbb{R}^m)$ such that $V_c(\mu_0, \mu_T) < +\infty$. Denote by $\mathcal{M}_p(\mathbb{R}^m)$ the space of Borel signed measures on \mathbb{R}^d with finite second moment, and equipped with the weak topology. We extend the function $V_c(\mu_0, \cdot)$ to be $+\infty$ outside $\mathcal{P}_p(\mathbb{R}^m)$, and still denote by V_c this extension. It is standard that the function $V_c(\mu_0, \cdot)$ remains convex and lower-semicontinuous on $\mathcal{M}_p(\mathbb{R}^m)$, recall Theorem 4.1 and Theorem 4.2. Thus, by Fenchel-Moreau theorem, see [60, Theorem 2.3.3] we have

$$V_c(\mu_0, \mu_T) = \sup_{\varphi \in C_p(\mathbb{R}^m)} (\langle \varphi, \mu_T \rangle - V_c^*(\varphi)) = \sup_{\varphi \in C_{b,p}(\mathbb{R}^m)} (-V_c^*(-\varphi) - \langle \varphi, \mu_T \rangle),$$

where the second equality follows by a monotone convergence argument, and V_c^* is the convex conjugate of V_c given by

$$V_c^*(\varphi) := \sup_{\mu_T \in \mathcal{M}_p(\mathbb{R}^m)} (\langle \varphi, \mu_T \rangle - V_c(\mu_0, \mu_T)), \quad \varphi \in C_{b,p}(\mathbb{R}^m).$$

The rest of the proof consists of showing that $-V_c^*(-\varphi) = \Psi^\varphi(\mu_0)$. To see that $-V_c^*(-\varphi) \geq \Psi^\varphi(\mu_0)$, note that

$$\begin{aligned} -V_c^*(-\varphi) &= \inf_{\mu_T \in \mathcal{P}_p(\mathbb{R}^m)} (V_c(\mu_0, \mu_T) + \langle \varphi, \mu_T \rangle) \\ &= \inf_{\mu_T \in \mathcal{P}_p(\mathbb{R}^m)} \inf_{\mathbb{P}^\alpha \in \mathcal{A}_c(\mu_0, \mu_T)} \left(\frac{1}{2} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + C(X) \right] + \langle \varphi, \mu_T \rangle \right) + \mathcal{I}_\ell(\mu_0 | \nu_0) \\ &\geq \inf_{\mu_T \in \mathcal{P}_p(\mathbb{R}^m)} \inf_{\mathbb{P}^\alpha \in \mathcal{A}_c(\mu_0, \mu_T)} \left(\frac{1}{2} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + Q_c \varphi(X_T) + C(X) \right] + \mathcal{I}_\ell(\mu_0 | \nu_0) \right) \\ &= \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu_0)} \frac{1}{2} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + Q_c \varphi(X_T) + C(X) \right] + \mathcal{I}_\ell(\mu_0 | \nu_0) \\ &= \Psi^\varphi(\mu_0) \end{aligned}$$

where the second equality follows from (4.3), the inequality follows from (4.1) since $\mathbb{P}^\alpha \in \mathcal{A}_c(\mu_0, \mu_T)$, the third equality follows since $\mathcal{A}(\mu_0) = \bigcup_{\mu \in \mathcal{P}_p(\mathbb{R}^m)} \mathcal{A}_c(\mu_0, \mu)$, and the last one is due to Theorem 4.5. The reversed inequality is argued as in [33] following ideas from [2, Lemma 2.1]. \square

5.2 Dual characterization of primal optimizers

Proof of Theorem 3.6. Let us assume that Theorem 3.5.(ii) holds, i.e., \mathcal{I}_ℓ being superadditive. Since φ^* attains the maximum in the dual problem and \mathbb{Q}^* attains the minimum in the primal problem, it holds

$$\mathbb{E}^{\mathbb{Q}^*}[C(X)] + \mathcal{I}_\ell(\mathbb{Q}^* | \mathbb{P}) = V_c(\mu_0, \mu_T) = \Psi^{\varphi^*}(\mu_0) - \langle \varphi^*, \mu_T \rangle. \quad (5.1)$$

We claim that $\langle \varphi^*, \mu_T \rangle = \mathbb{E}^{\mathbb{Q}^*}[Q_c \varphi^*(X_T)]$. Indeed, since \mathbb{Q}^* is feasible for the primal problem we have that $\mathcal{W}_c(\mathbb{Q}^* \circ X_T^{-1}, \mu_T) = 0$ and thus it follows from (4.1) that $\mathbb{E}^{\mathbb{Q}^*}[Q_c \varphi^*(X_T)] \leq \langle \varphi^*, \mu_T \rangle$. Suppose that $\mathbb{E}^{\mathbb{Q}^*}[Q_c \varphi^*(X_T)] - \langle \varphi^*, \mu_T \rangle < -\varepsilon$, for some $\varepsilon > 0$. Since $\mathbb{Q}^* \in \mathcal{A}(\mu_0)$, thanks to Theorem 4.5 and (5.1) we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^*}[C(X)] + \mathcal{I}_\ell(\mathbb{Q}^* | \mathbb{P}) &= \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu_0)} \left(\mathcal{I}_\ell(\mathbb{P}^\alpha | \mathbb{P}) + \mathbb{E}^{\mathbb{P}^\alpha}[Q_c \varphi^*(X_T) + C(X)] \right) - \langle \varphi^*, \mu_T \rangle \\ &\leq \mathcal{I}_\ell(\mathbb{Q}^* | \mathbb{P}) + \mathbb{E}^{\mathbb{Q}^*}[Q_c \varphi^*(X_T) + C(X)] - \langle \varphi^*, \mu_T \rangle \\ &< \mathcal{I}_\ell(\mathbb{Q}^* | \mathbb{P}) + \mathbb{E}^{\mathbb{Q}^*}[C(X)] - \varepsilon, \end{aligned}$$

which is a contradiction. Thus, $\langle \varphi^*, \mu_T \rangle = \mathbb{E}^{\mathbb{Q}^*}[Q_c \varphi^*(X_T)]$.

Consequently, (5.1) and (4.4) imply that

$$\begin{aligned} \mathcal{I}_\ell(\mathbb{Q}^* | \mathbb{P}) + \mathbb{E}^{\mathbb{Q}^*}[Q_c \varphi^*(X_T) + C(X)] &= \Psi^{\varphi^*}(\mu_0) \\ &= - \int_{\mathbb{R}^m} \sup_{\mathbb{Q} \ll \mathbb{P}_x} \left(\mathbb{E}^{\mathbb{Q}}[-Q_c \varphi^*(X_T) - C(X)] - \mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}_x) \right) \mu_0(dx) + \mathcal{I}_\ell(\mu_0 | \nu_0). \end{aligned} \quad (5.2)$$

Moreover, notice that because \mathcal{I}_ℓ is superadditive relative to \mathbb{P} , we also have

$$\begin{aligned} -\mathcal{I}_\ell(\mathbb{Q}^*|\mathbb{P}) - \mathbb{E}^{\mathbb{Q}^*}[Q_c\varphi^*(X_T) + C(X)] &\leq \int_{\mathbb{R}^m} \left(\mathbb{E}^{\mathbb{Q}_x^*}[-Q_c\varphi^*(X_T) - C(X)] - \mathcal{I}_\ell(\mathbb{Q}_x^*|\mathbb{P}_x) \right) \mu_0(dx) - \mathcal{I}_\ell(\mu_0|\nu_0) \\ &\leq \int_{\mathbb{R}^m} \sup_{\mathbb{Q} \ll \mathbb{P}_x} \left(\mathbb{E}^{\mathbb{Q}}[-Q_c\varphi^*(X_T) - C(X)] - \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}_x) \right) \mu_0(dx) - \mathcal{I}_\ell(\mu_0|\nu_0), \end{aligned} \quad (5.3)$$

which together with (5.2) implies that

$$\int_{\mathbb{R}^m} \sup_{\mathbb{Q} \ll \mathbb{P}_x} \left(\mathbb{E}^{\mathbb{Q}}[-Q_c\varphi^*(X_T) - C(X)] - \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}_x) \right) \mu_0(dx) = \int_{\mathbb{R}^m} \left(\mathbb{E}^{\mathbb{Q}_x^*}[-Q_c\varphi^*(X_T) - C(X)] - \mathcal{I}_\ell(\mathbb{Q}_x^*|\mathbb{P}_x) \right) \mu_0(dx).$$

It then follows from Theorem A.1 that

$$\frac{d\mathbb{Q}_x^*}{d\mathbb{P}_x} = \partial_x \ell^*(-Q_c\varphi^*(X_T) - C(X) - \psi_x^*), \quad \mathbb{Q}_x^*\text{-a.s., for } \mu_0\text{-a.e. } x \in \mathbb{R}^m, \quad (5.4)$$

where $\psi_x^* \in \mathbb{R}$ denotes the unique real number satisfying $\mathbb{E}^{\mathbb{P}_x}[\partial_x \ell^*(-Q_c\varphi^*(X_T) - C(X) - \psi_x^*)] = 1$.

We now show that the map $\iota : \mathbb{R}^m \ni x \mapsto \psi_x^*$ is $\mathcal{B}(\mathbb{R}^m)$ -measurable. To do so, recall that because $(\mathbb{P}_x)_{x \in \mathbb{R}^m}$ denotes the r.c.p.d. of \mathbb{P} given $\mathcal{F}_0 = \mathcal{B}(\mathbb{R}^m)$, it holds that for any $B \in \mathcal{F}_T$, the function $\mathbb{R}^m \ni x \mapsto \mathbb{E}^{\mathbb{P}_x}[\mathbf{1}_{\{B\}}]$ is $\mathcal{B}(\mathbb{R}^m)$ -measurable, and that, under Theorem 2.1, ℓ^* is continuously differentiable. We thus consider the measurable map $\pi : \mathbb{R}^m \times \mathbb{R} \ni (x, \psi) \mapsto \mathbb{E}^{\mathbb{P}_x}[\partial_x \ell^*(-Q_c\varphi^*(X_T) - C(X) - \psi)] \in \mathbb{R}$. It remains to note that for any measurable set $B \in \mathcal{B}(\mathbb{R}^m)$, $\pi^{-1}(\cdot, B)(1) = \iota^{-1}(B)$ is measurable.

Thus, we may write (5.4) as

$$\frac{d\mathbb{Q}_x^*}{d\mathbb{P}_x} = \partial_x \ell^*(-Q_c\varphi^*(X_T) - C(X) - \psi^*(x)), \quad \mathbb{Q}_x^*\text{-a.s., for } \mu_0\text{-a.e. } x \in \mathbb{R}^m \quad (5.5)$$

for a measurable function $\psi : \mathbb{R}^m \rightarrow \mathbb{R}$. Let $A \in \mathcal{F}_T$ and note $A = \bigcup_{x \in \mathbb{R}^m} A_x$, where $A_x := \{\omega \in A : X_0(\omega) = x\}$. Then by desintegration and (5.5), we have

$$\begin{aligned} \mathbb{Q}^*(A) &= \int_{\mathbb{R}^m} \int_{A_x} \mathbb{Q}_x^*(d\omega) \mu_0(dx) \\ &= \int_{\mathbb{R}^m} \frac{d\mu_0}{d\nu_0}(x) \int_{A_x} \mathbb{Q}_x^*(d\omega) \nu_0(dx) \\ &= \int_{\mathbb{R}^m} \frac{d\mu_0}{d\nu_0}(x) \int_{A_x} \partial_x \ell^*(-Q_c\varphi^*(X_T) - C(X) - \psi^*(X_0))(\omega) \mathbb{P}_x(d\omega) \nu_0(dx) \\ &= \int_A \frac{d\mu_0}{d\nu_0}(X_0) \partial_x \ell^*(-Q_c\varphi^*(X_T) - C(X) - \psi^*(X_0)) \mathbb{P}(d\omega). \end{aligned}$$

This establishes the result.

Alternatively, if Theorem 3.5.(i) holds, i.e., $\mu_0 = \nu_0$, then (5.3) now follows thanks to Theorem 4.4. This is the only step where superadditivity of \mathcal{I}_ℓ was needed. Thus, the result follows as above. \square

Proof of Theorem 3.8. We start by showing (i). Recall that, see Theorem A.3, for $c(x, \rho) = \int_{\mathbb{R}^m} \mathbf{1}_{\{x \neq y\}} \rho(dy)$, it holds $Q_c\hat{\varphi}(x) = \hat{\varphi}(x)$.

First, we show that

$$\frac{d\mathbb{Q}_x}{d\mathbb{P}_x} = \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(x)), \quad \mathbb{Q}_x\text{-a.s., for } \mu_0\text{-a.e. } x \in \mathbb{R}^m.$$

Let $f \in \mathbb{L}^0(\mathbb{Q})$. By definition of \mathbb{Q} , we have

$$\begin{aligned} \int_{\Omega} f(\omega) \mathbb{Q}(d\omega) &= \int_{\Omega} \frac{d\mu_0}{d\nu_0}(X_0) \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0)) f(\omega) \mathbb{P}(d\omega) \\ &= \int_{\mathbb{R}^m} \frac{d\mu_0}{d\nu_0}(x) \int_{\Omega_x} \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0)) f(\omega) \mathbb{P}_x(d\omega) \nu_0(dx) \\ &= \int_{\mathbb{R}^m} \int_{\Omega_x} \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(x)) f(\omega) \mathbb{P}_x(d\omega) \mu_0(dx), \end{aligned}$$

but also,

$$\int_{\Omega} f(\omega) \mathbb{Q}(\mathrm{d}\omega) = \int_{\mathbb{R}^m} \int_{\Omega_x} f(\omega) \mathbb{Q}_x(\mathrm{d}\omega) \mu_0(\mathrm{d}x).$$

The claim follows because by Radon-Nikodym theorem we have

$$\int_{\Omega_x} \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(x)) f(\omega) \mathbb{P}_x(\mathrm{d}\omega) = \int_{\Omega_x} f(\omega) \mathbb{Q}_x(\mathrm{d}\omega), \text{ for } \mu_0\text{-a.e. } x \in \mathbb{R}^m$$

so that applying Radon-Nikodym again yields the result.

Second, thanks to Theorem 3.1, we have that

$$\begin{aligned} V_c(\mu_0, \mu_T) &\geq - \int_{\mathbb{R}^m} \Phi_{\mathbb{P}_x}(-\hat{\varphi}(X_T) - C(X)) \mu_0(\mathrm{d}x) + \mathcal{I}_{\ell}(\mu_0|\nu_0) - \langle \hat{\varphi}, \mu_T \rangle \\ &= \mathbb{E}^{\mathbb{Q}}[\hat{\varphi}(X_T) + C(X)] - \langle \hat{\varphi}, \mu_T \rangle + \int_{\mathbb{R}^m} \mathcal{I}_{\ell}(\mathbb{Q}_x|\mathbb{P}_x) \mu_0(\mathrm{d}x) + \mathcal{I}_{\ell}(\mu_0|\nu_0) \\ &= \int_{\mathbb{R}^m} \mathcal{I}_{\ell}(\mathbb{Q}_x|\mathbb{P}_x) \mu_0(\mathrm{d}x) + \mathcal{I}_{\ell}(\mu_0|\nu_0) \geq \mathbb{E}^{\mathbb{Q}}[C(X)] + \mathcal{I}_{\ell}(\mathbb{Q}|\mathbb{P}) \geq V_c(\mu_0, \mu_T) \end{aligned}$$

where the first equality follows from the first step and (A.1). The second equality follows since $\mathbb{Q} \circ X_T^{-1} = \mu_T$, as $\hat{\mathbb{Q}} \in \mathcal{P}_c(\mu_0, \mu_T)$ for the choice of c in the statement, and thus $\mathbb{E}^{\mathbb{Q}}[\hat{\varphi}(X_T)] = \langle \hat{\varphi}, \mu_T \rangle$. The third step follows from Theorem 4.4 in the case $\mu_0 = \nu_0$ or by the subadditivity of \mathcal{I}_{ℓ} . The last step follows by the definition of $V_c(\mu_0, \mu_T)$. The equality to $V_c(\mu_0, \mu_T)$ of the second term implies that $\hat{\varphi}$ is dual optimal and equality to $\mathbb{E}^{\mathbb{Q}}[C(X)] + \mathcal{I}_{\ell}(\mathbb{Q}|\mathbb{P})$ implies \mathbb{Q} is primal optimal.

We now argue (ii). To this end, we first show that for \mathbb{Q} as in the statement, we have that

$$\frac{d\hat{\mathbb{Q}}}{d\hat{\mathbb{P}}} = \frac{d\hat{\mathbb{Q}}_0}{d\hat{\mathbb{P}}_0}(\hat{X}_0) \partial_x \ell^*(-\hat{\varphi}(\hat{X}_T) - C(\hat{X}) - \hat{\psi}(\hat{X}_0)). \quad (5.6)$$

First, letting $\{\hat{\mathbb{Q}}_x\}_{x \in \mathbb{R}^m}$ denote the r.c.p.d. of $\hat{\mathbb{Q}}$ with respect to $\sigma(\hat{X}_0)$, we claim that

$$(\hat{\mathbb{Q}})_x = \mathbb{Q}_x \circ \hat{\mathcal{T}}. \quad (5.7)$$

On the one hand, note that by definition of $(\hat{\mathbb{Q}})_x$ we have that for any measurable $\xi : \Omega \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow [0, \infty)$

$$\mathbb{E}^{\hat{\mathbb{Q}}}[\mathbb{E}^{\hat{\mathbb{Q}}_{\hat{X}_0}}[\xi]f(\hat{X}_0)] = \mathbb{E}^{\hat{\mathbb{Q}}}[\mathbb{E}^{\hat{\mathbb{Q}}}[\xi|\sigma(\hat{X}_0)]f(\hat{X}_0)] = \mathbb{E}^{\hat{\mathbb{Q}}}[\xi f(\hat{X}_0)] = \mathbb{E}^{\mathbb{Q}}[\xi \circ \hat{\mathcal{T}}^{-1}f(X_0)].$$

On the other hand, since $\mathbb{E}^{\mathbb{Q}_x \circ \hat{\mathcal{T}}}[\xi] = \mathbb{E}^{\mathbb{Q}_x}[\xi \circ \hat{\mathcal{T}}^{-1}]$, we have that

$$\mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}_{X_0} \circ \hat{\mathcal{T}}}[\xi]f(X_0)] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}_{X_0}}[\xi \circ \hat{\mathcal{T}}^{-1}]f(X_0)] = \mathbb{E}^{\mathbb{Q}}[\xi \circ \hat{\mathcal{T}}^{-1}f(X_0)]$$

That is

$$\mathbb{E}^{\hat{\mathbb{Q}}}[\mathbb{E}^{\hat{\mathbb{Q}}_{\hat{X}_0}}[\xi]f(\hat{X}_0)] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}_{X_0} \circ \hat{\mathcal{T}}}[\xi]f(X_0)]$$

The claim follows from the arbitrariness of ξ and f .

We now establish (5.6). Notice that, as in the proof of Theorem 3.8, we may deduce that

$$\frac{d\mathbb{Q}_x}{d\hat{\mathbb{P}}_x} = \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(x)), \text{ } \mathbb{Q}_x\text{-a.s., for } \mu_0\text{-a.e. } x \in \mathbb{R}^m. \quad (5.8)$$

Let $A \subseteq \Omega$, and note that

$$\begin{aligned} \hat{\mathbb{Q}}(A) &= \int_{\Omega} \mathbf{1}_A(\omega) \hat{\mathbb{Q}}(\mathrm{d}\omega) = \int_{\mathbb{R}^m} \int_{\Omega} \mathbf{1}_A(\omega) (\hat{\mathbb{Q}})_x(\mathrm{d}\omega) \hat{\mathbb{Q}}_0(\mathrm{d}x) \\ &= \int_{\mathbb{R}^m} \int_{\Omega} \mathbf{1}_A(\omega) \mathbb{Q}_x \circ \hat{\mathcal{T}}(\mathrm{d}\omega) \hat{\mathbb{Q}}_0(\mathrm{d}x) \\ &= \int_{\mathbb{R}^m} \int_{\Omega} \mathbf{1}_A(\hat{\mathcal{T}}^{-1}(\omega)) \partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0)) \mathbb{P}_x(\mathrm{d}\omega) \hat{\mathbb{Q}}_0(\mathrm{d}x) \\ &= \int_{\mathbb{R}^m} \frac{d\hat{\mathbb{Q}}_0}{d\hat{\mathbb{P}}_0}(x) \int_{\Omega} \mathbf{1}_A(\omega) \partial_x \ell^*(-\hat{\varphi}(X_T \circ \hat{\mathcal{T}}) - C(\hat{X}) - \hat{\psi}(X_0 \circ \hat{\mathcal{T}})) \mathbb{P}_x \circ \hat{\mathcal{T}}(\mathrm{d}\omega) \hat{\mathbb{P}}_0(\mathrm{d}x) \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^m} \int_{\Omega} \mathbf{1}_A(\omega) \frac{d\overleftarrow{\mathbb{Q}}_0}{d\overleftarrow{\mathbb{P}}_0}(x) \partial_x \ell^*(-\hat{\varphi}(\overleftarrow{X}_T) - C(\overleftarrow{X}) - \hat{\psi}(\overleftarrow{X}_0)) (\overleftarrow{\mathbb{P}})_x(d\omega) \overleftarrow{\mathbb{P}}_0(dx) \\
&= \int_{\Omega} \mathbf{1}_A(\omega) \frac{d\overleftarrow{\mathbb{Q}}_0}{d\overleftarrow{\mathbb{P}}_0}(\overleftarrow{X}_0) \partial_x \ell^*(-\hat{\varphi}(\overleftarrow{X}_T) - C(\overleftarrow{X}) - \hat{\psi}(\overleftarrow{X}_0)) \overleftarrow{\mathbb{P}}(d\omega).
\end{aligned}$$

where the third and sixth steps follow from (5.6), the fifth step is due to (5.7).

We now argue (3.3). The first equation follows since $\mathbb{Q}_0 = \mu_0$ using the density of \mathbb{Q} with respect to \mathbb{P} . Let us argue the second equation in (3.3). Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be bounded continuous and recall that because $\mathbb{Q}_T = \mu_T$, it follows from (5.6) that

$$\begin{aligned}
\int_{\mathbb{R}} f(x) \mu_T(dx) &= \mathbb{E}^{\mathbb{Q}}[f(X_T)] = \mathbb{E}^{\overleftarrow{\mathbb{P}}}[f(X_0)] = \mathbb{E}^{\overleftarrow{\mathbb{P}}}\left[f(X_0) \frac{d\overleftarrow{\mathbb{Q}}}{d\overleftarrow{\mathbb{P}}}\right] \\
&= \int_{\mathbb{R}} f(x) \frac{d\overleftarrow{\mathbb{Q}}_0}{d\overleftarrow{\mathbb{P}}_0}(x) \int_{\Omega} \partial_x \ell^*(-\hat{\varphi}(\overleftarrow{X}_T) - C(\overleftarrow{X}) - \hat{\psi}(\overleftarrow{X}_0)) \overleftarrow{\mathbb{P}}_x(d\omega) \overleftarrow{\mathbb{P}}_0(dx) \\
&= \int_{\mathbb{R}} f(x) \int_{\Omega} \partial_x \ell^*(-\hat{\varphi}(\overleftarrow{X}_T) - C(\overleftarrow{X}) - \hat{\psi}(\overleftarrow{X}_0)) \overleftarrow{\mathbb{P}}_x(d\omega) \mu_T(dx) \\
&= \int_{\mathbb{R}} f(x) \mathbb{E}^{\overleftarrow{\mathbb{P}}}[\partial_x \ell^*(-\hat{\varphi}(\overleftarrow{X}_T) - C(\overleftarrow{X}) - \hat{\psi}(\overleftarrow{X}_0)) | \overleftarrow{X}_0 = x] \mu_T(dx).
\end{aligned}$$

This ends the proof. \square

Proof of Theorem 3.9. If, in addition, \mathbb{P} is reversible, we claim that $\mathbb{E}^{\mathbb{P}}[\xi|\sigma(X_0)] = \mathbb{E}^{\mathbb{P}}[\xi|\sigma(X_T)]$ for any measurable function ξ . Assuming the claim for a moment, we have that

$$\begin{aligned}
&\int_{\mathbb{R}} f(x) \mathbb{E}^{\overleftarrow{\mathbb{P}}}[\partial_x \ell^*(-\hat{\varphi}(\overleftarrow{X}_T) - C(\overleftarrow{X}) - \hat{\psi}(\overleftarrow{X}_0)) | \overleftarrow{X}_0 = x] \mu_T(dx) \\
&= \int_{\mathbb{R}} f(x) \mathbb{E}^{\overleftarrow{\mathbb{P}}_x \circ \overleftarrow{\mathcal{T}}}[\partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0))] \mu_T(dx) \\
&= \int_{\mathbb{R}} f(x) \mathbb{E}^{\mathbb{P}^x}[\partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0))] \mu_T(dx) \\
&= \int_{\mathbb{R}} f(x) \mathbb{E}^{\mathbb{P}}[\partial_x \ell^*(-\hat{\varphi}(X_T) - C(X) - \hat{\psi}(X_0) | X_T = x)] \mu_T(dx)
\end{aligned}$$

It remains to show the $\mathbb{E}^{\mathbb{P}}[\xi|\sigma(X_0)] = \mathbb{E}^{\mathbb{P}}[\xi|\sigma(X_T)]$ for any measurable function ξ . This is obtained by

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\xi|\sigma(X_0)] f(X_0)] &= \mathbb{E}^{\mathbb{P}}[\xi f(X_0)] = \mathbb{E}^{\overleftarrow{\mathbb{P}}}[\xi f(X_0)] = \mathbb{E}^{\mathbb{P}}[\xi \circ \overleftarrow{\mathcal{T}}^{-1} f(X_T)] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\xi \circ \overleftarrow{\mathcal{T}}^{-1} |\sigma(X_T)] f(X_T)] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}_{X_T} \circ \overleftarrow{\mathcal{T}}}[\xi] f(X_T)] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{(\overleftarrow{\mathbb{P}})_{X_T}}[\xi] f(X_T)] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}_{X_T}}[\xi] f(X_T)] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\xi|\sigma(X_T)] f(X_T)]
\end{aligned}$$

where the second and last steps follow from the reversibility of \mathbb{P} and the sixth step is due to (5.7) since $\sigma(X_T) = \sigma(\overleftarrow{X}_0)$. This concludes the proof. \square

5.3 Dual characterization of the marginals

The proof of Theorem 3.6 will build upon elements of optimal stochastic control as well as properties of time-reversed diffusions. Before presenting the proof, let us derive some intermediate results from stochastic optimal control theory. In the rest of the paper, we assume $C = 0$ and $b(t, x) = -\partial_x U(x)/2$.

5.3.1 Tidbits of stochastic optimal control theory

In this subsection we take a closer look at the stochastic optimal control problem with value

$$\Psi^\varphi(\mu) - \mathcal{I}_\ell(\mu|\nu_0) = \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^T \frac{1}{2} \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + Q_c \varphi(X_T) \right], \quad dZ_t = Z_t \alpha_t \cdot dW_t, \quad Z_0 = 1. \quad (5.9)$$

We have already seen that for every $\varphi \in C_{b,p}(\mathbb{R}^m)$, we have

$$\Psi^\varphi(\mu) - \mathcal{I}_\ell(\mu|\nu_0) = \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu)} \left(\mathcal{I}_\ell(\mathbb{P}^\alpha | \mathbb{P}) + \mathbb{E}^{\mathbb{P}^\alpha}[Q_c \varphi(X_T)] \right),$$

see Theorem 4.5. Since φ (and thus $Q_c\varphi(X_T)$) is bounded from below, it follows exactly as in the proof of Theorem 4.2 that an optimal measure $\mathbb{Q} \in \mathcal{P}(\mu)$ exists. Thus, by Theorem 4.3 we deduce that the problem (5.9) admits an optimal $\mathbb{P}^{\alpha^*} \in \mathcal{A}(\mu)$ for a control process $\alpha^* \in \mathbb{H}_{loc}^2$. Our goal in this section is the further analyze properties of the optimizer α^* .

Proposition 5.1. *Let Assumption 2.1 be satisfied and put $\ell(x) = x \log(x) - x + 1$. Then for any $\varphi \in B_{b,p}(\mathbb{R}^m)$ the problem (5.9) admits a Markovian optimal control α^* satisfying*

$$\alpha_t^* = -\mathcal{Z}_t = -u(t, X_t) \quad \mathbb{P} \otimes dt\text{-a.s.}$$

where $(\mathcal{Y}, \mathcal{Z})$ is a pair of progressively measurable processes satisfying

$$\mathcal{Y}_t = Q_c\varphi(X_T) + \int_t^T \frac{1}{2}|\mathcal{Z}_s|^2 ds - \int_t^T \mathcal{Z}_s dW_s \quad (5.10)$$

and $u : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ a Borel-measurable function.

If in addition $Q_c\varphi$ and $\partial_x U$ are continuously differentiable, then $\alpha_t^* = \partial_x v(t, X_t)$ for all $t \in [0, T]$ where v is the classical solution of the PDE

$$\begin{cases} \partial_t v(t, x) - \frac{1}{2}\partial_x v(t, x)\partial_x U(x) + \frac{1}{2}\partial_{xx}v(t, x) - \frac{1}{2}|\partial_x v(t, x)|^2 = 0 \\ v(T, x) = Q_c\varphi(x). \end{cases} \quad (5.11)$$

Proof. If $\ell(x) = x \log(x) - x + 1$, then the stochastic optimal control problem becomes

$$\Psi^\varphi(\mu) - \mathcal{I}_\ell(\mu|\nu_0) = \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\mu)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\frac{1}{2} \int_0^T \|\alpha_t\|^2 dt + Q_c\varphi(X_T) \right].$$

In this case, it is classical that the optimal control is Markovian. See e.g. [35, Theorem 2.3]. The representation $\alpha_t^* = -\mathcal{Z}_t$ with $(\mathcal{Y}, \mathcal{Z})$ satisfying (5.10) follows e.g. by [49, Theorem A.1].

Let M be the continuous martingale $M_t := \mathbb{E}[e^{Q_c\varphi(X_T)} | \mathcal{F}_t]$. Let N_t be the unique process such that $M_t = Q_c\varphi(X_T) - \int_t^T N_s dW_s$. Then, it follows by Itô's formula applied to $\mathcal{Y}_t = \log(M_t)$ that $\mathcal{Z}_t = N_t/M_t$. Moreover, by [17, Lemma 3.3] there are two Borel-measurable functions $e : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $d : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^d$ such that $M_t = e(t, X_t)$ and $N_t = d(t, X_t)$. Thus it holds that $\mathcal{Y}_t = v(t, X_t)$ and $\mathcal{Z}_t = u(t, X_t)$ for two Borel-measurable functions u and v .

If $Q_c\varphi$ and b are continuously differentiable, then it follows by [41, Theorem 3.1] that $\alpha_t^* = \partial_x v(t, X_t)$ where v solves the PDE (5.11). \square

Proposition 5.2. *Let Theorem 2.1 be satisfied and put $\ell(x) = \frac{(x-1)^2}{2}$ for $x \geq 0$ and $\ell(x) = +\infty$ for $x < 0$. Assume that $Q_c\varphi$ and $\partial_x U$ are continuously differentiable with bounded derivatives. Then for any $\varphi \in B_{b,p}(\mathbb{R}^m)$ the problem (5.9) admits a Markovian optimal control α^* satisfying*

$$\alpha_t^* = -\frac{\partial_x v(t, X_t)}{\mathcal{Z}_t} \quad \mathbb{P} \otimes dt\text{-a.s.} \quad (5.12)$$

where v and \tilde{v} are classical solutions of the respective PDEs

$$\begin{cases} \partial_t v(t, x) - \frac{1}{2}\partial_x v(t, x)\partial_x U(x) + \frac{1}{2}\partial_{xx}v(t, x) - \frac{1}{2}|\partial_x v(t, x)|^2 = 0 \\ v(T, x) = 0 \end{cases} \quad (5.13)$$

and

$$\begin{cases} \partial_t \tilde{v}(t, x) - \frac{1}{2}\partial_x \tilde{v}(t, x)\partial_x U(x) + \frac{1}{2}\partial_{xx}\tilde{v}(t, x) = 0 \\ \tilde{v}(T, x) = Q_c\varphi(x). \end{cases} \quad (5.14)$$

Remark 5.3. *The optimal control process α^* is indeed Markovian because X and Z are both state processes. The crucial difference is that α^* does not depend on Z in the entropic case. We believe that this is the only case where α^* does not depend on Z .*

Proof. The Hamilton-Jacobi-Bellman (HJB) equation associated to the stochastic control problem (5.9) is given by

$$\begin{cases} \partial_t V - \frac{1}{2}\partial_x U\partial_x V + \frac{1}{2}\partial_{xx}V + \inf_{a \in \mathbb{R}^m} \left\{ \frac{1}{2}z^2|a|^2 + \frac{1}{2}(2za\partial_{xz}V + z^2|a|^2\partial_{zz}V) \right\} = 0 \\ V(T, x, z) = zQ_c\varphi(x). \end{cases} \quad (5.15)$$

By the standard verification theorem in optimal stochastic control theorem, see e.g., [19; 56], we know that the optimal control is given by $\alpha^* = \arg \min_{a \in \mathbb{R}^m} \left\{ \frac{1}{2} Z_t^2 |a|^2 + \frac{1}{2} (2Z_t a \partial_{xz} V(t, X_t, Z_t) + Z_t^2 |a|^2 \partial_{zz} V(t, X_t, Z_t)) \right\}$ provided that V is a classical solution of the HJB (5.15). Thus if V is a classical solution, then we have

$$\alpha^* = -\frac{\partial_{xz} V(t, X_t, Z_t)}{Z_t(1 + \partial_{zz} V(t, X_t, Z_t))}.$$

To solve the HJB equation, we make the ansatz $V(t, x, z) = v(t, x) + z\tilde{v}(t, x)$ with $z > 0$ and $(t, x) \in [0, T] \times \mathbb{R}^m$. It then follows that V solves the HJB equation if v and \tilde{v} satisfy (5.13) and (5.13). It follows from Friedman [22, Theorems 1.7.12 and 2.4.10] that equation (5.14) admits a unique classical solution \tilde{v} . This solution has Lipschitz derivative. In fact, it follows by Feynman-Kac formula that $\tilde{v}(t, x) = \mathbb{E}[Q_c \varphi(X_T^{t,x})]$ where $X^{t,x}$ solves the SDE (2.1) with $X_t = x$. Thus, we have that $\partial_x \tilde{v}(t, x) = \mathbb{E}[\partial_x Q_c(X_T^{t,x}) \nabla X_T^{t,x}]$ where $\nabla X^{t,x}$ is the variational process satisfying $d\nabla X_s^{t,x} = -\frac{1}{2} \partial_x U(X_s^{t,x}) \nabla X_s^{t,x} ds$. Since $\partial_x U$ is Lipschitz continuous and $\partial_x Q_c \varphi$ is bounded, it is direct to check that $\partial_x v$ is Lipschitz. Then applying Friedman [22, Theorems 1.7.12 and 2.4.10] again, (5.14) also admits a unique classical solution. This shows that V solves the HJB equation and therefore that an optimal control is given by $\alpha_t = -\frac{\partial_x v(t, X_t)}{Z_t}$. \square

Lemma 5.4. *Let Assumptions 2.1 and 3.5 be satisfied. Assume that $\varphi^* \in C_{b,p}(\mathbb{R}^m)$ is a dual optimizer for $V_c(\mu_0, \mu_T)$ and α^* is optimal for $\Psi^{\varphi^*}(\mu_0)$. Then the measure \mathbb{P}^{α^*} is the primal optimizer for $V_c(\mu_0, \mu_T)$. In particular, $\mathbb{P}^{\alpha^*} = \mathbb{Q}^*$.*

Proof. Let us first assume that (i) in Assumption 3.5 holds. If $\varphi^* \in C_{b,p}(\mathbb{R}^m)$ is dual optimal for $V_c(\mu_0, \mu_T)$ and α^* is optimal for $\Psi^{\varphi^*}(\mu_0) - \mathcal{I}_\ell(\mu_0|\nu_0)$, then subsequently using Corollary 4.4 and the desintegration theorem, optimality of α^* , definition of $\Psi^{\varphi^*}(\mu_0)$ and Theorem A.1, we have

$$\begin{aligned} \int_{\mathbb{R}^m} \left(\mathcal{I}_\ell(\mathbb{P}_x^{\alpha^*} | \mathbb{P}_x) + \mathbb{E}^{\mathbb{P}_x^{\alpha^*}} [Q_c \varphi^*(X_T)] \right) \mu_0(dx) &= \mathcal{I}_\ell(\mathbb{P}^{\alpha^*} | \mathbb{P}) + \mathbb{E}^{\mathbb{P}^{\alpha^*}} [Q_c \varphi^*(X_T)] \\ &= \Psi^{\varphi^*}(\mu_0) = - \int_{\mathbb{R}^m} \Phi_{\mathbb{P}_x}(-Q_c \varphi^*(X_T)) \mu_0(dx) \\ &= - \int_{\mathbb{R}^m} \sup_{\mathbb{Q} \ll \mathbb{P}_x} (\mathbb{E}^{\mathbb{Q}}[-Q_c \varphi^*(X_T)] - \mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}^x)) \mu_0(dx). \end{aligned}$$

If we instead have (ii) in Assumption 3.5, then by superadditivity of \mathcal{I}_ℓ and arguing as above

$$\begin{aligned} \int_{\mathbb{R}^m} \left(\mathcal{I}_\ell(\mathbb{P}_x^{\alpha^*} | \mathbb{P}_x) + \mathbb{E}^{\mathbb{P}_x^{\alpha^*}} [Q_c \varphi^*(X_T)] \right) \mu_0(dx) &\leq \mathcal{I}_\ell(\mathbb{P}^{\alpha^*} | \mathbb{P}) + \mathbb{E}^{\mathbb{P}^{\alpha^*}} [Q_c \varphi^*(X_T)] \\ &= - \int_{\mathbb{R}^m} \sup_{\mathbb{Q} \ll \mathbb{P}_x} (\mathbb{E}^{\mathbb{Q}}[-Q_c \varphi^*(X_T)] - \mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}^x)) \mu_0(dx). \end{aligned}$$

Since $\mathbb{P}_x^{\alpha^*} \ll \mathbb{P}_x$, this yields

$$\int_{\mathbb{R}^m} \left(\mathcal{I}_\ell(\mathbb{P}_x^{\alpha^*} | \mathbb{P}_x) + \mathbb{E}^{\mathbb{P}_x^{\alpha^*}} [Q_c \varphi^*(X_T)] \right) \mu_0(dx) = - \int_{\mathbb{R}^m} \sup_{\mathbb{Q} \ll \mathbb{P}_x} (\mathbb{E}^{\mathbb{Q}}[-Q_c \varphi^*(X_T)] - \mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}^x)) \mu_0(dx).$$

This implies that $\mathcal{I}_\ell(\mathbb{P}_x^{\alpha^*} | \mathbb{P}_x) + \mathbb{E}^{\mathbb{P}_x^{\alpha^*}} [Q_c \varphi^*(X_T)] = - \sup_{\mathbb{Q} \ll \mathbb{P}_x} (\mathbb{E}^{\mathbb{Q}}[-Q_c \varphi^*(X_T)] - \mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}^x))$ μ_0 -a.s. Thus, it follows by uniqueness of the optimizer in Theorem A.1 that $\frac{d\mathbb{P}_x^{\alpha^*}}{d\mathbb{P}_x} = \partial_x \ell^*(-Q_c \varphi^*(X_T) - \phi_x^*)$ $\mathbb{P}_x^{\alpha^*}$ -a.s., μ_0 -a.s with ϕ_x^* as in (5.4). Arguing as in the proof of Theorem 3.6, this shows that $\mathbb{P}^{\alpha^*} = \mathbb{Q}^*$, which concludes the proof. \square

5.3.2 Time reversal of diffusions and proofs of the characterizations of the marginal distribution

Recall the time reversal mapping $\overleftarrow{\mathcal{T}}$ defined in the introduction. Observe in particular that with respect to the canonical process X we have

$$X_t(\overleftarrow{\mathcal{T}}(\omega)) := X_{T-t}(\omega) = \omega_{T-t}.$$

Recall that a probability measure is said to be reversible if $\overleftarrow{\mathbb{Q}} = \mathbb{Q}$. It is a classical result, see Pavliotis [46, Proposition 4.5], that whenever $b(x) = -\partial_x U(x)/2$ for smooth U , the SDE (2.1) admits an invariant measure λ and taking $\nu_0 = \lambda$ it follows that \mathbb{P} , the weak solution to (2.1), is reversible.

Proposition 5.5. *Let Theorem 2.1 be satisfied, $\nu_0 = \lambda$, and assume that c is such that $\mathcal{W}_c(\mu, \nu) = 0$ if and only if $\mu = \nu$ and $b = -\partial_x U/2$ for some differentiable function $U : \mathbb{R}^m \rightarrow \mathbb{R}$. Let Φ^φ be given by (3.2). Then the following hold:*

(i) The optimal transport problem with value $V_c(\mu_T, \mu_0)$ admits the convex dual representation

$$V_c(\mu_T, \mu_0) = \sup_{\varphi \in C_{b,p}(\mathbb{R}^m)} (\Psi^\varphi(\mu_T) + \langle \varphi, \mu_0 \rangle). \quad (5.16)$$

(ii) If $\mathcal{P}_c(\mu_T, \mu_0) \neq \emptyset$, then $V_c(\mu_T, \mu_0)$ admits a unique primal optimizer $\hat{\mathbb{Q}}$ and it holds $\hat{\mathbb{Q}} = \overleftarrow{\mathbb{Q}}^*$.

(iii) Assume that $V_c(\mu_0, \mu_T)$ admits a dual optimizer $\varphi^* \in C_{b,p}(\mathbb{R}^m)$ and $V_c(\mu_T, \mu_0)$ admits a dual optimizer $\hat{\varphi} \in C_{b,p}(\mathbb{R}^m)$. Then there are two progressively measurable processes α^* and $\hat{\alpha}$ such that

$$\mathbb{E}^{\mathbb{Q}^*} \left[\alpha_t^* + \hat{\alpha}_{1-t} \circ \overleftarrow{\mathcal{T}} | X_t = x \right] - \partial_x U(x) = \nabla \log(\mathbb{Q}_t^*(x)). \quad (5.17)$$

In particular, α^* and $\hat{\alpha}$ are the optimal control of the stochastic optimal control problem with value $\Psi^{\varphi^*}(\mu_0) - \mathcal{I}_\ell(\mu_0|\nu)$ and $\Psi^{\hat{\varphi}}(\mu_T) - \mathcal{I}_\ell(\mu_T|\nu_0)$, respectively.

Proof. (i): The representation (5.16) follows from the convex duality Theorem 3.1(ii).

(ii): If $\mathcal{P}_c(\mu_T, \mu_0) \neq \emptyset$, then the existence of a unique primal optimizer $\hat{\mathbb{Q}}$ follows by Theorem 3.1(i).

Since $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is clearly a measurable function of $\overleftarrow{\mathcal{T}}$ we obtain by Theorem A.2 that $\mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) = \mathcal{I}_\ell(\overleftarrow{\mathbb{Q}}|\overleftarrow{\mathbb{P}})$. In addition, using that $\mathcal{W}_c(\mu, \nu) = 0$ if and only if $\mu = \nu$, it is directly checked that $\mathbb{Q} \in \mathcal{P}_c(\mu_T, \mu_0)$ if and only if $\overleftarrow{\mathbb{Q}} \in \mathcal{P}_c(\mu_0, \mu_T)$. Furthermore, since the drift is a time-independent gradient, it follows e.g. by [46, Proposition 4.5.] that \mathbb{P} is reversible. This allows us to get

$$\begin{aligned} \mathcal{I}_\ell(\overleftarrow{\mathbb{Q}}^*|\mathbb{P}) &\geq V_c(\mu_T, \mu_0) = \inf_{\mathbb{Q} \in \mathcal{P}_c(\mu_T, \mu_0)} \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) = \inf_{\mathbb{Q} \in \mathcal{P}_c(\mu_T, \mu_0)} \mathcal{I}_\ell(\overleftarrow{\mathbb{Q}}|\overleftarrow{\mathbb{P}}) \\ &= \inf_{\mathbb{Q} \in \mathcal{P}_c(\mu_0, \mu_T)} \mathcal{I}_\ell(\mathbb{Q}|\mathbb{P}) = \mathcal{I}_\ell(\mathbb{Q}^*|\mathbb{P}) = \mathcal{I}_\ell(\overleftarrow{\mathbb{Q}}^*|\overleftarrow{\mathbb{P}}) = \mathcal{I}_\ell(\overleftarrow{\mathbb{Q}}^*|\mathbb{P}), \end{aligned}$$

showing by uniqueness that $\hat{\mathbb{Q}} = \overleftarrow{\mathbb{Q}}^*$.

(iii) We have already argued (just before Proposition 5.1) that there exists a control process α^* that is optimal for $\Psi^{\varphi^*}(\mu_0) - \mathcal{I}_\ell(\mu_0|\nu_0)$. And arguing the same way but replacing $Q_c \varphi$ by $\overleftarrow{\varphi} \in C_{b,p}(\mathbb{R}^d)$, it follows that $\Psi^{\hat{\varphi}}(\mu_T) - \mathcal{I}_\ell(\mu_T|\nu_0)$ admits an optimal control $\hat{\alpha}$. Using Theorem 5.4 now allows to get that $\mathbb{P}^{\alpha^*} = \mathbb{Q}^*$ and $\mathbb{P}^{\hat{\alpha}} = \hat{\mathbb{Q}} = \overleftarrow{\mathbb{Q}}^*$. Thus, $\mathbb{P}^{\hat{\alpha}} = \mathbb{P}^{\alpha^*}$. By Girsanov's theorem the canonical process X hence satisfies

$$\begin{cases} dX_t = \alpha_t^* - \frac{1}{2} \partial_x U(X_t) dt + dW_t^{\alpha^*} & \mathbb{Q}^*\text{-a.s. with } \mathbb{Q}^* \circ X_0^{-1} = \mu_0 \\ dZ_t^{\alpha^*} = |\alpha_t^*|^2 Z_t^{\alpha^*} dt + \alpha_t^* Z_t^{\alpha^*} dW_t^{\alpha^*} & \mathbb{Q}^*\text{-a.s.} \end{cases} \quad (5.18)$$

and

$$\begin{cases} dX_t = \hat{\alpha}_t - \frac{1}{2} \partial_x U(X_t) dt + dW_t^{\hat{\alpha}} & \overleftarrow{\mathbb{Q}}^*\text{-a.s. with } \overleftarrow{\mathbb{Q}}^* \circ X_0^{-1} = \mu_T \\ dZ_t^{\hat{\alpha}} = |\hat{\alpha}_t|^2 Z_t^{\hat{\alpha}} dt + \hat{\alpha}_t Z_t^{\hat{\alpha}} dW_t^{\hat{\alpha}} & \overleftarrow{\mathbb{Q}}^* \end{cases} \quad (5.19)$$

where W^{α^*} and $W^{\hat{\alpha}}$ are \mathbb{Q}^* - and $\overleftarrow{\mathbb{Q}}^*$ -Brownian motions, respectively. Thus, by [20, Theorem 3.10], the law $\mathbb{Q}_t^* := \mathbb{Q}^* \circ X_t^{-1}$ is absolutely continuous and its density $\mathbb{Q}_t^*(\cdot)$ satisfies

$$\mathbb{E}^{\mathbb{Q}^*} \left[\alpha_t^* - \frac{1}{2} \partial_x U(X_t) + \{\hat{\alpha}_{T-t} - \frac{1}{2} \partial_x U(X_{T-t})\} \circ \overleftarrow{\mathcal{T}} | X_t = x \right] = \nabla \log(\mathbb{Q}_t^*(x))$$

for almost all t and for \mathbb{Q}_t^* -almost all x . Therefore, we obtain (5.17). \square

Proofs of Theorem 3.10 and Theorem 3.11. To conclude the proof of Theorem 3.10, observe that (3.4) was derived in Theorem 5.5.

Let $\ell(x) = x \log(x) - x + 1$, then it follows by Theorem 5.1 that $\alpha_t^* = f_t(X_t)$ and $\hat{\alpha}_t = g_t(X_t)$ for two Borel measurable functions. In particular, α^* and $\hat{\alpha}$ do not depend on Z^{α^*} or $Z^{\hat{\alpha}}$. Plugging this in Equation (3.4) directly yields Theorem 3.11(i). Under the additional assumptions in Theorem 3.11(ii), it follows by Theorem 5.1 that the optimal control α^* is given by $\alpha_t^* = \partial_x v(t, X_t)$ where v is the classical solution of the PDE (5.11). Arguing exactly as in Theorem 5.1 with $Q_c \varphi$ therein replaced by $\overleftarrow{\varphi}$, we have again that $\hat{\alpha}_t = \partial_x \overleftarrow{v}(t, X_t)$ where \overleftarrow{v} satisfies the PDE (5.11) but with terminal condition $\overleftarrow{\varphi}$. Plugging these values of α^* and $\hat{\alpha}$ in Equation (3.4) yields the result. \square

Proof of Theorem 3.13. Let $\ell(x) = (x-1)^2/2$ if $x \geq 0$ and $+\infty$ else; then we know by Theorem 5.2 that the optimal control in (5.9) is given by (5.12). Thus, the equation 5.18 becomes

$$\begin{cases} dX_t = -\frac{\partial_x v(t, X_t)}{Z_t^{\alpha^*}} - \frac{1}{2} \partial_x U(X_t) dt + dW_t^{\alpha^*} & \mathbb{Q}^*\text{-a.s. with } \mathbb{Q}^* \circ X_0^{-1} = \mu_0 \\ dZ_t^{\alpha^*} = \frac{|\partial_x v(t, X_t)|^2}{Z_t^{\alpha^*}} dt - \partial_x v(t, X_t) \cdot dW_t^{\alpha^*} & \mathbb{Q}^*\text{-a.s.} \end{cases} \quad (5.20)$$

That is, the measure \mathbb{Q}^* solves the martingale problem with drift B and volatility Σ , where

$$B_t(x, z) = \begin{pmatrix} -\frac{\partial_x v(t, x)}{z} - \frac{1}{2} \partial_x U(x) \\ \frac{|\partial_x v(t, x)|^2}{z} \end{pmatrix} \quad \text{and } \Sigma_t(x, z) = \begin{pmatrix} I_m \\ -\partial_x v(t, x)^\top \end{pmatrix} \quad (5.21)$$

and I_m being the identity matrix of $\mathbb{R}^{m \times m}$. That is, the canonical process \mathcal{Y} satisfies

$$d\mathcal{Y}_t = B_t(\mathcal{Y}_t)dt + \Sigma_t(\mathcal{Y}_t)dW_t, \quad \mathbb{Q}^*\text{-a.s.} \quad (5.22)$$

By construction, we have $\mathcal{H}(\mathbb{Q}^* | \mathbb{P}) < \infty$, where \mathbb{P} solves the martingale problem with drift \mathcal{B} and volatility Σ , with

$$\mathcal{B}(x) = \begin{pmatrix} -\frac{1}{2} \partial_x U(x) \\ 0 \end{pmatrix}.$$

Since the functions $\partial_x U$ and $\partial_x v$ are Lipschitz-continuous, \mathbb{P} is unique. Thus, by [11, Theorem 1.3], the time reversal $\overleftarrow{\mathbb{P}}$ of \mathbb{P} satisfies the martingale problem with drift $\overleftarrow{\mathcal{B}}_{T-t}(x, z) = -\mathcal{B}(x) + \nabla \cdot \Sigma \Sigma_t^\top(x, z) + \Sigma \Sigma_t^\top(x, z) \nabla \log \mathbb{P}_t(x, z)$ and volatility $\overleftarrow{\Sigma}_t(x, z) = \Sigma_{T-t}(x, z)$, where $\mathbb{P}_t = \mathbb{P} \circ (X_t, Z_t)^{-1}$. Note that,

$$\nabla \cdot \Sigma \Sigma_t^\top(x, z) = \begin{pmatrix} 0_m \\ -\Delta v(t, x) \end{pmatrix}.$$

Applying Cattiaux et al. [11, Theorem 1.16] again it follows that the time reversal $\overleftarrow{\mathbb{Q}}^*$ of \mathbb{Q}^* solves the martingale problem with drift $\overleftarrow{\mathcal{B}}$ and volatility $\overleftarrow{\Sigma}$, with

$$\overleftarrow{\mathcal{B}}_{T-t}(x, z) = -B_t(x, z) + \mathcal{B}(x) + \overleftarrow{\mathcal{B}}_{T-t}(x, z) + \Sigma \Sigma_t^\top(x, z) \nabla \log \rho_t \quad dt \mathbb{Q}_t^*\text{-a.s.}$$

with $\rho_t = d\mathbb{Q}_t^*/d\mathbb{P}_t$ and $\mathbb{Q}_t^* = \mathbb{Q}^* \circ (X_t, Z_t)^{-1}$, and where the derivative is in the sense of distributions. Spelling this out, using the fact that $\log(\rho_t) = \log(\mathbb{Q}_t^*) - \log(\mathbb{P}_t)$, we have

$$\overleftarrow{\mathcal{B}}_{T-t}(x, z) = \begin{pmatrix} \frac{\partial_x v(t, x)}{z} + \frac{1}{2} \partial_x U(x) \\ -\frac{|\partial_x v(t, x)|^2}{z} - \Delta v(t, x) \end{pmatrix} + \Sigma \Sigma_t^\top(x, z) \nabla \log \mathbb{Q}_t^*(x, z)$$

On the other hand, now specializing Equation 5.19 to the current choice of ℓ yields

$$\begin{cases} dX_t = -\frac{\partial_x \overleftarrow{v}(t, X_t)}{Z_t} - \frac{1}{2} \partial_x U(X_t) dt + dW_t^{\hat{\alpha}} & \overleftarrow{\mathbb{Q}}^* \text{-a.s. with } \overleftarrow{\mathbb{Q}}^* \circ X_0^{-1} = \mu_T \\ dZ_t^{\hat{\alpha}} = \frac{|\sigma^\top \partial_x \overleftarrow{v}(t, X_t)|^2}{Z_t^{\hat{\alpha}}} dt - \partial_x \overleftarrow{v}(t, X_t) \cdot dW_t^{\hat{\alpha}} & \overleftarrow{\mathbb{Q}}^*. \end{cases}$$

That is, the probability measure $\overleftarrow{\mathbb{Q}}^*$ solves the martingale problem with drift $\widehat{\mathcal{B}}$ and volatility $\widehat{\Sigma}$ given by

$$\widehat{\mathcal{B}}_t(x, z) = \begin{pmatrix} -\frac{\partial_x \overleftarrow{v}(t, x)}{z} - \frac{1}{2} \partial_x U(x) \\ \frac{|\partial_x \overleftarrow{v}(t, x)|^2}{z} \end{pmatrix} \quad \text{and } \widehat{\Sigma}_t(x, z) = \begin{pmatrix} I_m \\ \partial_x \overleftarrow{v}(t, x) \end{pmatrix}. \quad (5.23)$$

In other words, under $\overleftarrow{\mathbb{Q}}^*$ the canonical process \mathcal{Y} on $\mathcal{C}_m \times \mathcal{C}_1$ satisfies the SDE

$$d\mathcal{Y}_t = \widehat{\mathcal{B}}_t(\mathcal{Y}_t)dt + \widehat{\Sigma}_t(\mathcal{Y}_t)dW_t, \quad \overleftarrow{\mathbb{Q}}^*\text{-a.s.} \quad (5.24)$$

By (5.24) and (5.22) we thus deduce that $\widehat{\mathcal{B}} = \overleftarrow{\mathcal{B}}$ and $\widehat{\Sigma} = \overleftarrow{\Sigma}$. That is,

$$\begin{pmatrix} -\frac{\partial_x \overleftarrow{v}(T-t, x)}{z} - \frac{1}{2} \partial_x U(x) \\ \frac{|\partial_x \overleftarrow{v}(T-t, x)|^2}{z} \end{pmatrix} = \begin{pmatrix} \frac{\partial_x v(t, x)}{z} + \frac{1}{2} \partial_x U(x) \\ -\frac{|\partial_x v(t, x)|^2}{z} - \Delta v(t, x) \end{pmatrix} + \Sigma \Sigma_t^\top(x, z) \nabla \log \mathbb{Q}_t^*(x, z),$$

which is tantamount to the desired result. \square

A Appendix

In this appendix we provide full justification or references for some steps in the proofs. Some of the results might be well-known, we provide details because we could not find a directly citable reference. We begin by recalling the duality properties of divergences.

Lemma A.1. Let ℓ satisfy Theorem 2.1.(i). For every $\bar{\mathbb{P}} \in \text{Prob}(\Omega)$, the functions $\mathcal{I}_\ell(\cdot|\bar{\mathbb{P}})$ and $\Phi_{\bar{\mathbb{P}}}$ are conjugate of one another. More precisely, it holds

$$\mathcal{I}_\ell(\mathbb{Q}|\bar{\mathbb{P}}) = \sup_{\xi \in \mathbb{L}^0} (\mathbb{E}^\mathbb{Q}[\xi] - \Phi_{\bar{\mathbb{P}}}(\xi)) \quad \text{for all } \mathbb{Q} \ll \bar{\mathbb{P}} \quad (\text{A.1})$$

and

$$\Phi_{\bar{\mathbb{P}}}(\xi) = \sup_{\mathbb{Q} \ll \bar{\mathbb{P}}} (\mathbb{E}^\mathbb{Q}[\xi] - \mathcal{I}_\ell(\mathbb{Q}|\bar{\mathbb{P}})), \quad \text{for all } \xi \text{ such that } \mathbb{E}^{\bar{\mathbb{P}}}[\ell^*(\xi^+)] < \infty.$$

Moreover, for each $\xi \in \mathbb{L}^0$ such that $\mathbb{E}^{\bar{\mathbb{P}}}[\ell^*(\xi^+)] < \infty$, there is $\mathbb{Q}^* \ll \bar{\mathbb{P}}$ such that $\Phi_{\bar{\mathbb{P}}}(\xi) = \mathbb{E}^{\mathbb{Q}^*}[\xi] - \mathcal{I}_\ell(\mathbb{Q}^*|\bar{\mathbb{P}})$. Such an optimizer is given by

$$\frac{d\mathbb{Q}^*}{d\bar{\mathbb{P}}} = \partial_x \ell^*(\xi - r^*), \quad (\text{A.2})$$

where $r^* \in \mathbb{R}$ is uniquely defined by $\mathbb{E}^{\bar{\mathbb{P}}}[\partial_x \ell^*(\xi - r^*)] = 1$. This constant also satisfies $\Phi_{\bar{\mathbb{P}}}(\xi) = \mathbb{E}^{\bar{\mathbb{P}}}[\ell^*(\xi - r^*)] + r^*$.

Proof. The duality statement is the variational representation of f -divergences, see [6, Theorem 4.4] or [48, Theorem 7.27]. The existence and characterization of optimizers is [13, Theorem 2.3]. \square

The following result can be seen as (an instance of) the data processing inequality, well known in the entropic case. In the context of mathematical statistics, it pertains to the so-called sufficiency of a statistic $f : \Omega \rightarrow \Omega$. The following result appeared first in [40], though in a slightly different setting due to the formulation of divergences. For the sake of completeness and the reader's convenience, we present a proof.

Lemma A.2. For any measurable map $f : \Omega \rightarrow \Omega$ we have that

$$\mathcal{I}_\ell(\mathbb{Q} \circ f^{-1} | \mathbb{P} \circ f^{-1}) \leq \mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}) \quad (\text{A.3})$$

and whenever $\mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}) < \infty$, the previous inequality is tight if and only if $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is a measurable function of f .

Proof. Let us first argue (A.3). Note that when $\mathbb{Q} \ll \mathbb{P}$ does hold, (A.3) is clear. Suppose now $\mathbb{Q} \ll \mathbb{P}$ and let $\hat{\xi} = \xi \circ f$ for $\xi \in \mathbb{L}^\infty$ and note that by definition $\Phi_{\mathbb{P}}(\xi \circ f) = \Phi_{\mathbb{P} \circ f^{-1}}(\xi)$. Thus, thanks to (A.1),

$$\mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}) \geq \mathbb{E}^{\mathbb{Q} \circ f^{-1}}[\xi] - \Phi_{\mathbb{P} \circ f^{-1}}(\xi).$$

Taking the supremum over $\xi \in \mathbb{L}^0$, we deduce (A.3).

Suppose now that $\frac{d\mathbb{Q}}{d\mathbb{P}} = h \circ f$ for some measurable function h . We claim that $\frac{d\mathbb{Q} \circ f^{-1}}{d\mathbb{P} \circ f^{-1}} = h$. Indeed, for all ξ we have

$$\int_\Omega \xi(\omega) \mathbb{Q} \circ f^{-1}(d\omega) = \int_\Omega \xi(f(\omega)) \mathbb{Q}(d\omega) = \int_\Omega \xi(f(\omega)) h(f(\omega)) \mathbb{P}(d\omega) = \int_\Omega \xi(\omega) h(\omega) \mathbb{P} \circ f^{-1}(d\omega)$$

where the second equality follows by the Radon-Nikodym theorem. Consequently,

$$\mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}) = \mathbb{E}^\mathbb{P}\left[\ell\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = \mathbb{E}^\mathbb{P}[\ell(h \circ f)] = \mathbb{E}^{\mathbb{P} \circ f^{-1}}[\ell(h)] = \mathbb{E}^{\mathbb{P} \circ f^{-1}}\left[\ell\left(\frac{d\mathbb{Q} \circ f^{-1}}{d\mathbb{P} \circ f^{-1}}\right)\right] = \mathcal{I}_\ell(\mathbb{Q} \circ f^{-1} | \mathbb{P} \circ f^{-1}).$$

Let us now assume that $\mathcal{I}_\ell(\mathbb{Q} \circ f^{-1} | \mathbb{P} \circ f^{-1}) = \mathcal{I}_\ell(\mathbb{Q} | \mathbb{P}) < \infty$. Since $\mathcal{I}_\ell(\mathbb{Q} \circ f^{-1} | \mathbb{P} \circ f^{-1}) < \infty$, we deduce that there exists h such that for any ξ , $\int_\Omega \xi(\omega) \mathbb{Q} \circ f^{-1}(d\omega) = \int_\Omega \xi(\omega) h(\omega) \mathbb{P} \circ f^{-1}(d\omega)$, and thus

$$\int_\Omega \xi(f(\omega)) \mathbb{Q}(d\omega) = \int_\Omega \xi(\omega) \mathbb{Q} \circ f^{-1}(d\omega) = \int_\Omega \xi(\omega) h(\omega) \mathbb{P} \circ f^{-1}(d\omega) = \int_\Omega \xi(f(\omega)) h(f(\omega)) \mathbb{P}(d\omega).$$

By the \mathbb{P} -a.s. uniqueness of $\frac{d\mathbb{Q}}{d\mathbb{P}}$, it follows again by Radon-Nikodym theorem that it is a measurable function of f . \square

Lemma A.3. Let $c(x, \rho) = \int_{\mathbb{R}^m} \mathbf{1}_{\{x \neq y\}} \rho(dy)$. Then $\mathcal{W}_c(\mu, \mu_T) = \text{TV}(\mu, \mu_T)$ and $Q\varphi(x) = \varphi(x)$.

Proof. That $\mathcal{W}_c(\mu, \mu_T) = \text{TV}(\mu, \mu_T)$ follows directly by definition of WOT, see (2.2). We now argue that, without loss of generality, we may assume that $Q\varphi(x) = \varphi(x)$. First, note that since c is bounded, we have that the set $C_{b,p}$ in the dual formulation (3.1) corresponds to φ bounded. Second, by definition of $Q_c\varphi(x)$, see (3.2), for any $k \in \mathbb{R}$, $Q_c(\varphi + k) = Q_c\varphi + k$ and $Q_c\varphi(x) = \min\{1 + \inf_{y \neq x} \varphi(y), \varphi(x)\}$. Thus, thanks to Theorem A.1 we see that

$$\Psi^{\varphi+k}(\mu_0) - \langle \varphi + k, \mu_T \rangle = \Psi^\varphi(\mu_0) - \langle \varphi, \mu_T \rangle.$$

Thus, we can always add a constant to φ , without increasing \mathcal{W}_c and thus $Q_c\varphi(x) = \min\{0, \varphi(x)\}$. \square

Lemma A.4. *The functional*

$$\mathcal{P}(\mathbb{R}^m) \ni \nu \longmapsto \Phi(\nu) := \inf_{\mathbb{P}^\alpha \in \mathcal{A}(\nu)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\frac{1}{2} \int_0^T \ell''(Z_t) Z_t \|\alpha_t\|^2 dt + Q_c \varphi(X_T) + C(X) \right]$$

satisfies

$$\Phi(\nu) = \int_{\mathbb{R}^m} \Phi(\delta_x) \nu(dx).$$

Proof. Note that $\Phi(\nu)$ denotes the value of a stochastic control problem. For control problems in the so-called weak formulation in the canonical space, the statement follows from [33, Lemma 4.5] or [55, Lemma 3.5] for mean-field and classical control problems, respectively. In either setting, the argument rests on a measurable selection argument. In the context of this manuscript, we may enlarge the canonical space so that both α and Z are canonical processes and minimize over probability laws of (X, Z, α) instead of over $\mathcal{A}(\nu)$. That the value remains unchanged with such enlargement follows from [16, Theorem 4.10]. \square

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