

Asymmetry in Spectral Graph Theory: Harmonic Analysis on Directed Networks via Biorthogonal Bases

(Random-Walk Laplacian Formulation)

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Abstract

The operator-theoretic dichotomy underlying diffusion on directed networks is *symmetry versus non-self-adjointness* of the Markov transition operator. In the reversible (detailed-balance) regime, a directed random walk P is self-adjoint in a stationary π -weighted inner product and admits orthogonal spectral coordinates; outside reversibility, P is genuinely non-self-adjoint (often non-normal), and stability is governed by biorthogonal geometry and eigenvector conditioning. In this paper we develop an original harmonic-analysis framework for directed graphs anchored on the random-walk transition matrix $P = D_{\text{out}}^{-1}A$ and the random-walk Laplacian $L_{\text{rw}} = I - P$. Using biorthogonal left/right eigenvectors we define a *Biorthogonal Graph Fourier Transform* (BGFT) adapted to directed diffusion, propose a diffusion-consistent frequency ordering based on decay rates $\Re(1 - \lambda)$, and derive operator-norm stability bounds for iterated diffusion and for BGFT spectral filters. We prove sampling and reconstruction theorems for P -bandlimited (equivalently L_{rw} -bandlimited) signals and quantify noise amplification through the conditioning of the biorthogonal eigenbasis. A simulation protocol on directed cycles and perturbed non-normal digraphs demonstrates that asymmetry alone does not dictate instability, whereas non-normality and eigenvector ill-conditioning drive reconstruction sensitivity, making BGFT the correct analytical language for directed diffusion processes.

Key words: directed graphs; random walks; non-normal matrices; biorthogonal eigenvectors; graph Fourier transform; sampling; reversibility.

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1 Introduction

1.1 Symmetry vs. non-self-adjointness: Markov operators on directed networks

A directed network naturally carries a *one-step evolution operator*: the random-walk (Markov) transition matrix

$$P = D_{\text{out}}^{-1}A, \quad P\mathbf{1} = \mathbf{1},$$

and its generator $L_{\text{rw}} = I - P$. From the operator-theoretic viewpoint, the central dichotomy is not “directed vs. undirected” per se, but *symmetry vs. non-self-adjointness* of the Markov operator.

The *symmetry regime* is the reversible (detailed-balance) case: there exists a stationary distribution π with $\Pi = \text{diag}(\pi)$ such that

$$\Pi P = P^\top \Pi,$$

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equivalently P is self-adjoint in the weighted Hilbert space $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_\pi)$. In that regime, P is similar to a *symmetric* matrix $S = \Pi^{1/2} P \Pi^{-1/2}$, hence the spectrum is real and there is an orthonormal eigenbasis in the π -metric. This is precisely the mechanism by which a directed diffusion can *retain symmetry* (in a stationary metric), recovering Parseval-type identities and a clean variational frequency ordering. The *asymmetry regime* is non-reversibility, where P is genuinely non-self-adjoint and may be non-normal; then orthogonality is lost, spectral coordinates can be ill-conditioned, and stability is governed by eigenvector conditioning and non-normal effects [7, 6].

Our goal is to build a harmonic-analysis calculus that is *native to the Markov operator* P : it should reduce to the classical orthogonal theory in the reversible (symmetric) regime, and it should remain exact and analyzable in the non-reversible (non-self-adjoint) regime. The correct language here is *biorthogonality*: left/right eigenvectors provide an exact analysis/synthesis pair even when P is not normal.

1.2 Position relative to graph Fourier analysis

Graph Fourier analysis is often introduced through symmetric operators (undirected Laplacians/adjacencies), which guarantee orthogonal eigenvectors and stable spectral coordinates [2, 3, 16]. Directed graph settings typically replace symmetry by alternative constructions: optimization-based directed transforms and directed Laplacians associated with random walks and Cheeger-type theory [4, 1], as well as survey-level treatments of directed-graph signal processing [19]. Our approach is complementary and more “operator first”: we start from the canonical diffusion operator P and develop an *exact* biorthogonal Fourier calculus for directed diffusion, with a transparent symmetry/asymmetry interpretation in terms of reversibility [9, 11].

1.3 Main contributions

Main contributions (original).

1. **(Markov-operator BGFT)** We define the *Biorthogonal Graph Fourier Transform* (BGFT) for the random-walk operator P (equivalently $L_{rw} = I - P$) via left/right eigenvectors, yielding exact analysis/synthesis identities and diagonal dynamics for diffusion iterates.
2. **(Symmetry principle via reversibility)** We identify reversibility (detailed balance) as the precise *symmetry* notion for directed diffusion: in the π -metric, reversible P becomes self-adjoint, restoring orthogonality/Parseval identities and an exact diffusion-variational frequency ordering.
3. **(Diffusion-consistent frequency)** We propose a diffusion-consistent frequency ordering based on the decay rate $\Re(1 - \lambda)$ (and magnitude alternatives), aligning with the symmetry limit and the long-time behavior of $x_{t+1} = Px_t$.
4. **(Stability theorems for non-self-adjoint diffusion)** We prove operator-norm bounds for diffusion iterates P^t and for BGFT spectral filters $h(P)$, explicitly separating eigenvalue decay from eigenvector conditioning, the key instability driver in non-normal settings.
5. **(Sampling and reconstruction)** We prove sampling/reconstruction theorems for P -bandlimited signals and quantify noise amplification through $\sigma_{\min}(P_M V_\Omega)$ and conditioning of the biorthogonal eigenbasis.
6. **(Asymmetry vs. non-normality: numerical separation)** We introduce simple indices for directedness and departure from normality and provide experiments (directed cycle vs. perturbed non-normal digraphs) showing that *asymmetry alone* need not cause instability, whereas non-normality and eigenvector ill-conditioning do.

1.4 Organization

Section 2 introduces directed diffusion operators and asymmetry/non-normality indices. Section 3 presents reversibility as the symmetry regime in the stationary metric. Sections 4–5 develop BGFT and stability bounds for diffusion and filtering. Section 6 proves sampling and reconstruction results, followed by algorithms and illustrative experiments.

2 Preliminaries: directed diffusion operators

2.1 Directed graphs, adjacency, and out-degree

Let $G = (V, E, w)$ be a directed weighted graph with $|V| = n$ and adjacency $A \in \mathbb{R}^{n \times n}$:

$$A_{ij} = \begin{cases} w(i, j), & (i, j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Define out-degrees $d_i^{\text{out}} = \sum_j A_{ij}$ and $D_{\text{out}} = \text{diag}(d_1^{\text{out}}, \dots, d_n^{\text{out}})$.

2.2 Transition matrix and random-walk Laplacian

Definition 2.1 (Random-walk transition matrix). Assume $d_i^{\text{out}} > 0$ for all i (no sinks). Define

$$P := D_{\text{out}}^{-1} A.$$

Then P is row-stochastic: $P\mathbf{1} = \mathbf{1}$.

Definition 2.2 (Random-walk Laplacian). Define

$$L_{\text{rw}} := I - P.$$

Proposition 2.3 (Basic properties). (i) $P\mathbf{1} = \mathbf{1}$ and $L_{\text{rw}}\mathbf{1} = 0$. (ii) If P is irreducible and aperiodic, then the diffusion $x_{t+1} = Px_t$ converges to the stationary component (Markov mixing perspective).

Proof. (i) Row-stochasticity gives $P\mathbf{1} = \mathbf{1}$, hence $(I - P)\mathbf{1} = 0$.

(ii) This is standard Markov chain theory; see [6, 9, 11].. \square

2.3 Asymmetry and non-normality indices

Definition 2.4 (Asymmetry index). For any matrix M , define $\alpha(M) := \|M - M^\top\|_F / \|M\|_F$ (with $\alpha(0) = 0$).

Definition 2.5 (Departure from normality). For any matrix M , define $\delta(M) := \|MM^* - M^*M\|_F / \|M\|_F^2$ (with $\delta(0) = 0$).

Such non-normality measures (and related bounds) are classical in matrix analysis; see [12, 14, 13, 7].

We will use these for $M = P$ and $M = L_{\text{rw}}$ to separate structural directedness from numerical instability drivers.

3 Reversibility as the symmetry regime for directed diffusion

Let $P \in \mathbb{R}^{n \times n}$ be row-stochastic ($P\mathbf{1} = \mathbf{1}$). Assume P has a stationary distribution $\pi \in \mathbb{R}^n$ with $\pi_i > 0$ and $\pi^\top P = \pi^\top$. Let $\Pi := \text{diag}(\pi)$.

Define the π -weighted inner product and norm by

$$\langle x, y \rangle_\pi := x^\top \Pi y, \quad \|x\|_\pi^2 := \langle x, x \rangle_\pi.$$

The adjoint of P with respect to $\langle \cdot, \cdot \rangle_\pi$ is

$$P^\dagger := \Pi^{-1} P^\top \Pi, \quad \text{so that} \quad \langle Px, y \rangle_\pi = \langle x, P^\dagger y \rangle_\pi.$$

Definition 3.1 (Reversibility / detailed balance). P is *reversible* (w.r.t. π) if

$$\Pi P = P^\top \Pi, \quad \text{equivalently} \quad P = P^\dagger.$$

This detailed-balance condition is standard in reversible Markov chain theory; see [10, 9, 6].

Theorem 3.2 (Weighted symmetry equivalences). *The following are equivalent:*

- (i) P is reversible: $\Pi P = P^\top \Pi$.
- (ii) P is self-adjoint in $\langle \cdot, \cdot \rangle_\pi$: $P = P^\dagger$.
- (iii) The similarity transform $S := \Pi^{1/2} P \Pi^{-1/2}$ is symmetric: $S = S^\top$.

In this case, P has a complete π -orthonormal eigenbasis, and all eigenvalues are real.

Proof. (i) \Leftrightarrow (ii) is the definition of P^\dagger . For (i) \Rightarrow (iii), multiply $\Pi P = P^\top \Pi$ on the left by $\Pi^{-1/2}$ and on the right by $\Pi^{-1/2}$ to get $\Pi^{1/2} P \Pi^{-1/2} = (\Pi^{1/2} P \Pi^{-1/2})^\top$. Conversely, (iii) \Rightarrow (i) follows by reversing the steps. If S is symmetric, it is orthogonally diagonalizable with real eigenvalues, hence so is P by similarity. \square

See also [9, 11] for related equivalences and consequences.

Remark 3.3 (Symmetry/asymmetry interpretation for this paper). Undirected diffusion is symmetric in the standard Euclidean inner product. Directed diffusion can still be symmetric in the *weighted* π -inner product exactly in the reversible regime. Non-reversibility is the correct notion of *asymmetry* for random-walk harmonic analysis.

4 Biorthogonal Graph Fourier Transform (BGFT) for random-walk diffusion

4.1 Left/right eigenvectors and BGFT

Assumption 4.1 (Diagonalizability). *Assume P is diagonalizable over \mathbb{C} :*

$$P = V \Lambda V^{-1}, \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),$$

with right eigenvectors $V = [v_1 \ \cdots \ v_n]$.

Define $U^* := V^{-1}$ so $U^* V = I$ and $u_k^* v_\ell = \delta_{k\ell}$.

Definition 4.2 (BGFT (diffusion version)). For a graph signal $x \in \mathbb{C}^n$, define BGFT coefficients

$$\widehat{x} := U^* x, \quad \widehat{x}_k = u_k^* x, \tag{1}$$

and synthesis

$$x = V \widehat{x} = \sum_{k=1}^n v_k \widehat{x}_k. \tag{2}$$

Theorem 4.3 (Perfect reconstruction). *Under Assumption 4.1, for all $x \in \mathbb{C}^n$,*

$$I = \sum_{k=1}^n v_k u_k^*, \quad x = \sum_{k=1}^n v_k u_k^* x.$$

Proof. Since $U^*V = I$, we have $VU^* = I$; expand VU^* in columns/rows. \square

4.2 Diffusion dynamics are diagonal in BGFT coordinates

Theorem 4.4 (BGFT-domain diffusion). *Let $x_{t+1} = Px_t$ with $x_0 \in \mathbb{C}^n$. Then*

$$\widehat{x}_t = U^*x_t = \Lambda^t \widehat{x}_0, \quad x_t = V\Lambda^t U^*x_0.$$

Equivalently, for $L_{\text{rw}} = I - P$,

$$\widehat{(L_{\text{rw}}x)} = (I - \Lambda)\widehat{x}.$$

Proof. Use $P = V\Lambda U^*$ and $U^*V = I$. Then $U^*(Px) = \Lambda(U^*x)$ and iterate. \square

4.3 Diffusion-consistent frequency ordering

For diffusion, the mode with eigenvalue λ evolves as λ^t . If $|\lambda| < 1$, it decays; if $\lambda \approx 1$, it is slowly varying (low frequency). We define the *diffusion decay rate*:

$$\omega_{\text{diff}}(\lambda) := \Re(1 - \lambda).$$

Low ω_{diff} corresponds to persistent/slow modes; high ω_{diff} corresponds to fast decay. In the symmetric undirected limit (where $\lambda \in [-1, 1]$ for normalized settings), this ordering aligns with classical low/high frequency intuition.

5 Directed diffusion filtering and stability bounds

5.1 Spectral filters

Definition 5.1 (BGFT spectral filter for diffusion). Let $h : \mathbb{C} \rightarrow \mathbb{C}$. Define

$$H := V h(\Lambda) U^*.$$

Proposition 5.2 (Diagonal action in BGFT domain). *For $\widehat{x} = U^*x$,*

$$\widehat{Hx} = U^*Hx = h(\Lambda)\widehat{x}.$$

Proof. Compute $U^*Vh(\Lambda)U^*x = h(\Lambda)\widehat{x}$. \square

5.2 Operator-norm stability: diffusion and filtering

Theorem 5.3 (Norm bound for diffusion iterates). *Assume $P = V\Lambda V^{-1}$. Then for every $t \in \mathbb{N}$,*

$$\|P^t\|_2 \leq \text{cond}(V) \max_k |\lambda_k|^t, \quad \text{cond}(V) = \|V\|_2 \|V^{-1}\|_2.$$

Proof. $P^t = V\Lambda^t V^{-1}$, hence $\|P^t\|_2 \leq \|V\|_2 \|\Lambda^t\|_2 \|V^{-1}\|_2 = \text{cond}(V) \max_k |\lambda_k|^t$. \square

Theorem 5.4 (Norm bound for spectral filters). *Let $H = Vh(\Lambda)V^{-1}$. Then*

$$\|H\|_2 \leq \text{cond}(V) \max_k |h(\lambda_k)|.$$

Proof. $\|H\|_2 \leq \|V\|_2 \|h(\Lambda)\|_2 \|V^{-1}\|_2 = \text{cond}(V) \max_k |h(\lambda_k)|$. \square

Remark 5.5 (Symmetry/asymmetry interpretation). When P is normal and diagonalizable by a unitary basis, $\text{cond}(V) = 1$ and the bounds become tight and symmetry-like. For non-normal P , $\text{cond}(V)$ can be large, creating instability even if $|\lambda_k| \leq 1$. Sharper growth control for non-normal matrices can be expressed via pseudospectral/Kreiss-type constants; see [15, 7].

6 BGFT energy in the stationary metric and its symmetry limit

Assume P is diagonalizable over \mathbb{C} : $P = V\Lambda V^{-1}$ and define $U^* := V^{-1}$. Let $\hat{x} := U^*x$ be BGFT coefficients so that $x = V\hat{x}$.

Theorem 6.1 (π -metric Parseval identity). *For any $x \in \mathbb{C}^n$,*

$$\|x\|_\pi^2 = \hat{x}^* G_\pi \hat{x}, \quad G_\pi := V^* \Pi V.$$

Proof. Since $x = V\hat{x}$, $\|x\|_\pi^2 = x^* \Pi x = \hat{x}^* (V^* \Pi V) \hat{x}$. \square

Corollary 6.2 (Two-sided bounds via conditioning in π). *Let $W := \Pi^{1/2}V$. Then*

$$\sigma_{\min}(W)^2 \|\hat{x}\|_2^2 \leq \|x\|_\pi^2 \leq \sigma_{\max}(W)^2 \|\hat{x}\|_2^2.$$

Equivalently, energy distortion is controlled by $\kappa(W) = \sigma_{\max}(W)/\sigma_{\min}(W)$.

6.1 Diffusion variation and frequency ordering

Define the random-walk Laplacian $L_{\text{rw}} := I - P$ and the diffusion variation

$$\text{TV}_\pi(x) := \|L_{\text{rw}}x\|_\pi^2 = \|(I - P)x\|_\pi^2.$$

Theorem 6.3 (BGFT-domain bounds for diffusion variation). *With $x = V\hat{x}$ and $W = \Pi^{1/2}V$,*

$$\sigma_{\min}(W)^2 \sum_{k=1}^n |1 - \lambda_k|^2 |\hat{x}_k|^2 \leq \text{TV}_\pi(x) \leq \sigma_{\max}(W)^2 \sum_{k=1}^n |1 - \lambda_k|^2 |\hat{x}_k|^2.$$

Proof. $(I - P)x = V(I - \Lambda)\hat{x}$. Then $\|(I - P)x\|_\pi = \|\Pi^{1/2}V(I - \Lambda)\hat{x}\|_2 = \|W(I - \Lambda)\hat{x}\|_2$. Apply $\sigma_{\min}(W)\|z\|_2 \leq \|Wz\|_2 \leq \sigma_{\max}(W)\|z\|_2$ to $z = (I - \Lambda)\hat{x}$ and square. \square

Remark 6.4 (Exact symmetry limit). If P is reversible, one can choose V π -orthonormal, hence $W = \Pi^{1/2}V$ is unitary and $\sigma_{\min}(W) = \sigma_{\max}(W) = 1$. Then the inequalities become equalities and $|1 - \lambda_k|$ becomes an exact diffusion frequency.

7 Sampling and reconstruction for diffusion-bandlimited signals

Let $\Omega \subset \{1, \dots, n\}$ with $|\Omega| = K$ represent the “low diffusion-frequency” modes (e.g. smallest $\omega_{\text{diff}}(\lambda)$ or largest $\Re(\lambda)$). Let $V_\Omega \in \mathbb{C}^{n \times K}$ contain $\{v_k\}_{k \in \Omega}$.

Definition 7.1 (Diffusion-bandlimited signals). A signal x is Ω -bandlimited (relative to P) if $x = V_\Omega c$ for some $c \in \mathbb{C}^K$.

Bandlimited sampling on graphs has a substantial literature; see, e.g., [18, 17].

Let $M \subset V$, $|M| = m$, and $P_M \in \{0, 1\}^{m \times n}$ be the restriction operator.

Theorem 7.2 (Exact recovery). *If $x = V_\Omega c$ and $P_M V_\Omega$ has full column rank K , then x is uniquely determined by samples $y = P_M x$ and recovered by*

$$\hat{c} = (P_M V_\Omega)^\dagger y, \quad \hat{x} = V_\Omega \hat{c}.$$

Proof. Full column rank makes $P_M V_\Omega$ injective; solve the linear system in least squares. \square

Related sampling-set conditions and reconstruction stability on graphs are discussed in [18, 17].

Theorem 7.3 (Noise sensitivity). *If $y = P_M x + \eta$, then the least-squares reconstruction satisfies*

$$\|\hat{x} - x\|_2 \leq \|V_\Omega\|_2 \left\| (P_M V_\Omega)^\dagger \right\|_2 \|\eta\|_2 = \|V_\Omega\|_2 \frac{\|\eta\|_2}{\sigma_{\min}(P_M V_\Omega)}.$$

Proof. $\hat{c} - c = (P_M V_\Omega)^\dagger \eta$ and $\hat{x} - x = V_\Omega(\hat{c} - c)$. \square

8 Algorithms

Algorithm 1 BGFT for random-walk diffusion

Require: A (directed adjacency), D_{out} invertible, signal x

Ensure: BGFT coefficients \hat{x} , eigenpairs (Λ, V)

- 1: $P \leftarrow D_{\text{out}}^{-1}A$, $L_{\text{rw}} \leftarrow I - P$
 - 2: Compute eigendecomposition $P = V\Lambda V^{-1}$ (complex arithmetic)
 - 3: $U^* \leftarrow V^{-1}$
 - 4: $\hat{x} \leftarrow U^*x$
 - 5: **return** (\hat{x}, Λ, V)
-

Algorithm 2 Diffusion filtering and bandlimited reconstruction

Require: $P = V\Lambda V^{-1}$, response $h(\cdot)$, bandlimit Ω , sample set M , samples y

Ensure: filtered signal Hx or reconstructed \hat{x}

- 1: **Filtering:** $H \leftarrow Vh(\Lambda)V^{-1}$, output $Hx \leftarrow Hx$
 - 2: **Reconstruction:** form V_Ω , solve $\hat{c} \leftarrow \arg \min_c \|P_M V_\Omega c - y\|_2^2$
 - 3: $\hat{x} \leftarrow V_\Omega \hat{c}$
-

9 Experiments: directed cycle vs perturbed non-normal digraphs

9.1 Graphs

Use $n \in \{32, 64, 128\}$ and compare:

1. **Undirected cycle** C_n (convert to diffusion by symmetrizing and normalizing).
2. **Directed cycle** \overrightarrow{C}_n : P is a permutation shift (asymmetric but normal/unitary).
3. **Perturbed directed cycle** $\overrightarrow{C}_n^{(\varepsilon)}$: add a directed chord then renormalize rows to keep P stochastic; this typically yields non-normal P and large $\text{cond}(V)$.

9.2 Tasks and metrics

- **Diffusion filtering:** low-pass via $h(\lambda) = \exp(-\tau(1 - \lambda))$ (BGFT-defined), compare smoothing strength on the three graphs.
- **Forecasting:** iterate diffusion $x_{t+1} = Px_t$ and compare $\|x_t\|_2$ trends with Theorem 5.3.
- **Sampling/reconstruction:** generate Ω -bandlimited signals and recover from m samples; report RelErr and $\sigma_{\min}(P_M V_\Omega)$.

Report tables/figures:

$$\alpha(P), \delta(P), \text{cond}(V), \text{cond}(P_M V_\Omega), \text{RelErr} = \frac{\|\hat{x} - x\|_2}{\|x\|_2}.$$

Observed separation. The directed cycle is asymmetric ($\alpha(P) > 0$) but normal ($\delta(P) = 0$) with a well-conditioned eigenbasis ($\kappa(V) = 1$), whereas the perturbed digraph remains asymmetric but becomes non-normal ($\delta(P) > 0$) and strongly ill-conditioned ($\kappa(V) \gg 1$), leading to markedly larger reconstruction error, consistent with the stability bounds.

Table 1: Minimal numerical illustration for the transition-operator BGFT.

Graph	$\alpha(P)$	$\delta(P)$	$\kappa(V)$	$\kappa(P_M V_\Omega)$	RelErr
Undirected cycle A_{und}	0	0	1.2453204511204569	36.59492056037454	0.0000031215500108761767
Directed cycle A_{\rightarrow}	1.4142135623730951	0	1	93.04681515171424	0.000007080903224516353
Perturbed A_ε ($\varepsilon = 20$)	1.414213562373095	0.02987165083714049	28.011585066632986	352.8935063092261	0.00002523914083929862

10 Conclusion

We developed an original diffusion-centered harmonic analysis for directed graphs using the random-walk transition matrix P and Laplacian $L_{\text{rw}} = I - P$. The BGFT provides exact analysis/synthesis, diagonalizes diffusion dynamics, motivates a diffusion-consistent frequency ordering, and yields explicit stability bounds for iterated diffusion and spectral filtering governed by eigenvector conditioning. Sampling and reconstruction theorems quantify how non-normality amplifies noise through $\sigma_{\min}(P_M V_\Omega)$ and $\text{cond}(V)$. This establishes a principled symmetry/asymmetry narrative: symmetry yields orthogonality and stability; asymmetry forces biorthogonal geometry; non-normality determines practical robustness.

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Author Contributions

The author is solely responsible for conceptualization, methodology, analysis, software, validation, and writing.

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Data Availability Statement

No external datasets were used. The code that generates the reported numerical table/figures will be provided as a reproducible script and (upon acceptance) via a public repository link.

Conflicts of Interest

The author declares no conflicts of interest.

A Reference Python code (P and Lrw, BGFT, filtering, reconstruction)

```
import numpy as np

def directed_cycle_A(n):
    A = np.zeros((n,n), dtype=float)
```

```

for i in range(n):
    A[i, (i+1)%n] = 1.0
return A

def undirected_cycle_A(n):
    A = np.zeros((n,n), dtype=float)
    for i in range(n):
        A[i, (i+1)%n] = 1.0
        A[(i+1)%n, i] = 1.0
    return A

def add_directed_chord(A, eps=0.2, i=0, j=None):
    n = A.shape[0]
    if j is None:
        j = n//2
    B = A.copy()
    B[i, j] += eps
    return B

def D_out(A):
    d = A.sum(axis=1)
    if np.any(d == 0):
        raise ValueError("Found sink node (out-degree 0). Fix by adding small outgoing weight.")
    return np.diag(d)

def transition_P(A):
    D = D_out(A)
    return np.linalg.inv(D) @ A

def L_rw(P):
    n = P.shape[0]
    return np.eye(n) - P

def asymmetry_index(M):
    den = np.linalg.norm(M, ord='fro')
    if den == 0:
        return 0.0
    return np.linalg.norm(M - M.T, ord='fro') / den

def departure_from_normality(M):
    Mf = np.linalg.norm(M, ord='fro')
    if Mf == 0:
        return 0.0
    MMstar = M @ M.conj().T
    MstarM = M.conj().T @ M
    return np.linalg.norm(MMstar - MstarM, ord='fro') / (Mf**2)

def bgft_decomposition(P):
    lam, V = np.linalg.eig(P)
    Vinv = np.linalg.inv(V)
    Ustar = Vinv
    return lam, V, Ustar

def diffusion_filter_matrix(P, tau=2.0):
    #  $h(\lambda) = \exp(-\tau(1-\lambda))$ 
    lam, V, Ustar = bgft_decomposition(P)
    h = np.exp(-tau*(1.0 - lam))

```

```

H = V @ np.diag(h) @ Ustar
return H

def sample_operator(n, M):
    m = len(M)
    Pm = np.zeros((m,n), dtype=float)
    for r, idx in enumerate(M):
        Pm[r, idx] = 1.0
    return Pm

def reconstruct_bandlimited(P, Omega, M, x, noise=0.0, seed=0):
    np.random.seed(seed)
    lam, V, _ = bgft_decomposition(P)
    V_0 = V[:, Omega]
    Pm = sample_operator(P.shape[0], M)
    y = Pm @ x
    if noise > 0:
        y = y + noise * np.random.randn(*y.shape)
    B = Pm @ V_0
    c_hat, *_ = np.linalg.lstsq(B, y, rcond=None)
    x_hat = V_0 @ c_hat
    relerr = np.linalg.norm(x_hat - x) / np.linalg.norm(x)
    condB = np.linalg.cond(B)
    return x_hat, relerr, condB

if __name__ == "__main__":
    n = 64

    A_und = undirected_cycle_A(n)
    A_dir = directed_cycle_A(n)
    A_per = add_directed_chord(A_dir, eps=20, i=0, j=n//2)

    for name, A in [("undirected", A_und), ("directed", A_dir), ("perturbed", A_per)]:
        P = transition_P(A)
        lam, V, _ = bgft_decomposition(P)
        print(name,
              "alpha(P)=", asymmetry_index(P),
              "delta(P)=", departure_from_normality(P),
              "cond(V)=", np.linalg.cond(V),
              "rho(P)~=", np.max(np.abs(lam)))

    # Create a bandlimited signal on perturbed digraph
    P = transition_P(A_per)
    lam, V, _ = bgft_decomposition(P)
    K = 8

    # Choose "low diffusion-frequency": largest Re(lambda) (closest to 1)
    Omega = np.argsort(-np.real(lam))[:K]

    c = np.random.randn(K) + 1j*np.random.randn(K)
    x = V[:, Omega] @ c

    m = 20
    M = np.sort(np.random.choice(n, size=m, replace=False))
    x_hat, relerr, condB = reconstruct_bandlimited(P, Omega, M, x, noise=0.0)

    print("RelErr=", relerr, "cond(P_M\cup V_Omega)=", condB)

```

```

# Diffusion smoothing example
H = diffusion_filter_matrix(P, tau=2.0)
x_smooth = H @ x
print("||x||_2 = ", np.linalg.norm(x), "||Hx||_2 = ", np.linalg.norm(x_smooth))

```

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