

Partitioned robustness analysis of networks with uncertain links

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Abstract—An input-output model for networks with link uncertainty is developed. The main result presents a set of integral quadratic constraints (IQC)s that collectively imply robust stability of the uncertain network dynamics. The model dependency of each IQC is localized according to an edge-based partition of the network graph. The class of admissible network partitions affords scope for trading-off scalability against conservativeness. This is illustrated by numerical example.

Index Terms—Input-Output Methods, Robust Stability, Scalable Analysis

I. INTRODUCTION

Motivated by problems in power and water distribution, transportation, ecology, and economics, the study of large networks of dynamical systems has a long history in the systems and control literature; e.g., see [1], [2] for state-space methods, and [3], [4] for input-output methods.

An input-output approach is pursued in this paper. The specific aim is to progress scalable robustness analysis of dynamic networks with uncertain links. As in the preliminary work [5], [6], the proposed network model is oriented towards considering link uncertainty relative to ideal (unity gain) links, which is common in the study of cyber-physical systems [7], [8]. The structure of the network is encoded by a static ‘routing’ permutation, without restriction, as also seen in [9] for example. Application of the well-known integral quadratic constraint (IQC) robust stability theorem [10] underpins the proposed approach. The exploitation of structure to decompose the resultant monolithic robustness certificate is related to work in [11]–[14], and most closely [15]. As elaborated in [5], this existing work is not directly applicable to the particular model structure considered here.

The main contribution stems from a new structured coprime factorization of a model of the network with ideal links. This leads to a structured reformulation of the monolithic IQC robust stability certificate obtained via [10], that is in

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turn amenable to decomposition according to admissible edge-based partitions of the network graph. Each element of the resulting collection of IQCs only depends on localized model parameters. The admissible class of network partitions affords scope for trading-off scalability against conservativeness of the distributed certificates. The neighbourhood partition from [6] is a special case.

The paper is organized as follows. Various preliminaries are presented next. The structured input-output model of a network with uncertain links and monolithic IQC-based robust stability certificate from [5], [6], are extended in Section III to accommodate unstable agent dynamics. The aforementioned structured coprime factorization, and IQC decomposition, are developed in Section IV. A numerical example is given in Section V, followed by a brief conclusion in Section VI.

II. PRELIMINARIES

A. Basic notation

\mathbb{N} and \mathbb{R} denote the natural and real numbers. $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\mathbb{R}_{\bullet\theta} := \{\vartheta \in \mathbb{R} \mid \vartheta \bullet \theta\}$ for $\bullet \in \{<, \leq, \geq, >\}$. Ordered $[i:j] := \{k \in \mathbb{N}_0 \mid i \leq k \leq j\}$, abbreviated $[j]$ when $i = 1$, is empty if $j < i$. \mathbb{R}^r denotes the vectors with $r \in \mathbb{N}$ real co-ordinates, and $\mathbb{R}^{r \times q}$ real matrices with $r \in \mathbb{N}$ rows and $q \in \mathbb{N}$ columns. Given $X \in \mathbb{R}^{r \times q}$ and $(i, j) \in [r] \times [q]$, the j -th column is $X_{(\cdot, j)} \in \mathbb{R}^{r \times 1} \sim \mathbb{R}^r$, the i -th row is $X_{(i, \cdot)} \in \mathbb{R}^{1 \times q}$, and $X_{(i, j)} \in \mathbb{R}$ is the scalar in row i , column j . Given ordered index set $\mathcal{I} = [i_1, \dots, i_{|\mathcal{I}|}] \subset \mathbb{N}$ and $X_i \in \mathbb{R}^{q_i \times r_i}$ for $i \in \mathcal{I}$, block diagonal $\bigoplus_{i \in \mathcal{I}} X_i := X_{i_1} \oplus \dots \oplus X_{i_{|\mathcal{I}|}} \in \mathbb{R}^{r \times q}$, where $q = \sum_{i \in \mathcal{I}} q_i$ and $r = \sum_{i \in \mathcal{I}} r_i$. The identity is $I_r \in \mathbb{R}^{r \times r}$, all entries of $\mathbf{0}_{r,q} \in \mathbb{R}^{r \times q}$ are zero, all entries of $\mathbf{1}_{r,q} \in \mathbb{R}^{r \times q}$ are one. $X' \in \mathbb{R}^{q \times r}$ is the transpose of $X \in \mathbb{R}^{r \times q}$, and $Y^{-1} \in \mathbb{R}^{r \times r}$ is the inverse of non-singular $Y \in \mathbb{R}^{r \times r}$. Given $W = W', Z = Z' \in \mathbb{R}^{r \times r}$, the respective matrix inequalities $Z \succeq W$ and $Z \succ W$ mean $(\forall v \in \mathbb{R}^r) v'(Z - W)v \geq 0$ and $(\exists \epsilon \in \mathbb{R}_{>0}) (\forall v \in \mathbb{R}^r) v'(Z - W)v \geq \epsilon v'v$.

B. Signals and systems

\mathbf{L}_2^r denotes the space of measurable signals $v : \mathbb{T} \rightarrow \mathbb{R}^r$ with finite energy $\|v\| := \langle v, v \rangle^{1/2}$, where $\mathbb{T} = \mathbb{R}_{\geq 0}$ in continuous time or $\mathbb{T} = \mathbb{N}_0$ in discrete time, and $\langle v, u \rangle := \int_{\mathbb{T}} v(t)'u(t) dm$ for the Lebesgue measure or the counting measure m , respectively. Either way it is a Hilbert space. The extended space $\mathbf{L}_{2e}^r \supset \mathbf{L}_2^r$ comprises measurable $v : \mathbb{T} \rightarrow \mathbb{R}^r$ such that $\pi_{\tau}(v) \in \mathbf{L}_2^r$ for all $\tau \in \mathbb{T}$, where $(\pi_{\tau}(v))(t) := f(t)$ for $t \in \mathbb{T}_{\tau}$, and 0 otherwise, with $\mathbb{T}_{\tau} = \mathbb{R}_{\geq 0} \cap \mathbb{R}_{<\tau}$ in continuous time or $\mathbb{T}_{\tau} = [0 : \tau]$ in discrete time. While it

matters not whether the setup is continuous or discrete time, it is fixed to be one or the other throughout.

The map $G : \mathbf{L}_{2e}^q \rightarrow \mathbf{L}_{2e}^r$ is called bounded if

$$(\exists \gamma \in \mathbb{R}_{>0}) (\forall u \in \mathbf{L}_2^q) \|G(u)\| \leq \gamma \|u\|.$$

In this case $G(u) \in \mathbf{L}_2^r$ for $u \in \mathbf{L}_2^q$. The corresponding restriction is not distinguished from G . The composition of maps $F : \mathbf{L}_{2e}^r \rightarrow \mathbf{L}_{2e}^s$ and $G : \mathbf{L}_{2e}^q \rightarrow \mathbf{L}_{2e}^r$ is denoted by $F \circ G := (v \mapsto F(G(v)))$, and the direct sum by $F \oplus G := ((u, v) \mapsto (F(u), G(v)))$. Similarly, given naturally ordered $\mathcal{I} = [i_1, \dots, i_{|\mathcal{I}|}] \subset \mathbb{N}$, and $G_i \in \mathbf{L}_{2e}^{q_i} \rightarrow \mathbf{L}_{2e}^{r_i}$ for $i \in \mathcal{I}$, the direct sum $\bigoplus_{i \in \mathcal{I}} G_i := G_{i_1} \oplus \dots \oplus G_{i_{|\mathcal{I}|}} : \mathbf{L}_{2e}^q \rightarrow \mathbf{L}_{2e}^r$ with $q = \sum_{i \in \mathcal{I}} q_i$ and $r = \sum_{i \in \mathcal{I}} r_i$, under the equivalences $\mathbf{L}_{2e}^q \sim \mathbf{L}_{2e}^{q_1} \times \dots \times \mathbf{L}_{2e}^{q_{|\mathcal{I}|}}$ and $\mathbf{L}_{2e}^r \sim \mathbf{L}_{2e}^{r_1} \times \dots \times \mathbf{L}_{2e}^{r_{|\mathcal{I}|}}$. When $G : \mathbf{L}_{2e}^r \rightarrow \mathbf{L}_{2e}^r$ is bijective, the inverse is $G^{-1} : \mathbf{L}_{2e}^r \rightarrow \mathbf{L}_{2e}^r$.

If $G(\alpha u + \beta v) = \alpha G(u) + \beta G(v)$ for all $\alpha, \beta \in \mathbb{R}$, and $u, v \in \mathbf{L}_{2e}^q$, then $G : \mathbf{L}_{2e}^q \rightarrow \mathbf{L}_{2e}^r$ is called linear, and $G(v)$ may be abbreviated to Gv , with \circ often dropped when composing linear maps. The action of linear G corresponds to the action of $p \cdot q$ scalar systems $G_{(i,j)} : \mathbf{L}_{2e} \rightarrow \mathbf{L}_{2e}$, $(i, j) \in [r] \times [q]$, on the coordinates of the signal vector input; i.e., $(Gv)_i = \sum_{j=1}^q G_{(i,j)} v_j$. Matrix notation may be used to reflect the structure. For convenience, the linear map of pointwise (in time) multiplication by a matrix is not distinguished in notation from the matrix. If linear G is bounded, then the (unique) Hilbert adjoint $G^* : \mathbf{L}_2^r \rightarrow \mathbf{L}_2^q$ exists such that $(\forall (y, u) \in \mathbf{L}_2^r \times \mathbf{L}_2^q) \langle y, Gu \rangle = \langle G^* y, u \rangle$.

The map $G : \mathbf{L}_{2e}^q \rightarrow \mathbf{L}_{2e}^r$, is called a *system* if it is *causal* in the sense $\pi_\tau(G(u)) = \pi_\tau(G(\pi_\tau(u)))$ for all $\tau \in \mathbb{T}$, and $G(0) = 0$. It is called *stable* if G is also bounded. The composition of stable systems is therefore stable. The feedback interconnection of G with the system $\Delta : \mathbf{L}_{2e}^r \rightarrow \mathbf{L}_{2e}^q$ is *well-posed* if for all $(d_y, d_u) \in \mathbf{L}_{2e}^r \times \mathbf{L}_{2e}^q$, there exists unique $(y, u) \in \mathbf{L}_{2e}^r \times \mathbf{L}_{2e}^q$ such that

$$y = G(u) + d_y, \quad u = \Delta(y) + d_u, \quad (1)$$

and $\llbracket G, \Delta \rrbracket := ((d_y, d_u) \mapsto (y, u))$ is causal; i.e., $R - G \oplus \Delta$ is bijective with causal inverse $(R - G \oplus \Delta)^{-1} = R^{-1} \circ \llbracket G, \Delta \rrbracket$, where $R := ((u, y) \mapsto (y, u))$. With $(d_y, d_u) = (0, d_u) \in \{0\} \times \mathbf{L}_{2e}^q$ and $(d_y, d_u) = (d_y, 0) \in \mathbf{L}_{2e}^r \times \{0\}$, it follows that $(I_r - G \circ \Delta)$ and $(I_q - \Delta \circ G)$ are bijective with causal inverses. If well-posed $\llbracket G, \Delta \rrbracket$ is bounded, then the closed-loop is called stable. In this case, it follows that $(I_q - \Delta \circ G)^{-1}$, $G \circ (I_q - \Delta \circ G)^{-1}$, $\Delta \circ G \circ (I_q - \Delta \circ G)^{-1}$, $(I_r - G \circ \Delta)^{-1}$, $\Delta \circ (I_r - G \circ \Delta)^{-1}$, and $G \circ \Delta \circ (I_r - G \circ \Delta)^{-1}$, are all stable systems. The following well-known integral quadratic constraint (IQC) feedback stability theorem is from [10].

Theorem 1: Given stable system $\Delta : \mathbf{L}_{2e}^r \rightarrow \mathbf{L}_{2e}^q$ and bounded self-adjoint multiplier $\Pi : \mathbf{L}_2^r \times \mathbf{L}_2^q \rightarrow \mathbf{L}_2^r \times \mathbf{L}_2^q$ (i.e., linear $\Pi = \Pi^*$), suppose

$$(\forall (y, \alpha) \in \mathbf{L}_2^r \times [0, 1]) \langle (y, u), \Pi(y, u) \rangle \geq 0, \quad u = \alpha \Delta(y).$$

Further, given stable system $G : \mathbf{L}_{2e}^q \rightarrow \mathbf{L}_{2e}^r$, suppose

$$(\exists \epsilon \in \mathbb{R}_{>0}) (\forall u \in \mathbf{L}_2^q) \langle (y, u), \Pi(y, u) \rangle \leq -\epsilon \|u\|^2, \quad y = G(u).$$

and $\llbracket G, \alpha \Delta \rrbracket$ is well-posed for every $\alpha \in [0, 1]$. Then, $\llbracket G, \Delta \rrbracket$ is stable.

C. Graphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple (self-loopless and undirected) connected graph, where $\mathcal{V} = [n]$ is the set of $n \in \mathbb{N} \setminus \{1\}$ vertices, and $\mathcal{E} \subset \{\{i, j\} \mid i \neq j \in \mathcal{V}\}$ is the set of $m := |\mathcal{E}| \in \mathbb{N}$ edges. The elements of $\mathcal{N}_i := \{j \in \mathcal{V} \mid \{i, j\} \in \mathcal{E}\}$ are the neighbours of $i \in \mathcal{V}$, and $\mathcal{E}_i := \{\{i, j\} \mid j \in \mathcal{N}_i\}$ is the corresponding neighbourhood edge set; $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$ and $m_i := |\mathcal{N}_i| = |\mathcal{E}_i| \geq 1$ for all $i \in \mathcal{V}$ by connectedness. Given $\mathcal{F} \subset \mathcal{E}$, the corresponding edge-induced sub-graph is $\mathcal{G}[\mathcal{F}] := (\mathcal{V}_{\mathcal{F}}, \mathcal{F})$, where $\mathcal{V}_{\mathcal{F}} := \{i \in \mathcal{V} \mid \mathcal{E}_i \cap \mathcal{F} \neq \emptyset\}$.

A cardinality $c \in \mathbb{N}$ *localized edge partition* is any collection $\{\mathcal{F}_1, \dots, \mathcal{F}_c\} \subset 2^{\mathcal{E}}$ such that $(\forall p \in [c]) \mathcal{F}_p \neq \emptyset$,

- $(\forall p \in [c]) \mathcal{G}[\mathcal{F}_p]$ is connected, and
- $\mathcal{E} = \bigcup_{p \in [c]} \mathcal{F}_p$.

Overlap is allowed; i.e., $\mathcal{F}_p \cap \mathcal{F}_q \neq \emptyset$ is possible for $p \neq q$. Connectedness makes each element of the partition localized.

The bijective map $\kappa : \mathcal{E} \rightarrow \mathcal{M}$, with $\mathcal{M} := [m]$, fixes an enumeration of the edge set \mathcal{E} . Similarly, bijective map $\kappa_i : \mathcal{N}_i \rightarrow \mathcal{M}_i$, with $\mathcal{M}_i := [m_i]$, fixes an enumeration of the neighbours, and thus, edges associated with $i \in \mathcal{V}$. These bijections are subsequently used for indexing matrix structured representations of corresponding networks.

III. A NETWORK MODEL FOR ROBUSTNESS ANALYSIS

Consider a network of $n \in \mathbb{N} \setminus \{1\}$ dynamic agents coupled according to the simple connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The vertex set $\mathcal{V} = [n]$ is a fixed enumeration of the agents. The edge set is such that $\{i, j\} \in \mathcal{E}$ if there is a link to share the output of agent $i \in \mathcal{V}$ as an input to agent $j \in \mathcal{V}$, or *vice versa*. The number of edges $m := |\mathcal{E}|$. For each $i \in \mathcal{V}$, the number of neighbours $m_i := |\mathcal{N}_i| = |\mathcal{E}_i|$ is non-zero by connectedness, and $\sum_{i \in \mathcal{V}} m_i = 2m$. Let $\mathcal{M} := [m]$ and $\mathcal{M}_i := [m_i]$.

Each agent $i \in \mathcal{V}$ has a single (in order tame the notation) output with dynamic dependence on inputs from its neighbours. This possibly unstable linear dependence is modelled by $H_i : \mathbf{L}_{2e}^{m_i} \rightarrow \mathbf{L}_{2e}$. The order of the m_i input coordinates is fixed by the neighbour enumeration $\kappa_i : \mathcal{N}_i \rightarrow \mathcal{M}_i$. On the other hand, the stable but possibly non-linear time-varying dynamics of the directed link for sharing the output of agent $i \in \mathcal{V}$ with its neighbour $j = \kappa_i^{-1}(k) \in \mathcal{N}_i$ is modelled by the uncertain $\Lambda_{i,k} : \mathbf{L}_{2e} \rightarrow \mathbf{L}_{2e}$, for $k \in \mathcal{M}_i$. As such, while the links all have the single agent output as a common input, the dynamics (e.g., variable delay) of each link can be different. The network model and unstructured robust stability analysis presented in this section originate from [5], [6]. Modest generalization of this preliminary work is made here to accommodate unstable agent dynamics.

The proposed networked system model is $\llbracket P, \Lambda \circ T \circ H \rrbracket$, as shown on the top left of Figure 1, where

$$\Lambda := \bigoplus_{i \in \mathcal{V}} \left(\bigoplus_{k \in \mathcal{M}_i} \Lambda_{i,k} \right) : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^{2m}, \quad (2a)$$

$$T := \bigoplus_{i \in \mathcal{V}} \mathbf{1}_{m_i, 1} : \mathbf{L}_{2e}^n \rightarrow \mathbf{L}_{2e}^{2m}, \quad (2b)$$

$$H := \bigoplus_{i \in \mathcal{V}} H_i : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^n, \quad (2c)$$

and $P : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^{2m}$ is pointwise multiplication by a permutation matrix structured according to $\mathcal{G} = (\mathcal{V}, \mathcal{E})$; more specifically, for $i \in \mathcal{V}$, $k \in \mathcal{M}_i$, $r = (\sum_{h \in [i-1]} m_h + k) \in [2m]$,

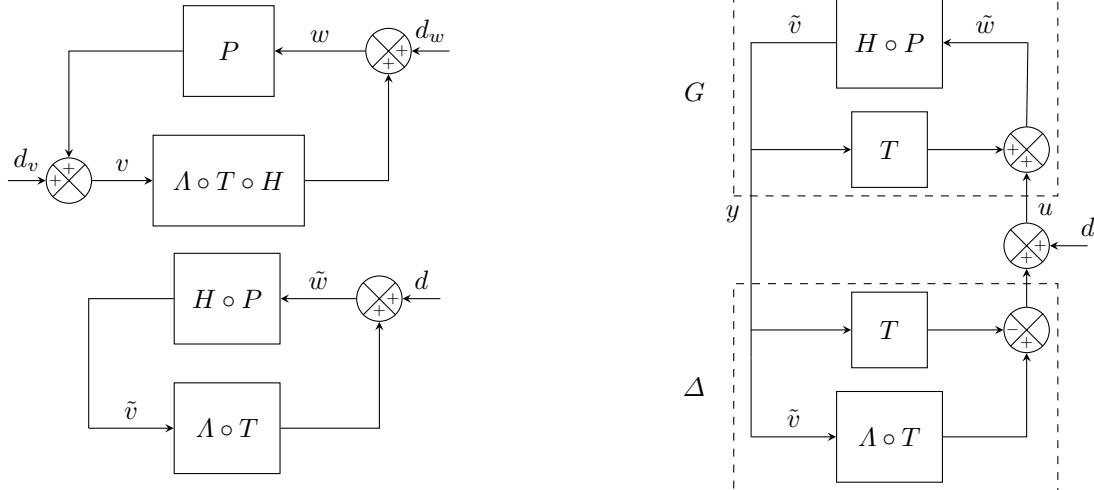


Fig. 1. Networked system model $\llbracket P, A \circ T \circ H \rrbracket$ and loop transformations for robust stability analysis.

and $q \in [2m]$, the corresponding entry of this ‘routing’ matrix is given by

$$P_{(r,q)} = \begin{cases} 1 & \text{if } q = \sum_{h \in [j-1]} m_h + \kappa_j(i) \\ & \quad \text{with } j = \kappa_i^{-1}(k), \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

(Recall that pointwise multiplication by a matrix is not distinguished in notation from the matrix.) The disturbance input d_v perturbs agents away from equilibrium. Invertibility of P means this can be set from $d = d_w + P^{-1}d_v$, as also shown in Figure 1. In addition, d_w may reflect noisy communication. Both could be used to model other sources of uncertainty or performance requirements, although robust performance lies beyond the current scope, which is focused on *structured* robust stability analysis of $\llbracket P, A \circ T \circ H \rrbracket$ in Section IV. In preparation, the rest of this section develops a monolithic robust stability result without direct regard for network structure.

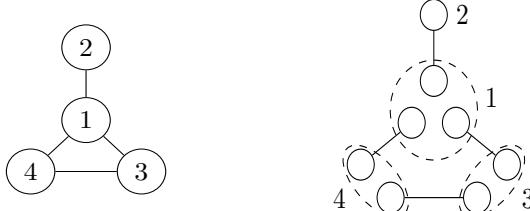


Fig. 2. Example network graph \mathcal{G} (left) and corresponding 1-regular sub-system graph $\mathcal{G}^* = \bigcup_{e \in \mathcal{E}} \mathcal{G}[e]$ with adjacency matrix P (right).

As defined, the ‘routing’ matrix $P = P' = P^{-1}$ is the adjacency matrix of the 1-regular graph $\mathcal{G}^* := (\mathcal{V}^*, \mathcal{E}^*)$, with $2m$ vertices and m edges, obtained by taking the disjoint union of the two-vertex sub-graphs $\mathcal{G}[e]$ induced by each edge $e \in \mathcal{E}$ of the network graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$; see Figure 2. In particular,

$$P = I_{2m} - L \quad (4)$$

in which the degree matrix is I_{2m} because \mathcal{G}^* is 1-regular, and the Laplacian $L = L' \in \mathbb{R}^{2m \times 2m}$ decomposes as

$$L = \sum_{k \in \mathcal{M}} L_k := \sum_{k \in \mathcal{M}} B_{(\cdot,k)} B'_{(\cdot,k)} = BB', \quad (5)$$

where the incidence matrix $B \in \mathbb{R}^{2m \times m}$ is defined by $B_{(r,k)} = 1 = -B_{(s,k)}$ for $r = \sum_{h \in [i-1]} m_h + \kappa_i(j)$ and $s = \sum_{h \in [j-1]} m_h + \kappa_j(i)$ with $i = \min \kappa^{-1}(k)$ and $j = \max \kappa^{-1}(k)$, and $B_{(t,k)} = 0$ for $t \in [2m] \setminus \{r, s\}$, over $k \in \mathcal{M}$. This corresponds to an enumeration of \mathcal{V}^* that aligns with the neighbour enumerations κ_i and κ_j in (3).

Assumption 1: The network with ideal links (i.e., $\Lambda = I_{2m}$) is stable; i.e., the linear system $\llbracket P, TH \rrbracket$ is stable.

Assumption 1 is used in stability analysis of the networked system model $\llbracket P, A \circ T \circ H \rrbracket$ by considering link uncertainty in Λ relative to ideal unity gain links. As illustrated in Figure 1, it is sufficient to verify that $\llbracket G, \Delta \rrbracket$ is stable, where

$$\Delta := (A - I_{2m}) \circ T, \quad (6)$$

with A and T as per (2), and the linear system

$$G := H(P - TH)^{-1} = HP(I_{2m} - THP)^{-1}, \quad (7)$$

with H as per (2), and P as per (4). Given Assumption 1, $I_{2m} - THP : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^{2m}$ is bijective, and the equality in (7) holds by linearity and the relation $P = P^{-1}$. The nominal stability assumption also gives $THP(I_{2m} - THP)^{-1} = TG$ is stable. Therefore, $G = (\bigoplus_{i \in \mathcal{V}} \frac{1}{m_i}) \circ T' TG$ is stable, since pointwise multiplication by a constant matrix is bounded. The uncertain perturbation $\Delta : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^{2m}$ from ideal links is stable by the standing assumption that A is stable.

Theorem 2: If $\llbracket G, \Delta \rrbracket$ is stable, with G as per (7) and Δ as per (6), then $\llbracket P, A \circ T \circ H \rrbracket$ is stable.

Proof: Given $d \in \mathbf{L}_{2e}^{2m}$, suppose $u \in \mathbf{L}_{2e}^{2m}$ is such that

$$\Delta(y) + d = u \quad \text{and} \quad y = Gu. \quad (8)$$

Such u exists uniquely, with causal dependence on d , and $(\exists c_1 \in \mathbb{R}_{>0})(\forall d \in \mathbf{L}_{2e}^{2m}) \|y, u\| \leq c_1 \|d\|$ by the hypothesized stability of $\llbracket G, \Delta \rrbracket$. Now let

$$\tilde{w} := (I_{2m} - THP)^{-1}u, \quad (9)$$

noting $(\exists c_2 \in \mathbb{R}_{>0})(\forall d \in \mathbf{L}_{2e}^{2m}) \|\tilde{w}\| \leq c_2 \|u\| \leq c_2 \cdot c_1 \|d\|$ by Assumption 1. Further, let

$$\tilde{v} := HP\tilde{w}, \quad (10)$$

noting from (7) that $\tilde{v} = Gu = y$, and thus, $\|\tilde{v}\| \leq c_1\|d\|$ if $d \in \mathbf{L}_2^{2m}$. Both \tilde{v} and \tilde{w} exhibit causal dependence on d . From (6), (8), (9), and (10), $\tilde{w} - T\tilde{v} = \tilde{w} - THP\tilde{w} = u = \Delta(y) + d = (\Lambda \circ T)(\tilde{v}) - T\tilde{v} + d$, whereby

$$\tilde{w} = (\Lambda \circ T)(\tilde{v}) + d. \quad (11)$$

Given any $d_v, d_w \in \mathbf{L}_{2e}^{2m}$, let $d := d_w + P^{-1}d_v \in \mathbf{L}_{2e}^{2m}$. With $w := \tilde{w} - P^{-1}d_v$, and $v := Pw + d_v$, it is immediate that $v = P\tilde{w}$, and therefore, $\tilde{v} = Hv$ from (10). As such, $w = (\Lambda \circ T \circ H)(v) + d_w$ from (11), and by the triangle inequality, $\|w\| \leq \|\tilde{w}\| + \|d_w\| \leq \sqrt{2} \cdot (c_1 \cdot c_2 + 1)\|(d_w, d_v)\|$, and $\|v\| = \|\tilde{v}\| \leq c_2 \cdot c_1\|d\| \leq \sqrt{2} \cdot c_2 \cdot c_1\|(d_w, d_v)\|$, since $P = P' = P^{-1}$ is an isometry, $\|\tilde{w}\| \leq c_2 \cdot c_1\|d\|$, and $\|d\| \leq \|d_w\| + \|d_v\| \leq \sqrt{2}\|(d_w, d_v)\|$. \square

Remark 1: As established in [5, Theorem 1], when H is also stable, and Λ is linear, the converse of Theorem 2 holds; i.e., if $\llbracket P, \Lambda \circ T \circ H \rrbracket$ is stable, then $\llbracket G, \Delta \rrbracket$ is stable.

Towards combining Theorems 1 and 2, consider the following characterization of the link uncertainty Δ in (6). For each $i \in \mathcal{V}$, suppose bounded linear self-adjoint

$$\Pi_i := \begin{bmatrix} \Pi_{1,i} & \Pi_{2,i} \\ \Pi_{2,i}^* & \Pi_{3,i} \end{bmatrix} : \mathbf{L}_2 \times \mathbf{L}_2^{m_i} \rightarrow \mathbf{L}_2 \times \mathbf{L}_2^{m_i},$$

is such that

$$(\forall (y_i, \alpha) \in \mathbf{L}_2 \times [0, 1]) \quad \langle (y_i, u_i), \Pi_i(y_i, u_i) \rangle \geq 0, \quad u_i = \alpha \Delta_i(y_i), \quad (12)$$

where $\Delta_i := (\bigoplus_{k \in \mathcal{M}_i} \Lambda_{i,k} - I_{m_i}) \circ \mathbf{1}_{m_i,1}$. Noting that $\Delta = \bigoplus_{i \in \mathcal{V}} \Delta_i$, it follows that

$$(\forall (y, \alpha) \in \mathbf{L}_2^n \times [0, 1]) \quad \langle (y, u), \Pi(y, u) \rangle \geq 0, \quad u = \alpha \Delta(y), \quad (13)$$

where $\Pi = \Pi^* : (\mathbf{L}_2^n \times \mathbf{L}_2^{2m}) \rightarrow (\mathbf{L}_2^n \times \mathbf{L}_2^{2m})$ is given by

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^* & \Pi_3 \end{bmatrix} = \begin{bmatrix} \bigoplus_{i \in \mathcal{V}} \Pi_{1,i} & \bigoplus_{i \in \mathcal{V}} \Pi_{2,i} \\ \bigoplus_{i \in \mathcal{V}} \Pi_{2,i}^* & \bigoplus_{i \in \mathcal{V}} \Pi_{3,i} \end{bmatrix}. \quad (14)$$

Theorem 3: Given G in (7), and $\Pi_{\{1,2,3\}}$ in accordance with (14) and (12), suppose

$$(\exists \epsilon > 0) \quad (\forall u \in \mathbf{L}_2^{2m}) \quad \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^* & \Pi_3 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle \leq -\epsilon \|u\|^2, \quad y = Gu. \quad (15)$$

Then, the uncertain networked system $\llbracket P, \Lambda \circ T \circ H \rrbracket$ is stable.

Proof: Stability of $\llbracket G, \Delta \rrbracket$ follows from Theorem 1, since (13) for the given Π . Therefore, $\llbracket P, \Lambda \circ T \circ H \rrbracket$ is stable by Theorem 2. \square

Remark 2: In (15), the structure of $\Pi_{\{1,2,3\}} = \bigoplus_{i \in \mathcal{V}} \Pi_{\{1,2,3\},i}$ is the same as $\Delta = \bigoplus_{i \in \mathcal{V}} \Delta_i$. On the other hand, the stable linear system $G : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^n$ (see Assumption 1) typically lacks structure, as $(P - T \circ H)^{-1}$ in (7) may be unstructured. However, as elaborated in Section IV-A, a coprime factorization exists that leads to an equivalent structured robust stability certificate.

IV. MAIN RESULTS

This section contains the main contributions, including the distributed robust stability certificate for $\llbracket P, \Lambda \circ T \circ H \rrbracket$ presented in Section IV-B. As elaborated next, this is enabled by a structured coprime factorization of G in (7).

A. An equivalent structured robust stability certificate

For each agent $i \in \mathcal{V}$, suppose

$$H_i = N_i D_i^{-1}, \quad (16)$$

where stable linear systems $N_i : \mathbf{L}_{2e}^{m_i} \rightarrow \mathbf{L}_{2e}$ and $D_i : \mathbf{L}_{2e}^{m_i} \rightarrow \mathbf{L}_{2e}^{m_i}$ satisfy $U_i N_i + V_i D_i = I_{m_i}$ for some stable linear systems $U_i : \mathbf{L}_{2e} \rightarrow \mathbf{L}_{2e}^{m_i}$ and $V_i : \mathbf{L}_{2e}^{m_i} \rightarrow \mathbf{L}_{2e}^{m_i}$; i.e., $N_i D_i^{-1}$ is a coprime factorization of H_i . Then, the pair of linear systems

$$N := \bigoplus_{i \in \mathcal{V}} N_i, \quad (17a)$$

$$M := P(\bigoplus_{i \in \mathcal{V}} D_i) - \bigoplus_{i \in \mathcal{V}} (\mathbf{1}_{m_i,1} N_i), \quad (17b)$$

is coprime, and $G = NM^{-1}$ in (7). In particular, $((\bigoplus_{i \in \mathcal{V}} U_i) + (\bigoplus_{i \in \mathcal{V}} V_i)PT)N + ((\bigoplus_{i \in \mathcal{V}} V_i)P)M = I_{2m}$, whereby $(y, u) \in \mathbf{L}_2^{2m} \times \mathbf{L}_2^n$ satisfies $y = Gu$ if, and only if, there exists $z \in \mathbf{L}_2^{2m}$ such that $y = Nz$ and $u = Mz$. Further, under Assumption 1, M^{-1} is stable. For more about coprime factorization in the analysis of feedback systems see [16]. To facilitate the subsequent development, let $D := \bigoplus_{i \in \mathcal{V}} D_i$, whereby $M = PD - TN$.

The potential lack of structure in G , as noted in Remark 2, translates to lack of apparent structure in the centralized robust stability certificate given by Theorem 3. However, with the structured coprime factors N and M in (17), it follows that (15) is equivalent to

$$(\exists \epsilon \in \mathbb{R}_{>0}) \quad (\forall z \in \mathbf{L}_2^{2m}) \quad \left\langle \begin{bmatrix} N \\ M \end{bmatrix} z, \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^* & \Pi_3 \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} z \right\rangle \leq -\epsilon \|z\|^2, \quad (18)$$

since M and M^{-1} are both stable systems, whereby $-\epsilon \|z\|^2 \leq -\epsilon \|Mz\|^2 / \|M\|^2$ and $-\epsilon \|Mz\|^2 \leq -\epsilon \|z\|^2 / \|M^{-1}\|^2$.

For the monolithic IQC (18), network structure is apparent in the stable coprime factor $M = J - K$, where

$$J := D - TN = \bigoplus_{i \in \mathcal{V}} (D_i - \mathbf{1}_{m_i,1} N_i), \quad (19a)$$

$$K := LD = L(\bigoplus_{i \in \mathcal{V}} D_i), \quad (19b)$$

and $L = \sum_{k \in \mathcal{M}} L_k$ is pointwise multiplication by the subsystem graph Laplacian given in (5). Note that N , J , and $\Pi_{\{1,2,3\}}$ are all agent-wise block diagonal.

The following lemma enables decomposition of (18) according to edge-based partitions of the network graph structure apparent in M through K , as pursued further in Section IV-B. The result is inspired by the proof of [15, Theorem 1] for the somewhat differently structured IQC therein.

Lemma 1: Given $\Pi = \Pi^* : (\mathbf{L}_2^n \times \mathbf{L}_2^{2m}) \rightarrow (\mathbf{L}_2^n \times \mathbf{L}_2^{2m})$, stable linear systems $N : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^n$ and $M : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^{2m}$, with $M = J - K$ and K stable, for each $p \in [c]$ suppose there exist $\epsilon_p \in \mathbb{R}_{>0}$, $\mathbf{0}_{2m,2m} \preceq W_p = W_p' \in \mathbb{R}^{2m \times 2m}$, bounded linear $X_p = X_p^* : \mathbf{L}_2^{2m} \rightarrow \mathbf{L}_2^{2m}$, $Y_p : \mathbf{L}_2^{2m} \rightarrow \mathbf{L}_2^{2m}$, $Z_p = Z_p^* : \mathbf{L}_2^{2m} \rightarrow \mathbf{L}_2^{2m}$, and stable linear $K_p : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^{2m}$, such that $(\forall z \in \mathbf{L}_2^{2m})$

$$\left\langle \begin{bmatrix} I_{2m} \\ K_p \end{bmatrix} z, \begin{bmatrix} X_p + \epsilon_p W_p & Y_p \\ Y_p^* & Z_p \end{bmatrix} \begin{bmatrix} I_{2m} \\ K_p \end{bmatrix} z \right\rangle \leq 0, \quad (20a)$$

$$\langle z, \Psi_1 z \rangle \leq \sum_{p \in [c]} \langle z, X_p z \rangle, \quad (20b)$$

$$\langle z, (K^* \Psi_2^* + \Psi_2 K) z \rangle \leq \sum_{p \in [c]} \langle z, (K_p^* Y_p + Y_p K_p) z \rangle, \quad (20c)$$

$$\langle z, K^* \Psi_3 K z \rangle \leq \sum_{p \in [c]} \langle z, K_p^* Z_p K_p z \rangle, \quad (20d)$$

and $W := \sum_{p \in [c]} W_p \succ \mathbf{0}_{2m,2m}$, where

$$\begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^* & \Psi_3 \end{bmatrix} := \begin{bmatrix} N^* & J^* \\ 0 & -I_{2m} \end{bmatrix} \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^* & \Pi_3 \end{bmatrix} \begin{bmatrix} N & 0 \\ J & -I_{2m} \end{bmatrix}. \quad (21)$$

Then, the IQC (18) holds.

Proof: First note that for all $z \in \mathbf{L}_2^{2m}$ the constraint in (18) can be written as

$$\left\langle \begin{bmatrix} I_{2m} \\ K \end{bmatrix} z, \begin{bmatrix} \Psi_1 + \epsilon I_{2m} & \Psi_2 \\ \Psi_2^* & \Psi_3 \end{bmatrix} \begin{bmatrix} I_{2m} \\ K \end{bmatrix} z \right\rangle \leq 0. \quad (22)$$

Now, observe that (20a) implies

$$\begin{aligned} \langle z, X_p z \rangle + \langle z, (K_p^* Y_p^* + Y_p K_p) z \rangle + \langle z, K_p^* Z_p K_p z \rangle \\ \leq -\epsilon_p \langle z, W_p z \rangle, \quad p \in [c]. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{p \in [c]} (\langle z, X_p z \rangle + \langle z, (K_p^* Y_p^* + Y_p K_p) z \rangle + \langle z, K_p^* Z_p K_p z \rangle) \\ \leq -\sum_{p \in [c]} \epsilon_p \langle z, W_p z \rangle \leq -\epsilon \|z\|_2^2, \end{aligned} \quad (23)$$

where $\epsilon = (\min_{p \in [c]} \epsilon_p) \cdot (\min_{x \in \mathbb{R}^{2m}} x' W x / x' x) > 0$ because $W := \sum_{p \in [c]} W_p \succ \mathbf{0}_{2m,2m}$. Combining (20b), (20c), (20d), and (23), gives

$$\begin{aligned} \langle z, (\Psi_1 + \epsilon I_{2m}) z \rangle + \langle z, (K^* \Psi_2^* + \Psi_2 K) z \rangle + \langle z, K^* \Psi_3 K z \rangle \\ \leq 0, \end{aligned}$$

which is (22). \square

B. Decomposition by localized edge partitions

Given a suitable (see Assumption 2) localized edge partition $\{\mathcal{F}_1, \dots, \mathcal{F}_c\} \subset 2^{\mathcal{E}}$ for the network graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, it shown here how to correspondingly select

- $W_p = W'_p \in \mathbb{R}^{2m \times 2m}$,
- $X_p = X_p^* : \mathbf{L}_2^{2m} \rightarrow \mathbf{L}_2^{2m}$, $Y_p : \mathbf{L}_2^{2m} \rightarrow \mathbf{L}_2^{2m}$, $Z_p = Z_p^* : \mathbf{L}_2^{2m} \rightarrow \mathbf{L}_2^{2m}$, and
- $K_p : \mathbf{L}_{2e}^{2m} \rightarrow \mathbf{L}_{2e}^{2m}$,

for each $p \in [c]$ in Lemma 1 with Π as per (14), (N, M) as per (17), and (J, K) as per (19), to establish a localized decomposition of (18). The development of this main result is facilitated by some additional notation.

Given a localized edge partition $\{\mathcal{F}_1, \dots, \mathcal{F}_c\} \subset 2^{\mathcal{E}}$, the associated vertex partition $\{\mathcal{U}_1, \dots, \mathcal{U}_c\} \subset 2^{\mathcal{V}}$ is defined by the *naturally ordered* edge-induced sub-graph vertex sets

$$\mathcal{U}_p := [i \in \mathcal{V} \mid \mathcal{E}_i \cap \mathcal{F}_p \neq \emptyset] = [i \in \mathcal{V} \mid p \in \mathcal{J}_i], \quad p \in [c],$$

where $\mathcal{J}_i := \{p \in [c] \mid \mathcal{E}_i \cap \mathcal{F}_p \neq \emptyset\} = \{p \in [c] \mid i \in \mathcal{U}_p\}$ collects the indexes of vertex partition elements that contain agent $i \in \mathcal{V}$. Since $\bigcup_{p \in [c]} \mathcal{F}_p = \mathcal{E}$ by definition, and since \mathcal{G} is connected by hypothesis, it follows that $\bigcup_{p \in [c]} \mathcal{U}_p = \mathcal{V}$ and $\bigcup_{i \in \mathcal{V}} \mathcal{J}_i = [c]$, whereby

$$\{(p, i) \mid p \in [c], i \in \mathcal{U}_p\} = \{(p, i) \mid i \in \mathcal{V}, p \in \mathcal{J}_i\}. \quad (24)$$

Further, recalling the edge-set enumeration $\kappa : \mathcal{E} \rightarrow \mathcal{M} := [m]$, where $m := |\mathcal{E}|$, the associated partition $\{\mathcal{K}_1, \dots, \mathcal{K}_c\} \subset 2^{\mathcal{M}}$ of the edge index set is defined by

$$\mathcal{K}_p := \{\kappa(e) \mid e \in \mathcal{F}_p\} \subset \mathcal{M}, \quad p \in [c]. \quad (25)$$

Also, $\mathcal{O}_k := \{p \in [c] \mid \kappa^{-1}(k) \in \mathcal{F}_p\} = \{p \in [c] \mid k \in \mathcal{K}_p\}$ collects the partition indexes that contain the edge indexed by $k \in \mathcal{M}$. Since $\bigcup_{p \in [c]} \mathcal{F}_p = \mathcal{E}$ by definition, it follows that $\bigcup_{p \in [c]} \mathcal{K}_p = \mathcal{M}$ and $\bigcup_{k \in \mathcal{M}} \mathcal{O}_k = [c]$, whereby

$$\{(p, k) \mid p \in [c], k \in \mathcal{K}_p\} = \{(p, k) \mid k \in \mathcal{M}, p \in \mathcal{O}_k\}. \quad (26)$$

Finally,

$$\mathcal{L}_k := \{\kappa(e) \mid e \in (\mathcal{E}_i \cup \mathcal{E}_j), \{i, j\} = \kappa^{-1}(k)\} \setminus \{k\} \quad (27)$$

collects the edge indexes associated with the two vertices that define the edge indexed by $k \in \mathcal{M}$, excluding the latter, and $\mathcal{Q}_{k,\ell} := \{p \in \mathcal{O}_k \mid \ell \in \mathcal{K}_p\} = \{p \in [c] \mid \{k, \ell\} \subset \mathcal{K}_p\}$ collects the indexes of the partition elements that contain both edges $\kappa^{-1}(k)$ and $\kappa^{-1}(\ell)$, for $\ell \in \mathcal{L}_k$.

Assumption 2: The given localized edge partition $\{\mathcal{F}_1, \dots, \mathcal{F}_c\}$ satisfies $(\forall k \in \mathcal{M}) \bigcup_{p \in \mathcal{O}_k} \mathcal{K}_p \setminus \{k\} \supset \mathcal{L}_k$.

Under Assumption 2, for all $k \in \mathcal{M}$ and $\ell \in \mathcal{L}_k$ the partition index set $\mathcal{Q}_{k,\ell}$ is non-empty. As such, for $k \in \mathcal{M}$,

$$\{(p, \ell) \mid p \in \mathcal{O}_k, \ell \in \mathcal{K}_p \cap \mathcal{L}_k\} = \{(p, \ell) \mid \ell \in \mathcal{L}_k, p \in \mathcal{Q}_{k,\ell}\}. \quad (28)$$

The equalities (24), (26), and (28), each play a role in establishing the main result, along with correspondingly localized decompositions of:

- the sub-system graph Laplacian $L = \sum_{k \in \mathcal{M}} L_k$ in (5), where $L_k := B_{(\cdot,k)} B'_{(\cdot,k)}$, and B is incidence matrix;
- the stable system $D := \bigoplus_{i \in \mathcal{V}} D_i$ in accordance with the coprime factorizations (16) of the agent dynamics; and
- the structured bounded self-adjoint multiplier Ψ in (21), which also bears dependence on the factorizations (16), as well as the IQC model of link uncertainty in (13).

More specifically, let $s(0) := 0$, and $s(i) := \sum_{h \in [i]} m_h$ for $i \in \mathcal{V}$. For each $p \in [c]$ and $k \in \mathcal{K}_p$, define

$$\hat{m}_p := \sum_{i \in \mathcal{U}_p} m_i, \quad (29a)$$

$$\hat{B}_{p,k} := (\bigoplus_{i \in \mathcal{U}_p} (\bigoplus_{j \in [s(i-1)+1:s(i)]} B_{(j,k)})) \mathbf{1}_{\hat{m}_p,1}, \quad (29b)$$

$$\hat{L}_{p,k} := \hat{B}_{p,k} (\hat{B}_{p,k})'. \quad (29c)$$

Further, bearing in mind the block diagonal structure of each block of Ψ in (21) with Π as per (14), N as per (17a), and J as per (19a), define $\hat{\Psi}_{\{1,2,3\},p} := \bigoplus_{i \in \mathcal{U}_p} \Psi_{\{1,2,3\},i}$, $\hat{D}_p := \bigoplus_{i \in \mathcal{U}_p} D_i$, and $\hat{L}_p := \sum_{k \in \mathcal{K}_p} \hat{L}_{p,k} = \hat{L}'_p$. Finally, define

$$\hat{K}_p := \hat{L}_p \hat{D}_p, \quad (30a)$$

$$\hat{W}_p := \bigoplus_{i \in \mathcal{U}_p} \omega_i I_{m_i}, \quad (30b)$$

$$\hat{X}_p := \bigoplus_{i \in \mathcal{U}_p} \xi_i \Psi_{1,i}, \quad (30c)$$

$$\hat{Y}_p := \hat{\Psi}_{2,p} \left(\sum_{k \in \mathcal{K}_p} \eta_k \hat{L}_{p,k} \right), \quad (30d)$$

$$\begin{aligned} \hat{Z}_p := \sum_{k \in \mathcal{K}_p} \zeta_k \hat{L}_{p,k} \hat{\Psi}_{3,p} \hat{L}_{p,k} \\ + \sum_{k \in \mathcal{K}_p} \sum_{\ell \in \mathcal{K}_p \cap \mathcal{L}_k} \theta_{k,\ell} \hat{L}_{p,k} \hat{\Psi}_{3,p} \hat{L}_{p,\ell}, \end{aligned} \quad (30e)$$

where

$$\omega_i = \xi_i = 1/|\mathcal{J}_i|, \quad i \in \mathcal{V}, \quad (31a)$$

$$\eta_k = \zeta_k = 1/|\mathcal{O}_k|, \quad k \in \mathcal{M}, \quad (31b)$$

$$\theta_{k,\ell} = 1/|\mathcal{Q}_{k,\ell}|, \quad k \in \mathcal{M}, \ell \in \mathcal{L}_k. \quad (31c)$$

Theorem 4: Under Assumption 2, for each $p \in [c]$ let $(\hat{K}_p, \hat{W}_p, \hat{X}_p, \hat{Y}_p, \hat{Z}_p)$ be as given in (30) with (31). Suppose $(\forall p \in [c]) (\exists \epsilon_p > 0) (\forall \hat{z} \in \mathbf{L}_2^{\hat{m}_p})$

$$\left\langle \begin{bmatrix} I_{\hat{m}_p} \\ \hat{K}_p \end{bmatrix} \hat{z}, \begin{bmatrix} \hat{X}_p + \epsilon_p \hat{W}_p & \frac{1}{2} \hat{Y}_p \\ \frac{1}{2} \hat{Y}_p^* & \frac{1}{4} \hat{Z}_p \end{bmatrix} \begin{bmatrix} I_{\hat{m}_p} \\ \hat{K}_p \end{bmatrix} \hat{z} \right\rangle \leq 0. \quad (32)$$

Then, the uncertain networked system $\llbracket P, \Lambda \circ T \circ H \rrbracket$ is stable.

Proof: By Lemma 5 in the Appendix, given $p \in [c]$ and corresponding $\epsilon_p \in \mathbb{R}_{>0}$, (32) for all $\hat{z} \in \mathbf{L}_2^{\hat{m}_p}$ if, and only if,

$$\left\langle \begin{bmatrix} I_{2m} \\ K_p \end{bmatrix} z, \begin{bmatrix} X_p + \epsilon_p W_p & \frac{1}{2} Y_p \\ \frac{1}{2} Y_p^* & \frac{1}{4} Z_p \end{bmatrix} \begin{bmatrix} I_{2m} \\ K_p \end{bmatrix} z \right\rangle \leq 0, \quad (33)$$

for all $z \in \mathbf{L}_2^{2m}$, with

$$K_p := \left(\sum_{k \in \mathcal{K}_p} L_k \right) D \quad (34a)$$

$$W_p := \sum_{i \in \mathcal{U}_p} \omega_i \left(\bigoplus_{t \in [2m]} T_{(t,i)} \right), \quad (34b)$$

$$X_p := \left(\sum_{i \in \mathcal{U}_p} \xi_i \left(\bigoplus_{t \in [2m]} T_{(t,i)} \right) \right) \Psi_1, \quad (34c)$$

$$Y_p := \Psi_2 \left(\sum_{k \in \mathcal{K}_p} \eta_k L_k \right), \quad (34d)$$

$$Z_p := \sum_{k \in \mathcal{K}_p} \zeta_k L_k \Psi_3 L_k + \sum_{k \in \mathcal{K}_p} \sum_{\ell \in \mathcal{K}_p \cap \mathcal{L}_k} \theta_{k,\ell} L_k \Psi_3 L_\ell, \quad (34e)$$

where $\Psi_{\{1,2,3\}}$ is given in (21), and T in (2b). It follows directly that $W_p \succeq 0$. Further, (20a) holds in view of (33). As such, in order to establish the claimed result via Lemma 1, the equivalence of (18) and (15), and Theorem 3, it remains to verify $W := \sum_{p \in [c]} W_p \succ 0_{2m,2m}$, and (20b)–(20d).

With (34b),

$$\begin{aligned} \sum_{p \in [c]} W_p &= \sum_{p \in [c]} \sum_{i \in \mathcal{U}_p} \omega_i \left(\bigoplus_{t \in [2m]} T_{(t,i)} \right) \\ &= \sum_{i \in \mathcal{V}} \sum_{p \in \mathcal{J}_i} \omega_i \left(\bigoplus_{t \in [2m]} T_{(t,i)} \right) \\ &= \sum_{i \in \mathcal{V}} \left(\bigoplus_{t \in [2m]} T_{(t,i)} \right) = I_{2m} \succ 0_{2m,2m}. \end{aligned}$$

The second equality above holds by (24), and the second last equality holds because (31a) implies $\sum_{p \in \mathcal{J}_i} \omega_i = 1$ for every $i \in \mathcal{V}$. Similarly, with (34c),

$$\sum_{p \in [c]} X_p = \sum_{p \in [c]} \sum_{i \in \mathcal{U}_p} \left(\bigoplus_{t \in [2m]} \xi_{p,i} T_{(t,i)} \right) \Psi_1 = \Psi_1,$$

since $\sum_{p \in \mathcal{J}_i} \xi_{p,i} = 1$, and thus, (20b) holds with equality.

With (34a) and (34d), $\frac{1}{2} Y_p K_p = Y_p D$ by Lemma 2 in the Appendix. As such,

$$\begin{aligned} \sum_{p \in [c]} \frac{1}{2} Y_p K_p &= \Psi_2 \left(\sum_{p \in [c]} \sum_{k \in \mathcal{K}_p} \eta_k L_k \right) D \\ &= \Psi_2 \left(\sum_{k \in \mathcal{M}} \sum_{p \in \mathcal{O}_k} \eta_k L_k \right) D \\ &= \Psi_2 \left(\sum_{k \in \mathcal{M}} L_k \right) D = \Psi_2 K. \end{aligned}$$

The second equality above holds by (26), the third because (31b) implies $\sum_{p \in \mathcal{O}_k} \eta_k = 1$, and last in view of (5) and (19b). Therefore, (20c) holds with equality.

Finally, with (34a) and (34e), $\frac{1}{4} K_p^* Z_p K_p = D^* Z_p D$ by Lemma 2 in the Appendix. As such, again by (26),

$$\begin{aligned} \sum_{p \in [c]} \frac{1}{4} K_p^* Z_p K_p &= D^* \left(\sum_{k \in \mathcal{M}} \sum_{p \in \mathcal{O}_k} \zeta_k L_k \Psi_3 L_k \right. \\ &\quad \left. + \sum_{k \in \mathcal{M}} \sum_{p \in \mathcal{O}_k} \sum_{\ell \in \mathcal{K}_p \cap \mathcal{L}_k} \theta_{k,\ell} L_k \Psi_3 L_\ell \right) D. \quad (35) \end{aligned}$$

The first term inside the brackets is equal to $\sum_{k \in \mathcal{M}} L_k \Psi_3 L_k$ since (31b) implies $\sum_{p \in \mathcal{O}_k} \zeta_k = 1$. By (28), the second equals $\sum_{k \in \mathcal{M}} \sum_{\ell \in \mathcal{L}_k} \sum_{p \in \mathcal{Q}_{k,\ell}} \theta_{k,\ell} L_k \Psi_3 L_\ell = \sum_{k \in \mathcal{M}} \sum_{\ell \in \mathcal{L}_k} L_k \Psi_3 L_\ell$, where this last equality holds because (31c) implies $\sum_{p \in \mathcal{Q}_{k,\ell}} \theta_{k,\ell} = 1$. By Lemma 3 in the Appendix, $\sum_{k \in \mathcal{M}} L_k \Psi_3 L_k + \sum_{k \in \mathcal{M}} \sum_{\ell \in \mathcal{L}_k} L_k \Psi_3 L_\ell = L \Psi_3 L$, and as such,

$$\sum_{p \in [c]} \frac{1}{4} K_p^* Z_p K_p = D^* L \Psi_3 L D = K^* \Psi_3 K,$$

where last equality holds in view of (5) and (19b). Therefore, (20d) also holds with equality. \square

Remark 3: The neighbourhood-wise decomposition considered in [6] corresponds to the localized edge partition $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ with $\mathcal{F}_p = \mathcal{E}_p$, $p \in [n]$, where $n = |\mathcal{V}|$. The elements of the associated vertex partition satisfy $|\mathcal{U}_p| = 1 + |\mathcal{N}_p|$, $p \in [n]$. Further, it is readily verified that Assumption 2 holds. In particular, $|\mathcal{O}_k| = 2$ and $\mathcal{L}_k = (\mathcal{K}_i \setminus \{k\}) \cup (\mathcal{K}_j \setminus \{k\})$ with $\{i, j\} = \kappa^{-1}(k)$, for all $k \in \mathcal{M}$. On the other hand, the link-wise decomposition originally considered in [5] involves the localized edge partition $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_m\}$ with $\mathcal{F}_p = \kappa^{-1}(p)$, $p \in [m]$, where $m = |\mathcal{E}|$. The elements of the associated vertex partition satisfy $|\mathcal{U}_p| = 2$, $p \in [m]$. In this case, the selection of \hat{Z}_p associated with (30e) and (34e) does not enable verification of (20d). Instead, a very conservative and non-unique diagonal plus semi-definite splitting of Ψ_3 is used to this end in [5].

Remark 4: Given state-space models for the coprime factors in (17), and static multipliers for the uncertain links, the localized IQC corresponding to the existence of $\epsilon_p \in \mathbb{R}_{>0}$ such (32) for all $z \in \mathbf{L}_2^{\hat{m}_p}$ can be verified via the Kalman-Yakubovic-Popov (KYP) lemma [17]. The resulting Linear Matrix Inequality (LMI) condition can be verified by solving a semi-definite program (SDP), as elaborated in the numerical example section. Towards reformulating the form of (32) to be directly amenable to the KYP lemma, let

$$\hat{N}_p := \bigoplus_{i \in \mathcal{U}_p} N_i, \quad \hat{T}_p := \bigoplus_{i \in \mathcal{U}_p} \mathbf{1}_{m_i,1} \text{ and } \hat{J}_p := \bigoplus_{i \in \mathcal{U}_p} J_i.$$

Then, $[\hat{N}_p^* \quad \hat{J}_p^* \quad -\hat{D}_p^*] = [\hat{N}_p^* \quad \hat{D}_p^*] \hat{S}_p^*$ with

$$\hat{S}_p := \begin{bmatrix} I_{n_p} & \mathbf{0}_{n_p, \hat{m}_p} \\ -\hat{T}_p & I_{\hat{m}_p} \\ \mathbf{0}_{\hat{m}_p, n_p} & -I_{\hat{m}_p} \end{bmatrix}$$

and $n_p := |\mathcal{U}_p|$. Further, let

$$\hat{\Xi}_p := ((\bigoplus_{i \in \mathcal{U}_p} \xi_i) \oplus (\bigoplus_{i \in \mathcal{U}_p} \xi_i I_{m_i})),$$

$$\hat{H}_p := \left(\sum_{k \in \mathcal{K}_p} \eta_k \hat{L}_{p,k} \right) \text{ and } \hat{\Pi}_p := \begin{bmatrix} \hat{\Pi}_{1,p} & \hat{\Pi}_{2,p} \\ \hat{\Pi}_{2,p}^* & \hat{\Pi}_{3,p} \end{bmatrix},$$

where $\hat{\Pi}_{\{1,2,3\},p} := \bigoplus_{i \in \mathcal{U}_p} \Pi_{\{1,2,3\},i}$ according to (12). Then, $\hat{Y}_p = -[\hat{N}_p^* \quad \hat{J}_p^*] [\hat{\Pi}_{2,p}^* \quad \hat{\Pi}_{3,p}]^* \hat{H}_p$, and $\hat{X}_p = [\hat{N}_p^* \quad \hat{J}_p^*] \tilde{\Pi}_p [\hat{N}_p^* \quad \hat{J}_p^*]^*$, where $\tilde{\Pi}_p := \hat{\Xi}_p^{1/2} \hat{\Pi}_p \hat{\Xi}_p^{1/2}$. Since $\frac{1}{2} \hat{Y}_p \hat{K}_p = \hat{Y}_p \hat{D}_p$ and $\frac{1}{4} \hat{K}_p^* \hat{Z}_p \hat{K}_p = \hat{D}_p^* \hat{Z}_p \hat{D}_p$ by Lemma 2, for all $\hat{z} \in \mathbf{L}_2^{\hat{m}_p}$ the inequality (32) is therefore equivalent to

$$\left\langle \begin{bmatrix} \hat{N}_p \\ \hat{D}_p \\ I_{\hat{m}_p} \end{bmatrix} \hat{z}, \begin{bmatrix} \hat{\Phi}_{1,p} & \hat{\Phi}_{2,p} & \mathbf{0}_{n_p, \hat{m}_p} \\ \hat{\Phi}_{2,p}^* & \hat{\Phi}_{3,p} & \mathbf{0}_{\hat{m}_p, \hat{m}_p} \\ \mathbf{0}_{n_p, \hat{m}_p} & \mathbf{0}_{\hat{m}_p, \hat{m}_p} & \epsilon_p \hat{W}_p \end{bmatrix} \begin{bmatrix} \hat{N}_p \\ \hat{D}_p \\ I_{\hat{m}_p} \end{bmatrix} \hat{z} \right\rangle \leq 0, \quad (36)$$

where

$$\begin{bmatrix} \hat{\Phi}_{1,p} & \hat{\Phi}_{2,p} \\ \hat{\Phi}_{2,p}^* & \hat{\Phi}_{3,p} \end{bmatrix} = \hat{S}_p^* \begin{bmatrix} \tilde{H}_p & \begin{bmatrix} \hat{\Pi}_{2,p} \\ \hat{\Pi}_{3,p} \end{bmatrix} \hat{H}_p \\ \hat{H}_p^* [\hat{\Pi}_{2,p}^* \quad \hat{\Pi}_{3,p}^*] & \tilde{Z}_p \end{bmatrix} \hat{S}_p.$$

With a static multiplier in (36), and all dynamics limited to \hat{N}_p and \hat{D}_p , the standard KYP lemma applies. Extension to classes of dynamic multipliers is possible, but more involved.

V. NUMERICAL EXAMPLE

Consider an interconnected system with twelve dynamic agents and path-graph network topology. The graph vertices $\mathcal{V} = [12] = \{1, 2, \dots, 12\}$ represent the agents and the eleven edges $\mathcal{E} = \{e_1, \dots, e_{11}\} := \{\{1, 2\}, \{2, 3\}, \dots, \{11, 12\}\}$ represent information exchange, which is bi-directional, but non-necessarily symmetrical. As such, the edges represent local feedback loops. Each agent $i \in \{2, \dots, 11\}$ has linear time-invariant dynamics $H_i = g_i \mathbf{1}_{1,2}$ corresponding to the SISO transfer function g_i as given in Table I. For agents 1 and 12, the dynamics $H_1 = g_1$ and $H_{12} = g_{12}$ are SISO.

For all agents, the uncertainty in each link from the single output to its neighbours is taken to have a common IQC characteristic; this does not mean the links are identical. Specifically, it is assumed that all link uncertainties lie in the sector $\{(u, y) \mid u \in \mathbb{R}, y \in \mathbb{R}_{\geq \beta u} \cap \mathbb{R}_{\leq \alpha u}\}$ for given parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. As such, (12) holds with

$$\Pi_1 = \varphi_1 \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix}, \quad \Pi_{12} = \varphi_{12} \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix},$$

and

$$\Pi_i = \begin{bmatrix} -2\alpha\beta(\varphi_i + \vartheta_i) & (\alpha + \beta)\varphi_i & (\alpha + \beta)\vartheta_i \\ (\alpha + \beta)\varphi_i & -2\varphi_i & 0 \\ (\alpha + \beta)\vartheta_i & 0 & -2\vartheta_i \end{bmatrix}$$

for $i \in [2 : 11]$, where $\varphi_i \in \mathbb{R}_{>0}$ and $\vartheta_i \in \mathbb{R}_{>0}$ are free multiplier variables.

Consider the five edge-localized partitions \mathcal{P}_c , each with cardinality $c \in [5]$, as given in Table II, where $e_{[i:i+j]}$ denotes $\{e_i, e_{i+1}, \dots, e_{i+j}\}$. For each partition, network stability is verified via Theorem 4. Given $\mathcal{F}_p \in \mathcal{P}_c$, the dimension of the KYP-based LMI associated with (36) is $\tilde{m}_p := (\hat{m}_p + \hat{n}_p)$, with \hat{m}_p as in (29a) and $\hat{n}_p := \sum_{i \in \mathcal{U}_p} \nu_i$, where ν_i is the state dimension of a minimal realization of g_i . The number

agent	1, 5, 7, 11	2, 6, 8, 12	3, 9	4, 10
$g_i(s)$	$\frac{-1}{s^2 + s + 2}$	$\frac{5}{s + 10}$	$\frac{-2}{s^2 + s + 5}$	$\frac{4}{s + 20}$

TABLE I
AGENT DYNAMICS

	$\mathcal{F}_1, \tilde{n}_1, \tilde{m}_1$	$\mathcal{F}_2, \tilde{n}_2, \tilde{m}_2$	$\mathcal{F}_3, \tilde{n}_3, \tilde{m}_3$	$\mathcal{F}_4, \tilde{n}_4, \tilde{m}_4$	$\mathcal{F}_5, \tilde{n}_5, \tilde{m}_5$
\mathcal{P}_1	$e_{[1:11]}, 194, 40$	—	—	—	—
\mathcal{P}_2	$e_{[1:6]}, 80, 24$	$e_{[6:11]}, 69, 23$	—	—	—
\mathcal{P}_3	$e_{[1:5]}, 57, 20$	$e_{[4:8]}, 58, 21$	$e_{[7:11]}, 57, 20$	—	—
\mathcal{P}_4	$e_{[1:4]}, 46, 17$	$e_{[4:7]}, 39, 17$	$e_{[5:8]}, 47, 18$	$e_{[8:11]}, 38, 16$	—
\mathcal{P}_5	$e_{[1:3]}, 29, 13$	$e_{[3:5]}, 30, 14$	$e_{[5:7]}, 30, 14$	$e_{[7:9]}, 30, 14$	$e_{[9:11]}, 29, 13$

TABLE II
PARTITIONS \mathcal{P}_c AND CORRESPONDING \tilde{n}_p AND \tilde{m}_p VALUES $p \in [c]$

of LMI variables is $\tilde{n}_p := \frac{1}{2}(\hat{n}_p^2 + \hat{n}_p) + \hat{m}_p + 1$. The first term corresponds to the KYP-lemma variables, the second to the multiplier variables, and the last to ϵ_p . The LMIs associated with partition components that overlap are coupled via multiplier variables. Fixing these decouples the LMIs, as well as reducing the number of variables, at the expense of increased conservativeness.

For each partition \mathcal{P}_c , an SDP is formulated to test feasibility of the cardinality c collection of LMIs given the common sector parameters $(\alpha, \beta) = (\tan(\theta_2), \tan(\theta_1))$, over a (θ_1, θ_2) grid, with $-90 < \theta_1, \theta_2 < 90$ degrees. For fixed multiplier variables, this decomposes into c independent SDPs. Each incurs $O(\tilde{n}_p \tilde{m}_p^3 + \tilde{n}_p^2 \tilde{m}_p^2 + \tilde{n}_p^3)$ computational complexity per Newton step for an interior-point method based on the standard logarithmic barrier function, where $\tilde{n}_p := (\tilde{n}_p - \hat{m}_p)$ is the reduced number of variables, $p \in [c]$; the first term corresponds to computation of the gradient, the second to computation of the Hessian, and the last to inversion of the Hessian in determining the Newton step [18]. These independent SDPs could be solved in parallel, but overall the complexity is no worse than $O(\sum_{p \in [c]} \tilde{n}_p \tilde{m}_p^3 + \tilde{n}_p^2 \tilde{m}_p^2 + \tilde{n}_p^3)$ per step. On the other hand, with free multiplier variables, the SDP does not decompose. Computation of the gradient and the Hessian of the standard logarithmic barrier function for the coupled LMIs increases in complexity due to the increase in the number of variables, but to no worse than $O(\sum_{p \in [c]} \tilde{n}_p \tilde{m}_p^3)$ and $O(\sum_{p \in [c]} \tilde{n}_p^2 \tilde{m}_p^2)$, respectively. Indeed, the Hessian permutes to a block arrow-head structure, with $c + 1$ diagonal blocks. For each $p \in [c]$, the corresponding diagonal block has dimension bounded by \tilde{n}_p . The remaining arrow-head block has dimension bounded by the number \tilde{n} of multiplier variables that couple the component LMIs. The complexity of inverting the Hessian is therefore no worse than $O(\tilde{n}^3 + \sum_{p \in [c]} \tilde{n}_p^3)$. As such, with free multiplier variables, computational advantage also arises from partitioning. An elaboration of this is in preparation as a companion paper.

Without explicitly exploiting structure, the aforementioned SDP is solved using the YALMIP [19] parser and SEDUMI [20] solver; the results were also validated using SDPT3 [21]. These are shown in Figure 3, where the enclosed ‘stability regions’ are where robust stability has been validated with the multiplier variables free (solid) and fixed to 1 (dashed). Note that no ‘stability region’ could be verified for the partition $\mathcal{P} := \{\mathcal{F}_p \mid p \in \{1, \dots, 10\}\}$ with $\mathcal{F}_p := e_{[p:p+1]}$. The trade-off between conservativeness and potential scalability is evident in Figure 3. On the left it can be seen that a reduction in complexity achieved by reducing the size of partition components whilst maintaining one-edge overlap, incurs increased conservativeness of the verified ‘stability region’. Similar results for more than one-edge overlap partitions (all two in \mathcal{P}_3 , mid three in \mathcal{P}_4) are shown on the right.

VI. CONCLUSION

The main result is a partitioned robust stability certificate for networks with uncertain links. This result provides scope trading-off scalability and conservativeness as preliminarily explored in a numerical example. Further elaboration of this trade-off is left as future work.

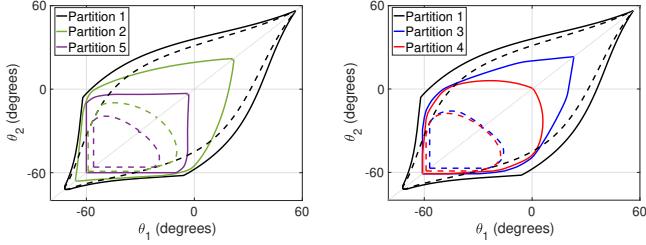


Fig. 3. Stability verified for (θ_1, θ_2) values enclosed by curves shown.

APPENDIX

Lemma 2: For $k \in \mathcal{M}$, let $L_k = B_{(\cdot,k)}B'_{(\cdot,k)}$ as in (5). Then, $L_k L_\ell = 2L_k$ if $\ell = k$, and $\mathbf{0}_{2m,2m}$ otherwise. Further, if $\ell \in \mathcal{M} \setminus (\mathcal{L}_k \cup \{k\})$, with \mathcal{L}_k as defined in (27), then $L_k \Phi L_\ell = \mathbf{0}_{2m,2m}$ for every linear $\Phi = \bigoplus_{i \in \mathcal{V}} \Phi_i$ with $\Phi_i : \mathbf{L}_2^{m_i} \rightarrow \mathbf{L}_2^{m_i}$.

Proof: The result follows by direct calculation, as also observed in [5], [6]. In particular, $B'_{(\cdot,k)} B_{(\cdot,\ell)} = 0$ if $k \neq \ell$, and 2 otherwise, for all $k, \ell \in \mathcal{M}$. Further, $L_k \Phi L_\ell$ is given by

$$B_{(\cdot,k)} \left(\bigoplus_{i \in \mathcal{V}} B'_{((s(i-1)+1):(s(i),k)} \Phi_i B_{((s(i-1)+1):(s(i),\ell)} \right) B'_{(\cdot,\ell)},$$

where $s(i) := \sum_{h \in [i]} m_h$. Since $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ is 1-regular, $\ell \notin (\mathcal{L}_k \cup \{k\})$ implies at least one of $B_{((s(i-1)+1):(s(i),k)}$ or $B_{((s(i-1)+1):(s(i),\ell)}$ is zero, and thus, $L_k \Phi L_\ell = \mathbf{0}_{2m,2m}$. \square

Lemma 3: Given $\Phi = \bigoplus_{i \in \mathcal{V}} \Phi_i$, with $\Phi_i : \mathbf{L}_2^{m_i} \rightarrow \mathbf{L}_2^{m_i}$ linear for each $i \in \mathcal{V}$, and $L = \sum_{k \in \mathcal{M}} L_k$ as per (5),

$$L \Phi L = \sum_{k \in \mathcal{M}} L_k \Phi L_k + \sum_{k \in \mathcal{M}} \sum_{\ell \in \mathcal{L}_k} L_k \Phi L_\ell,$$

where \mathcal{L}_k is defined in (27) for $k \in \mathcal{M}$.

Proof: By Lemma 2, $L_k \Phi L_\ell = \mathbf{0}_{2m,2m}$ for all $k \in \mathcal{M}$ and $\ell \in \mathcal{M} \setminus (\mathcal{L}_k \cup \{k\})$. Therefore,

$$\begin{aligned} L \Phi L &= (\sum_{k \in \mathcal{M}} L_k) \Phi (\sum_{\ell \in \mathcal{M}} L_\ell) \\ &= \sum_{k \in \mathcal{M}} L_k \Phi (L_k + \sum_{\ell \in \mathcal{L}_k} L_\ell + \sum_{\ell \in \mathcal{M} \setminus (\mathcal{L}_k \cup \{k\})} L_\ell) \\ &= \sum_{k \in \mathcal{M}} L_k \Phi L_k + \sum_{k \in \mathcal{M}} \sum_{\ell \in \mathcal{L}_k} L_k \Phi L_\ell, \end{aligned}$$

as claimed. \square

Lemma 4: For $p \in [c]$ and $k \in \mathcal{K}_p$, with \mathcal{K}_p as per (25), let $\hat{L}_{p,k} = \hat{B}_{p,k}(\hat{B}_{p,k})'$ as in (29). Then, $\hat{L}_{p,k} \hat{L}_{p,\ell} = 2\hat{L}_{p,k}$ if $\ell = k$, and $\mathbf{0}_{\hat{m}_p, \hat{m}_p}$ otherwise. Further, for $\mathcal{K}_{p,1}, \mathcal{K}_{p,2} \subseteq \mathcal{K}_p$, $(\sum_{k \in \mathcal{K}_{p,1}} \hat{L}_{p,k})(\sum_{\ell \in \mathcal{K}_{p,2}} \hat{L}_{p,\ell}) = 2 \sum_{k \in \mathcal{K}_{p,1} \cap \mathcal{K}_{p,2}} \hat{L}_{p,k}$.

Proof: By definition, $\hat{B}'_{p,k} \hat{B}_{p,\ell} = B'_{(\cdot,k)} B_{(\cdot,\ell)}$, and thus, the result holds by Lemma 2. \square

Lemma 5: Given $p \in [c]$ and $\epsilon_p \in \mathbb{R}_{>0}$, the following holds: (32) for all $\hat{z} \in \mathbf{L}_2^{\hat{m}_p} \Leftrightarrow$ (33) for all $z \in \mathbf{L}_2^{2m}$.

Proof: Recalling (30), $\frac{1}{2}\hat{Y}_p \hat{K}_p = \hat{Y}_p \hat{D}_p$ and $\frac{1}{4}K_p \hat{Z}_p \hat{K}_p = \hat{D}_p^* \hat{Z}_p \hat{D}_p$ by Lemma 4. As such, (32) is $\langle \hat{z}, \hat{\Xi}_p \hat{z} \rangle \leq 0$, where

$$\begin{aligned} \hat{\Xi}_p &:= \bigoplus_{i \in \mathcal{U}_p} (\xi_i \hat{\Psi}_{1,i} + \epsilon_p \omega_i I_{m_i}) + \hat{\Psi}_{2,p} (\sum_{k \in \mathcal{K}_p} \eta_k \hat{L}_{p,k}) \hat{D}_p \\ &\quad + (\hat{\Psi}_{2,p} (\sum_{k \in \mathcal{K}_p} \eta_k \hat{L}_{p,k}) \hat{D}_p)^* + \hat{D}_p^* \hat{Z}_p \hat{D}_p. \end{aligned}$$

Similarly, recalling (34a), $\frac{1}{2}Y_p K_p = Y_p D_p$ and $\frac{1}{4}K_p Z_p K_p = D_p^* Z_p D_p$ by Lemma 2, and thus, (33) is $\langle z, \Xi_p z \rangle \leq 0$, where

$$\begin{aligned} \Xi_p &= \sum_{i \in \mathcal{U}_p} (\xi_i (\bigoplus_{t \in [2m]} T_{(t,i)}) \Psi_1 + \epsilon_p \omega_i (\bigoplus_{t \in [2m]} T_{(t,i)})) \\ &\quad + \Psi_2 (\sum_{k \in \mathcal{K}_p} \eta_k L_k) D + (\Psi_2 (\sum_{k \in \mathcal{K}_p} \eta_k L_k) D)^* + D^* Z_p D. \end{aligned}$$

Given any $p \in [c]$ and $k \in \mathcal{K}_p$, $B_{((s(i-1)+1):(s(i),k)}$ in (29) is non-zero only if $i \in \mathcal{U}_p$. Therefore, there exists a permutation $\Omega_p = \Omega'_p = \Omega_p^{-1} \in \mathbb{R}^{2m \times 2m}$ for which $\Omega_p B_{(\cdot,k)} = (\hat{B}_{p,k} \oplus \mathbf{0}_{2m-\hat{m}_p}) \mathbf{1}_{2m,1}$, whereby $\Omega_p L_k \Omega'_p = \hat{L}_{p,k} \oplus \mathbf{0}_{2m-\hat{m}_p, 2m-\hat{m}_p}$. One can then show by direct calculation that $\Omega_p \Xi_p \Omega'_p = \hat{\Xi}_p \oplus \mathbf{0}_{2m-\hat{m}_p, 2m-\hat{m}_p}$, and the claimed equivalence follows. \square

REFERENCES

- [1] D. D. Siljak, *Large-Scale Dynamic Systems: Stability and Structure*. North-Holland, 1978.
- [2] M. Arcak, C. Meissen, and A. Packard, *Networks of Dissipative Systems: Compositional Certification of Stability, Performance, and Safety*. Springer, 2016.
- [3] P. Moylan and D. Hill, "Stability criteria for large-scale systems," *IEEE Transactions on Automatic Control*, vol. 23, pp. 143–149, 1978.
- [4] M. Vidyasagar, *Input-Output Analysis of Large-Scale Interconnected Systems: Decomposition, Well-posedness and Stability*. Springer, 1981.
- [5] S. Mariano and M. Cantoni, "Structured stability analysis of networked systems with uncertain links," in *2024 European Control Conference (ECC)*, pp. 1222–1227, 2024.
- [6] S. Mariano and M. Cantoni, "Neighbourhood conditions for network stability with link uncertainty," in *63rd IEEE Conference on Decision and Control (CDC)*, pp. 4418–4423, 2024.
- [7] W. Heemels, A. Teel, N. van de Wouw, and D. Nešić, "Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance," *IEEE Transactions on Automatic control*, vol. 55, no. 8, pp. 1781–1796, 2010.
- [8] M. Cantoni, C.-Y. Kao, and M. Fabbri, "Integral quadratic constraints for asynchronous sample-and-hold links," *IEEE Transactions on Automatic Control*, vol. 66, no. 10, pp. 4888–4894, 2021.
- [9] C. Langbort, R. Chandra, and R. D’Andrea, "Distributed control design for systems interconnected over an arbitrary graph," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1502–1519, 2004.
- [10] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 819–830, 1997.
- [11] I. Lestas and G. Vinnicombe, "Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 10, pp. 1613–1625, 2006.
- [12] U. T. Jönsson and C.-Y. Kao, "A scalable robust stability criterion for systems with heterogeneous LTI components," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2219–2234, 2010.
- [13] M. Andersen, S. Pakazad, A. Hansson, and A. Rantzer, "Robust stability analysis of sparsely interconnected uncertain systems," *IEEE Transactions on Automatic Control*, vol. 59, pp. 2151–2156, 2014.
- [14] S. Khong and A. Rantzer, "Scalable stability conditions for heterogeneous networks via integral quadratic constraints," in *2014 European Control Conference (ECC)*, pp. 2863–2867, IEEE, 2014.
- [15] R. Pates and G. Vinnicombe, "Scalable design of heterogeneous networks," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2318–2333, 2016.
- [16] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. MIT Press, 1985.
- [17] A. Rantzer, "On the Kalman—Yakubovich—Popov lemma," *Systems & Control Letters*, vol. 28, no. 1, pp. 7–10, 1996.
- [18] C. Kao and A. Megretski, "On the new barrier function and specialized algorithms for a class of semidefinite programs," *SIAM Journal on Control and Optimization*, vol. 46, no. 2, pp. 468–495, 2007.
- [19] J. Löfberg, "YALMIP : A toolbox for modeling and optimization in MATLAB," in *In Proceedings of the CACSD Conference*, (Taipei, Taiwan), 2004.
- [20] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11, no. 1–4, pp. 625–653, 1999.
- [21] K. C. Toh, M. J. Todd, and R. H. Tütüncü, "SDPT3 — A MATLAB software package for semidefinite programming, version 1.3," *Optimization Methods and Software*, vol. 11, no. 1–4, pp. 545–581, 1999.