

Heralded Linear Optical Generation of Dicke States

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Abstract

Entanglement is a fundamental feature of quantum mechanics and a key resource for quantum information processing. Among multipartite entangled states, Dicke states $|D_n^k\rangle$ are distinguished by their permutation symmetry, which provides robustness against particle loss and enables applications for quantum communication and computation. Although Dicke states have been realized in various platforms, most optical implementations rely on postselection, which destroys the state upon detection and prevents its further use. A heralded optical scheme is therefore highly desirable. Here, we present a linear-optical heralded scheme for generating arbitrary Dicke states $|D_n^k\rangle$ with $3n+k$ photons through the framework of the linear quantum graph (LQG) picture. By mapping the scheme design into the graph-finding problem, and exploiting the permutation symmetry of Dicke states, we overcome the structural complexity that has hindered previous approaches. Our results provide a resource-efficient pathway toward practical heralded preparation of Dicke states for quantum technologies.

1 Introduction

Entanglement is a fundamental feature of quantum mechanics, representing correlations with no classical reference. It not only plays a central role in deepening our understanding of the foundations of quantum theory, but also serves as the critical resource behind the advantages of quantum information processing [1]. Multipartite entanglement extends quantum correlations across many systems, enabling richer structures and more powerful applications than bipartite entanglement [2]. Such states are indispensable in areas including quantum cryptography [3–5], quantum teleportation [6–8], quantum dense coding [9–11], quantum error correction [12], and quantum computation [13, 14]. Over the past two decades, multipartite entanglement has been realized in diverse physical platforms—from trapped ions [15–17] and neutral atoms [18–20] to superconducting circuits [21–23], and photonic systems [24–27]—but scalable and resource-efficient generation of large entangled states still remains a central challenge.

Different classes of genuinely multipartite entangled states are characterized by distinct structures and applications [28, 29]. Dicke states $|D_n^k\rangle$ form one such class, defined as the equal-amplitude super-

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position of all computational basis states of n -qubits with Hamming weight k [30]:

$$|D_n^k\rangle = \sum_{\substack{\mathbf{x} \in \{0,1\}^{\otimes n} \\ \text{hw}(\mathbf{x})=k}} \binom{n}{k}^{-1/2} |\mathbf{x}\rangle. \quad (1)$$

Here, $\text{hw}(\mathbf{x})$ denotes the Hamming weight of the bitstring \mathbf{x} . Dicke states have a fixed number of qubit states ($n - k$ qubits of state 0 and k qubits of state 1) and are symmetric under qubit permutations, making them compactly representable within the symmetric subspace and resilient against particle loss. Their symmetry underlies applications across quantum networking [31], high-precision metrology [32–34], game theory [35], and error correction [36,37]. More recently, they have been recognized as efficient building blocks for variational quantum eigensolver (VQE) ansätze [38], enabling accurate simulations of many-body Hamiltonians with reduced resources.

Dicke states have been demonstrated experimentally in trapped ions [39–42], atomic systems [43–45], superconducting circuits [46] and optics [31,47–50]. In photonics, they have been generated using diverse sources such as spontaneous parametric down-conversion (SPDC) [31,47,48], Cross-Kerr non-linearity [49], and single photon [50]. However, most previous works on Dicke state generation have relied on postselected methods—sorting out successful outcomes by detecting all photons—which irreversibly destroys the entanglement. Such states cannot then serve as resources in further quantum protocols. This limitation makes heralded optical generation of Dicke states an important open challenge.

Heralded schemes address this issue by using ancillary photons and modes as success signals, identifying valid runs without disturbing the target state. In principle, this preserves entanglement as a usable quantum resource. Yet, designing heralded schemes is significantly more difficult than postselected ones, since the required ancillas and correlations add structural complexity. To date, heralded generation of Dicke states has therefore remained elusive.

In this work, we propose a linear-optical heralded scheme that generates arbitrary Dicke states $|D_n^k\rangle$ within the linear quantum graph (LQG) picture [15,38,39]. The LQG framework maps physical components of entanglement-generating circuits into graph elements, reducing the problem of scheme design to that of graph construction. This approach has successfully yielded heralded schemes for GHZ, W, GHZ-W superpositions [39,40], caterpillar graph states [40], and qudit entangled states [41,42]. Here, we extend the LQG picture to Dicke states, embedding their intrinsic permutation symmetry directly into the graph structure. This reduces design complexity and enables systematic heralded constructions of arbitrary Dicke states. The resulting heralded Dicke state is compatible with existing photonic quantum platforms and can be employed as a genuine quantum resource.

This work is organized as follows: Section 2 reviews the LQG picture of sculpting protocol, the central tool we employ to construct our heralded scheme. Section 3 introduces the Dicke graph in LQG picture, which corresponds to a sculpting operator that generates $|D_n^k\rangle$. Section 4 demonstrates how the Dicke graph is implemented as a linear optical circuit that generates the Dicke state $|D_n^k\rangle$. Section 5 provides concluding remarks and discussions.

2 Review: LQG picture of the sculpting protocol

In this section, we review the concept of sculpting protocol as a operational process to generate multipartite entanglement, which can be mapped to balanced bipartite graphs or equivalently directed graphs [51,52].

Sculpting protocol.— We can generate n -partite entangled states in sculpting protocol by applying spatially overlapped $n+k$ single-boson annihilation operators (called “sculpting operators”) to a $2n+k$ boson initial state, where k is the number of ancillary modes. In our setup, each boson has an $n+k$ -dimensional spatial state and a two-dimensional internal state, hence creation and annihilation operators are expressed as $\hat{a}_{j,s}^\dagger$ and $\hat{a}_{j,s}$ ($j \in \{1, 2, \dots, n+k\}$, $s \in \{+, -\}$) respectively. Here we choose the computational basis in the x-direction for later convenience, which is expressed in the z-directional computation basis $\{|0\rangle, |1\rangle\}$ as $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

We consider a setup with n qubit systems and k ancillary systems. The initial state is given by

$$|\Psi_{\text{init}}\rangle_{n,k} = \prod_{j=1}^n \hat{a}_{j,+}^\dagger \hat{a}_{j,-}^\dagger \prod_{l=1}^k \hat{a}_{n+l,+}^\dagger |\text{vac}\rangle, \quad (2)$$

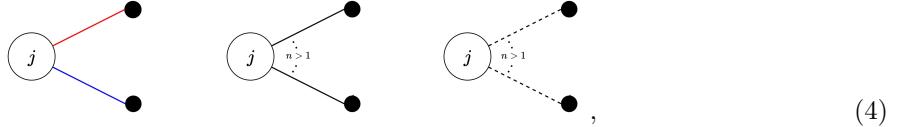
i.e., the n parties contain two bosons while k parties contain one boson. We apply a sculpting operator \hat{A}_{n+k} that subtracts $n+k$ bosons from $|\Psi_{\text{init}}\rangle_{n,k}$. The sculpting operator \hat{A}_{n+k} has the following form:

$$\hat{A}_{n+k} = \prod_{l=1}^{n+k} \hat{A}^{(l)} \equiv \prod_{l=1}^{n+k} \left(\sum_{j=1}^n (\alpha_j^{(l)} \hat{a}_{j,0} + \beta_j^{(l)} \hat{a}_{j,1}) + \sum_{m=1}^k \gamma_l^{(l)} \hat{a}_{n+l,+} \right) \quad (3)$$

with $\sum_{j=1}^n (|\alpha_j^{(l)}|^2 + |\beta_j^{(l)}|^2) + \sum_{m=1}^k |\gamma_l^{(m)}|^2 = 1$. For the above sculpting operator to generate a multipartite qubit state, the sculpting protocol must satisfy the *no-bunching restriction* [51], by which exactly one boson must be removed from each spatial mode by the operation of \hat{A}_{n+k} . Once we find a sculpting operator that generates an entangled state, it can be implemented in linear optics by translation rules [53].

LQG picture of sculpting protocols.— The LQG picture [51] maps all the essential elements in the sculpting protocol into graph elements. The same sculpting operator in the LQG picture can be described by two representations, the *undirected bigraph representation* (denoted as G_{ub}) and *directed unipartite graph representation* (denoted as G_{du}) [52]. Table 1 displays the correspondence relations from the physical elements of sculpting operator.

First, in the undirected bigraph representation G_{ub} , a sculpting operator is represented as an undirected balanced bipartite graph, which we call *sculpting bigraph*. For a sculpting operator to satisfy the no-bunching restriction, the final state must be completely determined by the superposition of the perfect matchings¹ of the corresponding sculpting bigraph [51]. Ref. [51] proposed a convenient subset of bigraphs—named *effective perfect matching bigraphs (EPM bigraphs)*—which inherently satisfy the no-bunching restriction by construction [51]. A bigraph is an EPM bigraph if all its edge-to-circle attachments match one of the configurations shown below:



where the edge colors {Solid Black, Dashed Black, Red, Blue} represent internal states $\{|+\rangle, |-\rangle, |0\rangle, |1\rangle\}$ ($|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$). For an EPM bigraph in the LQG picture, we can show that the final state after the application of the corresponding sculpting operator is represented as the *superposition of all the perfect matchings of the EPM bigraph*.

¹A perfect matching of a graph $G = (V, E)$ is a subset of edges $M \subseteq E$ such that every vertex in V is incident to exactly one edge in M .

Table 1: Correspondence relations between a sculpting operator in bosonic systems and graphs in LQG picture. Since we work in the two computational bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ ($|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$), the qubit states are represented as colors {Red, Blue} and {Solid Black, Dashed Black}.

Bosonic Systems with Sculpting Operator	Bipartite Graph $G = (U \cup V, E)$	Directed Graph $G = (W, E)$
Spatial modes j	Labelled circles $(\circ j)$	Labelled circles $(\circ j)$
$\hat{A}^{(l)}$	Unlabelled dots (\bullet)	Labelled circles $(\circ l)$
Spatial distributions of \hat{A}_i	Undirected edges $\in E$	Directed edges $\in E$
Probability amplitude $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,k}$	Edge weight $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,k}$	Edge weight $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,k}$
Qubit state ψ	Edge weight ψ (or color)	Edge weight ψ (or color)

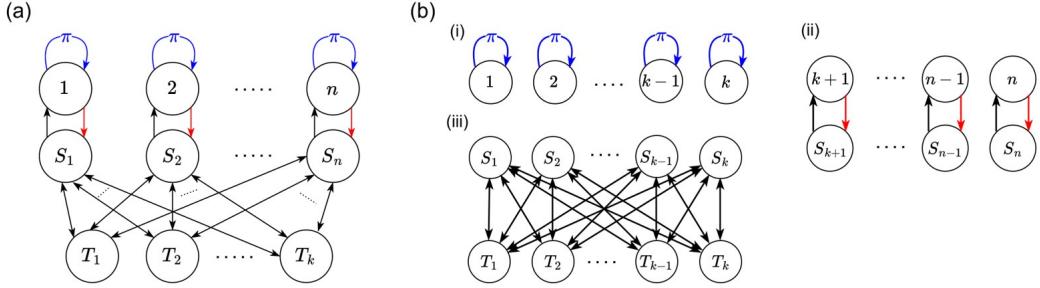
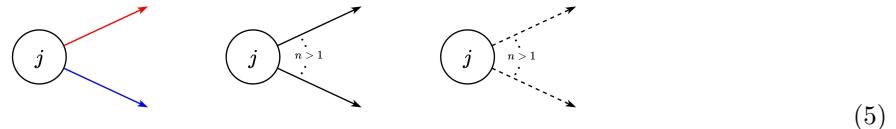


Figure 1: Dicke digraph D_n^k and its Directed Cycle Covers (DCCs). (a) The Dicke digraph D_n^k . Edge weights are omitted, implying that all outgoing edges from a vertex have equal amplitude weights. Two-headed arrows for $S_s \leftrightarrow A_a$ represent the combination of the two directed edges $S_s \rightarrow A_a$ and $A_a \rightarrow S_s$. (b) DCCs of D_n^k shown in (a). These include three types of cycles: (i) a system self-loop ($j \rightarrow j$); (ii) a 2-cycle on a system–ancilla pair ($S_j \rightarrow j \rightarrow S_j$); or (iii) an alternating $S \leftrightarrow T$ cycle of even length. The alternating cycles arise from the remaining vertices U and V forming a complete directed bipartite subgraph, where each choice of perfect matchings $S \rightarrow T$ and $T \rightarrow S$ specifies an alternating cycle cover.

Second, in the directed unipartite representation G_{du} , the same operator is represented as a directed graph, which we call *sculpting digraph*. A key concept in the digraph representation is the disjoint cycle cover² (DCC), which corresponds to a perfect matching in G_{ub} [26, 52, 54]. The final state is hence expressed as a superposition of all possible disjoint cycle covers in G_{du} . EPM bigraphs in G_{ub} are mapped to a special set of digraphs, of which edges are connected to circles as one of the following forms:



While G_{ub} is advantageous for intuitively illustrating sculpting protocols, G_{du} is more appropriate for visualizing complex entanglement structures, as we will see in the next section.

3 Dicke digraph D_n^k

In this section, we introduce the *Dicke digraph* D_n^k , which corresponds to a sculpting operator for generating Dicke states. We present it in the digraph representation G_{du} , because it directly reveals the symmetry of the operator in its structure.

Dicke digraph D_n^k is an EPM digraph that consists of n system vertices (denoted as $\{1, 2, \dots, n\}$) and $n + k$ ancillary vertices (denoted as S_j and T_l , where $j = 1, 2, \dots, n$ and $l = 1, 2, \dots, k$). We define two sets of ancillary vertices as $S \equiv \{S_j\}_{j=1}^n$ and $T \equiv \{T_l\}_{l=1}^k$. The ancillary vertices form a complete directed bipartite graph between S and T . And each system vertex j are connected to the corresponding ancillary vertex S_j . The most general form of D_n^k is shown in Fig. 1 (a). A key property of D_n^k is its symmetry: permutations of the pairs (j, S_j) among different j or of the T_l among l leave the graph invariant. Consequently, all disjoint cycle covers of D_n^k inherit this symmetry, ensuring that the generated quantum state shares the permutation symmetry of the Dicke state $|D_n^k\rangle$.

From Dicke graph to Dicke state.— From the structure of D_n^k , we can show that the final state generated by the corresponding sculpting operator is the Dicke state $|D_n^k\rangle$. To demonstrate this, we first check from Fig. 1 (a) that the Dicke digraph D_n^k is an EPM digraph (see (5)), hence the final state is the superposition of all its DCCs. Every cycle in D_n^k falls into one of three classes: a system self-loop

²A disjoint cycle cover of a digraph G is a collection of directed cycles such that every vertex of G lies in exactly one cycle.

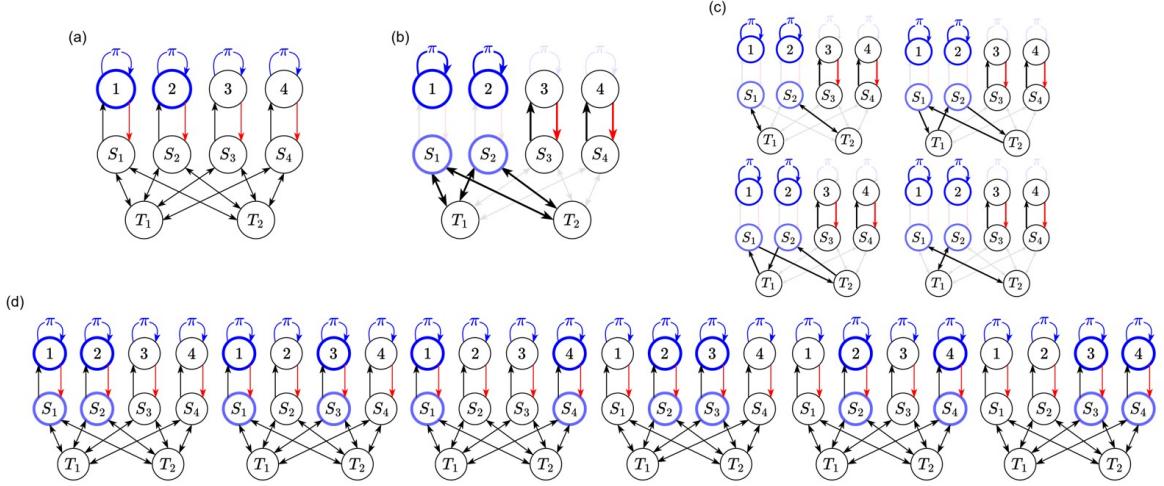


Figure 2: Example of the mechanism for the Dicke digraph with $(n, k) = (4, 2)$. (a) (b) The corresponding system vertices take self-loops, while the remaining system vertices form 2-cycles. $U = \{S_1, S_2\}$ and $W = \{T_1, T_2\}$ form a complete balanced directed bipartite subgraph. (c) Four disjoint cycle covers are possible in the subgraph, which correspond to the same operator monomial and generate $|1100\rangle$. (d) By the permutation symmetry among (j, S_j) ($j \in \{1, 2, 3, 4\}$) of the Dicke digraph, we can see that there are six more subgraphs that give permuted states of $|1100\rangle$, hence Dicke state $|D_4^2\rangle$.

$(j \rightarrow j)$, a 2-cycle $(j \rightarrow S_j \rightarrow j)$, or an cycle alternating through S and T , i.e., $S \leftrightarrow T$. Since system vertices are only connected to $\{S_j\}_{j=1}^n$ or itself, any cycle through j is either $(j \rightarrow j)$ or $(j \rightarrow S_j \rightarrow j)$; and each T_l is only connected to S , hence any cycle including T must be in $S \leftrightarrow T$. Then we can see that *all the DCCs contain $n - k$ self-loops* ($j \rightarrow j$), which is straightforward given that the elements in T are only connected to those in S and hence k elements in S are always included in the cycle $S \leftrightarrow T$.

Now let us consider the case when the system vertices $1, 2, \dots, k$ of the DCC have self-loops (see Fig. 1 (b)). Then the other system vertices $k+1, k+2, \dots, n$ are automatically included in $(j \rightarrow S_j \rightarrow j)$. The remaining cycles are those alternating between $U = \{S_1, \dots, S_k\}$ and $V = \{T_1, \dots, T_k\}$. Since the subgraph including S and T constructs a complete balanced bipartite graph, we can see that it has $(k!)^2$ distinct DCCs in it. From Table 1, we can directly see that these $(k!)^2$ DCCs with the same self-loops correspond to the same operator.

Indeed, the DCCs of the subgraph in Fig. 1 (b) are all written in the operator form as

$$\left(\prod_{m=1}^k -\hat{a}_{m,1} \right) \left(\prod_{p=k+1}^n \hat{a}_{p,0} \right) \left(\prod_{q=1}^n \hat{a}_{S_q,+} \right) \left(\prod_{l=1}^k \hat{a}_{T_l,+} \right). \quad (6)$$

This operator is applied to the $(3n + k)$ -particle initial state $|\Psi_{\text{init}}\rangle_{n,n+k}$ to obtain the state

$$\left(\prod_{m=1}^k \hat{a}_{m,1}^\dagger \right) \left(\prod_{p=k+1}^n \hat{a}_{p,0}^\dagger \right) |\text{vac}\rangle = |\underbrace{1, 1, \dots, 1}_k, \underbrace{0, 0, \dots, 0}_{n-k}\rangle, \quad (7)$$

which is directly derived using the identity

$$\hat{a}_{j,0} \hat{a}_{j,+}^\dagger \hat{a}_{j,-}^\dagger |\text{vac}\rangle = \hat{a}_{j,0}^\dagger |\text{vac}\rangle, \quad \hat{a}_{j,1} \hat{a}_{j,+}^\dagger \hat{a}_{j,-}^\dagger |\text{vac}\rangle = -\hat{a}_{j,1}^\dagger |\text{vac}\rangle. \quad (8)$$

By the permutation symmetry among the exchange of system qubits, the total final state becomes proportional to $|D_n^k\rangle$. By recovering all the normalization factors, the final state $|\Psi_{\text{fin}}\rangle$ is given by

$$|\Psi_{\text{fin}}\rangle = \hat{A}_{2n+k} |\Psi_{\text{init}}\rangle \sim |D_n^k\rangle |\text{vac}\rangle. \quad (9)$$

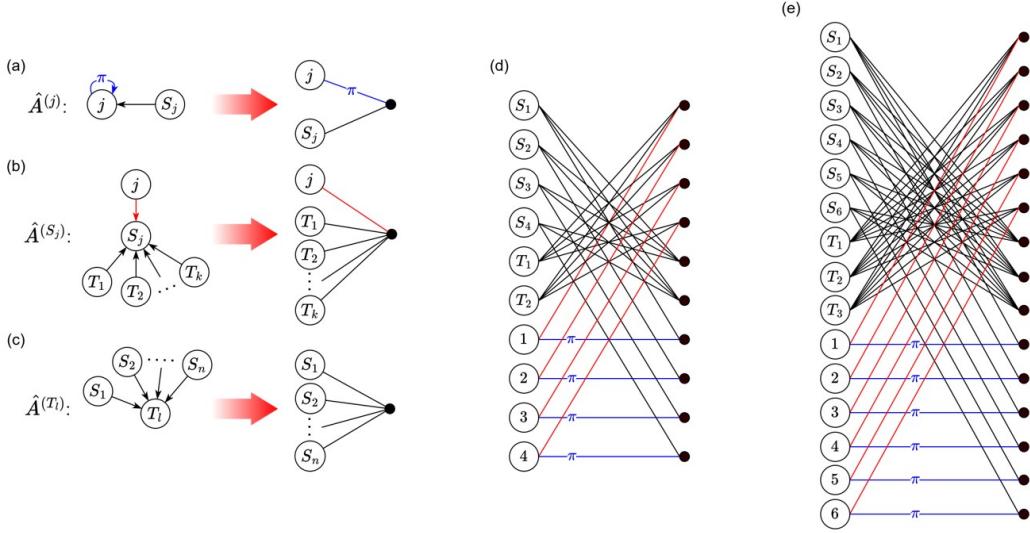


Figure 3: Conversion of a Dicke digraph into a sculpting bigraph. (a–c) Local subgraph transformation from G_{du} to G_{ub} . Node indices, colors and edge weights (including signs) are preserved. (d) Bigraphs D_4^2 and (e) D_6^3 constructed from the conversion.

D_4^2 example.— As a proof of concept, we analyze the simplest non-trivial $(n, k) = (4, 2)$ case here (see Fig. 2). First, we consider DCCs that have self-loops on 1 and 2 (Fig. 2 (a)). Then the other system vertices 3 and 4 pair with their corresponding ancillas to form the 2-cycles $(S_3 \rightarrow 3 \rightarrow S_3)$ and $(S_4 \rightarrow 4 \rightarrow S_4)$ (Fig. 2 (b)). Now the remaining vertices are $U = \{S_1, S_2\}$ and $V = \{T_1, T_2\}$; these induce a complete balanced bipartite directed graph. In $U \cup V$, a disjoint cycle cover is obtained by choosing one perfect matching for $S \rightarrow T$ and another independent perfect matching for $T \rightarrow S$, giving $(2!)^2 = 4$ possibilities (Fig. 2 (c)). These four DCCs all yield the same system operator monomial

$$\left(\prod_{m=1}^2 -\hat{a}_{m,1} \right) \left(\prod_{p=3}^4 \hat{a}_{p,0} \right) \left(\prod_{q=1}^4 \hat{a}_{S_q,+} \right) \left(\prod_{l=1}^2 \hat{a}_{T_l,+} \right). \quad (10)$$

Applying the above operator to the initial state

$$|\Psi_{\text{init}}\rangle_{4,6} = \prod_{j=1}^4 \hat{a}_{j,+}^\dagger \hat{a}_j^\dagger - \prod_{l=1}^4 \hat{a}_{S_l,+}^\dagger \prod_{m=1}^2 \hat{a}_{T_m,+}^\dagger |\text{vac}\rangle, \quad (11)$$

we obtain $|1100\rangle$. By the permutation symmetry among (j, S_j) ($j \in \{1, 2, 3, 4\}$) of the Dicke digraph, we can see that the final state is a superposition of the qubit permutation of $|1100\rangle$, hence becomes the Dicke state $|D_4^2\rangle$ (Fig. 2 (d)).

4 Heralded generation of Dicke states from Dicke graphs

In this section, based on the structure of D_n^k , we provide a physical setup that generates heralded Dicke states $|D_n^k\rangle$. While the sculpting protocol we propose can be implemented in any bosonic system with linear operations in principle, we present a linear optical circuit in this work. Following the procedure in Ref. [53], we design the circuit by first converting the Dicke digraph in G_{du} into an EPM bigraph in G_{ub} , and then mapping bigraph elements to optical elements via a set of translation rules. Those elements are assembled based on the structure of D_n^k , yielding the heralded circuit for our target state.

The transformation of D_n^k from G_{du} to G_{ub} is given in Fig. 3. The digraph D_n^k is decomposed into three elements denoted as $\hat{A}^{(j)}$, $\hat{A}^{(S_j)}$, and $\hat{A}^{(T_l)}$ in Fig. 3 (a)–(c). From the second column of Table 1, each elements are replaced with a dot connected to the relevant circles in G_{ub} . Those three elements completely compose the bigraph form of D_n^k for arbitrary n and k . Fig. 3 (d) displays the D_4^2 example.

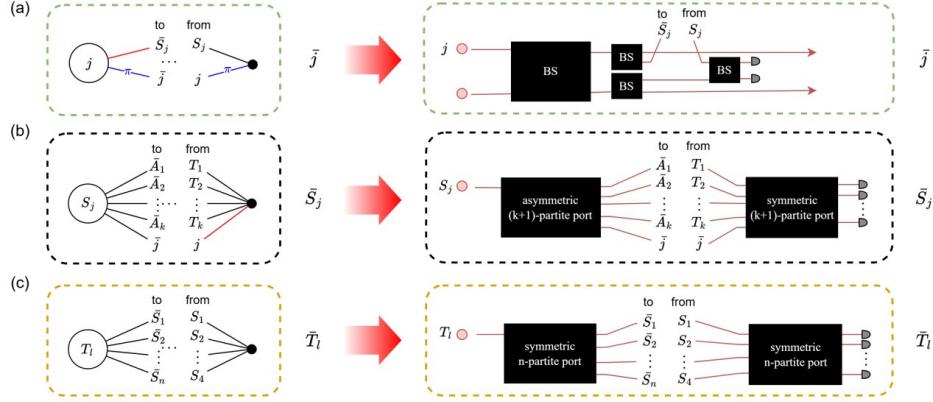


Figure 4: Transformation rules from bigraphs to linear optical networks. We decomposed D_n^k into circle-dot pairs on the same line in the bigraph. Each dashed box on the LHS is an open subgraph, whose open edges are attached to dots or circles following the designated labels. Then we obtain the circuit elements in the dotted boxes on the RHS from the translation rules in Fig. 2 of Ref. [53]. By connecting the open wires in the circuit elements, we can uniquely construct the linear optical circuit that generates $|D_n^k\rangle$ by heralding.

Then we can construct a linear optical setup following the translation rules from the bigraph elements to linear optical elements [53]. We adopt a dual-rail implementation using asymmetric and symmetric n -partite multiport interferometers, where the symmetric ones install the n -level discrete Fourier transformations (note that $n = 2$ multiport interferometer is the BS). In this architecture, each system mode m is represented as a dual-rail qubit composed of two spatial paths $(m, 0)$ and $(m, 1)$, corresponding to the logical states $|0_L\rangle$ and $|1_L\rangle$, respectively, while all ancillary and interface modes are treated as single-rail optical modes³. We then translate each circle-dot pair into linear optical components as shown in Fig. 4. Then the dashed boxes \bar{j} , \bar{S}_j and \bar{T}_l are connected to each other following the bigraph structure of D_n^k , hence we can uniquely construct a linear optical circuit for $|D_n^k\rangle$. For the simplest nontrivial $|D_4^2\rangle$ case, the circuit is as in Fig. 5.

Now we directly check that our circuit actually generates the target state, and derive the success probability by heralding. As denoted in Fig. 5, we divide the process into 5 steps:

Step 1. Preparation of the initial state as

$$\left(\prod_{j=1}^n \hat{a}_{S_j}^\dagger \right) \left(\prod_{l=1}^k \hat{a}_{T_l}^\dagger \right) \left(\prod_{m=1}^n \hat{a}_{m0}^\dagger \hat{a}_{m1}^\dagger \right). \quad (12)$$

Step 2. Division of the photon paths with BSs, asymmetric and symmetric multi-partite ports. Note that the number of paths divided from the ancillary modes corresponds to the number of edges for each ancillary vertex in the bigraph representation. The initial state as in Eq. (12) is transformed to

$$\frac{1}{2^n n^{k/2}} \left(\prod_{j=1}^n \left(\alpha \sum_{s=1}^k \hat{a}_{S_j s}^\dagger + \beta \hat{a}_{S_j k+1}^\dagger \right) \right) \cdot \left(\prod_{l=1}^k \left(\sum_{t=1}^n \hat{a}_{T_l t}^\dagger \right) \right) \cdot \left(\prod_{m=1}^n (\hat{a}_{m0}^{\dagger 2} - \hat{a}_{m1}^{\dagger 2}) \right). \quad (13)$$

Note that the probability amplitudes α and β ($k|\alpha|^2 + |\beta|^2 = 1$) from asymmetric multiports are set to satisfy the symmetry of the circuit. We can control them so that the entire circuit can have the maximal success probability. We denote the creation operator of a mode split by the N -partite port or the BS from a given mode by adding additional indices, i.e., from $\hat{a}_{S_j}^\dagger$ to $\hat{a}_{S_j s}^\dagger$ with $s = 1, \dots, k+1$, and from $\hat{a}_{T_l}^\dagger$ to $\hat{a}_{T_l t}^\dagger$ with $t = 1, \dots, n$.

³Although the main derivation in the manuscript adopts this dual-rail implementation, the same construction can be equivalently realized using polarization encoding.

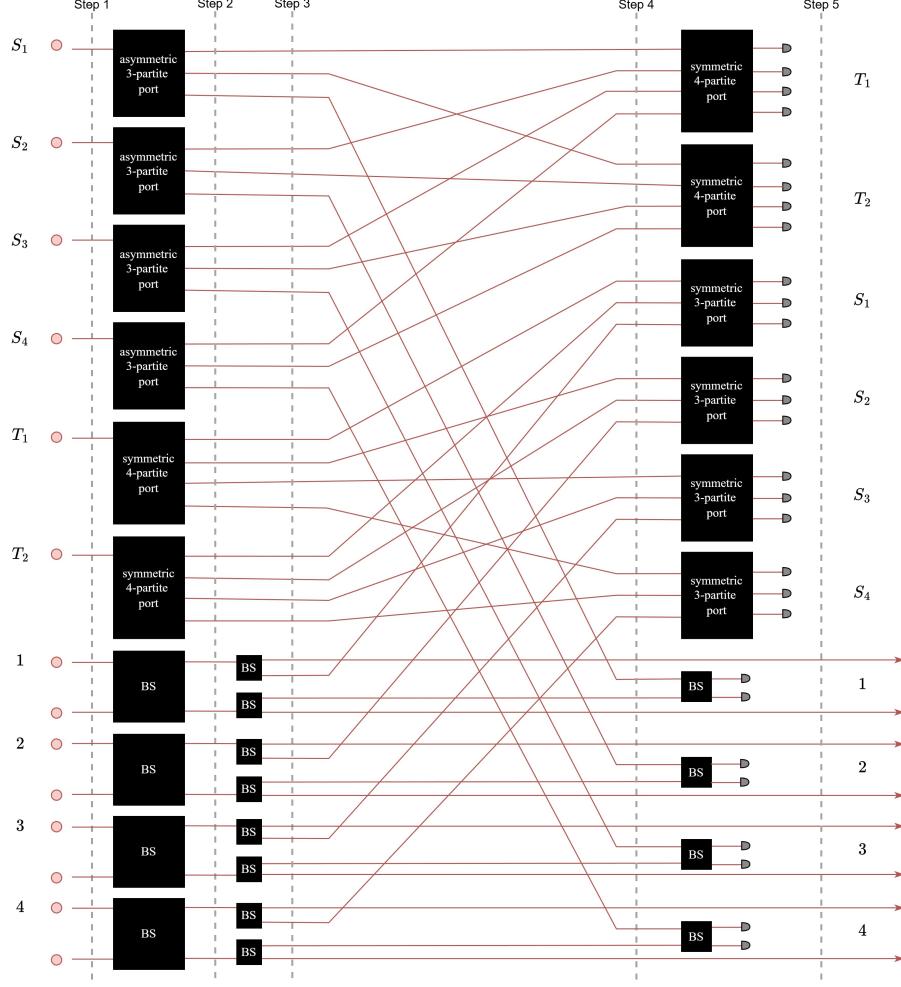


Figure 5: Linear optical network from Dicke digraph D_4^2 , split by several steps. Note that this figure is drawn with the positions of the 4-partite port and the 3-partite port swapped for clarity.

Step 3. Splitting two-photon states with BSs. By denoting the operator $\hat{a}_{j\sigma}^\dagger$ after the BS operation as $\hat{a}_{j\sigma\sigma'}^\dagger$ with $\sigma' = 0, 1$, the state evolves into

$$\begin{aligned} & \frac{1}{2^{2n} n^{k/2}} \left(\prod_{j=1}^n \left(\alpha \sum_{s=1}^k \hat{a}_{S_j s}^\dagger + \beta \hat{a}_{S_j k+1}^\dagger \right) \right) \cdot \left(\prod_{l=1}^k \left(\sum_{t=1}^n \hat{a}_{T_l t}^\dagger \right) \right) \\ & \times \left(\prod_{m=1}^n \left((\hat{a}_{m 0 0}^\dagger + \hat{a}_{m 0 1}^\dagger)^2 - (\hat{a}_{m 1 0}^\dagger - \hat{a}_{m 1 1}^\dagger)^2 \right) \right). \end{aligned} \quad (14)$$

Step 4. Permutation of wires. State (14) evolves into

$$\begin{aligned} & \frac{1}{2^{2n} n^{k/2}} \left(\prod_{j=1}^n \left(\alpha \hat{a}_{T_1 j}^\dagger + \dots + \alpha \hat{a}_{T_k j}^\dagger + \beta \hat{a}_{j 0 1}^\dagger \right) \right) \cdot \left(\prod_{l=1}^k \left(\hat{a}_{S_1 l}^\dagger + \dots + \hat{a}_{S_n l}^\dagger \right) \right) \\ & \times \left(\prod_{m=1}^n \left((\hat{a}_{m 0 0}^\dagger + \hat{a}_{S_m k+1}^\dagger)^2 - (\hat{a}_{m 1 0}^\dagger - \hat{a}_{m 1 1}^\dagger)^2 \right) \right). \end{aligned} \quad (15)$$

Step 5. Application of the n -partite port and BS before detection. Then (15) evolves into

$$\frac{1}{2^{2n} n^{k/2}} \left(\prod_{j=1}^n \left(\alpha \sum_{p=1}^n (U_n)_{pj} \hat{a}_{T_1 p}^\dagger + \cdots + \alpha \sum_{p=1}^n (U_n)_{pj} \hat{a}_{T_k p}^\dagger + \frac{\beta}{\sqrt{2}} (\hat{a}_{j01}^\dagger + \hat{a}_{j10}^\dagger) \right) \right) \quad (16)$$

$$\times \left(\prod_{l=1}^k \left(\sum_{q=1}^{k+1} (U_{k+1})_{ql} \hat{a}_{S_1 q}^\dagger + \cdots + \sum_{q=1}^{k+1} (U_{k+1})_{ql} \hat{a}_{S_n q}^\dagger \right) \right) \quad (17)$$

$$\times \left(\prod_{m=1}^n \left(\left(\hat{a}_{m00}^\dagger + \sum_{q=1}^{k+1} (U_{k+1})_{q k+1} \hat{a}_{S_m q}^\dagger \right)^2 - \left(\frac{1}{\sqrt{2}} (\hat{a}_{m01}^\dagger - \hat{a}_{m10}^\dagger) - \hat{a}_{m11}^\dagger \right)^2 \right) \right). \quad (18)$$

Here, U_d denotes the discrete Fourier transform matrix corresponding to a symmetric d -partite multiport.

After the postselection, all creation operators associated with the detected (upper-most) modes are removed, and only the non-detected modes remain. Collecting all $\binom{n}{k}$ possible combinations where k of the n modes correspond to the latter case gives the normalized post-selected state:

$$\binom{n}{k}^{-1/2} \sum_{\substack{M \subset \{1, \dots, n\} \\ |M|=k}} \left(\prod_{m \in M} \hat{a}_{m11}^\dagger \right) \left(\prod_{m \notin M} \hat{a}_{m00}^\dagger \right) |\text{vac}\rangle \quad (19)$$

with amplitude

$$\binom{n}{k}^{1/2} \frac{k!^2}{2^{\frac{3n}{2}} n^k (k+1)^{\frac{n}{2}}} \beta^{n-k} \alpha^k. \quad (20)$$

The factor $\beta^{n-k} \alpha^k$ attains its maximum value when

$$\beta = \sqrt{\frac{n-k}{n}}, \quad \alpha = \sqrt{\frac{1}{n}}. \quad (21)$$

Including the feed-forward factor $2^n n(k+1)$, the total success probability becomes

$$P_{\text{suc}} = \binom{n}{k} \frac{(k!)^4 (n-k)^{n-k}}{2^{2n} n^{n+2k-1} (k+1)^{n-1}}. \quad (22)$$

The success probabilities for $k = 2, 3, 4$ and n up to 10 are illustrated in Fig. 6 on a logarithmic scale. For $(n, k) = (4, 2)$, the success probability is about 3.4×10^{-6} which is compatible with current photonic technologies. The generation rates can be enhanced using temporal or spatial multiplexing of photon sources, parallelized detection, and fast feed-forward switching [55, 56]. These techniques make the scheme experimentally feasible within photonic quantum technologies.

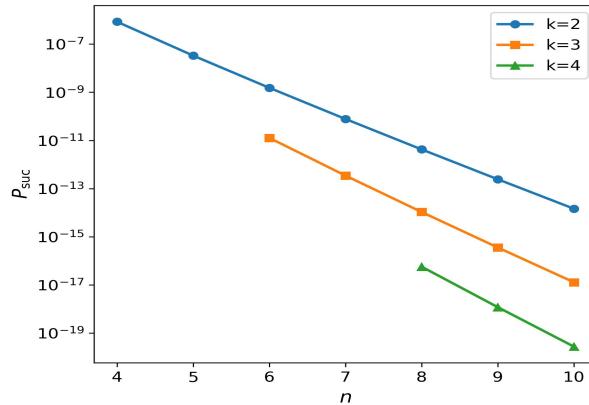


Figure 6: The success probability P_{suc} as a function of the n for $k = 2, 3, 4$. The probabilities are plotted on a logarithmic scale.

5 Conclusions

We have introduced a linear-optical heralded scheme for generating Dicke states within the linear quantum graph picture, overcoming the limitations of postselected approaches. By encoding the inherent permutation symmetry of Dicke states into graph structures, we substantially simplify the design problem and obtain systematic heralded schemes. Our result is compatible with current photonic technologies, with significantly higher success probabilities above detector dark-count levels. To our knowledge, no prior heralded linear optical scheme for generating Dicke states with feedforward has been reported.

The intrinsic robustness of Dicke states to particle loss indicates the possibility of developing loss-tolerant or application-optimized variants tailored to quantum networking, distributed sensing, and photonic variational algorithms. These directions position our scheme as a stepping stone toward practical multi-photon resource generation in future quantum technologies. Our work also highlights the usefulness of the LQG framework for discovering heralded entanglement generation schemes. Future research extends this approach to other highly symmetric and practically relevant resource states, including families of stabilizer states and absolutely maximally entangled states.

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