

Characterizing quantum synchronization in the van der Pol oscillator via tomogram and photon correlation

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We access the quantum synchronization (QS) in the steady state of a driven quantum van der Pol oscillator (vdPo) using two distinct figures of merit: (i) the nonclassical area δ and (ii) the second-order correlation function $g^{(2)}(0)$, which are both viable in experimental architectures. The nonclassical area quantifier rooted in homodyne tomography, allows us to assess the nonclassical nature of the vdPo's state directly from the tomogram without requiring full state reconstruction or the Wigner function negativity. Within a well-defined parameter regime of drive strength and detuning, both δ and $g^{(2)}(0)$ exhibit pronounced signatures of synchronization that complements the phase coherence between the drive and the vdPo. We derive an analytical expression for the steady-state density matrix and the corresponding tomogram of the system, valid for arbitrary strengths of the harmonic drive. Analysis of the quantum tomogram uncovers clear phase-locking behavior, enabling the identification of the synchronization region (Arnold tongue) directly in terms of experimentally measurable quantities. Furthermore, the behaviour of $g^{(2)}(0)$ provides a statistical perspective that reinforces the tomographic signatures of QS. By analyzing the interplay between these metrics, we can gain more profound insights into the underlying mechanisms that govern QS in such systems.

I. INTRODUCTION

Synchronization [1, 2] is a fundamental phenomenon wherein interacting systems adjust their rhythms to exhibit coherent dynamics. This behavior is ubiquitous in classical nonlinear systems—from pendulum clocks to neural networks—and is typically driven by the interplay between nonlinearity and nonequilibrium features, leading to gain and loss processes in coupled systems. Beyond its classical roots, synchronization has also been identified in quantum mechanical systems, where it has garnered significant interest for its potential to deepen our understanding of complex dynamics and to enable technological innovations in areas such as quantum metrology [3], communication [4, 5], and computation [6, 7]. Thus in the quantum regime, quantum synchronization (QS) [7–18] offers insight into how collective coherence arises from the continuous competition between coherent driving and dissipation. However, QS is not merely a scaled-down version of its classical counterpart—it also involves distinctly quantum features, such as quantum fluctuations, coherence, and entanglement [19], which make its characterization a nontrivial task. The phenomenon of QS has been studied extensively in various engineered quantum systems [15, 20–24] and quantum analogs of classical oscillators [16, 25]. Among these van der Pol oscillator (vdPo) [26] stands out as a prototypical model to explore QS. It provides a minimal yet effective framework where gain and loss mechanisms, characteristic of open quantum systems [17, 18, 27, 28], are balanced with nonlinearity to produce synchronized steady states.

The driven vdPo is a powerful model to study the quantum-classical boundary which in the classical limit show limit cycles and synchronization due to external driving while in quantum domain retains these characteristics but influenced by the quantum features. Several works [15–17, 24, 29–31] have analyzed the behaviour of a single driven quantum vdPo and its synchronization to an external frequency of the drive. Early studies demonstrated that a driven quantum vdPo can exhibit phase locking and synchronization plateaus despite quantum noise [15, 16], while frequency entrainment and locking were identified through power spectral analysis [17]. Classical-quantum links in synchronization have been identified in the exact dynamics of a quantum vdPo [11]. The emergence of metastable phases were observed for a squeezed-driven [18] quantum vdPo where system intermittently switches between different synchronized states and is unique to quantum regime [29]. With Kerr nonlinearity [32], the quantum asymptotic phase revealed signature of quantum torus synchronization where multiple frequencies lock rather than a limit cycle in the deep quantum regime [30]. Topologically coupled vdP network showed edge-state synchronization robust to noise and disorder [31]. Recently, an experimental implementation of a driven quantum vdPo using $^{40}\text{Ca}^+$ atom inside a Paul trap has been proposed, which can be a breakthrough in bringing QS to trapped ion platforms [24].

In quantum systems, synchronization is often visualized via Arnold tongue [33] structures, with various measures proposed [12, 17, 25, 28, 34] to quantify it. Tools like the Wigner function [35] or Husimi function [36] and frequency entrainment are commonly used but require reconstruction from measured observables, which is error-prone and computationally demanding for large Hilbert spaces. Recent advances suggest that quantum tomograms [37, 38] can directly capture state features

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without requiring full state reconstruction. In this work, we develop a tomographic framework to analyze the dynamics of a driven quantum vdPo and characterize synchronization behavior. As part of this, we use the *nonclassical area* (δ) [39] as a synchronization-sensitive quantifier, constructed from the phase-dependent quadrature variance extracted from the quantum tomogram. Nonclassical area measures deviations from the vacuum or coherent-state benchmark, which represents the most classical-like quantum states and saturates the minimum uncertainty bound. Thus it quantifies the effective area projected by a tomogram on the tomographic plane referenced against that of a vacuum state. Owing to its tomographic definition, δ serves as a useful tool for probing synchronization in the quantum regime as phase locking between the drive and the oscillator manifests through characteristic angular modulations of quadrature fluctuations. Moreover, quantum state tomography [40–42] enables the practical estimation of δ by measuring the rotated quadrature operator and constructing the corresponding tomogram as a positive probability distribution [43].

In this manuscript, we propose a new measure of QS that builds upon insight from quantum tomograms to quantum optics. The synchronization of the vdPo to the external drive is an excellent framework to investigate universal synchronization behavior. To measure synchronization in a driven quantum vdPo, we employ the nonclassical area δ as a diagnostic figure of merit. In particular, we investigate the emergence of nonclassical Arnold tongue structures, paralleling the classical Arnold tongue in the quantum vdPo, through both the nonclassical area and the equal-time second-order correlation function [44]. In this work, we study the synchronization of qvdP into two extreme limits, the classical (characterized by $\kappa_2/\kappa_1 = 0$) and deep quantum regimes ($\kappa_2/\kappa_1 \geq 10$), as per the classification in Ref. [27]. We benchmark the QS measured from both figures of merit for the driven vdPo under evidence of the Arnold tongue shape. Finally, in the landscape of Qs measured in the deep quantum limit ($\kappa_2/\kappa_1 \rightarrow \infty$), we demonstrate the construction of the quantum state in the quadrature plane with the tomogram and in phase space with the Wigner distribution. Furthermore, to achieve direct experimental access, we reformulate the master equation of driven vdPo in terms of the quantum tomogram, avoiding full state reconstruction.

The content of this paper is structured as follows: In Sec. II, we develop a model of driven-dissipative vdPo. In Sec. II.1, we discuss the assessment of QS on the prospects of the tomographic representation of quantum states and introduce the nonclassical area as an indicator of nonclassicality. In addition, another figure of merit—two-particle correlation—will be used to quantitatively compare the synchronization behavior of quantum vdPo. We display our results regarding the appearance of QS in Sec. III. In Section IV, we examine how quantum tomograms evolve at steady state within the framework of an Arnold tongue in the deep quantum regime. Finally, Section V summarizes the key findings of this work with the prospects of an experimental feasibility of QS. Relevant analytical details are included as Appendices A–B.

II. THEORETICAL FRAMEWORK AND FIGURE OF MERITS

A quantum vdPo serves as an ideal platform for studying quantum synchronization with an external drive on account of unavoidable environmental degrees of freedom. The dynamics of the whole dissipative system in the rotating frame of the external drive are governed by the following master equation for the density matrix ρ , which has the standard form [16, 17]:

$$\frac{d\rho}{dt} = -i[\Delta a^\dagger a + F(a + a^\dagger), \rho] + \kappa_1 \mathcal{D}[a^\dagger]\rho + \kappa_2 \mathcal{D}[a^2]\rho, \quad (1)$$

where $a(a^\dagger)$ is the annihilation (creation) operator of the single-mode oscillator, Δ is the detuning of the drive's frequency ω_d from the natural frequency of the oscillator ω_0 , and F denotes the drive strength. The Lindblad dissipator is defined as $\mathcal{D}[\mathcal{O}]\rho = (2\mathcal{O}\rho\mathcal{O}^\dagger - \mathcal{O}^\dagger\mathcal{O}\rho - \rho\mathcal{O}^\dagger\mathcal{O})/2$, which accounts for environment-induced dissipation, while κ_1 and κ_2 represent the rates of linear pumping and nonlinear damping, respectively. $\mathcal{D}[a^\dagger]\rho$ assemble single excitation into the system while $\mathcal{D}[a^2]\rho$ removes a pair of excitation from the system. The competition between linear pumping and nonlinear damping gives rise to self-sustained oscillations (limit cycle) with stable amplitude, and is crucial for the emergence of synchronization behavior in the quantum regime. The numerical simulations of the master Eq. (1) are done at the steady state ($t \rightarrow \infty$) in QuTip [45] to obtain the steady-state density matrix $\rho^{ss} = \rho(t \rightarrow \infty)$. Notably, all system parameters are scaled by the linear damping rate κ_1 , unless otherwise specified.

Strictly speaking, we do not explore the synchronization behavior across different dynamical regimes; instead, we focus on two extreme limiting cases by tuning κ_2 in Eq. (1). These regimes correspond to the classical limit, realized by setting $\kappa_2 = 0$ and the deep quantum limit, achieved by taking $\kappa_2 \rightarrow \infty$. In the classical limit, the absence of nonlinear damping eliminates two-photon loss, and the system dynamics are governed primarily by linear gain and external driving, thereby suppressing quantum features and yielding behavior close to that of a classical oscillator. Conversely, in the deep quantum limit, the nonlinear damping term dominates thus playing a major role in characterizing the discrete level structure of vdPo. Physically, large κ_2 corresponds to extremely strong two-photon loss, which rapidly de-excites higher photon-number states at rates proportional to $\kappa_2 n(n-1)$. As a result, population in Fock states with $n \geq 2$ is strongly suppressed. Here κ_2 effectively restricts the system's evolution to the lowest Fock states, mostly to the subspace spanned by the vacuum ($|0\rangle$) and single-photon ($|1\rangle$) states, by strongly suppressing higher photon-number excitations.

II.1. Measure of quantum synchronization

Unlike the synchronization in classical systems, the QS cannot be addressed directly through conventional means, since the fluctuations of the phase space variables strictly maintain the Heisenberg principle. A main open question is about the possibility of measuring QS in the presence of a different kind of environment coupling that causes dephasing rather than dissipation. In this study, we address this question by discussing the assessment of two distinct figures of merit: (i) quantum tomogram and (ii) quantum correlation, in a framework of the quantum vdPo. The first one encoded peculiar features in quantum measurement that are directly related to experimentally accessible probability distributions [43, 46] (e.g., quadrature measurements in quantum optics) and reveal nonclassical features like squeezing, superposition, and entanglement. In addition, the tomographic representation exploits universal applicability across systems and is neither basis dependent nor linked to any particular operator basis (like photon number or position). The latter figure of merit, $g^{(2)}(0)$ [44, 47], is directly measurable in experiments [48] and provides an invasive signature on novel many-body phenomena, including quantum phase transitions that account for fundamental quantum fluctuations in promising physical setups such as Bose-Hubbard, Dicke model, circuit-QED, etc. By analyzing both of these figures of merit, we can gain more profound insights into the underlying processes that facilitate synchronization in our quantum network. To the best of our knowledge, the development of a theory-experimental bridge with the aforementioned figure of merit in the context of QS has yet to be discussed.

II.1.A. Tomographic framework

Having detailed the driven-dissipative quantum vdPo we plan to investigate in our work, let us now highlight the figure of merit for quantifying synchronization in tomographic pictures. This picture offers a practical and robust alternative to probe synchronization particularly for experimental implementations. Here we characterize the quantum synchronization in the vdPo through its quantum tomogram. The rotated quadrature operators form a quorum of observables sufficient to reconstruct any single-mode quantum state. The single mode rotated quadrature operator is given by

$$\mathbb{X}_\theta = \frac{1}{\sqrt{2}}(ae^{-i\theta} + a^\dagger e^{i\theta}), \quad (2)$$

where $\theta \in [0, 2\pi]$ is the rotation angle in phase space that determines which field quadrature is being measured and $[a, \mathbb{X}_\theta] = e^{i\theta}/\sqrt{2}$. The quantum tomogram is the probability distribution of measurement outcomes for \mathbb{X}_θ . For quantum state described by the density matrix ρ the tomogram is defined as [37, 49]

$$\omega(X_\theta, \theta) = \langle X_\theta, \theta | \rho | X_\theta, \theta \rangle. \quad (3)$$

Here, $\mathbb{X}_\theta | X_\theta, \theta \rangle = X_\theta | X_\theta, \theta \rangle$ with

$$\begin{aligned} |X_\theta, \theta\rangle &= \frac{1}{\pi^{1/4}} \exp \left[-\frac{X_\theta^2}{2} - \frac{1}{2}e^{i2\theta}a^{\dagger 2} + \sqrt{2}e^{i\theta}a^\dagger X_\theta \right] |0\rangle \\ &= \sum_{nm} \mathcal{A}_{nm} |n+2m\rangle, \end{aligned} \quad (4)$$

$$\text{where } \mathcal{A}_{nm} = \frac{2^{\frac{n}{2}-m}\sqrt{(n+2m)!}}{\pi^{\frac{1}{4}} n! m!} X_\theta^n e^{[i(n+2m)\theta + im\pi - \frac{X_\theta^2}{2}]}.$$

Nonclassicality quantifier derived directly from the tomogram offers a practical framework to investigate QS. For a better understanding of the nonclassical area of a single-mode field, it is defined as [39]

$$\delta(\rho) = \int_0^{2\pi} d\theta \Delta X_\theta - \sqrt{2}\pi, \quad (5)$$

where ΔX_θ is the standard deviation in the measurement of the rotated quadrature operator \mathbb{X}_θ . Defined through quadrature fluctuations, $\delta = 0$ for classical states and $\delta \neq 0$ for genuinely nonclassical states, with higher values indicating stronger nonclassical character. Unlike indicators that may also respond to classical statistical mixtures, δ effectively filters out classicality and captures intrinsic quantum features.

II.1.B. Quantum correlation

Another commonly used indicator of nonclassicality, especially in photon statistics, is the equal-time second-order correlation function, $g^{(2)}(0)$, which characterizes the nature of correlations between two quanta (photons, excitations, etc.) emitted at the

Synchronization regime	$g^{(2)}(0)$	Statistical indicator
Unsynchronized, random	< 1	Irregular, anti-bunching
Synchronized, single-quantum regime	≈ 1	Regular, phase-locked, random
Synchronized, collective bursts	> 1	Highly regular, bunching

TABLE I. Synchronization performance of the quantum vdPo with statistical fingerprint. The results are bound by either maximum or minimum synchronization. This classification indicates that the efficiency of the synchronization process can vary significantly depending on the specific conditions and parameters involved in our study.

same time. At the steady state, for a single mode vdP oscillator, $g^{(2)}(0)$ is defined as [44],

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a \rangle_{\rho_{ss}}}{\langle a^\dagger a \rangle_{\rho_{ss}}^2}. \quad (6)$$

It is a measure of quantum degree of coherence encompassing information about photon correlations in the field. This insight reveals the non-universal connection between $g^{(2)}(0)$ and QS—it strongly depends on the physical system, parameter regime, and how synchronization can reach the fundamental limit imposed by the laws of quantum mechanics. This dependence highlights the importance of tailoring experimental setups to explore the nuances of QS.

III. NONCLASSICAL MANIFESTATIONS OF SYNCHRONIZATION

Our investigation of synchronization measure employs the nonclassical area (δ) formalism as a powerful tool for visualizing the Arnold tongue with nonclassicality. In particular, we examine the behavior of δ as a function of the driving strength F and detuning Δ , across two distinct dynamical regimes defined by the nonlinear damping parameter κ_2 . We consider a comparative analysis of the system dynamics in both deep quantum and classical regimes, with particular emphasis on the phase locking phenomenon. The Arnold tongue is a characteristic structure in the parameter space of driven nonlinear systems, often used to illustrate regions of synchronization. It delineates the range of parameters, typically drive strength and detuning, over which the system exhibits phase locking.

Figure 1 displays the Arnold tongue shape of the nonclassical area δ given in Eq. (5) as functions of F and Δ . Panels (a) and (b) correspond to the classical limit ($\kappa_2 = 0$) and the deep quantum limit ($\kappa_2 = 10^3$), respectively. Since δ is unbounded by any upper limit, an increase in δ indicates enhancement in nonclassicality. In both panels, red (blue) regions indicate maximal (minimal) values of δ , representing strong (weak) nonclassicality and hence stronger (weaker) synchronization signatures. In our study, the driving strength F is considered up to an order of 10, consistent with values already realized in experiments [24, 50]. In the classical regime [Fig. 1(a)], δ exhibits a pronounced dependence on both F and Δ , forming a distinct Arnold tongue structure centered around $\Delta = 0$. This tongue-shaped feature marks the region in parameter space where the oscillator locks to the drive frequency, indicating enhanced synchronization. In classical limit $\kappa_2 = 0$, the weak drive strength $F < 1$ always capture the maximum synchronization (red) but strong drive only measure the synchronization around the resonance. However, in $(F - \Delta)$ plane, a certain region (blue) indicates absence of any synchronization. For higher driving the nonclassical area reaches values as high as $\delta \sim 26$ in the vicinity of resonance.

In the deep quantum limit, characterized by strong nonlinear damping ($\kappa_2 = 10^3$), the dynamics of the quantum vdPo are confined to the lowest Fock levels, specifically $|0\rangle$ and $|1\rangle$. As shown in Fig. 1(b), the nonclassical area remains significantly reduced with values saturating around $\delta \sim 1.8$. Despite this reduction in nonclassicality, the system still exhibits a well-defined synchronization region. The Arnold tongue structure, though less sharp, becomes more spread out, forming a broad triangular region centered at resonance ($\Delta = 0$). This indicates a smooth crossover from the unsynchronized to synchronized regime as the driving strength F increases. Unlike the classical case where synchronization onset is abrupt and accompanied by a sharp increase in δ , the deep quantum regime displays a more gradual phase-locking behaviour with relatively modest changes in the nonclassical area. The key conclusion drawn from panels 1(a) and (b) is that the Arnold tongue illustrates the synchronization region with phase locking in parameter space by plotting the nonclassical area measure against the driving strength F and detuning Δ .

Figure 1(c) helps us understand the synchronization as well as nonclassical measure smoothly from the classical to deep quantum regime, whereas the top panels 1(a, b) are in two extreme regimes of the quantum vdPo in the $(F - \Delta)$ plane. The observation is that the classical region inherently shows strong synchronization with $\delta > 1$; conversely, in the quantum limit, the system becomes less synchronized. As a result, the classical limit $\kappa_2 = 0$ captures strong synchronization with the higher nonclassicality,

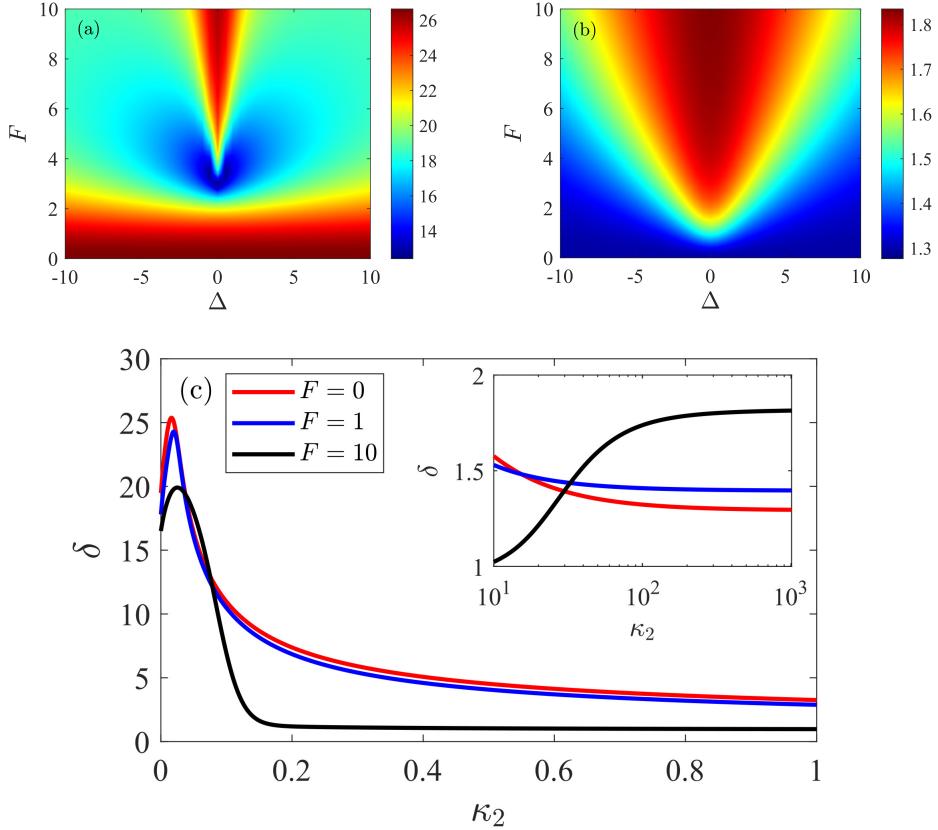


FIG. 1. (Color online) Different behaviors of the phase locking measure δ at the steady state. The nonclassical area δ measures synchronization at the steady state of an externally driven van der Pol oscillator. In panels (a) and (b), red (blue) indicates maximum (minimum) synchronization as measured by δ . The damping ratio is set to (a) $\kappa_2 = 0$, (b) $\kappa_2 = 10^3$. Inside the tongue: strong phase-locking, collective dynamics ($\delta > 1$). Outside the tongue: Weak or no phase-locking, less collective emission ($\delta > 1$). The lower panel (c) represents the synchronization measure across various regimes of the quantum vdp, as mentioned in Reference [27] in Table 1. The different drive strengths F , indicated in the legend, are selected from the top panels, implying a scenario of undriven to driven quantum vdPo, with $\Delta = 2$. Increasing the driving strength is seen to produce qualitative nonclassicality up to the quantum regime ($\kappa_2 = 1$). The inset illustrates that the behavior of nonclassicality in the deep quantum regime ($\kappa_2 \gg 1$) results in synchronization similar to that observed in the classical limit.

whereas the deep quantum regime is exactly opposite in it. Importantly, in all these cases, our results show that there is typically an inverse relationship between the nonclassical area and QS in the vdPo.

To gain a more comprehensive understanding of the synchronization measure, we study the quantum correlation at steady state, which is expected to affect phase locking significantly. In this regard, in Figure 2, the $g^{(2)}(0)$ -function is drawn as functions of F and Δ for different values of the κ_2 . In Figure 2(a), the vdPo is strongly synchronized in the classical limit $\kappa_2 = 0$ with the drive; it can exhibit phase lock, collective oscillations, leading to bursty or bunched emission ($g^{(2)}(0) > 1$). Here, enhanced synchronization is linked with bunching. It is observed that for large drive strength or detuning, synchronization typically increases. The intriguing feature in the quantum regime ($\kappa_2 = 1$) is that mirror symmetry of the Arnold tongues with respect to ($\Delta = 0$) arises from the system's symmetry: synchronization can occur for drive frequencies both above and below the natural frequency by the same amount (in terms of $|\Delta|$), as shown in Fig. 2(b). On the other hand, it is observed that at the deep quantum regime $\kappa_2 = 10^3$, synchronization is not captured, resulting in $g^{(2)}(0) \rightarrow 0$ due to the strong quantum fluctuation in the phase of vdPo. The stark signature of $g^{(2)}(0)$ emphasizes that the sensitivity of synchronization is dependent on how well the operating regime of quantum vdPo's is controlled through careful selection of system parameters. This is a distinctive signature of how QS manifests in nonlinear, driven-dissipative quantum systems like the vdP oscillator.

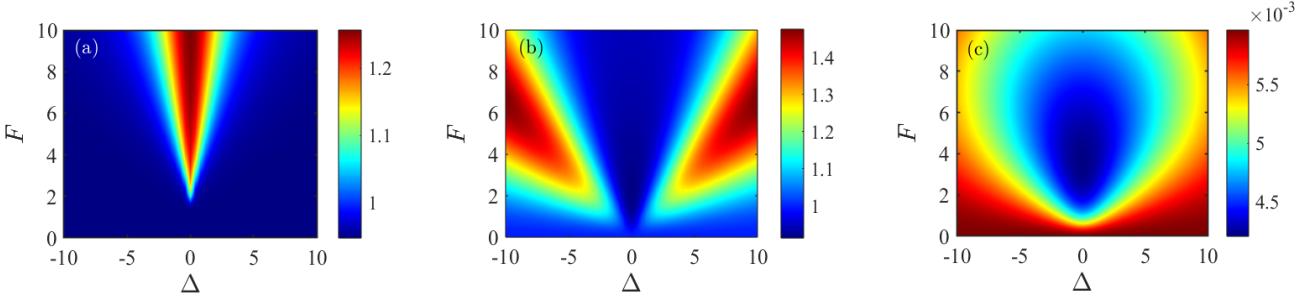


FIG. 2. (color online). At steady-state, the equal-time second-order correlation $g^{(2)}(0)$ exhibits phase locking as a function of driving strength F and detuning Δ . In panels (a, b), the red (blue) region indicates the maximum (minimum) synchronization measurement. Inside the tongue: strong phase-locking indicates bunching with collective dynamics ($g^{(2)}(0) > 1$). Outside the tongue: Weak or no phase-locking indicates antibunching with nonclassicality due to ($g^{(2)}(0) < 1$). In panel (c), the absence of Arnold's tongue shape, indicated by $g^{(2)}(0) \rightarrow 0$, highlights a strikingly diminishing correlation with strong nonclassicality. The damping ratio is set to (a) $\kappa_2 = 0$, (b) $\kappa_2 = 1$, and (c) $\kappa_2 = 10^3$.

IV. ANALYTICS OF TOMOGRAPH AND PHASE COHERENCE

In our study, we focus on the deep quantum limit, where the mean excitation number is small and only the lowest Fock states are significantly populated. For driving strengths up to $F \leq 10$, which are experimentally accessible [27], the one-photon coherent drive primarily couples two consecutive energy eigenstates of the undriven vdPo. The existence of coherence terms in ρ^{ss} depends on the driving strength F , while their magnitude is governed by the nonlinear damping rate κ_2 . Consequently, we neglect higher-order coherence terms in ρ^{ss} beyond the deep quantum regime, and the relevant Hilbert space can be truncated to $|0\rangle$, $|1\rangle$, and $|2\rangle$, with coherences only between $|0\rangle$ and $|1\rangle$ for $F \leq 10$. Based on this, we adopt the following ansatz for the steady-state density matrix ρ^{ss} of the driven vdP oscillator in the deep quantum regime:

$$\rho^{ss} = \begin{pmatrix} \rho_{00} & \rho_{01} & 0 \\ \rho_{10} & \rho_{11} & 0 \\ 0 & 0 & \rho_{22} \end{pmatrix}, \quad (7)$$

where the nonvanishing off-diagonal elements ρ_{01} and $\rho_{10} = \rho_{01}^*$ explicitly account for the coherence between the lowest-energy states $|0\rangle$ and $|1\rangle$. This structure follows from solving the master equation (1) for the steady state, with the explicit expressions for the matrix elements given in Appendix B. In the limit $\kappa_2 \rightarrow \infty$, the maximum population is bound up to $|2\rangle$ and neglects all coherences beyond the state $|0\rangle - |1\rangle$ manifold. Without higher-order coherences, analytical calculations are much more straightforward as well as negligible in exact numerical simulations. In the absence of drive, $F = 0$, and in the limit $\kappa_2 \rightarrow \infty$, we are able to write the elements of Eq. (7) after simplification from the elements (B2) of them in Appendix B as $\rho_{10} = \rho_{22} = 0$, $\rho_{00} = 2/3$, and $\rho_{11} = 1/3$. Correspondingly, the steady-state given in Eq. (7) reduces to a limit cycle [15, 16]:

$$\rho_{\kappa_2 \rightarrow \infty; F=0}^{ss} = \frac{2}{3} |0\rangle \langle 0| + \frac{1}{3} |1\rangle \langle 1|, \quad (8)$$

reveals a statistical mixture of the vacuum and single photon state, with no phase coherence existing between the Fock levels owing to the vanishing off-diagonal terms in Eq. (8). Since higher Fock levels are suppressed by the large κ_2 , the system predominantly relaxes to $|0\rangle$ while retaining a finite probability of occupying $|1\rangle$ because of the dissipative dynamics. In principle, $\rho_{\kappa_2 \rightarrow \infty}^{ss}$ is therefore a mixed state with nonzero entropy and no phase information, precluding the possibility of phase synchronization in this regime. Notably, this limit cycle [16] has a ring-shaped positive Wigner function in the phase space.

The full expression for the steady-state tomogram $\omega(X_\theta, \theta)$ with the advent of density matrix elements B2 is given by

$$\begin{aligned} \omega(X_\theta, \theta) = & \frac{e^{-X_\theta^2}}{2\sqrt{\pi}R} \left(X_\theta^4 (R - 24F^2 - 12\Delta^2 - 27\kappa_1^2) - 2X_\theta^2 (R - 48F^2 - 20\Delta^2 - 45\kappa_1^2) \right. \\ & \left. - 8\sqrt{2}FX_\theta(3\kappa_1 \sin \theta + 2\Delta \cos \theta) + (4\Delta^2 + 9\kappa_1^2 + R) \right), \end{aligned} \quad (9)$$

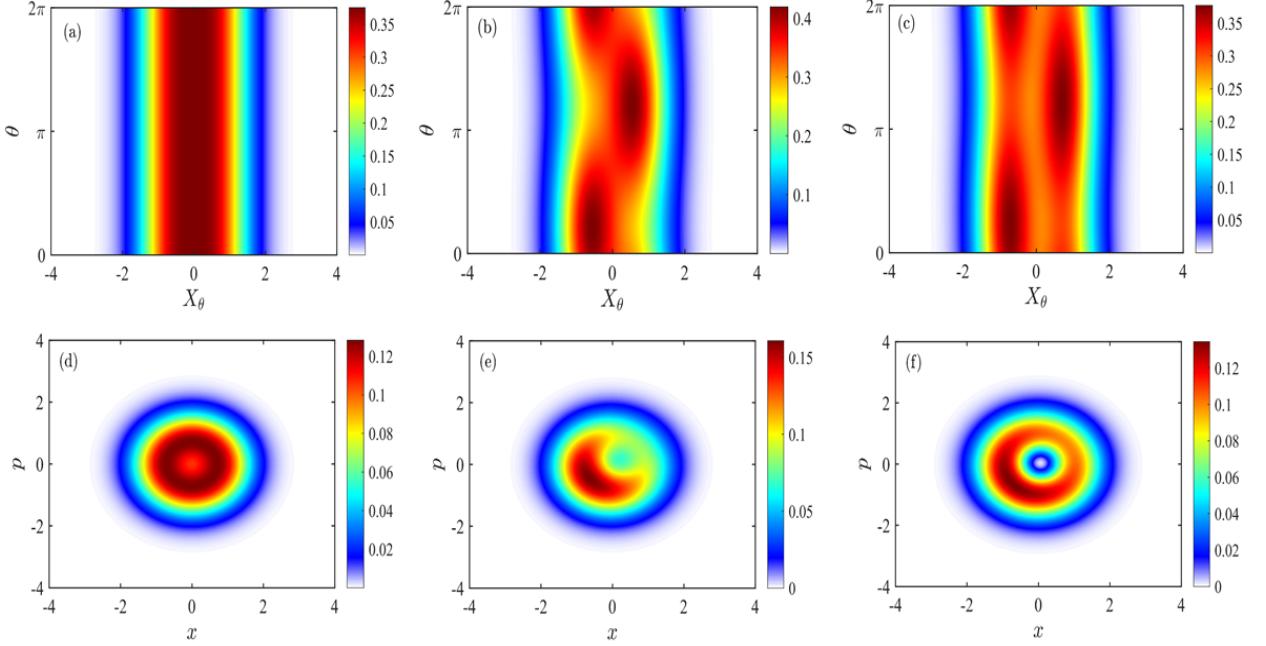


FIG. 3. (color online). Top row: Tomographic probability distributions $\omega(X_\theta, \theta)$ at steady state of the driven quantum vdPo in the deep quantum regime, shown for different drive strengths: (a) weak drive $F = 0$, (b) intermediate drive $F = 1$, and (c) strong drive $F = 10$. For weak driving [panel (a)], the tomogram remains broadly distributed, indicating low phase preference and weak synchronization. At intermediate driving [panel (b)], the distribution becomes anisotropic with visible deformation, signaling the onset of partial phase locking. Under strong driving [panel (c)], the tomogram exhibits a pronounced localized shape, characteristic of enhanced phase concentration and stronger synchronization. Bottom row: Wigner functions (d-f) corresponding to the tomograms in panels (a-c). At weak drive (d), the Wigner function retains near-rotational symmetry, consistent with the absence of synchronization. For intermediate drive (e), angular modulation emerges, marking the onset of rotational symmetry breaking. Under strong drive (f), the Wigner function shows pronounced phase-space asymmetry, mirroring the tomogram's localization and providing clear evidence of QS. Here, we choose $\Delta = 2$ for all panels.

and we can simplify the above expressions for the undriven vdPo in limit $\kappa_2 \rightarrow \infty$ as

$$\omega(X_\theta)_{\kappa_2 \rightarrow \infty; F=0} = \frac{2}{3\sqrt{\pi}}(1 + X_\theta^2)e^{-X_\theta^2}. \quad (10)$$

The Wigner distribution in phase space summarizes the striking physical insight of $\omega(X_\theta, \theta)$ that defines the different measures of synchronization in the quadrature plane. This approach provides a comprehensive understanding of the visualization (see Fig. 3) of quantum states and their features in the perspective of QS. Figure 3 exhibits the behaviour of the steady-state quantum tomograms 3(a)-3(c) and their corresponding Wigner functions 3(d)-3(f) at varying driving strengths in the deep quantum limit. Let us now be more precise.

In the absence of drive ($F = 0$), the tomogram 3(a) remains essentially independent of the quadrature angle θ , reflecting full angular symmetry and the absence of synchronization. In phase space, the associated Wigner function 3(d) displays a rotationally symmetric distribution (disk shape) centered at the origin, indicating no preferred phase and thus no phase locking to the external drive. This symmetry reflects complete phase indeterminacy: the system does not prefer any particular phase. At weak drive $F = 1$, the tomogram 3(b) develops clear oscillations along θ , signifying the onset of phase localization. This oscillation reflects the emergence of phase sensitivity in the quadrature outcomes due to partial phase locking to the external drive. As expected, QS induces a spontaneous symmetry breaking with weak driving in the Wigner function (see Fig. 3(e)), forcing the previously circular (rotationally symmetric) Wigner function to become deformed. This deformation indicates that the system is developing a preferred phase—the hallmark of synchronization. For strong drive $F = 10$, Fig. 3(c) exhibits that the oscillations are sharper, showing that the system's quadratures are tightly locked to the external drive, which is a signature of strong quantum synchronization. The corresponding Wigner function in Fig. 3(f) shows pronounced phase-space localization (banana-like shape) shifted away from the center with reduced rotational symmetry. Thus, reflecting tomograms in the phase space configuration offers a clearer understanding of synchronization by revealing the emergence of phase-dependent quadrature statistics.

Finally, we analyze the phase coherence (S) that encodes the relative phase between vdPo and the external drive—indicating the

degree of phase locking inside the system. This analysis is crucial and describes how phase coherence can influence QS in the limit $\kappa_2 \rightarrow \infty$. It is straightforward to reach the explicit form of the relative phase with as follows

$$\begin{aligned} \mathcal{S}_{\kappa_2 \rightarrow \infty} &= |\rho_{01}| \\ &= \frac{2F\sqrt{4\Delta^2 + 9\kappa_1^2}}{24F^2 + 12\Delta^2 + 27\kappa_1^2}, \end{aligned} \quad (11)$$

which is easily found from the Appendix B solving for the steady-state density matrix elements of ρ^{ss} , as defined in Eq. (7). The connection between synchronization and coherence has already been reported in literature, for example, in Refs. [27, 51]. Notably, here we address the coherence in a fully quantum mechanical way by specifically targeting the off-diagonal element, ρ_{01} , which only survives in the limit $\kappa_2 \rightarrow \infty$ and is also restricted for the drive $F \leq 10$. A full characterization of the emergence of coherence ρ_{01} is given in Fig. 4(a) where, by varying both F and Δ , we calculate the maximum coherence $\mathcal{S}(F_c) \approx 0.12$ determined from the critical drive strength $F_c = \sqrt{(9\kappa_1^2 + 4\Delta^2)/8}$, as indicated by the black curve. This value highlights the optimization of coherence amplitude, which adequately supports the drive strength of $F \leq 10$, and also reveals the maximum phase locking in limit $\kappa_2 \rightarrow \infty$. As expected, the classical regime $\kappa_2 \rightarrow 0$ refers to lower coherence $\mathcal{S} \approx 0.006$ (maximum) due to discarding all quantum fluctuations. Since $\rho_{22} = 0$, the population imbalance $|\rho_{00} - \rho_{11}|$ can be fostered to optimize the coherence \mathcal{S} in parallel with ρ_{01} using the oscillator state $\rho_{\kappa_2 \rightarrow \infty}^{ss}$. Eventually, controlling the excitations with the drive force ($F \neq 0$) allows for a distribution that is restricted to two discrete eigenbases, $|0\rangle$ and $|1\rangle$ of the vdPo, which is represented as a conceptual two-level qubit. This distribution facilitates the manipulation of quantum states for a qubit, enabling fundamental operations through the unique properties of superposition and entanglement.

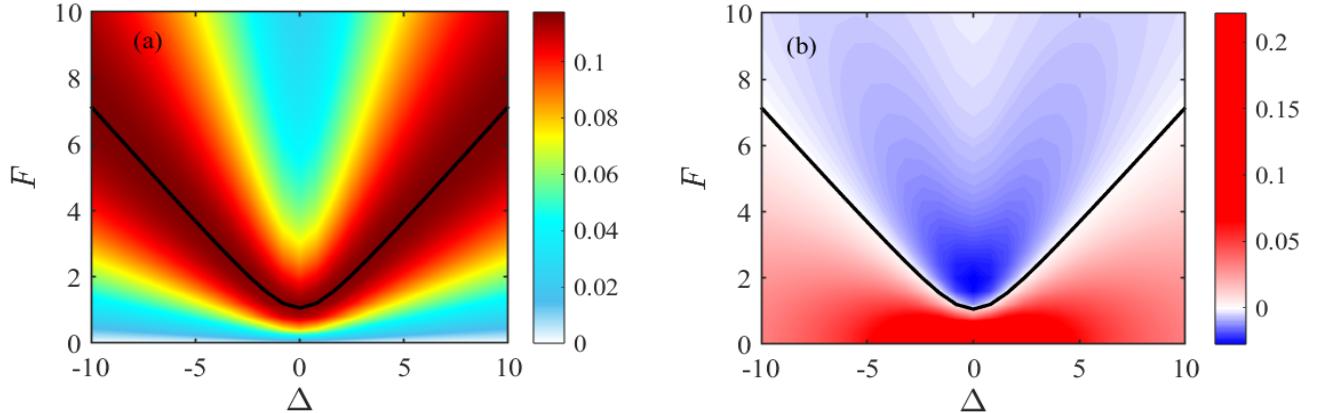


FIG. 4. (color online). (a) Steady state coherence $|\rho_{01}|$ as a function of driving strength F and detuning Δ for the quantum vdPo in the limit $\kappa_2 \rightarrow \infty$, showing sensitivity of phase locking. Outside of the red area, the vdPo exhibits more classical (mixed) behavior, loses quantum features, and fails to exhibit synchronization. (b) The gradient $\partial_F \mathcal{S}$ is used to locate the synchronization threshold, or the region where quantum control is most effective. By fine-tuning the parameters, the phase relationship between $|0\rangle$ and $|1\rangle$ is optimized, enabling more reliable qubit operations. In both panels, the black curve represents the critical drive F_c that describes the peak of maximum coherence $|\rho_{01}|$, which inherently demonstrates a two-level qubit.

The results in Figure 4(a) motivate us to comprehensively examine the sensitivity of coherence; we have made use of the interpretation of the gradient $\partial_F \mathcal{S}$ (see in Fig. 4(b)), which acts as an onset for the synchronization measure. In stark contrast to the qubit picture, which emerges from the coherence solely influenced by the drive F , we gain insights into the phase locking between the qubit and the external drive. The positive (negative) slope indicates an in-phase (anti-phase) relationship between $|0\rangle$ and $|1\rangle$, but the critical drive F_c describes maximum coherence between them—completely absent in the generic limit cycle $\rho_{\kappa_2 \rightarrow \infty; F=0}^{ss}$. The results indicate that simulating a driven vdPo as a two-level qubit in the limit $\kappa_2 \rightarrow \infty$ reveals a phase-locking aspect related to synchronization measurement.

V. CONCLUSIONS

In this work we have developed a tomographic framework to characterize synchronization in a driven quantum vdPo. By employing the nonclassical area as a quantifier, we identified Arnold tongue structures in the quadrature space that clearly reflect synchronization signatures in both the classical and quantum regimes. In the classical limit, the onset of synchronization is sharp, accompanied by a significant increase in nonclassicality, whereas in the deep quantum regime, the synchronization regions

become broader and smoother, with the nonclassical area saturating to modest values. We analyzed the steady-state tomograms for different driving strengths, which provided an experimentally accessible visualization of the underlying nonclassical features. A central theoretical contribution of this study is the explicit derivation of the steady-state density matrix elements at arbitrary driving strengths in the deep quantum regime, where nonlinear damping confines the oscillator to the lowest Fock levels. This analytic solution captures how drive strength and detuning shape the populations and coherences of the steady state. Likewise, our tomogram measure is a key achievement that delineates an underlying set of quantum states in the QS landscape. Importantly, we observe symmetry breaking and phase localization in the quantum tomograms, providing a direct and novel signature of the onset of quantum synchronization. This finding is complementary to the symmetry breaking in the Wigner function, linking phase-space asymmetry with measurable tomographic features of synchronization. Similarly, the nutshell of the two-level qubit is crucial for phase locking as coherence emerges in the deep quantum limit. In addition, we have reformulated (Eq. A2) the master equation in terms of the quantum tomogram, providing direct access to synchronization signatures from experimentally measurable probability distributions. These results offer a practical framework for robustness of nonclassical steady states in vdP oscillators with potential relevance for future experimental and technological applications, including quantum state engineering [52], error correction [53], protocol design [54], etc. Last but not least, this study contributes as a theory-experiment bridge apart from the framework of the vdPo, allowing for further investigation into the potential link between the two figures of merit mentioned here.

VI. ACKNOWLEDGMENTS

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Appendix A: Master equation of the driven vdPo

The master Eq. (1) describing the evolution of the density matrix ρ of the quantum vdPo. Here we provide this equation on the basis of the eigenmodes of the Hamiltonian as well as in the tomographic plane. In the Fock basis, the master equation (1) for the driven vdPo can be expressed in the matrix-element form as

$$\begin{aligned} \frac{d}{dt}\rho_{m,n} = & -i\Delta(m-n)\rho_{m,n} - iF(\sqrt{m}\rho_{m-1,n} + \sqrt{m+1}\rho_{m+1,n} - \sqrt{n}\rho_{m,n-1} - \sqrt{n+1}\rho_{m,n+1}) \\ & + \frac{\kappa_1}{2}(2\sqrt{mn}\rho_{m-1,n-1} - (m+n+2)\rho_{m,n}) \\ & + \frac{\kappa_2}{2}(2\sqrt{(m+1)(m+2)(n+1)(n+2)}\rho_{m+2,n+2} - (m^2 + n^2 - m - n)\rho_{m,n}) \end{aligned} \quad (\text{A1})$$

At steady-state, where $\frac{d}{dt}\rho_{m,n} = 0$, the density-matrix elements $\rho_{m,n}$ are calculated numerically in Qutip [45] and also analytically for the small system. In the limit $\kappa_2 \rightarrow \infty$, the further insights of the analytical solutions are highlighted in Appendix B.

In the quadrature plane $(X_\theta - \theta)$, the master Eq. (1) in the eigenbasis $|X_\theta, \theta\rangle$ of the single-mode rotated quadrature operator X_θ can be written as,

$$\begin{aligned} \frac{d}{dt}\rho_{p+2q;n+2m} = & -i\Delta((p+2q)\mathcal{A}_{pq}^* - (n+2m)\mathcal{A}_{nm}) \\ & - iF\mathcal{A}_{pq}^*(\sqrt{p+2q}\rho_{p+2q-1;n+2m} + \sqrt{p+2q+1}\rho_{p+2q+1;n+2m}) \\ & + iF\mathcal{A}_{nm}(\sqrt{n+2m}\rho_{p+2q;n+2m-1} + \sqrt{n+2m+1}\rho_{p+2q;n+2m+1}) \\ & + \frac{\kappa_1}{2}\mathcal{A}_{nm}\mathcal{A}_{pq}^*\left(2\sqrt{(n+2m)(p+2q)}\rho_{p+2q-1;n+2m-1} - (n+p+2m+2q+2)\rho_{p+2q;n+2m}\right) \\ & + \frac{\kappa_2}{2}\mathcal{A}_{nm}\mathcal{A}_{pq}^*\left(2\sqrt{(n+2m+1)(n+2m+2)(p+2q+1)(p+2q+2)}\rho_{p+2q+2;n+2m+2} - \right. \\ & \left. ((p+2q)(p+2q-1) - (n+2m)(n+2m-1))\rho_{p+2q;n+2m}\right) \end{aligned} \quad (\text{A2})$$

This tomogram-based master Eq. (A2) results in a significant corollary: it allows for a potential direct measurement of the synchronization with the nonclassical area δ of the quantum vdPo. It is important to note that this corollary is numerous and far-reaching after transforming the master Eq. (1) apart from vdPo.

Appendix B: Elements of the steady state density matrix in the deep quantum regime

For completeness, we present here the explicit analytical solutions for Eq. (A1) discussed in the main text. Imposing the steady-state condition $\dot{\rho} = 0$ on Eq. (A1), the following set of coupled linear equations:

$$\left. \begin{array}{l} 0 = -iF(\rho_{10} - \rho_{01}) - \kappa_1\rho_{00} + 2\kappa_2\rho_{22} \\ 0 = iF(\rho_{10} - \rho_{01}) + \kappa_1(\rho_{00} - 2\rho_{11}) \\ 0 = 2\kappa_1\rho_{11} - (3\kappa_1 + 2\kappa_2)\rho_{22} \\ 0 = (i\Delta - \frac{2}{3}\kappa_1)\rho_{01} - iF(\rho_{11} - \rho_{00}) \end{array} \right\}. \quad (\text{B1})$$

Solving the above coupled equations gives full expressions of the steady-state matrix elements:

$$\left. \begin{array}{l} \rho_{00} = \frac{2}{R}(6F^2 + 4\Delta^2 + 9\kappa_1^2) \\ \rho_{01} = \rho_{10}^* = \frac{2F}{R}(3i\kappa_1 - 2\Delta) \\ \rho_{11} = \frac{1}{R}(12F^2 + 4\Delta^2 + 9\kappa_1^2) \\ \rho_{22} = 1 - \frac{3}{R}(8F^2 + 4\Delta^2 + 9\kappa_1^2) \end{array} \right\}, \quad (\text{B2})$$

where the denominator $R = (12F^2 + 4\Delta^2 + 9\kappa_1^2)(3 + \frac{\kappa_1}{\kappa_2}) - 12F^2$.

We note that the observed tomogram $\omega(X_\theta, \theta = 0)$ in Fig. 5 from Eq. (10) is a Gaussian-modulated polynomial, which is completely symmetric under rotation, as shown earlier in Fig. 3(a). This typically corresponds to vacuum or thermal states, indicating no synchronization.

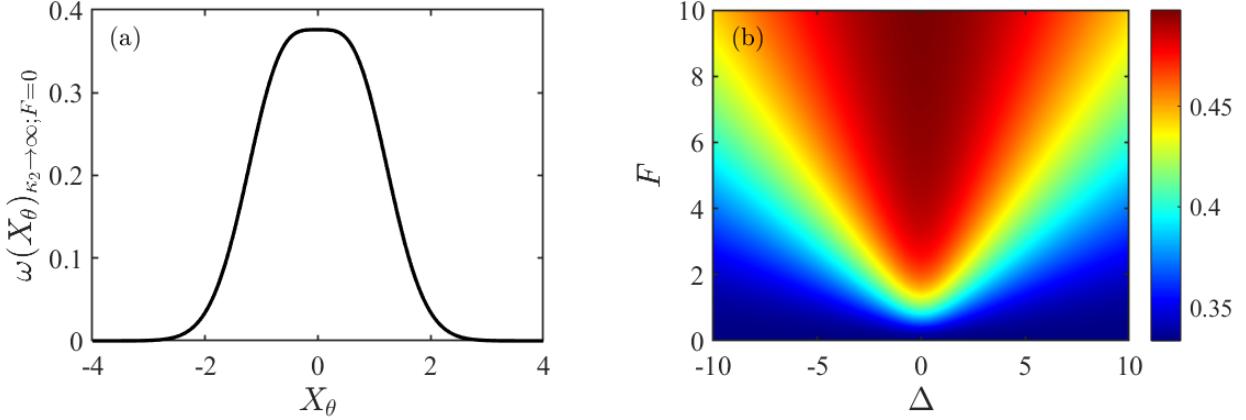


FIG. 5. (color online). (a) Quantum tomogram against quadrature eigenvalues X_θ for an undriven quantum vdPo showing rotational symmetry, i.e., no preferred phase direction. From the synchronization perspective, that corresponds to no phase locking, as shown in panel 3(a). The limit-cycle amplitude Eq. (B3) is shown in panel (b) at steady state. The color scale indicates the degree of phase locking: deep-blue regions correspond to weak synchronization, while the transition to red marks enhanced synchronization-near resonance ($\Delta \approx 0$) and grows with increasing drive strength.

Next, we discuss how the limit-cycle (Eq. (8)) amplitude is determined by the mean excitation number, $N = \langle a^\dagger a \rangle = \rho_{11} + 2\rho_{22}$ in the deep quantum limit ($\kappa_2 \rightarrow \infty$). The analytical expression is subsequently calculated using elements (B2) as follows:

$$N_{\kappa_2 \rightarrow \infty} = \frac{1}{3} + \frac{4F^2}{24F^2 + 12\Delta^2 + 27\kappa_1^2}. \quad (\text{B3})$$

The constant term $1/3$ sets the amplitude of the limit cycle, which corresponds to the intrinsic quantum fluctuations for the un-driven vdPo, as explained in Eq. (8). This baseline value agrees with the well-known finding that sheds light on the classification of the operating regime characterized by κ_2/κ_1 , as reported in Ref. [27], confirming that the undriven quantum vdPo sustains a nonzero excitation number due to quantum noise. To understand the parameter dependence of the limit cycle, we display it in Fig. (5), showing the Arnold tongue, which is a key signature of synchronization. The effect of drive enables slowly increasing the amplitude and prominently captures the maximum mean excitation at resonance $\Delta = 0$. In essence, in the deep quantum limit, the limit-cycle amplitude reflects a nontrivial interplay between quantum fluctuations, coherent drive, and dissipation.

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