

Energy-Gain Control of Time-Varying Systems: Receding Horizon Approximation

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Abstract—Standard formulations of prescribed worst-case disturbance energy-gain control policies for linear time-varying systems depend on all forward model data. In a discrete-time setting, this dependence arises through a backward Riccati recursion. The aim herein is to consider the infinite-horizon ℓ_2 gain performance of state feedback policies with only finite receding-horizon preview of the model parameters. The proposed synthesis of controllers subject to such a constraint leverages the strict contraction of lifted Riccati operators under uniform controllability and observability. The main approximation result establishes a sufficient number of preview steps for the performance loss to remain below any set tolerance, relative to the baseline gain bound of the associated infinite-preview controller. Aspects of the main result are explored in the context of a numerical example.

Index Terms—Finite preview, infinite-horizon performance, non-stationary systems, Riccati contraction

I. INTRODUCTION

The synthesis of feedback controllers for linear time-varying systems against infinite-horizon quadratic performance criteria has a long history [1]–[5]. Generally, the standard formulations are somewhat impractical, as the control policies depend on all future time-varying parameters. In the discrete-time setting of this paper, the stabilizing solution of a backward Riccati recursion encodes this dependence [6]–[9].

In periodic settings, including the time-invariant case, availability of the model data across one period suffices to obtain the infinite-horizon solution of the backward recursion via a finite-dimensional algebraic Riccati equation [10]–[13]. In more general time-varying settings, an infinite-horizon lifting, given all model data, also leads to algebraic characterization of the solution [14]–[16]. However, the practical concern of computation remains a challenge unless a finite structure underlies the parameter variation, e.g., periodicity [16], or more general switching within a finite set [17]. Furthermore, such infinite-horizon lifting would be infeasible in supervisory hierarchies involving online roll-out of finite-horizon plans that determine the model data relevant for low-level control.

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This paper concerns state feedback controller synthesis subject to finite receding-horizon preview of the time-varying model data without preview of the disturbance. The goal is to ensure that the closed-loop infinite-horizon energy gain does not exceed a specified worst-case bound. The proposed approach involves approximation of the standard infinite-preview formulation to within desired tolerance, via a Riccati contraction property that holds under uniform controllability and observability and an associated finite-horizon lifting. Related work on the disturbance-free optimal linear quadratic regulator can be found in [18], [19, Ch. 4].

To enable further elaboration of the contribution, the standard formulation of state feedback ℓ_2 gain controllers with infinite-horizon preview of the model data is recalled in Section I-A. This provides context for Section I-B, where the proposed finite receding-horizon synthesis of such control policies is outlined. Relevant literature is discussed in Section I-C, including well-known work on model predictive control, and more recent work on optimal regret, which by contrast focuses on performance loss relative to a policy with non-causal preview of the disturbance. The main technical developments are mapped out in Section I-D.

A. Basic notation and problem formulation

\mathbb{N} and \mathbb{R} denote the natural and real numbers, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, and $\mathbb{R}_{>\theta} := \{\vartheta \in \mathbb{R} : \vartheta > \theta\}$. \mathbb{R}^n denotes the space of vectors with $n \in \mathbb{N}$ real co-ordinates, and $\mathbb{R}^{n \times m}$ the space of matrices with $n \in \mathbb{N}$ rows and $m \in \mathbb{N}$ columns of real entries. The identity matrix is $I_n \in \mathbb{R}^{n \times n}$, all entries of $0_{m,n} \in \mathbb{R}^{m \times n}$ are zero, $X' \in \mathbb{R}^{m \times n}$ is the transpose of $X \in \mathbb{R}^{n \times m}$, and for non-singular $Y \in \mathbb{R}^{n \times n}$, the inverse is $Y^{-1} \in \mathbb{R}^{n \times n}$. Given the sets of symmetric matrices $\mathbb{S}^n := \{Z \in \mathbb{R}^{n \times n} \mid Z = Z'\}$, $\mathbb{S}_+^n := \{Z \in \mathbb{S}^n \mid (\forall v \in \mathbb{R}^n) v' Z v \geq 0\}$, and $\mathbb{S}_{++}^n := \{Z \in \mathbb{S}^n \mid (\forall v \in \mathbb{R}^n) v' Z v > 0\}$, $Y \preceq Z$ means $(Z - Y) \in \mathbb{S}_+^n$, and $Y \prec Z$ means $(Z - Y) \in \mathbb{S}_{++}^n$. For $Z \in \mathbb{S}_+^n$, the matrix square root is $Z^{1/2} \in \mathbb{S}_+^n$. For $Y \in \mathbb{S}^n$, $\lambda_{\min}(Y)$ and $\lambda_{\max}(Y)$ denote minimum and maximum eigenvalue.

Consider the linear time-varying system

$$x_{t+1} = A_t x_t + B_t u_t + w_t, \quad t \in \mathbb{N}_0, \quad (1)$$

with initial state $x_0 = 0 \in \mathbb{R}^n$, control and disturbance inputs $u_t \in \mathbb{R}^m$ and $w_t \in \mathbb{R}^n$, respectively, and performance output

$$z_t = [Q_t^{1/2} \quad 0_{n,m}]' x_t + [0_{m,n} \quad R_t^{1/2}]' u_t, \quad (2)$$

where $A_t \in \mathbb{R}^{n \times n}$, $B_t \in \mathbb{R}^{n \times m}$, $Q_t \in \mathbb{S}_+^n$, and $R_t \in \mathbb{S}_{++}^m$.

Assumption 1: The model data A_t, B_t, Q_t, R_t and the inverses A_t^{-1}, R_t^{-1} are uniformly bounded across $t \in \mathbb{N}_0$.

Note that A_t is non-singular whenever the model arises from discretization. Here it enables direct access to key results on Riccati operator contraction [20]. Generalizing to singular A_t (e.g., see [21]) is beyond the current scope.

Assumption 2: The system $(Q_t^{1/2}, A_t, B_t)_{t \in \mathbb{N}_0}$ is uniformly observable and controllable in the following sense [7]:

$$(\exists d \in \mathbb{N}) (\exists c \in \mathbb{R}_{>0}) (\forall t \in \mathbb{N}_0)$$

$$\left(\sum_{s=t}^{t+d-1} \Phi'_{s,t} Q_s \Phi_{s,t} \right) \succeq c I_n \preceq \left(\sum_{s=t}^{t+d-1} \Phi_{s,t} B_s B'_s \Phi'_{s,t} \right),$$

where $\Phi_{t,t} := I_n$, and $\Phi_{s,t} := A_{s-1} \cdots A_t$ for $s > t$.

While infinite-preview state feedback controllers with energy-gain guarantees exist under uniform stabilizability and detectability, Assumption 2 plays a role in the Riccati contraction based synthesis of finite receding-horizon approximations, as elaborated in Sections II and III-A.

The object of energy-gain control is to find a policy for $u = (u_t)_{t \in \mathbb{N}_0}$ such that the resulting map from the disturbance input $w = (w_t)_{t \in \mathbb{N}_0}$ to the performance output $z = (z_t)_{t \in \mathbb{N}_0}$ is input-output stable over the space of finite energy signals

$$\ell_2 := \{w = (w_t)_{t \in \mathbb{N}_0} \mid \|w\|_2^2 := \sum_{t \in \mathbb{N}_0} w'_t w_t < +\infty\},$$

with prescribed worst-case gain bound $\gamma \in \mathbb{R}_{>0}$ in the sense that $(\forall w \in \ell_2) J_\gamma(u, w) \leq 0$, where

$$J_\gamma(u, w) := \|z\|_2^2 - \gamma^2 \|w\|_2^2 = \sum_{t \in \mathbb{N}_0} z'_t z_t - \gamma^2 w'_t w_t, \quad (3)$$

with z_t as per (2) and (1) given $x_0 = 0$. This performance requirement amounts to $\sup_{0 \neq w \in \ell_2} \|z\|_2 / \|w\|_2 \leq \gamma$, whereby internal stability is implied under Assumptions 1 and 2.

The following result is the standard infinite-horizon formulation of a (strictly causal) state feedback ℓ_2 gain control policy; e.g., see [4], [5], [9], [19, Sec. 2.4.2].

Theorem 1: Given $\gamma \in \mathbb{R}_{>0}$, suppose the sequence $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_+^n$ satisfies the following:

$$(\exists \varepsilon \in \mathbb{R}_{>0}) (\forall t \in \mathbb{N}_0) P_{t+1} - \gamma^2 I_n \preceq -\varepsilon I_n; \quad \text{and} \quad (4)$$

$$(\forall t \in \mathbb{N}_0) P_t = \mathcal{R}_{\gamma,t}(P_{t+1}), \quad (5)$$

where the γ -dependent time-varying ℓ_2 gain Riccati operator for the system (1)–(2) is given by

$$\mathcal{R}_{\gamma,t}(P) := Q_t + A'_t P A_t - L_t(P)' (M_{\gamma,t}(P))^{-1} L_t(P), \quad (6)$$

with

$$\begin{aligned} L_t(P) &:= [B_t \quad I_n]' P A_t, \\ M_{\gamma,t}(P) &:= \begin{bmatrix} R_t + B'_t P B_t & B'_t P \\ P B_t & P - \gamma^2 I_n \end{bmatrix}. \end{aligned} \quad (7)$$

Further, in the system (1)–(2), let

$$u_t = u_{\gamma,t}^{\inf}(x_t) := -K_{\gamma,t}(P_{t+1}) x_t, \quad t \in \mathbb{N}_0, \quad (8)$$

where

$$K_{\gamma,t}(P) := (\nabla_{\gamma,t}(P))^{-1} B'_t (P + P(\gamma^2 I_n - P)^{-1} P) A_t, \quad (9)$$

$$\nabla_{\gamma,t}(P) := R_t + B'_t (P + P(\gamma^2 I_n - P)^{-1} P) B_t.$$

Then, with J_γ as per (3), $(\forall w \in \ell_2) J_\gamma(u, w) \leq 0$.

Related infinite-dimensional operator based formulations of feedback controllers with guaranteed ℓ_2 gain are given in [14]–[16]. As above, these also depend on all forward problem data. In particular, the state feedback policy in (8) depends on $(Q_s, A_s, B_s, R_s)_{s \geq t}$ via the recursion (5). The superscript inf emphasizes this *infinite preview* dependence.

Infinite-preview dependence of the policy (8) on the model parameters detracts from its practical applicability. Determining the hypothesized solution $(P_t)_{t \in \mathbb{N}_0}$ of (5) is a challenge unless there is known structure, such as periodic invariance [12], [13]. This motivates consideration of state feedback ℓ_2 gain control policy synthesis subject to finite preview of the model data in a receding-horizon fashion. The proposed approach is based on approximating P_{t+1} in (8), as outlined in Section I-B. It is established that the error can be made arbitrarily small by using a sufficient number of model preview steps in the construction of the approximation. As such, the development yields a practical method for computing the hypothesized solution of (5) to desired accuracy at each time $t \in \mathbb{N}_0$. Continuity of closed-loop ℓ_2 gain bound with respect to the approximation error then leads to the main receding-horizon controller synthesis result.

B. Main contributions

As indicated above, the main contribution is a finite receding-horizon preview synthesis of a state feedback controller that approximates the infinite-preview control policy (8) in Theorem 1. The prescribed baseline disturbance energy-gain performance bound $\gamma \in \mathbb{R}_{>0}$ for the latter is taken to be large enough for (5) to imply $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$, as elaborated in Remarks 1 and 7 in Sections II and III, respectively.

As detailed in Section II, given any performance loss tolerance $\beta \in \mathbb{R}_{>0}$, the proposed finite-preview synthesis involves approximating P_{t+1} in (8) by a positive definite $X_{t+1} \prec (\gamma + \beta)^2 I_n$, constructed by composing finitely many strictly contractive Riccati operators arising from a d -step lifting of (5), in alignment with Assumption 2. In this way, dependence on the model parameters in (1)–(2) is confined to a finite horizon ahead of $t \in \mathbb{N}_0$. The corresponding (strictly causal) state feedback control policy is then given by

$$u_t = u_{\gamma+\beta,t}^{\inf}(x_t) := -K_{\gamma+\beta,t}(X_{t+1}) x_t, \quad t \in \mathbb{N}_0, \quad (10)$$

where $K_{\alpha,t}(X) = (\nabla_{\alpha,t}(X))^{-1} B'_t (X^{-1} - \alpha^{-2} I_n)^{-1} A_t$ and $\nabla_{\alpha,t}(X) = R_t + B'_t (X^{-1} - \alpha^{-2} I_n)^{-1} B_t$, as per (9) since $X - X(X - \alpha^2 I_n)^{-1} X = (X^{-1} - \alpha^{-2} I_n)^{-1}$ for non-singular $X \prec \alpha^2 I_n$, by the Woodbury identity. The superscript fin emphasizes *finite preview* dependence on the model data.

The main result is formulated as Theorem 2, in Section II. It gives a sufficient number of preview steps in the proposed construction of each element of $(X_{t+1})_{t \in \mathbb{N}_0}$, for the resulting policy $u = (u_t)_{t \in \mathbb{N}_0} = (u_{\gamma+\beta,t}^{\inf}(x_t))_{t \in \mathbb{N}_0}$ in (10) to achieve $(\forall w \in \ell_2) J_{\gamma+\beta}(u, w) \leq 0$, with $J_{\gamma+\beta}$ as per (1)–(3); i.e., β bounded energy-gain performance loss, relative to the infinite-preview policy (8). The structured proof developed in Section III uses Theorem 3 on strict contraction of the lifted Riccati operators composed to form each X_{t+1} , and Theorem 4 on performance continuity. All other results, presented as lemmas, serve to establish these three main contributions.

C. Related work

A moving-horizon controller, for which an infinite-horizon energy-gain bound exists, is presented in [22] for time-varying linear continuous-time systems. In the complementary setting of discrete-time systems, the distinctive Riccati contraction based receding-horizon controller synthesis proposed here limits performance degradation relative to the infinite-preview policy (8), thereby ensuring a prescribed worst-case energy-gain bound. The receding-horizon controller in [17] also satisfies an ℓ_2 gain specification for a class of switched systems. The finite structure underlying parameter switching plays a key role in refining the results of [16] to this class of time-varying systems. By contrast, the subsequent developments do not rely on such structural assumptions.

In [23], [24], the setting is linear time-varying and discrete time, but the energy-gain performance horizon is finite. The focus is on performance degradation relative to non-causal preview of the disturbance. This so-called regret perspective is different to the setup here, where the preview constraint relates to the availability of the model data in strictly causal control policy synthesis with no preview of the disturbance. In [25], limited preview of the cost parameters is considered from a regret perspective in a two-player linear quadratic game, but again the overall problem horizon is finite.

In receding horizon approximations of infinite-horizon optimal control policies, a terminal state penalty is typically imposed in the finite-horizon optimization problem solved at each step; e.g., see [26]–[28]. For time-invariant nonlinear continuous-time systems and no disturbance, it is established in [29] that with zero terminal cost, there exists a finite prediction horizon for which stability is guaranteed. In [30], an infinite-horizon performance bound is also quantified for zero terminal penalty and set prediction horizon, with the added complication of state and input constraints, but nevertheless time-invariant model parameters and no disturbance.

A receding-horizon feedback control policy that achieves a given ℓ_2 gain bound appears in [31] for constrained time-invariant linear discrete-time systems. The online synthesis involves the optimization of feedback policies over a finite prediction horizon [32], with terminal ingredients constructed via the algebraic ℓ_2 gain Riccati equation for the unconstrained problem. This presumes constant problem data in a way that cannot be extended directly to a time-varying context without infinite preview of the model. The same applies to the terminal ingredients in the receding-horizon regret-optimal schemes of [33], [34] for time-invariant systems, where again regret relates to full preview of the disturbance.

In a stationary setting, a contraction analysis of the Riccati operator for discrete-time block-update risk-sensitive filtering is developed in [35]. Under controllability and observability, the Riccati operator is shown to be strictly contractive with respect to Thompson's part metric, for a range of risk-sensitivity parameter values. The block-update implementation of the risk-sensitive filter is related to the type of lifting employed here, where the time-varying setting is more general, the Riemannian metric is used for contraction analysis, and the context is ℓ_2 gain control.

D. Organization

The paper is organized as follows. The main result, outlined in Section I-B, is formulated as Theorem 2 in Section II. This section includes development of the underlying finite-horizon lifting used to obtain a one-step controllable and observable model, which enables direct application of an existing result on Riccati operator contraction in the synthesis of the approximating control policy. A structured proof of the main result, based on strict contraction of lifted ℓ_2 gain Riccati operators, is developed in Section III. A numerical example is considered in Section IV to explore aspects of the main result, followed by some concluding remarks in Section V.

II. MAIN RESULT

With regard to the proposed receding-horizon ℓ_2 gain controller synthesis, lifting the system model at each time $t \in \mathbb{N}_0$ enables finite-preview construction of a positive definite matrix X_{t+1} to approximate P_{t+1} in (8). Under Assumption 2, d -step lifting yields a one-step controllable and observable representation of the problem, unlocking existing theory on the contraction properties of Riccati operators [20]. This theory informs the construction of a suitable X_{t+1} from a finite-horizon preview of the model data. A sufficient number of preview steps is identified in the main result, which is formulated as Theorem 2, in terms of the lifted representation of the model (1)–(2) given in Lemmas 1 and 2 below.

Given any $d \in \mathbb{N}$ fixed in accordance with Assumption 2, and reference time $t \in \mathbb{N}_0$, for every $k \in \mathbb{N}_0$ define

$$x_{k|t} := x_{t+dk} \in \mathbb{R}^n, \quad (11)$$

the d -step lifted control input

$$u_{k|t} := (u_{t+dk}, \dots, u_{t+d(k+1)-1}) \in (\mathbb{R}^m)^d \sim \mathbb{R}^{md}, \quad (12)$$

the d -step lifted disturbance input

$$w_{k|t} := (w_{t+dk}, \dots, w_{t+d(k+1)-1}) \in (\mathbb{R}^n)^d \sim \mathbb{R}^{nd}, \quad (13)$$

and the d -step lifted performance output

$$z_{k|t} := (z_{t+dk}, \dots, z_{t+d(k+1)-1}) \in (\mathbb{R}^{n+m})^d \sim \mathbb{R}^{(n+m)d}. \quad (14)$$

The number of lifting steps d is fixed in subsequent analysis, and as such, for convenience, the dependence on d is suppressed in the notation. From (1),

$$A_{k|t} \begin{bmatrix} x_{t+dk} \\ \vdots \\ x_{t+d(k+1)-1} \\ x_{k+1|t} \end{bmatrix} = \begin{bmatrix} x_{k|t} \\ 0_{nd,1} \end{bmatrix} + \Xi_{k|t} u_{k|t} + \begin{bmatrix} 0_{n,nd} \\ I_{nd} \end{bmatrix} w_{k|t}, \quad (15)$$

where

$$A_{k|t} := I_{n(d+1)} - \begin{bmatrix} 0_{n,nd} & 0_{n,n} \\ \text{diag}(A_{t+dk}, \dots, A_{t+d(k+1)-1}) & 0_{nd,n} \end{bmatrix}, \quad (16)$$

$$\Xi_{k|t} := \begin{bmatrix} 0_{n,md} \\ \text{diag}(B_{t+dk}, \dots, B_{t+d(k+1)-1}) \end{bmatrix}. \quad (17)$$

Similarly, from (2),

$$z_{k|t} = \begin{bmatrix} \Gamma_{k|t} \\ 0_{md,n(d+1)} \end{bmatrix} \begin{bmatrix} x_{t+dk} \\ \vdots \\ x_{t+d(k+1)-1} \\ x_{k+1|t} \end{bmatrix} + \begin{bmatrix} 0_{n(d+1),md} \\ R_{k|t}^{1/2} \end{bmatrix} u_{k|t}, \quad (18)$$

where

$$\Gamma_{k|t} := \begin{bmatrix} \text{diag}(Q_{t+dk}^{1/2}, \dots, Q_{t+d(k+1)-1}^{1/2}) & 0_{nd,n} \end{bmatrix}, \quad (19)$$

$$R_{k|t} := \text{diag}(R_{t+dk}, \dots, R_{t+d(k+1)-1}). \quad (20)$$

On noting that $\Lambda_{k|t}$ in (16) is non-singular for all $t, k \in \mathbb{N}_0$, the next two lemmas are a direct consequence of (15).

Lemma 1: Given $u = (u_t)_{t \in \mathbb{N}_0}$ and $w = (w_t)_{t \in \mathbb{N}_0}$ in (1), for every $t \in \mathbb{N}_0$,

$$x_{k+1|t} = A_{k|t}x_{k|t} + B_{k|t}u_{k|t} + F_{k|t}w_{k|t}, \quad k \in \mathbb{N}_0, \quad (21)$$

with $x_{0|t} = x_t$, $u_{k|t}$, and $w_{k|t}$ as per (11)–(13), where

$$A_{k|t} := [0_{n,nd} \ I_n] \Lambda_{k|t}^{-1} \begin{bmatrix} I_n \\ 0_{nd,n} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (22)$$

$$B_{k|t} := [0_{n,nd} \ I_n] \Lambda_{k|t}^{-1} \Xi_{k|t} \in \mathbb{R}^{n \times md}, \quad (23)$$

$$F_{k|t} := [0_{n,nd} \ I_n] \Lambda_{k|t}^{-1} \begin{bmatrix} 0_{n,nd} \\ I_{nd} \end{bmatrix} \in \mathbb{R}^{n \times nd}, \quad (24)$$

$A_{k|t}$ is defined in (16), and $\Xi_{k|t}$ in (17).

In (23), $B_{k|t}$ is the d -step controllability matrix associated with the model data $(A_s, B_s)_{s \in \{t+dk, \dots, t+d(k+1)-1\}}$. Under Assumption 2, $(\exists c \in \mathbb{R}_{>0}) (\forall t, k \in \mathbb{N}_0) cI_n \preceq B_{k|t}B'_{k|t}$, making the lifted model (21) one-step controllable. Further,

$$F_{k|t} = [\Phi_{t+d(k+1),t+dk+1} \ \cdots \ \Phi_{t+d(k+1),t+d(k+1)-1} \ I_n],$$

and thus, $(\exists \underline{c}, \bar{c} \in \mathbb{R}_{>0}) (\forall t, k \in \mathbb{N}_0) \underline{c}I_n \preceq F_{k|t}F'_{k|t} \preceq \bar{c}I_n$. Moreover,

$$A_{k|t} = A_{t+d(k+1)-1} \ \cdots \ A_{t+dk} = \Phi_{t+d(k+1),t+dk},$$

$A_{k|t}^{-1}$, and $B_{k|t}$, are all uniformly bounded by Assumption 1.

Lemma 2: Given $u = (u_t)_{t \in \mathbb{N}_0}$ and $w = (w_t)_{t \in \mathbb{N}_0}$ in (1), for every $t, k \in \mathbb{N}_0$,

$$z_{k|t} = \begin{bmatrix} C_{k|t} \\ 0_{md,n(d+1)} \end{bmatrix} x_{k|t} + \begin{bmatrix} D_{k|t} \\ R_{k|t}^{1/2} \end{bmatrix} u_{k|t} + \begin{bmatrix} E_{k|t} \\ 0_{md,n(d+1)} \end{bmatrix} w_{k|t}, \quad (25)$$

with $u_{k|t}$, $w_{k|t}$, and $z_{k|t}$ as per (12)–(14) and (18), and $x_{k|t}$ as per (21), where $R_{k|t}$ is defined in (20),

$$C_{k|t} := \Gamma_{k|t} \Lambda_{k|t}^{-1} \begin{bmatrix} I_n \\ 0_{nd,n} \end{bmatrix} \in \mathbb{R}^{nd \times n}, \quad (26)$$

$$D_{k|t} := \Gamma_{k|t} \Lambda_{k|t}^{-1} \Xi_{k|t} \in \mathbb{R}^{nd \times md}, \quad (27)$$

$$E_{k|t} := \Gamma_{k|t} \Lambda_{k|t}^{-1} \begin{bmatrix} 0_{n,nd} \\ I_{nd} \end{bmatrix} \in \mathbb{R}^{nd \times nd}, \quad (28)$$

$A_{k|t}$ is defined in (16), $\Xi_{k|t}$ in (17), and $\Gamma_{k|t}$ in (19).

In (26), $C_{k|t}$ is the d -step observability matrix associated with the model data $(Q_s^{1/2}, A_s)_{s \in \{t+dk, \dots, t+d(k+1)-1\}}$. Under Assumption 2, $(\exists c \in \mathbb{R}_{>0}) (\forall t, k \in \mathbb{N}_0) cI_n \preceq C'_{k|t}C_{k|t}$ making the lifted model (21) with output (25) one-step observable.

Moreover, $C_{k|t}$, $D_{k|t}$, and $E_{k|t}$ are all uniformly bounded by Assumption 1.

The following finite-preview construction of a matrix X_{t+1} to approximate P_{t+1} in (8) involves a final transformation of the model data. This arises to accommodate the direct dependence of $z_{k|t}$ on $w_{k|t}$ in the supporting analysis elaborated in Section III-B. For $t, k \in \mathbb{N}_0$, with $A_{k|t}$, $B_{k|t}$, $C_{k|t}$, $F_{k|t}$, $D_{k|t}$, $E_{k|t}$, and $R_{k|t}$ as given in Lemmas 1 and 2, define

$$\tilde{B}_{k|t} := [B_{k|t} \ F_{k|t}], \quad (29)$$

$$\tilde{R}_{k|t} := \begin{bmatrix} R_{k|t} + D'_{k|t}D_{k|t} & D'_{k|t}E_{k|t} \\ E'_{k|t}D_{k|t} & E'_{k|t}E_{k|t} - \gamma^2 I_{nd} \end{bmatrix}, \quad (30)$$

with $\gamma \in \mathbb{R}_{>0}$ such that $\tilde{R}_{k|t}$ is non-singular,

$$\tilde{Q}_{k|t} := C'_{k|t}C_{k|t} - C'_{k|t}[D_{k|t} \ E_{k|t}] \tilde{R}_{k|t}^{-1} \begin{bmatrix} D'_{k|t} \\ E'_{k|t} \end{bmatrix} C_{k|t}, \quad (31)$$

$$\tilde{A}_{k|t} := A_{k|t} - [B_{k|t} \ F_{k|t}] \tilde{R}_{k|t}^{-1} \begin{bmatrix} D'_{k|t} \\ E'_{k|t} \end{bmatrix} C_{k|t}. \quad (32)$$

The dependence on γ here is suppressed for convenience. Note that non-singularity of $\tilde{R}_{k|t}$ implies non-singularity of $\tilde{A}_{k|t}$ by Lemma 13 in the Appendix. Finally, given $T \in \mathbb{N}$, let

$$\tilde{X}_{t+1} := \tilde{Q}_{T|t+1} + \tilde{A}'_{T|t+1}(\tilde{B}_{T|t+1}\tilde{R}_{T|t+1}^{-1}\tilde{B}'_{T|t+1})^{-1}\tilde{A}_{T|t+1}, \quad (33)$$

subject to non-singularity of $\tilde{B}_{T|t+1}\tilde{R}_{T|t+1}^{-1}\tilde{B}'_{T|t+1}$. If, in addition, $\tilde{B}_{T|t+1}\tilde{R}_{T|t+1}^{-1}\tilde{B}'_{T|t+1} \in \mathbb{S}_{++}^n$ and $\tilde{Q}_{T|t+1} \in \mathbb{S}_{++}^n$, which would be infeasible were the lifted model not one-step controllable and observable, then $\tilde{X}_{t+1} \in \mathbb{S}_{++}^n$. Given this, $(X_{t+1})_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$ when defined according to

$$X_{t+1} := \tilde{\mathcal{R}}_{\gamma,0|t+1} \circ \cdots \circ \tilde{\mathcal{R}}_{\gamma,T-1|t+1}(\tilde{X}_{t+1}), \quad (34)$$

with \tilde{X}_{t+1} as in (33), where the lifted Riccati operators involved are defined as follows:

$$\tilde{\mathcal{R}}_{\gamma,k|t}(X)$$

$$:= \tilde{Q}_{k|t} + \tilde{A}'_{k|t}(X - X\tilde{B}_{k|t}(\tilde{R}_{k|t} + \tilde{B}'_{k|t}X\tilde{B}_{k|t})^{-1}\tilde{B}'_{k|t}X)\tilde{A}_{k|t}. \quad (35)$$

This construction of X_{t+1} involves $d \cdot (T+1)$ steps of model data ahead of $t \in \mathbb{N}_0$. The main result formulated below identifies a sufficient number of lifted preview steps T for the corresponding state feedback control policy (10) to meet a specified performance loss bound $\beta \in \mathbb{R}_{>0}$, relative to the infinite-preview policy (8) with prescribed baseline energy-gain performance bound γ .

Theorem 2: Given $\gamma \in \mathbb{R}_{>0}$, suppose:

- 1) the hypothesis in Theorem 1 holds; and
- 2) there exist $\underline{c}, \bar{c} \in \mathbb{R}_{>0}$, such that for all $t \in \mathbb{N}_0$,
 - a) $\underline{c}I_{(m+n)d} \preceq \tilde{R}'_{0|t}\tilde{R}_{0|t} \preceq \bar{c}I_{(m+n)d}$,
 - b) $\underline{c}I_n \preceq B_{0|t}\tilde{R}_{0|t}^{-1}\tilde{B}'_{0|t} \preceq \bar{c}I_n$, and
 - c) $\underline{c}I_n \preceq \tilde{Q}_{0|t} \preceq \bar{c}I_n$,

with $\tilde{B}_{0|t}$ as per (29), and γ -dependent $\tilde{R}_{0|t}$ and $\tilde{Q}_{0|t}$ as per (30) and (31), respectively.

Further, given $\beta \in \mathbb{R}_{>0}$, let $u = (u_t)_{t \in \mathbb{N}_0} = (u_{\gamma+\beta,t}^{\text{fin}}(x_t))_{t \in \mathbb{N}_0}$ as per the finite-preview state feedback control policy (10)

for the system (1)–(2), with $(X_{t+1})_{t \in \mathbb{N}_0}$ constructed according to (34) for any selection of

$$T > \underline{T} := \log \left(\log \left((\underline{\alpha} + 1)^{1/\bar{\delta}} \right) \right) / \log(\bar{\rho}), \quad (36)$$

where $\underline{\alpha} := (\gamma^{-2} - (\gamma + \beta)^{-2}) \cdot \underline{\kappa}$,

$$\underline{\kappa} := \inf_{t \in \mathbb{N}_0} \lambda_{\min}(\tilde{Q}_{0|t}) > 0, \quad (37)$$

$$\begin{aligned} \bar{\delta} &:= \sqrt{n} \log \left(\sup_{t \in \mathbb{N}_0} \frac{\lambda_{\max}(\tilde{Q}_{0|t} + \tilde{A}'_{0|t}(\tilde{B}_{0|t}\tilde{R}_{0|t}^{-1}\tilde{B}'_{0|t})^{-1}\tilde{A}_{0|t})}{\lambda_{\min}(\tilde{Q}_{0|t})} \right) \\ &< +\infty, \end{aligned} \quad (38)$$

$$\bar{\rho} := \sup_{t \in \mathbb{N}_0} 1/(1 + \tilde{\omega}_t) < 1, \quad (39)$$

$$\tilde{\omega}_t := \frac{\lambda_{\min}(\tilde{Q}_{0|t} + \tilde{Q}_{0|t}\tilde{A}_{0|t}^{-1}\tilde{B}_{0|t}\tilde{R}_{0|t}^{-1}\tilde{B}'_{0|t}(\tilde{A}'_{0|t})^{-1}\tilde{Q}_{0|t})}{\lambda_{\max}(\tilde{Q}_{0|t} + \tilde{A}'_{0|t}(\tilde{B}_{0|t}\tilde{R}_{0|t}^{-1}\tilde{B}'_{0|t})^{-1}\tilde{A}_{0|t})}, \quad (40)$$

and $\tilde{A}_{0|t}$ is as defined in (32). Then, with $J_{\gamma+\beta}$ as per (3), $(\forall w \in \ell_2) J_{\gamma+\beta}(u, w) \leq 0$.

Remark 1: Parts 1) and 2) of the hypothesis in Theorem 2 amount to considering a sufficiently large gain bound γ for the baseline infinite-preview control policy (8). Given (4), the Schur complement of $M_{\gamma,t}(P_{t+1})$, as defined in (7), is given by $R_t + B'_t(P_{t+1} - P_{t+1}(P_{t+1} - \gamma^2 I_n)^{-1}P_{t+1})B_t = \nabla_{\gamma,t}(P_{t+1})$; cf. (9). Further, $P_{t+1} + P_{t+1}(\gamma^2 I_n - P_{t+1})^{-1}P_{t+1} \in \mathbb{S}_+^n$, whereby $\nabla_{\gamma,t}(P_{t+1}) \in \mathbb{S}_{++}^m$. Thus, $M_{\gamma,t}(P_{t+1})$ is non-singular for $t \in \mathbb{N}_0$; i.e., Riccati recursion (5) is well posed. In fact, $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$ under part 2) of the hypothesis, as elaborated in Remark 7, Section III-B. It is important to note that the construction of $(X_{t+1})_{t \in \mathbb{N}_0}$ according to (34) does not involve knowledge of $(P_t)_{t \in \mathbb{N}_0}$. The dependence on γ relates to the performance loss perspective used to assess the resulting finite-preview controller (10), relative to (8).

Remark 2: Under part 2) of the hypothesis in Theorem 2, $\tilde{X}_{t+1} \in \mathbb{S}_{++}^n$ in (33), and $\mathcal{R}_{\gamma,k|t}(X) \in \mathbb{S}_{++}^n$ over $X \in \mathbb{S}_{++}^n$, as elaborated in Remark 8, Section III-B. Thus, $X_{t+1} \in \mathbb{S}_{++}^n$ in (34). Using one-step controllability and observability of the lifted model, it is established in [19, Sec. 5.4.2] that the three components of this part of the hypothesis are necessary conditions when for every $t \in \mathbb{N}_0$, the lifted deadbeat policy

$$u_{k|t} = -B'_{k|t}(B_{k|t}B'_{k|t})^{-1}(A_{k|t}x_{k|t} + F_{k|t}w_{k|t}), \quad k \in \mathbb{N}_0, \quad (41)$$

achieves the baseline gain bound γ , given $u_s = 0$ and $w_s = 0$ for $s < t$. This full-information feedback control policy is not causal in the original time domain since in addition to model data it involves d -step preview of the disturbance.

Remark 3: The quantity $\underline{\kappa}$ in (37) depends on the fixed number of steps d used to lift the model to the form (21) and (25). This number is not uniquely determined by complying with Assumption 2, as any greater value also complies. Likewise, $\bar{\delta}$ in (38), which bounds the Riemannian distance between P_{t+dT+1} and \tilde{X}_{t+1} , and $\bar{\rho}$ in (39), which bounds the contraction rate of the lifted Riccati operator in (35), both depend on d . These quantities also all depend on γ .

Remark 4: In principle, the quantities (37)–(40) can be replaced by corresponding lower or upper estimates determined from uniform bounds on the problem data in accordance with

Assumption 1. For periodic systems, these quantities, and $(P_t)_{t \in \mathbb{N}_0}$ for that matter [12], [13], can be computed directly from the finite amount of model data. This facilitates numerical investigation of the kind presented in Section IV to assess the conservativeness of Theorem 2, which is only sufficient to guarantee the performance loss bound β , relative to a suitably large baseline gain bound γ .

III. PROOF OF THE MAIN RESULT

This section encompasses a structured proof of Theorem 2. This main result is established by combining Theorems 3 and 4, which are formulated below.

In particular, the proof relies on the *strict* contraction of the lifted Riccati operators composed in (34) to form $(X_{t+1})_{t \in \mathbb{N}_0}$. This key property holds under part 2) of the hypothesis on the baseline gain bound $\gamma \in \mathbb{R}_{>0}$ in Theorem 2, as summarized in Theorem 3, which is proved in Section III-A; recall that non-singularity of γ -dependent $\tilde{R}_{k|t}$ in (30) implies non-singularity of $\tilde{A}_{k|t}$ in (32), by Lemma 13 in the Appendix.

Theorem 3: Given $t, k \in \mathbb{N}_0$, and $\gamma \in \mathbb{R}_{>0}$, suppose $\tilde{R}_{k|t}$ in (30) is non-singular, and with reference to (29) and (31), suppose $\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \in \mathbb{S}_{++}^n$ and $\tilde{Q}_{k|t} \in \mathbb{S}_{++}^n$. Then, for any $X, P \in \mathbb{S}_{++}^n$,

$$\delta(\tilde{\mathcal{R}}_{\gamma,k|t}(X), \tilde{\mathcal{R}}_{\gamma,k|t}(P)) \leq \tilde{\rho}_{k|t} \cdot \delta(X, P) \quad (42)$$

with $\tilde{\rho}_{k|t} := \tilde{\zeta}_{k|t}/(\tilde{\zeta}_{k|t} + \tilde{\epsilon}_{k|t}) < 1$, where $\tilde{\mathcal{R}}_{\gamma,k|t}(\cdot)$ is defined in (35),

$$\tilde{\zeta}_{k|t} := 1/\lambda_{\min}(\tilde{Q}_{k|t} + \tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}(\tilde{A}'_{k|t})^{-1}\tilde{Q}_{k|t}), \quad (43)$$

$$\tilde{\epsilon}_{k|t} := 1/\lambda_{\max}(\tilde{Q}_{k|t} + \tilde{A}'_{k|t}(\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t}), \quad (44)$$

and $\delta(\cdot, \cdot)$ is the Riemannian metric on \mathbb{S}_{++}^n ; see (47).

The other key ingredient pertains to continuity of the baseline gain bound associated with the infinite-preview policy (8).

Theorem 4: Given $\gamma \in \mathbb{R}_{>0}$, suppose $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$ is such that (4) and (5) hold in addition to $\underline{\lambda} := \inf_{t \in \mathbb{N}_0} \lambda_{\min}(P_t) > 0$. Further, given $\beta \in \mathbb{R}_{>0}$, and $(X_{t+1})_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$, suppose

$$(\exists \varepsilon \in \mathbb{R}_{>0}) (\forall k \in \mathbb{N}_0)$$

$$P_{t+1} \prec X_{t+1} \wedge \delta(X_{t+1}, P_{t+1}) \leq \log((\eta - \varepsilon)\underline{\lambda} + 1), \quad (45)$$

where \wedge denotes logical conjunction,

$$\eta := \gamma^{-2} - (\gamma + \beta)^{-2} > 0, \quad (46)$$

and $\delta(\cdot, \cdot)$ is the Riemannian distance; see (47). Finally, let $u = (u_t)_{t \in \mathbb{N}_0} = (u_{\gamma+\beta,t}^{\text{fin}}(x_t))_{t \in \mathbb{N}_0}$ as per the finite-preview state feedback control policy (10) for the system (1)–(2). Then, with $J_{\gamma+\beta}$ as per (3), $(\forall w \in \ell_2) J_{\gamma+\beta}(u, w) \leq 0$.

Relationships between the Riccati operator (35) for the lifted model (21) with (25), and the ℓ_2 gain Riccati operator (6) for (1)–(2), are established in Section III-B. This forms the basis for the proof of Theorem 4 in Section III-C. The development of these relationships also highlights interesting links between Riccati recursions and Schur decompositions of quadratic forms in ℓ_2 gain analysis. In Section III-D, where Theorem 2 is proved, it is shown via Theorem 3 that with sufficiently large $T \in \mathbb{N}$, the construction of $(X_{t+1})_{t \in \mathbb{N}_0}$ in (34) satisfies the hypothesis (45).

A. Lifted Riccati operator contraction

By application of [20, Thm. 1.7], the hypothesis in Theorem 3 is sufficient for *strict* contraction of the lifted Riccati operator $X \mapsto \tilde{\mathcal{R}}_{\gamma,k|t}(X)$ given by (35). Without Assumption 2, the second part of the hypothesis regarding $\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}$ and $\tilde{Q}_{k|t}$ is infeasible, and the contraction may be non-strict. The Riemannian distance between $X, P \in \mathbb{S}_{++}^n$ is defined as follows [36]:

$$\delta(X, P) := \sqrt{\sum_{i \in \{1, \dots, n\}} (\log(\lambda_i))^2}, \quad (47)$$

where $\{\lambda_1, \dots, \lambda_n\} = \text{spec}(XP^{-1})$ is the spectrum of XP^{-1} (i.e., the collection of all eigenvalues, including multiplicity.) The latter coincides with $\text{spec}(P^{-1/2}XP^{-1/2}) \subset \mathbb{R}_{>0}$ because $\text{spec}(YZ) \cup \{0\} = \text{spec}(ZY) \cup \{0\}$ for all square Y, Z , and $\{0\} \cap \text{spec}(XP^{-1}) = \emptyset = \{0\} \cap \text{spec}(P^{-1/2}XP^{-1/2})$. Indeed, $\delta(X, P) = \delta(X^{-1}, P^{-1}) = \delta(P^{-1}, X^{-1}) = \delta(P, X)$ as $\lambda_i \in \text{spec}(XP^{-1}) = \text{spec}(P^{-1}X)$ implies $1/\lambda_i \in \text{spec}(X^{-1}P) = \text{spec}(PX^{-1})$, and $(\log(\lambda_i))^2 = (\log(1/\lambda_i))^2$.

Proof of Theorem 3: With $\tilde{R}_{k|t}$ and $\tilde{A}_{k|t}$ non-singular, from (60),

$$\begin{aligned} \tilde{\mathcal{R}}_{\gamma,k|t}(P) &= \tilde{Q}_{k|t} + \tilde{A}'_{k|t}P(\tilde{A}_{k|t}^{-1} + \tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}P)^{-1} \\ &= \left(\tilde{Q}_{k|t}(\tilde{A}_{k|t}^{-1} + \tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}P) + \tilde{A}'_{k|t}P \right) \\ &\quad \times \left(\tilde{A}_{k|t}^{-1} + \tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}P \right)^{-1} \\ &= (\tilde{E}_{k|t}P + \tilde{F}_{k|t})(\tilde{G}_{k|t}P + \tilde{H}_{k|t})^{-1} \end{aligned}$$

for $P \in \mathbb{S}_{++}^n$, where

$$\begin{aligned} \tilde{E}_{k|t} &:= \tilde{A}'_{k|t} + \tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \\ &= \tilde{A}'_{k|t}((\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1} + (\tilde{A}'_{k|t})^{-1}\tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1})\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}, \\ \tilde{F}_{k|t} &:= \tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1}, \quad \tilde{G}_{k|t} := \tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}, \text{ and } \tilde{H}_{k|t} := \tilde{A}_{k|t}^{-1}. \quad \text{The hypothesis that } \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \in \mathbb{S}_{++}^n \text{ and } \tilde{Q}_{k|t} \in \mathbb{S}_{++}^n, \text{ implies} \\ &((\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1} + (\tilde{A}'_{k|t})^{-1}\tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1}) \in \mathbb{S}_{++}^n. \end{aligned}$$

Therefore, $\tilde{E}_{k|t}$ is non-singular. Further,

$$\tilde{F}_{k|t}\tilde{E}'_{k|t} = \tilde{Q}_{k|t} + \tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}(\tilde{A}'_{k|t})^{-1}\tilde{Q}_{k|t} \in \mathbb{S}_{++}^n,$$

$$\begin{aligned} \tilde{E}'_{k|t}\tilde{G}_{k|t} &= \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \\ &\quad + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}(\tilde{A}'_{k|t})^{-1}\tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \in \mathbb{S}_{++}^n \end{aligned}$$

and

$$\tilde{G}_{k|t}\tilde{E}_{k|t}^{-1} = (\tilde{Q}_{k|t} + \tilde{A}'_{k|t}(\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t})^{-1} \in \mathbb{S}_{++}^n,$$

since

$$\begin{aligned} \tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}(\tilde{A}'_{k|t} + \tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1} \\ = \tilde{A}_{k|t}^{-1}((\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1} + (\tilde{A}'_{k|t})^{-1}\tilde{Q}_{k|t}\tilde{A}_{k|t}^{-1})^{-1}(\tilde{A}'_{k|t})^{-1}. \end{aligned}$$

As such, [20, Thm. 1.7] applies to give $\delta(\tilde{\mathcal{R}}_{\gamma,k|t}(X), \tilde{\mathcal{R}}_{\gamma,k|t}(P)) \leq \tilde{\rho}_{k|t} \cdot \delta(X, P)$ for all $X, P \in \mathbb{S}_{++}^n$, with $\tilde{\rho}_{k|t} = \tilde{\zeta}_{k|t}/(\tilde{\zeta}_{k|t} + \tilde{\epsilon}_{k|t}) < 1$, $\tilde{\zeta}_{k|t} = 1/\lambda_{\min}(\tilde{F}_{k|t}\tilde{E}'_{k|t})$, and $\tilde{\epsilon}_{k|t} = \lambda_{\min}(\tilde{G}_{k|t}\tilde{E}_{k|t}^{-1})$, as per the proof therein. \square

B. Riccati operator lifting

Consider the lifted model formulated in Lemmas 1 and 2, and the definitions of $A_{k|t}, B_{k|t}, F_{k|t}, D_{k|t}, E_{k|t}$, and $R_{k|t}$ therein, with $d \in \mathbb{N}$ fixed according to Assumption 2. Given $t, k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}_{>0}$, $v_{k|t} = (u_{k|t}, w_{k|t}) \in \mathbb{R}^{md} \times \mathbb{R}^{nd}$, $x_{k|t} \in \mathbb{R}^n$, and $P \in \mathbb{S}^n$, in view of (21) and (25),

$$\begin{aligned} z'_{k|t}z_{k|t} - \gamma^2w'_{k|t}w_{k|t} + x'_{k+1|t}Px_{k+1|t} \\ = \begin{bmatrix} x_{k|t} \\ v_{k|t} \end{bmatrix}' \begin{bmatrix} C'_{k|t}C_{k|t} + A'_{k|t}PA_{k|t} & L'_{k|t}(P) \\ L_{k|t}(P) & M_{\gamma,k|t}(P) \end{bmatrix} \begin{bmatrix} x_{k|t} \\ v_{k|t} \end{bmatrix}, \end{aligned} \quad (48)$$

$$\begin{aligned} \text{where } L_{k|t}(P) &:= [D_{k|t} \quad E_{k|t}]' C_{k|t} + \tilde{B}'_{k|t}PA_{k|t}, \\ M_{\gamma,k|t}(P) &:= \tilde{R}_{k|t} + \tilde{B}'_{k|t}P\tilde{B}_{k|t}, \end{aligned} \quad (49)$$

$\tilde{B}_{k|t}$ is defined in (29), and γ -dependent $\tilde{R}_{k|t}$ is defined in (30).

Lemma 3: Given $t \in \mathbb{N}_0$, $\gamma \in \mathbb{R}_{>0}$, and $(P_s)_{s \in \{t, \dots, t+d\}} \subset \mathbb{S}^n$, suppose for $s \in \{t, \dots, t+d-1\}$ that $M_{\gamma,s}(P_{s+1})$ as defined in (7) is non-singular, and $P_s = \mathcal{R}_{\gamma,s}(P_{s+1})$ in accordance with (6). Then, $M_{\gamma,0|t}(P_{t+d})$ as defined in (49) is non-singular.

Proof: If $d = 1$, then $R_{0|t} = R_t$, $B_{0|t} = B_t$, $F_{0|t} = I_n$, $C_{0|t} = Q_t^{1/2}$, $D_{0|t} = 0_{n,m}$, $E_{0|t} = 0_{n,n}$, and $F_{0|t} = I_n$, whereby $M_{\gamma,0|t}(P_{t+1}) = M_{\gamma,t}(P_{t+1})$. As such, the result follows by induction on noting that it holds with one additional lifting step irrespective of the value of d , as shown below.

Given $x_{0|t} \in \mathbb{R}^n$, $v_{0|t} = (u_{0|t}, w_{0|t}) \in \mathbb{R}^{md} \times \mathbb{R}^{nd}$, and $v_{t+d} = (u_{t+d}, w_{t+d}) \in \mathbb{R}^m \times \mathbb{R}^n$, and $x_{t+d} = x_{1|t} = A_{0|t}x_{0|t} + \tilde{B}_{0|t}v_{0|t}$ as per (21), it follows from (1) that

$$\begin{aligned} x_{t+d+1} &= A_{t+d}x_{t+d} + B_{t+d}u_{t+d} + w_{t+d} \\ &= A_{t+d}A_{0|t}x_{0|t} + A_{t+d}\tilde{B}_{0|t}v_{0|t} + [B_{t+d} \quad I_n]v_{t+d}. \end{aligned}$$

Furthermore, with $z_{0|t}$ as per (25), for any $P \in \mathbb{S}^n$,

$$\begin{aligned} z'_{0|t}z_{0|t} - \gamma^2w'_{0|t}w_{0|t} \\ + z'_{t+d}z_{t+d} - \gamma^2w'_{t+d}w_{t+d} + x'_{t+d+1}Px_{t+d+1} \\ = \begin{bmatrix} x_{0|t} \\ v_{0|t} \end{bmatrix}' \begin{bmatrix} Q_{0|t}^+(P) & (L_{0|t}^+(P))' \\ L_{0|t}^+(P) & M_{\gamma,0|t}^+(P) \end{bmatrix} \begin{bmatrix} x_{0|t} \\ v_{0|t} \end{bmatrix}, \end{aligned} \quad (50)$$

where $z_{t+d} = [Q_{t+d}^{1/2} \quad 0_{n,m}]' x_{1|t} + [0_{m,n} \quad R_{t+d}^{1/2}]' u_{t+d}$ as per (2), $v_{0|t}^\perp := (u_{0|t}, u_{t+d}, w_{0|t}, w_{t+d})^\top = \Pi(v_{0|t}, v_{t+d})$ for a corresponding permutation matrix $\Pi \in \mathbb{R}^{(m+n) \cdot (d+1) \times (m+n) \cdot (d+1)}$,

$$Q_{0|t}^+(P) := C'_{0|t}C_{0|t} + A'_{0|t}(Q_{t+d} + A'_{t+d}PA_{t+d})A_{0|t}, \quad (51)$$

$$L_{0|t}^+(P) := \Pi \begin{bmatrix} L_{0|t}^{1+}(P) \\ L_{0|t}^{2+}(P) \end{bmatrix}, \quad (52)$$

$$L_{0|t}^{1+}(P) := \begin{bmatrix} D'_{0|t} \\ E'_{0|t} \end{bmatrix} C_{0|t} + \tilde{B}'_{0|t}(Q_{t+d} + A'_{t+d}PA_{t+d})A_{0|t}, \quad (53)$$

$$L_{0|t}^{2+}(P) := \begin{bmatrix} B'_{t+d} \\ I_n \end{bmatrix} PA_{t+d}A_{0|t} = L_{t+d}(P)A_{0|t}, \quad (54)$$

and finally,

$$\begin{aligned} M_{\gamma,0|t}^+(P) &:= \\ \Pi \begin{bmatrix} \tilde{R}_{0|t} + \tilde{B}'_{0|t}(Q_{t+d} + A'_{t+d}PA_{t+d})\tilde{B}_{0|t} & \tilde{B}'_{0|t}L'_{t+d}(P) \\ L_{t+d}(P)\tilde{B}_{0|t} & M_{\gamma,t+d}(P) \end{bmatrix} \Pi', \end{aligned}$$

with $L_{t+d}(P)$ and $M_{\gamma,t+d}(P)$ as per (7). When $M_{\gamma,t+d}(P)$ is non-singular, Schur decomposition yields

$$\begin{aligned} M_{\gamma,0|t}^+(P) &= \\ \Pi \begin{bmatrix} I_{(m+n)d} & (L_{t+d}(P)\tilde{B}_{0|t})'(M_{\gamma,t+d}(P))^{-1} \\ 0_{m+n,(m+n)d} & I_{m+n} \end{bmatrix} \\ &\times \begin{bmatrix} M_{\gamma,0|t}(\mathcal{R}_{\gamma,t+d}(P)) & 0_{(m+n)d,m+n} \\ 0_{m+n,(m+n)d} & M_{\gamma,t+d}(P) \end{bmatrix} \\ &\times \begin{bmatrix} I_{(m+n)d} & 0_{(m+n)d,m+n} \\ (M_{\gamma,t+d}(P))^{-1}L_{t+d}(P)\tilde{B}_{0|t} & I_{m+n} \end{bmatrix} \Pi', \end{aligned} \quad (55)$$

with $\mathcal{R}_{\gamma,t+d}$ and $M_{\gamma,0|t}$ as per (6) and (49), respectively. Therefore, if $P = P_{t+d+1} \in \mathbb{S}^n$ with $M_{\gamma,t+d}(P_{t+d+1})$ non-singular, and $P_{t+d} = \mathcal{R}_{\gamma,t+d}(P_{t+d+1})$ with $M_{\gamma,0|t}(P_{t+d})$ non-singular, then $M_{\gamma,0|t}^+(P_{t+d+1})$ is non-singular, as required for the induction argument. \square

If $M_{\gamma,k|t}(P)$ is non-singular for given $t, k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}_{>0}$, and $P \in \mathbb{S}^n$, then Schur decomposition of (48) yields

$$\begin{aligned} z'_{k|t}z_{k|t} - \gamma^2 w'_{k|t}w_{k|t} + x'_{k+1|t}Px_{k+1|t} \\ = x'_{k|t}\mathcal{R}_{\gamma,k|t}(P)x_{k|t} \\ + (v_{k|t} - v_{\gamma,k|t})'M_{\gamma,k|t}(P)(v_{k|t} - v_{\gamma,k|t}), \end{aligned} \quad (56)$$

where the lifted ℓ_2 gain Riccati operator

$$\begin{aligned} \mathcal{R}_{\gamma,k|t}(P) &:= C'_{k|t}C_{k|t} + A'_{k|t}PA_{k|t} \\ &- L'_{k|t}(P)(M_{\gamma,k|t}(P))^{-1}L_{k|t}(P), \end{aligned} \quad (57)$$

and $v_{\gamma,k|t} := -(M_{\gamma,k|t}(P))^{-1}L_{k|t}(P)x_{k|t}$.

Lemma 4: Given $t \in \mathbb{N}_0$, $\gamma \in \mathbb{R}_{>0}$, and $(P_s)_{s \in \{t, \dots, t+d\}} \subset \mathbb{S}^n$, suppose for $s \in \{t, \dots, t+d-1\}$ that $M_{\gamma,s}(P_{s+1})$ as defined in (7) is non-singular, and $P_s = \mathcal{R}_{\gamma,s}(P_{s+1})$ in accordance with (6). Then, $\mathcal{R}_{\gamma,0|t}(P_{t+d}) = \mathcal{R}_{\gamma,t} \circ \dots \circ \mathcal{R}_{\gamma,t+d-1}(P_{t+d}) = P_t$.

Proof: Building upon the induction argument in the proof of Lemma 3 yields the result. If $d = 1$, then $\mathcal{R}_{\gamma,0|t}(P_{t+1}) = \mathcal{R}_{\gamma,t}(P_{t+1})$ by definition. Given this, it remains to show that if $\mathcal{R}_{\gamma,0|t}(P_{t+d}) = \mathcal{R}_{\gamma,t} \circ \dots \circ \mathcal{R}_{\gamma,t+d-1}(P_{t+d})$, then it is also true with one additional step in the lifting. To this end, observe that for $P \in \mathbb{S}^n$ such that $M_{\gamma,0|t}^+(P)$ in (55) is non-singular, the lifted Riccati operator with one additional step is given by the corresponding Schur complement

$$\mathcal{R}_{\gamma,0|t}^+(P) := Q_{0|t}^+(P) - (L_{0|t}^+(P))'(M_{\gamma,0|t}^+(P))^{-1}L_{0|t}^+(P) \quad (58)$$

of (50). Noting that

$$\begin{aligned} (M_{\gamma,0|t}^+(P))^{-1} &= \\ \Pi \begin{bmatrix} I_{(m+n)d} & 0_{(m+n)d,m+n} \\ -(M_{\gamma,t+d}(P))^{-1}L_{t+d}(P)\tilde{B}_{0|t} & I_{m+n} \end{bmatrix} \\ &\times \begin{bmatrix} (M_{\gamma,0|t}(\mathcal{R}_{\gamma,t+d}(P)))^{-1} & 0_{(m+n)d,m+n} \\ 0_{m+n,(m+n)d} & (M_{\gamma,t+d}(P))^{-1} \end{bmatrix} \\ &\times \begin{bmatrix} I_{(m+n)d} & -((M_{\gamma,t+d}(P))^{-1}L_{t+d}(P)\tilde{B}_{0|t})' \\ 0_{m+n,(m+n)d} & I_{m+n} \end{bmatrix} \Pi', \end{aligned}$$

if $M_{\gamma,t+d}(P)$ and $M_{\gamma,0|t}(P)$ as per (7) and (49) are non-singular, then from (6), (51)–(54), (57), and (58),

$$\begin{aligned} \mathcal{R}_{\gamma,0|t}^+(P) &= \\ C'_{0|t}C_{0|t} + A'_{0|t}\mathcal{R}_{\gamma,t+d}(P)A_{0|t} \\ - L_{0|t}(\mathcal{R}_{\gamma,t+d}(P))'(M_{\gamma,0|t}(\mathcal{R}_{\gamma,t+d}(P)))^{-1}L_{0|t}(\mathcal{R}_{\gamma,t+d}(P)) \\ &= \mathcal{R}_{\gamma,0|t}(\mathcal{R}_{\gamma,t+d}(P)). \end{aligned}$$

So, if $P = P_{t+d+1} \in \mathbb{S}^n$ with $M_{\gamma,t+d}(P_{t+d+1})$ non-singular, and $P_{t+d} = \mathcal{R}_{\gamma,t+d}(P_{t+d+1})$, then $\mathcal{R}_{\gamma,0|t}^+(P_{t+d+1}) = \mathcal{R}_{\gamma,0|t}(P_{t+d}) = \mathcal{R}_{\gamma,t} \circ \dots \circ \mathcal{R}_{\gamma,t+d-1}(P_{t+d}) = \mathcal{R}_{\gamma,t} \circ \dots \circ \mathcal{R}_{\gamma,t+d}(P_{t+d+1})$, as required for the induction argument. \square

Remark 5: As noted in Remark 1, with $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_+^n$ satisfying part 1) of the hypothesis in Theorem 2, $M_{\gamma,t}(P_{t+1})$ is non-singular, and $P_t = \mathcal{R}_{\gamma,t}(P_{t+1})$ for every $t \in \mathbb{N}_0$. As such, for all $t, k \in \mathbb{N}_0$, $M_{\gamma,0|t}(P_{t+d})$ and $M_{\gamma,k|t}(P_{t+d(k+1)})$ are non-singular by Lemma 3, since $M_{\gamma,k|t}(P_{t+d(k+1)}) = M_{\gamma,0|t+dk}(P_{t+dk+d})$. Further, $P_{t+dk} = \mathcal{R}_{\gamma,t+dk} \circ \dots \circ \mathcal{R}_{\gamma,t+d(k+1)-1}(P_{t+d(k+1)}) = \mathcal{R}_{\gamma,k|t}(P_{t+d(k+1)})$ by Lemma 4, as exploited subsequently.

Remark 6: Given $t, k \in \mathbb{N}_0$, suppose $\gamma \in \mathbb{R}_{>0}$ and $P \in \mathbb{S}_+^n$ are such that $M_{\gamma,k|t}(P)$ in (49) is non-singular. Then, in (57),

$$\begin{aligned} \mathcal{R}_{\gamma,k|t}(P) &= C'_{k|t}C_{k|t} + A'_{k|t}PA_{k|t} \\ &- \left(A'_{k|t}P\tilde{B}_{k|t} + C'_{k|t}[D_{k|t} \quad E_{k|t}] \right) \\ &\times (\tilde{R}_{k|t} + \tilde{B}'_{k|t}P\tilde{B}_{k|t})^{-1} \left(\tilde{B}'_{k|t}PA_{k|t} + \begin{bmatrix} D'_{k|t} \\ E'_{k|t} \end{bmatrix} C_{k|t} \right). \end{aligned}$$

If $\tilde{R}_{k|t} \in \mathbb{S}^{(m+n)d}$ is also non-singular, then in (35), $\tilde{\mathcal{R}}_{\gamma,k|t}(P) = \mathcal{R}_{\gamma,k|t}(P)$ by [37, Prop. 12.1.1]. Further,

$$\begin{aligned} \tilde{\mathcal{R}}_{\gamma,k|t}(P) &= \\ \tilde{Q}_{k|t} + \tilde{A}'_{k|t}P^{1/2}(I + P^{1/2}\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}P^{1/2})^{-1}P^{1/2}\tilde{A}_{k|t} \end{aligned} \quad (59)$$

by the Woodbury formula, whenever $\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \in \mathbb{S}_{++}^n$. As such, if $\tilde{Q}_{k|t} \in \mathbb{S}_{++}^{(m+n)d}$, in addition to the preceding conditions, then $\tilde{\mathcal{R}}_{\gamma,k|t}(P) \in \mathbb{S}_{++}^n$. Finally, from (59),

$$\tilde{\mathcal{R}}_{\gamma,k|t}(P) = \tilde{Q}_{k|t} + \tilde{A}'_{k|t}(P^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t} \quad (60)$$

for $P \in \mathbb{S}_{++}^n$, as leveraged in subsequent sections.

Remark 7: As noted in Remark 5, with $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_+^n$ as per part 1) of the hypothesis in Theorem 2, $M_{\gamma,k|t}(P_{t+d(k+1)})$ is non-singular for all $t, k \in \mathbb{N}_0$. With part 2) of the hypothesis, $\tilde{R}_{k|t} = \tilde{R}_{0|t+kd}$ is non-singular, $\tilde{Q}_{k|t} = \tilde{Q}_{0|t+kd} \in \mathbb{S}_{++}^n$, and $\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} = \tilde{B}_{0|t+kd}\tilde{R}_{0|t+kd}^{-1}\tilde{B}'_{0|t+kd} \in \mathbb{S}_{++}^n$. Thus, by Remark 6, $P_{t+dk} = \mathcal{R}_{\gamma,k|t}(P_{t+d(k+1)}) \in \mathbb{S}_{++}^n$; i.e., the hypothesized $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_+^n$ must be contained in \mathbb{S}_{++}^n as previously noted in Remark 1.

Remark 8: With part 2) of the hypothesis in Theorem 2, $\tilde{R}_{k|t} = \tilde{R}_{0|t+dk}$ is non-singular for all $t, k \in \mathbb{N}_0$, whereby $\tilde{A}_{k|t} = \tilde{A}_{0|t+dk}$ is non-singular by Lemma 13 in the Appendix. Furthermore, $\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} = \tilde{B}_{0|t+dk}\tilde{R}_{0|t+dk}^{-1}\tilde{B}'_{0|t+dk} \in \mathbb{S}_{++}^n$, and $\tilde{Q}_{k|t} = \tilde{Q}_{0|t+dk} \in \mathbb{S}_{++}^n$. Therefore, $\tilde{\mathcal{R}}_{\gamma,k|t}(X) \in \mathbb{S}_{++}^n$ over $X \in \mathbb{S}_{++}^n$, as noted in Remark 6, and $\tilde{\mathcal{R}}_{\gamma,k|t}(\cdot)$ is a strict contraction by Theorem 3.

C. Continuity of closed-loop ℓ_2 gain performance

The following lemmas lead to a proof of Theorem 4.

Lemma 5: Given $\gamma \in \mathbb{R}_{>0}$, suppose $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$ is such that (4) and (5) hold. Further, given $\beta \in \mathbb{R}_{>0}$, and bounded sequence $(X_{t+1})_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$, suppose

$$(\exists \varepsilon \in \mathbb{R}_{>0}) (\forall k \in \mathbb{N}_0)$$

$$(X_{t+1} - (\gamma + \beta)^2 I_n \preceq -\varepsilon I_n) \wedge (P_{t+1} \preceq X_{t+1}) \wedge (\mathcal{R}_{\gamma+\beta,t}(X_{t+1}) \preceq \mathcal{R}_{\gamma,t}(P_{t+1})),$$

in accordance with (6). Then, with $J_{\gamma+\beta}$ as per (3), the state feedback control policy $u_t = \mathbf{u}_{\gamma+\beta,t}^{\text{fin}}(x_t)$ defined in (10) for the system (1)–(2) achieves $(\forall w \in \ell_2) J_{\gamma+\beta}(u, w) \leq 0$.

Proof: Since $((\gamma + \beta)^2 I_n - X_{t+1}) \in \mathbb{S}_{++}^n$, and $R_t \in \mathbb{S}_{++}^n$,

$$\begin{aligned} & \nabla_{\gamma+\beta,t}(X_{t+1}) \\ &= R_t + B'_t(X_{t+1} + X_{t+1}((\gamma + \beta)^2 I_n - X_{t+1})^{-1} X_{t+1})B_t \\ &\in \mathbb{S}_{++}^m, \end{aligned}$$

This is the Schur complement of $M_{\gamma+\beta,t}(X_{t+1})$, as defined in (7), which is therefore non-singular. Given $x_t \in \mathbb{R}^n$, and $v_t = (u_t, w_t) \in \mathbb{R}^{m+n}$, it follows from (1)–(2) that

$$\begin{aligned} & z'_t z_t - (\gamma + \beta)^2 w'_t w_t + x'_{t+1} X_{t+1} x_{t+1} \\ &= x'_t \mathcal{R}_{\gamma+\beta,t}(X_{t+1}) x_t \\ &\quad + (v_t - \mathbf{v}_{\gamma+\beta,t}^{\text{fin}})' M_{\gamma+\beta,t}(X_{t+1})(v_t - \mathbf{v}_{\gamma+\beta,t}^{\text{fin}}), \end{aligned}$$

with $\mathbf{v}_{\gamma+\beta,t}^{\text{fin}} := -(M_{\gamma+\beta,t}(X_{t+1}))^{-1} L_t(X_{t+1}) x_t$, and $L_t(X_{t+1})$ as per (7); cf. derivation of (56) from (48) by Schur decomposition. Then, further decomposition of $M_{\gamma+\beta,t}(X_{t+1})$ in the same fashion leads to

$$\begin{aligned} & z'_t z_t - (\gamma + \beta)^2 w'_t w_t + x'_{t+1} X_{t+1} x_{t+1} \\ &= x'_t \mathcal{R}_{\gamma+\beta,t}(X_{t+1}) x_t \\ &\quad + (u_t - \mathbf{u}_{\gamma+\beta,t}^{\text{fin}})' \nabla_{\gamma+\beta,t}(X_{t+1})(u_t - \mathbf{u}_{\gamma+\beta,t}^{\text{fin}}) \\ &\quad + (w_t - \mathbf{w}_{\gamma+\beta,t}^{\text{fin}})' (X_{t+1} - (\gamma + \beta)^2 I_n)(w_t - \mathbf{w}_{\gamma+\beta,t}^{\text{fin}}), \end{aligned} \quad (61)$$

with $\mathbf{w}_{\gamma+\beta,t}^{\text{fin}} = -(X_{t+1} - (\gamma + \beta)^2 I_n)^{-1} X_{t+1} (A_t x_t + B_t u_t)$, and $\mathbf{u}_{\gamma+\beta,t}^{\text{fin}}$ as per (10).

Now given any $w = (w_t)_{t \in \mathbb{N}_0} \in \ell_2$, with $u_t = \mathbf{u}_{\gamma+\beta,t}^{\text{fin}}(x_t)$ in (1)–(2),

$$\begin{aligned} & \sum_{t=0}^N z'_t z_t - (\gamma + \beta)^2 w'_t w_t \\ &\leq \sum_{t=0}^N x'_t \mathcal{R}_{\gamma+\beta,t}(X_{t+1}) x_t - x'_{t+1} X_{t+1} x_{t+1} \\ &\leq \sum_{t=0}^N (x'_t \mathcal{R}_{\gamma+\beta,t}(X_{t+1}) x_t - x'_t \mathcal{R}_{\gamma,t}(P_{t+1}) x_t \\ &\quad + x'_t P_t x_t - x'_{t+1} P_{t+1} x_{t+1}) \\ &= x'_0 P_0 x_0 - x'_{N+1} P_{N+1} x_{N+1} \\ &\quad + \sum_{t=0}^N x'_t (\mathcal{R}_{\gamma+\beta,t}(X_{t+1}) - \mathcal{R}_{\gamma,t}(P_{t+1})) x_t \\ &\leq \sum_{t=0}^N x'_t (\mathcal{R}_{\gamma+\beta,t}(X_{t+1}) - \mathcal{R}_{\gamma,t}(P_{t+1})) x_t, \end{aligned}$$

for every $N \in \mathbb{N}$. The first inequality follows from (61) since $X_{t+1} - (\gamma + \beta)^2 I_n \preceq -\varepsilon I_n$. The second inequality follows from $P_t = \mathcal{R}_{\gamma,t}(P_{t+1})$ as per (5), and the hypothesis $P_{t+1} \preceq X_{t+1}$. The last inequality holds because $x_0 = 0$, and $P_{N+1} \in \mathbb{S}_{++}^n$. Therefore, with the hypothesis $\mathcal{R}_{\gamma+\beta,t}(X_{t+1}) \preceq \mathcal{R}_{\gamma,t}(P_{t+1})$,

$$(\forall N \in \mathbb{N}) \quad \sum_{t=0}^{N-1} z'_t z_t - (\gamma + \beta)^2 w'_t w_t \leq 0.$$

As such, $z \in \ell_2$ since $w \in \ell_2$, with $\|z\|_2^2 \leq (\gamma + \beta)^2 \|w\|_2^2$. \square

Lemma 6: Given $\gamma \in \mathbb{R}_{>0}$, and $P \in \mathbb{S}_{++}^n$ such that $P \prec \gamma^2 I_n$, the ℓ_2 gain Riccati operator can be expressed as

$$\mathcal{R}_{\gamma,t}(P) = Q_t + A'_t(P^{-1} - \gamma^{-2} I_n + B_t R_t^{-1} B'_t)^{-1} A_t \quad (62)$$

in accordance with (6).

Proof: Since $(P - \gamma^2 I_n)$ is non-singular by hypothesis, Schur decomposition of $M_{\gamma,t}(P)$, as defined in (7), yields

$$\begin{aligned} & \mathcal{R}_{\gamma,t}(P) \\ &= Q_t + A'_t P A_t \\ &\quad - \left[\begin{matrix} B'_t (P - P(P - \gamma^2 I_n)^{-1} P) A_t \\ P A_t \end{matrix} \right]' \\ &\quad \times \left[\begin{matrix} (\nabla_{\gamma,t}(P))^{-1} & 0_{m,n} \\ 0_{n,m} & (P - \gamma^2 I_n)^{-1} \end{matrix} \right] \\ &\quad \times \left[\begin{matrix} B'_t (P - P(P - \gamma^2 I_n)^{-1} P) A_t \\ P A_t \end{matrix} \right], \end{aligned} \quad (63)$$

where $\nabla_{\gamma,t}(P) = R_t + B'_t (P - P(P - \gamma^2 I_n)^{-1} P) B_t$. Thus, with $W := P - P(P - \gamma^2 I_n)^{-1} P$,

$$\mathcal{R}_{\gamma,t}(P) = Q_t + A'_t (W - W B_t (R_t + B'_t W B_t)^{-1} B'_t W) A_t.$$

As such, the expression (62) for $\mathcal{R}_{\gamma,t}(P)$ follows from the identities $W = (P^{-1} - \gamma^{-2} I_n)^{-1}$ and $(W - W B_t (R_t + B'_t W B_t)^{-1} B'_t W) = (W^{-1} + B_t R_t^{-1} B_t)^{-1}$, which both hold by the Woodbury formula. \square

Lemma 7: Given $\gamma, \beta \in \mathbb{R}_{>0}$, and $P, X \in \mathbb{S}_{++}^n$, if

$$(P \prec \gamma^2 I_n) \wedge (P^{-1} - X^{-1} \prec (\gamma^{-2} - (\gamma + \beta)^{-2}) I_n), \quad (64)$$

then $X \prec (\gamma + \beta)^2 I_n$ and $(\forall t \in \mathbb{N}_0) \mathcal{R}_{\gamma+\beta,t}(X) \preceq \mathcal{R}_{\gamma,t}(P)$.

Proof: From (64),

$$0 \prec P^{-1} - \gamma^{-2} I_n \prec X^{-1} - (\gamma + \beta)^{-2} I_n, \quad (65)$$

which implies $X \prec (\gamma + \beta)^2 I_n$. Given $B_t R_t^{-1} B'_t \in \mathbb{S}_+^n$, it also follows from (65) that

$$\begin{aligned} & (X^{-1} - (\gamma + \beta)^{-2} I_n + B_t R_t^{-1} B'_t)^{-1} \\ &\quad \prec (P^{-1} - \gamma^2 I_n + B_t R_t^{-1} B'_t)^{-1}. \end{aligned}$$

As such,

$$\begin{aligned} & Q_t + A'_t (X^{-1} - (\gamma + \beta)^{-2} I_n + B_t R_t^{-1} B'_t)^{-1} A_t \\ &\preceq Q_t + A'_t (P^{-1} - \gamma^2 I_n + B_t R_t^{-1} B'_t)^{-1} A_t, \end{aligned}$$

which is $\mathcal{R}_{\gamma+\beta,t}(X) \preceq \mathcal{R}_{\gamma,t}(P)$ in view of Lemma 6. \square

Proof of Theorem 4: First note that $0 \prec Y \prec Z \Leftrightarrow 0 \prec Z^{-1} \prec Y^{-1}$, given that $(Y^{-1} - Z^{-1})^{-1} = Y + Y(Z - Y)^{-1} Y$ and $(Z - Y)^{-1} = Z^{-1} + Z^{-1}(Y^{-1} - Z^{-1})^{-1} Z^{-1}$ for all

$Y, Z \in \mathbb{S}_{++}^n$, by the Woodbury formula. Since by hypothesis $0 \prec P_{t+1} \prec X_{t+1}$, it follows that $0 \prec X_{t+1}^{-1} \prec P_{t+1}^{-1}$, which yields

$$\lambda_{\max}(X_{t+1}^{-1}) < \lambda_{\max}(P_{t+1}^{-1}) = 1/\lambda_{\min}(P_{t+1}) \leq 1/\lambda \quad (66)$$

and $I_n \prec X_{t+1}^{1/2} P_{t+1}^{-1} X_{t+1}^{1/2}$, whereby $\{\lambda_1, \dots, \lambda_n\} = \text{spec}(X_{t+1}^{1/2} P_{t+1}^{-1} X_{t+1}^{1/2}) = \text{spec}(X_{t+1} P_{t+1}^{-1}) \subset \mathbb{R}_{>1}$. As such, $\log(\lambda_i) > 0$ for all $i \in \{1, \dots, n\}$, and

$$\begin{aligned} \delta(X_{t+1}, P_{t+1}) &= \sqrt{\sum_{i \in \{1, \dots, n\}} (\log(\lambda_i))^2} \\ &\geq \sqrt{\max_{i \in \{1, \dots, n\}} (\log(\lambda_i))^2} \\ &= \sqrt{(\max_{i \in \{1, \dots, n\}} \log(\lambda_i))^2} \\ &= \log(\lambda_{\max}(X_{t+1}^{1/2} P_{t+1}^{-1} X_{t+1}^{1/2})) > 0 \end{aligned} \quad (67)$$

because $\lambda \mapsto \lambda^2$ and $\lambda \mapsto \log(\lambda)$ are both increasing functions over $\lambda \in \mathbb{R}_{>0}$.

Given $\frac{\lambda_{\max}(YZ)}{\lambda_{\max}(Z^{1/2})} \leq \lambda_{\max}(Y)\lambda_{\max}(Z)$ and $\lambda_{\max}(Z^{1/2}) = \sqrt{\lambda_{\max}(Z)}$ for all $Y, Z \in \mathbb{S}_{++}^n$, it follows from (67), (66), (46), and (45), that

$$\begin{aligned} &\lambda_{\max}(P_{t+1}^{-1} - X_{t+1}^{-1}) \\ &\leq \lambda_{\max}(X_{t+1}^{-1/2})\lambda_{\max}(X_{t+1}^{1/2} P_{t+1}^{-1} X_{t+1}^{1/2} - I_n)\lambda_{\max}(X_{t+1}^{-1/2}) \\ &\leq \lambda_{\max}(X_{t+1}^{-1})(\exp(\delta(X_{t+1}, P_{t+1})) - 1) \\ &\leq \frac{1}{\lambda}(\exp(\delta(X_{t+1}, P_{t+1})) - 1) \\ &\leq (\gamma^{-2} - (\gamma + \beta)^{-2}) - \varepsilon. \end{aligned}$$

Therefore, uniformity of the hypothesis $P_{t+1} - \gamma^2 I_n \prec -\varepsilon I_n$ in (4) means that both parts of the sufficient conjunction (64) in Lemma 7 hold with some margin. Thus, $\mathcal{R}_{\gamma+\beta,t}(X_{t+1}) \preceq \mathcal{R}_{\gamma,t}(P_{t+1})$ and $X_{t+1} - (\gamma + \beta)^2 I_n \prec 0$ uniformly. Lemma 5 then applies to yield the stated closed-loop ℓ_2 gain bound. \square

D. Proof of Theorem 2

To complete the proof, the context here returns to the lifted domain of Sections III-A and III-B, starting with a collection of lemmas, which for convenience all refer to the following summary of the two parts of the hypothesis on the baseline gain bound $\gamma \in \mathbb{R}_{>0}$ in Theorem 2.

Hypothesis 1: Part 1) and Part 2) of the hypothesis in Theorem 2 hold, whereby (4) and (5) for some $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^n$ by Remark 7, in which $\tilde{B}_{k|t}$, $\tilde{R}_{k|t}$, $\tilde{Q}_{k|t}$, and $\tilde{A}_{k|t}$ are defined in (29), (30), (31), and (32).

Lemma 8: Under Part 2) of Hypothesis 1 on $\gamma \in \mathbb{R}_{>0}$, the lifted Riccati operator defined in (35) satisfies

$$\tilde{\mathcal{R}}_{\gamma,k|t}(P) \prec \tilde{Q}_{k|t} + \tilde{A}'_{k|t}(\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t}$$

for $P \in \mathbb{S}_{++}^n$ and $t, k \in \mathbb{N}_0$.

Proof: Given $P^{-1} \in \mathbb{S}_{++}^n$ and $\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \in \mathbb{S}_{++}^n$, note that $(P^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1} \prec (\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}$. As such, since $\tilde{A}_{k|t}$ is non singular by Lemma 13,

$$\begin{aligned} &\tilde{A}'_{k|t}(P^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t} \\ &\prec \tilde{A}'_{k|t}(\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t}. \end{aligned}$$

Thus, the result holds in view of (60). \square

Lemma 9: Under Part 2) of Hypothesis 1 on $\gamma \in \mathbb{R}_{>0}$,

$$P \prec X \implies \tilde{\mathcal{R}}_{\gamma,k|t}(P) \prec \tilde{\mathcal{R}}_{\gamma,k|t}(X)$$

for $X, P \in \mathbb{S}_{++}^n$ and $t, k \in \mathbb{N}_0$.

Proof: Given $0 \prec P \prec X$ and $\tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \in \mathbb{S}_{++}^n$, note that $0 \prec X^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t} \prec P^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t}$. Therefore, $(P^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1} \prec (X^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}$, whereby

$$\begin{aligned} \tilde{\mathcal{R}}_{\gamma,k|t}(P) &= \tilde{Q}_{k|t} + \tilde{A}'_{k|t}(P^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t} \\ &\prec \tilde{Q}_{k|t} + \tilde{A}'_{k|t}(X^{-1} + \tilde{B}_{k|t}\tilde{R}_{k|t}^{-1}\tilde{B}'_{k|t})^{-1}\tilde{A}_{k|t} = \tilde{\mathcal{R}}_{\gamma,k|t}(X) \end{aligned}$$

since $\tilde{A}_{k|t}$ is non-singular by Lemma 13. \square

Lemma 10: Under Hypothesis 1 on $\gamma \in \mathbb{R}_{>0}$,

$$(\forall t \in \mathbb{N}_0) \quad P_{t+1} \prec X_{t+1}$$

with $(X_{t+1})_{t \in \mathbb{N}_0}$ as defined in (34) given $T \in \mathbb{N}$.

Proof: By Remark 5, $P_{t+1+dT} = \mathcal{R}_{\gamma,t+1+dT} \circ \cdots \circ \mathcal{R}_{\gamma,t+d(T+1)}(P_{t+1+d(T+1)}) = \tilde{\mathcal{R}}_{\gamma,T|t+1}(P_{t+1+d(T+1)})$. Thus,

$$P_{t+1+dT} \prec \tilde{X}_{t+1} \quad (68)$$

by Lemma 8, where \tilde{X}_{t+1} is defined in (33). As such, $P_{t+1} = \mathcal{R}_{\gamma,t+1} \circ \cdots \circ \mathcal{R}_{\gamma,T-1|t+1}(P_{t+1+dT}) = \tilde{\mathcal{R}}_{\gamma,0|t+1} \circ \cdots \circ \tilde{\mathcal{R}}_{\gamma,T-1|t+1}(P_{t+1+dT}) \prec \tilde{\mathcal{R}}_{\gamma,0|t+1} \circ \cdots \circ \tilde{\mathcal{R}}_{\gamma,T-1|t+1}(\tilde{X}_{t+1}) = X_{t+1}$, where the last equality is (34), and the matrix inequality follows from (68) by Lemma 9. \square

A uniform bound on the Riemannian distance $\delta(\tilde{X}_{t+1}, P_{t+1+dT})$, as per Lemma 11, leads to the same for $\delta(X_{t+1}, P_{t+1})$ as per the subsequent Lemma 12.

Lemma 11: Under Hypothesis 1 on $\gamma \in \mathbb{R}_{>0}$,

$$(\forall t \in \mathbb{N}_0) \quad \delta(\tilde{X}_{t+1}, P_{t+1+dT}) \leq \bar{\delta},$$

with $(\tilde{X}_{t+1})_{t \in \mathbb{N}_0} \in \mathbb{S}_{++}^n$ as defined in (33) given $T \in \mathbb{N}$, where $\delta(\cdot, \cdot)$ is defined in (47), and $\bar{\delta} \in \mathbb{R}_{>0}$ is given in (38).

Proof: From (68), $0 \prec P_{t+1+dT} \prec \tilde{X}_{t+1}$. Therefore, using the same argument underlying (66) and (67), it follows that $\{\lambda_1, \dots, \lambda_n\} = \text{spec}(\tilde{X}_{t+1} P_{t+1+dT}^{-1}) \subset \mathbb{R}_{>1}$, and

$$\begin{aligned} \delta(\tilde{X}_{t+1}, P_{t+1+dT}) &= \sqrt{\sum_{i=1}^n (\log(\lambda_i))^2} \\ &\leq \sqrt{n \max_{i \in \{1, \dots, n\}} (\log(\lambda_i))^2} \\ &\leq \sqrt{n} \log(\lambda_{\max}(P_{t+1+dT}^{-1} \tilde{X}_{t+1} P_{t+1+dT})) \\ &\leq \sqrt{n} \log(\lambda_{\max}(\tilde{X}_{t+1}) / \lambda_{\min}(P_{t+1+dT})). \end{aligned}$$

From (60) and Remark 5, $\lambda_{\min}(P_{t+1+dT}) \geq \lambda_{\min}(\tilde{Q}_{0|t+1+dT}) \geq \inf_{t \in \mathbb{N}_0} \lambda_{\min}(\tilde{Q}_{0|t})$, and since

$$\begin{aligned} \tilde{X}_{t+1} &= \tilde{Q}_{0|t+1+dT} \\ &+ \tilde{A}'_{0|t+1+dT}(\tilde{B}_{0|t+1+dT}\tilde{R}_{0|t+1+dT}^{-1}\tilde{B}'_{0|t+1+dT})^{-1}\tilde{A}_{0|t+1+dT}, \end{aligned}$$

it follows that

$$\delta(\tilde{X}_{t+1}, P_{t+1+dT})$$

$$\leq \sqrt{n} \log \left(\sup_{t \in \mathbb{N}_0} \frac{\lambda_{\max}(\tilde{Q}_{0|t} + \tilde{A}'_{0|t}(\tilde{B}_{0|t}\tilde{R}_{0|t}^{-1}\tilde{B}'_{0|t})^{-1}\tilde{A}_{0|t})}{\lambda_{\min}(\tilde{Q}_{0|t})} \right).$$

This upper bound is finite under part 2) of Assumption 1. \square

Lemma 12: Under Hypothesis 1 on $\gamma \in \mathbb{R}_{>0}$,

$$(\forall t \in \mathbb{N}_0) \delta(X_{t+1}, P_{t+1}) \leq (\bar{\rho})^T \cdot \bar{\delta} \quad (69)$$

with $(X_{t+1})_{t \in \mathbb{N}_0} \in \mathbb{S}_+^n$ as defined in (34) given $T \in \mathbb{N}$, where $\bar{\delta}, \bar{\rho} \in \mathbb{R}_{>0}$ are given in (38) and (39), respectively.

Proof: Given $t \in \mathbb{N}_0$, $\delta(\tilde{X}_{t+1}, P_{t+1+dT}) \leq \bar{\delta}$ by Lemma 11. From (34), $X_{t+1} = \tilde{\mathcal{R}}_{\gamma,0|t+1} \circ \dots \circ \tilde{\mathcal{R}}_{\gamma,T-1|t+1}(\tilde{X}_{t+1})$, and by Lemma 4, $P_{t+1} = \tilde{\mathcal{R}}_{\gamma,0|t+1} \circ \dots \circ \tilde{\mathcal{R}}_{\gamma,T-1|t+1}(P_{t+1+dT})$. Since $\tilde{R}_{k|t} = \tilde{R}_{0|t+dk}$ is non-singular for all $k \in \mathbb{N}_0$, $\tilde{A}_{k|t} = \tilde{A}_{0|t+dk}$ is also non-singular by Lemma 13. Thus, repeated application of Theorem 3 gives

$$\delta(X_{t+1}, P_{t+1}) \leq (\bar{\rho})^T \cdot \delta(\tilde{X}_{t+1}, P_{t+1+dT}) \leq (\bar{\rho})^T \cdot \bar{\delta},$$

with $\bar{\rho} = \sup_{s \in \mathbb{N}_0} 1/(1 + \tilde{\omega}_{0|s}) \geq \tilde{\rho}_t = \sup_{k \in \mathbb{N}_0} 1/(1 + \tilde{\omega}_{k|t})$, where $\tilde{\omega}_{k|t} := \tilde{\epsilon}_{k|t}/\tilde{\zeta}_{k|t} = \tilde{\omega}_{0|t+kd}$, and $\tilde{\epsilon}_{k|t}, \tilde{\zeta}_{k|t} \in \mathbb{R}_{>0}$ are given in (44), and (43), respectively.

It remains to show $\bar{\rho} < 1$, which holds if $\inf_{s \in \mathbb{N}_0} \tilde{\omega}_{0|s} > 0$, as shown below. From (43) and (44), $\inf_{s \in \mathbb{N}_0} \tilde{\omega}_{0|s} \geq \epsilon_0/\zeta_0$, where

$$\zeta_0 := \sup_{s \in \mathbb{N}_0} \zeta_{0|s} \leq 1 / \inf_{s \in \mathbb{N}_0} \lambda_{\min}(\tilde{Q}_{0|s}),$$

because $\lambda_{\min}(\tilde{Q}_{0|s} + \tilde{Q}_{0|s}\tilde{A}_{0|s}^{-1}\tilde{B}_{0|s}\tilde{R}_{k|s}^{-1}\tilde{B}'_{k|s}(\tilde{A}'_{k|s})^{-1}\tilde{Q}_{k|s}) \geq \lambda_{\min}(\tilde{Q}_{0|s})$, and

$$\begin{aligned} \epsilon_0 &:= \inf_{s \in \mathbb{N}_0} \epsilon_{0|s} \\ &= 1 / \sup_{s \in \mathbb{N}_0} \lambda_{\max}(\tilde{Q}_{0|s} + \tilde{A}_{0|s}(\tilde{B}_{0|s}\tilde{R}_{0|s}^{-1}\tilde{B}'_{0|s})^{-1}\tilde{A}_{0|s}). \end{aligned}$$

By the uniform bounds in part 2) of the hypothesis in Assumption 1, $\zeta_0 < +\infty$ and $\epsilon_0 > 0$, whereby $\epsilon_0/\zeta_0 > 0$. \square

Proof of Theorem 2: Given the strict inequality (36), with η as defined in (46), direct calculation yields

$$(\exists \varepsilon \in \mathbb{R}_{>0}) (\bar{\rho})^T \cdot \bar{\delta} \leq \log((\eta - \varepsilon) \cdot \underline{\kappa} + 1).$$

By Lemma 12, $(\forall t \in \mathbb{N}_0) \delta(X_{t+1}, P_{t+1}) \leq (\bar{\rho})^T \cdot \bar{\delta}$. Thus, $(\exists \varepsilon \in \mathbb{R}_{>0}) (\forall t \in \mathbb{N}_0)$

$$\begin{aligned} \delta(X_{t+1}, P_{t+1}) &\leq \log((\eta - \varepsilon) \cdot \underline{\kappa} + 1) \\ &\leq \log((\eta - \varepsilon) \cdot \underline{\lambda} + 1), \end{aligned}$$

since $\inf_{t \in \mathbb{N}_0} \lambda_{\min}(\tilde{Q}_{0|t}) =: \underline{\kappa} \leq \underline{\lambda} := \inf_{t \in \mathbb{N}_0} \lambda_{\min}(P_t)$ in view of Remark 5 and (60). Further, $P_{t+1} \prec X_{t+1}$ by Lemma 10. Therefore, the sufficient condition in (45) is satisfied, and Theorem 4 applies to yield Theorem 2. \square

IV. NUMERICAL EXAMPLE

A periodic numerical example is presented here to explore aspects of the main result Theorem 2. As far as the authors are aware, this result admits the only available practical synthesis of a state feedback control policy guaranteed to achieve a worst-case ℓ_2 gain specification in a general time-varying setting. In the special case of a periodic system, the relevant constants $\underline{\kappa}, \bar{\delta}$, and $\bar{\rho}$ can be determined exactly, which enables assessment of the analytical conservativeness of the result, as noted in Remark 4, and elaborated below in the context of the linearization of a nonlinear system along a periodic trajectory.

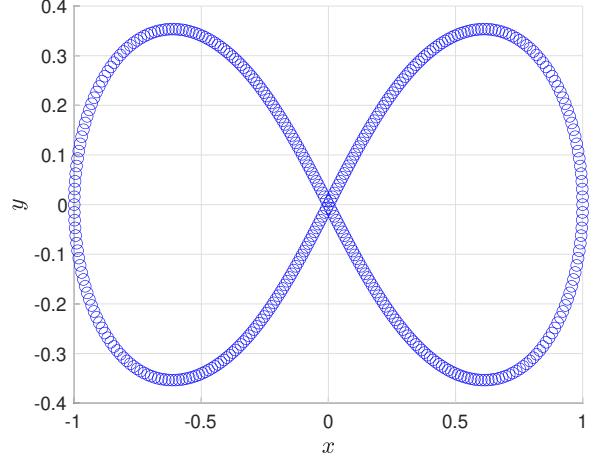


Fig. 1. Nominal trajectory.

With sampling interval $h \in \mathbb{R}_{>0}$, Euler discretization of the unicycle kinematics yields the state space model

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \psi_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + v_t \cos(\psi_t)h \\ y_t + v_t \sin(\psi_t)h \\ \psi_t + r_t h \end{bmatrix},$$

where (x_t, y_t) is the position of the robot in the plane, ψ_t is the yaw angle, v_t is the velocity along the yaw angle, and r_t is the yaw rate, at continuous times $t \cdot h \in \mathbb{R}_{\geq 0}$, for $t \in \mathbb{N}_0$.

Consider the nominal lemniscate state trajectory given by

$$\begin{aligned} x_t^* &= \frac{a \cos(k)}{1 + (\sin(k))^2}, & y_t^* &= \frac{a \sin(k) \cos(k)}{1 + (\sin(k))^2}, \\ \psi_t^* &= \arctan\left(\frac{y_{t+1}^* - y_t^*}{x_{t+1}^* - x_t^*}\right), \end{aligned}$$

for $t \in \mathbb{N}_0$, where $k = \frac{2\pi}{N}(t \bmod N)$, $a = 1$ is the half-width of the lemniscate, and $N \in \mathbb{N}$ is the period. The nominal trajectory in the xy -plane is shown in Figure 1 for $N = 400$. The corresponding nominal inputs are given by

$$\begin{aligned} v_t^* &= \sqrt{(x_{t+1}^* - x_t^*)^2 + (y_{t+1}^* - y_t^*)^2}/h, \\ r_t^* &= (\psi_{t+1}^* - \psi_t^*)/h. \end{aligned}$$

Linearization along the nominal trajectory yields the time-varying discrete time model

$$\begin{bmatrix} x_{t+1} - x_{t+1}^* \\ y_{t+1} - y_{t+1}^* \\ \psi_{t+1} - \psi_{t+1}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & -v_t^* \sin(\psi_t^*)h \\ 0 & 1 & v_t^* \cos(\psi_t^*)h \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t - x_t^* \\ y_t - y_t^* \\ \psi_t - \psi_t^* \end{bmatrix} + \begin{bmatrix} \cos(\psi_t^*)h & 0 \\ \sin(\psi_t^*)h & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} v_t - v_t^* \\ r_t - r_t^* \end{bmatrix} + h w_t,$$

in the form (1) with disturbance input $h w_t$.

For the performance output z_t in (2) with

$$Q_t = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \quad \text{and} \quad R_t = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix},$$

it can be verified with sampling interval $h = 0.05$, and baseline ℓ_2 gain bound $\gamma = 125$, that parts 1) and 2) of the hypothesis

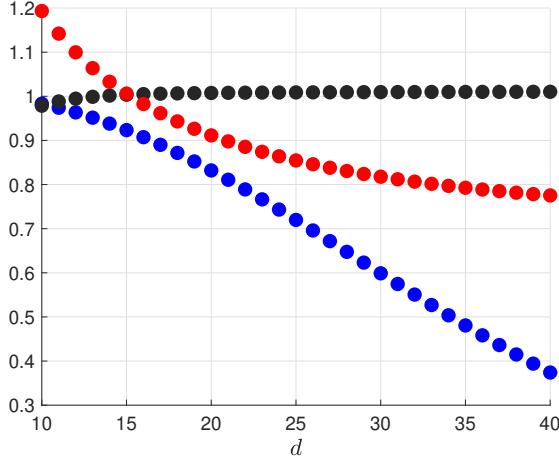


Fig. 2. $\underline{\kappa}$ (black), $0.1\bar{\delta}$ (red), and $\bar{\rho}$ (blue), in Theorem 2 for lifting steps $d \in [10 : 40]$.

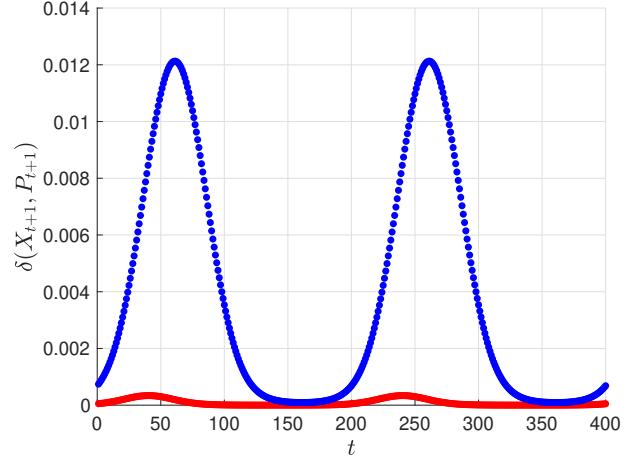


Fig. 4. $\delta(X_{t+1}, P_{t+1})$ for: (i) $T = 1, d = 40 \Rightarrow \bar{\rho}^T \bar{\delta} = 2.90$ (blue); and (ii) $T = 2, d = 40 \Rightarrow \bar{\rho}^T \bar{\delta} = 1.08$ (red).

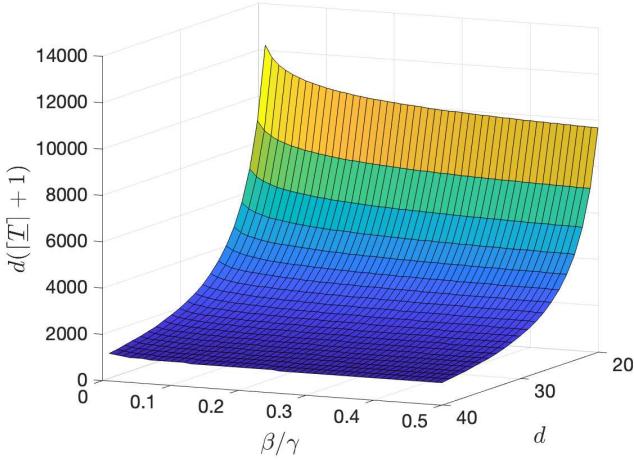


Fig. 3. Preview steps $d([\underline{T}] + 1)$ sufficient according to Theorem 2 for different performance loss bounds β relative to baseline gain bound $\gamma = 125$ (1 – 50%), and lifting steps $d \in [20 : 40]$.

in Theorem 2 hold for all lifting steps $d \in [10 : 40]$, set in accordance with Assumption 2. This amounts to checking the finite number of conditions across one period of the problem data for part 2), and exploiting the periodic structure of the Riccati recursion (5) as shown in, e.g., [12], [13]. In particular, it can be established that there exists periodic $(P_t)_{t \in \mathbb{N}_0} \subset \mathbb{S}_{++}^3$ such that (4) and (5) both hold, and the infinite-preview control policy (8) achieves $(\forall w \in \ell_2) \|z\|_2 \leq \gamma \|h w\|_2 = 6.25 \|w\|_2$.

In Figure 2, it is shown how $\underline{\kappa}$, $\bar{\delta}$, and $\bar{\rho}$ in Theorem 2 depend on the number of lifting steps $d \in [10 : 40]$; see Remark 3. In the periodic setting of this example it is possible to evaluate these quantities exactly; one period suffices to determine the sup and inf in (37)–(40). Increasing d decreases both the approximate Riccati solution error bound $\bar{\delta}$ from Lemma 11, and the Riccati contraction rate bound $\bar{\rho}$ in the proof of Lemma 12, whereas $\underline{\kappa}$ remains roughly constant.

The number $d([\underline{T}] + 1)$ of preview steps that is according to Theorem 2 sufficient for the control policy (10), with $(X_{t+1})_{t \in \mathbb{N}_0}$ as per (34), to achieve the performance loss bound

β , relative to the baseline gain bound $\gamma = 125$, is shown in Figure 3 for $d \in [20 : 40]$. When $d = 40$, this estimate of a sufficient number of preview steps lies between only 2 and 3 system periods (800 – 1200 steps) for all $\beta \in [0.01\gamma, 0.5\gamma]$, i.e., 1 – 50% performance loss. The measured approximation error $\delta(X_{t+1}, P_{t+1}) \leq (\bar{\rho})^T \bar{\delta}$ (see Lemma 12) is shown in Figure 4 for $T = 1$ and $T = 2$, which corresponds to just 80 and 120 preview horizon steps, respectively. In both cases, the measured Riemannian distance is two to three orders of magnitude smaller than the bound used to prove Theorem 2.

V. CONCLUSION

A finite receding-horizon preview approximation of the standard infinite-preview state feedback ℓ_2 gain control policy is considered for linear time-varying systems. Under uniform controllability and observability, and given a suitably large baseline gain bound, the strict contraction of lifted Riccati operators is exploited to construct a feedback gain at each step that depends on a finite-horizon preview of the problem data. The approximation achieves a prescribed infinite-horizon performance loss bound. As such, the proposed approach constitutes a practical controller synthesis method with a prescribed closed-loop ℓ_2 gain performance guarantee.

As future work, it would be of interest to extend the proposed synthesis to dynamic output feedback controllers for time-varying systems. In this case, the standard policy structure is more complicated, and the main challenge lies in effectively accounting for the coupling between the two Riccati recursions that arise in the analysis of performance loss. Relaxing controllability and observability to stabilizability and detectability could also be considered. Another direction is to apply the approach in formulating terminal ingredients for receding horizon schemes with hard input and state constraints, along the lines of [31], [33], [34] for time-invariant systems. Finally, investigating non-stationary structures that constrain variation in the model data to enable tighter estimates of analysis bounds would also be of interest.

APPENDIX

Lemma 13: Given $t, k \in \mathbb{N}_0$, if $\tilde{R}_{k|t}$ in (30) is non-singular, then $\tilde{A}_{k|t}$ in (32) is non-singular. Further, $\tilde{A}_{k|t}^{-1}$ is uniformly bounded if $\tilde{R}_{k|t}^{-1}$ is uniformly bounded.

Proof: Consider the matrix

$$H_{k|t} := \begin{bmatrix} A_{k|t} & [B_{k|t} \quad F_{k|t}] \\ \begin{bmatrix} D'_{k|t} \\ E'_{k|t} \end{bmatrix} C_{k|t} & \tilde{R}_{k|t} \end{bmatrix}, \quad (70)$$

where $A_{k|t}$, $B_{k|t}$, $F_{k|t}$, $C_{k|t}$, $D_{k|t}$, and $E_{k|t}$, are as defined in Lemmas 1 and 2. With $\tilde{R}_{k|t}$ non-singular, the corresponding Schur complement of $H_{k|t}$ is equal to $\tilde{A}_{k|t}$. As such, $H_{k|t}$ is non-singular if and only if $\tilde{A}_{k|t}$ is non-singular. By Assumption 1, $A_{k|t} = \Phi_{t+dt+d, t+dk}$ is also invertible. Thus, the alternative Schur complement of $H_{k|t}$ in (70) is given by

$$S_{k|t} = \tilde{R}_{k|t} - \begin{bmatrix} D'_{k|t} \\ E'_{k|t} \end{bmatrix} C_{k|t} A_{k|t}^{-1} [B_{k|t} \quad F_{k|t}] = \Upsilon_{k|t} + U_{k|t}, \quad (71)$$

where

$$\Upsilon_{k|t} := \begin{bmatrix} R_{k|t} & 0_{md,nd} \\ 0_{nd,md} & -\gamma^2 I_{nd} \end{bmatrix}$$

and

$$\begin{aligned} U_{k|t} &:= \begin{bmatrix} D'_{k|t} \\ E'_{k|t} \end{bmatrix} [D_{k|t} \quad E_{k|t}] - \begin{bmatrix} D'_{k|t} \\ E'_{k|t} \end{bmatrix} C_{k|t} A_{k|t}^{-1} [B_{k|t} \quad F_{k|t}] \\ &= \begin{bmatrix} \Xi_{k|t}' \\ 0_{nd,n} \quad I_{nd} \end{bmatrix} (\Lambda_{k|t}^{-1})' \Gamma_{k|t}' \\ &\times \Gamma_{k|t} (\Lambda_{k|t}^{-1} - \Lambda_{k|t}^{-1} \begin{bmatrix} I_n \\ 0_{nd,n} \end{bmatrix} A_{k|t}^{-1} [0_{n,nd} \quad I_n] \Lambda_{k|t}^{-1}) \\ &\times \begin{bmatrix} \Xi_{k|t}' \\ 0_{nd,n} \quad I_{nd} \end{bmatrix}' \end{aligned}$$

with $\Lambda_{k|t}$, $\Xi_{k|t}$, and $\Gamma_{k|t}$ as per in (16), (17), and (19), respectively. Therefore, bounded invertibility of $S_{k|t} = \Upsilon_{k|t} + U_{k|t}$ is equivalent to bounded invertibility of $\tilde{A}_{k|t}$.

Noting that

$$\Lambda_{k|t}^{-1} = \begin{bmatrix} G_{k|t} & \Delta_{k|t} \\ A_{k|t} & F_{k|t} \end{bmatrix},$$

where

$$G_{k|t} = \begin{bmatrix} I_n & \Phi'_{t+dk+1, t+dk} & \cdots & \Phi'_{t+d(k+1)-1, t+dk} \end{bmatrix}',$$

and $\Delta_{k|t} \in \mathbb{R}^{nd \times nd}$ is block lower triangular with $n \times n$ zero blocks on the diagonal, it follows by direct calculation that

$$\begin{aligned} (\Lambda_{k|t}^{-1} - \Lambda_{k|t}^{-1} \begin{bmatrix} I_n \\ 0_{nd,n} \end{bmatrix} A_{k|t}^{-1} [0_{n,nd} \quad I_n] \Lambda_{k|t}^{-1}) \\ = \begin{bmatrix} 0_{nd,n} & \Delta_{k|t} - G_{k|t} \Lambda_{k|t}^{-1} F_{k|t} \\ 0_{n,n} & 0_{n,nd} \end{bmatrix}, \end{aligned}$$

where $\Delta_{k|t} - G_{k|t} \Lambda_{k|t}^{-1} F_{k|t} = \Delta_{k|t} - G_{k|t} \Phi_{t+d(k+1), t+dk}^{-1} F_{k|t}$ is block upper triangular. Similarly,

$$\begin{aligned} &\left[\begin{bmatrix} \Xi_{k|t}' \\ 0_{nd,n} \quad I_{nd} \end{bmatrix} (\Lambda_{k|t}^{-1})' \Gamma_{k|t}' \right] \\ &= \left[\begin{bmatrix} 0_{nd,n} & \text{diag}(B'_{t+dk}, \dots, B'_{t+d(k+1)-1}) \\ 0_{nd,n} & I_{nd} \end{bmatrix} \right] \\ &\quad \times \begin{bmatrix} G'_{k|t} & A'_{k|t} \\ \Delta'_{k|t} & F'_{k|t} \end{bmatrix} \begin{bmatrix} \text{diag}((Q_{t+kd}^{1/2})', \dots, (Q_{t+d(k+1)-1}^{1/2})') \\ 0_{nd,n} \end{bmatrix} \\ &= \text{diag}(B'_{t+dk}, \dots, B'_{t+d(k+1)-1}) \\ &\quad \times \Delta'_{k|t} \text{diag}((Q_{t+kd}^{1/2}), \dots, (Q_{t+d(k+1)-1}^{1/2})), \end{aligned}$$

which is block upper triangular with $n \times n$ zero blocks along the diagonal. As such, $U_{k|t}$ is block upper triangular, with all zero diagonal blocks, and thus,

$$\begin{aligned} \text{spec}(S_{k|t}) &= \text{spec}(\Upsilon_{k|t} + U_{k|t}) \\ &= \text{spec}(\Upsilon_{k|t}) = \{\lambda_1(\Upsilon_{k|t}), \dots, \lambda_{(m+n)d}(\Upsilon_{k|t})\}, \end{aligned}$$

with real valued $\lambda_1(\Upsilon_{k|t}) \geq \dots \geq \lambda_{(m+n)d}(\Upsilon_{k|t})$, and

$$c := \min\{|\lambda_i(\Upsilon_{k|t})| : i \in \{1, \dots, (m+n)d\}\} \in \mathbb{R}_{>0}.$$

Application of Weyl's inequality for singular values [38, Thm. 3.3.2] then yields

$$\begin{aligned} 0 < (m+n)d \cdot c &\leq \left| \prod_{i=1}^{(m+n)d} \lambda_i(\Upsilon_{k|t}) \right| \\ &= \left| \prod_{i=1}^{(m+n)d} \lambda_i(S_{k|t}) \right| = \prod_{i=1}^{(m+n)d} \sqrt{\lambda_i(S'_{k|t} S_{k|t})}. \end{aligned}$$

Hence, the smallest eigenvalue of $S'_{k|t} S_{k|t}$ satisfies

$$\begin{aligned} \sqrt{\lambda_{(m+n)d}(S'_{k|t} S_{k|t})} &\geq \frac{((m+n)d \cdot c)}{((m+n)d - 1) \cdot \sqrt{\lambda_1(S'_{k|t} S_{k|t})}} \\ &\geq c \in \mathbb{R}_{>0}, \end{aligned}$$

uniformly because, in view of (71), the largest eigenvalue $\lambda_1(S'_{k|t} S_{k|t})$ is uniformly bounded. Therefore, $S_{k|t}$ is non-singular with uniformly bounded inverse, whereby the same holds for $\tilde{A}_{k|t}$, as claimed. \square

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