

# ERDŐS–WINTNER THEOREM FOR LINEAR RECURRENT BASES

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**ABSTRACT.** Let  $(G_n)_{n \geq 0}$  be a linear recurrence sequence defining a numeration system and satisfying mild structural hypotheses. For  $G$ -additive functions—that is, functions additive in the greedy  $G$ -digits—we establish an Erdős–Wintner-type theorem: convergence of two canonical series, a first-moment series and a quadratic digit-energy series, is necessary and sufficient for the existence of a limiting distribution along initial segments of the integers. In particular, the limiting characteristic function admits an infinite-product factorization whose local factors depend only on the underlying digit system. We also indicate conditional extensions of this two-series criterion to Ostrowski numeration systems with bounded partial quotients and to Parry  $\beta$ -expansions with Pisot–Vijayaraghavan base  $\beta$ .

## CONTENTS

1. Introduction	1
2. Linear recurrent sequences and distribution function	3
3. Erdős–Wintner theorem for linear recurrent bases	6
4. Examples	14
5. LRB of order 2	16
6. Stability under addition and small perturbations	17
7. Outlook: Ostrowski expansions and $\beta$ –PV systems	18
Appendix. Proofs of auxiliary lemmas	20
References	33

## 1. INTRODUCTION

In the classical setting, Erdős–Wintner [11] established an if-and-only-if criterion for additive arithmetical functions  $f$  (i.e.  $f(mn) = f(m) + f(n)$  for  $(m, n) = 1$ ) which admit a distribution function. A real-valued function  $f$  on  $\mathbb{N}$  is said to have a *distribution function*  $F$  [21] if there exists a non-decreasing, right-continuous function  $F : \mathbb{R} \rightarrow [0, 1]$  with  $F(-\infty) = 0$  and  $F(+\infty) = 1$  such that, for every continuity point  $x$  of  $F$ ,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \#\{0 \leq n < N : f(n) \leq x\} = F(x).$$

Equivalently, the associated empirical measures converge weakly to a probability measure on  $\mathbb{R}$ .

**Classical Erdős–Wintner (1939).** Let  $f : \mathbb{N} \rightarrow \mathbb{R}$  be additive. Then  $f$  admits a (limiting) distribution function if and only if the three series

$$\sum_{|f(p)| > 1} \frac{1}{p}, \quad \sum_{|f(p)| \leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| \leq 1} \frac{f(p)^2}{p}$$

converge. When these conditions hold, the limiting characteristic function admits an Euler product representation.

An effective version in the classical setting was proved by Tenenbaum and the present author [22]. Delange obtained an analogue in an integer base  $q \geq 2$  for  $q$ -additive functions [6], where  $f$  is determined by its digit-level values  $f(jq^k)$  for  $0 \leq j < q$  and  $k \geq 0$ , and is extended to  $\mathbb{N}$  by  $q$ -additivity. Distributional concentration phenomena for additive functions were studied by Erdős–Kátai [10].

For both the Delange setting and the Zeckendorf system (based on the Fibonacci sequence  $(F_n)$  defined by  $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ ), a complete Erdős–Wintner theorem with *effective*

bounds was proved by Drmota and the present author [9]. Every integer  $N \geq 0$  has a unique Zeckendorf expansion

$$N = \sum_{k \geq 2} e_k(N) F_k$$

with  $e_k(N) \in \{0, 1\}$  and no two consecutive 1's. A prototypical Zeckendorf-additive function is

$$f(N) = \sum_{k \geq 2} f(e_k(N) F_k),$$

so that  $f$  admits a distribution function if and only if two canonical series converge; in that case, the limiting characteristic function factors as an explicit infinite product, and one obtains an *effective* quantitative rate.

Our goal is to generalize these distributional results to a broad class of numeration systems defined by linear recurrent bases. In this setting we prove an Erdős–Wintner type theorem for  $G$ -additive functions: we obtain a genuine if-and-only-if criterion for the existence of a limiting distribution, expressed in terms of the convergence of two explicit canonical series, and we derive an explicit infinite-product representation for the limiting characteristic function. Effective rates of convergence in the general linear recurrence setting are left open in the present article, and corresponding refinements for specific subclasses of bases are deferred to future work.

Fix an integer  $d \geq 2$  and coefficients  $a_0, \dots, a_{d-1} \in \mathbb{N}$  with  $a_0 \geq 1$ , and consider the sequence  $(G_n)_{n \geq 0}$  defined by

$$G_{n+d} = a_0 G_{n+d-1} + \dots + a_{d-2} G_{n+1} + a_{d-1} G_n \quad (n \geq 0),$$

with initial conditions

$$G_0 = 1 \quad \text{and} \quad G_k = a_0 G_{k-1} + \dots + a_{k-1} G_0 + 1 \quad (0 < k < d).$$

We associate to this recurrence its *companion matrix*

$$A := \begin{pmatrix} a_0 & a_1 & \cdots & a_{d-2} & a_{d-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & 0 & \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

whose characteristic polynomial is  $X^d - a_0 X^{d-1} - \dots - a_{d-2} X - a_{d-1}$ . We set  $\alpha := \max_{0 \leq j < d} a_j$  and we say that  $(G_n)_{n \geq 0}$  is a *linear recurrence base* (LRB) if, in addition, the following properties hold:

- (i) every nonnegative integer  $N$  admits a unique greedy  $G$ -expansion, that is, there exist digits  $e_k(N) \in \{0, \dots, \alpha\}$ , all but finitely many equal to 0, such that  $N = \sum_{k \geq 0} e_k(N) G_k$ , and the expansion is obtained by the usual greedy algorithm;
- (ii) the companion matrix  $A$  is *primitive*, meaning that some power  $A^n$  has all entries strictly positive (in particular, it has a simple Perron–Frobenius eigenvalue  $\alpha > 1$ );
- (iii)  $\alpha$  is a Pisot–Vijayaraghavan (PV) number and  $G_n / \alpha^n \rightarrow \kappa$  for some  $\kappa > 0$ .

We call  $f$   *$G$ -additive* if it is determined by its digit-level values  $f(jG_k)$  for all *admissible* digits  $j$  (in particular  $f(0) = 0$ ), and extended by  $G$ -additivity:

$$f\left(\sum_{k \geq 0} e_k(N) G_k\right) = \sum_{k \geq 0} f(e_k(N) G_k),$$

where  $e_k(N)$  denotes the  $k$ -th digit in the  $G$ -expansion of  $N$ . The precise admissibility conditions are detailed in Section 2. The Fibonacci sequence corresponds to the order-2 case with  $a_0 = a_1 = 1$ .

**Informal main theorem.** Let  $(G_n)$  be an LRB. For any real  $G$ -additive  $f$ , the following are equivalent:

- $f$  admits a distribution function along  $\{0, \dots, N-1\}$ ;
- two canonical series (first-moment and quadratic digit-energy) converge.

In that case, the limiting characteristic function factorizes as an explicit infinite product.

To place our result in context, we compare in a table the various frameworks where an Erdős–Wintner theorem with an explicit product (and possibly an effective rate) is currently available.

Framework	Outcome	
	Distribution function & explicit product	Effective rate
Classical additive functions	Erdős–Wintner [11]	Tenenbaum–V. [22]
$q$ -additive, order 1	Delange [6]	Drmota–V. [9]
Zeckendorf (order 2, $a_0 = a_1 = 1$ )	Drmota–V. [9]	Drmota–V. [9]
General LRB, order $d \geq 2$	This paper	Open (in general LRB)

**Roadmap.** Section 2 introduces linear recurrence bases, the associated digit systems, and  $G$ -additive functions. Section 3 states and proves the general Erdős–Wintner theorem for LRBs and establishes the explicit infinite-product factorization of the limiting characteristic function. Section 4 gives two simple examples of the theorem and briefly discusses if an explicit description of the limiting law is available. Section 5 specializes the method to order-2 bases (including the Zeckendorf case) and records concrete criteria in this setting. Section 6 proves stability of the criterion under addition and under small digitwise perturbations. Finally, Section 7 discusses conditional extensions to Ostrowski numeration systems and Parry  $\beta$ -PV expansions, and formulates some directions for further work.

We finish this introduction by fixing some notation and conventions used throughout the paper.

- We use  $n, m, k$  for positive integers.
- The shorthand  $\sum_{j < m}$  stands for  $\sum_{0 \leq j < m}$ , and similarly  $\sum_{j \leq m}$  stands for  $\sum_{0 \leq j \leq m}$ .
- We use Vinogradov notation and write  $A \ll B$  (equivalently  $A = O(B)$ ) to mean that there exists a constant  $M > 0$  such that  $|A| \leq MB$  for all admissible values of the variables (in particular,  $B \geq 0$ ). If the implied constant may depend on auxiliary parameters  $\theta$ , we write  $A \ll_\theta B$  (for instance  $A \ll_\alpha$  or  $A \ll_{\alpha,p,c}$ ). We write  $A \gg B$  if  $B \ll A$ , and  $A \asymp B$  if both  $A \ll B$  and  $B \ll A$  hold. Implied constants may change from one occurrence to the next; unless explicitly stated otherwise, they may depend only on the fixed LRB data (such as the Perron root  $\alpha$  or the recurrence coefficients) and on the digit admissibility constraints.

## 2. LINEAR RECURRENT SEQUENCES AND DISTRIBUTION FUNCTION

For any increasing sequence  $(G_n)_{n \geq 0}$  of positive integers with  $G_0 = 1$ , every integer  $n \geq 0$  can be written as a finite expansion

$$(2.1) \quad n = \sum_{k \geq 0} e_k(n) G_k,$$

called the  **$G$ -ary expansion** of  $n$ , where the digits  $e_k(n)$  satisfy

$$e_k(n) \in \{0, 1, \dots, \lfloor G_{k+1}/G_k \rfloor - 1\}.$$

In particular, we have  $e_k(n) = 0$  for all sufficiently large  $k$ . The expansion (2.1) is unique if, for any  $K \geq 0$ ,

$$(2.2) \quad \sum_{k < K} e_k G_k < G_K,$$

and in that case the digits are computed by the greedy algorithm [12].

Fix integers  $a_0, \dots, a_{d-1} \geq 1$  and let  $(G_n)_{n \geq 0}$  be a linear recurrence sequence of order  $d \geq 2$ :

$$(2.3) \quad G_{n+d} = a_0 G_{n+d-1} + \dots + a_{d-1} G_n \quad (n \geq 0),$$

with initial conditions

$$(2.4) \quad G_0 = 1 \quad \text{and} \quad G_k = a_0 G_{k-1} + \dots + a_{k-1} G_0 + 1 \quad (0 < k < d).$$

We associate to this recurrence its *companion matrix*

$$A := \begin{pmatrix} a_0 & a_1 & \cdots & a_{d-2} & a_{d-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

whose characteristic polynomial is

$$X^d - a_0 X^{d-1} - \cdots - a_{d-2} X - a_{d-1}.$$

In this setting, the uniqueness condition (2.2) can be replaced by Parry's admissibility in the lexicographic order (see [17]): for all  $k \in \{1, \dots, d-1\}$ ,

$$(2.5) \quad (a_k, \dots, a_{d-1}) < (a_0, \dots, a_{d-1-k}),$$

and admissible digit blocks satisfy  $(e_k, \dots, e_{k-\ell+1}) < (a_0, \dots, a_{\ell-1})$  for all  $k \geq \ell-1$  and  $1 \leq \ell \leq d$ , where  $<$  denotes the lexicographic order (see, e.g., [12, 13, 17]). Under (2.5) the sequence  $(G_n)$  is a linearly recurrent numeration system in the sense of Jelínek [15, Def. 1.1]. Condition (2.5) is equivalent to the inequality, valid for all  $n \geq 0$  and  $1 \leq k < d$  (see [20]),

$$\sum_{k < i < d} a_i G_{n+d-1-i} < G_{n+d-1-k},$$

which is used in [7, 8].

We set  $\alpha := \max_{0 \leq j < d} a_j$ . The characteristic equation  $x^d - a_0 x^{d-1} - \cdots - a_{d-1} = 0$  has a unique dominant (Perron) real root  $\alpha$  with  $a_0 < \alpha < \alpha + 1$ , whose modulus strictly exceeds the moduli of the other roots.

We now record the structural assumptions on the base; all subsequent results rely only on these features.

**Definition 2.1** (Linear recurrent base (LRB)). *A sequence  $(G_n)_{n \geq 0}$  generated by the recurrence (2.3) with initial conditions (2.4) is called a **linear recurrent base (LRB)** if the following hold:*

- (i) *every nonnegative integer  $N$  admits a unique greedy  $G$ -expansion, that is, there exist digits  $e_k(N) \in \{0, \dots, \alpha\}$ , all but finitely many equal to 0, such that  $N = \sum_{k \geq 0} e_k(N) G_k$ , and the expansion is obtained by the usual greedy algorithm;*
- (ii) *the companion matrix  $A$  is primitive, meaning that some power  $A^n$  has all entries strictly positive (in particular, it has a simple Perron–Frobenius eigenvalue  $\alpha > 1$ );*
- (iii)  *$\alpha$  is a Pisot–Vijayaraghavan (PV) number and  $G_n/\alpha^n \rightarrow \kappa$  for some  $\kappa > 0$ .*

If the coefficients are positive and non-increasing, i.e.  $a_0 \geq \cdots \geq a_{d-1} \geq 1$ , then the three properties in Definition 2.1 hold automatically – see, for instance, [4, 12–15, 17] for proofs and for many standard examples of such linear recurrent bases.

The three items in Definition 2.1 are exactly the structural features that also appear in Jelínek's work on Gowers norms for linearly recurrent numeration systems [15]. Our terminology "LRB" is only meant to stress the role of  $(G_n)$  as a numeration base.

We now briefly comment on why each of the conditions (i)–(iii) is natural and essential for our arguments.

(i) *Greedy uniqueness.* The  $G$ -additive class is defined at the digit level, so a unique greedy expansion is needed to make  $f(n) = \sum_k f(e_k(n) G_k)$  well defined and independent of a normal form. Parry's lexicographic admissibility provides exactly this, together with a finite carry automaton, which is what our block factorization uses.

(ii) *Priminity / Perron–Frobenius.* Priminity of the companion matrix  $A$  supplies a simple top eigenvalue and drives uniform block frequencies for admissible digit patterns. Analytically, this aperiodicity rules out cyclic classes and ensures that the block ratios we linearize around the Perron direction admit a stable expansion. If  $A$  were reducible or imprimitive (period  $q > 1$ ), one typically sees oscillations along residue classes modulo  $q$ , and global distribution functions can fail unless one restricts to subsequences; the infinite-product factorization also breaks across the cycles.

(iii) *PV property.* The Pisot–Vijayaraghavan condition implies the exponential approximation

$$G_n = \kappa \alpha^n + O(\rho^n) \quad (\text{for some } \rho < \alpha),$$

so all non-Perron modes decay. This does two jobs in our proofs: it makes the first-order cancellation by the characteristic equation exact up to an exponentially small remainder (which is summable across levels), and it localizes carry interactions, which is crucial for the product factorization. Without PV (e.g. Salem or non-PV Perron roots), conjugates on or near the unit circle create long-range resonances and non-decaying remainders; uniqueness/greedy may remain, but the linearization errors cease to be summable and quantitative estimates generally fail.

In the sequel,  $(G_n)$  denotes an LRB (Definition 2.1). Our goal is to obtain necessary and sufficient conditions for a  $G$ -additive function  $f$  to *have a distribution function*  $F$ , i.e. for the empirical distribution functions to converge weakly to a function  $F$ :

$$\frac{1}{N} \# \{ 0 \leq n < N : f(n) \leq x \} \xrightarrow[N \rightarrow \infty]{} F(x) \quad \text{for every continuity point } x \text{ of } F.$$

Sufficient criteria are known under Parry admissibility (the lexicographic condition (2.5)) together with the digit-block constraints

$$(e_k, \dots, e_{k-\ell+1}) < (a_0, \dots, a_{\ell-1}) \quad (1 \leq \ell \leq d),$$

like in [12, 13, 17]. More general ergodic frameworks have been studied (*cf.* [2, 3]), which also explain, via a perturbative example, why one should not expect simple *necessary* conditions for non-constant coefficients.

By contrast, in the LRB setting of this paper we obtain a full analogue of the Erdős–Wintner theorem: a real-valued  $G$ -additive function  $f$  admits a distribution function if and only if two canonical series, built from the digit data of  $f$ , both converge. These series play the role of a first-moment drift and a quadratic digit energy, and their convergence gives necessary and sufficient conditions for the existence of a limiting distribution in this setting. The proof proceeds via an explicit infinite-product factorization of the limiting characteristic function.

We now introduce the analogues, in the  $G$ -additive setting, of the classical notions of additive and multiplicative functions from probabilistic number theory.

**Definitions 2.2.** A function  $f$  is  **$G$ -additive** if

$$f(n) = \sum_{k \geq 0} f(e_k(n) G_k).$$

A function  $g$  is  **$G$ -multiplicative** if

$$g(n) = \prod_{k \geq 0} g(e_k(n) G_k).$$

The following definitions are standard – see [21, Tome III]. For a real-valued  $G$ -additive function  $f$  and each  $N \geq 1$ , define

$$F_N(z) := \frac{1}{N} \# \{ n < N : f(n) \leq z \} \quad (z \in \mathbb{R}).$$

**Definitions 2.3.** i) A **distribution function** (abbreviated *d.f.*) is a nondecreasing, right-continuous function  $F : \mathbb{R} \rightarrow [0, 1]$  with  $F(-\infty) = 0$  and  $F(+\infty) = 1$ .

ii) A sequence  $(F_N)_{N \geq 1}$  of *d.f.*'s **converges weakly** to a function  $F$  if, for every real  $z$  which is a continuity point of  $F$ ,

$$\lim_{N \rightarrow \infty} F_N(z) = F(z).$$

iii) We say that  $f$  has a **d.f.**  $F$  if  $(F_N)_N$  converges weakly to a *d.f.*  $F$ .

iv) The **characteristic function** (*c.f.*)  $\Phi$  of a *d.f.*  $F$  is the Fourier transform of the Stieltjes measure  $dF$ :

$$\Phi(t) := \int_{-\infty}^{+\infty} e^{itz} dF(z) \quad (t \in \mathbb{R}).$$

It is uniformly continuous on  $\mathbb{R}$  and satisfies  $|\Phi(t)| \leq 1 = \Phi(0)$  for all  $t \in \mathbb{R}$ .

### 3. ERDŐS–WINTNER THEOREM FOR LINEAR RECURRENT BASES

Throughout,  $(G_n)$  is an LRB of order  $d \geq 2$ . For  $n \in \mathbb{N}$ ,  $t \in \mathbb{R}$  and a  $G$ -additive function  $f$ , set

$$g_t(n) := e^{itf(n)}, \quad H_n(t) := \sum_{m < G_n} g_t(m).$$

Define the block ratios by

$$r_0(t) := 1, \quad r_j(t) := \frac{H_j(t)}{H_{j-1}(t)} \quad (j \geq 1).$$

In particular,  $H_0(t) = g_t(0) = e^{itf(0)} = 1$ ; at  $t = 0$  one has  $g_0 \equiv 1$ , so  $H_n(0) = G_n$  and  $r_j(0) = G_j/G_{j-1}$ . We also introduce the two canonical series

$$(3.1) \quad \begin{aligned} (S_1) \quad & \sum_{n \geq 0} \sum_{j < d} \frac{1}{\alpha^j} \sum_{k < a_j} \left( f(k G_{n+d-j}) + \sum_{\ell < j} f(a_\ell G_{n+d-\ell}) \right), \\ (S_2) \quad & \sum_{n \geq 0} \sum_{k \leq a} f(k G_n)^2. \end{aligned}$$

With the block ratios  $r_j(t)$  and the canonical series (3.1) in place, we can now state the LRB analogue of the Erdős–Wintner theorem. The canonical series  $(S_1)$  and  $(S_2)$  control both the existence and the shape of the limiting distribution. In particular, we will show that the convergence of both series is not only sufficient but also necessary for a  $G$ -additive function  $f$  to admit a distribution function.

**Theorem 3.1** (Erdős–Wintner theorem for LRB). *Let  $(G_n)_n$  be an LRB and let  $f : \mathbb{N} \rightarrow \mathbb{R}$  be  $G$ -additive. Then the following are equivalent:*

- (1)  $f$  has a distribution function;
- (2) both series  $(S_1)$  and  $(S_2)$  in (3.1) converge.

In this case, the limiting characteristic function  $\Phi$  admits the infinite product factorization

$$(3.2) \quad \Phi(t) = \frac{1}{\kappa} \prod_{j \geq 1} \frac{r_j(t)}{\alpha}.$$

Moreover, the limiting law is purely atomic if and only if  $f(c G_j) = 0$  for all  $c \in \{1, \dots, a\}$  and all  $j \geq J$  for some  $J$ ; this criterion is classical in the digital/additive setting (see [2, Prop. 11]).

Heuristically, the two canonical series  $(S_1)$  and  $(S_2)$  arise by linearizing the block ratios  $\frac{r_j(t)}{\alpha}$  for small  $t$ : the first series governs the cumulative first-order drift in the logarithm of the product

$$\prod_{j \geq 1} \frac{r_j(t)}{\alpha},$$

while the second series controls the quadratic error coming from digit-level fluctuations.

As a first illustration, we record the specialization of Theorem 3.1 to multinacci bases, where all coefficients are equal to 1.

**Corollary 3.2** (Multinacci bases). *Assume that  $(G_n)$  is an LRB of order  $d \geq 2$  with  $a_0 = \dots = a_{d-1} = 1$ , so that*

$$G_{n+d} = G_{n+d-1} + \dots + G_n, \quad G_0 = 1, \quad G_k = G_{k-1} + \dots + G_0 + 1 \quad (0 < k < d).$$

Let  $\alpha > 1$  denote the dominant root of the characteristic polynomial

$$X^d - X^{d-1} - \dots - X - 1,$$

and let  $f$  be a real-valued  $G$ -additive function. Then  $f$  has a distribution function if and only if

$$\sum_{n \geq 0} \sum_{1 \leq j < d} \frac{1}{\alpha^j} \sum_{\ell < j} f(G_{n+d-\ell}) < \infty, \quad \sum_{n \geq 0} f(G_n)^2 < \infty.$$

Further special cases will be discussed below, including the order-2 situation in Section 5 and concrete examples of  $G$ -additive functions in Section 4. We outline the proof of Theorem 3.1. The argument is organized into four steps.

**Outline of the proof.**

- (1) Derive a  $d$ -step block recurrence for  $H_n$  and the induced product factorization for the ratios  $r_j$ .
- (2) Exploit an exact first-order cancellation along the Perron eigendirection, leaving a purely quadratic remainder.
- (3) Obtain a one-step bound for  $\varepsilon_j(t) := r_j(t) - \alpha$  using a suitable generating-function kernel.
- (4) Establish a telescoping identity relating the partial sums of the drift terms  $u_j(t)$  and the deviations  $\varepsilon_j(t)$ , which completes the proof of Theorem 3.1.

For  $q \geq 0$  and  $0 \leq \ell < d$ , define

$$\vartheta_{q,\ell} := \sum_{j < \ell} a_j G_{q-j}.$$

Following [2, Eq. (2.5)], for each  $n \geq 0$  every integer  $u < G_{n+d}$  admits a unique representation

$$u = \vartheta_{n+d-1,\ell} + k G_{n+d-1-\ell} + v,$$

with  $0 \leq \ell < d$ ,  $0 \leq k < a_\ell$  and  $0 \leq v < G_{n+d-1-\ell}$ . For  $q \geq 0$  and  $0 \leq \ell < d$ , also set

$$\sigma_{q,\ell}(t) := \sum_{h < a_\ell} g_t(h G_{q-\ell}).$$

By  $G$ -additivity, the function  $g_t$  factorizes over blocks, which yields the recurrence

$$(3.3) \quad H_{n+d}(t) = \sum_{\ell < d} g_t(\vartheta_{n+d-1,\ell}) \sigma_{n+d-1,\ell}(t) H_{n+d-1-\ell}(t) \quad (n \geq 0).$$

For  $k \geq 0$ , define the companion matrix

$$A_k(t) := \begin{pmatrix} \sigma_{k,0} & g_t(\vartheta_{k,1}) \sigma_{k,1} & \cdots & g_t(\vartheta_{k,d-1}) \sigma_{k,d-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then (3.3) is equivalent, for  $k \geq d-1$ , to

$$(3.4) \quad \begin{pmatrix} H_{k+1}(t) \\ H_k(t) \\ \vdots \\ H_{k-d+2}(t) \end{pmatrix} = A_k(t) \begin{pmatrix} H_k(t) \\ H_{k-1}(t) \\ \vdots \\ H_{k-d+1}(t) \end{pmatrix}.$$

Iterating this relation, we obtain, for  $k \geq d-1$ ,

$$H_k(t) = (1 \ 0 \ \cdots \ 0) A_{k-1}(t) \cdots A_{d-1}(t) \begin{pmatrix} H_{d-1}(t) \\ \vdots \\ H_1(t) \\ H_0(t) \end{pmatrix}.$$

Since  $r_m(t) = H_m(t)/H_{m-1}(t)$  for  $m \geq 1$ , it follows inductively that

$$H_k(t) = r_1(t) \cdots r_k(t) \quad (k \geq 1).$$

Thus, if  $f$  has a distribution function, then its characteristic function is

$$(3.5) \quad \Phi(t) = \lim_{k \rightarrow \infty} \frac{H_k(t)}{G_k} = \lim_{k \rightarrow \infty} \frac{H_k(t)/\alpha^k}{G_k/\alpha^k} = \frac{1}{\kappa} \prod_{j \geq 1} \frac{r_j(t)}{\alpha}.$$

Dividing (3.3) by  $H_{k-1}(t)$  (with  $k = n+d$ ) and using

$$\frac{H_{k-1}(t)}{H_{k-1-\ell}(t)} = \prod_{1 \leq s \leq \ell} r_{k-s}(t),$$

we obtain, for every  $k \geq d$ ,

$$(3.6) \quad r_k(t) = \sum_{\ell < d} g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) \prod_{1 \leq s \leq \ell} r_{k-s}(t)^{-1},$$

with the convention that an empty product equals 1. The right-hand side is well defined provided that, in each summand (for a given  $\ell$ ), one has  $r_{k-s}(t) \neq 0$  for  $s = 1, \dots, \ell$ . If  $r_j(t) = 0$  for only finitely many indices  $j$ , one may start the recurrence at any  $k$  larger than the last such index.

**3.1. Auxiliary lemmas.** This subsection collects the technical lemmas used in the proof of Theorem 3.1. We also indicate how these ingredients fit together to establish the theorem. The proofs are given in the appendices and may be read separately.

**Lemma 3.3.** *Assume that  $f$  is  $G$ -additive and satisfies*

$$(H_f) \quad f(cG_k) \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{for every } 1 \leq c \leq \alpha.$$

*Then, for every  $T_0 > 0$ ,*

$$r_k(t) \xrightarrow[k \rightarrow \infty]{} \alpha \quad \text{uniformly for } |t| \leq T_0.$$

See Appendix A for the proof of Lemma 3.3. In particular, for each fixed  $t_0 > 0$  we have

$$r_k(t) \xrightarrow[k \rightarrow \infty]{} \alpha \quad \text{uniformly for } |t| \leq t_0.$$

Hence there exists  $j_0 \geq 1$  such that  $\sup_{|t| \leq t_0} |r_j(t) - \alpha| \leq \alpha/2$  for all  $j \geq j_0$ . Then  $|r_j(t)| \geq \alpha/2$  by the reverse triangle inequality, and in particular  $r_j(t) \neq 0$ . Using  $H_k(t) = \prod_{m \leq k} r_m(t)$  for  $k \geq 1$ , we can write, for  $|t| \leq t_0$ ,

$$\Phi(t) = \lim_{k \rightarrow \infty} \frac{H_k(t)}{G_k} = \lim_{k \rightarrow \infty} \frac{H_{j_0}(t)}{G_k} \prod_{j_0 < j \leq k} r_j(t) = \frac{1}{\kappa} \frac{H_{j_0}(t)}{\alpha^{j_0}} \prod_{j > j_0} \frac{r_j(t)}{\alpha},$$

where we used  $\alpha^k/G_k \rightarrow 1/\kappa$ .

For  $k \geq j_0 + d$  we now apply the ratio recurrence (3.6), since all denominators  $r_{k-s}(t)$  in the products are nonzero for  $|t| \leq t_0$  and  $1 \leq s \leq \ell$ . Set

$$\varepsilon_k(t) := r_k(t) - \alpha, \quad k \geq 1,$$

and define, for  $k \geq 1$ ,

$$u_k(t) := \alpha^d \sum_{\ell < d} \frac{g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) - a_\ell}{\alpha^{\ell+1}}.$$

In informal terms,  $\varepsilon_k$  is the deviation of  $r_k$  from the limit ratio  $\alpha$ , and  $u_k$  is the corresponding error term in the one-step recursion for  $\varepsilon_k$ . Moreover, for  $\ell \in \{0, \dots, d-1\}$  write

$$\Pi_{k,\ell} := (\varepsilon_{k-d+1}(t) + \alpha) \cdots (\varepsilon_{k-\ell-1}(t) + \alpha),$$

and define

$$\widehat{\Pi}_{k,\ell} := \Pi_{k,\ell} - \alpha^{d-\ell-1}.$$

In particular,  $\widehat{\Pi}_{k,d-1} = 0$ . Then (3.6) is equivalent, for all  $k \geq j_0 + d$ , to

$$\begin{aligned} (3.7) \quad \varepsilon_k(t) &= \frac{1}{\Pi_{k,0}} \left( \sum_{\ell < d} g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) \Pi_{k,\ell} \right) - \alpha \\ &= \frac{1}{\Pi_{k,0}} \left( \sum_{\ell < d} g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) \Pi_{k,\ell} - \alpha \widehat{\Pi}_{k,0} - \alpha^d \right) \\ &= \frac{1}{\Pi_{k,0}} \left( \alpha^d \sum_{\ell < d} \frac{g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t)}{\alpha^{\ell+1}} - \alpha^d + \sum_{\ell < d} g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) \widehat{\Pi}_{k,\ell} - \alpha \widehat{\Pi}_{k,0} \right) \\ &= \frac{1}{\Pi_{k,0}} \left( u_k(t) + \sum_{\ell < d} g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) \widehat{\Pi}_{k,\ell} - \alpha \widehat{\Pi}_{k,0} \right). \end{aligned}$$

where we used the Perron identity  $\sum_{\ell < d} a_\ell / \alpha^{\ell+1} = 1$ . In particular, the convergence  $r_k(t) \rightarrow \alpha$  is equivalent to  $\varepsilon_k(t) \rightarrow 0$ .

**Lemma 3.4.** *Let  $g : \mathbb{N} \rightarrow \mathbb{C}$  be a  $G$ –multiplicative function with  $|g(n)| \leq 1$  for all  $n \in \mathbb{N}$ . If the limit*

$$\ell := \lim_{k \rightarrow \infty} \frac{1}{G_k} \sum_{n < G_k} g(n)$$

*exists, then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} g(n)$$

*also exists and is equal to  $\ell$ .*

Barat and Grabner [2, Lemma 3] record this implication as folklore and omit a proof. For completeness, we include a short self-contained argument in Appendix B. Related odometer-based viewpoints and mean-value estimates for  $G$ –multiplicative sequences can be found in [14]. The degenerate case  $\ell = 0$  is discussed in [5].

The next lemma is the technical core of the proof of Theorem 3.1. It provides  $\ell^2$ –control on  $u_k(t)$  and  $\varepsilon_k(t)$  under the second canonical series, and it shows that, under the same hypothesis, the convergence of  $\sum_k u_k(t)$  is equivalent to that of  $\sum_k \varepsilon_k(t)$ . This is a discrete summation criterion linking the two sequences.

**Lemma 3.5.** *Assume that  $f$  is  $G$ –additive and satisfies*

$$(H_f) \quad f(cG_n) \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{for all } 1 \leq c \leq \alpha.$$

*Let  $u_k(t)$  and  $\varepsilon_k(t)$  be defined as above. Then:*

(1) *If the second canonical series (S2) converges, then for each fixed  $t \in \mathbb{R}$  we have*

$$\sum_{k \geq 0} |u_k(t)|^2 < \infty.$$

(2) *For each fixed  $t \in \mathbb{R}$ ,*

$$\sum_{k \geq 0} |u_k(t)|^2 < \infty \implies \sum_{k \geq 0} |\varepsilon_k(t)|^2 < \infty.$$

(3) *If (S2) converges, then for each fixed  $t \in \mathbb{R}$ ,*

$$\sum_{k \geq 0} \varepsilon_k(t) \text{ converges} \iff \sum_{k \geq 0} u_k(t) \text{ converges.}$$

Hypothesis  $(H_f)$  is automatic under the convergence of the second series in (3.1). This assumption is used to justify uniform small– $x$  expansions and limit arguments; discarding a finite initial segment does not affect convergence. The proof of this lemma is long and technical, so we postpone it to Appendix C.

We have now established the auxiliary lemmas needed to prove the sufficiency part, namely that the convergence of the canonical series forces the existence of a limit law. To prove the converse implication (necessity), we state two further lemmas. The proof of the first one is given in Appendix D.

Fix  $T_0 > 0$ . For  $|t| \leq T_0$  and  $n \geq d - 1$ , we encode the  $d$ –step recurrence for  $(H_n(t))$  in block–matrix form. Set

$$\mathbf{H}_n(t) := (H_n(t), H_{n-1}(t), \dots, H_{n-d+1}(t))^{\top} \in \mathbb{C}^d,$$

so that

$$\mathbf{H}_{n+1}(t) = A_n(t) \mathbf{H}_n(t),$$

Throughout the remainder of the proof we consider frequencies  $t \in [-T_0, T_0]$ . To prove the sufficiency in Theorem 3.1, we will combine a blockwise Taylor expansion of the coefficients in the first row of the companion matrix  $A_n(t)$  defined in (3.4). We denote this first row by

$$(c_{n,0}(t), c_{n,1}(t), \dots, c_{n,d-1}(t)),$$

so that, explicitly,

$$c_{n,0}(t) := \sigma_{n,0}(t), \quad c_{n,\ell}(t) := g_t(\vartheta_{n,\ell}) \sigma_{n,\ell}(t) \quad (1 \leq \ell < d).$$

Fix once and for all a norm  $\|\cdot\|$  on  $\mathbb{C}^d$ . For  $n \geq d$  we also set the block energy

$$(3.8) \quad Q_n := \sum_{r < d} \sum_{1 \leq j \leq \alpha} f(jG_{n-r})^2.$$

**Lemma 3.6.** *Assume that  $f$  is  $G$ -additive and satisfies  $(H_f)$ , that is*

$$f(cG_m) \rightarrow 0 \quad (m \rightarrow \infty)$$

for each fixed digit  $1 \leq c \leq \alpha$ . Then, uniformly for  $|t| \leq T_0$  and  $n$  large, there exist  $\Lambda_{n,\ell}, \zeta_{n,\ell} \in \mathbb{C}$  and  $\mathcal{R}_{n,\ell}(t) \in \mathbb{C}$ , depending only on the block values  $f(jG_{n-r})$  with  $0 \leq r < d$  and  $1 \leq j \leq \alpha$ , such that

$$c_{n,\ell}(t) = a_\ell + t \Lambda_{n,\ell} + t^2 \zeta_{n,\ell} + \mathcal{R}_{n,\ell}(t), \quad 0 \leq \ell < d.$$

Moreover, if we set

$$\Lambda_n := (\Lambda_{n,0}, \dots, \Lambda_{n,d-1}), \quad \zeta_n := (\zeta_{n,0}, \dots, \zeta_{n,d-1}), \quad \mathcal{R}_n(t) := (\mathcal{R}_{n,0}(t), \dots, \mathcal{R}_{n,d-1}(t)),$$

then, as  $n \rightarrow \infty$ ,

$$\|\Lambda_n\| \ll \sum_{r < d} \sum_{j \leq \alpha} |f(jG_{n-r})|, \quad \|\zeta_n\| \ll Q_n.$$

Furthermore, there exists a sequence  $(\omega_n)_{n \geq 0}$  with  $\omega_n \rightarrow 0$  such that, uniformly for  $|t| \leq T_0$  and  $n$  large,

$$\|\mathcal{R}_n(t)\| \leq \omega_n t^2 Q_n.$$

The implied constants may depend on the digit system, on  $T_0$ , and on the chosen norm on  $\mathbb{C}^d$ , but not on  $n$  or  $t$ .

The next lemma turns this blockwise expansion into a quantitative dissipation estimate for the dominant eigenvalue of  $A_n(t)$ .

**Lemma 3.7.** *Assume  $(H_f)$  and, for  $n \geq d$ , define  $Q_n$  as in (3.8), namely*

$$Q_n := \sum_{r < d} \sum_{1 \leq c \leq \alpha} f(cG_{n-r})^2.$$

Let  $A$  be the companion matrix associated with the recurrence (2.3), and let  $\alpha > 1$  be its Perron–Frobenius eigenvalue. Denote by  $v, w > 0$  the associated right/left Perron–Frobenius eigenvectors, normalized by  $w^\top v = 1$ .

Then there exist constants  $T_1 \in (0, T_0]$ ,  $c_0 > 0$ ,  $n_0$  and  $\delta > 0$  such that, for all  $n \geq n_0$  and all  $|t| \leq T_1$ , the matrix  $A_n(t)$  has a simple eigenvalue  $\lambda_n(t)$  with  $\lambda_n(0) = \alpha$  and

$$(3.9) \quad |\lambda_n(t)| \leq \alpha \exp(-c_0 t^2 Q_n),$$

and all its other eigenvalues have modulus at most  $\alpha - \delta$ .

This lemma is proved in Appendix E. Now, let us show how all the auxiliary lemmas fit into the proof of Theorem 3.1.

### 3.2. Proof of Theorem 3.1.

*Step 1: Sufficiency.* Assume that both canonical series (S1) and (S2) in (3.1) converge. Fix  $T_0 > 0$  and  $t \in [-T_0, T_0]$ , and abbreviate  $u_k := u_k(t)$  and  $\varepsilon_k := \varepsilon_k(t)$ . By Lemma 3.5 (1) and (2) we have

$$\sum_{k \geq 0} |u_k|^2 < \infty \quad \text{and thus} \quad \sum_{k \geq 0} |\varepsilon_k|^2 < \infty.$$

Next we prove that  $\sum_{k \geq 0} u_k$  converges. Set

$$M_n := \sum_{j < d} \frac{1}{\alpha^j} \sum_{0 \leq k < a_j} \left( f(kG_{n+d-j}) + \sum_{\ell < j} f(a_\ell G_{n+d-\ell}) \right) \quad (n \geq 0),$$

so that the first canonical series (S1) is  $\sum_{n \geq 0} M_n$ . Set

$$S_2(m) := \sum_{1 \leq c \leq \alpha} f(cG_m)^2 \quad (m \geq 0),$$

so that the convergence of the second canonical series (S2) is exactly

$$\sum_{m \geq 0} S_2(m) < \infty.$$

For  $n \geq d$ , recall the associated  $d$ –block energy defined in (3.8)

$$Q_n = \sum_{r < d} \sum_{1 \leq c \leq a} f(cG_{n-r})^2 = \sum_{r=0}^{d-1} S_2(n-r).$$

Since  $d$  is fixed,  $\sum_{m \geq 0} S_2(m) < \infty$  is equivalent to  $\sum_{n \geq d} Q_n < \infty$  (finite shifts and finite sums preserve convergence). Moreover, by Cauchy–Schwarz and the finiteness of the digit set, we have the pointwise bound

$$|M_n|^2 \ll Q_{n+d} \quad (n \geq 0),$$

hence  $\sum_{n \geq 0} |M_n|^2 < \infty$  whenever (S2) holds.

We claim that, uniformly for  $t \in [-T_0, T_0]$  and all  $n \geq 0$ ,

$$(3.10) \quad u_{n+d+1} = it M_n + O(t^2 Q_{n+d}),$$

where the implied constant depends only on the digit system and on  $T_0$ . Indeed, by definition,

$$u_{n+d+1} = \sum_{j < d} \frac{1}{\alpha^j} \left( g_t(\vartheta_{n+d,j}) \sigma_{n+d,j}(t) - a_j \right).$$

Using the elementary bounds  $e^{iy} = 1 + iy + O(y^2)$  and  $e^{iy} - 1 = O(y)$ , valid for all real  $y$ , and using  $G$ –additivity, we obtain for each fixed  $j < d$

$$g_t(\vartheta_{n+d,j}) \sigma_{n+d,j}(t) = a_j + it \sum_{0 \leq k < a_j} \left( f(kG_{n+d-j}) + \sum_{\ell < j} f(a_\ell G_{n+d-\ell}) \right) + O(t^2 W_{n,j}),$$

where

$$W_{n,j} := \sum_{0 \leq k < a_j} \left( f(kG_{n+d-j}) + \sum_{\ell < j} f(a_\ell G_{n+d-\ell}) \right)^2.$$

Since  $d$  and the digits are fixed,  $(x_1 + \cdots + x_m)^2 \leq m(x_1^2 + \cdots + x_m^2)$  gives

$$W_{n,j} \ll \sum_{0 \leq k < a_j} f(kG_{n+d-j})^2 + \sum_{\ell < j} f(a_\ell G_{n+d-\ell})^2 \ll \sum_{r < d} \sum_{1 \leq c \leq a} f(cG_{n+d-r})^2 = Q_{n+d}.$$

Summing over  $j < d$  with weights  $\alpha^{-j}$  therefore yields (3.10).

Since  $\sum_{n \geq 0} M_n$  converges by (S1) and  $\sum_{n \geq d} Q_n$  converges by (S2), the estimate (3.10) implies that  $\sum_{k \geq 0} u_k(t)$  converges (for each fixed  $t \in [-T_0, T_0]$ ). Consequently, Lemma 3.5 (3) yields the convergence of  $\sum_{k \geq 0} \varepsilon_k(t)$ . Since  $\varepsilon_k \rightarrow 0$  and  $\log(1+x) = x + O(x^2)$  as  $x \rightarrow 0$ , the convergence of  $\sum_k \varepsilon_k$  and  $\sum_k \varepsilon_k^2$  implies that, for some  $k_0$ ,

$$\sum_{k \geq k_0+1} \log \left( 1 + \frac{\varepsilon_k}{\alpha} \right)$$

converges. Hence the infinite product  $\prod_{k \geq 1} r_k(t)/\alpha$  converges to a nonzero limit. Using  $H_k(t) = \prod_{j=1}^k r_j(t)$  and  $\alpha^k/G_k \rightarrow 1/\kappa$ , we obtain

$$\frac{H_k(t)}{G_k} = \left( \prod_{j=1}^k \frac{r_j(t)}{\alpha} \right) \cdot \frac{\alpha^k}{G_k} \xrightarrow{k \rightarrow \infty} \frac{1}{\kappa} \prod_{j \geq 1} \frac{r_j(t)}{\alpha} =: \Phi(t).$$

Since  $H_k(t) = \sum_{n < G_k} e^{itf(n)}$ , the left-hand side equals  $\Phi_{G_k}(t)$ . Therefore  $\Phi_{G_k}(t) \rightarrow \Phi(t)$ , and Lemma 3.4 implies that  $\Phi_N(t) := (1/N) \sum_{n < N} e^{itf(n)}$  converges to  $\Phi(t)$  as  $N \rightarrow \infty$ .

Finally, by the equicontinuity argument in [2, Theorem 4, Step 2] (which uses only (S2)), the limit  $\Phi$  is continuous at the origin. Lévy’s continuity theorem then shows that  $f$  admits a distribution function.

*Step 2 : Necessity.* Assume that  $f$  admits a distribution function with characteristic function  $\Phi$ . Since  $\Phi$  is continuous at the origin and  $\Phi(0) = 1$ , we may pick two small, nonzero, rationally independent frequencies  $t_1, t_2$  (i.e.  $t_2/t_1 \notin \mathbb{Q}$ ) such that  $|\Phi(t_i)| \geq \frac{1}{2}$ . For each fixed  $t_i$  we have

$$\frac{H_k(t_i)}{G_k} \longrightarrow \Phi(t_i) \neq 0 \implies \frac{r_k(t_i)}{G_k/G_{k-1}} = \frac{H_k(t_i)/H_{k-1}(t_i)}{G_k/G_{k-1}} \longrightarrow 1,$$

and since  $G_k/G_{k-1} \rightarrow \alpha$  (by Perron–Frobenius theory for the companion matrix), we conclude that  $r_k(t_i) \rightarrow \alpha$ .

We now identify some block coefficients and deduce the vanishing of the digit blocks. Recall the  $d$ -step ratio recurrence (3.6) for  $(r_k(t))$  and the block recurrence

$$H_{n+d}(t) = \sum_{\ell < d} c_{n+d-1,\ell}(t) H_{n+d-1-\ell}(t), \quad c_{m,\ell}(t) := g_t(\vartheta_{m,\ell}) \sigma_{m,\ell}(t).$$

From  $r_k(t_i) \rightarrow \alpha$  we obtain, for each fixed  $\ell$ ,

$$\frac{H_{n+d-1-\ell}(t_i)}{H_n(t_i)} = \prod_{j=0}^{d-2-\ell} r_{n+d-1-j}(t_i) = \alpha^{d-1-\ell} (1 + o(1)),$$

and similarly

$$\frac{H_{n+d}(t_i)}{H_n(t_i)} = \prod_{j=0}^{d-1} r_{n+d-j}(t_i) = \alpha^d (1 + o(1)).$$

Dividing the block recurrence by  $H_n(t_i)$  and letting  $n \rightarrow \infty$  yields the *limit linear form*

$$(3.11) \quad \lim_{n \rightarrow \infty} S_{n+d-1} := \lim_{n \rightarrow \infty} \sum_{\ell < d} c_{n+d-1,\ell}(t_i) \alpha^{d-1-\ell} = \alpha^d.$$

On the other hand,  $|g_{t_i}(\cdot)| = 1$  and  $|\sigma_{m,\ell}(t_i)| \leq a_\ell$ , hence  $|c_{m,\ell}(t_i)| \leq a_\ell$ . For any  $m \geq 1$  write

$$z_{m,\ell} := \frac{c_{m,\ell}(t_i)}{a_\ell}, \quad w_\ell := a_\ell \alpha^{d-1-\ell} > 0,$$

so that  $|z_{m,\ell}| \leq 1$  and (3.11) becomes

$$\lim_{m \rightarrow \infty} \sum_{\ell < d} w_\ell z_{m,\ell} = \sum_{\ell < d} w_\ell = \alpha^d.$$

By the triangle inequality,

$$\left| \sum_{\ell} w_\ell z_{m,\ell} \right| \leq \sum_{\ell} w_\ell |z_{m,\ell}| \leq \sum_{\ell} w_\ell,$$

and since the left-hand side tends to  $\alpha^d = \sum_\ell w_\ell$ , we obtain

$$\sum_{\ell < d} w_\ell (1 - |z_{m,\ell}|) \longrightarrow 0.$$

As the index set  $\{0, \dots, d-1\}$  is finite and each  $w_\ell > 0$ , it follows that

$$|z_{m,\ell}| \longrightarrow 1 \quad \text{for each } \ell.$$

Now set

$$S_m := \sum_{\ell < d} w_\ell z_{m,\ell}, \quad \sum_{\ell < d} w_\ell = \alpha^d.$$

An exact computation gives

$$\alpha^{2d} - |S_m|^2 = \left( \sum_{\ell < d} w_\ell \right)^2 - \left| \sum_{\ell < d} w_\ell z_{m,\ell} \right|^2 = \sum_{\ell, \ell'} w_\ell w_{\ell'} \left( 1 - |z_{m,\ell}| |z_{m,\ell'}| \cos(\theta_{m,\ell} - \theta_{m,\ell'}) \right),$$

where we write  $z_{m,\ell} = |z_{m,\ell}| e^{i\theta_{m,\ell}}$ . Since

$$S_m = \sum_{\ell < d} w_\ell z_{m,\ell} \longrightarrow \alpha^d,$$

we have  $|S_m| \rightarrow \alpha^d$ , hence  $\alpha^{2d} - |S_m|^2 \rightarrow 0$ . Together with  $|z_{m,\ell}| \rightarrow 1$ , this forces

$$\theta_{m,\ell} - \theta_{m,\ell'} \longrightarrow 0 \pmod{2\pi} \quad \text{for all } \ell, \ell' < d.$$

In other words, all the  $z_{m,\ell}$  share a common limiting phase: there exists a real sequence  $(\psi_m)_{m \geq 1}$  such that

$$z_{m,\ell} = |z_{m,\ell}| e^{i\psi_m} + o(1) = e^{i\psi_m} + o(1) \quad \text{uniformly in } \ell.$$

But

$$\frac{S_m}{\alpha^d} = \frac{1}{\alpha^d} \sum_{\ell < d} w_\ell z_{m,\ell} = e^{i\psi_m} + o(1),$$

and since the left-hand side converges to 1, we must have  $e^{i\psi_m} \rightarrow 1$ , hence  $\psi_m \rightarrow 0 \pmod{2\pi}$ . Therefore

$$z_{m,\ell} \rightarrow 1 \quad \text{for each } \ell, \quad \text{i.e.} \quad c_{m,\ell}(t_i) \rightarrow a_\ell.$$

In particular,

$$|\sigma_{m,\ell}(t_i)| = \frac{|c_{m,\ell}(t_i)|}{|g_{t_i}(\vartheta_{m,\ell})|} \rightarrow a_\ell.$$

We now convert this extremality into pointwise phase alignment inside each block. For any fixed  $\ell$ , the identity

$$0 \leq \frac{1}{2} \sum_{j,k < a_\ell} |e^{it_i f(jG_{m-\ell})} - e^{it_i f(kG_{m-\ell})}|^2 = a_\ell^2 - |\sigma_{m,\ell}(t_i)|^2$$

shows that  $|\sigma_{m,\ell}(t_i)| \rightarrow a_\ell$  forces

$$e^{it_i f(jG_{m-\ell})} - e^{it_i f(kG_{m-\ell})} \rightarrow 0 \quad (0 \leq j, k < a_\ell).$$

Since  $f(0) = 0$ , taking  $k = 0$  yields

$$|e^{it_i f(jG_{m-\ell})} - 1| \rightarrow 0 \quad (0 \leq j < a_\ell),$$

that is,

$$e^{it_i f(jG_{m-\ell})} \rightarrow 1 \quad (0 \leq j < a_\ell),$$

or equivalently,

$$t_i f(jG_{m-\ell}) \rightarrow 0 \pmod{2\pi}.$$

Finally, since

$$e^{it_1 f(jG_{m-\ell})} \rightarrow 1 \quad \text{and} \quad e^{it_2 f(jG_{m-\ell})} \rightarrow 1,$$

with  $t_2/t_1 \notin \mathbb{Q}$ , a standard Diophantine argument yields that  $f(jG_{m-\ell}) \rightarrow 0$  as a real number. Thus, for every fixed  $(\ell, j)$ ,

$$f(jG_{m-\ell}) \rightarrow 0.$$

Fix  $t$  with  $0 < |t| \leq T_0$  and  $|\Phi(t)| \geq \frac{1}{2}$ . From Step 1 we know that  $(H_f)$  holds, hence  $Q_n \rightarrow 0$  as  $n \rightarrow \infty$ . Fix once and for all a norm  $\|\cdot\|$  on  $\mathbb{C}^d$ . Applying Lemma 3.6 to the first row of  $A_n(t)$ , we obtain, uniformly for  $|t| \leq T_0$  and  $n$  large,

$$(c_{n,0}(t), \dots, c_{n,d-1}(t)) = (a_0, \dots, a_{d-1}) + t \Lambda_n + t^2 \zeta_n + \mathcal{R}_n(t),$$

where  $\Lambda_n, \zeta_n \in \mathbb{C}^d$  and  $\mathcal{R}_n(t) \in \mathbb{C}^d$  depend only on the finitely many block values  $f(jG_{n-r})$  with  $0 \leq r < d$  and  $1 \leq j \leq \mathfrak{a}$ , and

$$\|\Lambda_n\| \ll \sum_{r < d} \sum_{j \leq \mathfrak{a}} |f(jG_{n-r})|, \quad \|\zeta_n\| \ll Q_n, \quad \|\mathcal{R}_n(t)\| \leq \omega_n t^2 Q_n.$$

Since the index set  $\{0, \dots, d-1\} \times \{1, \dots, \mathfrak{a}\}$  is finite, Cauchy–Schwarz gives

$$\sum_{r < d} \sum_{j \leq \mathfrak{a}} |f(jG_{n-r})| \ll Q_n^{1/2},$$

so that  $\|\Lambda_n\| \ll Q_n^{1/2}$ .

Define matrices  $B_n, C_n$  and a remainder  $\mathcal{E}_n(t)$  by letting  $B_n$  (resp.  $C_n, \mathcal{E}_n(t)$ ) have first row  $\Lambda_n$  (resp.  $\zeta_n, \mathcal{R}_n(t)$ ) and all other rows equal to 0. Then

$$(3.12) \quad A_n(t) = A + t B_n + t^2 C_n + \mathcal{E}_n(t),$$

with

$$\|B_n\| \ll Q_n^{1/2}, \quad \|C_n\| \ll Q_n, \quad \|\mathcal{E}_n(t)\| \leq \omega_n t^2 Q_n,$$

as  $n \rightarrow \infty$ , uniformly for  $|t| \leq T_0$ .

Applying Lemma 3.7, we obtain constants  $T_1 \in (0, T_0]$ ,  $c_0 > 0$ ,  $n_0$  and  $\delta > 0$  such that, for all  $n \geq n_0$  and all  $|t| \leq T_1$ , the matrix  $A_n(t)$  has a simple eigenvalue  $\lambda_n(t)$  with  $\lambda_n(0) = \alpha$  and satisfying (3.9), and all its other eigenvalues have modulus at most  $\alpha - \delta$ .

The remaining task is to convert this *one-step* spectral dissipation into an upper bound for the cocycle

$$\mathbf{H}_N(t) = A_{N-1}(t) \cdots A_{n_1}(t) \mathbf{H}_{n_1}(t).$$

We use the following perturbative dominated-splitting estimate, proved in Appendix F.

**Lemma 3.8.** *Assume that  $A_n(t)$  satisfies the conclusions of Lemma 3.7 for all  $n \geq n_0$  and all  $|t| \leq T_1$ , with dominant eigenvalue  $\lambda_n(t)$  and spectral gap  $\delta > 0$ .*

*Then there exist an operator norm  $\|\cdot\|_\star$  on  $\mathbb{C}^d$ , an integer  $n_1 \geq n_0$ , and constants  $\delta_1 \in (0, \delta)$  and  $C \geq 1$  (depending only on the digit system) such that, for all  $N \geq n_1$  and all  $|t| \leq T_1$ ,*

$$\|\mathbf{H}_N(t)\|_\star \leq C \|\mathbf{H}_{n_1}(t)\|_\star \left( \prod_{n_1 \leq n < N} |\lambda_n(t)| + (\alpha - \delta_1)^{N-n_1} \right).$$

Applying Lemma 3.8 together with (3.9) yields that, for some  $c_0 > 0$  and all  $|t| \leq T_1$ ,

$$(3.13) \quad \|\mathbf{H}_N(t)\|_\star \ll \alpha^N \exp\left(-c_0 t^2 \sum_{n=n_1}^{N-1} Q_n\right) + \alpha^N \exp(-\eta(N - n_1)),$$

where  $\eta := -\log((\alpha - \delta_1)/\alpha) > 0$  and the implied constant is uniform in  $N$  and  $t$  (for  $|t| \leq T_1$ ).

On the other hand, our ratio limit from Step 1 gives

$$\frac{H_n(t)}{G_n} \rightarrow \Phi(t) \neq 0,$$

so  $|H_n(t)| \asymp G_n \asymp \alpha^n$  and therefore  $\|\mathbf{H}_N(t)\|_\star \asymp \alpha^N$  as  $N \rightarrow \infty$ . Since  $\eta > 0$ , the second term in (3.13) is  $o(\alpha^N)$ . Therefore, the lower bound  $\|\mathbf{H}_N(t)\|_\star \asymp \alpha^N$  can hold only if

$$\sum_{n \geq 1} Q_n < \infty,$$

which is precisely the convergence of the second canonical series (S2) in (3.1).

#### 4. EXAMPLES

Throughout this section we work in the setting of Definition 2.1, so in particular  $d \geq 2$  and  $(G_n)$  is a genuine linear recurrence base. We write

$$n = \sum_{k \geq 0} e_k(n) G_k, \quad 0 \leq e_k(n) \leq \mathfrak{a},$$

for the greedy  $G$ -expansion of  $n$ .

We illustrate Theorem 3.1 on two simple families of digit functions, with polynomial and geometric damping in the height of the digit.

(1) *Polynomially damped digit function.* Fix  $\beta > 1$  and a function

$$\varphi : \{0, \dots, \mathfrak{a}\} \rightarrow \mathbb{R}$$

with  $\varphi(0) = 0$  and  $\varphi \not\equiv 0$ , and define

$$f(jG_n) := \frac{\varphi(j)}{(n+1)^\beta}, \quad 0 \leq j \leq \mathfrak{a}, \quad n \geq 0.$$

By  $G$ -additivity this determines

$$f(n) = \sum_{k \geq 0} f(e_k(n)G_k) = \sum_{k \geq 0} \frac{\varphi(e_k(n))}{(k+1)^\beta}.$$

For the first canonical series (S1) in (3.1), the contribution of layer  $n$  can be written, for each fixed  $t \in \mathbb{R}$ , as

$$A_n(t) = \sum_{j=0}^{\mathfrak{a}} \alpha_{n,j}(t) f(jG_n),$$

where the coefficients  $\alpha_{n,j}(t)$  encode the local block structure at height  $n$  and are uniformly bounded:  $|\alpha_{n,j}(t)| \leq C(t)$ . Consequently,

$$|A_n(t)| \leq C(t) \max_{0 \leq j \leq \mathfrak{a}} |f(jG_n)| \ll_t \frac{1}{(n+1)^\beta},$$

and the series  $\sum_{n \geq 0} A_n(t)$  converges absolutely whenever  $\beta > 1$ .

In the second canonical series (S2) of (3.1), the  $n$ th summand is

$$\sum_{1 \leq j \leq \mathfrak{a}} f(jG_n)^2 = \sum_{1 \leq j \leq \mathfrak{a}} \frac{\varphi(j)^2}{(n+1)^{2\beta}} \ll \frac{1}{(n+1)^{2\beta}},$$

since  $\varphi$  takes only finitely many values. Hence

$$\sum_{n \geq 0} \sum_{1 \leq j \leq \mathfrak{a}} f(jG_n)^2 < \infty.$$

Thus, for  $\beta > 1$ , both canonical series in (3.1) converge absolutely for all  $t \in \mathbb{R}$ , and Theorem 3.1 provides a nondegenerate limit distribution for  $f(n)$ , supported in a compact interval and with finite moments of all orders.

- (2) *Geometrically damped digit function.* Fix  $\rho \in (-1, 1)$  and a function  $\varphi : \{0, \dots, \mathfrak{a}\} \rightarrow \mathbb{R}$  with  $\varphi(0) = 0$  and  $\varphi \not\equiv 0$ , and define

$$f(jG_n) := \rho^n \varphi(j), \quad 0 \leq j \leq \mathfrak{a}, \quad n \geq 0.$$

By  $G$ -additivity,

$$f(n) = \sum_{k \geq 0} \rho^k \varphi(e_k(n)).$$

For each fixed  $t \in \mathbb{R}$ , the  $n$ th term of the first canonical series (S1) has the form

$$A_n(t) = \sum_{j=0}^{\mathfrak{a}} \alpha_{n,j}(t) f(jG_n),$$

with uniformly bounded coefficients  $\alpha_{n,j}(t)$ , so that

$$|A_n(t)| \ll_t |\rho|^n$$

and  $\sum_{n \geq 0} |A_n(t)| < \infty$ .

In the second canonical series (S2) of (3.1), the  $n$ th summand is

$$\sum_{1 \leq j \leq \mathfrak{a}} f(jG_n)^2 = \sum_{1 \leq j \leq \mathfrak{a}} \rho^{2n} \varphi(j)^2 \ll |\rho|^{2n},$$

so that

$$\sum_{n \geq 0} \sum_{1 \leq j \leq \mathfrak{a}} f(jG_n)^2 < \infty \quad \text{since } |\rho| < 1.$$

Both canonical series in (3.1) thus converge absolutely, and Theorem 3.1 applies.

The corresponding limit law describes the asymptotic distribution of the weighted sums

$$\sum_{k \geq 0} \rho^k \varphi(e_k(n)),$$

for  $n$  uniform in  $\{0, \dots, N-1\}$  and  $N \rightarrow \infty$ . Its support is contained in a compact interval of size  $O((1 - |\rho|)^{-1})$  and all moments are finite.

Let us discuss the explicit limit law. For a fixed base  $(G_n)$ , the greedy expansions form a subshift of finite type, and the canonical block weights appearing in (3.6) are given by the Parry measure of the associated finite automaton. Even under additional restrictions such as  $\mathfrak{a} = 1$  (so that the greedy digits take values in  $\{0, 1\}$ ), the marginal distribution of the digit at height  $n$  is described in terms of left and right eigenvectors of the adjacency matrix of this automaton and has no simple closed form in general. Consequently, the factors  $r_n(t)$  inherit this dependence on the underlying automaton, and explicit expressions for  $r_n(t)$  and  $\Phi(t)$  are typically available only in very special families of bases  $(G_n)$  (for instance in the Fibonacci case treated separately in the next subsection). In the general LRB framework one should therefore not expect explicit formulas, but rather the qualitative description of the limit law provided by Theorem 3.1.

## 5. LRB OF ORDER 2

In this section we consider linear recurrence bases of order 2: we specialize Theorem 3.1, record the block recurrences for  $H_n$  and  $r_k$ , and rewrite the canonical series in this setting.

We start from a second-order linear recurrence with integer parameters  $a, b \geq 1$ . Set

$$G_0 = 1, \quad G_1 = a + 1, \quad G_{n+2} = aG_{n+1} + bG_n \quad (n \geq 0),$$

and let  $G = (G_n)_{n \geq 0}$  denote the resulting sequence.

The companion matrix is

$$A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix},$$

with characteristic polynomial  $x^2 - ax - b$  and real roots  $\alpha$  and  $\lambda_2$  satisfying  $\alpha > \lambda_2$ . Explicitly,

$$\alpha = \frac{a + \sqrt{a^2 + 4b}}{2} > 0, \quad \lambda_2 = \frac{a - \sqrt{a^2 + 4b}}{2}.$$

A short computation shows that

$$G_n = \kappa \alpha^n + (1 - \kappa) \lambda_2^n, \quad \kappa := \frac{a + 2 + \sqrt{a^2 + 4b}}{2\sqrt{a^2 + 4b}},$$

so that  $G_n/\alpha^n \rightarrow \kappa$  as  $n \rightarrow \infty$ .

Since  $a, b \geq 1$ , the matrix  $A$  has strictly positive entries in  $A^2$ , hence is primitive. Moreover, for integer parameters  $a, b \geq 1$  one has

$$|\lambda_2| < 1 \iff b \leq a,$$

so that  $\alpha$  is a Pisot–Vijayaraghavan number precisely when  $b \leq a$ . In this range the coefficients of the recurrence are positive and nonincreasing ( $a \geq b \geq 1$ ), so the discussion following Definition 2.1 applies: greedy  $G$ -expansions are well defined and unique, and  $G_n/\alpha^n \rightarrow \kappa > 0$ . Thus, for the second-order recurrence above, the sequence  $G$  is an LRB in the sense of Definition 2.1 if and only if  $b \leq a$ .

From now on in this section we assume  $b \leq a$ , so that  $G$  is an LRB of order 2 with companion matrix  $A$ , Perron root  $\alpha > 1$ , and  $\kappa = \lim_{n \rightarrow \infty} G_n/\alpha^n$ .

For a  $G$ -additive function  $f$ , we keep the notation

$$H_n(t) := \sum_{m < G_n} e^{itf(m)}, \quad r_k(t) := \frac{H_k(t)}{H_{k-1}(t)} \quad (k \geq 1).$$

Writing

$$\sigma_{m,0} := \sum_{j < a} g_t(jG_m), \quad \sigma_{m,1} := \sum_{j < b} g_t(jG_{m-1}), \quad g_t(n) := e^{itf(n)},$$

the  $d$ -step recursion (3.3) specializes to

$$(5.1) \quad H_{n+2}(t) = \sigma_{n+1,0} H_{n+1}(t) + g_t(aG_{n+1}) \sigma_{n+1,1} H_n(t).$$

In particular,

$$(5.2) \quad r_k(t) = \sigma_{k-1,0} + \frac{g_t(aG_{k-1}) \sigma_{k-1,1}}{r_{k-1}(t)} \quad (k \geq 2).$$

With  $a_0 = a$  and  $a_1 = b$ , the first canonical series (3.1) reduces to

$$(S_1) \quad \sum_{n \geq 0} \left( \sum_{k < a} f(kG_{n+2}) + \frac{1}{\alpha} \sum_{k < b} (f(kG_{n+1}) + f(aG_{n+2})) \right).$$

After the index shift  $n \mapsto n - 1$  (which does not affect convergence), we obtain the following corollary.

**Corollary 5.1.** *Let  $f$  be a real-valued  $G$ -additive function for an LRB of order 2 as above. Then  $f$  has a distribution function if and only if the two following series converge:*

$$\sum_{n \geq 0} \left( \sum_{k < a} f(kG_{n+1}) + \frac{1}{\alpha} \sum_{k < b} (f(kG_n) + f(aG_{n+1})) \right),$$

$$\sum_{n \geq 0} \sum_{1 \leq k \leq a} f(kG_n)^2.$$

In this case the limiting characteristic function exists for all  $t \in \mathbb{R}$  and satisfies

$$\Phi(t) = \frac{1}{\kappa} \prod_{j \geq 1} \frac{r_j(t)}{\alpha}.$$

The weight  $1/\alpha$  reflects the one-step recursion (5.2) and the cancellation on the dominant eigendirection. In the one-step majorization used in the proof of Lemma 3.5, the comparison constant is  $L = 1 - \frac{a}{\alpha} \in (0, 1)$ . Consequently the generating-function kernel reduces to

$$T(x) = 1 - Lx, \quad T(x)^{-1} = (1 - Lx)^{-1},$$

which makes the control of  $\sum_k |\varepsilon_k|^2$  particularly transparent when  $d = 2$ .

**Corollary 5.2.** *Let  $f$  be a real-valued  $G$ -additive function for an LRB of order 2 as above.*

(a) *If  $a = b$ , then  $f$  has a distribution function if and only if*

$$\sum_{n \geq 0} \left( (\alpha + 1) \sum_{k < a} f(kG_n) + a f(aG_n) \right) < \infty, \quad \sum_{n \geq 0} \sum_{1 \leq k \leq a} f(kG_n)^2 < \infty.$$

(b) *If  $b = 1$ , then  $f$  has a distribution function if and only if*

$$\sum_{n \geq 0} \left( \alpha \sum_{k < a} f(kG_n) + f(aG_n) \right) < \infty, \quad \sum_{n \geq 0} \sum_{1 \leq k \leq a} f(kG_n)^2 < \infty.$$

When  $a = b = 1$  (Zeckendorf's expansion), Corollary 5.1 specializes to [9, Theorem 7].

We now record a simple permanence property of the Erdős–Wintner criterion in the LRB setting. In particular, we show that the class of  $G$ -additive functions admitting a distribution function is stable under addition.

## 6. STABILITY UNDER ADDITION AND SMALL PERTURBATIONS

We now record a simple permanence property of the Erdős–Wintner criterion in the LRB setting: the class of  $G$ -additive functions admitting a distribution function is stable under addition and under small digitwise perturbations.

Throughout this section  $G$  is a fixed LRB and  $f, g : \mathbb{N} \rightarrow \mathbb{R}$  are  $G$ -additive functions.

**Proposition 6.1** (Stability under addition). *Assume that both  $f$  and  $g$  admit distribution functions. Then  $f + g$  admits a distribution function as well.*

The verification is a straightforward bookkeeping computation on the canonical series, but we include it for completeness.

*Proof.* If  $f$  and  $g$  admit distribution functions, then Theorem 3.1 applies to each of them and shows that the canonical series (S1) and (S2) converge for both  $f$  and  $g$ . Since the first canonical series is linear in  $f$  and the second one is quadratic, the corresponding series for  $f + g$  can be written as

$$(S1)[f + g] = (S1)[f] + (S1)[g],$$

$$(S2)[f + g] = (S2)[f] + (S2)[g] + 2 \sum_{n \geq 0} \sum_{1 \leq c \leq a} f(cG_n) g(cG_n).$$

The last series converges absolutely by Cauchy–Schwarz, since

$$\sum_{n \geq 0} \sum_{1 \leq c \leq a} |f(cG_n) g(cG_n)| \leq \left( \sum_{n,c} f(cG_n)^2 \right)^{1/2} \left( \sum_{n,c} g(cG_n)^2 \right)^{1/2} < \infty,$$

because the two series (S2) for  $f$  and  $g$  converge. Hence the canonical series for  $f + g$  converge, so Theorem 3.1 applies and  $f + g$  has a distribution function.  $\square$

**Remark 6.2.** *In general we do not attempt to identify the limiting law of  $f + g$ . Even if  $f$  and  $g$  admit distribution functions with limiting laws  $\mu_f$  and  $\mu_g$ , the limiting law of  $f + g$  need not be the convolution  $\mu_f * \mu_g$ , since  $f(n)$  and  $g(n)$  are evaluated on the same integer  $n$  and are typically not (asymptotically) independent.*

As an immediate consequence, adding a “small”  $G$ -additive function in the digitwise sense preserves the existence of a distribution function.

**Corollary 6.3** (Small digitwise perturbations). *Let  $f$  be a  $G$ -additive function that admits a distribution function, and let  $g$  be a  $G$ -additive function such that*

$$\sum_{n \geq 0} \sum_{1 \leq c \leq a} g(cG_n)^2 < \infty, \quad \sum_{n \geq 0} \sum_{1 \leq c \leq a} |g(cG_n)| < \infty.$$

*Then both canonical series (S1) and (S2) converge for  $g$ , and  $f + g$  admits a distribution function. In particular, this applies to the digit functions in items (1) and (2) of Section 4.*

*Proof.* By definition, the second canonical series (S2) for  $g$  is

$$(S2)[g] = \sum_{n \geq 0} \sum_{1 \leq c \leq a} g(cG_n)^2,$$

which converges by the first hypothesis.

For the first canonical series, fix  $t \in \mathbb{R}$ . The  $n$ th layer of (S1) can be written in the form

$$A_n(t) = \sum_{1 \leq c \leq a} \alpha_{n,c}(t) g(cG_n),$$

where the coefficients  $\alpha_{n,c}(t)$  encode the local block structure at height  $n$  and are uniformly bounded in  $n$  and  $c$  (for each fixed  $t$ ), say  $|\alpha_{n,c}(t)| \leq C(t)$ . Hence

$$|A_n(t)| \leq C(t) \sum_{1 \leq c \leq a} |g(cG_n)|.$$

By the second hypothesis, the series

$$\sum_{n \geq 0} \sum_{1 \leq c \leq a} |g(cG_n)|$$

converges, so  $\sum_n |A_n(t)| < \infty$  and (S1) converges absolutely for every fixed  $t$ .

Thus both canonical series (S1) and (S2) converge for  $g$ , so  $g$  admits a distribution function by Theorem 3.1. Applying Proposition 6.1 to  $f$  and  $g$  then shows that  $f + g$  has a distribution function.  $\square$

## 7. OUTLOOK: OSTROWSKI EXPANSIONS AND $\beta$ -PV SYSTEMS

In this section we briefly indicate how the two-series criterion is expected to extend beyond stationary linear recurrence bases, to numeration systems arising from primitive automata such as Ostrowski expansions and Parry  $\beta$ -PV systems. We only record the formal framework and a conditional analogue of Theorem 3.1; no proofs are included.

**Local place values and digit bounds.** There exists a canonical sequence of place values  $(W_n)_{n \geq 0}$  and digit bounds  $(B_n)_{n \geq 0}$  such that every integer  $m$  has a greedy  $\mathcal{S}$ -expansion

$$m = \sum_{n \geq 0} d_n W_n \quad (0 \leq d_n \leq B_n),$$

with local carry rules prescribed by the automaton. In the *Ostrowski* case one may take  $W_n = q_n$  (denominators of convergents) and  $B_n = a_{n+1}$ . In the  $\beta$ -*PV Parry* case,  $W_n$  are the canonical Parry weights arising from the  $\beta$ -shift automaton (so that  $W_n \asymp \beta^n$  and greedy admissibility holds); here  $B_n$  is the top admissible digit at level  $n$  given by the automaton. In both settings we have a dominant growth constant  $\Lambda > 1$  such that

$$\frac{W_n}{\Lambda^n} \longrightarrow \kappa_{\mathcal{S}} \in (0, \infty).$$

**$\mathcal{S}$ –additive functions.** A function  $f : \mathbb{N} \rightarrow \mathbb{R}$  is called  $\mathcal{S}$ –additive if, for the greedy expansion  $m = \sum_n d_n W_n$ , one has

$$f(m) = \sum_{n \geq 0} f(d_n W_n),$$

with the convention  $f(0) = 0$ . Equivalently, the collection  $\{f(kW_n) : 0 \leq k \leq B_n, n \geq 0\}$  determines  $f$  (carries are fixed by greediness).

**Natural cutoffs.** For distributional statements we use the truncation at a place value:  $F_{W_N}$  denotes the distribution function of  $f(m)$  for  $m < W_N$ . When comparing to a general cutoff  $X$ , we pick  $N$  by  $W_N \leq X < W_{N+1}$ ; all bounds are invariant under replacements with  $X \asymp W_N$ .

**Two canonical series.** There exists an integer  $d_0 \geq 1$  (local carry memory) and coefficients  $(\gamma_j)_{j < d_0}$ , depending only on  $\mathcal{S}$  and on the Perron normalization, such that we define

$$\begin{aligned} (\text{S1}_{\mathcal{S}}) \quad & \sum_{n \geq d_0} \sum_{j < d_0} \gamma_j \sum_{1 \leq k \leq B_{n-j}} f(kW_{n-j}), \\ (\text{S1}_{\mathcal{S}}, \text{S2}_{\mathcal{S}}) \quad & \sum_{n \geq 0} \sum_{1 \leq k \leq B_n} f(kW_n)^2. \end{aligned}$$

**Remark 7.1** (Calibration). *In the LRB case one has  $W_n = G_n$ ,  $B_n = a_0$ ,  $d_0 = d$ , and  $\gamma_j = \Lambda^{-j}$  after first-order normalization on the Perron direction;  $(\text{S1}_{\mathcal{S}}, \text{S2}_{\mathcal{S}})$  then recovers the two canonical series (S1) and (S2) (with the  $j$ -offset induced by local carries).*

**Dynamical hypotheses.** We assume the following standard conditions for the automaton/subshift and its incidence/transfer operator:

- (H1) *Primitivity & unique ergodicity:* the automaton/incidence cocycle is primitive (i.e. some power has all positive entries) and uniquely ergodic.
- (H2) *Spectral gap:* the transfer/incidence operator on a suitable Banach space has a spectral gap.
- (H3) *Greedy admissibility:* greedy expansions exist and are unique (Parry admissibility along the subshift).
- (H4) *Simple top direction:* the leading Lyapunov/Perron exponent is simple.

**On the hypotheses.** For Ostrowski systems with bounded partial quotients, the substitution/automaton is primitive and uniquely ergodic (see Queffélec [18]). For Parry  $\beta$ –systems with PV  $\beta$ , the associated  $\beta$ –shift is sofic (simple Parry gives an SFT); the Ruelle–Perron–Frobenius operator has a spectral gap on Hölder observables (see Ruelle [19] and Baladi [1]). Greedy admissibility is classical for Parry automata [17]. Simplicity of the top Lyapunov/Perron direction follows from primitivity.

Define  $H_n(t) := \sum_{m < W_n} e^{itf(m)}$  and block ratios  $r_j(t) := H_j(t)/H_{j-1}(t)$  (with the natural notion of “level” along the cocycle). Under (H1)–(H4), the Erdős–Wintner mechanism is expected to carry over as follows.

**Theorem 7.2** (Conditional Erdős–Wintner for Ostrowski and  $\beta$ –PV systems). *Assume (H1)–(H4). For any real  $\mathcal{S}$ –additive  $f$ , the following are equivalent:*

- (1)  $f$  admits a distribution function along the cutoffs  $W_N$ ;
- (2) the two series  $(\text{S1}_{\mathcal{S}})$  and  $(\text{S2}_{\mathcal{S}})$  in  $(\text{S1}_{\mathcal{S}}, \text{S2}_{\mathcal{S}})$  converge.

*In this case, with  $\kappa_{\mathcal{S}} := \lim_{n \rightarrow \infty} W_n/\Lambda^n \in (0, \infty)$ , the limiting characteristic function admits the infinite product*

$$\Phi(t) = \frac{1}{\kappa_{\mathcal{S}}} \prod_{j \geq 1} \frac{r_j(t)}{\Lambda},$$

*locally uniformly in  $t$ , and  $\Phi$  is continuous at the origin.*

**Remark 7.3** (Limitation: nonstationary perturbations). *Even in order  $d = 2$ , simple necessary criteria fail in genuinely nonstationary settings. Fix an LRB  $G_{n+2} = a_0 G_{n+1} + a_1 G_n$  with Perron root  $\alpha > 1$ , and define*

$$Z_{n+2} = \left(a_0 - \frac{1}{n}\right) Z_{n+1} + \left(a_1 + \frac{G_{n+1}}{n G_n}\right) Z_n, \quad Z_0 = G_0, \quad Z_1 = G_1.$$

A direct check gives

$$Z_{n+2} = \left( a_0 - \frac{1}{n} \right) G_{n+1} + \left( a_1 + \frac{G_{n+1}}{n G_n} \right) G_n = a_0 G_{n+1} + a_1 G_n = G_{n+2},$$

hence  $Z_n \equiv G_n$  and  $Z_n/G_n \equiv 1$ , while

$$\sum_{n \geq 1} \left| \left( a_0 - \frac{1}{n} \right) - a_0 \right| = +\infty, \quad \sum_{n \geq 1} \left| \left( a_1 + \frac{G_{n+1}}{n G_n} \right) - a_1 \right| = +\infty \quad (\text{since } G_{n+1}/G_n \rightarrow \alpha).$$

This is the perturbative phenomenon emphasized in [3, Remark after Lemma 2]: convergence may persist even when the coefficient drifts are not absolutely summable. It lies beyond the stationary hypotheses (H1)–(H4); under those, the conditional result above applies.

#### APPENDIX. PROOFS OF AUXILIARY LEMMAS

**A. Proof of Lemma 3.3.** Let  $T_0 > 0$  be arbitrary and work throughout with  $|t| \leq T_0$ . Recall that

$$g_t(n) := e^{itf(n)}, \quad H_n(t) := \sum_{m < G_n} g_t(m), \quad r_k(t) := \frac{H_k(t)}{H_{k-1}(t)} \quad (k \geq 1),$$

and set

$$\varepsilon_k(t) := r_k(t) - \alpha.$$

Define

$$u_k(t) := \alpha^d \sum_{\ell < d} \frac{g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) - a_\ell}{\alpha^{\ell+1}}.$$

Using the definitions of  $\Pi_{k,\ell}(t)$  and  $\widehat{\Pi}_{k,\ell}(t)$  from the main text (see (3.7)), we may rewrite the recursion for the ratios as

$$(A.1) \quad \varepsilon_k(t) = \frac{1}{\Pi_{k,0}(t)} \left( u_k(t) + \sum_{\ell < d} g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) \widehat{\Pi}_{k,\ell}(t) - \alpha \widehat{\Pi}_{k,0}(t) \right).$$

Assume that  $(H_f)$  holds, namely

$$f(c G_k) \longrightarrow 0 \quad (k \rightarrow \infty)$$

for every  $1 \leq c \leq \alpha$ . Fix  $\eta > 0$ . By  $(H_f)$  there exists  $k_0$  such that

$$|f(c G_m)| \leq \eta \quad \text{for all } m \geq k_0, \quad 1 \leq c \leq \alpha.$$

For  $|t| \leq T_0$  and any real  $y$  we have

$$|e^{ity} - 1| \leq |t| |y| \leq T_0 |y|.$$

Hence, for  $1 \leq c \leq \alpha$ ,

$$|g_t(c G_m) - 1| = |e^{itf(c G_m)} - 1| \leq T_0 |f(c G_m)|.$$

Using  $G$ -additivity and  $\vartheta_{k-1,\ell} = \sum_{j < \ell} a_j G_{k-1-j}$ , we obtain

$$|g_t(\vartheta_{k-1,\ell}) - 1| \leq T_0 |f(\vartheta_{k-1,\ell})| \leq T_0 \sum_{j < \ell} |f(a_j G_{k-1-j})| \leq T_0 \sum_{r < d} \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-r})|.$$

Moreover,

$$|\sigma_{k-1,\ell}(t) - a_\ell| \leq \sum_{1 \leq h < a_\ell} |g_t(h G_{k-1-\ell}) - 1| \leq T_0 \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-\ell})|.$$

Since  $|\sigma_{k-1,\ell}(t)| \leq a_\ell \leq \alpha$ , we deduce

$$\begin{aligned} |g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) - a_\ell| &\leq |g_t(\vartheta_{k-1,\ell}) - 1| |\sigma_{k-1,\ell}(t)| + |\sigma_{k-1,\ell}(t) - a_\ell| \\ &\leq T_0 (a_0 + 1) \sum_{r < d} \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-r})|. \end{aligned}$$

The right-hand side tends to 0 as  $k \rightarrow \infty$ , independently of  $t$  with  $|t| \leq T_0$ , so

$$(A.2) \quad u_k(t) \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{uniformly for } |t| \leq T_0.$$

From (A.1),

$$|\varepsilon_k(t)| \leq \frac{1}{|\Pi_{k,0}(t)|} \left( |u_k(t)| + \sum_{1 \leq \ell < d} a_\ell |\widehat{\Pi}_{k,\ell}(t)| + |(\alpha - \sigma_{k-1,0}(t))\widehat{\Pi}_{k,0}(t)| \right).$$

For any  $\delta_1 > 0$ ,  $0 < \delta_2 \leq \alpha^{d-1} - 1$ , there exists  $k_1 \geq k_0$  such that for  $k \geq k_1$ ,

$$|\alpha - \sigma_{k-1,0}(t)| \leq (\alpha - a_0 + \delta_1), \quad |\Pi_{k,0}(t)| \geq \alpha^{d-1} - \delta_2 \geq 1,$$

and, for  $d \geq 3$  and  $1 \leq \ell \leq d-2$ ,

$$|\widehat{\Pi}_{k,0}(t)| \leq \alpha^{d-2} \sum_{1 \leq j < d} |\varepsilon_{k-j}(t)|, \quad |\widehat{\Pi}_{k,\ell}(t)| \leq \frac{1}{2(d-1)} \alpha^{d-\ell-2} \sum_{\ell < j < d} |\varepsilon_{k-j}(t)|.$$

Combining these estimates and letting  $\delta_1, \delta_2 \rightarrow 0$  yields, for  $d \geq 3$ ,

$$|\varepsilon_k(t)| \leq |u_k(t)| + \frac{1}{2(d-1)} \left( 2 - \frac{2a_0}{\alpha} - \frac{a_{d-1}}{\alpha^d} \right) \sum_{1 \leq j < d} |\varepsilon_{k-j}(t)|,$$

and, for  $d = 2$ ,

$$|\varepsilon_k(t)| \leq |u_k(t)| + \left( 1 - \frac{a_0}{\alpha} \right) |\varepsilon_{k-1}(t)|.$$

Using the Perron identity

$$\sum_{\ell < d} \frac{a_\ell}{\alpha^{\ell+1}} = 1$$

we get, for  $d \geq 3$ ,

$$2 - \frac{2a_0}{\alpha} - \frac{a_{d-1}}{\alpha^d} = 2 \sum_{\ell=1}^{d-2} \frac{a_\ell}{\alpha^{\ell+1}} + \frac{a_{d-1}}{\alpha^d} > 0,$$

since at least one among  $a_1, \dots, a_{d-1}$  is positive (the recurrence has order  $\geq 2$ ). Therefore

$$0 < L := \frac{1}{2(d-1)} \left( 2 - \frac{2a_0}{\alpha} - \frac{a_{d-1}}{\alpha^d} \right) < \frac{1}{d-1}.$$

For  $d = 2$  one has  $L = 1 - a_0/\alpha \in (0, 1)$ , which proves the one-step bound

$$(A.3) \quad |\varepsilon_k(t)| \leq |u_k(t)| + L \sum_{1 \leq j < d} |\varepsilon_{k-j}(t)| \quad (k \geq k_1, |t| \leq T_0).$$

Define

$$E_k := \sup_{m \geq k} \sup_{|t| \leq T_0} |\varepsilon_m(t)| \quad (k \geq 0).$$

From (A.3) we infer that, for every  $m \geq k$  with  $k \geq k_1 + d - 1$  and every  $|t| \leq T_0$ ,

$$|\varepsilon_m(t)| \leq |u_m(t)| + L \sum_{1 \leq j < d} |\varepsilon_{m-j}(t)|.$$

If  $m \geq k$  and  $1 \leq j < d$ , then  $m - j \geq k - d + 1$ , hence  $|\varepsilon_{m-j}(t)| \leq E_{k-d+1}$  and

$$|\varepsilon_m(t)| \leq |u_m(t)| + L(d-1) E_{k-d+1}.$$

Taking the supremum over  $m \geq k$  and  $|t| \leq T_0$  gives

$$E_k \leq a_k + L(d-1) E_{k-d+1}, \quad a_k := \sup_{m \geq k, |t| \leq T_0} |u_m(t)|, \quad (k \geq k_1 + d - 1).$$

Set  $L_1 := L(d-1) \in (0, 1)$ ; the recurrence becomes

$$E_k \leq a_k + L_1 E_{k-d+1}, \quad (k \geq k_1 + d - 1).$$

By (A.2) we have  $a_k \rightarrow 0$  as  $k \rightarrow \infty$ . Let  $S := \limsup_{k \rightarrow \infty} E_k \geq 0$ . Passing to the  $\limsup$  in the inequality  $E_k \leq a_k + L_1 E_{k-d+1}$  yields

$$S \leq \limsup_{k \rightarrow \infty} a_k + L_1 \limsup_{k \rightarrow \infty} E_{k-d+1} = 0 + L_1 S.$$

Since  $L_1 \in (0, 1)$ , this forces  $(1 - L_1)S \leq 0$  and hence  $S = 0$ . Thus  $E_k \rightarrow 0$  as  $k \rightarrow \infty$ , i.e.

$$\varepsilon_k(t) = r_k(t) - \alpha \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{uniformly for } |t| \leq T_0.$$

Since  $T_0$  was arbitrary, this proves that  $r_k(t) \rightarrow \alpha$  locally uniformly in  $t$  and completes the proof of Lemma 3.3.

**B. Proof of Lemma 3.4.** For  $M \geq 1$  write  $S(M) := \sum_{n < M} g(n)$ . Let

$$N = \sum_{r \leq Q} e_r(N) G_r$$

be the greedy  $G$ -expansion (with  $Q = Q(N)$  and  $e_Q(N) > 0$ ). By greedy admissibility (Parry), for every integer  $n < N$  there exist unique  $q \in \{0, \dots, Q\}$  and unique integers  $j, t$  such that

$$n = \sum_{q < r \leq Q} e_r(N) G_r + j G_q + t, \quad 0 \leq j < e_q(N), \quad 0 \leq t < G_q.$$

Equivalently,

$$e_r(n) = e_r(N) \quad (r > q), \quad e_q(n) = j, \quad \text{and } (e_{q-1}(n), \dots, e_0(n)) \text{ encodes } t.$$

In particular,

$$\#\{(q, j, t) : q \leq Q, 0 \leq j < e_q(N), 0 \leq t < G_q\} = \sum_{q \leq Q} e_q(N) G_q = N,$$

so each  $n < N$  arises exactly once in this parametrization. Hence, by  $G$ -multiplicativity (and the convention  $g(0) = 1$ ),

$$g(n) = \left( \prod_{q < r \leq Q} g(e_r(N) G_r) \right) g(jG_q) g(t).$$

Summing first over  $t$  gives  $S(G_q) = \sum_{t < G_q} g(t)$ , and we obtain

$$S(N) = \sum_{q \leq Q} \left( \sum_{j < e_q(N)} \prod_{q < r \leq Q} g(e_r(N) G_r) g(jG_q) \right) S(G_q).$$

Dividing by  $N$ , we have

$$\frac{S(N)}{N} = \sum_{q \leq Q} \omega_q(N) \frac{S(G_q)}{G_q},$$

where the (generally complex) weights  $\omega_q(N)$  are given by

$$\omega_q(N) := \left( \sum_{j < e_q(N)} \prod_{q < r \leq Q} g(e_r(N) G_r) g(jG_q) \right) \frac{G_q}{N}.$$

Using  $|g| \leq 1$ , we have

$$|\omega_q(N)| \leq e_q(N) \frac{G_q}{N},$$

hence

$$\sum_{q \leq Q} |\omega_q(N)| \leq \frac{1}{N} \sum_{q \leq Q} e_q(N) G_q = 1.$$

We write

$$\frac{S(N)}{N} - \ell = \sum_{q \leq Q} \omega_q(N) \left( \frac{S(G_q)}{G_q} - \ell \right),$$

and therefore

$$\left| \frac{S(N)}{N} - \ell \right| \leq \sum_{q \leq Q} |\omega_q(N)| \left| \frac{S(G_q)}{G_q} - \ell \right|.$$

Fix  $\varepsilon > 0$  and choose  $K$  such that

$$\left| \frac{S(G_q)}{G_q} - \ell \right| \leq \varepsilon \quad \text{for all } q \geq K.$$

Splitting the sum at  $K$ , we obtain

$$\left| \frac{S(N)}{N} - \ell \right| \leq \sum_{q < K} |\omega_q(N)| \left| \frac{S(G_q)}{G_q} - \ell \right| + \varepsilon \sum_{q \geq K} |\omega_q(N)|.$$

For the first term, we bound by the maximum over  $q < K$  and use the estimate

$$|\omega_q(N)| \leq e_q(N) \frac{G_q}{N} \leq \alpha \frac{G_q}{N}.$$

Since  $K$  is fixed, we obtain

$$\sum_{q < K} |\omega_q(N)| \left| \frac{S(G_q)}{G_q} - \ell \right| \leq \left( \max_{q < K} \left| \frac{S(G_q)}{G_q} - \ell \right| \right) \frac{a_0}{N} \sum_{q < K} G_q \xrightarrow[N \rightarrow \infty]{} 0.$$

For the second term,  $\sum_{q \geq K} |\omega_q(N)| \leq 1$ , so

$$\limsup_{N \rightarrow \infty} \left| \frac{S(N)}{N} - \ell \right| \leq \varepsilon.$$

As  $\varepsilon > 0$  is arbitrary, we conclude that  $\lim_{N \rightarrow \infty} S(N)/N = \ell$ .

**C. Proof of Lemma 3.5.** Assume throughout that  $f$  is  $G$ –additive and satisfies  $(H_f)$ . We prove the three assertions of Lemma 3.5 in order.

Assertion (1) is proved in Subsection C.1: assuming the second canonical series (S2) converges, we obtain  $\sum_k |u_k(t)|^2 < \infty$  for each fixed  $t$  (take any  $t_0 > |t|$  in the argument).

Assertion (2) is proved at the beginning of Subsection C.3 by the domination and generating–function argument.

The auxiliary identity (C.1) linking  $u_k(t)$  and the block errors  $\varepsilon_k(t)$  is derived in Subsection C.2 and is used only in the proof of assertion (3).

Finally, under (S2), assertions (1)–(2) yield  $\sum_k |u_k(t)|^2 < \infty$  and  $\sum_k |\varepsilon_k(t)|^2 < \infty$ , and we conclude (3) at the end of Subsection C.3 using the relation (C.1).

### C.1. An upper bound and the $L^2$ criterion.

Fix  $t_0 > 0$ . In this subsection we show that the convergence of the second canonical series

$$\sum_{n \geq 0} \sum_{1 \leq c \leq \alpha} f(c G_n)^2 < \infty$$

implies that, for every fixed  $t$  with  $|t| \leq t_0$ , we have

$$\sum_{k \geq 0} |u_k(t)|^2 < \infty.$$

Recall that

$$u_k(t) = \alpha^d \sum_{\ell < d} \frac{g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) - a_\ell}{\alpha^{\ell+1}}.$$

For each  $\ell$  we decompose

$$g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) - a_\ell = (g_t(\vartheta_{k-1,\ell}) - 1) \sigma_{k-1,\ell}(t) + (\sigma_{k-1,\ell}(t) - a_\ell).$$

We first bound the two contributions separately. Since  $|g_t(\cdot)| = 1$  and  $f(0) = 0$  (by  $G$ –additivity),

$$|\sigma_{k-1,\ell}(t) - a_\ell| = \left| \sum_{j < a_\ell} (e^{itf(jG_{k-1-\ell})} - 1) \right| \leq \sum_{j < a_\ell} |e^{itf(jG_{k-1-\ell})} - 1|.$$

Using  $|e^{ix} - 1| \leq |x|$  for all real  $x$  and  $|t| \leq t_0$ , we obtain

$$|\sigma_{k-1,\ell}(t) - a_\ell| \leq |t| \sum_{j < a_\ell} |f(jG_{k-1-\ell})| \ll_{t_0} \sum_{1 \leq j \leq \alpha} |f(jG_{k-1-\ell})|.$$

Next, by definition of  $\vartheta_{q,\ell}$  we have

$$\vartheta_{k-1,\ell} = \sum_{r < \ell} a_r G_{k-1-r}.$$

By  $G$ –additivity it follows that

$$f(\vartheta_{k-1,\ell}) = \sum_{r < \ell} f(a_r G_{k-1-r}),$$

and hence, using  $a_r \leq \alpha$ ,

$$|f(\vartheta_{k-1,\ell})| \leq \sum_{r < \ell} |f(a_r G_{k-1-r})| \leq \sum_{r < d} \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-r})|.$$

Thus, for  $|t| \leq t_0$ ,

$$|g_t(\vartheta_{k-1,\ell}) - 1| = |\mathrm{e}^{itf(\vartheta_{k-1,\ell})} - 1| \leq |t| |f(\vartheta_{k-1,\ell})| \ll_{t_0} \sum_{r < d} \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-r})|.$$

Since  $|\sigma_{k-1,\ell}(t)| \leq a_\ell \leq \alpha$ , we deduce

$$|(g_t(\vartheta_{k-1,\ell}) - 1) \sigma_{k-1,\ell}(t)| \ll_{t_0} \sum_{r < d} \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-r})|.$$

Combining the two bounds, we find that, for every  $|t| \leq t_0$ ,

$$|g_t(\vartheta_{k-1,\ell}) \sigma_{k-1,\ell}(t) - a_\ell| \ll_{t_0} \sum_{r < d} \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-r})|.$$

Since the factor  $\alpha^d/\alpha^{\ell+1}$  and the number of indices  $0 \leq \ell < d$  are fixed and depend only on the digit system, we obtain

$$|u_k(t)| \ll_{t_0} \sum_{r < d} \sum_{1 \leq c \leq \alpha} |f(c G_{k-1-r})| \quad (|t| \leq t_0).$$

We now square and sum over  $k$ . By Cauchy–Schwarz on the finite index set  $\{0, \dots, d-1\} \times \{1, \dots, \alpha\}$ , we have

$$|u_k(t)|^2 \ll_{t_0} \sum_{r < d} \sum_{1 \leq c \leq \alpha} f(c G_{k-1-r})^2,$$

with an implied constant depending only on  $d$  and  $\alpha$ . Summing over  $k \geq 0$  and performing the change of variables  $n = k - 1 - r$  (which affects only finitely many indices) shows that

$$\sum_{k \geq 0} |u_k(t)|^2 \ll_{t_0} \sum_{n \geq 0} \sum_{1 \leq c \leq \alpha} f(c G_n)^2, \quad (|t| \leq t_0).$$

In particular, if the second canonical series converges, then for every fixed  $t$  with  $|t| \leq t_0$  we have

$$\sum_{k \geq 0} |u_k(t)|^2 < \infty.$$

This proves assertion (1) of Lemma 3.5.

### C.2. Derivation of the $u_k$ -relation.

We need to link  $u_k$  and  $\sum_{j < d} \varepsilon_{k-j}$  in order to prove the two last assertions of the lemma. For  $k \geq k_0$  and  $j, \ell \in \{0, \dots, d-1\}$ , define

$$\begin{aligned} \tau_{k,j}(t) &:= \sum_{m < j} \frac{g_t(\vartheta_{k-1,m}) \sigma_{k-1,m}(t)}{\alpha^m}, \\ \widehat{\Pi}_{k,\ell}^{(2)}(t) &:= \sum_{2 \leq m < d-\ell} \alpha^{d-\ell-m-1} \sum_{\{j_1, \dots, j_m\} \subset \{\ell+1, \dots, d-1\}} \varepsilon_{k-j_1}(t) \cdots \varepsilon_{k-j_m}(t), \end{aligned}$$

Recall that  $\vartheta_{q,\ell} = \sum_{r < \ell} a_r G_{q-r}$ . By (H<sub>f</sub>), for each fixed  $r$  we have  $f(a_r G_{k-1-r}) \rightarrow 0$  as  $k \rightarrow \infty$  (note that  $a_r \leq \alpha$ ), hence for each fixed  $m$ ,

$$f(\vartheta_{k-1,m}) = \sum_{r < m} f(a_r G_{k-1-r}) \xrightarrow[k \rightarrow \infty]{} 0, \quad \text{so} \quad g_t(\vartheta_{k-1,m}) \xrightarrow[k \rightarrow \infty]{} 1.$$

Moreover, since

$$\sigma_{k-1,m}(t) = \sum_{r < a_m} g_t(r G_{k-1-m}) = \sum_{r < a_m} \exp(itf(r G_{k-1-m})),$$

and (H<sub>f</sub>) gives  $f(r G_{k-1-m}) \rightarrow 0$  for each fixed  $0 \leq r < a_m$ , we have  $\sigma_{k-1,m}(t) \rightarrow a_m$ . Therefore, for each fixed  $j$  and  $t$ ,

$$\tau_{k,j}(t) \xrightarrow[k \rightarrow \infty]{} \sum_{m < j} \frac{a_m}{\alpha^m}.$$

Moreover, every monomial in  $\widehat{\Pi}_{k,\ell}^{(2)}(t)$  involves at least two factors  $\varepsilon_{k-j}(t)$ , so  $R_k(t)$  is a finite linear combination of products of at least two such terms. In particular, whenever  $\varepsilon_k(t) \rightarrow 0$ , we also have  $R_k(t) \rightarrow 0$ . From (3.7) we get

$$u_k(t) = \Pi_{k,0}(t) \varepsilon_k(t) + \alpha \widehat{\Pi}_{k,0}(t) - \sum_{m < d} g_t(\vartheta_{k-1,m}) \sigma_{k-1,m}(t) \widehat{\Pi}_{k,m}(t).$$

We define the error term

$$R_k(t) := (\alpha + \varepsilon_k(t)) \widehat{\Pi}_{k,0}^{(2)}(t) + \alpha^{d-2} \left( \sum_{1 \leq j < d} \varepsilon_{k-j}(t) \right) \varepsilon_k(t) - \sum_{m < d} g_t(\vartheta_{k-1,m}) \sigma_{k-1,m}(t) \widehat{\Pi}_{k,m}^{(2)}(t).$$

Using  $\widehat{\Pi}_{k,d-1}(t) = 0$  and

$$\widehat{\Pi}_{k,\ell}(t) = (\varepsilon_{k-d+1}(t) + \alpha) \cdots (\varepsilon_{k-\ell-1}(t) + \alpha) - \alpha^{d-\ell-1} = \alpha^{d-\ell-2} \sum_{\ell < j < d} \varepsilon_{k-j}(t) + \widehat{\Pi}_{k,\ell}^{(2)}(t),$$

together with  $\Pi_{k,0}(t) = \widehat{\Pi}_{k,0}(t) + \alpha^{d-1}$ , we obtain the useful relation

$$(C.1) \quad u_k(t) = \alpha^{d-2} \sum_{j < d} (\alpha - \tau_{k,j}(t)) \varepsilon_{k-j}(t) + R_k(t).$$

### C.3. Generating functions and the $L^2$ bound.

We use repeatedly the last identity (C.1) and the one-step bound (A.3) proved in Appendix A. Fix  $t \in \mathbb{R}$ . We now prove assertion (2): assume that  $\sum_{k \geq 0} |u_k(t)|^2 < \infty$ , and we show that  $\sum_{k \geq 0} |\varepsilon_k(t)|^2 < \infty$ .

Let  $L$  be the constant of (A.3). Define a nonnegative upper bounding sequence  $(\tilde{\varepsilon}_k)_{k \geq k_1}$  by imposing equality in (A.3) for  $k \geq k_1 + d - 1$  and taking  $\tilde{\varepsilon}_k := |\varepsilon_k(t)|$  for  $k_1 \leq k \leq k_1 + d - 2$ . Then  $|\varepsilon_k(t)| \leq \tilde{\varepsilon}_k$  for all  $k \geq k_1$ . For convenience, set  $\tilde{\varepsilon}_k := 0$  for  $k < k_1$ . Moreover, set<sup>1</sup>

$$E(x) := \sum_{k \geq k_1} \tilde{\varepsilon}_k x^k, \quad U(x) := \sum_{k \geq k_1} |u_k(t)| x^k.$$

For  $k_1 \leq \ell \leq k_1 + d - 2$ , define

$$s_\ell := \tilde{\varepsilon}_\ell - |u_\ell(t)| - L \sum_{1 \leq j < d} \tilde{\varepsilon}_{\ell-j},$$

and

$$S(x) := \sum_{k_1 \leq \ell \leq k_1 + d - 2} s_\ell x^\ell, \quad T(x) := 1 - L \sum_{1 \leq j < d} x^j = 1 - L(x + \cdots + x^{d-1}).$$

By construction and a standard index shift, we obtain

$$E(x) = \frac{U(x) + S(x)}{T(x)}.$$

Write

$$T(x)^{-1} = \sum_{n \geq 0} b_{n,d} x^n, \quad b_{n,d} := 0 \text{ for } n < 0,$$

so that  $b_{n,d} \geq 0$  (indeed,  $T(x)^{-1} = \sum_{m \geq 0} L^m (x + \cdots + x^{d-1})^m$  as a formal series). Extend  $v_k := |u_k(t)|$  by 0 for  $k < k_1$ , and extend  $(\tilde{\varepsilon}_k)$  by 0 for  $k < k_1$ . Extracting coefficients and using the Cauchy product yields, for all  $k \geq k_1$ ,

$$\tilde{\varepsilon}_k = (b * v)_k + w_k, \quad (b * v)_k := \sum_{\ell \geq 0} b_{\ell,d} v_{k-\ell},$$

where  $w_k := \sum_{k_1 \leq \ell \leq k_1 + d - 2} b_{k-\ell,d} s_\ell$  is the boundary contribution (note that  $(s_\ell)$  is finitely supported).

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<sup>1</sup>The author thanks the contributors to Chris Jones's question on MathOverflow (Stack Exchange network) for expressions for  $b_{n,d}$ , and Greg Martin and Alex Ravsky for helpful comments on MathOverflow (Stack Exchange network).

We claim that  $(b_{\ell,d})_{\ell \geq 0} \in \ell^1(\mathbb{N})$  whenever  $L(d-1) < 1$ . Indeed, for  $|x| \leq 1$  we have  $|x + \cdots + x^{d-1}| \leq (d-1)|x|$ , hence

$$\sum_{m \geq 0} L^m |x + \cdots + x^{d-1}|^m \leq \sum_{m \geq 0} (L(d-1))^m < \infty,$$

so  $T(x)^{-1}$  converges absolutely on  $|x| \leq 1$  and in particular

$$\sum_{n \geq 0} b_{n,d} = \sum_{n \geq 0} |b_{n,d}| = T(1)^{-1} = \frac{1}{1 - L(d-1)} < \infty.$$

Therefore, by Young's inequality for convolutions ( $\ell^1 * \ell^2 \rightarrow \ell^2$ ),

$$\sum_{k \geq k_1} (b * v)_k^2 \leq \left( \sum_{n \geq 0} b_{n,d} \right)^2 \sum_{k \geq k_1} |u_k(t)|^2.$$

Moreover, since  $(s_\ell)$  is supported on  $[k_1, k_1 + d - 2]$ , another application of Young's inequality gives

$$\sum_{k \geq k_1} w_k^2 \leq \left( \sum_{n \geq 0} b_{n,d} \right)^2 \sum_{k_1 \leq \ell \leq k_1 + d - 2} s_\ell^2 < \infty.$$

Since  $\tilde{\varepsilon}_k = (b * v)_k + w_k$ , we have

$$\sum_{k \geq k_1} \tilde{\varepsilon}_k^2 \leq 2 \sum_{k \geq k_1} (b * v)_k^2 + 2 \sum_{k \geq k_1} w_k^2 < \infty,$$

hence  $\sum_{k \geq k_1} |\varepsilon_k(t)|^2 \leq \sum_{k \geq k_1} \tilde{\varepsilon}_k^2 < \infty$ . Since the range  $0 \leq k < k_1$  is finite, this proves  $\sum_{k \geq 0} |\varepsilon_k(t)|^2 < \infty$ , i.e. assertion (2) of Lemma 3.5.

We now assume that (S2) converges. By assertion (1) we have  $\sum_k |u_k(t)|^2 < \infty$ , hence by assertion (2) also  $\sum_k |\varepsilon_k(t)|^2 < \infty$ . We prove assertion (3).

Using the decomposition obtained from (C.1), we have

$$\sum_{P \leq k \leq Q} \sum_{j < d} \varepsilon_{k-j}(t) = \frac{1}{\alpha^{d-1}} \sum_{P \leq k \leq Q} u_k(t) - \frac{1}{\alpha^{d-1}} \sum_{P \leq k \leq Q} R_k(t) + \frac{1}{\alpha} \sum_{j < d} \sum_{P \leq k \leq Q} \tau_{k,j}(t) \varepsilon_{k-j}(t).$$

Splitting

$$\tau_{k,j}(t) = (\tau_{k,j}(t) - c_j) + c_j, \quad c_j := \sum_{m < j} \frac{a_m}{\alpha^m},$$

and setting

$$d_j := 1 - \frac{c_j}{\alpha} \quad (0 \leq j < d), \quad D := \sum_{j < d} d_j,$$

we obtain the rearranged identity

$$(C.2) \quad \sum_{P \leq k \leq Q} \sum_{j < d} d_j \varepsilon_{k-j}(t) = \frac{1}{\alpha^{d-1}} \sum_{P \leq k \leq Q} u_k(t) - \frac{1}{\alpha^{d-1}} \sum_{P \leq k \leq Q} R_k(t) + \frac{1}{\alpha} \sum_{j < d} \sum_{P \leq k \leq Q} (\tau_{k,j}(t) - c_j) \varepsilon_{k-j}(t).$$

We have  $\sum_{k \geq 0} |\varepsilon_k(t)|^2 < \infty$ , and under (S2) we have, for each fixed  $j$ ,

$$\sum_{k \geq 0} |\tau_{k,j}(t) - c_j|^2 < \infty.$$

Fix such a  $j$ . Since a finite shift preserves  $\ell^2$ , we also have

$$\sum_{k \geq 0} |\varepsilon_{k-j}(t)|^2 < \infty$$

Hence, by Cauchy-Schwarz,

$$\sum_{k \geq 0} |(\tau_{k,j}(t) - c_j) \varepsilon_{k-j}(t)| \leq \left( \sum_{k \geq 0} |\tau_{k,j}(t) - c_j|^2 \right)^{1/2} \left( \sum_{k \geq 0} |\varepsilon_{k-j}(t)|^2 \right)^{1/2} < \infty,$$

so  $\sum_{k \geq 0} (\tau_{k,j}(t) - c_j) \varepsilon_{k-j}(t)$  converges absolutely.

Likewise,  $\sum_k R_k(t)$  converges absolutely, since

$$|R_k(t)| \ll \left( \sum_{1 \leq j < d} |\varepsilon_{k-j}(t)| \right)^2$$

and  $\sum_k |\varepsilon_k(t)|^2 < \infty$ . We note that  $c_j \leq c_{d-1} = \sum_{m < d-1} a_m / \alpha^m = \alpha - a_{d-1} / \alpha^{d-1} < \alpha$  (since  $a_{d-1} > 0$ ), hence each  $d_j = 1 - c_j / \alpha$  is strictly positive and in particular  $D > 0$ .

We can now conclude (3). If  $\sum_k \varepsilon_k(t)$  converges, then  $\sum_k u_k(t)$  converges by summing (C.1) thanks to the convergence of  $\sum_k R_k(t)$  and of  $\sum_k (\tau_{k,j}(t) - c_j) \varepsilon_{k-j}(t)$ .

Conversely, assume that  $\sum_k u_k(t)$  converges. Fix  $\varepsilon > 0$  and set  $\eta := D\varepsilon/2$ . From (C.2) and the previous paragraph (Cauchy's criterion applied to the convergent series  $\sum_k u_k(t)$ ,  $\sum_k R_k(t)$  and  $\sum_k (\tau_{k,j}(t) - c_j) \varepsilon_{k-j}(t)$ , and using that there are only finitely many  $j < d$ ), there exists  $P_*$  such that for all  $Q \geq P \geq P_*$ ,

$$\left| \sum_{P \leq k \leq Q} \sum_{j < d} d_j \varepsilon_{k-j}(t) \right| < \eta.$$

Writing  $A_{P,Q} := \sum_{k=P}^Q \varepsilon_k(t)$ , we have  $\sum_{k=P}^Q \varepsilon_{k-j}(t) = A_{P-j,Q-j}$ , hence

$$\left| \sum_{j < d} d_j A_{P-j,Q-j} \right| < \eta \quad (Q \geq P \geq P_*).$$

Next, since  $\varepsilon_k(t) \rightarrow 0$ , there exists  $N_\varepsilon$  such that  $|\varepsilon_n(t)| < \varepsilon/(4(d-1))$  for all  $n \geq N_\varepsilon$ . Let  $P^* := \max(P_*, N_\varepsilon + d)$ . Then for all  $Q \geq P \geq P^*$  and all  $0 \leq j < d$ , the explicit bound gives

$$|A_{P-j,Q-j} - A_{P,Q}| \leq \sum_{1 \leq r < d} (|\varepsilon_{P-r}(t)| + |\varepsilon_{Q+1-r}(t)|) < \frac{\varepsilon}{2}.$$

Therefore, for all  $Q \geq P \geq P^*$ ,

$$\begin{aligned} D |A_{P,Q}| &= \left| \sum_{j < d} d_j A_{P,Q} \right| \\ &\leq \left| \sum_{j < d} d_j A_{P-j,Q-j} \right| + \sum_{j < d} d_j |A_{P-j,Q-j} - A_{P,Q}| \\ &< \eta + D \cdot \frac{\varepsilon}{2} = D\varepsilon. \end{aligned}$$

Since  $D > 0$ , this implies  $|A_{P,Q}| < \varepsilon$  for all  $Q \geq P \geq P^*$ . By Cauchy's criterion, the series  $\sum_k \varepsilon_k(t)$  converges. This completes the proof of assertion (3) and hence of Lemma 3.5.

#### D. Proof of Lemma 3.6.

By definition we have

$$c_{n,\ell}(t) = g_t(\vartheta_{n,\ell}) \sigma_{n,\ell}(t),$$

where

$$\vartheta_{n,\ell} := \sum_{j < \ell} a_j G_{n-j}, \quad \sigma_{n,\ell}(t) = \sum_{j < a_\ell} e^{itf(jG_{n-\ell})}.$$

Set

$$\delta_n := \max_{0 \leq r < d, 1 \leq j \leq a} |f(jG_{n-r})|.$$

By (H<sub>f</sub>) and finiteness of the index set, we have  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . For each real  $x$  and each  $|t| \leq T_0$  we have the Taylor expansion

$$e^{itx} - 1 = itx - \frac{1}{2}t^2x^2 + R(t,x), \quad |R(t,x)| \leq C|t|^3|x|^3,$$

for some constant  $C > 0$  depending only on  $T_0$ . Applying this with  $x = f(jG_{n-\ell})$  and summing over  $0 \leq j < a_\ell$  gives

$$\sigma_{n,\ell}(t) = a_\ell + it \sum_{j < a_\ell} f(jG_{n-\ell}) - \frac{1}{2}t^2 \sum_{j < a_\ell} f(jG_{n-\ell})^2 + E_{n,\ell}(t),$$

where

$$|E_{n,\ell}(t)| \leq C|t|^3 \sum_{j < a_\ell} |f(jG_{n-\ell})|^3.$$

Next applying the same Taylor expansion with  $x = f(\vartheta_{n,\ell})$  yields

$$g_t(\vartheta_{n,\ell}) = 1 + itF_{n,\ell} - \frac{1}{2}t^2 F_{n,\ell}^2 + \tilde{E}_{n,\ell}(t), \quad F_{n,\ell} := f(\vartheta_{n,\ell}),$$

with

$$|\tilde{E}_{n,\ell}(t)| \leq C' |t|^3 |F_{n,\ell}|^3$$

for some constant  $C'$  depending only on  $T_0$  and the digit system. Set

$$S_{1,n,\ell} := \sum_{j < a_\ell} f(jG_{n-\ell}), \quad S_{2,n,\ell} := \sum_{j < a_\ell} f(jG_{n-\ell})^2.$$

Multiplying the two expansions, we obtain

$$\begin{aligned} c_{n,\ell}(t) &= g_t(\vartheta_{n,\ell}) \sigma_{n,\ell}(t) \\ &= \left(1 + itF_{n,\ell} - \frac{1}{2}t^2 F_{n,\ell}^2 + \tilde{E}_{n,\ell}(t)\right) \left(a_\ell + itS_{1,n,\ell} - \frac{1}{2}t^2 S_{2,n,\ell} + E_{n,\ell}(t)\right). \end{aligned}$$

Collecting the constant, linear and quadratic terms in  $t$  gives

$$c_{n,\ell}(t) = a_\ell + t \Lambda_{n,\ell} + t^2 \zeta_{n,\ell} + \mathcal{R}_{n,\ell}(t),$$

with

$$\Lambda_{n,\ell} := i(a_\ell F_{n,\ell} + S_{1,n,\ell}), \quad \zeta_{n,\ell} := -\frac{1}{2}a_\ell F_{n,\ell}^2 - \frac{1}{2}S_{2,n,\ell} - F_{n,\ell} S_{1,n,\ell},$$

and where  $\mathcal{R}_{n,\ell}(t)$  collects all terms of order at least  $|t|^3$ :

$$\mathcal{R}_{n,\ell}(t) = O\left(|t|^3 |F_{n,\ell}|^3 + |t|^3 |F_{n,\ell}| S_{2,n,\ell} + |t|^3 |F_{n,\ell}|^2 |S_{1,n,\ell}| + |E_{n,\ell}(t)| + |\tilde{E}_{n,\ell}(t)|\right).$$

Since  $\vartheta_{n,\ell} = \sum_{j < \ell} a_j G_{n-j}$  and  $f$  is  $G$ -additive, we have

$$F_{n,\ell} = f(\vartheta_{n,\ell}) = \sum_{j < \ell} f(a_j G_{n-j}),$$

hence  $|F_{n,\ell}| \ll \delta_n$ . Moreover, since  $a_\ell \leq \alpha$ , both  $S_{1,n,\ell}$  and  $S_{2,n,\ell}$  are finite sums of values  $f(jG_{n-r})$  and  $f(jG_{n-r})^2$  with  $0 \leq r < d$  and  $1 \leq j \leq \alpha$ , so

$$|S_{1,n,\ell}| \ll \delta_n, \quad S_{2,n,\ell} \leq Q_n.$$

Also,

$$\sum_{r < d} \sum_{j \leq \alpha} |f(jG_{n-r})|^3 \leq \delta_n Q_n,$$

and since  $Q_n \geq \delta_n^2$  whenever  $\delta_n > 0$ , we have  $|F_{n,\ell}|^3 \ll \delta_n^3 \leq \delta_n Q_n$ . Therefore, uniformly for  $|t| \leq T_0$  and  $n$  large,

$$|\mathcal{R}_{n,\ell}(t)| \ll |t|^3 \delta_n Q_n \leq (T_0 \delta_n) t^2 Q_n.$$

Passing to any fixed norm  $\|\cdot\|$  on  $\mathbb{C}^d$  and using finiteness of the index set  $\{0, \dots, d-1\}$  yields

$$\|\Lambda_n\| \ll \sum_{r < d} \sum_{j \leq \alpha} |f(jG_{n-r})|, \quad \|\zeta_n\| \ll Q_n,$$

and

$$\|\mathcal{R}_n(t)\| \leq \omega_n t^2 Q_n \quad (|t| \leq T_0),$$

with  $\omega_n := C'' \delta_n \rightarrow 0$ . This proves the lemma.

**E. Proof of Lemma 3.7.** Throughout this appendix we work with the  $w$ –weighted  $\ell^1$ –norm

$$\|x\|_w := \sum_{j=0}^{d-1} w_j |x_j|, \quad x = (x_0, \dots, x_{d-1})^\top \in \mathbb{C}^d,$$

and with the induced operator norm on  $\mathbb{C}^{d \times d}$ ,

$$\|M\|_w := \sup_{x \neq 0} \frac{\|Mx\|_w}{\|x\|_w}.$$

*Step 1: A uniform perturbative eigenvalue and a uniform gap.* Let  $A$  be the companion matrix from Definition 2.1. By Perron–Frobenius,  $A$  has a simple dominant eigenvalue  $\alpha > 1$  with strictly positive right and left eigenvectors  $v, w > 0$ , normalized by  $w^\top v = 1$ , and all other eigenvalues  $\beta \neq \alpha$  satisfy  $|\beta| \leq \alpha - \delta_0$  for some  $\delta_0 > 0$ .

Fix a positively oriented circle  $\Gamma$  around  $\alpha$  contained in the open annulus  $\{z : |z - \alpha| < \delta_0/2\}$ , so that  $\Gamma$  encloses no other eigenvalue of  $A$ . For each fixed  $n$ , the entries of  $A_n(t)$  are finite linear combinations of exponentials  $e^{itf(\cdot)}$ , hence the map  $t \mapsto A_n(t)$  is holomorphic on  $\mathbb{C}$ . By the Riesz–Dunford calculus (see [16, Chapter VII, Theorems 1.7 and 1.8]), for each fixed  $n$  there exists  $T(n) > 0$  such that, for  $|t| < T(n)$ , the rank–one spectral projector

$$P_n(t) := -\frac{1}{2\pi i} \int_\Gamma (A_n(t) - zI)^{-1} dz$$

is holomorphic, and it projects onto a simple eigenvalue  $\lambda_n(t)$  of  $A_n(t)$ , with  $\lambda_n(0) = \alpha$ . Moreover, the remaining eigenvalues of  $A_n(t)$  stay outside  $\Gamma$  for  $|t| < T(n)$ .

Under  $(H_f)$ , all block parameters  $f(jG_{n-r})$  with  $0 \leq r < d$  and  $1 \leq j \leq \mathfrak{a}$  tend to 0 as  $n \rightarrow \infty$ . Hence, for every fixed  $T_0 > 0$ ,

$$\sup_{|t| \leq T_0} \|A_n(t) - A\|_w \xrightarrow{n \rightarrow \infty} 0.$$

Consequently, shrinking the radius of  $\Gamma$  if necessary, there exist  $T_1 \in (0, T_0]$ ,  $n_0$  and  $\delta \in (0, \delta_0/2)$  such that, for all  $n \geq n_0$  and all  $|t| \leq T_1$ , the spectrum of  $A_n(t)$  consists of one simple eigenvalue  $\lambda_n(t)$  inside  $\Gamma$  and  $d - 1$  eigenvalues in the closed disk  $\{z : |z| \leq \alpha - \delta\}$ .

*Step 2: Modulus dissipation via the entrywise absolute value.* For  $n \geq n_0$  and  $|t| \leq T_1$ , set

$$\tilde{A}_n(t) := |\lambda_n(t)|,$$

where  $|\cdot|$  is taken entrywise. Since  $A_n(t)$  is a companion matrix, all entries of  $\tilde{A}_n(t)$  are nonnegative. By the triangle inequality,

$$\rho(A_n(t)) \leq \rho(\tilde{A}_n(t)),$$

and since  $|\lambda_n(t)| \leq \rho(A_n(t))$ , we obtain

$$(E.1) \quad |\lambda_n(t)| \leq \rho(\tilde{A}_n(t)).$$

Write  $\rho_n(t) := \rho(\tilde{A}_n(t))$ . As  $\tilde{A}_n(0) = A$ , we have  $\rho_n(0) = \alpha$ , and  $\alpha$  is a simple Perron–Frobenius eigenvalue of  $\tilde{A}_n(0)$ . Moreover, for each fixed  $n$  and each  $\ell < d$ , the function  $t \mapsto c_{n,\ell}(t)$  is holomorphic with  $c_{n,\ell}(0) = a_\ell > 0$ . Hence  $t \mapsto |c_{n,\ell}(t)| = \sqrt{c_{n,\ell}(t)c_{n,\ell}(-t)}$  is real–analytic in a neighborhood of 0, and so is  $t \mapsto \tilde{A}_n(t)$ . Standard perturbation of a simple eigenvalue for real–analytic matrix families (see [16, Chapter II, §1]) therefore yields a real–analytic function  $t \mapsto \rho_n(t)$  near 0, with  $\rho'_n(0) = 0$  and

$$\rho''_n(0) = w^\top \tilde{A}''_n(0) v.$$

Only the first row of  $\tilde{A}_n(t)$  depends on  $t$ , so

$$\tilde{A}''_n(0) = \begin{pmatrix} (|c_{n,0}(t)|'')|_{t=0} & \cdots & (|c_{n,d-1}(t)|'')|_{t=0} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

and therefore

$$(E.2) \quad \rho_n''(0) = w_0 \sum_{\ell < d} v_\ell (|c_{n,\ell}(t)|)''|_{t=0}.$$

We now bound  $(|c_{n,\ell}(t)|)''|_{t=0}$  from above. Fix  $\ell < d$  and write  $x_j := f(jG_{n-\ell})$  for  $0 \leq j < a_\ell$ . Recall  $\sigma_{n,\ell}(t) := \sum_{j < a_\ell} e^{itx_j}$  and  $|c_{n,\ell}(t)| = |\sigma_{n,\ell}(t)|$ . A direct Taylor expansion of  $|\sigma_{n,\ell}(t)|$  at  $t = 0$  gives

$$|\sigma_{n,\ell}(t)| = a_\ell - \frac{a_\ell}{2} \text{Var}(x) t^2 + O(|t|^3 \|x\|_2^3),$$

where  $\text{Var}(x)$  denotes the variance of  $(x_0, \dots, x_{a_\ell-1})$  under the uniform measure and  $\|x\|_2^2 := \sum_{j < a_\ell} x_j^2$ . Since  $x_0 = f(0) = 0$ , we have

$$\text{Var}(x) = \frac{1}{a_\ell} \sum_{j < a_\ell} x_j^2 - \left( \frac{1}{a_\ell} \sum_{j < a_\ell} x_j \right)^2 \geq \frac{1}{a_\ell^2} \sum_{1 \leq j < a_\ell} x_j^2,$$

and therefore

$$(E.3) \quad (|c_{n,\ell}(t)|)''|_{t=0} \leq -\frac{1}{a_\ell} \sum_{1 \leq j < a_\ell} f(jG_{n-\ell})^2.$$

Combining (E.2) and (E.3), and using that  $v, w > 0$ , we obtain a constant  $\kappa > 0$  depending only on the digit system such that

$$(E.4) \quad \rho_n''(0) \leq -\kappa Q_n,$$

where  $Q_n$  is defined in (3.8).

Finally, Lemma 3.6 gives a uniform control of the third-order remainder in the Taylor expansion of  $\tilde{A}_n(t)$  in terms of the block energy  $Q_n$ . Concretely, there exist  $C \geq 1$  and a sequence  $\omega_n \rightarrow 0$  such that, uniformly for  $|t| \leq T_0$  and  $n$  large,

$$\rho_n(t) = \alpha + \frac{t^2}{2} \rho_n''(0) + O(\omega_n |t|^2 Q_n).$$

Shrinking  $T_1$  if necessary and enlarging  $n_0$ , we may assume that the error term is bounded in modulus by  $\frac{\kappa}{4} t^2 Q_n$  whenever  $n \geq n_0$  and  $|t| \leq T_1$ . Using (E.4), we then obtain, for such  $n, t$ ,

$$\rho_n(t) \leq \alpha - \frac{\kappa}{4} t^2 Q_n \leq \alpha \exp\left(-\frac{\kappa}{4} t^2 Q_n\right).$$

Together with (E.1), this yields (3.9) (with  $c_0 := \kappa/4$ ). This completes the proof.

## F. Proof of Lemma 3.8.

Fix  $|t| \leq T_1$  and suppress the parameter  $t$  from the notation. For  $n \geq n_0$ , set  $A_n := A_n(t)$  and let  $\lambda_n := \lambda_n(t)$  be the dominant simple eigenvalue given by Lemma 3.7. The remaining eigenvalues of  $A_n$  have modulus at most  $\alpha - \delta$ .

We keep the contour  $\Gamma$  fixed as in Appendix E, Step 1. In particular, for all  $n \geq n_0$  and all  $|t| \leq T_1$ , the contour  $\Gamma$  surrounds  $\lambda_n$  and contains no other eigenvalue of  $A_n$ . We define the associated Riesz projector

$$P_n := -\frac{1}{2\pi i} \int_{\Gamma} (A_n - zI)^{-1} dz, \quad R_n := I - P_n.$$

Then  $\text{rank}(P_n) = 1$  and

$$A_n P_n = P_n A_n = \lambda_n P_n, \quad A_n R_n = R_n A_n.$$

Let  $v, w > 0$  be the right/left Perron–Frobenius eigenvectors of the limit matrix  $A$ , normalized by  $w^\top v = 1$ , and let

$$P := -\frac{1}{2\pi i} \int_{\Gamma} (A - zI)^{-1} dz, \quad Q := I - P.$$

Since  $A_n(t) \rightarrow A$  as  $n \rightarrow \infty$  uniformly for  $|t| \leq T_1$ , the resolvents converge uniformly on  $\Gamma$ , hence

$$P_n \rightarrow P \quad (n \rightarrow \infty),$$

uniformly for  $|t| \leq T_1$ . In particular,  $P$  has rank one and satisfies  $Px = (w^\top x)v$ .

Let  $\delta_0 > 0$  be such that every eigenvalue  $\beta \neq \alpha$  of  $A$  satisfies  $|\beta| \leq \alpha - \delta_0$  (as in Appendix E, Step 1). Choose  $\delta_1 \in (0, \min(\delta, \delta_0)/10)$  and set  $\gamma := \alpha - 3\delta_1$ . Since  $Q$  is a spectral projector of  $A$ ,

we have  $AQ = QA$ , and the eigenvalues of  $AQ$  are precisely the eigenvalues of  $A$  different from  $\alpha$ . Hence the spectral radius of  $AQ$  satisfies  $\rho(AQ) \leq \alpha - \delta_0 < \gamma$ .

We take  $\|\cdot\|_0 := \|\cdot\|_w$  and denote by  $\|\cdot\|_0$  also its induced operator norm. By Gelfand's formula

$$\rho(B) = \lim_{m \rightarrow \infty} \|B^m\|_0^{1/m},$$

there exists  $m \geq 1$  such that  $\|(AQ)^m\|_0 \leq \gamma^m$ . Define

$$\|x\|_* := \sum_{k=0}^{m-1} \gamma^{-k} \|(AQ)^k x\|_0.$$

Then  $\|\cdot\|_*$  is a norm equivalent to  $\|\cdot\|_0$  (hence to  $\|\cdot\|_w$ ), and it satisfies

$$(F.1) \quad \|AQ\|_* \leq \gamma = \alpha - 3\delta_1.$$

Since  $P_n \rightarrow P$  and  $A_n \rightarrow A$  uniformly for  $|t| \leq T_1$ , we also have  $R_n \rightarrow Q$  and  $A_n R_n \rightarrow AQ$  uniformly for  $|t| \leq T_1$ . Fix  $\varepsilon > 0$  (to be chosen later). After increasing  $n_1 \geq n_0$  (and shrinking  $T_1$  if necessary), we may assume that, for all  $n \geq n_1$  and all  $|t| \leq T_1$ ,

$$(F.2) \quad |\lambda_n| \geq \alpha - \frac{\delta_1}{2},$$

$$(F.3) \quad \|P_{n+1} - P_n\|_* \leq \varepsilon,$$

$$(F.4) \quad \|A_n R_n\|_* \leq \alpha - 2\delta_1.$$

Indeed, (F.2) follows from  $\lambda_n(t) \rightarrow \alpha$  as  $n \rightarrow \infty$  uniformly for  $|t| \leq T_1$ ; (F.3) follows from  $P_n \rightarrow P$ ; and (F.4) is obtained by combining (F.1) with

$$\|A_n R_n\|_* \leq \|AQ\|_* + \|A_n R_n - AQ\|_*$$

and the fact that  $\sup_{|t| \leq T_1} \|A_n R_n - AQ\|_* \rightarrow 0$ .

Since  $\text{rank}(P_n) = 1$ , we may choose  $v_n \in \mathbb{C}^d$  and  $\psi_n \in (\mathbb{C}^d)^*$  such that

$$P_n x = \psi_n(x) v_n, \quad \psi_n(v_n) = 1.$$

A convenient choice is

$$v_n := P_n v, \quad \psi_n(x) := \frac{w^\top P_n x}{w^\top P_n v} \quad (x \in \mathbb{C}^d),$$

for which  $P_n x = \psi_n(x) v_n$  and  $\psi_n(v_n) = 1$ . Since  $P_n \rightarrow P$  uniformly for  $|t| \leq T_1$ , we have  $v_n \rightarrow v$  and  $w^\top P_n v \rightarrow w^\top P v = w^\top v = 1$  uniformly; in particular,  $w^\top P_n v \neq 0$  for  $n$  large, and moreover

$$\psi_{n+1}(v_n) \rightarrow \frac{w^\top P v}{w^\top v} = 1$$

uniformly for  $|t| \leq T_1$ .

After possibly increasing  $n_1$ , we may therefore renormalize the pairs  $(v_n, \psi_n)$  (for  $n \geq n_1$ ) by scalar factors so as to enforce the *intertwining normalization*

$$(F.5) \quad \psi_{n+1}(v_n) = 1 \quad (n \geq n_1).$$

Indeed, multiplying  $v_n$  by  $\theta_n \neq 0$  and  $\psi_n$  by  $\theta_n^{-1}$  leaves  $P_n$  unchanged; choosing  $\theta_{n+1} := \theta_n \psi_{n+1}(v_n)$  enforces (F.5).

Set  $F_n := \ker(\psi_n)$ , so that  $\mathbb{C}^d = \mathbb{C}v_n \oplus F_n$  and  $R_n$  is the projection onto  $F_n$ . For  $x_n := \mathbf{H}_n(t)$  write

$$x_n = u_n v_n + s_n, \quad u_n := \psi_n(x_n) \in \mathbb{C}, \quad s_n := R_n x_n \in F_n.$$

Since  $v_n \rightarrow v \neq 0$  and  $\psi_n \rightarrow w^\top$  uniformly for  $|t| \leq T_1$ , the coordinate maps  $(u, s) \mapsto uv_n + s$  and their inverses have operator norms bounded uniformly for  $n \geq n_1$ . In particular,

$$(F.6) \quad \|x_n\|_* \asymp |u_n| + \|s_n\|_* \quad (n \geq n_1),$$

with implicit constants independent of  $n$  and  $t$ .

Using  $x_{n+1} = A_n x_n$  and  $A_n v_n = \lambda_n v_n$ , and applying  $\psi_{n+1}$ , we get

$$u_{n+1} = \psi_{n+1}(A_n(u_n v_n + s_n)) = \lambda_n u_n \psi_{n+1}(v_n) + \psi_{n+1}(A_n s_n) = \lambda_n u_n + \psi_{n+1}(A_n s_n),$$

where we used (F.5) in the last step. Since  $P_n A_n s_n = A_n P_n s_n = 0$ , we have  $P_{n+1} A_n s_n = (P_{n+1} - P_n) A_n s_n$ , hence by (F.3) and the uniform boundedness of  $\|A_n\|_\star$  for  $n \geq n_1$ ,

$$\|P_{n+1} A_n s_n\|_\star \leq \|P_{n+1} - P_n\|_\star \|A_n\|_\star \|s_n\|_\star \ll \varepsilon \|s_n\|_\star.$$

As  $P_{n+1} A_n s_n = \psi_{n+1}(A_n s_n) v_{n+1}$  and  $\|v_{n+1}\|_\star \asymp 1$  uniformly, we deduce  $|\psi_{n+1}(A_n s_n)| \ll \varepsilon \|s_n\|_\star$ . Therefore, for some  $C_1 \geq 1$ ,

$$(F.7) \quad |u_{n+1}| \leq |\lambda_n|(|u_n| + C_1 \varepsilon \|s_n\|_\star) \quad (n \geq n_1).$$

Similarly,

$$s_{n+1} = R_{n+1} x_{n+1} = R_{n+1} A_n (u_n v_n + s_n) = \lambda_n u_n R_{n+1} v_n + R_{n+1} A_n s_n.$$

Since  $R_{n+1} v_n = (I - P_{n+1}) v_n = -(P_{n+1} - P_n) v_n$ , we have  $\|R_{n+1} v_n\|_\star \leq \varepsilon \|v_n\|_\star \ll \varepsilon$ . Moreover, writing  $R_{n+1} A_n s_n = (R_{n+1} - R_n) A_n s_n + R_n A_n s_n$  and using  $R_n A_n s_n = A_n R_n s_n = A_n s_n$ , we obtain

$$\|R_{n+1} A_n s_n\|_\star \leq \|R_{n+1} - R_n\|_\star \|A_n\|_\star \|s_n\|_\star + \|A_n R_n\|_\star \|s_n\|_\star \ll \varepsilon \|s_n\|_\star + (\alpha - 2\delta_1) \|s_n\|_\star,$$

where we used (F.4) and the fact that  $\|R_{n+1} - R_n\|_\star = \|P_{n+1} - P_n\|_\star$ . Choosing  $\varepsilon > 0$  sufficiently small (depending only on the digit system and on  $\delta_1$ ), we may assume that the right-hand side is bounded by  $(\alpha - \delta_1) \|s_n\|_\star$ . Thus there exists  $C_2 \geq 1$  such that

$$(F.8) \quad \|s_{n+1}\|_\star \leq (\alpha - \delta_1) \|s_n\|_\star + C_2 \varepsilon |u_n| \quad (n \geq n_1).$$

Set

$$\lambda_* := \inf_{n \geq n_1, |t| \leq T_1} |\lambda_n|.$$

By (F.2) we have  $\lambda_* \geq \alpha - \delta_1/2$ . Define

$$r := \frac{\alpha - \delta_1}{\lambda_*}, \quad \Theta_{m,n} := \prod_{k=n}^{m-1} |\lambda_k| \quad (m \geq n),$$

with the convention  $\Theta_{n,n} := 1$ . Note that  $0 < r < 1$ , since

$$r \leq \frac{\alpha - \delta_1}{\alpha - \delta_1/2} < 1.$$

From (F.8) we get, by a standard induction (repeated substitution of the recurrence), that for every  $n \geq n_1$ ,

$$(F.9) \quad \|s_n\|_\star \leq (\alpha - \delta_1)^{n-n_1} \|s_{n_1}\|_\star + C_2 \varepsilon \sum_{m=n_1}^{n-1} (\alpha - \delta_1)^{n-1-m} |u_m|.$$

Set

$$U_n := \frac{|u_n|}{\Theta_{n,n_1}} \quad (n \geq n_1), \quad \Omega_n := \max_{n_1 \leq m \leq n} U_m.$$

Dividing (F.7) by  $\Theta_{n+1,n_1} = \Theta_{n,n_1} |\lambda_n|$  gives, for  $n \geq n_1$ ,

$$(F.10) \quad U_{n+1} \leq U_n + C_1 \varepsilon \frac{\|s_n\|_\star}{\Theta_{n,n_1}}.$$

We now estimate  $\|s_n\|_\star / \Theta_{n,n_1}$  using (F.9). First,

$$\frac{(\alpha - \delta_1)^{n-n_1}}{\Theta_{n,n_1}} \leq \frac{(\alpha - \delta_1)^{n-n_1}}{\lambda_*^{n-n_1}} = r^{n-n_1}.$$

Next, for  $m \leq n-1$  we factor  $\Theta_{n,n_1} = \Theta_{n,m} \Theta_{m,n_1}$ , hence

$$\frac{|u_m|}{\Theta_{n,n_1}} = \frac{|u_m|}{\Theta_{m,n_1}} \cdot \frac{1}{\Theta_{n,m}} = U_m \cdot \frac{1}{\Theta_{n,m}} \leq U_m \lambda_*^{-(n-m)},$$

because  $\Theta_{n,m} = \prod_{k=m}^{n-1} |\lambda_k| \geq \lambda_*^{n-m}$ . Therefore, dividing (F.9) by  $\Theta_{n,n_1}$  yields

$$(F.11) \quad \frac{\|s_n\|_\star}{\Theta_{n,n_1}} \leq r^{n-n_1} \|s_{n_1}\|_\star + C_2 \varepsilon \sum_{m=n_1}^{n-1} r^{n-1-m} U_m.$$

Plugging (F.11) into (F.10) gives, for  $n \geq n_1$ ,

$$U_{n+1} \leq U_n + C_1 \varepsilon r^{n-n_1} \|s_{n_1}\|_\star + C_1 C_2 \varepsilon^2 \sum_{m=n_1}^{n-1} r^{n-1-m} U_m.$$

Since  $U_m \leq \Omega_n$  for  $n_1 \leq m \leq n$ , we have

$$\sum_{m=n_1}^{n-1} r^{n-1-m} U_m \leq \Omega_n \sum_{j \geq 0} r^j = \frac{\Omega_n}{1-r}.$$

Hence, for all  $n \geq n_1$ ,

$$(F.12) \quad U_{n+1} \leq U_n + C_1 \varepsilon r^{n-n_1} \|s_{n_1}\|_\star + C_1 C_2 \varepsilon^2 \frac{\Omega_n}{1-r}.$$

Taking the maximum of (F.12) over  $n_1 \leq n \leq N-1$  and using again  $\sum_{j \geq 0} r^j = 1/(1-r)$  yields, for every  $N \geq n_1$ ,

$$\Omega_N \leq U_{n_1} + \frac{C_1 \varepsilon}{1-r} \|s_{n_1}\|_\star + C_1 C_2 \varepsilon^2 \frac{\Omega_N}{1-r}.$$

Choose  $\varepsilon > 0$  so small that  $C_1 C_2 \varepsilon^2 / (1-r) \leq \frac{1}{2}$ . Then  $\Omega_N \ll U_{n_1} + \|s_{n_1}\|_\star$  uniformly in  $N$ , hence for all  $n \geq n_1$ ,

$$(F.13) \quad |u_n| = \Theta_{n,n_1} U_n \leq \Theta_{n,n_1} \Omega_n \ll \Theta_{n,n_1} (|u_{n_1}| + \|s_{n_1}\|_\star).$$

Insert (F.13) into (F.9). Using  $\Theta_{m,n_1} \leq \Theta_{n,n_1} \lambda_*^{-(n-m)}$  (since  $\Theta_{n,m} \geq \lambda_*^{n-m}$ ) and summing the resulting geometric series, we obtain

$$\|s_n\|_\star \ll (\alpha - \delta_1)^{n-n_1} \|s_{n_1}\|_\star + \varepsilon \Theta_{n,n_1} (|u_{n_1}| + \|s_{n_1}\|_\star) \quad (n \geq n_1).$$

Together with (F.13) and (F.6), and with  $\|x_{n_1}\|_\star \asymp |u_{n_1}| + \|s_{n_1}\|_\star$ , this gives for all  $N \geq n_1$ ,

$$\|x_N\|_\star \ll \|x_{n_1}\|_\star \left( \Theta_{N,n_1} + (\alpha - \delta_1)^{N-n_1} \right),$$

which is exactly the claimed estimate.

## REFERENCES

- [1] V. Baladi. *Positive Transfer Operators and Decay of Correlations*, volume 16 of *Advanced Series in Nonlinear Dynamics*. World Scientific, 2000.
- [2] G. Barat and P. J. Grabner. Distribution properties of  $G$ -additive functions. *Journal of Number Theory*, 60:103–123, 1996.
- [3] G. Barat and P. J. Grabner. Limit distribution of  $Q$ -additive functions from an ergodic point of view. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae / Sectio computatorica*, 28:55–78, 2008.
- [4] A. Brauer. On algebraic equations with all but one root in the interior of the unit circle. *Math. Nachr.*, 4:250–257, 1951.
- [5] J. Coquet, G. Rhin, and P. Toffin. Représentations des entiers naturels et indépendance statistique. II. *ANN. Inst. Fourier*, 31:1–15, 1981.
- [6] H. Delange. Sur les fonctions  $q$ -additives ou  $q$ -multiplicatives. *Acta Arithmetica*, 21:285–298, 1972.
- [7] M. Drmota and J. Gajdosik. The parity of the sum-of-digits-function of generalized Zeckendorf representations. *Fibonacci Quarterly*, 36, 1998.
- [8] M. Drmota and J. Gajdosik. The distribution of the sum-of-digits function. *Journal de Théorie des Nombres de Bordeaux*, 10(1):17–32, 1998.
- [9] M. Drmota and J. Verwee. Effective Erdős–Wintner theorems for digital expansions. *Journal of Number Theory*, 229:218–260, 2021.
- [10] P. Erdős and I. Kátai. On the distribution of additive arithmetic functions. *Acta Arith.*, 34:201–220, 1979.
- [11] P. Erdős and A. Wintner. Additive arithmetical functions and statistical independence. *Amer. J. Math.*, 61:713–721, 1939.
- [12] A. S. Fraenkel. Systems of Numeration. *The American Mathematical Monthly*, 92(2):105–114, 1985.
- [13] P. Grabner and R. Tichy. Contributions to digit expansions with respect to linear recurrences. *Journal of Number Theory*, 36:160–169, 1990.
- [14] P. J. Grabner, P. Liardet, and R. F. Tichy. Odometers and systems of numeration. *Acta Arithmetica*, 70(2):103–123, 1995.
- [15] P. Jelinek. Gowers norms for linearly recurrent numeration systems, 2025. Preprint.
- [16] T. Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 2nd edition (1976).
- [17] W. Parry. On the  $\beta$ -expansion of real numbers. *Acta Math. Acad. Sci. Hung.*, 2(11):401–416, 1960.

- [18] M. Queffélec. *Substitution Dynamical Systems — Spectral Analysis*, volume 1294 of *Lecture Notes in Mathematics*. Springer, 2nd edition, 2010.
- [19] D. Ruelle. *Thermodynamic Formalism*. Addison–Wesley, 1978.
- [20] W. Steiner. Digital expansions and the distribution of related functions. Master’s thesis, TU Wien, Vienna, 1998.
- [21] G. Tenenbaum. *Introduction à la Théorie Analytique et Probabiliste des Nombres (quatrième édition mise à jour)*. Belin, 2015.
- [22] G. Tenenbaum and J. Verwee. Effective Erdős-Wintner theorems. *Proc. Steklov Inst. Math.*, 314:264–278, 2021.