Sharpness of the percolation phase transition for weighted random connection models

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Abstract

We establish the sharpness of the percolation phase transition for a class of infinite-range weighted random connection models. The vertex set is given by a marked Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$, where each vertex carries an independent weight. Pairs of vertices are then connected independently with a probability that depends on both their spatial displacement and their respective weights. It is well known that such models undergo a phase transition in λ with respect to the existence of an infinite cluster (under suitable assumptions on the connection probabilities and the weight distribution). We prove that in the subcritical regime the cluster-size distribution has exponentially decaying tails, whereas in the supercritical regime the percolation probability grows at least linearly with respect to λ near criticality. Our proof follows the approach of Duminil-Copin, Raoufi, and Tassion, applying the OSSS inequality to a finite-lattice approximation of the continuum model in order to derive a new differential inequality, which we then analyze and pass to the limit. In addition to the classical random connection model, we consider weighted models with unbounded weights satisfying the min-reach condition under which the neighborhood of each vertex is deterministically bounded by a radius depending solely on its weight. Notably, finite range is not assumed—that is, we allow unbounded edge lengths—but the weight distribution is required to satisfy appropriate moment conditions. We expect that our method extends to a broad class of weighted random connection models.

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1   Introduction

1.1   Motivation

The Gilbert graph, introduced by Gilbert in [16], is generally considered the first mathematical model of continuum percolation. In this model, points are distributed according to a homogeneous Poisson point process on $\mathbb{R}^d$, and any two points are connected by an edge if their Euclidean distance is less than a fixed threshold. Although Gilbert's original motivation lay in modeling communication networks, the model introduced key probabilistic and geometric ideas that are now central to the study of continuum percolation. The *random connection model* (RCM), introduced by Penrose in [31], extends this framework by replacing deterministic connections with probabilistic ones: any two points are connected independently with a probability determined by an *adjacency function* (also called a connection function), typically depending on their spatial distance and decaying with it. This model sits conceptually between Poisson-Boolean percolation, where randomness is encoded in the geometry (e.g., in the radii of balls), and Bernoulli percolation, where edges of a lattice graph are retained independently.

Penrose showed that the RCM exhibits a non-trivial phase transition: there exists a critical intensity $\lambda_T \in (0, \infty)$ such that for $\lambda < \lambda_T$, the expected size of the cluster containing a typical point is finite, while for $\lambda > \lambda_T$, this expectation diverges. He also considered a second critical threshold λ_c, defined as the infimum value of λ for which a typical vertex belongs to an infinite cluster with positive probability, and showed that $\lambda_T \leq \lambda_c < \infty$. Meester [27] extended this analysis by adapting the differential inequality method of Aizenman and Barsky [2] from Bernoulli percolation, and proved that $\lambda_c = \lambda_T$ for isotropic, non-increasing adjacency functions, thus establishing the sharpness of the phase transition.

Further progress came with the work of Heydenreich, van der Hofstad, Last, and Matzke [18], who, in sufficiently high dimensions ($d > 6$), established convergence of the lace expansion for the RCM, derived the triangle condition, proved an infrared bound, and computed the critical exponent $\gamma = 1$, confirming mean-field behavior near criticality. These results were extended to the *marked random connection model* by Heydenreich and Dickson [10], where each vertex is assigned an independent mark influencing its connection behavior. This setting allowed Caicedo and Dickson [5] to prove sharpness and compute additional critical exponents by extending the framework of [3]. However, all of these results required a uniform bound on the marks; in the context of Poisson-Boolean percolation, this is analogous to assuming bounded radii.

This is where our contribution lies — apart from treating the classical RCM with potentially infinite range under minimal assumption upon the adjacency function. We establish the sharpness of the phase transition for a class of infinite-range weighted random connection models in which no uniform bound is imposed on the marks (which we refer to as weights to emphasize their ordering). The vertex set is given by a marked Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$, and edges are included independently according to an adjacency function that depends on the spatial displacement and the respective weights of the points. We allow unbounded weights, assuming their distribution is light-tailed in the sense that it satisfies a moment condition sufficient for our analysis. Our results cover a class of models where there is no global bound on the range of interaction but individual connectivity remains locally controlled; to be more precise, we require the neighborhood of each vertex to be deterministically contained in a ball whose radius depends solely on its weight. This extends the scope of continuum percolation theory beyond previously established limits. We refer to such model as *min-reach RCM* because whether an edge between two vertices of a fixed distance can be drawn with positive probability depends only on the minimum of the weights.

Our proof draws on both classical and modern techniques. On the classical side, we employ a finite-volume, discrete approximation of the continuum model, adapting the approach of Meester and Roy from the 1990s [28]. This allows us to control the geometry and combinatorics of the system, and to reduce the analysis to a discrete setup where Boolean function tools can be applied. On the modern side, we build on recent developments in the analysis of Boolean functions, particularly based on the OSSS inequality [29], which has proven to be a powerful tool in percolation theory by Duminil-Copin, Raoufi, and Tassion. Among several other settings [12, 11], they applied it in [13] to establish sharpness of the phase transition for Poisson-Boolean percolation with unbounded radii. At the same time, Hutchcroft [21] developed a related algorithmic approach to analyze the random-cluster model on graphs, providing a discrete analogue that generalizes well to long-range settings. Hutchcroft's algorithm, while developed in a discrete setting, extends naturally to our model and provides the backbone of our proof. However, his analysis does not address the influence of rare but heavy vertices. For this, we adapt the influence bounds developed in [13] — originally specific to Poisson-Boolean percolation — which generalize effectively to our setting. The interplay between the classical finite-volume approximation and these modern techniques is essential to deriving a differential inequality that captures the weighted, long-range structure of the model. We combine these two strands of work, along with original arguments tailored to our setting, which ultimately allows us to prove exponential decay of cluster sizes in the subcritical regime, and establish the sharpness of the phase transition. A more nuanced discussion of our work in the context of recent developments will be given in Subsection 1.4 — after stating our main result Theorem 1.8.

1.2   Framework

The weighted random connection model.   We now give an informal description of the model; the reader is referred to [10, Section 3] for the formal construction. Our vertex set is given by a Poisson point process η in $\mathbb{X} = \mathbb{R}^d \times [1, \infty)$ with intensity measure $\lambda\nu$, where $\lambda > 0$ and $\nu = \mathrm{Leb} \otimes \pi$; Leb refers to the standard Lebesgue measure on $\mathbb{R}^d$ and π is a probability measure on $[1, \infty)$ — our vertices have a spatial coordinate and a weight. We assume $d \geq 2$ (to ensure that a phase transition is possible under reasonable assumptions). For simplicity, we also assume that the weight distribution π has a non-increasing Lebesgue density or is discrete with non-increasing weight probabilities. The probability of an edge to appear will be determined by a measurable adjacency (or connection) function $\varphi : \mathbb{X}^2 \to [0, 1]$: Two vertices $x, y \in \eta$ will be connected by an edge with probability $\varphi(x, y)$, with distinct edges being sampled independently.

We want to impose some general assumptions upon the adjacency function φ.

Assumption (A). *The adjacency function φ satisfies the following properties:*

(A.1) For any $(\bar{x}, a), (\bar{y}, b) \in \mathbb{X}$, we have

$$\varphi\big((\bar{x}, a), (\bar{y}, b)\big) = \varphi(|\bar{x} - \bar{y}|; a, b).$$

(A.2) For every $a, b \geq 1$, the map $r \mapsto \varphi(r; a, b)$ is non-increasing.

(A.3) For every $r \geq 0$ and $a, b \geq 1$, we have $\varphi(r; a, b) = \varphi(r; b, a)$, and $\hat{a} \mapsto \varphi(r; \hat{a}, b)$ is non-decreasing (hence $\hat{b} \mapsto \varphi(r; a, \hat{b})$ is also non-decreasing).

Furthermore, in order to guarantee the existence of a non-trivial phase transition, we impose the following two-sided neighborhood bounds:

Assumption (NB). *There exist constants $\upsilon, \Upsilon > 0$ such that for all $a \geq 1$ we have*

$$\upsilon \leq \int_0^\infty \left(\int_1^\infty \varphi(r; a, b) \, \pi(\mathrm{d}b) \right) r^{d-1} \mathrm{d}r \leq \Upsilon. \tag{NB}$$

We then denote a realization of the full random graph by ξ, which means that ξ is a realization of the Poisson point process η together with the realization of the edges sampled via φ as described above. Note that since both the distribution of the vertex set η and the adjacency function φ are spatially translation invariant, the whole random graph ξ is also spatially translation invariant.

Augmentation by fixed points. Since our intensity measure ν is non-atomic, any given point $x \in \mathbb{X}$ is almost surely not a vertex of ξ. Therefore, for studying most interesting events, it is necessary to consider augmented versions of ξ. Given a vertex set η and a point $x \in \mathbb{X}$, we denote by $\eta \cup \{x\}$ the vertex set where we place a vertex at position x. Given a graph ξ, we obtain ξ^x by augmenting the vertex set η to $\eta \cup \{x\}$ as above and letting ξ^x inherit all the edges that were present in ξ. The edge set of ξ^x is completed by additionally drawing edges from x to the vertices of η, independently of everything else, with probabilities prescribed by the same adjacency function φ. This can be easily generalized to obtain $\eta \cup \{x_1, \ldots, x_m\}$ and $\xi^{x_1, \ldots, x_m}$ for any $m \in \mathbb{N}$ and any set of points $x_1, \ldots, x_m \in \mathbb{X}$.

Connection events. Given $x, y \in \eta$, we say that x and y are *adjacent* in ξ, denoted $x \sim y$, if there exists an edge with endpoints x and y. We say that x and y are connected in ξ, denoted $x \longleftrightarrow y$ in ξ, if there exists a finite sequence of adjacent vertices in ξ starting at x and ending at y. That is, either $x = y$ or there exist $n \geq 2$ and $x_1, \ldots, x_n \in \eta$ such that $x \sim x_1$, $y \sim x_n$, and $x_i \sim x_{i+1}$ in ξ for all $i \in \{1, \ldots, n-1\}$. Next, given $x \in \eta$ and a (possibly random) set $B \subset \mathbb{X}$, we say that $x \longleftrightarrow B$ in ξ if there exists $y \in \eta \cap B$ such that $x \longleftrightarrow y$ in ξ. Furthermore, because of the non-atomicity of ν, almost surely any fixed point $x \in \mathbb{X}$ will not be present in η, therefore (in most cases) the relevant events to consider are of the form $x \longleftrightarrow y$ in $\xi^{x,y}$ or $x \longleftrightarrow B$ in ξ^x.

Probability measure and expectation. All random objects introduced above are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P}_\lambda)$, where the parameter $\lambda > 0$ denotes the intensity of the underlying marked Poisson point process. The stationary law $\mathbb{P}_\lambda$ governs both the random vertex set η and the independent edge configuration determined by the adjacency function φ. We choose to handle Palm expectations implicitly via augmented configurations ξ^x rather than by explicitly changing measure. Expectation with respect to $\mathbb{P}_\lambda$ is denoted by $\mathbb{E}_\lambda$.

1.3 Main Result

We start by defining the relevant quantities in order to state our main result, Theorem 1.8. First and foremost, we are interested in the set of all points connected by a path to a given vertex of our graph, i.e., in the cluster of that vertex.

Definition 1.1 (Cluster of a point). *For a given point $x \in \mathbb{X}$, we define the cluster $\mathscr{C}(x)$ of x in ξ^x by*

$$\mathscr{C}(x) = \mathscr{C}(x, \xi^x) := \{y \in \eta \mid x \longleftrightarrow y \text{ in } \xi^x\}. \tag{1.1}$$

Remark 1.2. *Since a fixed point $x \in \mathbb{X}$ is almost surely not contained in ξ, it does not generally make sense to replace ξ^x by ξ in the above definition.*

In our analysis, we focus on the cluster of a point present at the origin (denoted by $\bar{0}$) and force this point to have the minimal weight 1. This choice serves as a convenient normalization and provides a fixed reference point throughout the analysis. The definitions introduced below could equally well be formulated by assigning a different deterministic weight to the point at the origin, or by sampling its weight according to the underlying weight distribution. Under the general regularity assumptions imposed on the model, the local connectivity properties of a point are uniformly controlled across admissible weights, and the qualitative behavior of the associated cluster does not depend sensitively on the particular choice made at the origin. In this sense, the quantities defined below are expected to reflect the typical behavior of clusters in the model.

We now introduce the main objects of interest.

Definition 1.3 (Tail function, percolation probability, expected susceptibility). *For $\lambda \geq 0$, we define*

- *the tail function $\theta_\lambda \colon \mathbb{N} \to [0,1]$ by*

$$\theta_\lambda(k) := \mathbb{P}_\lambda\left(|\mathscr{C}(\bar{0},1)| \geq k\right), \tag{1.2}$$

- *the percolation probability $\theta(\lambda)$ by*

$$\theta(\lambda) := \lim_{k \to \infty} \theta_\lambda(k) = \mathbb{P}_\lambda\left(|\mathscr{C}(\bar{0},1)| = \infty\right), \tag{1.3}$$

- *the expected susceptibility χ_λ by*

$$\chi_\lambda := \mathbb{E}_\lambda\left[|\mathscr{C}(\bar{0},1)|\right]. \tag{1.4}$$

We use these quantities to describe the behavior of the model in different regimes; the transition between those is governed by the following critical intensities:

Definition 1.4 (Critical intensities). *We define the critical intensities*

$$\lambda_T := \inf\{\lambda \geq 0 \colon \mathbb{E}_\lambda[|\mathscr{C}(\bar{0},1)|] = \infty\},$$
$$\lambda_c := \sup\{\lambda \geq 0 \colon \theta(\lambda) = 0\}.$$

A first result states that a non-trivial phase transition in terms of the expected cluster size occurs in the model:

Lemma 1.5 (Non-triviality of the phase transition). *Under the assumptions* (A) *and* (NB), *the critical intensities λ_T and λ_c defined above satisfy $0 < \lambda_T \leq \lambda_c < \infty$.*

The proof of Lemma 1.5 is deferred to Appendix II.

In the above framework, the adjacency function φ may depend on the vertex weights in a monotone way. This general setting covers both the classical *non-weighted* random connection model, where the connection probability depends only on spatial distance, and more general *weighted* models in which the connection range or intensity varies with the weights. To make this distinction explicit, and since both cases play a central role in our analysis, we now introduce two subclasses of the general model. Both satisfy Assumptions (A) and (NB). The first is the *non-weighted random connection model*, obtained when the connection function is independent of the vertex weights, and the second is the *min-reach random connection model*, which retains weight dependence but imposes a deterministic geometric control on connections via a reach function R depending on the minimum of the two weights.

Model (Non-weighted RCM). *Let a model satisfy Assumptions* (A). *We refer to the model as the* non-weighted random connection model *if the adjacency function φ is independent of the vertex weights, that is, if there exists a measurable non-increasing function $\phi : [0, \infty) \to [0, 1]$ such that*

$$\varphi(r; a, b) = \phi(r) \qquad \text{for all } r \geq 0, \ a, b \geq 1. \tag{NonW}$$

Remark 1.6. *This model coincides with the classical random connection model introduced by Penrose [31]. In particular, we allow $\phi(r) > 0$ for all $r > 0$, so that the connection range may be unbounded.*
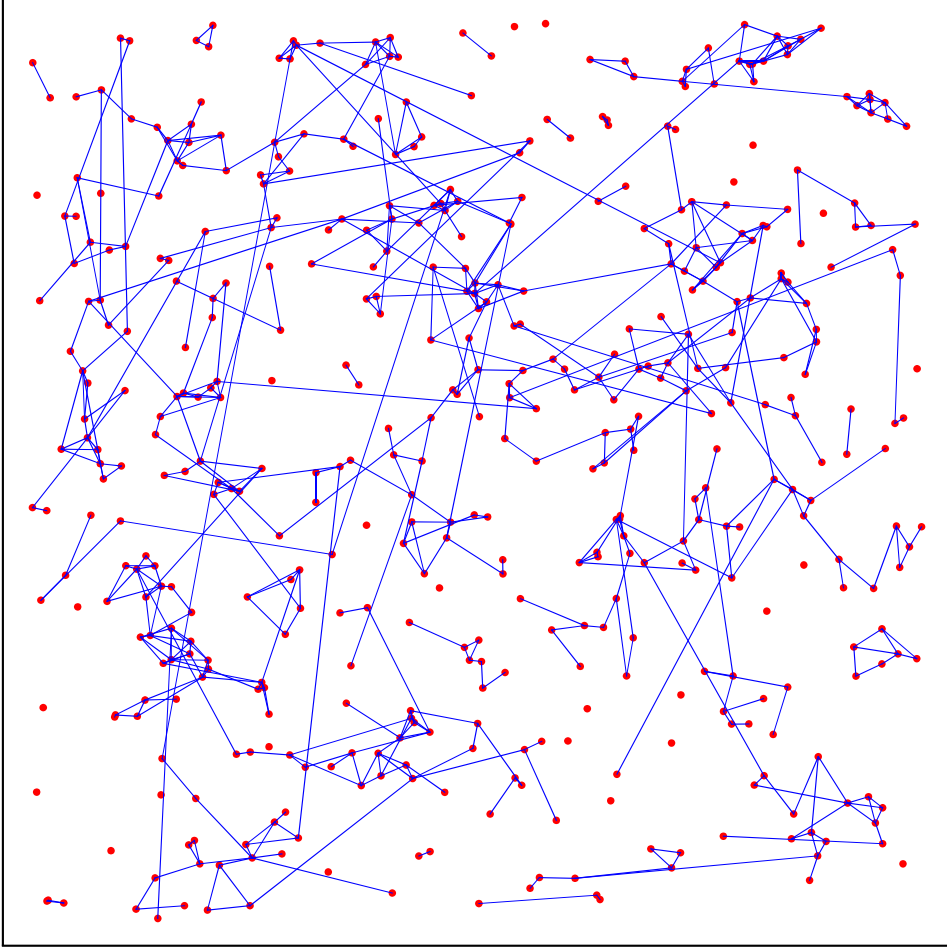

Figure 1: Realization of a non-weighted random connection model in $\mathbb{R}^2$ with the adjacency function given by $\phi(r) = 1 - \exp(-r^{-3})$.

Model (Min-reach RCM). *Let a model satisfy Assumptions* (A). *We refer to it as* min-reach random connection model *if the following additional assumption is satisfied: There exists a non-decreasing function $R : [1, \infty) \mapsto (0, \infty)$ such that for every $a, b \geq 1$*

$$\varphi(r; a, b) = 0 \text{ if } r > R(\min\{a, b\}) \text{ and } \varphi(r; 1, 1) > 0 \text{ if } r < R(1). \tag{MinR}$$

Remark 1.7. *The function R — which we call the reach function — prescribes a deterministic geometric upper bound on the possible interaction range of each vertex to its weight. The min-reach condition ensures that two vertices can not connect if their Euclidean distance exceeds the minimum of their respective reach radii.*

Furthermore, in order to derive the sharpness of the phase transition, we impose an additional assumption on the min-reach RCM throughout the paper. Specifically, we assume an exponential moment condition (EMC) on the weight distribution — ensuring, together with (MinR), that long edges in the percolation graph are highly improbable: There exist constants $C > 0$ and $\varepsilon > 0$ such that

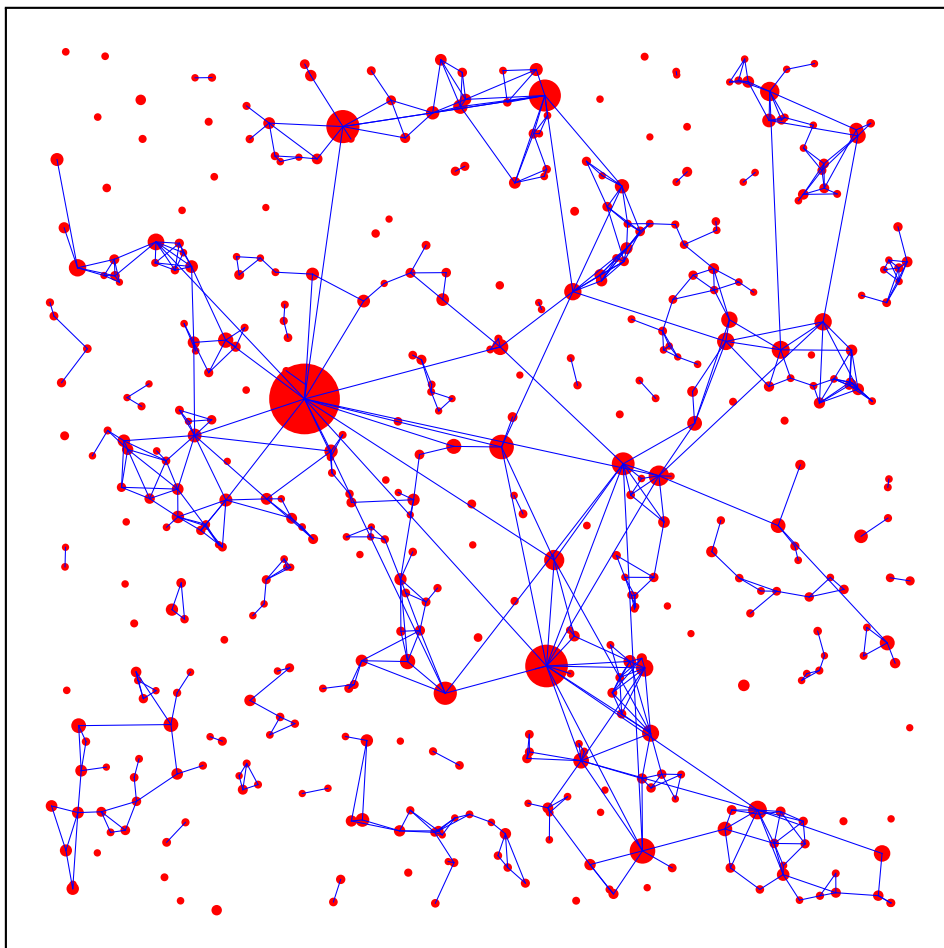$$\int_1^\infty \exp\left\{CR(m)^{d+\varepsilon}\right\} \pi(\mathrm{d}m) < \infty. \tag{EMC}$$

Figure 2: Realization of a min-reach random connection model in $\mathbb{R}^2$ with a Pareto-tailed weight distribution π. Displayed node radii are proportional to the weights. The adjacency function is given by $\varphi(r, a, b) = \mathbf{1}_{\{r \leq \min(a,b)\}} \left(1 - e^{-ab/r^3}\right)$.

Standing assumptions. Throughout the paper, all results are derived under Assumptions (A) and (NB). For the min-reach model we additionally impose the exponential moment condition (EMC). The condition $\varphi(r; 1, 1) > 0$ for $r < R(1)$ ensures that the *lower* bound in (NB) holds automatically, while the *upper* bound in (NB) follows from the integrability guaranteed by (EMC). Together, assumptions (MinR) and (EMC) therefore imply the full neighborhood bound (NB).

We are now ready to state and discuss our main result:

Theorem 1.8 (Sharpness of the phase transition)**.** *Assume that the adjacency function φ satisfies Assumptions (A) and (NB). Suppose further that either*

(i) *the model is* non-weighted, *that is,* φ *is independent of the vertex weights* (assumption (NonW) *is verified*), *or*

(ii) *the model is* min-reach *in the sense of* (MinR), *and the weight distribution* π *satisfies the exponential moment condition* (EMC).

Then the following hold:

(a) *(Subcritical regime.) For any* $\lambda < \lambda_T$, *there exists a constant* $C_\lambda > 0$ *such that*

$$\theta_\lambda(k) \leq \exp(-C_\lambda k) \qquad \text{for all } k \in \mathbb{N}.$$

(b) *(Supercritical regime.) For any* $\lambda_0 > \lambda_T$, *there exists a constant* $c_0 > 0$ *such that for all* $\lambda \in [\lambda_T, \lambda_0]$,
$$\theta(\lambda) \geq c_0 \left(\lambda - \lambda_T\right).$$

In particular, the critical parameters coincide:

$$\lambda_c = \lambda_T. \tag{1.5}$$

Remark 1.9. *Notice that identity* (1.5) *is known for the non-weighted RCM (see [27]) and in the Poisson-Boolean case (see [13]) under polynomial moment conditions for the radii distribution. Very recently, for the non-weighted RCM, the exponential decay from the first part of the statement was shown by Higgs [19] using an alternative method.*

1.4 Discussion

In this section, we first comment on the assumptions and scope of our framework — before discussing the main results and their relation to previous work. We begin by examining the hypotheses, since they delineate the class of models to which our methods apply and highlight the points where further generalization is possible.

The broad structural setting in which our analysis takes place is established by Assumption (A) together with the neighborhood bound (NB). The former encodes isotropy, symmetry, and monotonicity of the adjacency function, while the latter provides uniform bounds on the expected neighborhood size of a vertex. These hypotheses are deliberately formulated in a minimal and flexible way. They capture the essential geometric and probabilistic properties needed for the OSSS analysis and for the existence of a non-trivial phase transition, while remaining broad enough to include a large class of continuum percolation models. In particular, Assumptions (A)–(NB) are satisfied by the classical (non-weighted) random connection model [31], by the Poisson–Boolean model [13, 24], by finite-range random connection models such as those analyzed in [14], and by their marked or weighted versions [10, 5, 15]. Several relaxations would remain compatible with our proofs. For instance, spatial isotropy in (A.1) could be weakened to mere translation invariance, allowing the connection probability to depend on the relative orientation of vertices, and the monotonicity assumptions in (A.2)–(A.3) could hold only asymptotically without affecting the overall argument.

Assumption (NB) plays a key technical role in our framework. It provides two-sided uniform control on the expected number of potential neighbors of any vertex. The upper bound allows us to bound the local cluster exploration from above by suitable Galton–Watson branching processes of finite expectation and thereby to obtain the positivity of λ_T. The lower bound, on the other hand, allows us to view our random graph as an augmentation of an unweighted finite-range random connection model. Since the latter is known to percolate, we can deduce the finiteness of λ_c. The uniform neighborhood bound (NB) is therefore tailored

to yield a two-sided domination for a point neighborhood; it is sufficient but not always necessary for the existence of a non-trivial phase transition. In particular, for scale-free or inhomogeneous long-range models (as in [8]), the upper bound in (NB) is not satisfied, yet a phase transition has been established by different means. A detailed argument establishing the coupling is provided in Appendix II.

The additional *min-reach condition* (MinR) marks the specific subclass of weighted RCM for which we establish sharpness in full detail. Within this setting we manage to control the influences and revealment probabilities of vertices with arbitrarily large weights, relating them quantitatively through the geometry imposed by the weights. This control is essential for adapting the OSSS framework to weighted continuum percolation and for deriving the key differential inequality that drives our proof. Although somewhat restrictive, the min-reach condition represents a natural starting point for a systematic study of continuum percolation with unbounded weights. In particular, it includes Poisson-Boolean models with random unbounded radii. In future work, one could aim to replace the deterministic cutoff by a quantitative decay condition on the connection function, for instance

$$\frac{1}{r^{\alpha_1}} \leq \varphi(r;1,1) \leq \frac{1}{r^{\alpha_2}} \qquad \text{for some } \alpha_1 \geq \alpha_2 > d,$$

or to impose the neighborhood control only in expectation rather than almost surely. Such generalizations would bring the present framework closer to that of long-range or scale-free percolation and could likely be handled by suitable refinements of our arguments.

In our definition of the min-reach model we did not assume (NB) explicitly. However, this condition is in fact implied by the geometric constraint (MinR) together with a mild integrability assumption on the weight distribution. Indeed, one may impose the moment condition

$$\int_1^\infty R(m)^d \, \pi(\mathrm{d}m) < \infty,$$

which, combined with (MinR), entails (NB) and thus guarantees the non-triviality of the phase transition. More precisely, the geometric cutoff (MinR) automatically provides the lower bound in (NB), while the exponential moment condition (EMC) ensures the corresponding upper bound, so that together they yield the full two-sided neighborhood control.

It is important to emphasize that the OSSS-based analysis developed in Section 2 is valid under Assumptions (A) and (NB) alone; it does not require the min-reach hypothesis. The latter condition is introduced to obtain uniform bounds on influences and revealments when weights become large, ensuring that the technical assumptions behind our differential inequality can be verified. In particular, the OSSS exploration and the derivative formula are formulated and proved in a considerably more general setting than the one ultimately used for sharpness.

Relation to previous work. The present paper contributes to a long line of research on continuum percolation models, from the classical Gilbert graph [16] and Penrose's random connection model [31] (see also [26, 23, 18]) to modern extensions incorporating marks or weights [10, 5]. Penrose already considered unbounded connection ranges but without random weights; in contrast, weighted or marked models such as those studied by Dickson and Heydenreich [10] and by Caicedo and Dickson [5] assume a uniformly bounded influence of marks. Similarly, Pabst [30] recently investigated higher-dimensional simplicial structures in marked RCMs, but again under bounded-range assumptions. Our work establishes sharpness of the phase transition for random connection models that allow for both *infinite range of interaction* and *unbounded weights*. Together with the very recent work of Chebunin and Last [7],

this places the present results among the first sharpness results in such settings, extending the class of weighted models for which sharpness is known beyond the Poisson–Boolean case [13].

From a methodological standpoint, our approach also extends the scope of several key analytical tools in modern percolation theory. We build on the OSSS inequality of O'Donnell, Saks, Schramm, and Servedio [29], which Duminil-Copin, Raoufi, and Tassion adapted to the continuum setting to prove sharpness for Poisson–Boolean percolation with unbounded radii [13]. At the same time, we draw on Hutchcroft's algorithmic exploration method [21], which provides a discrete analogue of OSSS arguments for the random-cluster model. Our proof merges these two lines of reasoning: Hutchcroft's exploration algorithm is extended to a weighted, long-range continuum model, while the influence bounds from [13] are generalized to control rare vertices with very large weights. This combination allows us not only to prove sharpness under new structural conditions, but also to demonstrate that the OSSS framework and Hutchcroft's algorithmic analysis remain robust tools even beyond discrete or finite-range settings.

Using a kindred line of reasoning, applying OSSS after discretizing space, Faggionato and Mimun [14] obtained exponential decay for the diameter of clusters in finite-range RCM and Poisson–Boolean models. It is instructive to compare such use of the OSSS inequality with that of Last, Peccati, and Yogeshwaran [24]. The latter work introduces stopping-set techniques that extend the OSSS inequality directly to continuum models, thereby avoiding any need for spatial discretization. This approach is particularly elegant because it translates the notion of decision trees and revealments — originally defined for discrete Boolean functions — into a geometric and measure-theoretic framework, providing a conceptual bridge between discrete and continuum structures. Despite the fascinating methodology, the sharpness results therein also apply only to finite-range percolation models.

Infinite-range models remained out of reach until, very recently, a completely different method was used by Higgs [19] to show exponential decay of the cluster size in the infinite-range RCM — under minimal assumption on the adjacency function. We learned about this exciting alternative approach — building upon a stochastic comparison technique due to Vanneuville [33] — while finalizing this work. In parallel, Chebunin and Last [7] independently obtained sharp phase transition results for random connection models, including weighted settings, by a different approach.

Another interesting development is due to Hirsch, Jahnel, and Muirhead [20], who proved sharp phase transition for a particular class of Cox percolation models via the OSSS inequality in combination with a coarse-graining technique. Thereby they extended the OSSS framework to random environments with spatial dependence, where the vertex process itself is a Cox process, i.e., a Poisson process driven by a random intensity measure.

Meanwhile, the lace-expansion program for high-dimensional random connection models [18] and its extension to the marked case [10] established mean-field exponents but required bounded marks. Our work complements these approaches by providing a unified algorithmic framework that captures sharpness in an infinite-range, unbounded-marks setting, while remaining consistent with the general theory of marked, inhomogeneous percolation models developed in [15, 8]. We hope that this work serves as a blueprint for analyzing a broad class of weighted RCMs, illustrating how sharpness results can be established in such general settings — with the min-reach assumption providing a natural sufficient condition under which sharpness persists as a particular example.

1.5   Proof structure

The paper is organized as follows: In Section 2.1, we approximate our model by looking at a finite window of the Euclidean space and cutting that weighted space into small boxes. We

then consider a Bernoulli percolation on the complete graph on those boxes, which are open with the probability of containing at least one Poisson point scattered with intensity λ; an edge is open with a probability governed by our adjacency function φ. In that setting, we derive a Russo-type derivative formula in Section 2.2, Corollary 2.5 (which is non-trivial since the underlying graph is infinite: we discretize but do not truncate the weights). In Sections 2.3 and 2.4, following an ingenious idea of Duminil-Copin, Raoufi and Tassion, we employ an exploration scheme and the corresponding OSSS inequality (2.15), to get an estimate on the probability of a large cluster containing the origin. The exploration scheme is a modification of Hutchcroft's algorithm from [21] that accounts for the specifics of our Bernoulli percolation. In Section 3, we combine the OSSS estimate with the derivative formula to obtain a differential inequality for the tail of the cluster size (Proposition 3.6). The key step in the proof of the proposition is to show that a technical assumption is satisfied (Assumption 3.3) which we show for the min-reach RCM in Section 4. In Section 5, we show in the finite-volume discrete setting that the differential equation implies exponential decay in the subcritical regime $\lambda < \lambda_T$ (Lemma 5.1) and the first part of our main result Theorem 1.8 follows by taking the appropriate limits (Proposition 5.3). Similar arguments in the spirit of [21] allow us to show a positive probability of the infinite cluster containing the origin in the supercritical regime $\lambda > \lambda_T$ (Proposition 5.5).

2 Discretizing, employing an exploration scheme and using the OSSS inequality

Before deriving the differential inequality leading to our main result, we first introduce a lattice approximation of our model. This allows us to apply the exploration-scheme approach of Hutchcroft [21], which is naturally formulated in a discrete setting. In particular, we discretize both the spatial positions and the weight marks.

One might expect that a continuum analogue of Hutchcroft's arguments is obtained in the spirit of Last, Peccati, and Yogeshwaran [24]. Their framework, however, relies on finite-range interactions, while our adjacency function is long-range, and the corresponding adaptation is not presently available. Since the discretized version already provides the ingredients we need for our proof, we follow this simpler and more direct route.

2.1 Approximation of lattice models

We now describe the discrete version of our model, obtained by replacing space and weights by suitable grids and defining a product Bernoulli measure on the resulting graph.

We follow the classical idea of finite lattice approximation, similar to the approach in Meester (Section 3 of [27]), but in our case we also need to discretize (but not truncate) the weights. Indeed, we consider two integers $L, n \in \mathbb{N}$ and define the cylinder $\Lambda_n^L = [-L - 2^{-n-1}, L + 2^{-n-1}]^d \times [1, \infty)$ which we then divide into small boxes of side length 2^{-n}. Afterwards, we put a vertex inside each box, which we consider to be open independently, with a probability equal to the probability that the Poisson point process of intensity λ has at least one point present inside that box. Next, we draw edges between vertices according to the original probabilities prescribed by φ, independently for every pair of vertices. This leads to a Bernoulli percolation model on both the vertices and the edges of a complete graph.

More precisely, we consider the (infinite) graph $G_n^L = (V_n^L, E_n^L)$, characterized as follows: $V_n^L := U_n^L \times M_n$ is the set of vertices, where

$$U_n^L := \left\{ \bar{u} \in 2^{-n} \mathbb{Z}^d : \bar{u}_i \in [-L, L] \text{ for all } i \in \{1, \ldots, d\} \right\} \tag{2.1}$$

and

$$M_n := \{m \in 2^{-n}\mathbb{N} \mid m = 1 + \frac{l}{2^n}, \, l \in \mathbb{N}\} \tag{2.2}$$

if π is absolutely continuous; if π is discrete, we set M_n to be its support. Moreover, for $m \in M_n$, let us denote $\pi([m, m + 2^{-n}])$ by $\Pi(m, n)$ if π is absolutely continuous and set $\Pi(m, n) = \pi(\{m\})$ if π is discrete. Recall that the map $m \mapsto \Pi(m, n)$ is non-increasing on $[1, \infty)$, for every $n \in \mathbb{N}$, by our assumptions on π. The edge set E_n^L are all possible distinct pairs of vertices, i.e., we consider the complete graph on V_n^L.

Next, we construct a probability measure on the configuration space $\{0, 1\}^{V_n^L \cup E_n^L}$ as follows: For a given realization $\omega \in \{0, 1\}^{V_n^L \cup E_n^L}$ and a vertex $(\bar{u}, m) \in V_n^L$, we say that $(\bar{u}, m)$ is open in ω if $\omega((\bar{u}, m)) = 1$ and we say that it is closed if $\omega((\bar{u}, m)) = 0$. We use the same terminology for the edges and define $P_\lambda^{L,n}$ as the probability measure on $\{0, 1\}^{V_n^L \cup E_n^L}$ given by the inhomogeneous Bernoulli percolation on both sites and edges such that, for every $u \in V_n^L$ and $e \in E_n^L$, we have

$$P_\lambda^{L,n}\left(u = (\bar{u}, m) \text{ is open}\right) = 1 - e^{-\lambda 2^{-nd} \times \Pi(m,n)},$$

$$P_\lambda^{L,n}\left(e = ((\bar{u}_1, m_1), (\bar{u}_2, m_2)) \text{ is open}\right) = \varphi(|\bar{u}_1 - \bar{u}_2|, m_1, m_2).$$

Notice that the states of all vertices and edges are stochastically independent of one another (in particular, the openness of an edge does not depend on whether its endpoints are open). Moreover, the probability of an edge to be open is constant in the intensity λ.

For a vertex $v \in V_n^L$, we denote by $\mathcal{C}(v) := \mathcal{C}_n^L(v, \omega)$ the (open) cluster of v which is defined as the set of all open vertices v' connected to it by an open path in $\omega \in \{0, 1\}^{V_n^L \cup E_n^L}$. More precisely, $v' \in \mathcal{C}(v)$ if there exists an $l \in \mathbb{N}$ and a sequence of open vertices $(v_1, \ldots, v_l = v')$ such that with the convention $v_0 = v$ the edges $e_i = (v_{i-1}, v_i)$ are also open, for each $i \in \{1, .., l\}$. Note the slightly unusual convention that we do not require the vertex v itself to be open but always include it in the cluster. We omit the indices to lighten the cluster notation; nevertheless, we use a different calligraphy so it is not confused with its continuum counterpart.

Furthermore, for $k \in \mathbb{N}$, we define

$$\theta_\lambda^{L,n}(k) = P_\lambda^{L,n}(|\mathcal{C}(\bar{0}, 1)| \geq k). \tag{2.3}$$

Before proceeding, two important remarks about this construction deserve consideration.

Remark 2.1. *Working in finite volume breaks translation invariance: points near the center and near the boundary behave differently with respect to connectivity. One could correct for this, as in [27], by modifying the connection function — for example, defining*

$$1 - \varphi_L(\bar{x}, a, b) = 1 - \prod_{\bar{z} \in \mathbb{Z}^d} \left(1 - \varphi(\bar{x} + 2L\bar{z}, a, b)\right).$$

We choose not to adopt this modification, as translation invariance is not required for our analysis. Indeed, Assumption (A) implies $E_\lambda^{L,n}[|\mathcal{C}(\bar{0}, m)|] \geq E_\lambda^{L,n}[|\mathcal{C}(\bar{u}, m)|]$ for all $L, n \in \mathbb{N}$, $\bar{u} \in U_n^L$, $m \in M_n$ and $\lambda > 0$, which is sufficient for our results. Furthermore, note that our arguments still work if this inequality holds only up to a positive multiplicative constant, which may prove to be useful in slightly different settings.

Remark 2.2. *From a measure-theoretic perspective, introducing a new probability measure for the discretized model is not strictly necessary. However, the analysis presented in this section might be of independent interest for researchers working on discrete percolation, and it is more natural to formulate it in the traditional discrete setting. Furthermore, to deduce our continuum results, it suffices to establish convergence of specific quantities rather than full convergence of probability measures; this is carried out in Appendix III.*

2.2 Derivative formula

The graph $G_n^L = (V_n^L, E_n^L)$ is an infinite graph; however, under the measure $P_\lambda^{L,n}$, its open components are almost surely finite for any $\lambda > 0$. Indeed, even though the number of open edges can be infinite, the number of open vertices is almost surely finite, which allows us to obtain a Margulis-Russo-type differential formula (compare to the formula in Appendix I).

The idea is to first consider a finite graph in which we delete all vertices with weight above a certain threshold, apply a derivative formula for local functions, and then take limits to derive the formula on the infinite graph.

Fix $\lambda > 0$ and $L, n \in \mathbb{N}$. Let $H \in \mathbb{N}$ with $H \geq 2$, and consider the finite graph $G_H = (V_H, E_H)$, where

$$V_H := U_n^L \times M_H, \qquad M_H := \{m \in M_n \mid m \leq H\},$$

and E_H is the set of all possible pairs of distinct vertices in V_H.

Define the configuration space

$$\Omega_H := \{0, 1\}^{V_H \cup E_H},$$

and define the probability measure P_H as the Bernoulli product measure on Ω_H such that, for any $x \in V_H \cup E_H$, we have

$$P_H(x \text{ is open}) = P_\lambda^{L,n}(x \text{ is open}).$$

For any $\omega \in \Omega_H$, we may write

$$P_H(\omega) = \left[\prod_{(\bar{u},m) \in U_n^L \times M_H} p_\lambda(\bar{u}, m)^{\omega((\bar{u},m))} (1 - p_\lambda(\bar{u}, m))^{1-\omega((\bar{u},m))} \right] \alpha(\omega), \qquad (2.4)$$

where

$$p_\lambda(\bar{u}, m) = 1 - e^{-\lambda 2^{-nd} \Pi(m,n)},$$

and

$$\alpha(\omega) = \prod_{e \in E_H} P_H(e \text{ open})^{\omega(e)} P_H(e \text{ closed})^{1-\omega(e)}.$$

Note that $\alpha(\omega)$ does not depend on λ.

We set $I_{(\bar{u},m)} = \omega((\bar{u}, m))$, denote by E_{P_H} the expectation under P_H, and by Cov_{P_H} the corresponding covariance.

Lemma 2.3. *Let $F : \Omega_H \to \mathbb{R}$. Then for every $\lambda > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E_{P_H}[F] = \sum_{(\bar{u},m) \in U_n^L \times M_H} \frac{2^{-nd} \Pi(m,n)}{p_\lambda(\bar{u},m)} \text{Cov}_{P_H}(F, I_{(\bar{u},m)}). \qquad (2.5)$$

Remark 2.4. *Since we consider percolation on both edges and vertices, the measure P_H corresponds to the product ($q = 1$) case of the random-cluster measure from [22]. Formula (2.5) is the analogue of [22, Eq. (2.1)]. For this proof, replicated here for the reader's convenience, we follow the ideas from [4, Proposition 4].*

Proof. For each $(\bar{u}, m) \in U_n^L \times M_H$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} p_\lambda(\bar{u}, m) = 2^{-nd} \Pi(m,n) \big(1 - p_\lambda(\bar{u}, m)\big).$$

The derivatives of the two factors in (2.4) are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} p_\lambda(\bar{u},m)^{I_{(\bar{u},m)}} = I_{(\bar{u},m)}\, 2^{-nd}\Pi(m,n)(1-p_\lambda(\bar{u},m))p_\lambda(\bar{u},m)^{I_{(\bar{u},m)}-1},$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(1-p_\lambda(\bar{u},m))^{1-I_{(\bar{u},m)}} = -(1-I_{(\bar{u},m)})\, 2^{-nd}\Pi(m,n)(1-p_\lambda(\bar{u},m))^{1-I_{(\bar{u},m)}}.$$

Combining these contributions and using that $\alpha(\omega)$ is λ-independent yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}P_H(\omega) = \sum_{(\bar{u},m)} 2^{-nd}\Pi(m,n)\left(\frac{I_{(\bar{u},m)}}{p_\lambda(\bar{u},m)}-1\right)P_H(\omega). \tag{2.6}$$

Differentiating under the sum yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E_{P_H}[F] = \sum_{\omega\in\Omega_H} F(\omega)\frac{\mathrm{d}}{\mathrm{d}\lambda}P_H(\omega)$$

$$= \sum_{(\bar{u},m)} 2^{-nd}\Pi(m,n)\, E_{P_H}\left[F\left(\frac{I_{(\bar{u},m)}}{p_\lambda(\bar{u},m)}-1\right)\right].$$

Finally,

$$E_{P_H}\left[F\left(\frac{I_{(\bar{u},m)}}{p_\lambda(\bar{u},m)}-1\right)\right] = \frac{1}{p_\lambda(\bar{u},m)}\operatorname{Cov}_{P_H}(F,I_{(\bar{u},m)}),$$

which gives (2.5). $\qquad\square$

For a big class of functions, we can extend the preceding lemma to the infinite graph $G_n^L = (V_n^L, E_n^L)$. In particular, if we fix $k \in \mathbb{N}$, the vertex $(\bar{0},1) \in V_n^L$ and consider the function $\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}$, then we obtain the following result.

Corollary 2.5. *For any $k \in \mathbb{N}$ and $\lambda > 0$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\, P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| \geq k\big) = \sum_{(\bar{u},m)\in V_n^L} \frac{2^{-nd}\Pi(m,n)}{p_\lambda(\bar{u},m)}\operatorname{Cov}_{P_\lambda^{L,n}}\big(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)}\big). \tag{2.7}$$

Proof. Define

$$N_H := \{(\bar{u},m) \in U_n^L \times M_n : m > H\}, \qquad A_H := \{\omega : \omega(\bar{u},m) = 0\ \forall\,(\bar{u},m) \in N_H\}.$$

Since the states of all vertices are independent under $P_\lambda^{L,n}$ and

$$P_\lambda^{L,n}\big((\bar{u},m)\text{ is open}\big) = p_\lambda(\bar{u},m) = 1 - e^{-\lambda 2^{-nd}\Pi(m,n)},$$

we obtain

$$P_\lambda^{L,n}(A_H) = \prod_{(\bar{u},m)\in N_H}\big(1-p_\lambda(\bar{u},m)\big)$$

$$= \prod_{(\bar{u},m)\in U_n^L\times M_n,\ m>H} e^{-\lambda 2^{-nd}\Pi(m,n)}$$

$$= \left(\prod_{m\in M_n,\ m>H} e^{-\lambda 2^{-nd}\Pi(m,n)}\right)^{|U_n^L|}$$

$$= \exp\left(-\lambda 2^{-nd}|U_n^L|\sum_{m\in M_n,\ m>H}\Pi(m,n)\right)$$

$$= \exp\big(-\lambda 2^{-nd}|U_n^L|\,\pi[H,\infty)\big) \xrightarrow[H\to\infty]{} 1.$$

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We now work under the conditional measure $P_\lambda^{L,n}(\,\cdot\mid A_H)$. On the event A_H, all vertices with $m > H$ are closed, so the cluster $\mathcal{C}(\bar{0},1)$ in the infinite graph coincides with the cluster $\mathcal{C}_H(\bar{0},1)$ in the finite graph G_H. By the product structure, the conditional law of the configuration on $V_H \cup E_H$ given A_H coincides with P_H. Hence,

$$P_H\big(|\mathcal{C}_H(\bar{0},1)| \geq k\big) = P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| \geq k \mid A_H\big),$$

and, for $(\bar{u},m) \in V_H$,

$$\mathrm{Cov}_{P_H}\big(\mathbb{1}_{\{|\mathcal{C}_H(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)}\big) = \mathrm{Cov}_{P_\lambda^{L,n}}\big(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)} \mid A_H\big). \tag{2.8}$$

Applying Lemma 2.3 with $F(\omega) = \mathbb{1}_{\{|\mathcal{C}_H(\bar{0},1)|\geq k\}}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} P_H\big(|\mathcal{C}_H(\bar{0},1)| \geq k\big) = \sum_{(\bar{u},m)\in U_n^L\times M_H} \frac{2^{-nd}\Pi(m,n)}{p_\lambda(\bar{u},m)} \mathrm{Cov}_{P_H}\big(\mathbb{1}_{\{|\mathcal{C}_H(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)}\big). \tag{2.9}$$

Plugging (2.8) into (2.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| \geq k \mid A_H\big) = \sum_{(\bar{u},m)\in U_n^L\times M_n} \frac{2^{-nd}\Pi(m,n)}{p_\lambda(\bar{u},m)} \mathrm{Cov}_{P_\lambda^{L,n}}\big(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)} \mid A_H\big),$$

where the terms with $m > H$ vanish by definition of A_H. As $H \to \infty$, we have $P_\lambda^{L,n}(A_H) \to 1$, so

$$P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| \geq k \mid A_H\big) \xrightarrow[H\to\infty]{} P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| \geq k\big),$$

and, for each $(\bar{u},m) \in V_n^L$,

$$\mathrm{Cov}_{P_\lambda^{L,n}}\big(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)} \mid A_H\big) \xrightarrow[H\to\infty]{} \mathrm{Cov}_{P_\lambda^{L,n}}\big(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)}\big).$$

Moreover, if $m \leq H$, conditioning on A_H leads to

$$\mathbb{E}_{P_\lambda^{L,n}}\big[I_{(\bar{u},m)} \mid A_H\big] = p_\lambda(\bar{u},m),$$

on the other hand, if $m > H$, we have

$$\mathbb{E}_{P_\lambda^{L,n}}\big[I_{(\bar{u},m)} \mid A_H\big] = 0.$$

Since F_H and $I_{(\bar{u},m)}$ are indicator functions, in both cases

$$\big|\mathrm{Cov}_{P_\lambda^{L,n}}(F_H, I_{(\bar{u},m)} \mid A_H)\big| \leq \mathbb{E}_{P_\lambda^{L,n}}\big[I_{(\bar{u},m)} \mid A_H\big] \leq p_\lambda(\bar{u},m),$$

and therefore

$$\sum_{(\bar{u},m)\in U_n^L\times M_n} \frac{2^{-nd}\Pi(m,n)}{p_\lambda(\bar{u},m)} \big|\mathrm{Cov}_{P_\lambda^{L,n}}\big(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)} \mid A_H\big)\big| \leq \sum_{(\bar{u},m)\in U_n^L\times M_H} 2^{-nd}\Pi(m,n)$$

$$\leq 2^{-nd}\sum_{\bar{u}\in U_n^L} 1 = L < \infty,$$

which implies, by dominated convergence, that we can pass to the limit $H \to \infty$ in the derivative. This yields (2.7). $\qquad\square$

Remark 2.6. *Notice that for all $(\bar{u}, m) \in V_n^L$,*

$$\text{Cov}_{P_\lambda^{L,n}}\big(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}, I_{(\bar{u},m)}\big) = p_\lambda(\bar{u},m)\big(1 - p_\lambda(\bar{u},m)\big) \times$$
$$\times P_\lambda^{L,n}\Big((\bar{u},m) \text{ is pivotal for } \{|\mathcal{C}(\bar{0},1)| \geq k\}\Big),$$

where we call a vertex v pivotal for an increasing event A if A occurs when v is open but not when it is closed, with all other states unchanged. This yields the alternative formula

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| \geq k\big) = \sum_{(\bar{u},m) \in V_n^L} \frac{\mathrm{d}}{\mathrm{d}\lambda} P_\lambda^{L,n}\Big((\bar{u},m) \text{ is open}\Big) \times$$
$$\times P_\lambda^{L,n}\Big((\bar{u},m) \text{ is pivotal for } \{|\mathcal{C}(\bar{0},1)| \geq k\}\Big).$$

2.3 OSSS inequality

As previously stated, in order to derive the differential inequality present in the next section, we make use of the OSSS inequality. More precisely, we use its generalization to covariances from [12] extended to decision forests in [21]. Before we state it, we need to make some auxiliary definitions. In the following, S denotes an arbitrary countable set.

Definition 2.7. *A probability measure μ on $\{0,1\}^S$ is said to be monotonic if*

$$\mu(\omega(s) = 1 \mid \omega|_F = \xi) \geq \mu(\omega(s) = 1 \mid \omega|_F = \zeta)$$

whenever $F \subset S$, $s \in S$ and $\xi, \zeta \in \{0,1\}^F$ are such that $\xi \geq \zeta$ (coordinatewise). Here $\omega|_F$ denotes the restriction of $\omega \in \{0,1\}^S$ to F.

Notice that any product measure on $\{0,1\}^S$ is monotonic. Furthermore, if μ_1 is a monotonic measure on $\{0,1\}^{S_1}$ and μ_2 is a monotonic measure on $\{0,1\}^{S_2}$, then the product $\mu_1 \otimes \mu_2$ is monotonic on $\{0,1\}^{S_1 \sqcup S_2}$. Here $S_1 \sqcup S_2$ denotes the disjoint union of the sets S_1 and S_2.

Definition 2.8. *A decision tree is a function $T : \{0,1\}^S \to S^{\mathbb{N}}$ satisfying the following property: $T_1(\omega) = s_1$ for some fixed $s_1 \in S$ and for every $r \geq 2$ there exists a function $f_r : (S \times \{0,1\})^{r-1} \to S$ such that*

$$T_r(\omega) = f_r[(T_i, \omega(T_i))_{i=1}^{r-1}].$$

This means that T is a deterministic procedure that first queries the value of $\omega(s_1)$ and at each subsequent step it chooses which element of S to investigate next based on the information it has gathered so far.

Let μ be a probability measure on $\{0,1\}^S$ and let ω be a random variable with law μ. Given a decision tree T and $r \geq 1$, let $\mathcal{F}_r(T)$ be the σ-algebra generated by the random variables $\{T_j(\omega) : 1 \leq j \leq r\}$ and let $\mathcal{F}(T) = \bigcup_r \mathcal{F}_r(T)$. For a measurable function $f : \{0,1\}^S \to [-1,1]$, we say that T computes f if $f(\omega)$ is measurable with respect to the μ-completion of $\mathcal{F}(T)$.

Then, for any $s \in S$ we define the *revealment probability*

$$\delta_s(T) := \mu(\exists r \geq 1 \text{ such that } T_r(\omega) = s). \tag{2.10}$$

Following Hutchcroft [21, Eq. (2.3)], which in turn adopts the notation introduced in [29], we define, for measurable functions $f, g : \{0,1\}^S \to \mathbb{R}$,

$$\text{Covr}_\mu[f, g] := \mu \otimes \mu\big(|f(\omega_1) - g(\omega_2)|\big) - \mu\big(|f(\omega_1) - g(\omega_1)|\big), \tag{2.11}$$

where ω_1, ω_2 are independent μ-distributed samples. If f and g are $\{0,1\}$-valued then

$$\mathrm{Covr}_\mu[f,g] = 2\mathrm{Cov}_\mu[f,g]. \tag{2.12}$$

Now, we extend this setup to the case of forests instead of trees, in the same way as in [21].

Definition 2.9. *A decision forest is a set of decision trees $F = \{T^i : i \in I\}$ for a countable set I. Given a decision forest F, let $\mathcal{F}(F)$ the smallest σ-algebra containing all of the σ-algebras $\mathcal{F}(T^i)$. Given a measure μ on $\{0,1\}^S$, a function $f : \{0,1\}^S \to \mathbb{R}$ and a decision forest F, we say that F computes f if f is measurable with respect to the μ-completion of the σ-algebra $\mathcal{F}(F)$. We also define the revealment probability $\delta_s(F,\mu)$ to be the probability under μ that there exists $i \in I$ and $r \geq 1$ such that $T_r^i(\omega) = s$.*

We are finally ready to state the generalization of the OSSS inequality, named after O'Donnell–Saks–Schramm–Servedio, Corollary 2.4 in [21].

Proposition 2.10 (OSSS inequality)**.** *Let S be a finite or countably infinite set and let μ be a monotonic measure on $\{0,1\}^S$. Then for every pair of measurable, μ-integrable functions $f, g : \{0,1\}^S \to \mathbb{R}$ with f increasing and every decision forest F computing g we have that*

$$\frac{1}{2}|\mathrm{Covr}_\mu[f,g]| \leq \sum_{s \in S} \delta_s(F,\mu)\mathrm{Cov}_\mu[f,\omega(s)]. \tag{2.13}$$

We use this inequality in a particular way: in our case the measure μ will always be a product measure, and it is therefore convenient to work with influences rather than covariances. Given a product measure μ, a $\{0,1\}$-valued function f, and $s \in S$, we define the influence of the s-th coordinate by

$$\mathrm{INF}_s^\mu(f) = \mu(f(\omega) \neq f(\tilde{\omega})), \tag{2.14}$$

where $\tilde{\omega}$ is obtained from ω by resampling the s-th coordinate independently according to the marginal μ_s.

Corollary 2.11. *Let S be a finite or countably infinite set and let μ be a product measure on $\{0,1\}^S$. Then for every pair of measurable, μ-integrable functions $f, g : \{0,1\}^S \to \{0,1\}$ with f increasing and every decision forest F computing g we have*

$$|\mathrm{Cov}_\mu[f,g]| \leq \frac{1}{2} \sum_{s \in S} \delta_s(F,\mu)\,\mathrm{INF}_s^\mu(f). \tag{2.15}$$

Proof. Since f and g are $\{0,1\}$–valued, identity (2.12) gives

$$\frac{1}{2}\mathrm{Covr}_\mu[f,g] = \mathrm{Cov}_\mu[f,g].$$

Next, because μ is a product measure and f is increasing in $s \in S$, we can use the standard pivotal identity

$$\mathrm{Cov}_\mu(f,\omega(s)) = \frac{1}{2}\mathrm{INF}_s^\mu(f),$$

compare also with the first statement in Remark 2.6. Inserting these into the OSSS inequality (2.13) from the preceding proposition,

$$\tfrac{1}{2}\,|\mathrm{Covr}_\mu[f,g]| \leq \sum_{s \in S} \delta_s(F,\mu)\,\mathrm{Cov}_\mu(f,\omega(s)),$$

yields (2.15). $\qquad\square$

2.4 Cluster exploration scheme

In this subsection, we construct an augmented variant of the graph G_n^L. The purpose of this augmentation is to enable a coupling between a magnetization-based argument and the OSSS inequality. To that end, we introduce a specific cluster–exploration procedure (formulated as a tailored decision forest) together with the associated Boolean functions. These will serve as the objects to which we subsequently apply the OSSS inequality.

Algorithm setup. Fix $L, n, k \in \mathbb{N}$ and let $\tilde{\gamma} > 0$ be an intensity parameter. We introduce an auxiliary Bernoulli field, via a set of "copy vertices", to decide which original vertices of G_n^L are declared "green". More precisely, we add a copy vertex $\tilde{v}$ for each $v \in V_n^L$. Each copy vertex $\tilde{v}$ is open with probability $h := 1 - \exp\{-\tilde{\gamma}\}$, independently of everything else. We then declare the original vertex v to be green whenever its copy $\tilde{v}$ is open. Finally, a vertex $u \in G_n^L$ is said to be in the "green component" whenever it is connected in G_n^L to at least one green vertex (i.e., one whose copy is open).

More formally, let $\omega \in \{0,1\}^{V_n^L \cup E_n^L}$ be distributed according to $P = P_\lambda^{L,n}$. Let $\tilde{V}_n^L$ be a disjoint copy of V_n^L, and for each $v \in V_n^L$ denote by $\tilde{v}$ its copy. Let $\rho \in \{0,1\}^{\tilde{V}_n^L}$ be independent of ω and distributed according to the Bernoulli product measure

$$Q = Q_{\tilde{\gamma}}^{L,n},$$

so that $\rho(\tilde{v}) = 1$ with probability h independently for all $v \in V_n^L$. Set

$$\mu = \mu_{\lambda,\tilde{\gamma}}^{L,n} := P \otimes Q,$$

which is a product measure on $\{0,1\}^{V_n^L \cup \tilde{V}_n^L \cup E_n^L}$.

Next, fix a vertex $v \in V_n^L$ and define the increasing functions $f, g : \{0,1\}^{V_n^L \cup E_n^L \cup \tilde{V}_n^L} \to \{0,1\}$ by

$$f(\omega, \rho) = \mathbb{1}(|\mathcal{C}(v)| \geq k), \qquad g(\omega, \rho) = \mathbb{1}\big(\rho(\tilde{u}) = 1 \text{ for some } u \in \mathcal{C}(v)\big). \qquad (2.16)$$

For our application, the choice $v = (\bar{0}, 1)$ will be the relevant one. The function g is the one computed by the exploration scheme described below.

Informal description. Before introducing the algorithm for the exploration scheme, we first convey the informal idea behind the procedure. Our goal is to explore the clusters of all green vertices in the complete graph simultaneously, according to the following rules. For each vertex in the graph, we initiate an exploration only if the vertex is green, and also open. In that case, its open cluster is successively revealed — including both edges and vertices. An edge is revealed only if it is incident to a vertex that has already been revealed to be open, and not incident to any vertex that has already been revealed to be closed. Once an edge is revealed to be open, the scheme immediately reveals its second endpoint, provided it has not yet been revealed. The decision tree rooted at any open vertex terminates once the entire open cluster of that vertex has been determined.

The revealment probability of any edge is thus by design controlled by the probability that an endpoint of the edge is open. This is crucial, because it ensures that the total contribution of the edge influences vanishes in the continuum limit — as established in Appendix III.

Formal description. Let us describe the procedure precisely now. A decision forest $F = \{T^u \mid u \in V_n^L\}$ which computes g is defined as follows:

For the vertex v, the decision tree T^v just reveals the status of $\rho(\tilde{v})$ and then halts, i.e., $T_n^v(\omega, \rho) = \tilde{v}$ for $n \in \mathbb{N}$.

Next, fix any vertex $u \in V_n^L$, $u \neq v$, and fix an arbitrary total order on the set of edges E_n^L. Set $T_0^u(\omega, \rho) = \tilde{u}$. If $\rho(\tilde{u}) = 0$, set $T_r^u(\omega, \rho) = \tilde{u}$ for every $r \geq 1$, i.e., halt the exploration. Otherwise, if $\rho(\tilde{u}) = 1$, set $T_1^u(\omega, \rho) = u$. If $\omega(u) = 0$, set $T_r^u(\omega, \rho) = u$ for every $r \geq 2$, i.e., halt the exploration.

Finally, if we found $\rho(\tilde{u}) = 1$ and $\omega(u) = 1$, define $T_r^u(\omega, \rho)$, for $r \geq 2$, the following way: At each step of the decision tree, we will have a set of revealed open vertices A_r^u, a set of revealed closed vertices B_r^u, a set of revealed open edges O_r^u and a set of revealed closed edges C_r^u. Moreover, we will denote by I_r^u the set of edges with at least one endpoint in A_r^u that are contained in $O_r^u \cup C_r^u$ (i.e., that are revealed until step r). We initialize by setting $A_1^u = u$ and $B_1^u = O_1^u = C_1^u = \emptyset$. Assume that $r \geq 1$ and that we have computed $(A_s^u, B_s^u, O_s^u, C_s^u, T_s^u)$ for every $s \leq r$. We are now ready to formulate the algorithm:

- <u>Case 1</u>: We have $T_r^u \in E_n^L$, $\omega(T_r^u) = 1$ and an endpoint of T_r^u is not in $A_r^u \cup B_r^u$.

  We set T_{r+1}^u to be the endpoint of T_r^u that is not in $A_r^u \cup B_r^u$. If $\omega(T_{r+1}^u) = 1$, we set $A_{r+1}^u = A_r^u \cup \{T_{r+1}^u\}$, set $B_{r+1}^u = B_r^u$, set $O_{r+1}^u = O_r^u$ and set $C_{r+1}^u = C_r^u$. Otherwise, $\omega(T_{r+1}^u) = 0$ and we set $A_{r+1}^u = A_r^u$, set $B_{r+1}^u = B_r^u \cup \{T_{r+1}^u\}$, set $O_{r+1}^u = O_r^u$ and set $C_{r+1}^u = C_r^u$.

- <u>Case 2</u>: One of the following three scenarios occurs: (i) $T_r^u \in E_n^L$, $\omega(T_r^u) = 1$, and both endpoints of T_r^u lie in A_r^u; (ii) $T_r^u \in E_n^L$ and $\omega(T_r^u) = 0$; or (iii) $T_r^u \in V_n^L$.

  If I_r^u is empty, halt the exploration. Otherwise, set T_{r+1}^u to be the edge minimal in I_r^u. If $\omega(T_{r+1}^u) = 1$, we set $A_{r+1}^u = A_r^u$, set $B_{r+1}^u = B_r^u$, set $O_{r+1}^u = O_r^u \cup \{T_{r+1}^u\}$ and set $C_{r+1}^u = C_r^u$. Otherwise, $\omega(T_{r+1}^u) = 0$ and we set $A_{r+1}^u = A_r^u$, set $B_{r+1}^u = B_r^u$, set $O_{r+1}^u = O_r^u$ and set $C_{r+1}^u = C_r^u \cup \{T_{r+1}^u\}$.

The above procedure defines a decision tree T^u such that

$$\{x \in V_n^L \cup E_n^L \cup \tilde{V}_n^L \mid T_n^u(\omega, \rho) = x \text{ for some } n \in \mathbb{N}\}$$
$$= \begin{cases} \{\tilde{u}\}, & \rho(\tilde{u}) = 0, \\ \{\tilde{u}\} \cup \{u\}, & \rho(\tilde{u}) = 1 \cap \omega(u) = 0, \\ \{u\} \cup \mathcal{C}_u(\omega) \cup E(\mathcal{C}_u(\omega)), & \rho(u) = 1 \cap \omega(u) = 1. \end{cases} \quad (2.17)$$

We now proceed to make use of the OSSS inequality, Corollary 2.11, applying it to the decision forest F defined above and the associated Boolean functions f, g. To do so, we consider the set $S = V_n^L \cup \tilde{V}_n^L \cup E_n^L$ and the measure

$$\mu = P \otimes Q = P_\lambda^{L,n} \otimes Q_{\tilde{\gamma}}^{L,n}, \quad (2.18)$$

which is a product measure on $\{0,1\}^S$.

Fact 2.12 (Identity for revealment probabilities of vertices). *Let $\delta_{\tilde{\gamma}}(\bar{u}, m) = \delta_{\tilde{\gamma}}^{L,n}(\bar{u}, m)$ denote the revealment probability of the vertex $(\bar{u}, m)$ with respect to μ in the above exploration scheme. We can identify this probability by noticing that a vertex $(\bar{u}, m)$ is explored if and only if it is connected to an open green vertex by a path of open edges and vertices. To be more precise: Let $\lambda, \tilde{\gamma} > 0$, $L, n \in \mathbb{N}$ and $(\bar{u}, m) \in V_n^L$, then*

$$\delta_{\tilde{\gamma}}(\bar{u}, m) = \mu\Big((\bar{u}, m) \in \mathcal{C}(v) \text{ for an open green vertex } v\Big)$$
$$= E\left[1 - e^{-\tilde{\gamma}|\mathcal{C}(\bar{u},m)|}\right].$$

Remark 2.13 (Connection to magnetization)**.** *The above identity expresses the revealment probability as the magnetization at $(\bar{u}, m)$ under a reparametrized external field. In the exploration scheme, a vertex is* revealed *precisely when the algorithm uncovers its state, which occurs if it is connected to a green (ghost) vertex. This mirrors the standard representation of magnetization in random–cluster models (cf. Aizenman [1]). Hence, up to the change of variables in the field parameter, revealment probabilities coincide with magnetization values, providing the natural bridge between the algorithmic and physical formulations.*

Notice that Corollary 2.11 yields the following inequality for the choice of functions f, g as in (2.16):

$$\text{Cov}_\mu[f, g] \leq \sum_{u \in V_n^L} \delta_{\tilde{\gamma}}(u)\text{INF}_u^P(f) + \sum_{e \in E_n^L} \delta_{\tilde{\gamma}}(e)\text{INF}_e^P(f) + \sum_{\tilde{u} \in \tilde{V}_n^L} \delta_{\tilde{\gamma}}(\tilde{u})\text{INF}_{\tilde{u}}^Q(f)$$

$$= \sum_{u \in V_n^L} \delta_{\tilde{\gamma}}(u)\text{INF}_u^P(f) + \sum_{e \in E_n^L} \delta_{\tilde{\gamma}}(e)\text{INF}_e^P(f). \tag{2.19}$$

The sum over the copy vertices vanishes because the value of f does not depend on the states of copy vertices by definition (and therefore $\text{INF}_{\tilde{u}}^Q(f) = 0$ for any $\tilde{u} \in \tilde{V}_n^L$). Hence, we get the following proposition:

Proposition 2.14. *Let $L, n \in \mathbb{N}$. We have*

$$\sum_{u \in V_n^L} \delta_{\frac{\gamma}{k}}(u)\text{INF}_u^P(f)$$

$$\geq \left(1 - e^{-\gamma} - E\left[1 - \exp\left\{-\frac{\gamma|\mathcal{C}(v)|}{k}\right\}\right]\right) P(|\mathcal{C}(v)| \geq k) - \sum_{e \in E_n^L} \delta_{\frac{\gamma}{k}}(e)\text{INF}_e^P(f) \tag{2.20}$$

for all $v \in V_n^L$, $k \geq 1$ and $\gamma > 0$.

Proof. Let $\tilde{\gamma} = \frac{\gamma}{k}$ and denote by $\rho(A)$ the number of "green" vertices in $A \subset V_n^L$. Then we can bound the left side of (2.19) as follows

$$\text{Cov}_\mu[f, g] = \mu(|\mathcal{C}(v)| \geq k, \rho(\mathcal{C}(v)) \geq 1) - \mu(\rho(\mathcal{C}_v) \geq 1)P(|\mathcal{C}(v)| \geq k)$$

$$= E\left[(1 - \exp\{-\gamma|\mathcal{C}(v)|/k\})\mathbb{1}_{\{|\mathcal{C}(v)| \geq k\}}\right] - E\left(1 - \exp\{-\gamma|\mathcal{C}(v)|/k\}\right)P(|\mathcal{C}(v)| \geq k)$$

$$\geq (1 - e^{-\gamma})P(|\mathcal{C}(v)| \geq k) - E\left[1 - e^{-\gamma|\mathcal{C}(v)|/k}\right]P(|\mathcal{C}(v)| \geq k),$$

so that the claim follows directly from (2.19). $\qquad\qquad\square$

3 Derivation of differential inequalities.

In this section, we use Proposition 2.14 in combination with the variant of Russo's formula from Subsection 2.2 to derive differential equations for the tail function of the cluster size $|\mathcal{C}(\bar{0}, 1)|$ — here in the discrete finite-volume setup given by our approximation of the random connection model. We do not need to restrict ourselves to the min-reach RCM here, instead we consider a set of technical assumptions for the discrete finite-volume setup, in particular to have some uniform control over ratios of vertex influences (naturally, we will later verify that the min-reach RCM satisfy these technical assumptions, see Section 4). Taking the continuum limit $n \to \infty$ and the infinite-volume limit $L \to \infty$ for those differential equations will provide our main result in Section 5 — under suitable moment conditions for the weight distribution π.

First of all, we want to make some simple observations with regard to the revealment probabilities and influences appearing on the left side of (2.20) in Proposition 2.14 and introduce some additional notation in the process. We will continue to write $P = P_\lambda^{L,n}$ in the following.

Notation 3.1 (Pivotality events and their probabilities)**.** *We let $A^k(\bar{u}, m)$ denote the event that $(\bar{u}, m)$ is closed and pivotal for the size of the open cluster of $(\bar{0}, 1)$ being at least k; let $\mathscr{P}^k(\bar{u}, m)$ denote the probability of this event. To be more precise: Let $\lambda > 0$, $L, n, k \in \mathbb{N}$ and $(\bar{u}, m) \in V_n^L$, then*

$$\mathscr{P}^k(\bar{u}, m) = P\Big(A^k(\bar{u}, m)\Big),$$

where

$$A^k(\bar{u}, m) = \{(\bar{u}, m) \text{ is closed and pivotal for the event } |\mathcal{C}(\bar{0}, 1)| \geq k\}.$$

Recall, that we call a vertex v pivotal for an increasing event A if A occurs when v is open but not when it is closed, with all other states unchanged.

Fact 3.2 (Identity for the influences of vertices)**.** *Let $\lambda > 0$, $L, n, k \in \mathbb{N}$ and $(\bar{u}, m) \in V_n^L$. Then, using the notation above, we have*

$$\begin{aligned} &\mathrm{INF}_{(\bar{u},m)}^P \left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}\right) \\ =&2\Big(1 - e^{-\lambda 2^{-nd}\Pi(m,n)}\Big) \mathscr{P}^k(\bar{u}, m), \end{aligned} \tag{3.1}$$

since for any vertex $v \in V_n^L$ the pivotality of v is independent from its closedness and

$$\begin{aligned} &\mathrm{INF}_v^P \left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}\right) \\ =&2 P\left((v \text{ is open}\right) P\left((v \text{ is closed}\right) P\left((v \text{ is pivotal for } |\mathcal{C}(\bar{0},1)| \geq k\right), \end{aligned}$$

compare also with the first statement in Remark 2.6.

Before we can apply our variant of Russo's formula from Corollary 2.5 to the expression on the left side of (2.20), we still need some additional input. The following assumption is essential for the proof of Proposition 3.6, the main result of this section:

Assumption 3.3. *Both the influences and the revealment probabilities for vertices of different weights are comparable in our discrete setup — in the following sense:*

a) *For every $\lambda > 0$, there exists a measurable function $\mathfrak{R}_\lambda : [1, \infty) \to [1, \infty)$ such that for all $L, n \in \mathbb{N}$ sufficiently large, all $(\bar{u}, m) \in V_n^L$ and all $\tilde{\gamma} > 0$ sufficiently small, we have*

$$\frac{\delta_{\tilde{\gamma}}^{L,n}(\bar{u}, m)}{\delta_{\tilde{\gamma}}^{L,n}(\bar{u}, 1)} \leq \mathfrak{R}_\lambda(m). \tag{3.2}$$

b) *For every $\lambda > 0$, there exists a measurable function $\mathfrak{r}_\lambda : [1, \infty) \to [1, \infty)$ such that for all $k \in \mathbb{N}$, for sufficiently large $L, n \in \mathbb{N}$ and all $m \geq 1$, we have*

$$\frac{\sum_{\bar{u} \in U_n^L} \mathscr{P}^k(\bar{u}, m)}{\sum_{\bar{u} \in U_n^L} \mathscr{P}^k(\bar{u}, 1)} \leq \mathfrak{r}_\lambda(m). \tag{3.3}$$

Moreover, we assume that the functions $\lambda \mapsto \mathfrak{r}_\lambda(m)$ and $\lambda \mapsto \mathfrak{R}_\lambda(m)$ are non-increasing for every $m \geq 1$.

Remark 3.4. *The assumption holds trivially when the connection function φ does not depend on the weights* (NonW), *with $\mathfrak{r}_\lambda = \mathfrak{R}_\lambda = 1$. Moreover, in Section 4, we prove the assumption to hold for the min-reach RCM* (MinR) *with exponential upper bounds. However, we expect Assumption 3.3 to be satisfied in a very general class of weighted RCM with infinite range and unbounded weights, at least with exponential upper bounds. Although we will use the additional assumption that the functions $\mathfrak{r}_\lambda, \mathfrak{R}_\lambda$ are non-increasing in λ in the proofs from Section 5, it can be considerably weakened without changing those proofs in any major way.*

Next, let us introduce useful notation to adopt from now on throughout the paper.

Notation 3.5. *(The relation $\lesssim, \gtrsim$) We will denote inequalities up to positive multiplicative constants by $\lesssim, \gtrsim$ in the obvious fashion. Unless otherwise specified, these constant will not depend on any parameter (apart from model assumptions like the dimension d), in particular the constants do not depend on the intensity $\lambda > 0$ or the approximation parameters $L, n \in \mathbb{N}$.*

Finally, we are ready to state the following proposition which is the main result of this section:

Proposition 3.6. *Let $\lambda > 0$ and suppose that the estimates from Assumption 3.3 hold for some functions $\mathfrak{r}_\lambda$ and $\mathfrak{R}_\lambda$. Then, for all $k \in \mathbb{N}$, for sufficiently large $L, n \in \mathbb{N}$ and sufficiently small $\tilde{\gamma} > 0$, we have*

$$
\sum_{(\bar{u},m) \in V_n^L} \delta_{\tilde{\gamma}}(\bar{u}, m) \mathrm{INF}_{(\bar{u},m)}^P \left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)| \geq k\}} \right)
$$

$$
\lesssim \lambda \delta_{\tilde{\gamma}}(\bar{0}, 1) \left(\sum_{m \in M_n} \Pi(m, n) \mathfrak{r}_\lambda(m) \mathfrak{R}_\lambda(m) \right) \frac{\mathrm{d}}{\mathrm{d}\lambda} \theta_\lambda^{L,n}(k).
$$

Proof. Fix $\lambda > 0$ and $k \in \mathbb{N}$. First, we use Assumption 3.3a) — and the fact that the estimate $\delta_{\tilde{\gamma}}(\bar{u}, 1) \lesssim \delta_{\tilde{\gamma}}(\bar{0}, 1)$ holds for $L, n \in \mathbb{N}$ sufficiently large — to obtain

$$
\sum_{(\bar{u},m) \in V_n^L} \delta_{\tilde{\gamma}}(\bar{u}, m) \mathrm{INF}_{(\bar{u},m)}^P \left(\mathbb{1}_{|\mathcal{C}(\bar{0},1)| \geq k} \right)
$$

$$
\lesssim \delta_{\tilde{\gamma}}(\bar{0}, 1) \sum_{m \in M_n} \mathfrak{R}_\lambda(m) \sum_{\bar{u} \in U_n^L} \mathrm{INF}_{(\bar{u},m)}^P \left(\mathbb{1}_{|\mathcal{C}(\bar{0},1)| \geq k} \right)
$$

$$
= \delta_{\tilde{\gamma}}(\bar{0}, 1) \sum_{m \in M_n} \mathfrak{R}_\lambda(m) \sum_{\bar{u} \in U_n^L} 2 \left(1 - \mathrm{e}^{-\lambda 2^{-nd} \Pi(m,n)} \right) \mathscr{P}^k(\bar{u}, m),
$$

for sufficiently large $L, n \in \mathbb{N}$ and sufficiently small $\tilde{\gamma} > 0$, where for the last equality we used identity (3.1).

Assumption 3.3b) yields

$$
\delta_{\tilde{\gamma}}(\bar{0}, 1) \sum_{m \in M_n} \mathfrak{R}_\lambda(m) \left(1 - \mathrm{e}^{-\lambda 2^{-nd} \Pi(m,n)} \right) \sum_{\bar{u} \in U_n^L} \mathscr{P}^k(\bar{u}, m)
$$

$$
\lesssim \delta_{\tilde{\gamma}}(\bar{0}, 1) \sum_{m \in M_n} \mathfrak{r}_\lambda(m) \mathfrak{R}_\lambda(m) \left(1 - \mathrm{e}^{-\lambda 2^{-nd} \Pi(m,n)} \right) \sum_{\bar{u} \in U_n^L} \mathscr{P}^k(\bar{u}, 1). \tag{3.4}
$$

For any $\bar{u} \in U_n^L$, we can write, setting $A(k) = \{|\mathcal{C}(\bar{0}, 1)| \geq k\}$,

$$
\mathscr{P}^k(\bar{u}, 1) = \left(\sum_{m \in M_n} \Pi(m, n) \right) \mathscr{P}^k(\bar{u}, 1)
$$

$$
= \sum_{m \in M_n} \Pi(m, n) \mathrm{e}^{-\lambda 2^{-nd} \Pi(1,n)} P_\lambda^{L,n} \left((\bar{u}, 1) \text{ is } A(k)\text{-pivotal} \right).
$$

Observe that — by Assumption (A.3) — the probability of a vertex $(\bar{u}, m)$ to be $A(k)$-pivotal is non-decreasing in the weight m of the vertex. Therefore, for all $L, n \in \mathbb{N}$ and $m \in M_n$

$$P_\lambda^{L,n}\Big((\bar{u}, 1) \text{ is } A(k)\text{-pivotal}\Big) \leq P_\lambda^{L,n}\Big((\bar{u}, m) \text{ is } A(k)\text{-pivotal}\Big)$$

and, moreover, the estimate

$$e^{-\lambda 2^{-nd}\Pi(1,n)} \leq e^{-\lambda 2^{-nd}\Pi(m,n)}$$

holds for all $n \in \mathbb{N}$ and $m \in M_n$ by our assumption on the weight distribution π. Thus, we get

$$\mathscr{P}^k(\bar{u}, 1) \leq \sum_{m \in M_n} \Pi(m, n)e^{-\lambda 2^{-nd}\Pi(m,n)} P_\lambda^{L,n}\Big((\bar{u}, m) \text{ is } A(k)\text{-pivotal}\Big).$$

Plugging that estimate into the upper bound from (3.4) yields

$$\delta_{\tilde{\gamma}}(\bar{0}, 1) \sum_{m \in M_n} \mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m) \sum_{\bar{u} \in U_n^L} \Big(1 - e^{-\lambda 2^{-nd}\Pi(m,n)}\Big)\mathscr{P}^k(\bar{u}, m)$$

$$\lesssim \delta_{\tilde{\gamma}}(\bar{0}, 1) \left(\sum_{m \in M_n} \mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m)\frac{1 - e^{-\lambda 2^{-nd}\Pi(m,n)}}{2^{-nd}}\right) \sum_{\bar{u} \in U_n^L} \sum_{m \in M_n} 2^{-nd}\Pi(m, n)\mathscr{P}^k(\bar{u}, m)$$

and — since $\frac{1 - e^{-\lambda 2^{-nd}\Pi(m,n)}}{2^{-nd}} \leq \frac{\lambda 2^{-nd}\Pi(m,n)}{2^{-nd}} = \lambda\Pi(m, n)$ — we can conclude that altogether

$$\sum_{(\bar{u}, m) \in V_n^L} \delta_{\tilde{\gamma}}(\bar{0}, 1)\mathrm{INF}_{(\bar{u}, m)}^P\Big(\mathbb{1}_{\{|\mathcal{C}(\bar{0}, 1)| \geq k\}}\Big)$$

$$\lesssim \lambda\delta_{\tilde{\gamma}}(\bar{u}, m) \left(\sum_{m \in M_n} \mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m)\Pi(m, n)\right) \sum_{\bar{u} \in U_n^L} \sum_{m \in M_n} 2^{-nd}\Pi(m, n)\mathscr{P}^k(\bar{u}, m).$$

The claim of the proposition follows since by Russo's formula (Corollary 2.5 and Remark 2.6)

$$\sum_{\bar{u} \in U_n^L} \sum_{m \in M_n} 2^{-nd}\Pi(m, n)\mathscr{P}^k(\bar{u}, m) = \sum_{v \in V_n^L} \frac{\mathrm{d}}{\mathrm{d}\lambda}P_\lambda^{L,n}\Big(v \text{ is open}\Big)P_\lambda^{L,n}\Big(v \text{ is } A(k)\text{-pivotal}\Big)$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}P_\lambda^{L,n}\Big(A(k)\Big) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_\lambda^{L,n}(k).$$

$\square$

Following Hutchcroft's strategy in [21], we combine the obtained result with (2.20) to immediately obtain the following inequalities:

Corollary 3.7. *Let $\lambda > 0$ and suppose that the estimates from Assumption 3.3 hold for some functions $\mathfrak{r}_\lambda$ and $\mathfrak{R}_\lambda$. Then, we have*

$$\left(\frac{1 - e^{-\tilde{\gamma}k}}{\delta_{\tilde{\gamma}}(\bar{0}, 1)} - 1\right)\theta_\lambda^{L,n}(k) - \frac{1}{\delta_{\tilde{\gamma}}(\bar{0}, 1)}\sum_{e \in E_n^L} \delta_{\tilde{\gamma}}(e)\mathrm{INF}_e^P(\mathbb{1}_{\{|\mathcal{C}(\bar{0}, 1)| \geq k\}})$$

$$\lesssim \left(\lambda \sum_{m \in M_n} \Pi(m, n)\mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m)\right)\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_\lambda^{L,n}(k)$$

for all $k \geq 1$, for $L, n \in \mathbb{N}$ sufficiently large and for $\tilde{\gamma} > 0$ sufficiently small.

Proof. The inequality follows directly from Proposition 2.14 and Proposition 3.6 by choosing $\gamma = k\tilde\gamma$. $\qquad\qquad\square$

Corollary 3.8. *Let $\lambda > 0$ and suppose that the estimates from Assumption 3.3 hold for some functions $\mathfrak{r}_\lambda$ and $\mathfrak{R}_\lambda$. Then, we have*

$$\left(\frac{k}{E_\lambda^{L,n}[|\mathcal{C}(\bar{0},1)|]} - 1\right)\theta_\lambda^{L,n}(k) - \sup_{\tilde\gamma > 0}\frac{1}{\delta_{\tilde\gamma}(\bar{0},1)}\sum_{e \in E_n^L}\delta_{\tilde\gamma}(e)\mathrm{INF}_e^P(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)| \geq k\}})$$
$$\lesssim \left(\lambda\sum_{m \in M_n}\Pi(m,n)\mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m)\right)\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_\lambda^{L,n}(k) \tag{3.5}$$

for any $k \in \mathbb{N}$ and sufficiently large $L, n \in \mathbb{N}$. Moreover, we have

$$\left(\frac{k\left(1 - e^{-1}\right)}{\sum_{i=1}^k \theta_\lambda^{L,n}(i)} - 1\right)\theta_\lambda^{L,n}(k) - \sup_{\tilde\gamma > 0}\frac{1}{\delta_{\tilde\gamma}(\bar{0},1)}\sum_{e \in E_n^L}\delta_{\tilde\gamma}(e)\mathrm{INF}_e^P(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)| \geq k\}})$$
$$\lesssim \left(\lambda\sum_{m \in M_n}\Pi(m,n)\mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m)\right)\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_\lambda^{L,n}(k) \tag{3.6}$$

for sufficiently large $L, n, k \in \mathbb{N}$.

Remark 3.9. *These differential inequalities correspond to the one from [22, Proposition 1.4]. The main difference in our estimates is the presence of the expression given as the sum of edge influences, but in Appendix III we show that it vanishes in the limit $n \to \infty$, allowing us to recover Hutchcroft's estimates.*

Proof. We set $\tilde\gamma = \frac{\gamma}{k}$. Using the estimate

$$\delta_{\frac{\gamma}{k}}(\bar{0},1) \leq \frac{\gamma}{k}\sum_{i=1}^{\left\lceil\frac{k}{\gamma}\right\rceil}\theta_\lambda^{L,n}(i)$$

and setting $\gamma = 1$, inequality (3.6) follows directly from Corollary 3.7. On the other hand, inequality (3.5) follows from Corollary 3.7 by taking the limit $\gamma \to 0$ — using the estimates

$$\delta_{\frac{\gamma}{k}}(\bar{0},1) \leq \frac{\gamma}{k}E_\lambda^{L,n}[|\mathcal{C}(\bar{0},1)|]$$

and

$$\lim_{\gamma \to 0}\frac{1 - e^{-\gamma}}{\delta_{\frac{\gamma}{k}}(\bar{0},1)} \geq \lim_{\gamma \to 0}\frac{1 - e^{-\gamma}}{\gamma}\frac{k}{E_\lambda^{L,n}[|\mathcal{C}(\bar{0},1)|]} = \frac{k}{E_\lambda^{L,n}[|\mathcal{C}(\bar{0},1)|]}.$$

Notice that inequality (3.5) holds for any $k \in \mathbb{N}$. $\qquad\qquad\square$

4 Min-reach RCM satisfies Assumptions 3.3

In this section, we show that the min-reach RCM (MinR) satisfies Assumptions 3.3, allowing us to apply the general results of Section 5 under suitable moment conditions. The growth rates of the bounds $\mathfrak{r}_\lambda$ and $\mathfrak{R}_\lambda$ determine the required moment assumptions. Using the estimates obtained here, which are exponential in the reach of a vertex, we obtain the moment condition (EMC). Any improvement in the order of these bounds would yield a corresponding weakening of (EMC).

4.1 Preparations: Notation and definitions

Let the adjacency function φ satisfy Assumption (MinR), i.e., for all $x_1, x_2 \in \mathbb{R}^d$ and for all $m_1, m_2 \geq 1$, we have

$$\varphi((x_1, m_1), (x_2, m_2)) = 0 \text{ for } ||x_1 - x_2|| > R(\min\{m_1, m_2\})$$

and

$$\varphi((x_1, 1), (x_2, 1)) > 0 \text{ for } ||x_1 - x_2|| < R(1)$$

for some non-decreasing function $R : [1, \infty) \to (0, \infty)$.

For $r \in \mathbb{R}$ and $\bar{u} \in U_n^L$, we denote by $B_r(\bar{u})$ the ball of radius r around $\bar{u}$, i.e., $B_r(\bar{u}) = \{\bar{v} \in U_n^L : ||\bar{v} - \bar{u}|| \leq r\}$. Furthermore, we denote by $Z_r(\bar{u})$ the cylinder set of radius r around $\bar{u}$, i.e., $Z_r(\bar{u}) = B_r(\bar{u}) \times M_n$. We generally say that Z is a cylinder set if $Z = B \times M_n$ for some $B \subset U_n^L$.

For $L, n \in \mathbb{N}$, let $(\bar{u}, m) \in V_n^L$. Notice that $\mathcal{N}(\bar{u}, m) \subset Z_{R(m)}(x)$ holds almost surely by (MinR). Let $\mathfrak{C}(\bar{u}, m)$ denote the random set of connected components of $\mathcal{C}(\bar{u}, m) \backslash \{(\bar{u}, m)\}$ and let $\tilde{\mathfrak{C}}(\bar{u}, m)$ denote the random set of connected components of $\mathcal{C}(\bar{u}, m) \setminus \{(\bar{u}, m)\}$ that do not contain $(\bar{0}, 1)$. Then $|\tilde{\mathfrak{C}}(\bar{u}, m)| \leq |\mathfrak{C}(\bar{u}, m)| \leq |\mathcal{N}(\bar{u}, m)|$ holds deterministically. Let

- $\mathfrak{Q}(\bar{u}, m) := \{Q \subset \mathcal{C}(\bar{u}, m)| \; \forall C \in \mathfrak{C}(\bar{u}, m) \exists! \; x \in Q : x \in C\}$

- $\tilde{\mathfrak{Q}}(\bar{u}, m) := \{Q \subset \mathcal{C}(\bar{u}, m)| \; \forall C \in \tilde{\mathfrak{C}}(\bar{u}, m) \exists! \; x \in Q : x \in C\}$.

The main idea behind these definitions is the following: Take any $Q \in \mathfrak{Q}(\bar{u}, m)$. Given that the vertex $(\bar{u}, m)$ is connected to an open green vertex, we can guarantee that $(\bar{u}, 1)$ is connected to an open green vertex by connecting to every vertex in Q. The set $\tilde{\mathfrak{Q}}(\bar{u}, m)$ plays a similar role for the pivotality of closed vertices.

Notice that $|Q| = |\mathfrak{C}(\bar{u}, m)|$ for every $Q \in \mathfrak{Q}(\bar{u}, m)$ and that there exists a $Q \in \mathfrak{Q}(\bar{u}, m)$ with $Q \subset \mathcal{N}(\bar{u}, m)$. Next, we want to define the event that will help us formalize the fact that we can locally control the cardinality of Q (with high probability). For $\ell \in \mathbb{N}$ and $Z \subset Z_{R(m)}(\bar{u})$, we introduce the events

- $Q_\ell^Z(\bar{u}, m) := \{|Q \cap Z| \leq \ell \text{ holds for any } Q \in \mathfrak{Q}(\bar{u}, m)\}$,

- $\tilde{Q}_\ell^Z(\bar{u}, m) := \{|Q \cap Z| \leq \ell \text{ holds for any } Q \in \tilde{\mathfrak{Q}}(\bar{u}, m)\}$.

Next, let us introduce the event

$$G(\bar{u}, m) := \{(\bar{u}, m) \in \mathcal{C}(v) \text{ for a green open vertex } v\}$$

and recall the definition of the events $A^k(\bar{u}, m)$ from the previous section. Again, we lighten the notation by fixing $\lambda, \tilde{\gamma} > 0$ as well as sufficiently large parameters $L, n \in \mathbb{N}$ and denoting (as in the previous section)

$$P = P_\lambda^{L,n}, \quad \mu = P_\lambda^{L,n} \otimes Q_{\tilde{\gamma}}^{L,n}.$$

Definition 4.1. *Fix $0 < \varepsilon < 1$ and set $\underline{R} = \epsilon R(1)$. Furthermore, set*

$$\hat{\ell} = \min \left\{ \ell \in \mathbb{N} : \prod_{i=1}^{\infty} \left(1 - q^{\ell+i}\right) \geq \frac{1}{2} \right\},$$

where $q := 1 - \varphi(\underline{R}, 1, 1)$.

4.2 Treating the revealment probabilities

First, we establish uniform control over ratios of revealment probabilities for vertices of varying weights, as required in Assumption 3.3.

Lemma 4.2. *Let $\ell \geq \hat{\ell}$ (given by Definition 4.1), then — for any cylinder Z satisfying $\max_{x,y \in Z} \operatorname{dist}(x,y) \leq \underline{R}$ — we have*

$$\mu\left(G(\bar{u}, m)\right) \leq 2\mu\left(G(\bar{u}, m) \cap Q_\ell^Z(\bar{u}, m)\right). \tag{4.1}$$

Moreover, we obtain — for any partition of $Z_{R(m)}(\bar{u})$ into disjoint cylinders $Z_1, \ldots, Z_M$ with $\max_{x,y \in Z_k} \operatorname{dist}(x,y) \leq \underline{R}$ for all $1 \leq k \leq M$ — the estimate

$$\mu\left(G(\bar{u}, m)\right) \leq 2^M \mu\left(G(\bar{u}, m) \cap \bigcap_{1 \leq k \leq M} Q_\ell^{Z_k}(\bar{u}, m)\right). \tag{4.2}$$

Proof. Consider a cylinder set Z as in the hypothesis of the lemma. First of all we notice that, conditioned on $\mathcal{C}(\bar{u}, m) \cap Z$, the events $G(\bar{u}, m)$ and $Q_\ell^Z(\bar{u}, m)$ are stochastically independent with respect to μ. Moreover, we can estimate the probability of $(Q_\ell^Z(\bar{u}, m))^c$ from above by the probability of $\mathcal{C}(\bar{u}, m) \cap Z$ containing an independent vertex set of size at least $\ell + 1$ (in the sense that no two vertices from that set are connected by an edge). Conditioned on $|\mathcal{C}(\bar{u}, m) \cap Z| = \ell + j$ for any $j \in \mathbb{N}$, the latter probability is bounded from above by $1 - \prod_{i=1}^{j}\left(1 - q^{\ell+i}\right)$. Altogether, we get

$$\mu\left(G(\bar{u}, m) \cap (Q_\ell^Z(\bar{u}, m))^c\right)$$

$$= \sum_{j=1}^{\infty} \mu\left(G(\bar{u}, m) \cap (Q_\ell^Z(\bar{u}, m))^c \mid |\mathcal{C}(\bar{u}, m) \cap Z| = \ell + j\right)\mu(|\mathcal{C}(\bar{u}, m) \cap Z| = \ell + j)$$

$$\leq \sum_{j=1}^{\infty} \mu(G(\bar{u}, m) \mid |\mathcal{C}(\bar{u}, m) \cap Z| = \ell + j)\left(\left(1 - \prod_{i=1}^{j}\left(1 - q^{\ell+i}\right)\right)\mu(|\mathcal{C}(\bar{u}, m) \cap Z| = \ell + j)\right)$$

$$\leq \left(\left(1 - \prod_{i=1}^{\infty}\left(1 - q^{\ell+i}\right)\right)\sum_{j=1}^{\infty} \mu(G(\bar{u}, m) \mid |\mathcal{C}(\bar{u}, m) \cap Z| = \ell + j)\mu(|\mathcal{C}(\bar{u}, m) \cap Z| = \ell + j)\right)$$

$$\leq \left(\left(1 - \prod_{i=1}^{\infty}\left(1 - q^{\ell+i}\right)\right)\mu(G(\bar{u}, m)).$$

and, by our choice of ℓ, we have

$$\left(\left(1 - \prod_{i=1}^{\infty}\left(1 - q^{\ell+i}\right)\right) \leq \frac{1}{2}$$

which concludes the proof for the estimate (4.1). Given a partition $Z_1, \ldots, Z_M$ as in the hypothesis of the lemma, we use the fact that the cylinders $Z_1, \ldots, Z_k$ are disjoint (and, conditioned on $\mathcal{C}(\bar{u}, m)$, the events $Q_\ell^{Z_1}(\bar{u}, m), \ldots, Q_\ell^{Z_M}(\bar{u}, m)$ are therefore stochastically independent) to successively re-iterate the above argument M times — obtaining the estimate (4.2). $\square$

Now we are ready to show that Assumption 3.3a) is satisfied for the min-reach RCM and specify the corresponding family of functions $(\mathfrak{R}_\lambda)_{\lambda > 0}$:

Theorem 4.3. *There exists a constant $C(\lambda) \geq 1$ such that the min-reach RCM satisfies Assumption 3.3a) for the choice*

$$\mathfrak{R}_\lambda(m) = C(\lambda)^{R(m)^d},$$

in particular, $\lambda \mapsto C(\lambda)$ is non-increasing on $(0, \infty)$.

Proof. Let $Z_1, \ldots, Z_M$ be a partition of $Z_{R(m)}(\bar{u})$ into cylinder sets such that

1. $\max_{x,y \in Z_q} \operatorname{dist}(x, y) \leq \underline{R}$ for all $1 \leq q \leq M$,

2. $(\bar{u}, 1) \in Z_1$ and $\max_{x \in Z_q, y \in Z_{q+1}} \operatorname{dist}(x, y) \leq \underline{R}$ for all $1 \leq q < M$,

3. $P\left(\exists v \in Z_q \mid v \text{ is open}\right) \geq 1 - \mathrm{e}^{-\lambda C_1}$ for some constant $C_1 > 0$ and all $1 \leq q \leq M$.

4. $M \leq C_2 R(m)^d$ for some constant $C_2 > 0$.

Clearly, such a partition exists and we have, for any $(\bar{u}, m) \in V_n^L$,

$$\delta_{\bar{\gamma}}(\bar{u}, 1) = \mu(G(\bar{u}, 1)) \geq \mu\left(G(\bar{u}, 1) \cap G(\bar{u}, m) \cap \bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_q}(\bar{u}, m)\right).$$

Next, let us fix some rule to choose $Q \in \mathfrak{Q}(\bar{u}, m) \cap \mathcal{N}(\bar{u}, m)$. Now, on the event $G(\bar{u}, m) \cap \left(\bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_q}(\bar{u}, m)\right)$, the vertex $(\bar{u}, 1)$ is connected to a green open vertex v if the event Θ occurs which we define as follows:

1. for every $1 \leq q \leq M$, the cylinder set Z_q contains an open vertex v_q and the path $(\bar{u}, 1), v_1, \ldots, v_M$ consists of open edges, and

2. for every $1 \leq q \leq M$, each vertex in $Q \cap Z_q$ is connected to v_q by a direct edge.

We have

$$\delta_{\bar{\gamma}}(\bar{u}, 1) \geq \mu\left(\Theta \cap G(\bar{u}, m) \cap \bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_q}(\bar{u}, m)\right).$$

Now (since conditioned on Q the events Θ, $G(\bar{u}, m)$ are both increasing and we therefore can use the FKG-inequality for the corresponding conditional product measure, see [17]) we obtain

$$\mu\left(\Theta \cap G(\bar{u}, m) \cap \bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_q}(\bar{u}, m)\right)$$

$$\geq \mu\left(\Theta \mid \bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_1}(\bar{u}, m)\right) \mu\left(G(\bar{u}, m) \cap \bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_q}(\bar{u}, m)\right)$$

Left to observe is that, by our assumptions on the partition $Z_1, \ldots, Z_M$, we have the estimate

$$\mu\left(\Theta \mid \bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_q}(\bar{u}, m)\right) \geq \left(1 - \mathrm{e}^{-\lambda C_1}\right)^M \varphi(\underline{R}, 1, 1)^M \varphi(\underline{R}, 1, 1)^{\hat{\ell} M} \tag{4.3}$$

and that, by Lemma 4.2, we have

$$\mu\left(G(\bar{u}, m) \cap \bigcap_{1 \leq q \leq M} Q_{\hat{\ell}}^{Z_q}(\bar{u}, m)\right) \geq \left(\frac{1}{2}\right)^M \mu\left(G(\bar{u}, m)\right).$$

Combining these two observations and setting $p := \frac{1}{2}\varphi(\underline{R}, 1, 1)^{\hat{\ell}+1}$ yields

$$\mu(G(\bar{u}, 1)) \geq \mu\left(G(\bar{u}, m)\right)\left(p - p\mathrm{e}^{-\lambda C_1}\right)^M \geq \mu\left(G(\bar{u}, m)\right)\left(p - p\mathrm{e}^{-\lambda C_1}\right)^{C_2 R(m)^d}.$$

Left to notice is that $\lambda \mapsto C(\lambda) := (p - p\mathrm{e}^{-\lambda C_1})^{-C_2}$ is non-increasing on $(0, \infty)$. The claim of the theorem follows readily. □

Remark 4.4. *Naturally, one can also easily obtain a bound of the form*

$$\mathfrak{R}_\lambda(m) = \hat{C}(\lambda)^{R(m)},$$

but we choose to prove the rougher bound in order to outline the corresponding proof for the influences.

4.3 Treating the influences

Next, we treat the ratios of the pivotality probabilities in a very similar fashion. However, the argument is slightly more involved. First, we state a result that is analogous to Lemma 4.2:

Lemma 4.5. *Let $\ell \geq \hat{\ell}$ (given by Definition 4.1) and $k \in \mathbb{N}$, then — for any cylinder Z satisfying the bound $\max_{x,y \in Z} \mathrm{dist}(x, y) \leq \underline{R}$ — we have*

$$P\left(A^k(\bar{u}, m)\right) \leq 2P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell^Z(\bar{u}, m)\right). \tag{4.4}$$

Moreover, we obtain — for any partition of $Z_{R(m)}(\bar{u})$ into disjoint cylinders $Z_1, \ldots, Z_M$ with $\max_{x,y \in Z_q} \mathrm{dist}(x, y) \leq \underline{R}$ for all $1 \leq q \leq M$ — the estimate

$$P\left(A^k(\bar{u}, m)\right) \leq 2^M P\left(A^k(\bar{u}, m) \cap \bigcap_{1 \leq q \leq M} \tilde{Q}_\ell^{Z_q}(\bar{u}, m)\right). \tag{4.5}$$

Proof. The proof is analogous to the proof of Lemma 4.2: Consider a cylinder set Z as in the hypotheses of the lemma. Let $\mathcal{C}_0$ denote the vertex set of the connected components $\mathcal{C}(\bar{u}, m) \setminus \{(\bar{u}, m)\}$ that do not contain $(\bar{0}, 1)$. Then, conditioned on $\mathcal{C}_0 \cap Z$, the events $A^k(\bar{u}, m)$ and $\tilde{Q}_\ell^Z(\bar{u}, m)$ are stochastically independent with respect to P, for every $k \in \mathbb{N}$. Moreover, we can estimate the probability of $(\tilde{Q}_\ell^Z(\bar{u}, m))^c$ from above by the probability of $\mathcal{C}_0 \cap Z$ containing an independent vertex set of size at least $\ell + 1$. Conditioned on $|\mathcal{C}_0 \cap Z| = \ell + j$ for any $j \in \mathbb{N}$, the latter probability is bounded from above by $1 - \prod_{i=1}^{j} (1 - q^{\ell+i})$. Altogether, we get, as in the proof of Lemma 4.2,

$$P\left(A^k(\bar{u}, m) \cap (\tilde{Q}_\ell^Z(\bar{u}, m))^c\right)$$
$$\leq \left(\left(1 - \prod_{i=1}^{\infty} \left(1 - q^{\ell+i}\right)\right) P(A^k(\bar{u}, m)).$$

and, by our choice of ℓ, we have

$$\left(\left(1 - \prod_{i=1}^{\infty} \left(1 - q^{\ell+i}\right)\right) \leq \frac{1}{2}$$

which concludes the proof for the estimate (4.4). Given a partition $Z_1, \ldots, Z_M$ as in the hypothesis of the lemma, we use the fact that the cylinders $Z_1, \ldots, Z_k$ are disjoint (and, conditioned on $\mathcal{C}_0$, the events $\tilde{Q}_\ell^{Z_1}(\bar{u}, m), \ldots, \tilde{Q}_\ell^{Z_M}(\bar{u}, m)$ are therefore stochastically independent) to successively re-iterate the above argument M times — obtaining the estimate (4.5). $\qquad\square$

Left to show is the analogue of Theorem 4.3.

Theorem 4.6. *There exists a constant $C(\lambda) \geq 1$ such that the min-reach RCM satisfies Assumption 3.3b) for the choice*

$$\mathfrak{r}_\lambda(m) = C(\lambda)^{R(m)^d},$$

in particular, $\lambda \mapsto C(\lambda)$ is non-increasing on $(0, \infty)$.

Proof. Let us start by giving a brief informal description of the proof idea: Given that $(\bar{u}, m)$ is a pivotal vertex, any closed vertex that connects to every connected component of $\mathcal{C}(\bar{u}, m) \setminus \{(\bar{u}, m)\}$ is itself pivotal and we can control the number of those connected components uniformly in k by Lemma 4.5. Some difficulty, however, stems from the fact that for any vertex of weight 1 some connected components of $\mathcal{C}(\bar{u}, m) \setminus \{(\bar{u}, m)\}$ might be out of reach to connect by a direct edge due to the nature of our model (as its name suggests). Therefore, one needs to be careful in order to build paths of open vertices without destroying the pivotality of $(\bar{u}, m)$ (since pivotality is not an increasing event, we cannot use an FKG-type inequality like in Theorem 4.3).

Let $Z_1, \ldots, Z_M$ be a partition of $Z_{R(m)}(\bar{u})$ into cylinder sets as in the proof of Theorem 4.3. We need to introduce some additional notation. First of all, we want to choose a $Q \in \tilde{\mathfrak{Q}}(\bar{u}, m) \cap \mathcal{N}(\bar{u}, m)$ by a deterministic rule and refer to this random configuration of vertices by $\hat{Q}$. Then, we set for some fixed $\ell \geq \hat{\ell}$ given as in Definition 4.1,

$$\tilde{Q}_\ell(\bar{u}, m) := \bigcap_{1 \leq q \leq M} \tilde{Q}_\ell^{Z_q}(\bar{u}, m) \subset \bigcap_{1 \leq q \leq M} \{|\hat{Q} \cap Z_q| \leq \ell\}.$$

Next, for every $1 \leq q \leq M$, we fix a position $\bar{u}_q$ from Z_q by a deterministic rule and define

$$\widetilde{\Gamma}_q = \{(\bar{u}_q, 1) \text{ would be pivotal if adjacent to all of } \hat{Q} \cap Z_q\}$$

and

$$\Gamma_q = \widetilde{\Gamma}_q \cap \{(\bar{u}_q, 1) \text{ is closed}\}.$$

Furthermore, we define

$$\Theta_q = \{(\bar{u}_q, 1) \text{ is open and adjacent to all of } \hat{Q} \cap Z_q\}$$

and, for $1 \leq q \leq M - 1$, introduce the events

$$\overline{\Theta}_q = \{\Theta_1 \cap \ldots \cap \Theta_q \text{ occurs and the path } (\bar{u}_1, 1), \ldots, (\bar{u}_{q+1}, 1) \text{ is open}\}.$$

We want to employ an iterative estimate based on the following observation: Without loss of generality, assume that $\tilde{Q} \cap Z_1 \neq \varnothing$. In a first step, we notice that either $(\bar{u}_1, 1)$ would be pivotal if one connected it to every vertex in $\hat{Q} \cap Z_1$ by an open edge or we can open

$(\bar{u}_1, 1)$ and connect it to every vertex in $\hat{Q} \cap Z_1$ without losing the pivotality of $(\bar{u}, m)$. So, we decompose

$$P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m)\right)$$
$$= P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \Gamma_1\right) + P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \Gamma_1^c\right)$$

and notice that

$$P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \Gamma_1\right) \lesssim \varphi(\underline{R}, 1, 1)^{-\ell} P\left(A^k(\bar{u}_1, 1)\right)$$

as well as

$$P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \Gamma_1^c\right)$$
$$\lesssim P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \Theta_1\right) \varphi(\underline{R}, 1, 1)^{-\ell} P\left((\bar{u}_1, 1) \text{ is open}\right)^{-1}.$$

That last estimate holds by the following consideration: Let $P^Q(\cdot) := P(\cdot \mid \tilde{Q}_\ell(\bar{u}, m))$, then

$$P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \widetilde{\Gamma}_1^c \cap \Theta_1\right) = P^Q\left(A^k(\bar{u}, m) \mid \widetilde{\Gamma}_1^c\right) P^Q\left(\Theta_1 \mid \widetilde{\Gamma}_1^c\right) P^Q(\widetilde{\Gamma}_1^c) P\left(\tilde{Q}_\ell(\bar{u}, m)\right)$$
$$= P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \widetilde{\Gamma}_1^c\right) P^Q\left(\Theta_1\right),$$

where

- for the first equality, we used that $A^k(\bar{u}, m)$ and Θ_1 are independent conditioned on $\widetilde{\Gamma}_1^c$ with respect to P^Q,

- for the second equality, we used that Θ_1 is independent of $\widetilde{\Gamma}_1^c$ with respect to P^Q.

So, using the notation $\mathfrak{p} = 1 - \mathrm{e}^{-\lambda 2^{-nd} \Pi(1, n)} = P\left((\bar{u}, 1) \text{ is open}\right)$ and noticing that the estimate $P^Q(\Theta_1) \le \varphi(\underline{R}, 1, 1)^{\ell+1} \mathfrak{p}$ holds, we get

$$P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m)\right) \lesssim P\left(A^k(\bar{u}_1, 1)\right) \varphi(\underline{R}, 1, 1)^{-\ell}$$
$$+ P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \overline{\Theta}_1 \cap \Gamma_2\right) \varphi(\underline{R}, 1, 1)^{-\ell-1} \mathfrak{p}^{-1}$$
$$+ P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \overline{\Theta}_1 \cap \Gamma_2^c\right) \varphi(\underline{R}, 1, 1)^{-\ell-1} \mathfrak{p}^{-1}.$$

In a second step, we consider the relevant case where $(\bar{u}_1, 1)$ is open, connects to every vertex in $\hat{Q} \cap Z_1$ by an open edge and is not pivotal. Now, we can again make the analogous case distinction: Either $(\bar{u}_2, 1)$ would be pivotal if one connected it to every vertex in $\hat{Q} \cap Z_2$ and to $(\bar{u}_1, 1)$ by an open edge, or we can open $(\bar{u}_2, 1)$ and connect it to every vertex in $\hat{Q} \cap Z_1$ as well as to $(\bar{u}_1, 1)$ without losing the pivotality of $(\bar{u}, m)$. Based on that consideration, we can estimate the probability

$$P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m) \cap \overline{\Theta}_1 \cap \Gamma_2^c\right)$$

as above.

If we re-iterate this case distinction and the corresponding estimate for $u_2, \ldots, u_{M-1}$, we obtain — using that $P\left(A^k(\bar{u}, m)\right) \lesssim P\left(A^k(\bar{u}, m) \cap \tilde{Q}_\ell(\bar{u}, m)\right)$ holds by Lemma 4.5 —

$$
\begin{aligned}
P\left(A^k(\bar{u}, m)\right) \lesssim & P\left(A^k(\bar{u}_1, 1)\right) \varphi(\underline{R}, 1, 1)^{-\ell} \\
& + \sum_{q=1}^{M-1} P\left(A^k(\bar{u}, m) \cap \overline{\Theta}_q \cap \Gamma_{q+1}\right) \varphi(\underline{R}, 1, 1)^{-(\ell+1)q} \mathfrak{p}^{-q} \\
& + P\left(A^k(\bar{u}, m) \cap \overline{\Theta}_{M-1} \cap \Gamma_M^c\right) \varphi(\underline{R}, 1, 1)^{-(\ell+1)(M-1)} \mathfrak{p}^{-(M-1)} \\
\leq & P\left(A^k(\bar{u}_1, 1)\right) \varphi(\underline{R}, 1, 1)^{-\ell} \\
& + \sum_{q=1}^{M-1} P\left(A^k(\bar{u}_q, 1)\right) \varphi(\underline{R}, 1, 1)^{-(\ell+1)q-\ell} \mathfrak{p}^{-q} \\
& + P\left(A^k(\bar{u}, m) \cap \overline{\Theta}_{M-1} \cap \Gamma_M^c\right) \varphi(\underline{R}, 1, 1)^{-(\ell+1)(M-1)} \mathfrak{p}^{-(M-1)}.
\end{aligned}
$$

Now, it suffices to observe that

$$
P\left(A^k(\bar{u}, m) \cap \overline{\Theta}_{M-1} \cap \Gamma_M^c\right) = 0 \tag{4.6}
$$

holds — since we can force the vertex $(\bar{u}_M, 1)$ in the last cylinder Z_M to become pivotal by attaching it to $\hat{Q} \cap Z_M$ and to $(\bar{u}_{M-1}, 1)$ (the latter vertex is connected to the remaining points of $\hat{Q}$). Altogether, we get

$$
P\left(A^k(\bar{u}, m)\right) \lesssim \sum_{q=0}^{M-1} P\left(A^k(\bar{u}_{q+1}, 1)\right) \left(\varphi(\underline{R}, 1, 1)^{-(\ell+1)} \mathfrak{p}^{-1}\right)^q \tag{4.7}
$$

and thus

$$
\sum_{u \in U_n^L} P\left(A^k(\bar{u}, m)\right) \lesssim M \left(\varphi(\underline{R}, 1, 1)^{-(\ell+1)} \mathfrak{p}^{-1}\right)^M \sum_{u \in U_n^L} P\left(A^k(\bar{u}, 1)\right). \tag{4.8}
$$

We complete the estimate by recalling that $M \lesssim R(m)^d$ holds by assumption.

Left to notice is that the constant $\mathfrak{p}^{-1} = \left(1 - e^{-\lambda 2^{-nd} \Pi(1,n)}\right)^{-1}$ diverges as $n \to \infty$; however, the proof is easily modified to replace $\mathfrak{p}$ by a constant that does not depend on n. The key modification is the following: Instead of considering an a priori fixed set of positions $\bar{u}_1, \ldots, \bar{u}_M$, we can choose the positions in every step of the above procedure randomly. Replace the events Γ_q by the events that there exists a closed weight-one vertex in Z_q which would be pivotal if adjacent to every vertex in $Z_q \cap \hat{Q}$. On the complement of this event, there exists a deterministic $\epsilon > 0$ such that, with high probability, a sub-cylinder $B_\epsilon \times [1, 1+\epsilon] \subset Z_q$ has a positive fraction of closed vertices that would not be pivotal if adjacent to every vertex in $Z_q \cap \hat{Q}$. Replicating the proof above, we may force exactly one of these vertices open and connect it to $Z_q \cap \hat{Q}$ without destroying the pivotality of $(\bar{u}, m)$, at a cost asymptotically of order $\lambda |B_\epsilon| \pi([1, 1+\epsilon])$ as $n \to \infty$. Thus $\mathfrak{p}$ can be replaced by $\lambda |B_\epsilon| \pi([1, 1+\epsilon])$, or, equivalently, by λ in the corresponding estimates. $\qquad \square$

5 Infinite-volume differential inequalities and the proof of Theorem 1.8

Throughout this section, we will assume that Assumption 3.3 holds for some families of functions $(\mathfrak{r}_\lambda)_{\lambda>0}$ and $(\mathfrak{R}_\lambda)_{\lambda>0}$ as well as that the moment condition

$$F(\lambda) := \int_1^\infty \mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m)\pi(\mathrm{d}m) < \infty \tag{5.1}$$

holds for every $\lambda > 0$. Naturally, to show exponential decay in the subcritical regime $\lambda < \lambda_T$, it would be sufficient that the moment condition (5.1) holds subcritically. Moreover, notice that for any compact interval $[\lambda_1, \lambda_2]$ and $\lambda \in [\lambda_1, \lambda_2]$, we have

$$1 \leq F(\lambda_2) \leq F(\lambda) \leq F(\lambda_1) < \infty. \tag{5.2}$$

We now turn to the continuum system and derive the differential inequalities that will ultimately be used in infinite volume. Since our main results concern the infinite-volume quantities $\theta_\lambda(k)$ and $\mathbb{E}_\lambda[|\mathscr{C}(\bar{0}, 1)|]$, we must first introduce the *finite-volume continuum* versions of these objects. These will arise as limits of the finite-volume discrete approximations constructed before.

The results of this chapter are obtained by combining ideas from the proof of [21, Theorem 1.5] with the limiting procedures established in Appendix III. Using Corollary 3.7, we begin by taking the limit $n \to \infty$ in our discrete approximation in order to obtain the differential inequalities for the continuum system in a finite box.

For $\lambda > 0$ and $L, k \in \mathbb{N}$, we work under the infinite-volume measure $\mathbb{P}_\lambda$, but we define the *finite-volume cluster* $\mathscr{C}_L(\bar{0}, 1)$ to be the connected component of $(\bar{0}, 1)$ in the graph obtained from ξ by *removing all edges that leave the cylinder* $\Lambda^L := [-L, L]^d \times [1, \infty)$. Equivalently, vertices outside of this box are treated as absent. This is the standard finite-volume restriction in continuum percolation (see, e.g., [28, Chapter 3], [32, Chapter 6]).

We then set

$$\theta_\lambda^L(k) := \mathbb{P}_\lambda\big(|\mathscr{C}_L(\bar{0}, 1)| \geq k\big), \tag{5.3}$$

the finite-volume cluster-tail function. In the discrete approximation constructed before, we similarly defined

$$\theta_\lambda^{L,n}(k) := P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0}, 1)| \geq k\big). \tag{5.4}$$

The continuum limit proved in Lemma III.4 yields

$$\theta_\lambda^L(k) = \lim_{n\to\infty} \theta_\lambda^{L,n}(k),$$

for every fixed $\lambda > 0$, $L \in \mathbb{N}$, and $k \in \mathbb{N}$. This identity provides the differential equation for the finite-volume continuum system, which is the starting point for the infinite-volume analysis carried out below.

Lemma 5.1. *Let $0 < \lambda < \lambda_T$. For any L sufficiently large and for all $k \in \mathbb{N}$, we have*

$$c(\lambda)\left(\frac{k}{\mathbb{E}_\lambda[|\mathscr{C}_L(\bar{0}, 1)|]} - 1\right)\theta_\lambda^L(k) \lesssim \frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_\lambda^L(k), \tag{5.5}$$

where the constant $c(\lambda) > 0$ is given by

$$c(\lambda) = \left(\lambda \int_1^\infty \mathfrak{r}_\lambda(m)\mathfrak{R}_\lambda(m)\pi(\mathrm{d}m)\right)^{-1}. \tag{5.6}$$

Proof. The claim follows from the first estimate (3.5) claimed in Corollary 3.8 by taking the limit $n \to \infty$ and using Lemma III.4. We have

$$\sum_{m \in M_n} \Pi(m, n) \mathfrak{r}_\lambda(m) \mathfrak{R}_\lambda(m) \to \int_1^\infty \mathfrak{r}_\lambda(m) \mathfrak{R}_\lambda(m) \pi(\mathrm{d}m) < \infty, \quad n \to \infty,$$

since the sums on the left side are precisely the Riemann sums corresponding to the integral on the right side.

The remainder of the proof of the theorem is essentially given by Lemma III.1 from Appendix III since the contribution of the edge influences on the right side of (3.5) vanishes in the limit $n \to \infty$. $\qquad\square$

Instead of taking the infinite-volume limit $L \to \infty$ now, we first derive an exponential bound for θ_λ^L in finite volume:

Lemma 5.2. *Let $0 < \lambda < \lambda_0 < \lambda_T$. Then, for L sufficiently large, there exist constants $c(\lambda, \lambda_0) > 0$ and $C(\lambda, \lambda_0) > 0$ such that, for all $k \in \mathbb{N}$, we have*

$$\theta_\lambda^L(k) \lesssim C(\lambda, \lambda_0) \exp\left\{ -\frac{c(\lambda, \lambda_0)}{\mathbb{E}_{\lambda_0}[|\mathscr{C}_L(\bar{0}, 1)|]} \times k \right\}.$$

Proof. Fix $\lambda_0 < \lambda_T$ and $\lambda < \lambda_0$. Let L be sufficiently large, then the inequality (5.5) — which holds by Lemma 5.1 — is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log \theta_\lambda^L(k) \gtrsim c(\lambda) \left(\frac{k}{\mathbb{E}_\lambda[|\mathscr{C}_L(\bar{0}, 1)|]} - 1 \right). \tag{5.7}$$

Integrating inequality (5.7) over $[\lambda, \lambda_0]$, exploiting the obvious monotonicity $E_\lambda^L[|\mathscr{C}(\bar{0}, 1)|] \leq \mathbb{E}_{\lambda_0}[|\mathscr{C}_L(\bar{0}, 1)|]$ and using

$$\int_\lambda^{\lambda_0} c(\lambda) \mathrm{d}\lambda \geq \frac{\lambda_0 - \lambda}{\lambda_0} \frac{1}{F(\lambda)} =: c(\lambda, \lambda_0) > 0, \tag{5.8}$$

we obtain

$$\log \theta_{\lambda_0}^L(k) - \log \theta_\lambda^L(k) \gtrsim c(\lambda, \lambda_0) \left(\frac{k}{\mathbb{E}_{\lambda_0}[|\mathscr{C}_L(\bar{0}, 1)|]} - 1 \right) \mathrm{d}\lambda. \tag{5.9}$$

Equivalently, the estimate (5.9) can be formulated as

$$\log \left(\frac{\theta_{\lambda_0}^L(k)}{\theta_\lambda^L(k)} \right) \gtrsim c(\lambda, \lambda_0) \left(\frac{k}{\mathbb{E}_{\lambda_0}[|\mathscr{C}_L(\bar{0}, 1)|]} - 1 \right)$$

and taking the exponential on both sides yields the claim of Lemma 5.2 with the choice $C(\lambda, \lambda_0) = \mathrm{e}^{c(\lambda, \lambda_0)}$. $\qquad\square$

Now, we are ready to get our main result in the subcritical regime — by taking the infinite-volume limit $L \to \infty$:

Proposition 5.3. *Let $0 < \lambda < \lambda_0 < \lambda_T$. There exist constants $c(\lambda, \lambda_0) > 0$ and $C(\lambda, \lambda_0) > 0$ such that for all $k \in \mathbb{N}$ we have*

$$\theta_\lambda(k) \lesssim C(\lambda, \lambda_0) \exp\left\{ -\frac{c(\lambda, \lambda_0)}{\mathbb{E}_{\lambda_0}[|\mathscr{C}(\bar{0}, 1)|]} \times k \right\}. \tag{5.10}$$

Proof. Given Lemma III.5 and Lemma III.7, the claim follows immediately from the inequality in Lemma 5.2 by taking the infinite-volume limit $L \to \infty$. Notice that $\mathbb{E}_{\lambda_0}[|\mathscr{C}(\bar{0}, 1)|] < \infty$ holds since $\lambda_0 < \lambda_T$. $\qquad\square$

In the subcritical case, we obtained the desired result as a continuum and infinite-volume limit of (3.5) under suitable moment conditions — but it does not provide a meaningful estimate in the supercritical case. We turn to the more universally useful bound (3.6) next.

Proposition 5.4. *Let $\lambda > 0$. We have*

$$c(\lambda) \left(\frac{k\,(1 - e^{-1})}{\sum_{i=1}^{k} \theta_\lambda(i)} - 1 \right) \theta_\lambda(k) \lesssim \frac{\mathrm{d}}{\mathrm{d}\lambda} \theta_\lambda(k), \tag{5.11}$$

with the constant $c(\lambda) > 0$ given by (5.6). Moreover, let $0 < \lambda_1 \leq \lambda_0$, then, for every $k \geq 1$, we have

$$\theta_{\lambda_1}(k) \lesssim C(\lambda_1, \lambda_0) \exp\left\{ -\frac{(1 - e^{-1})c(\lambda_1, \lambda_0)k}{\sum_{i=1}^{k} \theta_{\lambda_0}(i)} \right\}, \tag{5.12}$$

where the constants $c(\lambda_1, \lambda_0)$, $C(\lambda_1, \lambda_0) > 0$ are as in the proof of Lemma 5.2.

Proof. Starting from the estimate (3.6) in Corollary 3.7, we proceed by taking the limit $n \to \infty$ (using Lemma III.4) and the limit $L \to \infty$ (using Lemma III.5) to obtain inequality (5.11). Integrating (5.11) over $[\lambda_1, \lambda_0]$ and taking the exponential on both sides yields (5.12). $\qquad\square$

We now move on to the supercritical regime where we closely follow the proof strategy for the second part of [21, Theorem 1.5] to derive the following result:

Proposition 5.5. *Let $\lambda > \lambda_T$, then we have*

$$\theta(\lambda) = \lim_{k \to \infty} \theta_\lambda(k) > 0$$

and thus $\lambda_T \geq \lambda_c$ holds. To be more precise, for any $\lambda_0 > \lambda_T$, there exists a constant $c_0 > 0$ such that for any $\lambda \in [\lambda_T, \lambda_0]$ we have $\theta(\lambda) > c_0(\lambda - \lambda_T)$.

Proof. We start by defining an auxiliary critical parameter

$$\widetilde{\lambda}_c := \sup\{\lambda \geq 0 : \exists \alpha, \beta > 0 \text{ such that } \theta_\lambda(k) \leq \alpha k^{-\beta} \text{ for every } k \geq 1\}$$
$$= \inf\left\{\lambda \geq 0 : \limsup_{n \to \infty} \frac{\log \theta_\lambda(n)}{\log n} \geq 0\right\}$$

Trivially, we have $\widetilde{\lambda}_c \leq \lambda_c$ and $\lambda_T \leq \widetilde{\lambda}_c$ holds by Proposition 5.3 (which gives us equivalence of exponential decay of the tail function and finiteness of the expectation for the cluster size). The proof strategy is to show first $\widetilde{\lambda}_c = \lambda_T$ and then $\widetilde{\lambda}_c = \lambda_c$. Indeed, we actually have exponential decay below $\widetilde{\lambda}_c$ which we prove in the following: Fix $\lambda_0 < \widetilde{\lambda}_c$ so that there exist $\alpha, \beta > 0$ such that $\theta_{\lambda_0}(k) \leq \alpha k^{-\beta}$ for any $k \geq 1$. Without loss of generality, we can assume that $\beta < 1$ holds. Once again, we use the notation $\lesssim$ to denote inequalities up to a multiplicative constant (here we allow these constants to depend only on α, β, λ_0 and model assumptions like the dimension $d \geq 1$). For every $k \geq 1$, we get

$$\sum_{i=1}^{k} \theta_{\lambda_0}(i) \lesssim k^{1-\beta}$$

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since $\sum_{i=1}^{k} i^{-\beta} \leq 1 + \frac{1}{1-\beta}k^{1-\beta}$ holds for $\beta < 1$. By (5.12), we then get

$$\theta_{\lambda_1}(k) \lesssim k^{-\beta} \exp\left\{-(1 - e^{-1})c(\lambda_1, \lambda_0)k^{\beta}\right\}, \tag{5.13}$$

for some $c_1 > 0$, any $0 < \lambda_1 \leq \lambda_0$ and $k \geq 1$. Summing inequality (5.13) over k leads to

$$\mathbb{E}_{\lambda_1}[|\mathscr{C}(\bar{0}, 1)|] \lesssim \sum_{k=1}^{\infty} k^{1-\beta} \exp(-(1 - e^{-1})c(\lambda_1, \lambda_0)k^{\beta})$$
$$\lesssim (1 - e^{-1})^{1-1/\beta} c(\lambda_1, \lambda_0)^{1-1/\beta} < \infty$$

for any $0 \leq \lambda_1 \leq \lambda_0$. Therefore, we get $\lambda_0 < \lambda_T$ and conclude $\lambda_T \geq \widetilde{\lambda}_c$.

Next, we prove that $\theta(\lambda) > 0$ holds for every $\lambda > \widetilde{\lambda}_c$. Consider, for $\lambda > 0$,

$$T_N(\lambda) = \frac{1}{\log N} \sum_{i=1}^{N} \frac{1}{i} \theta_{\lambda}(i), \quad N \geq 2,$$

and notice that we have $\lim_{N \to \infty} T_N(\lambda) = \theta(\lambda)$. We use the notation $\Sigma_k(\lambda) = \sum_{i=0}^{k-1} \theta_{\lambda}(i)$ and, by applying (5.11) from Proposition 5.4, obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} T_N(\lambda) \gtrsim \frac{c(\lambda)}{\log N} \sum_{k=1}^{N} \left[\frac{\theta_{\lambda}(k)}{\Sigma_k(\lambda)} - \frac{\theta_{\lambda}(k)}{k}\right], \tag{5.14}$$

for every $\lambda > 0$ and $N \geq 2$, where $c(\lambda)$ is given by (5.6). Notice now that we have

$$\frac{\theta_{\lambda}(k)}{\Sigma_k(\lambda)} \geq \int_{\Sigma_k(\lambda)}^{\Sigma_{k+1}(\lambda)} \frac{1}{x} \mathrm{d}x = \log \Sigma_{k+1}(\lambda) - \log \Sigma_k(\lambda) \tag{5.15}$$

and combining (5.14) with (5.15) yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} T_N(\lambda) \gtrsim \frac{c(\lambda) \log \Sigma_{N+1}(\lambda)}{\log N} - c(\lambda) T_N(\lambda),$$

for every $\lambda > 0$ and $N \geq 2$. Fixing $\widetilde{\lambda}_c < \lambda_1 < \lambda_2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} T_N(\lambda) \gtrsim \frac{c(\lambda) \log \Sigma_{N+1}(\lambda_1)}{\log N} - c(\lambda) T_N(\lambda_2),$$

for every $\lambda_1 \leq \lambda \leq \lambda_2$ and $N \geq 2$. Furthermore, notice the relation

$$\frac{\log \Sigma_{N+1}(\lambda)}{\log N} \geq \frac{\log(N\theta_{\lambda}(N))}{\log N} = 1 + \frac{\log(\theta_{\lambda}(N))}{\log N}, \quad N \geq 2,$$

which — since $\lambda_1 > \widetilde{\lambda}_c$ — implies

$$\limsup_{N \to \infty} \inf_{\lambda_1 \leq \lambda \leq \lambda_2} \frac{\mathrm{d}}{\mathrm{d}\lambda} T_N(\lambda) \gtrsim c(\lambda)(1 - \theta(\lambda_2)).$$

Integrating this last inequality and plugging in the lower bound from (5.8) leads to

$$\theta(\lambda_2) \geq \limsup_{n \to \infty} \int_{\lambda_1}^{\lambda_2} \frac{\mathrm{d}}{\mathrm{d}\lambda} T_n(\lambda) \mathrm{d}\lambda \gtrsim \frac{\lambda_2 - \lambda_1}{\lambda_2} \frac{1}{F(\lambda_1)} [1 - \theta(\lambda_2)]$$

which obviously implies

$$\theta(\lambda_2) > 0$$

and, to be more precise,

$$\theta(\lambda_2) \gtrsim \frac{1}{\lambda_2 F(\lambda_1)}(\lambda_2 - \lambda_1) \geq \frac{1}{\lambda_0 F(\lambda_T)}(\lambda_2 - \lambda_1) \tag{5.16}$$

for all $\lambda_1 > \lambda_T$ and $\lambda_2 \in [\lambda_1, \lambda_0]$. Notice that the choice of $\lambda_2 > \lambda_1 > \widetilde{\lambda}_c$ was arbitrary. This implies that $\widetilde{\lambda}_c > \lambda_c$, therefore we deduce that $\widetilde{\lambda}_c = \lambda_c$ and the claim follows readily. $\qquad \square$

The proof of our main result Theorem 1.8 — postulating the sharpness of the phase transition for the min-reach RCM and the classical RCM with infinite range — now follows immediately from the results of this section:

Proof of Theorem 1.8. We have shown in Section 4 that the min-reach RCM satisfies Assumption 3.3 for the choice

$$\mathfrak{r}_\lambda = \mathfrak{R}_\lambda = C(\lambda)^{R(m)^d}, \tag{5.17}$$

with $C(\lambda) > 0$ such that $\lambda \mapsto C(\lambda)$ is non-increasing on $(0, \infty)$. Therefore, the moment conditions (5.1) — required for all the results stated in this section — are satisfied by our Assumption (EMC), so that the claims of Proposition 5.3 and Proposition 5.5 hold for the min-reach RCM satisfying (EMC). Naturally, the non-weighted RCM (NonW) also satisfies the assumptions of Proposition 5.3 and Propsitions 5.5 for the choice

$$\mathfrak{r}_\lambda = \mathfrak{R}_\lambda \equiv 1. \tag{5.18}$$

The claim of Theorem 1.8 thus follows simply from combining the claims of Proposition 5.3 and Proposition 5.5.

$\qquad \square$

Appendix

I   Tool box

In order to prove results presented later in the appendix, we will require several well-known technical tools. For proofs and further discussion of the following theorems, the reader is referred to Last and Penrose [25]. Some of the definitions and formulas below are quoted verbatim from that reference.

Mecke's Formula.   Given $m \in \mathbb{N}$ and a measurable function $f \colon \mathbf{N} \times \mathbb{X}^m \to \mathbb{R}_{\geq 0}$, the Mecke equation for ξ states that

$$\mathbb{E}_\lambda \left[\sum_{\vec{x} \in \eta^{(m)}} f(\xi, \vec{x}) \right] = \lambda^m \int \mathbb{E}_\lambda \left[f\left(\xi^{x_1, \ldots, x_m}, \vec{x}\right) \right] \nu^{\otimes m}(\mathrm{d}\vec{x}), \tag{I.1}$$

where $\vec{x} = (x_1, \ldots, x_m)$, $\eta^{(m)} = \{(x_1, \ldots, x_m) \colon x_i \in \eta, x_i \neq x_j \text{ for } i \neq j\}$, and $\nu^{\otimes m}$ is the m-product measure of ν on $\mathbb{X}^m$.

Slivnyak's theorem and conditional independence. Let η be a Poisson point process on a measurable space $(\mathbb{X}, \mathcal{X})$ with intensity measure $\lambda\nu$. For any fixed finite collection of points $\{x_1, \ldots, x_k\} \subset \mathbb{X}$, the conditional law of η given that it contains these points is the same as the law of

$$\{x_1, \ldots, x_k\} \cup \eta',$$

where η' is an independent Poisson point process on $\mathbb{X}$ with the same intensity measure $\lambda\nu$. In particular, removing a finite number of known points from η leaves the remainder as an independent Poisson process with the same intensity. This property, often referred to as Slivnyak's theorem, underlies the Galton–Watson exploration argument used in Lemma 1.5.

Restriction of a Poisson process. Let η be a Poisson point process on a measurable space $(\mathbb{X}, \mathcal{X})$ with intensity measure $\lambda\nu$, and let $A \subset \mathbb{X}$ be measurable. Then the restricted process $\eta \cap A$ is a Poisson point process with intensity $\lambda\nu|_A$. If $(A_i)_{i \in I}$ are disjoint measurable subsets of $\mathbb{X}$, the restrictions $\eta \cap A_i$ are independent.

Projection (mapping theorem). Let η be a Poisson point process on $\mathbb{X} \times \mathbb{Y}$ with intensity $\mu(\mathrm{d}x)\,\nu(\mathrm{d}y)$. Under the projection map $(x, y) \mapsto x$, the image of η is a Poisson point process on $\mathbb{X}$ with intensity measure $\mu(\mathrm{d}x)\,\nu(\mathbb{Y})$. More generally, the image of a Poisson point process under any measurable map is a Poisson process with the pushed-forward intensity measure.

Furthermore, we are also going to need the following Russo formula proved in a recent paper by Last and Chebunin [6, Remark 9.5], which works directly in infinite volume.

Margulis–Russo formula. Let η be the underlying Poisson point process on $\mathbb{X} = \mathbb{R}^d \times (1, \infty)$ with intensity measure $\lambda\nu$, and let ξ denote the random geometric graph constructed from η according to the connection function φ. For $x \in \mathbb{X}$, we write ξ^x for the configuration obtained from ξ by adding the point x (with its mark and all the corresponding incident edges).

Assume that $f(\xi) < \infty$ almost surely and imposes no compact-support assumption on f, which is crucial for our purposes:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathbb{E}_\lambda\big[f(\xi)\big] \;=\; \int_\mathbb{X} \mathbb{E}_\lambda\big[f(\xi^x) - f(\xi)\big]\,\nu(\mathrm{d}x). \tag{I.2}$$

In particular, (I.2) applies whenever f is an indicator function of an event depending on the infinite-volume graph ξ, such as $\{|\mathscr{C}(\bar{0}, 1)| \geq k\}$.

For completeness, we recall that in the classical situation where f depends only on ξ_Λ for some ν-finite $\Lambda \subset \mathbb{X}$, a finite-volume Margulis–Russo formula is well known; see, e.g., [26, Theorem 3.2].

II Existence of phase transition

In this section, we provide a proof of Lemma 1.5, which guarantees the existence of a phase transition under assumptions (A) and (NB).

The subcritical part follows the classical Galton–Watson comparison method used in continuum percolation, as developed in the monographs by Meester and Roy [28] and Penrose [32]; see also Last and Penrose [25] for the modern formulation via Mecke's formula and Slivnyak's theorem, and related developments in Deijfen and Häggström [9].

For the supercritical part, we use a different comparison argument. We first restrict the PPP to a positive–measure set of weights and to a finite interval of radii on which the

connection probability admits a uniform positive lower bound. By the restriction and mapping properties of the PPP (see Section I), the restricted configuration is again a marked PPP but with controlled weights and spatial intensity. On this process, we construct a finite–range unweighted random connection model whose connection function is bounded from above by the restricted marked kernel. Thus, the unweighted model is stochastically dominated by the restricted marked model. Since unweighted finite–range RCMs in dimensions $d \geq 2$ have a non–trivial phase transition (see Meester and Roy [28, Chapter 3]), percolation of the unweighted RCM implies percolation of the restricted model, and hence of the original one.

We now give the details.

Proof of Lemma 1.5. We begin by defining a few quantities that will be used throughout the proof. For $r \geq 0$ and $a \geq 1$, set

$$\iota(r;a) = \int_1^\infty \varphi(r;a,b)\,\pi(\mathrm{d}b), \qquad I_{\mathrm{inf}} = \int_0^\infty r^{d-1}\iota(r;1)\,\mathrm{d}r, \qquad I_{\mathrm{sup}} = \operatorname*{ess\,sup}_{a \geq 1} \int_0^\infty r^{d-1}\iota(r;a)\,\mathrm{d}r.$$

By Assumption (A), the map $a \mapsto \iota(r;a)$ is non-decreasing for every r, hence

$$\int_0^\infty r^{d-1}\iota(r;a)\,\mathrm{d}r \geq I_{\mathrm{inf}} \qquad \text{for all } a \geq 1. \tag{II.1}$$

Assumption (NB) states precisely that $0 < I_{\mathrm{inf}} \leq I_{\mathrm{sup}} < \infty$.

Subcritical regime. We first show that the expected cluster size $\chi_\lambda = \mathbb{E}_\lambda[|\mathscr{C}(\bar{0},1)|]$ is finite for sufficiently small $\lambda > 0$.

We explore the cluster $\mathscr{C}(\bar{0},1)$ in the graph $\xi^{(\bar{0},1)}$ in a breadth–first manner. For $n \geq 0$, let $\mathcal{Z}_n$ denote the set of vertices discovered at generation n and set $Z_n = |\mathcal{Z}_n|$. The exploration starts with $\mathcal{Z}_0 = \{(\bar{0},1)\}$, hence $Z_0 = 1$. For $n \geq 0$, the next generation $\mathcal{Z}_{n+1}$ consists of those points of η that, in $\xi^{(\bar{0},1)}$, are adjacent to some vertex in $\mathcal{Z}_n$ but do not belong to $\mathcal{Z}_0 \cup \cdots \cup \mathcal{Z}_n$.

We define

$$\mathcal{E}_n = \bigcup_{k=0}^n \mathcal{Z}_k, \qquad \mathcal{R}_n = \eta \setminus \mathcal{E}_n,$$

so that $\mathcal{E}_n$ represents the vertices already discovered up to generation n, and $\mathcal{R}_n$ the remaining unexplored vertices. By Slivnyak's theorem, conditional on $\mathcal{E}_n$, the process $\mathcal{R}_n$ is again a Poisson point process on $\mathbb{X}$ with intensity $\lambda\nu$, independent of the previously discovered part. Fix a discovered vertex $(\bar{x},a) \in \mathcal{Z}_n$. Conditionally on the current exploration, each point $(\bar{y},b) \in \mathcal{R}_n$ is connected to $(\bar{x},a)$ independently with probability $\varphi((\bar{x},a),(\bar{y},b))$.

By Mecke's formula, the number of potential neighbors of a vertex with mark a in the full graph $\xi^{(\bar{0},1)}$ follows a Poisson law with mean

$$m(a) = \lambda s_d \int_0^\infty r^{d-1}\iota(r;a)\,\mathrm{d}r, \tag{II.2}$$

where s_d denotes the surface area of the unit sphere in $\mathbb{R}^d$.

Conditional on $\mathcal{E}_n$, let $(a_i)_{i \leq Z_n}$ denote the weights of the vertices in $\mathcal{Z}_n$. By construction and by (II.2), the expected number of points discovered at the next step satisfies

$$\mathbb{E}_\lambda[Z_{n+1} \mid \mathcal{E}_n] \leq \sum_{i=1}^{Z_n} m(a_i), \tag{II.3}$$

since each vertex of mark a_i gives rise on average to $m(a_i)$ potential neighbors, and some of these may already have been discovered or may coincide for different parents.

Using Assumption (NB) and (II.2), we have

$$m(a_i) \leq \lambda s_d I_{\sup}, \quad i \leq Z_n, \tag{II.4}$$

and therefore

$$\mathbb{E}_\lambda[Z_{n+1} \mid \mathcal{E}_n] \leq (\lambda s_d I_{\sup}) Z_n.$$

Taking expectations and iterating over n yields

$$\mathbb{E}_\lambda[Z_n] \leq (\lambda s_d I_{\sup})^n.$$

Since the cluster size satisfies $|\mathscr{C}(\bar{0}, 1)| = \sum_{n \geq 0} Z_n$, we obtain, whenever $\lambda s_d I_{\sup} < 1$ holds,

$$\chi_\lambda = \sum_{n \geq 0} \mathbb{E}_\lambda[Z_n] \leq \sum_{n \geq 0} (\lambda s_d I_{\sup})^n < \infty.$$

Hence, the expected cluster size is finite in this regime, and consequently

$$\lambda_T \geq \frac{1}{s_d I_{\sup}} > 0.$$

Supercritical regime. We now show that $\theta(\lambda) > 0$ holds for all λ sufficiently large, and thus $\lambda_c < \infty$.

Assumption (NB) states that

$$I_{\inf} = \int_0^\infty r^{d-1} \iota(r; 1) \, \mathrm{d}r > 0,$$

where we set

$$\iota(r; 1) = \int_1^\infty \varphi(r; 1, b) \, \pi(\mathrm{d}b).$$

Since the integrand in $I_{\inf}$ is non-negative, the positivity of $I_{\inf}$ implies that $\iota(r; 1) > 0$ on a set of radii of positive Lebesgue measure. By Assumption (A.2), which states that $\varphi(r; a, b)$ is non-increasing in the distance variable r, we may find $r_2 \in (0, \infty)$ such that $\varphi(r; 1, b)$ is bounded away from zero on $(0, r_2]$ for a positive mass set of weights b. More precisely, by measurability of φ and Fubini's theorem, there exists a measurable set $B \subset (1, \infty)$ with $\pi(B) > 0$ and a constant $\varepsilon > 0$ such that

$$\varphi(r; 1, b) \geq \varepsilon \qquad \text{for all } r \in (0, r_2] \text{ and all } b \in B. \tag{II.5}$$

By Assumption (A.3), which states that $\varphi(r; a, b)$ is non-decreasing in both weights, this bound extends to all $a \in B$:

$$\varphi(r; a, b) \geq \varepsilon \qquad \text{for all } r \in (0, r_2], \ a \in B, \ b \in B. \tag{II.6}$$

Let $\eta_B := \eta \cap (\mathbb{R}^d \times B)$ be the restriction of the marked PPP to the set of weights B. By the restriction property of Poisson point processes (see Appendix I), η_B is a PPP on $\mathbb{R}^d \times B$ with intensity measure $\lambda \, \mathrm{d}\bar{x} \, \pi|_B(\mathrm{d}a)$. Projecting η_B onto the spatial coordinate yields a homogeneous PPP on $\mathbb{R}^d$ of intensity $\lambda_B := \lambda \pi(B)$. In the comparison below, we consider only vertices whose weights lie in B; the distinguished point $(\bar{0}, 1)$ may be added on top of this configuration, but this does not affect the existence of an infinite cluster.

We now define an auxiliary unweighted random connection model on the spatial process associated to η_B. Let $g \colon [0, \infty) \to [0, 1]$ be the finite range adjacency function

$$g(r) := \varepsilon \, \mathbf{1}_{\{0 < r \leq r_2\}}(r).$$

Consider the unweighted random connection model on the Poisson process of intensity λ_B in $\mathbb{R}^d$ with the adjacency function g. By (II.6), for any two points $(\bar{x}, a)$ and $(\bar{y}, b)$ with $a, b \in B$ and $|\bar{x} - \bar{y}| \leq r_2$, we have

$$g(|\bar{x} - \bar{y}|) \leq \varphi((\bar{x}, a), (\bar{y}, b)).$$

Hence, every edge present in the unweighted model is also present in the restricted marked model if we couple the models accordingly. In other words, the unweighted RCM (λ_B, g) is stochastically dominated by the restricted marked RCM. In particular, we get the implications

percolation of the unweighted RCM$(\lambda_B, g) \implies$ percolation in the restricted marked RCM,

percolation in the restricted marked RCM $\implies$ percolation in the full marked RCM. (II.7)

Since g has finite range and $d \geq 2$, the unweighted RCM with the adjacency function g features a non-trivial phase transition: There exists $\lambda_c(g) \in (0, \infty)$ such that the model percolates whenever the spatial intensity exceeds $\lambda_c(g)$ (see Meester and Roy [28, Chapter 3]). Therefore, if

$$\lambda_B = \lambda \, \pi(B) > \lambda_c(g),$$

then the unweighted RCM percolates, and so, by (II.7), the full marked RCM percolates as well. Hence, percolation occurs for

$$\lambda > \frac{\lambda_c(g)}{\pi(B)}.$$

This shows $\lambda_c < \infty$. Combining that statement with the previously derived lower bound for λ_T and using $\lambda_T \leq \lambda_c$, we conclude that $0 < \lambda_T \leq \lambda_c < \infty$.

$\square$

III Taking limits

In this appendix section we collect the limit transitions needed for passing from the discrete, finite-volume setting to the continuous, infinite-volume framework. The finite-volume discrete differential inequalities used earlier (see Corollaries 3.7 and 3.8) are establsihed on a bounded domain. To obtain their continuous and infinite-volume counterparts, we first take the continuum limit $n \to \infty$ at fixed volume L, and subsequently the infinite-volume limit $L \to \infty$. The results below formalize these two steps and justify the transition to the infinite-volume continuous differential inequalities in Section 5.

III.1 Continuum limit

In this subsection we take the continuum limit $n \to \infty$ while keeping the volume L fixed. This provides the passage from the discrete approximation of our model to its finite-volume continuum version.

The next result confirms that the contribution of edge influences vanishes as $n \to \infty$ (by design of our exploration scheme), a fact that is crucial for reducing the discrete differential inequality to its continuum form and for proceeding in the spirit of [22].

Lemma III.1. *Let $\lambda > 0$ and let the moment condition*

$$\int_1^\infty \mathfrak{R}_\lambda(m)\pi(\mathrm{d}m) < \infty \tag{III.1}$$

hold. Then, for every $L, n \in \mathbb{N}$ sufficiently large,

$$I_\lambda^{n,L,\tilde{\gamma}} := \frac{1}{\delta_{\tilde{\gamma}}(\bar{0},1)} \sum_{e \in E_n^L} \delta_{\tilde{\gamma}}(e) \mathrm{INF}_e^P \left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)| \geq k\}} \right)$$

is bounded from above uniformly in $\tilde{\gamma}$. Moreover, for every $\lambda > 0$ and L sufficiently large, we have

$$\limsup_{n \to \infty} \sup_{\tilde{\gamma} > 0} I_\lambda^{n,L,\tilde{\gamma}} = 0.$$

Proof. For $L, n \in \mathbb{N}$, let $\hat{E}_n^L$ denote the set of edges in E_n^L incident to $(\bar{0}, 1)$ and $\tilde{E}_n^L := E_n^L \setminus \hat{E}_n^L$. For every $e = \{v, (\bar{0}, 1)\} \in \hat{E}_n^L$ such that $v = (\bar{u}, m)$ we have

$$\delta_{\tilde{\gamma}}(e) \leq \delta_{\tilde{\gamma}}(v) P_\lambda^{L,n}(v \text{ is open}), \tag{III.2}$$

since the edge e is only revealed if v is both revealed and open. Similarly, for every $e = \{v_1, v_2\} \in \tilde{E}_n^L$ such that $v_1 = (\bar{u}_1, m_1)$ and $v_2 = (\bar{u}_2, m_2)$ we have

$$\delta_{\tilde{\gamma}}(e) \leq \delta_{\tilde{\gamma}}(v_1) P_\lambda^{L,n}(v_1 \text{ is open}) + \delta_{\tilde{\gamma}}(v_2) P_\lambda^{L,n}(v_2 \text{ is open}), \tag{III.3}$$

since $e = \{v_1, v_2\} \in \tilde{E}_n^L$ is only revealed if at least one of the vertices v_1, v_2 is both revealed and open. Since (for n, L sufficiently large) we have $\delta_{\tilde{\gamma}}((\bar{u}, 1)) \lesssim \delta_{\tilde{\gamma}}((\bar{0}, 1))$ for every $u \in U_n^L$ and using Assumption 3.3a), we get:

- For $e = \{v, (\bar{0}, 1)\} \in \hat{E}_n^L$ such that $v = (\bar{u}, m)$

$$\frac{\delta_{\tilde{\gamma}}(e)}{\delta_{\tilde{\gamma}}((\bar{0},1))} \lesssim P_\lambda^{L,n}(v \text{ is open}) \mathfrak{R}_\lambda(m)$$

$$\lesssim \left(1 - e^{-\lambda 2^{-nd}} \right) \mathfrak{R}_\lambda(m).$$

- For $e = \{v_1, v_2\} \in \tilde{E}_n^L$ such that $v_1 = (\bar{u}_1, m_1)$ and $v_2 = (\bar{u}_2, m_2)$

$$\frac{\delta_{\tilde{\gamma}}(e)}{\delta_{\tilde{\gamma}}((\bar{0},1))} \lesssim P_\lambda^{L,n}(v_1 \text{ is open}) \mathfrak{R}_\lambda(m_1) + P_\lambda^{L,n}(v_2 \text{ is open}) \mathfrak{R}_\lambda(m_2)$$

$$\lesssim \left(1 - e^{-\lambda 2^{-nd}} \right) (\mathfrak{R}_\lambda(m_1) + \mathfrak{R}_\lambda(m_2)).$$

Therefore, we obtain the estimate

$$\frac{1}{\delta_{\tilde{\gamma}}(\bar{0},1)} \sum_{e \in E_n^L} \delta_{\tilde{\gamma}}(e) \mathrm{INF}_e^P \left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)| \geq k\}} \right)$$

$$\lesssim \sum_{e = \{(\bar{u},m),(\bar{0},1)\} \in \hat{E}_n^L} \mathfrak{R}_\lambda(m) \left(1 - e^{-\lambda 2^{-nd}} \right) \mathrm{INF}_e^P \left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)| \geq k\}} \right)$$

$$+ \sum_{e = \{(\bar{u}_1,m_1),(\bar{u}_2,m_2)\} \in \tilde{E}_n^L} (\mathfrak{R}_\lambda(m_1) + \mathfrak{R}_\lambda(m_2)) \left(1 - e^{-\lambda 2^{-nd}} \right) \mathrm{INF}_e^P \left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)| \geq k\}} \right). \tag{III.4}$$

We use the following estimates for the influences of the edges: For $e = \{(\bar{0}, 1), v\} \in \hat{E}_n^L$ such that $v = (\bar{u}, m)$, we have

$$\mathrm{INF}_e^P(f) \leq 2 \left(1 - e^{-\lambda 2^{-nd} \Pi(m,n)} \right) \lesssim \lambda 2^{-nd} \Pi(m, n)$$

and for $e \in \tilde{E}_n^L$ we have

$$\mathrm{INF}_e^\mu(f) \leq \left(1 - \mathrm{e}^{-\lambda 2^{-nd}\Pi(m_1,n)}\right)\left(1 - \mathrm{e}^{-\lambda 2^{-nd}\Pi(m_2,n)}\right) \leq \lambda^2 2^{-2nd}\Pi(m_1,n)\Pi(m_2,n).$$

Plugging those estimates into (III.4) and using the notation $|\hat{E}|$ (resp. $|\tilde{E}|$) for the number of unweighted edges in $\hat{E}_n^L$ (resp. $\tilde{E}_n^L$), we get

$$\frac{1}{\delta_{\tilde{\gamma}}(\bar{0},1)} \sum_{e \in E_n^L} \delta_{\tilde{\gamma}}(e)\mathrm{INF}_e^P\left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}\right)$$

$$\lesssim \left(\sum_{m \in M_n} \mathfrak{R}_\lambda(m)\Pi(m,n)\right) \lambda 2^{-nd}\left(1 - \mathrm{e}^{-\lambda 2^{-nd}}\right)|\hat{E}|$$

$$+ \left(\sum_{m_1 \in M_n} \sum_{m_2 \in M_n} \left(\mathfrak{R}_\lambda(m_1) + \mathfrak{R}_\lambda(m_2)\right)\Pi(m_1,n)\Pi(m_2,n)\right)\left(1 - \mathrm{e}^{-\lambda 2^{-nd}}\right)\lambda^2 2^{-2nd}|\tilde{E}|.$$

Moreover, we have

$$|\hat{E}| \leq 2^{nd}C(L,d), \qquad |\tilde{E}| \leq 2^{2nd}C(L,d)^2$$

for a constant $C(L,d) > 0$ which only depends on L and d. Altogether, we get

$$\frac{1}{\delta_{\tilde{\gamma}}(\bar{0},1)} \sum_{e \in E_n^L} \delta_{\tilde{\gamma}}(e)\mathrm{INF}_e^P\left(\mathbb{1}_{\{|\mathcal{C}(\bar{0},1)|\geq k\}}\right)$$

$$\lesssim \left[\lambda C(L,d) + \lambda^2 C(L,d)^2\right]\left(\sum_{m \in M_n} \mathfrak{R}_\lambda(m)\Pi(m,n)\right)\left(1 - \mathrm{e}^{-\lambda 2^{-nd}}\right)$$

and the claim follows immediately under the moment condition

$$\int_1^\infty \mathfrak{R}_\lambda(m)\pi(\mathrm{d}m) < \infty.$$

$\square$

We next show that the discrete quantities appearing in the finite-volume differential inequalities converge, as the mesh size tends to zero, to their continuous counterparts on $\Lambda^L = [-L, L]^d \times [1, \infty)$. To do so it is convenient to construct an auxiliary discrete model coupled to the continuum Poisson configuration.

Coupling (Auxiliary discretized model)**.** *For each n, we consider the partition of $\Lambda_n^L = [-L - 2^{-n-1}, L + 2^{-n-1}]^d \times [1, \infty)$ into cells $\{R_u : u = (\bar{u}, m) \in V_n^L\}$. The cells are given, as already hinted at in the definition of the finite lattice approximation in Section 2, by*

$$R_u = \bigotimes_{i=1}^d [\bar{u}_i - 2^{-n-1}, \bar{u}_i + 2^{-n-1}] \times [m, m + 2^{-n}]$$

for $u = (\bar{u}, m) \in V_n^L$. The boundary points form a ν-null set and are negligible for our purposes, but we can also choose boxes half-open in each coordinate to define a partition. Given a realization of the continuum marked Poisson graph $\xi^{(\bar{0},1)}$ in Λ_n^L, we associate to it a discrete graph whose vertices are the cells $\{R_u : u \in V_n^L\}$ as follows.

A vertex R_u (with $u = (\bar{u}, m) \in V_n^L$) is declared open if the cell R_u contains at least one point of $\xi^{(\bar{0},1)}$, and closed otherwise. Two vertices R_{u_1} and R_{u_2} (with $u_i = (\bar{u}_i, m_i)$) are joined

by an edge in the discrete graph if there exists at least one edge in $\xi^{(\bar{0},1)}$ between a point of $\xi^{(\bar{0},1)} \cap R_{u_1}$ and a point of $\xi^{(\bar{0},1)} \cap R_{u_2}$. In other words, the discrete graph on $\{R_u \colon u \in V_n^L\}$ records which cells are occupied by Poisson points and which pairs of occupied cells are linked by at least one continuum edge between points inside them.

We then define $\mathscr{C}_L^n(\bar{0}, 1)$ to be the (random) cluster of the vertex $R_{(\bar{0},1)}$ in this discrete graph, that is, the set of all vertices R_u that are connected to $R_{(\bar{0},1)}$ by an open path of edges.

Remark III.2 (Poisson cells with at most one point). *Let $N_{R_{(\bar{u},m)}}$ be the number of Poisson points in the cell $R_{(\bar{u},m)}$. Define*

$$K_n := \bigcap_{(\bar{u},m) \in V_n^L} \{N_{R_{(\bar{u},m)}} \leq 1\}.$$

A Poisson estimate gives $\mathbb{P}_\lambda(N_{R_{(\bar{u},m)}} \geq 2) \leq \lambda^2 \nu(R_{(\bar{u},m)})^2$, hence

$$\mathbb{P}_\lambda(K_n^c) \leq \lambda^2 \sum_{(\bar{u},m) \in V_n^L} \nu(R_{(\bar{u},m)})^2 \xrightarrow[n \to \infty]{} 0.$$

Thus with high probability each cell contains at most one Poisson point. This estimate will be used to control the error terms arising when we compare the discrete model $P_\lambda^{L,n}$ with the discretized continuum model described above.

Remark III.3 (Connectivity in the auxiliary model). *For two distinct cells $R_u, R_v \subset \Lambda_n^L$, we denote by*

$$\widehat{\varphi}(R_u, R_v) := \mathbb{P}_\lambda(R_u \longleftrightarrow R_v \text{ in } \xi \mid R_u \text{ and } R_v \text{ are open})$$

the probability that they are joined by an edge in the auxiliary discrete graph, already conditioned on both cells being open. In general, this quantity does not admit a simple closed expression, because an open cell may contain more than one Poisson point.

On the high–probability event K_n, the definition becomes explicit: conditioning on K_n and on both cells being open forces them to contain exactly one point each, with independent conditional distributions proportional to ν on their respective cells. Hence,

$$\widehat{\varphi}(R_u, R_v) = \frac{1}{\nu(R_u)\,\nu(R_v)} \int_{R_u} \int_{R_v} \varphi(\|\bar{x} - \bar{y}\|; m_x, m_y)\, \nu(\mathrm{d}x)\, \nu(\mathrm{d}y) \qquad \text{on } K_n. \tag{III.5}$$

Notice that the expression on the right side of (III.5) does not depend on λ. On K_n, the cell–cell connection probability is essentially the average of the continuum connection function over the two cells. Since $\mathbb{P}_\lambda(K_n^c) \to 0$, this representation is key for the convergence arguments in the proof of the next result Lemma III.4.

By construction, $\mathscr{C}_L^n(\bar{0}, 1)$ is a measurable function of the full continuum configuration $\xi^{(\bar{0},1)}$. We introduce the notation

$$\widehat{\theta}_\lambda^{L,n}(k) := \mathbb{P}_\lambda\big(|\mathscr{C}_L^n(\bar{0}, 1)| \geq k\big), \qquad k \in \mathbb{N}. \tag{III.6}$$

This auxiliary quantity will be used in the proof of the next lemma to relate the asymptotic behavior of the discrete tail probabilities $\theta_\lambda^{L,n}(k)$ and the continuum tail probabilities $\theta_\lambda^L(k)$.

Lemma III.4. *Let $\lambda > 0$ and $k \in \mathbb{N}$. As n goes to infinity, the following limits hold:*

(1) $\theta_\lambda^{L,n}(k) \to \theta_\lambda^L(k)$,

(2) $\frac{d}{d\lambda}\theta_\lambda^{L,n}(k) \to \frac{d}{d\lambda}\theta_\lambda^L(k)$,

(3) $E_\lambda^{L,n}[|\mathcal{C}(\bar{0},1)|] \to \mathbb{E}_\lambda[|\mathscr{C}_L(\bar{0},1)|]$.

Proof. To compare the auxiliary model to the original continuous one, we follow ideas from Meester [27], while accounting for the additional parameter arising from the weights. The comparison of our discrete inhomogeneous Bernoulli percolation to the auxiliary model follows a more original and analytical approach with a cluster expansion argument.

Reduction to a uniformly continuous connection function. Throughout the proof we may and shall assume that the connection function $\varphi(r;a,b)$ is uniformly continuous, jointly in (r,a,b) on $[0,2L] \times [1,\infty)^2$, without altering the law of the model. This can be justified as follows:

Continuity in each variable. By Assumption (A.2), for each pair of weights (a,b) the map $r \mapsto \varphi(r;a,b)$ is monotone and hence has at most countably many discontinuities. Since the spatial positions of the PPP are absolutely continuous with respect to Lebesgue measure and Λ^L contains almost surely only finitely many points, the probability that the distance between any two points falls on a discontinuity is zero; redefining connection function on this ν-null set does not affect the law, and we may assume that $r \mapsto \varphi(r;a,b)$ is continuous for all (a,b). By a similar argument using Assumption (A.3), we may without loss of generality assume that the connection function is continuous in the weight variables as well.

Joint uniform continuity. By the assumed separate continuity and monotonicity properties, the connection function φ is jointly continuous in its variables. Since possible non-uniformity can only arise from the unbounded weight coordinates, a standard weight truncation argument (as in the proof of Corollary 2.5) allows us to assume joint uniform continuity.

Proof of Part (1). We first show that

$$\widehat{\theta}_\lambda^{L,n}(k) \xrightarrow{n\to\infty} \theta_\lambda^L(k) \tag{III.7}$$

and then that

$$\theta_\lambda^{L,n}(k) - \widehat{\theta}_\lambda^{L,n}(k) \xrightarrow{n\to\infty} 0. \tag{III.8}$$

Combining (III.7) and (III.8) gives the claim.

Step 1: Convergence of the discretized continuum cluster. First of all, we couple the Bernoulli percolation on the vertices in our finite lattice approximation to the same underlying marked PPP $\eta \cup \{(\bar{0},1)\}$ in the obvious way, i.e., by letting a vertex $(\bar{u},m) \in V_n^L$ being open if and only if the corresponding cell is occupied by at least one point from $\eta \cup \{(\bar{0},1)\}$.

Next, fix a realization $\xi^{(\bar{0},1)}$. Since η almost surely has finitely many points in Λ^L and the cell diameters tend to zero, there almost surely exists a finite $n_0(\xi)$ such that for all $n \geq n_0(\xi)$ each point lies in a distinct cell of the partition. By the construction of the auxiliary model and coupling Bernoulli percolation on the vertices to the same underlying PPP in our finite lattice approximation in the obvious way (i.e., letting a vertex $(\bar{u},m) \in V_n^L$ being open if and only if the corresponding cell is occupied by Poisson points), both clusters $\mathscr{C}_L^n(\bar{0},1)$ and $\mathscr{C}_L(\bar{0},1)$ are the same from a graph-theoretic perspective (are isomorphic). Consequently, for all $n \geq n_0(\xi)$, we get

$$\left|\mathscr{C}_L^n(\bar{0},1)\right| = \left|\mathscr{C}_L(\bar{0},1)\right|.$$

Hence

$$\mathbb{1}_{\{|\mathscr{C}_L^n(\bar{0},1)| \geq k\}} \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{1}_{\{|\mathscr{C}_L(\bar{0},1)| \geq k\}}.$$

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By dominated convergence,
$$\widehat{\theta}_\lambda^{L,n}(k) \xrightarrow{n\to\infty} \theta_\lambda^L(k).$$

Step 2: Comparing $\theta_\lambda^{L,n}$ and $\widehat{\theta}_\lambda^{L,n}$. We now prove

$$\theta_\lambda^{L,n}(k) - \widehat{\theta}_\lambda^{L,n}(k) \xrightarrow{n\to\infty} 0. \tag{III.9}$$

We write the tail probability as a sum over the exact cluster size:

$$\theta_\lambda^{L,n}(k) = \sum_{s \geq k-1} P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| = s+1\big), \qquad \widehat{\theta}_\lambda^{L,n}(k) = \sum_{s \geq k-1} \mathbb{P}_\lambda\big(|\mathscr{C}_L^n(\bar{0},1)| = s+1\big).$$

Note the following uniform upper bound on the expected cluster size:

$$E_\lambda^{L,n}\big[|\mathcal{C}(\bar{0},1)|\big] \leq \sum_{(\bar{u},m)\in V_n^L} p_\lambda(\bar{u},m) \leq \sum_{(\bar{u},m)\in V_n^L} \lambda\,\nu_{(\bar{u},m)} \leq \lambda\,\nu(\Lambda_0^L) < \infty. \tag{III.10}$$

The same uniform estimate holds for the discretized continuum cluster $\mathscr{C}_L^n(\bar{0},1)$ and, by Markov's inequality, for any $s \geq 1$,

$$\sup_n P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| \geq s\big) \leq \frac{\lambda\,\nu(\Lambda_0^L)}{s}, \qquad \sup_n \mathbb{P}_\lambda\big(|\mathscr{C}_L^n(\bar{0},1)| \geq s\big) \leq \frac{\lambda\,\nu(\Lambda_0^L)}{s}.$$

Thus, we have control over the tails in s — uniformly in n. To obtain (III.9), it then suffices to show that for each fixed $s \in \mathbb{N}$,

$$P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| = s+1\big) - \mathbb{P}_\lambda\big(|\mathscr{C}_L^n(\bar{0},1)| = s+1\big) \xrightarrow{n\to\infty} 0.$$

Notice that, for any $s \in \mathbb{N}$, we have

$$P_\lambda^{L,n}\big(|\mathcal{C}(\bar{0},1)| = s+1\big) \tag{III.11}$$
$$= \frac{1}{s!} \sum_{\substack{u_1,\ldots,u_s \in V_n^L \setminus \{u_0\} \\ \text{all distinct}}} \sum_{G \in \mathrm{Gr}_{s+1}} P_\lambda^{L,n}\big(\mathcal{E}_G(u_0,\ldots,u_s)\big) P_\lambda^{L,n}\big(\partial(\{u_0,\ldots,u_s\}) = \varnothing\big) \prod_{i=1}^s p_\lambda(u_i),$$

where $u_0 = (\bar{0},1)$, the outer sum runs over all possible locations of the remaining s vertices of the cluster, and Gr_{s+1} denotes the set of simple connected graphs on $\{0,1,\ldots,s\}$. The event $\mathcal{E}_G(u_0,\ldots,u_s)$ specifies that the induced subgraph on $\{u_0,\ldots,u_s\}$ is exactly G, while the event $\partial(\{u_0,\ldots,u_s\}) = \varnothing$ denotes the external-isolation event (i.e., the event that the vertices $u_0,\ldots,u_s$ do not connect to any other vertex). Finally, the factor $\prod_{i=1}^s p_\lambda(u_i)$ is the probability that the vertices $u_1,\ldots,u_s$ are open.

We obtain an almost identical expansion for the probability $\mathbb{P}_\lambda\big(|\mathscr{C}_L^n(\bar{0},1)| = s+1\big)$, with exactly the same sums and same factors $\prod_{i=1}^s p_\lambda(u_i)$; the only difference being that the "subgraph configuration" and "external isolation" probabilities, taken under continuum law $\mathbb{P}_\lambda$, might differ.

Fix $s \in \mathbb{N}$, a graph $G \in \mathrm{Gr}_{s+1}$, and distinct sites $u_1,\ldots,u_s \in V_n^L \setminus \{u_0\}$. In the two models, the probabilities that $u_0,\ldots,u_s$ are open and that the cells $R_{u_0},\ldots,R_{u_s}$ are occupied are the same, to be more precise

$$P_\lambda^{L,n}(u_i \text{ open}) = 1 - e^{-\lambda\nu_{u_i}} = \mathbb{P}_\lambda(N_{R_{u_i}} \geq 1),$$

Thus, after conditioning on the set of occupied cells, the comparison reduces to two factors:

(i) the probability that the subgraph induced by $\{u_0, \ldots, u_s\}$ is equal to G;

(ii) the probability that $\{u_0, \ldots, u_s\}$ is externally isolated.

Subgraph configuration. Fix $s \in \mathbb{N}$ and $G \in \mathrm{Gr}_{s+1}$. Condition in both models on the event that the vertices $u_0, \ldots, u_s$ are open / the cells $R_{u_0}, \ldots, R_{u_s}$ are occupied.

In our discrete Bernoulli percolation model, the probability of an edge between u_i and u_j is $\varphi(\|\bar{u}_i - \bar{u}_j\|; m_{u_i}, m_{u_j})$. In the auxiliary model, the corresponding edge probability is $\widehat{\varphi}(R_{u_i}, R_{u_j})$ from Remark III.3. On the event K_n, we have the representation (III.5) and, since the cell diameters tend to 0 and φ is uniformly continuous on $\Lambda^L \times \Lambda^L$, this implies

$$\sup_{u \neq v} \left| \widehat{\varphi}(R_u, R_v) - \varphi(\|\bar{u} - \bar{v}\|; m_u, m_v) \right| \xrightarrow[n \to \infty]{} 0.$$

For fixed G and $(u_0, \ldots, u_s)$, the event $\mathcal{E}_G(u_0, \ldots, u_s)$ is specified by finitely many edges and non-edges on $\{u_0, \ldots, u_s\}$, so its probability is a polynomial in the corresponding edge probabilities. The uniform bound above therefore yields

$$\sup_{\substack{u_1,\ldots,u_s \in V_n^L \setminus \{u_0\} \\ \text{all distinct}}} \sup_{G \in \mathrm{Gr}_{s+1}} \left| P_\lambda^{L,n}\big(\mathcal{E}_G(u_0, \ldots, u_s)\big) - \mathbb{P}_\lambda\big(\mathcal{E}_G(R_{u_0}, \ldots, R_{u_s}) \mid K_n\big) \right| \xrightarrow[n \to \infty]{} 0.$$

Since $\mathbb{P}_\lambda(K_n^c) \to 0$ by Remark III.2, restricting to K_n does not affect the limit, and the subgraph configuration probabilities coincide asymptotically in the two models.

External isolation. Unlike the subgraph configuration, the external–isolation event depends on the state of every other open cell in Λ^L, which are almost surely finite. Thus, to compare the two models we first condition on the full set of occupied cells in the auxiliary model (respectively, the full set of open sites in the finite lattice approximation), and only afterwards restrict to the event K_n, so that this conditioning does not change any marginals.

Under this conditioning, the external–isolation probability of $\{u_0, \ldots, u_s\}$ is determined only by the collection of edge probabilities between these $s + 1$ sites and all other occupied/open cells.

The external–isolation probability is therefore a function of finitely many edge probabilities. Since these vertex–opening probabilities coincide in the two models, continuity of this function in its arguments yields pointwise convergence of the isolation probabilities for every possible set of open cells, which suffices here because the convergence of edge probabilities is already uniform in n. Hence,

$$\sup_{\substack{u_1,\ldots,u_s \in V_n^L \setminus \{u_0\} \\ \text{all distinct}}} \left| P_\lambda^{L,n}\big(\partial(\{u_0, \ldots, u_s\}) = \varnothing\big) - \mathbb{P}_\lambda\big(\partial(\{R_{u_0}, \ldots, R_{u_s}\}) = \varnothing \mid K_n\big) \right| \xrightarrow[n \to \infty]{} 0.$$

Since $\mathbb{P}_\lambda(K_n^c) \to 0$ by Remark III.2, restricting to K_n does not affect the limit, and the external–isolation probabilities coincide asymptotically in the two models.

Conclusion. Fix $s \in \mathbb{N}$ and notice that we have the following bound (when bounding the probability of subgraph configuration and external isolation by 1)

$$\frac{1}{s!} \sum_{x_1,\ldots,x_s \in V_n^L \setminus \{x_0\}} \sum_{G \in \mathrm{Gr}_{s+1}} \prod_{i=1}^{s} p_\lambda(x_i) \leq \frac{|\mathrm{Gr}_{s+1}|}{s!} \left(\sum_{x \in V_n^L} p_\lambda(x) \right)^s \leq \frac{|\mathrm{Gr}_{s+1}|}{s!} \big(\lambda \nu(\Lambda_0^L)\big)^s < \infty.$$

Using the fact that the vertex factors $\prod_{i=1}^{s} p_\lambda(x_i)$ are the same in both expansions, as well as both pointwise convergences and the upper bound above, allows to conclude that, for each $s \in \mathbb{N}$, we have

$$\left| P_\lambda^{L,n}(|\mathcal{C}(\bar{0}, 1)| = s + 1) - \mathbb{P}_\lambda(|\mathscr{C}_L^n(\bar{0}, 1)| = s + 1) \right| \to 0 \quad \text{as } n \to \infty,$$

which yields the claim in Part (1). $\qquad\square$

Proof of Part (2). We prove that for every $\lambda > 0$ and $k \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\, \theta_\lambda^{L,n}(k) \xrightarrow[n \to \infty]{} \frac{\mathrm{d}}{\mathrm{d}\lambda}\, \theta_\lambda^L(k).$$

The argument proceeds in two separate steps.

Step 1: Continuum versus discretized continuum. Let $A := \{|\mathscr{C}_L(\bar{0},1)| \geq k\}$ and $A_n := \{|\mathscr{C}_L^n(\bar{0},1)| \geq k\}$. Russo's formula in the continuum gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\, \theta_\lambda^L(k) = \int_{\Lambda^L} \Delta(x)\, \nu(\mathrm{d}x),$$

where $\Delta(x)$ denotes the probability that x is pivotal for A under Palm insertion/removal, i.e.,

$$\Delta(x) := \mathbb{P}_\lambda(\mathbb{1}_A(\xi^x) \neq \mathbb{1}_A(\xi)) = \mathbb{P}_\lambda(\xi^x \in A) - \mathbb{P}_\lambda(\xi \in A). \tag{III.12}$$

By the same argument as in Step 1 of Part (1), for almost every Poisson realization ξ, the clusters in A and A_n coincide as soon as $n \geq n_0(\xi)$ for some finite $n_0(\xi)$. Hence, for almost every $x \in \Lambda^L$,

$$\mathbb{1}_{\{x \text{ pivotal for } A_n\}}(\xi) = \mathbb{1}_{\{x \text{ pivotal for } A\}}(\xi) \qquad \text{for } n \geq n_0(\xi),$$

and therefore $\Delta_n(x) \to \Delta(x)$ a.e., where $\Delta_n(x)$ is the probability that x is pivotal for A_n (defined analogously to (III.12)). Since $0 \leq \Delta_n(x) \leq 1$ and $\Lambda_n^L \downarrow \Lambda^L$, dominated convergence yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\, \widehat{\theta}_\lambda^{L,n}(k) = \int_{\Lambda_n^L} \Delta_n(x)\, \nu(\mathrm{d}x) \xrightarrow[n \to \infty]{} \frac{\mathrm{d}}{\mathrm{d}\lambda}\, \theta_\lambda^L(k).$$

Step 2: Discretized continuum versus finite lattice apporximation. For the discrete Bernoulli model $P_\lambda^{L,n}$, Russo's formula (see Remark 2.6) yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\, \theta_\lambda^{L,n}(k) = \sum_{v \in V_n^L} \nu_v\, (1 - p_\lambda(v))\, \pi_v^{(n)}, \tag{III.13}$$

where $p_\lambda(v) = 1 - e^{-\lambda \nu_v}$ and $\pi_v^{(n)}$ denotes the probability that the vertex v is pivotal for the event $\{|\mathcal{C}(\bar{0},1)| \geq k\}$. We will show

$$\int_{\Lambda_n^L} \Delta_n(x)\, \nu(\mathrm{d}x) - \sum_{v \in V_n^L} \nu_v\, (1 - p_\lambda(v))\, \pi_v^{(n)} \xrightarrow[n \to \infty]{} 0. \tag{III.14}$$

Cell–averaged pivotality. Decomposing the integral on the left side of (III.14) cell by cell, we define for each $v \in V_n^L$

$$\widehat{\pi}_v^{(n)} := \frac{1}{\nu_v} \int_{R_v} \Delta_n(x)\, \nu(\mathrm{d}x), \qquad \nu_v := \nu(R_v). \tag{III.15}$$

Then

$$\int_{\Lambda_n^L} \Delta_n(x)\, \nu(\mathrm{d}x) = \sum_{v \in V_n^L} \nu_v\, \widehat{\pi}_v^{(n)}.$$

Note that $\Delta_n(x)$ is not constant on R_v, even on K_n, so $\widehat{\pi}_v^{(n)}$ is a genuine average of point pivotalities over the cell.

Cell pivotality. We next introduce an intermediate pivotality notion. Let $\widetilde{\pi}_v^{(n)}$ be the *cell pivotality* of R_v for A_n, defined as the probability that A_n occurs when the cell R_v is forced open, minus the probability that A_n occurs when R_v is forced closed, with all other cells sampled according to the discretized continuum law. Equivalently, $\widetilde{\pi}_v^{(n)}$ is the probability that the indicator of A_n changes when the state of the cell R_v is switched from closed to open.

First comparison: averaged point pivotality versus cell pivotality. Recall that $\widehat{\pi}_v^{(n)}$ is the average, over $x \in R_v$, of the Palm pivotality of a *point* inserted at x, whereas $\widetilde{\pi}_v^{(n)}$ measures the effect of switching the *state of the cell* R_v from closed to open.

The only possible discrepancy between these two notions arises from the fact that, in the Palm experiment defining $\Delta_n(x)$, the cell R_v may already contain a Poisson point. On the event K_n, this can only happen when $N_{R_v} = 1$. On the complementary event $\{N_{R_v} = 0\} \cap K_n$, inserting a point at $x \in R_v$ produces exactly one point in R_v with distribution proportional to ν, which coincides with the law of the unique point created when the cell R_v is forced open. In this case, the Palm pivotality at x and the cell pivotality agree.

Since both pivotality indicators take values in $[0,1]$, we may therefore bound

$$\left| \widehat{\pi}_v^{(n)} - \widetilde{\pi}_v^{(n)} \right| \leq \mathbb{P}_\lambda(N_{R_v} = 1) + \mathbb{P}_\lambda(K_n^c).$$

Multiplying by ν_v and summing over $v \in V_n^L$, we obtain

$$\sum_{v \in V_n^L} \nu_v \left| \widehat{\pi}_v^{(n)} - \widetilde{\pi}_v^{(n)} \right| \leq \sum_{v \in V_n^L} \nu_v \, p_\lambda(v) + \nu(\Lambda_0^L) \, \mathbb{P}_\lambda(K_n^c).$$

Since $p_\lambda(v) = O(\nu_v)$ and $\sum_v \nu_v^2 \to 0$, while $\mathbb{P}_\lambda(K_n^c) \to 0$, the right-hand side converges to zero as $n \to \infty$, which yields

$$\sum_{v \in V_n^L} \nu_v \left| \widehat{\pi}_v^{(n)} - \widetilde{\pi}_v^{(n)} \right| \xrightarrow[n \to \infty]{} 0. \tag{III.16}$$

Second comparison: cell pivotality versus Bernoulli pivotality. We now compare $\widetilde{\pi}_v^{(n)}$ with the Bernoulli pivotality $\pi_v^{(n)}$. In both models, pivotality of v implies that forcing v closed yields a finite cluster (S, G_0) with $|S| < k$, while forcing v open yields a larger finite cluster (T, G_1) with $|T| \geq k$, both clusters being externally isolated. Thus, both $\widetilde{\pi}_v^{(n)}$ and $\pi_v^{(n)}$ admit cluster expansions indexed by quadruples (S, T, G_0, G_1).

For $M \in \mathbb{N}$, let $\widetilde{\pi}_{v,M}^{(n)}$ and $\pi_{v,M}^{(n)}$ denote the contributions restricted to $|T| \leq M$, and let the corresponding remainders be $\widetilde{R}_{v,M}^{(n)}$ and $R_{v,M}^{(n)}$. For each fixed (S, T, G_0, G_1) with $|T| \leq M$, the associated probabilities in the two models differ only through internal edge probabilities and a single external–isolation factor, and these differences vanish as $n \to \infty$ by the results of Part (1). Since the truncated sums are finite, we obtain

$$\widetilde{\pi}_{v,M}^{(n)} - \pi_{v,M}^{(n)} \xrightarrow[n \to \infty]{} 0 \qquad \text{for each fixed } v \text{ and } M.$$

The remainders are controlled uniformly by the expected cluster size estimate: if $|T| > M$, forcing v open produces a cluster of size $> M$, so by Markov's inequality,

$$\sup_{n,v} \widetilde{R}_{v,M}^{(n)} + \sup_{n,v} R_{v,M}^{(n)} \leq \frac{C(\lambda, L)}{M}.$$

Letting first $n \to \infty$ at fixed M, and then $M \to \infty$, yields

$$\sum_{v \in V_n^L} \nu_v \left| \widetilde{\pi}_v^{(n)} - \pi_v^{(n)} \right| \xrightarrow[n \to \infty]{} 0. \tag{III.17}$$

Conclusion. Combining (III.16) and (III.17), we obtain

$$\int_{\Lambda_n^L} \Delta_n(x)\,\nu(\mathrm{d}x) - \sum_{v \in V_n^L} \nu_v\,\pi_v^{(n)} \xrightarrow[n \to \infty]{} 0.$$

Finally, since $1 - p_\lambda(v) = e^{-\lambda \nu_v} = 1 + O(\nu_v)$ and $\sum_v \nu_v^2 \to 0$, we may replace $(1 - p_\lambda(v))$ by 1 in (III.13) at vanishing cost. Comparing with (III.14) and Step 1 completes the proof of Part (2). $\qquad\square$

Proof of Part (3). Finally,

$$E_\lambda^{L,n}\big[|\mathcal{C}(\bar{0},1)|\big] = \sum_{k=1}^\infty \theta_\lambda^{L,n}(k), \qquad \mathbb{E}_\lambda\big[|\mathscr{C}_L(\bar{0},1)|\big] = \sum_{k=1}^\infty \theta_\lambda^L(k).$$

By Part (1) we have $\theta_\lambda^{L,n}(k) \to \theta_\lambda^L(k)$ for each fixed k. Moreover,

$$\begin{aligned}
E_\lambda^{L,n}\big[|\mathcal{C}(\bar{0},1)|\big] &\le E_\lambda^{L,n}\big[\#\{\text{open vertices in } \Lambda_n^L\}\big] \\
&\le \sum_{(\bar{u},m) \in V_n^L} p_\lambda(\bar{u},m) \\
&\le \sum_{(\bar{u},m) \in V_n^L} \lambda\,\nu_{(\bar{u},m)} = \lambda\,\nu(\Lambda_0^L),
\end{aligned}$$

so the expectations are uniformly bounded in n. Dominated convergence then yields the claim in Part (3). $\qquad\square$

III.2 Infinite-volume limit

To finish, we explain why under our assumptions the continuum models in finite volume converges to the appropriate quantities in infinite volume.

Lemma III.5. *Let $\lambda > 0$ and $k \in \mathbb{N}$. Under Assumptions (A) and (NB) the following limits hold:*

$$\theta_\lambda^L(k) \to \theta_\lambda(k), \tag{III.18}$$

$$\frac{\mathrm{d}\theta_\lambda^L(k)}{\mathrm{d}\lambda} \to \frac{\mathrm{d}\theta_\lambda(k)}{\mathrm{d}\lambda} \tag{III.19}$$

Proof. For this it suffices to apply the Lemma 3.5 from [5], or, more precisely, the corresponding result that appears in its proof. Indeed, for the models that we consider, the assumption **(D.1-)** from [5] is implied by our assumption (NB). $\qquad\square$

Remark III.6. *One important thing to notice for any extension of this last result to more general models is that the assumption **(D.1-)** from [5] is not necessary to get (III.18) and a simpler integrability assumption like*

$$\int_{(1,\infty)} \int_{(1,\infty)} \int_{\mathbb{R}^d} \varphi(\bar{x}, m_0, m)\mathrm{d}\bar{x}\pi(\mathrm{d}m_0)\pi(\mathrm{d}m) < \infty \tag{III.20}$$

*would be sufficient. Furthermore, although the assumption **(D.1 -)** appears somewhat naturally in the proof of (III.19) (and we believe it might be necessary for some specific models), it is possible to proceed without that convergence. Indeed, one could consider Dini-derivatives which always exist, like Hutchcroft does in [22], since our results only require one sided bounds.*

Lemma III.7. *Let $\lambda > 0$ such that $\lambda < \lambda_T$. Under Assumptions* (A) *and* (NB) *as L goes to infinity, the following limit holds:*

$$\mathbb{E}_\lambda[|\mathscr{C}_L(\bar{0}, 1)|] \to \mathbb{E}_\lambda[|\mathscr{C}(\bar{0}, 1)|], \tag{III.21}$$

Proof. Since we assume $\lambda < \lambda_T$ we have $\mathbb{E}_\lambda[|\mathscr{C}(\bar{0}, 1)|] < \infty$. Next, notice that for any $L > 0$ we have $\mathbb{E}_\lambda[|\mathscr{C}_L(\bar{0}, 1)|] \le \mathbb{E}_\lambda[|\mathscr{C}(\bar{0}, 1)|]$. Therefore, we can conclude by using the formula

$$\mathbb{E}_\lambda[|\mathscr{C}_L(\bar{0}, 1)|] = \sum_{k=1}^\infty \theta_\lambda^L(k), \tag{III.22}$$

together with the preceding result Lemma III.5. $\qquad\square$

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