

# Variance-Refined In-Diameter Lower Bound for the First Dirichlet Eigenvalue

Thomas Schürmann

Düsseldorf, Germany

## Abstract

Let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold with nonempty boundary and  $n \geq 2$ . Assume that  $\text{Ric}(M) \geq (n - 1)K$  for some  $K > 0$  and that  $\partial M$  has nonnegative mean curvature with respect to the outward unit normal. Denote by  $\lambda$  the first Dirichlet eigenvalue of the Laplacian. Ling's gradient-comparison method [4] provides an explicit lower bound for  $\lambda$  in terms of  $K$  and the in-diameter  $\tilde{d}$  (twice the maximal distance from a point of  $M$  to  $\partial M$ ). We isolate the only step in Ling's argument that loses quantitative information: a Jensen-Hölder averaging that replaces a nonconstant one-dimensional comparison function by its mean. Using the uniform strong convexity of  $x \mapsto x^{-1/2}$  on  $(0, 1]$ , we refine this averaging by a variance term and thereby retain part of the discarded oscillation. This yields an explicit closed-form in-diameter bound that is strictly stronger than Ling's estimate for every  $K > 0$ .

## 1 Introduction and main result

Let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold with nonempty boundary  $\partial M$ . Assume that the Ricci curvature satisfies

$$\text{Ric}(M) \geq (n - 1)K \tag{1}$$

for some constant  $K > 0$ , and that the mean curvature of  $\partial M$  with respect to the outward unit normal is nonnegative. Let  $\lambda$  denote the first Dirichlet eigenvalue of the Laplacian on  $M$ .

A classical result of Reilly [1] yields the Lichnerowicz-type estimate

$$\lambda \geq nK. \tag{2}$$

This estimate contains no diameter information and becomes trivial in the limiting case  $K = 0$ . For closed manifolds, the case  $K = 0$  in (1) corresponds to nonnegative Ricci curvature; in that setting Li-Yau [2] and Zhong-Yang [3] obtained sharp diameter-type lower bounds.

In [4], Ling proved a unified in-diameter estimate. Writing

$$\tilde{d} = 2 \sup_{x \in M} \text{dist}(x, \partial M), \tag{3}$$

Ling's main theorem yields

$$\lambda \geq \frac{(n - 1)K}{2} + \frac{\pi^2}{\tilde{d}^2}. \tag{4}$$

The proof is based on a refined gradient comparison and introduces an auxiliary function  $\xi$  on  $[-\pi/2, \pi/2]$ . After reducing the eigenvalue estimate to a one-dimensional integral inequality, Ling applies Hölder's inequality to replace the nonconstant comparison function  $z(t)$  by its mean. This step discards information about the oscillation of  $z$ .

The aim of this note is to retain this oscillation quantitatively via a variance refinement. We obtain an explicit strengthening of Ling's bound in closed form.

**Theorem 1.1** (Variance-refined in-diameter bound). *Let  $(M, g)$  satisfy the assumptions above and let  $\lambda$  be the first Dirichlet eigenvalue. Set*

$$\alpha = \frac{(n-1)K}{2}, \quad D = \frac{\pi^2}{\tilde{d}^2}, \quad V = 4\zeta(3) - \frac{1}{3}(\pi^2 + 4). \quad (5)$$

Then

$$\lambda \geq \frac{(\alpha + D) + \sqrt{(\alpha + D)^2 + V\alpha^2}}{2}. \quad (6)$$

In particular, since  $V = \text{Var}(\xi) > 0$  (see Remark 4.2), one has the strict improvement

$$\lambda > \alpha + D \quad \text{whenever } K > 0. \quad (7)$$

**Remark 1.2.** *The bound (6) is derived from the same one-dimensional comparison inequality as in [4] and is obtained in closed form, without taking a maximum with the estimate  $\lambda \geq nK$ . We use Reilly's estimate  $\lambda \geq nK$  only to ensure that the parameter  $\delta = \alpha/\lambda$  lies in  $[0, 1/2]$ .*

## 2 The comparison inequality from Ling's argument

We briefly recall the one-dimensional inequality that concludes Ling's proof of Theorem 1. The full gradient comparison argument can be found in [4]; for our purposes we only need the resulting integral inequality and the explicit auxiliary function  $\xi$ .

**Lemma 2.1** (The auxiliary function  $\xi$ ). *Define  $\xi : [-\pi/2, \pi/2] \rightarrow \mathbb{R}$  by*

$$\xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \pi^2/4}{\cos^2 t}. \quad (8)$$

*Then  $\xi$  is smooth and even on  $[-\pi/2, \pi/2]$ , satisfies  $\xi(\pm\pi/2) = 0$ , and*

$$\int_0^{\pi/2} \xi(t) dt = -\frac{\pi}{2}. \quad (9)$$

*Moreover  $\xi(t) \leq 0$  for  $t \in [0, \pi/2]$ .*

*Proof.* These properties are established in Lemma 5 of [4], where  $\xi$  is constructed explicitly and shown to satisfy a linear ODE. For completeness, we derive (9) from the first-order relation

$$(\xi(t) \cos^2 t)' = 4t \cos^2 t \quad (10)$$

together with the boundary value  $\xi(\pi/2) = 0$ . Integrating (10) from  $t$  to  $\pi/2$  gives

$$-\xi(t) \cos^2 t = \int_t^{\pi/2} 4s \cos^2 s ds, \quad \text{hence} \quad \xi(t) = -4 \sec^2 t \int_t^{\pi/2} s \cos^2 s ds.$$

Using Fubini's theorem, we compute

$$\begin{aligned} \int_0^{\pi/2} \xi(t) dt &= -4 \int_0^{\pi/2} \sec^2 t \left( \int_t^{\pi/2} s \cos^2 s ds \right) dt \\ &= -4 \int_0^{\pi/2} s \cos^2 s \left( \int_0^s \sec^2 t dt \right) ds = -4 \int_0^{\pi/2} s \cos^2 s \tan s ds \\ &= -4 \int_0^{\pi/2} s \sin s \cos s ds = -2 \int_0^{\pi/2} s \sin(2s) ds = -\frac{\pi}{2}, \end{aligned}$$

which proves (9). □

**Lemma 2.2** (Ling's integral inequality). *Let  $\lambda$  be the first Dirichlet eigenvalue, set  $\alpha = (n - 1)K/2$  and  $\delta = \alpha/\lambda$ . Define*

$$z(t) = 1 + \delta \xi(t), \quad t \in [0, \pi/2]. \quad (11)$$

Then

$$\sqrt{\lambda} \frac{\tilde{d}}{2} \geq \int_0^{\pi/2} \frac{dt}{\sqrt{z(t)}}. \quad (12)$$

*Proof.* This is [4, (44)], obtained by integrating a gradient comparison inequality along a minimizing geodesic from a maximum point of the first Dirichlet eigenfunction to the boundary and then letting the normalization parameter  $b \downarrow 1$ . By definition of the in-diameter, the length of such a geodesic is at most  $\tilde{d}/2$ . The function  $z$  is the explicit comparison function used in [4, (35)].  $\square$

**Remark 2.3.** Ling derives his explicit bound by applying Hölder's inequality (equivalently, Jensen's inequality for the convex function  $x \mapsto x^{-1/2}$  with respect to the normalized measure  $\frac{2}{\pi} dt$ ) to (12):

$$\int_0^{\pi/2} \frac{dt}{\sqrt{z(t)}} \geq \frac{\left( \int_0^{\pi/2} dt \right)^{3/2}}{\left( \int_0^{\pi/2} z(t) dt \right)^{1/2}} = \frac{\pi}{2} \cdot \frac{1}{\sqrt{1 - \delta}}, \quad (13)$$

because  $\frac{2}{\pi} \int_0^{\pi/2} z(t) dt = 1 - \delta$  by (9). This step ignores that  $z$  is nonconstant. Equality in this Jensen/Hölder step would force  $z$  to be constant almost everywhere; since  $\xi$  is not constant, this cannot occur when  $\delta > 0$ . We replace it by a variance-sensitive estimate.

### 3 A strong-convexity refinement

The key observation is that  $x \mapsto x^{-1/2}$  is uniformly strongly convex on  $(0, 1]$ .

**Proposition 3.1** (Variance improvement for  $x^{-1/2}$ ). *Let  $z : [0, \pi/2] \rightarrow (0, 1]$  be measurable and set*

$$\mu = \frac{2}{\pi} \int_0^{\pi/2} z(t) dt. \quad (14)$$

Then

$$\frac{2}{\pi} \int_0^{\pi/2} \frac{dt}{\sqrt{z(t)}} \geq \frac{1}{\sqrt{\mu}} + \frac{3}{8} \text{Var}(z), \quad (15)$$

where

$$\text{Var}(z) = \frac{2}{\pi} \int_0^{\pi/2} (z(t) - \mu)^2 dt. \quad (16)$$

*Proof.* Let  $f(x) = x^{-1/2}$  on  $(0, 1]$ . Then

$$f''(x) = \frac{3}{4} x^{-5/2} \geq \frac{3}{4} \quad \text{for all } x \in (0, 1]. \quad (17)$$

Fix  $\mu \in (0, 1]$  and use the second-order Taylor expansion of  $f$  at  $\mu$  with integral remainder. Using the lower bound on  $f''$ , we obtain for every  $x \in (0, 1]$ :

$$f(x) \geq f(\mu) + f'(\mu)(x - \mu) + \frac{3}{8}(x - \mu)^2. \quad (18)$$

Apply this pointwise with  $x = z(t)$  and integrate over  $t \in [0, \pi/2]$ . The linear term vanishes because  $\mu$  is the mean of  $z$ :

$$\frac{2}{\pi} \int_0^{\pi/2} (z(t) - \mu) dt = 0. \quad (19)$$

This yields (15).  $\square$

We now apply Proposition 3.1 to the specific choice  $z(t) = 1 + \delta\xi(t)$  from (11). Note that  $z(t) \leq 1$  on  $[0, \pi/2]$  since  $\xi(t) \leq 0$  by Lemma 2.1. Moreover, (10) implies the pointwise lower bound  $\xi(t) \geq -2$  on  $[0, \pi/2]$ : indeed, integrating (10) from  $t$  to  $\pi/2$  gives

$$-\xi(t) \cos^2 t = \int_t^{\pi/2} 4s \cos^2 s \, ds,$$

hence

$$-\xi(t) = 4 \sec^2 t \int_t^{\pi/2} s \cos^2 s \, ds.$$

Setting  $I(t) = \int_t^{\pi/2} s \cos^2 s \, ds$  and  $F(t) = \frac{1}{2} \cos^2 t - I(t)$ , we have  $F(\pi/2) = 0$  and

$$F'(t) = -\cos t \sin t + t \cos^2 t = \cos^2 t (t - \tan t) \leq 0,$$

since  $\tan t \geq t$  for  $t \in [0, \pi/2]$ . Thus  $F(t) \geq 0$  and  $I(t) \leq \frac{1}{2} \cos^2 t$ , which yields  $-\xi(t) \leq 2$ . Since Reilly's estimate gives  $\delta = \alpha/\lambda \leq (n-1)/(2n) < 1/2$ , we obtain  $z(t) = 1 + \delta\xi(t) \geq 1 - 2\delta > 0$ . Therefore  $z(t) \in (0, 1]$  and Proposition 3.1 applies.

**Lemma 3.2** (Mean and variance of  $z(t) = 1 + \delta\xi(t)$ ). *Let  $z(t) = 1 + \delta\xi(t)$  on  $[0, \pi/2]$ , where  $\xi$  is given by (8). Then*

$$\mu = \frac{2}{\pi} \int_0^{\pi/2} z(t) \, dt = 1 - \delta, \quad (20)$$

and

$$\text{Var}(z) = \delta^2 \text{Var}(\xi), \quad \text{Var}(\xi) = \mathbb{E}[\xi^2] - 1, \quad \mathbb{E}[\xi^2] = \frac{2}{\pi} \int_0^{\pi/2} \xi(t)^2 \, dt. \quad (21)$$

*Proof.* The identity (20) follows immediately from (9). Since  $\mathbb{E}[\xi] = \frac{2}{\pi} \int_0^{\pi/2} \xi(t) \, dt = -1$ , we have

$$z(t) - \mu = 1 + \delta\xi(t) - (1 - \delta) = \delta(\xi(t) + 1), \quad (22)$$

and hence (21).  $\square$

Combining Proposition 3.1 and Lemma 3.2 yields the refined lower bound on the integral in (12):

**Proposition 3.3** (Variance-refined integral estimate). *Let  $z(t) = 1 + \delta\xi(t)$  as in (11). Then*

$$\int_0^{\pi/2} \frac{dt}{\sqrt{z(t)}} \geq \frac{\pi}{2} \left( \frac{1}{\sqrt{1-\delta}} + \frac{3}{8} \text{Var}(\xi) \delta^2 \right). \quad (23)$$

*In particular, if  $\text{Var}(\xi) > 0$  and  $\delta > 0$ , the right-hand side is strictly larger than  $\frac{\pi}{2}(1-\delta)^{-1/2}$ .*

*Proof.* This is (15) with  $\mu = 1 - \delta$  and  $\text{Var}(z) = \delta^2 \text{Var}(\xi)$ .  $\square$

## 4 Evaluation of $\text{Var}(\xi)$

We now compute the constant  $\text{Var}(\xi)$  in closed form. Set

$$V := \text{Var}(\xi). \quad (24)$$

**Lemma 4.1** (Second moment of  $\xi$ ). *For  $\xi$  defined by (8),*

$$\int_0^{\pi/2} \xi(t)^2 \, dt = \pi \left( 2\zeta(3) - \frac{\pi^2 + 1}{6} \right). \quad (25)$$

*Consequently,*

$$V = 4\zeta(3) - \frac{1}{3}(\pi^2 + 4). \quad (26)$$

*Proof.* A detailed reduction of  $\int_0^{\pi/2} \xi(t)^2 dt$  to a short list of logarithmic integrals is given in Appendix A. We record the remaining (standard) Fourier-series evaluations. First, rewrite (8) as

$$\xi(t) = 1 + 2t \tan t + \left( t^2 - \frac{\pi^2}{4} \right) \sec^2 t. \quad (27)$$

Introduce  $L(t) = \log(\cos t)$  on  $(0, \pi/2)$ , so that  $L'(t) = -\tan t$  and  $L''(t) = -\sec^2 t$ . Then (27) becomes

$$\xi(t) = 1 - 2t L'(t) - \left( t^2 - \frac{\pi^2}{4} \right) L''(t). \quad (28)$$

Expanding  $\xi(t)^2$  and integrating by parts repeatedly reduces  $\int_0^{\pi/2} \xi(t)^2 dt$  to a linear combination of the three classical integrals

$$\int_0^{\pi/2} L(t) dt, \quad \int_0^{\pi/2} t L(t) dt, \quad \int_0^{\pi/2} t^2 L(t) dt, \quad (29)$$

plus elementary polynomial integrals. The singular boundary terms cancel because  $\xi(\pi/2) = 0$  and  $\xi(t) \cos^2 t$  is smooth up to  $t = \pi/2$ .

It therefore suffices to evaluate the three logarithmic integrals above. For  $|t| < \pi/2$  one has the absolutely convergent Fourier series

$$\log(2 \cos t) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos(2kt)}{k}. \quad (30)$$

Integrating term-by-term yields

$$\int_0^{\pi/2} \log(\cos t) dt = -\frac{\pi}{2} \log 2. \quad (31)$$

A second integration against  $t$  and  $t^2$ , using

$$\int_0^{\pi/2} t \cos(2kt) dt = \frac{(-1)^k - 1}{4k^2}, \quad \int_0^{\pi/2} t^2 \cos(2kt) dt = \frac{\pi(-1)^k}{4k^2}, \quad (32)$$

shows that

$$\int_0^{\pi/2} t \log(\cos t) dt = -\frac{\pi^2}{8} \log 2 - \frac{7}{16} \zeta(3), \quad (33)$$

and

$$\int_0^{\pi/2} t^2 \log(\cos t) dt = -\frac{\pi^3}{24} \log 2 - \frac{\pi}{4} \zeta(3), \quad (34)$$

where we use the classical identity for the alternating zeta value

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^3} = (1 - 2^{-2}) \zeta(3) = \frac{3}{4} \zeta(3). \quad (35)$$

Substituting these evaluations into the reduction from Appendix A yields (25). Finally, since  $\mathbb{E}[\xi] = -1$ , we have

$$V = \mathbb{E}[\xi^2] - \mathbb{E}[\xi]^2 = \mathbb{E}[\xi^2] - 1 = \frac{2}{\pi} \int_0^{\pi/2} \xi(t)^2 dt - 1, \quad (36)$$

which gives (26).  $\square$

**Remark 4.2.** *The constant  $V$  is a variance, hence nonnegative by definition. Moreover  $V > 0$  because  $\xi$  is not (a.e.) constant: using (27) one has  $\xi(0) = 1 - \pi^2/4 < 0$  (since  $\pi > 2$ ), while  $\xi(\pi/2) = 0$  by Lemma 2.1. For reference,  $V \approx 0.1850261456$ .*

## 5 From the refined integral to an explicit eigenvalue bound

We now combine the refined integral estimate with a simple one-root majorization to obtain the explicit bound (6).

**Lemma 5.1** (One-root lower bound). *Let  $V > 0$  be as in (26). For every  $\delta \in [0, 1/2]$  one has*

$$\frac{1}{\sqrt{1-\delta}} + \frac{3}{8}V\delta^2 \geq \frac{1}{\sqrt{1-\delta - \frac{V}{4}\delta^2}}. \quad (37)$$

*Proof.* Fix  $\delta \in [0, 1/2]$  and consider the function

$$h(s) = (1 - \delta - s\delta^2)^{-1/2} \quad \text{for } s \in \left[0, \frac{V}{4}\right]. \quad (38)$$

Then  $h$  is increasing and convex in  $s$ . By convexity,

$$h\left(\frac{V}{4}\right) - h(0) \leq \frac{V}{4} h'\left(\frac{V}{4}\right). \quad (39)$$

Since

$$h'(s) = \frac{\delta^2}{2}(1 - \delta - s\delta^2)^{-3/2}, \quad (40)$$

we obtain

$$h\left(\frac{V}{4}\right) \leq \frac{1}{\sqrt{1-\delta}} + \frac{V}{8}\delta^2 \left(1 - \delta - \frac{V}{4}\delta^2\right)^{-3/2}. \quad (41)$$

For  $\delta \in [0, 1/2]$  we have the uniform lower bound

$$1 - \delta - \frac{V}{4}\delta^2 \geq \frac{1}{2} - \frac{V}{16}. \quad (42)$$

We now bound this factor uniformly using only explicit analytic inequalities (in particular, without inserting a decimal approximation for  $V$ ).

Recall from (26) that  $V = 4\zeta(3) - \frac{1}{3}(\pi^2 + 4)$ . We first claim that  $V < \frac{1}{4}$ . By the integral test,

$$\zeta(3) = \sum_{m=1}^{\infty} \frac{1}{m^3} < \sum_{m=1}^5 \frac{1}{m^3} + \int_5^{\infty} x^{-3} dx = \left(1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125}\right) + \frac{1}{50} = \frac{260423}{216000}. \quad (43)$$

Hence  $4\zeta(3) < \frac{260423}{54000}$ . On the other hand, the classical bound  $\pi > \frac{223}{71}$  implies  $\pi^2 > \left(\frac{223}{71}\right)^2 = \frac{49729}{5041} > \frac{493}{50}$ , so

$$\frac{\pi^2 + 4}{3} > \frac{\frac{493}{50} + 4}{3} = \frac{231}{50}. \quad (44)$$

Consequently,

$$V = 4\zeta(3) - \frac{1}{3}(\pi^2 + 4) < \frac{260423}{54000} - \frac{231}{50} = \frac{10943}{54000} < \frac{1}{4}. \quad (45)$$

Using  $V < \frac{1}{4}$  we get  $\frac{1}{2} - \frac{V}{16} > \frac{1}{2} - \frac{1}{64} = \frac{31}{64}$ , and therefore

$$\left(\frac{1}{2} - \frac{V}{16}\right)^{-3/2} < \left(\frac{31}{64}\right)^{-3/2} = \left(\frac{64}{31}\right)^{3/2} < 3, \quad (46)$$

since  $\left(\frac{64}{31}\right)^{3/2} < 3 \iff \left(\frac{64}{31}\right)^3 < 9$ , and indeed  $\left(\frac{64}{31}\right)^3 = \frac{262144}{29791} < 9$  because  $9 \cdot 29791 = 268119 > 262144$ . Therefore

$$\left(1 - \delta - \frac{V}{4}\delta^2\right)^{-3/2} \leq 3, \quad (47)$$

and hence

$$h\left(\frac{V}{4}\right) \leq \frac{1}{\sqrt{1-\delta}} + \frac{3V}{8} \delta^2, \quad (48)$$

which is exactly (37).  $\square$

**Remark 5.2.** Lemma 5.1 is used only to turn the additive variance correction in (23) into the simple denominator in (52), and hence into the closed-form bound (6). If one keeps (23) without Lemma 5.1, one gets a slightly stronger (but implicit) lower bound for  $\lambda$ .

**Lemma 5.3** (Range of  $\delta$ ). *Under the standing geometric assumptions,*

$$0 \leq \delta = \frac{\alpha}{\lambda} \leq \frac{n-1}{2n} < \frac{1}{2}. \quad (49)$$

*Proof.* By Reilly's estimate  $\lambda \geq nK$  and  $\alpha = (n-1)K/2$  we obtain

$$\delta = \frac{\alpha}{\lambda} \leq \frac{(n-1)K/2}{nK} = \frac{n-1}{2n}. \quad (50)$$

$\square$

*Proof of Theorem 1.1.* Set  $\delta = \alpha/\lambda$ . By Lemma 2.2, Proposition 3.3, Lemma 4.1, and Lemmas 5.1–5.3, we have

$$\sqrt{\lambda} \frac{\tilde{d}}{2} \geq \int_0^{\pi/2} \frac{dt}{\sqrt{z(t)}} \geq \frac{\pi}{2} \cdot \frac{1}{\sqrt{1-\delta - \frac{V}{4}\delta^2}}. \quad (51)$$

Squaring and writing  $D = \pi^2/\tilde{d}^2$  yields

$$\lambda \geq \frac{D}{1-\delta - \frac{V}{4}\delta^2}. \quad (52)$$

Substitute  $\delta = \alpha/\lambda$  into (52) and clear denominators:

$$\lambda \left(1 - \frac{\alpha}{\lambda} - \frac{V}{4} \frac{\alpha^2}{\lambda^2}\right) \geq D. \quad (53)$$

Equivalently,

$$\lambda^2 - (\alpha + D)\lambda - \frac{V}{4}\alpha^2 \geq 0. \quad (54)$$

Since  $\lambda > 0$ , this quadratic inequality implies

$$\lambda \geq \frac{(\alpha + D) + \sqrt{(\alpha + D)^2 + V\alpha^2}}{2}, \quad (55)$$

which is (6).

Finally,  $V > 0$  by (26), so the square-root term is strictly larger than  $\alpha + D$  whenever  $\alpha > 0$ , proving the strict improvement when  $K > 0$ .  $\square$

## 6 Concluding remark

In order to make the size of the refinement more transparent, let

$$B_{\text{Ling}} := \alpha + D \quad \text{and} \quad B_{\text{var}} := \frac{(\alpha + D) + \sqrt{(\alpha + D)^2 + V \alpha^2}}{2}$$

denote, respectively, Ling's lower bound and the variance-refined bound (6). A direct computation yields

$$\frac{B_{\text{var}}}{B_{\text{Ling}}} = \frac{1 + \sqrt{1 + V \left( \frac{\alpha}{\alpha+D} \right)^2}}{2},$$

so in particular

$$1 < \frac{B_{\text{var}}}{B_{\text{Ling}}} \leq \frac{1 + \sqrt{1 + V}}{2} \approx 1.0443.$$

Thus the improvement over Ling's estimate is universally bounded by about 4.5% in relative terms. The gain is governed by the dimensionless ratio

$$\frac{\alpha}{D} = \frac{(n-1)K \tilde{d}^2}{2\pi^2},$$

and becomes more pronounced when the curvature term  $\alpha$  dominates the diameter term  $D$ . As a model case, for a geodesic hemisphere of the round  $n$ -sphere scaled so that  $\text{Ric} = (n-1)K$ , one has  $\tilde{d} = \pi/\sqrt{K}$  and hence  $D = K$ , so that  $\alpha/(\alpha+D) = (n-1)/(n+1)$ ; this yields a relative gain in the lower bound of about 3% already for  $n = 10$ , and it approaches the universal limit  $(1 + \sqrt{1 + V})/2 - 1 \approx 4.4\%$  as  $n \rightarrow \infty$ .

In summary the improvement (6) is small but uniform: it depends only on the universal constant  $V = \text{Var}(\xi) > 0$  that measures the nonconstancy of the one-dimensional comparison function  $z(t)$ . It shows that the Hölder/Jensen reduction in [4] is not optimal and can be sharpened while retaining a closed-form dependence on  $K$  and  $d$ .

## A Details for Lemma 4.1

Write  $a := \pi/2$  and  $A := \pi^2/4$ , and set  $P(t) := t^2 - A$ . For  $0 < \varepsilon < a$  define  $u := a - \varepsilon$  and  $I_\varepsilon := \int_0^u \xi(t)^2 dt$ , where  $\xi$  is given by (27). Since  $u < a$ , all functions below are smooth on  $[0, u]$ , so the integrations by parts are classical; we take the limit  $\varepsilon \downarrow 0$  at the end.

### A.1. Reduction to logarithmic integrals

Expanding (27) gives

$$\xi(t)^2 = 1 + 4t^2 \tan^2 t + P(t)^2 \sec^4 t + 4t \tan t + 2P(t) \sec^2 t + 4tP(t) \tan t \sec^2 t. \quad (56)$$

Using  $\tan^2 t = \sec^2 t - 1$ , we rewrite the first two terms on the right as  $1 + 4t^2 \tan^2 t = 1 - 4t^2 + 4t^2 \sec^2 t$  and hence

$$\xi(t)^2 = 1 - 4t^2 + 4t \tan t + 2(3t^2 - A) \sec^2 t + P(t)^2 \sec^4 t + 4tP(t) \tan t \sec^2 t. \quad (57)$$

We now integrate each group in (57) over  $[0, u]$ . First, integrating the  $\sec^2$ -term by parts using  $(\tan t)' = \sec^2 t$  yields

$$\int_0^u 2(3t^2 - A) \sec^2 t dt = 2(3u^2 - A) \tan u - 12 \int_0^u t \tan t dt. \quad (58)$$

Second, we treat the  $\sec^4$ -term using the elementary identity

$$(\tan t \sec^2 t)' = 3 \sec^4 t - 2 \sec^2 t, \quad (59)$$

which follows by direct differentiation. Multiplying (59) by  $P(t)^2$  and integrating by parts gives

$$\int_0^u P(t)^2 \sec^4 t \, dt = \frac{1}{3} P(u)^2 \tan u \sec^2 u - \frac{1}{3} \int_0^u (P(t)^2)' \tan t \sec^2 t \, dt + \frac{2}{3} \int_0^u P(t)^2 \sec^2 t \, dt. \quad (60)$$

Since  $(P(t)^2)' = 4tP(t)$ , the middle integral in (60) combines with the last term in (57). Using moreover

$$\tan t \sec^2 t = \frac{1}{2} (\sec^2 t)' \quad (61)$$

and integrating by parts once more, one arrives at the identity

$$I_\varepsilon = u - \frac{4}{3}u^3 + B_\varepsilon + \frac{2\pi^2}{3} \int_0^u \log(\cos t) \, dt - 8 \int_0^u t^2 \log(\cos t) \, dt, \quad (62)$$

where the boundary term is

$$\begin{aligned} B_\varepsilon := & \frac{2}{3}(3u^2 - A) \tan u + \frac{1}{3}P(u)^2 \tan u \sec^2 u \\ & + \frac{4}{3}uP(u) \sec^2 u + \frac{2}{3}P(u)^2 \tan u + \frac{8}{3}(u^3 - Au) \log(\cos u). \end{aligned} \quad (63)$$

The algebraic manipulations leading to (62) use only (58)–(61) and repeated integration by parts.

## A.2. Passage to the endpoint

As  $\varepsilon \downarrow 0$  one has  $u \uparrow a$  and  $\tan u = \cot \varepsilon$ ,  $\sec^2 u = \csc^2 \varepsilon$ ,  $\log(\cos u) = \log(\sin \varepsilon)$ . Using  $P(u) = u^2 - A = -\pi\varepsilon + O(\varepsilon^2)$  and the standard expansions

$$\cot \varepsilon = \frac{1}{\varepsilon} - \frac{\varepsilon}{3} + O(\varepsilon^3), \quad \csc^2 \varepsilon = \frac{1}{\varepsilon^2} + \frac{1}{3} + O(\varepsilon^2), \quad \log(\sin \varepsilon) = \log \varepsilon + O(\varepsilon^2), \quad (64)$$

at 0, one checks that all singular contributions in (63) cancel and

$$\lim_{\varepsilon \downarrow 0} B_\varepsilon = -\frac{2\pi}{3}. \quad (65)$$

Letting  $\varepsilon \downarrow 0$  in (62) therefore yields the reduction

$$\int_0^{\pi/2} \xi(t)^2 \, dt = \frac{\pi}{2} - \frac{\pi^3}{6} - \frac{2\pi}{3} + \frac{2\pi^2}{3} \int_0^{\pi/2} \log(\cos t) \, dt - 8 \int_0^{\pi/2} t^2 \log(\cos t) \, dt. \quad (66)$$

Substituting the two logarithmic integrals evaluated in the proof of Lemma 4.1 into (66) gives (25).

## References

- [1] R. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. **26** (1977), 459–472.
- [2] P. Li and S.-T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, in *Geometry of the Laplace operator* (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., Vol. XXXVI, Amer. Math. Soc., Providence, RI, 1980, pp. 205–239.
- [3] J. Q. Zhong and H. C. Yang, *On the estimate of the first eigenvalue of a compact Riemannian manifold*, Sci. Sinica Ser. A **27** (1984), no. 12, 1265–1273.
- [4] J. Ling, *A lower bound of the first Dirichlet eigenvalue of a compact manifold with positive Ricci curvature*, International Journal of Mathematics **17** (2006), no. 5, 605–617.