

$$x(1 - t(x + x^{-1}))F(x; t) = x - tF(0; t)$$

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The purpose of these notes is to introduce some of the problems the enumeration of lattice walks is dedicated to and familiarize with some of the arguments they can be addressed with. We will discuss the enumeration of lattice walks, their generating functions, and the functional equations they satisfy. We will focus on algebraic methods for manipulating and solving these equations. Elementary power series algebra will play a prominent role, computer algebra too, but we will repeatedly digress and present ideas and methods of different kind whenever it seems appropriate. The exposition is organized around the most simple yet non-trivial problem in the enumeration of lattice walks. The intention is to illustrate different techniques without getting technical.

The purpose of the notes is by no means to give an exhaustive overview of the literature on the enumeration of lattice walks. The subject is much too vast. As just pointed out, we limit ourselves to the study of functional equations and the algebraic (and related) methods that (could have) originated from that. The pointers to the literature we give are therefore limited to works that are relevant in this context. For a broader and more detailed exposition of the subject we refer to the surveys by Krattenthaler [37] and Humphreys [30].

At the time of writing it was not clear at all that these notes would grow to such a size. It was a surprise to see how things worked out, and it was fun to work them out. We hope the reader can feel some of the joy this was done with.

The notes are organized in 14 sections. Section 1 provides the basic definitions and introduces lattice walks, their generating functions, and the functional equations they satisfy. In Sections 2, 3, 4, 5 and 6 we solve such equations using methods that rely on elementary power series algebra only. We discuss the kernel method, Wiener-Hopf factorization, the orbit-sum method, compositional inverses and Lagrange inversion as well as the invariant method. These methods give rise to expressions for the generating functions that beg for combinatorial explanations. Such explanations are given in Sections 7, 8, 9 and 10 and relate them to continued fractions, the reflection principle, a combinatorial factorization of lattice walks and the cycle lemma. In Section 11 and Section 12 we illustrate the paradigm of guess and prove and discuss the usefulness of the notion of D-finiteness. They connect enumerative combinatorics with computer and differential algebra. Section 13 discusses asymptotics and the usefulness of generalized series solutions of linear recurrences. The references much of this text relies on are presented in Section 14. The methods illustrated here were invented there, or appeared there in much more generality. Some methods for the

enumeration of lattice walks are not presented here though they do fall into the framework of this text. Still, we do not want to leave them unmentioned here, but give pointers to the literature. They too can be found in Section 14.

0 Notation

We denote by \mathbb{N} , \mathbb{Z} , \mathbb{Q} and \mathbb{C} the non-negative integers, the integers and the rational and complex numbers, respectively. We denote by x and t variables, and for notational convenience we write $\bar{x} := x^{-1}$. The set of powers series in t whose coefficients are Laurent polynomials in x over \mathbb{Q} is denoted by $\mathbb{Q}[x, \bar{x}][[t]]$. Similarly, we write $\mathbb{Q}[x][[t]]$ and $\mathbb{Q}[\bar{x}][[t]]$ for the subset of series whose coefficients are polynomials in x and \bar{x} , respectively, and $\mathbb{Q}[[t]]$ for those series which do not involve x . Given a series $F(x; t)$ in x and t and an integer n , we denote by $[t^n]F(x; t)$ the coefficient of t^n , and we define the *non-negative part* of $F(x; t)$ by

$$[x^{\geq}]F(x; t) := \sum_{n \geq 0} x^n [t^n]F(x; t).$$

1 Generating functions and functional equations

Definition 1. A *lattice walk* is a sequence P_0, \dots, P_n of points of \mathbb{Z}^d , $d \in \mathbb{N}$. The points P_0 and P_n are its starting and end point, respectively, the consecutive differences $P_{i+1} - P_i$ are its steps, and n is its length. Fixing $R, S \subseteq \mathbb{Z}^d$, we denote by $f(Q; n)$ the number of walks in R that start at the origin, consist of n steps all of which are taken from S , and end in $Q \in \mathbb{Z}^d$. The series

$$F(x; t) := \sum_{n \geq 0} \left(\sum_{Q \in R} f(Q; n) x^Q \right) t^n$$

is called the *generating function* of the sequence $(f(Q; n))$.

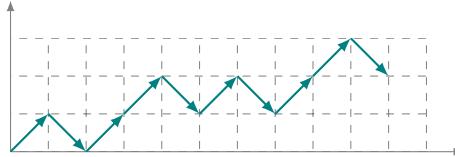


Figure 1: A graphical representation of a lattice walk from $(0, 0)$ to $(10, 1)$ whose steps are either $(1, 1)$ or $(1, -1)$.

Enumerative combinatorics asks for the value of $f(Q; n)$, whether there is a “nice” formula for it in terms of Q and n , and what its asymptotic behavior is as n goes to infinity. A first step in answering these questions is to consider the associated generating function and to locate it in the hierarchy of rational, algebraic, D-finite and D-algebraic functions. As soon as a representation of the generating function in terms of an algebraic or a differential equation is known, there are classes of algorithms that can be applied to derive information about its coefficient sequence. The starting point of the investigation is a functional equation for the generating function. In general this equation can be complicated, but sometimes it is easily written down.

Let $F(x; t)$ be the generating function of walks in \mathbb{N} which start at 0 and whose steps are either 1 or -1 . A lattice walk of positive length that ends at i is either a walk to $i - 1$ followed by a right step, or a walk to $i + 1$ followed by a left step. Hence,

$$f(i; n) = f(i - 1; n - 1) + f(i + 1; n - 1), \quad n \geq 1. \quad (1)$$

Furthermore, there is only one walk of length 0, and there are no walks that end on $i < 0$. Thus,

$$f(i; 0) = \delta_{i, 0} \quad \text{and} \quad f(i; n) = 0 \quad \text{if } i < 0. \quad (2)$$

Multiplying equation (1) by $x^i t^n$ and summing over all $i \geq 0$ and $n \geq 1$, it easily follows that

$$F(x; t) = 1 + t(x + \bar{x})F(x; t) - t\bar{x}F(0; t). \quad (3)$$

This equation uniquely determines $F(x; t)$ as it implies

$$\begin{aligned} [t^0]F(x; t) &= 1, \\ [t^{n+1}]F(x; t) &= (x + \bar{x})[t^n]F(x; t) - \bar{x}[t^n]F(0; t), \quad n \geq 0, \end{aligned}$$

for the coefficient sequence $([t^n]F(x; t))$. The first terms of $F(x; t)$ are easily computed based thereon. We have

$$F(x; t) = 1 + xt + (1 + x^2)t^2 + (2x + x^3)t^3 + (2 + 3x^2 + x^4)t^4 + O(t^5)$$

from which we can again read off some of the values of $f(i; n)$.

The existence and uniqueness of a solution of the equation in $\mathbb{Q}[x][[t]]$ also follows from Banach's fixed point theorem. For any $F, G \in \mathbb{Q}[x][[t]]$, define their distance from each other by

$$d(F, G) := 2^{-\text{val}(F - G)},$$

where val denotes the valuation

$$\text{val}(F) := \min\{n : [t^n]F \neq 0\}.$$

It turns $\mathbb{Q}[x][[t]]$ into a metric space on which the operator

$$F(x; t) \mapsto 1 + t(x + \bar{x})F(x; t) - t\bar{x}F(0; t)$$

acts as a contraction. Hence, the equation has a unique solution in $\mathbb{Q}[x][[t]]$, and the sequence of iterates of any element in $\mathbb{Q}[x][[t]]$ converges to it.

To find a closed form expression for $F(x; t)$ it is convenient to rewrite equation 3 in the form

$$x(1 - t(x + \bar{x}))F(x; t) = x - tF(0; t). \quad (4)$$

This equation will be referred to as the *kernel equation*. The coefficient of $F(x; t)$ is the *kernel polynomial*. The latter will play, as the name indicates, an important role in the different solution strategies subsumed under what is known as the *kernel method*.

2 Classical kernel method

The kernel equation involves two unknowns one of which involves only one of the variables. The classical kernel method is based on the idea of eliminating one of the unknowns by coupling x and t without altering the other unknown.

Viewed as a polynomial in x , the kernel polynomial

$$K(x, t) := x(1 - t(x + \bar{x}))$$

has two roots,

$$x_0(t) = \frac{1 - \sqrt{1 - 4t^2}}{2t} \quad \text{and} \quad x_1(t) = \frac{1 + \sqrt{1 - 4t^2}}{2t},$$

only one of which, namely $x_0(t)$, can be interpreted as an element of $\mathbb{Q}[[t]]$. The composition $F(x_0(t), t)$ is therefore well-defined, and the substitution of $x_0(t)$ for x in (4) results in

$$0 = x_0(t) - tF(0; t).$$

Hence,

$$F(0; t) = \frac{x_0(t)}{t} \quad \text{and} \quad F(x; t) = \frac{1 - \bar{x}x_0(t)}{1 - t(x + \bar{x})}.$$

This can also be interpreted as follows: the right-hand side of equation (4) is a polynomial in x over $\mathbb{Q}((t))$. Its degree is 1, and so is its leading coefficient, and its only root is $tF(0; t)$. Since $x_0(t)$ is a root of $K(x, t)$, and since it is a power series in t , and hence can be substituted for x into $F(x; t)$, it is also a root of the left-hand side of the equation, and therefore necessarily also a root of the right-hand side of the equation. Consequently,

$$tF(0; t) = x_0(t),$$

from which we deduce the expressions for $F(0; t)$ and $F(x; t)$ from before.

3 Wiener-Hopf factorization

Instead of solving equation (4) by eliminating the unknown on the left-hand side of the equation one can just as well eliminate $F(0; t)$ on the right-hand side. There is more than one way of doing so. The one we present here relates to what is known as Wiener-Hopf factorization. It is based on the observation that the kernel polynomial factors into

$$K(x, t) = -t(x - x_0)(x - x_1),$$

where

$$(x - x_0)^{-1} \in \mathbb{Q}[\bar{x}]((t)) \quad \text{and} \quad (x - x_1) \in \mathbb{Q}[x]((t)).$$

This can be exploited by dividing the kernel equation by $x - x_0$. It results in

$$-t(x - x_1)F(x; t) = \frac{1 - \bar{x}tF(0; t)}{1 - \bar{x}x_0}$$

whose left-hand side lies in $\mathbb{Q}[x][[t]]$, and whose right-hand side is an element of $\mathbb{Q}[\bar{x}][[t]]$ whose constant term with respect to x is 1. Discarding all terms which involve negative powers of x gives an equation which does not involve $F(0; t)$ and from which we deduce that

$$F(x; t) = -\frac{1}{t(x - x_1)}.$$

4 Orbit-sum method

Alternatively one can exploit that $1 - t(x + \bar{x})$ is invariant under the natural action of the group generated by the transformation $x \mapsto \bar{x}$. The orbit of the kernel equation under this action consists of two equations, equation (4) and

$$\bar{x}(1 - t(x + \bar{x}))F(\bar{x}; t) = \bar{x} - tF(0; t).$$

Subtracting the latter from the first and dividing by $x(1 - t(x + \bar{x}))$ results in

$$F(x; t) - \bar{x}^2 F(\bar{x}; t) = \frac{1 - \bar{x}^2}{1 - t(x + \bar{x})}.$$

As $F(x; t)$ only involves non-negative powers of x , while $\bar{x}^2 F(\bar{x}; t)$ only involves negative powers of x , we can eliminate $\bar{x}^2 F(\bar{x}; t)$ by discarding all negative powers of x and conclude that

$$F(x; t) = [x^{\geq}] \frac{1 - \bar{x}^2}{1 - t(x + \bar{x})}.$$

The expression of $F(x; t)$ as the non-negative part of a rational function allows us to recover the expression from before. A partial fraction decomposition with respect to x results in

$$\frac{1 - \bar{x}^2}{1 - t(x + \bar{x})} = \frac{\bar{x}}{t} - \frac{\bar{x}^2 x_0}{t(1 - \bar{x}x_0)} + \frac{\bar{x}}{t(1 - x\bar{x}_1)}$$

to which $[x^{\geq}]$ easily applies when $\frac{\bar{x}^2 x_0}{1 - \bar{x}x_0}$ and $\frac{\bar{x}}{1 - x\bar{x}_1}$ are interpreted as series that are elements of $\mathbb{Q}[x, \bar{x}][[t]]$.

5 Compositional inverses

So far we have considered the kernel polynomial as a polynomial in x , but we can just as well consider it as a polynomial in t . Its only root is then

$$G(x) = \frac{x}{1 + x^2}.$$

Interpreted as an element of $\mathbb{Q}[[x]]$ it has valuation 1. So the composition $F(x; G(x))$ is well-defined, and the substitution of $G(x)$ for t in (4) results in

$$\frac{x}{1 + x^2} F\left(0; \frac{x}{1 + x^2}\right) = x. \quad (5)$$

It follows that $G(x)$ is the compositional right-inverse of $xF(0; x)$. We next show how this implies that $F(0; t)$, and consequently $F(x; t)$, is algebraic.

Definition 2. A series $F(x; t) \in \mathbb{Q}[x][[t]]$ is *algebraic* over $\mathbb{Q}(x, t)$, if there is a non-zero polynomial $P \in \mathbb{Q}(x, t)[Y]$ such that

$$P(x, t, F(x; t)) = 0.$$

If P has leading coefficient 1, and its degree is minimal among all non-zero polynomials in $\mathbb{Q}(x, t)[Y]$ which have $F(x; t)$ as a root, then P is called the *minimal polynomial* of $F(x; t)$.

Let

$$P(x, Y) = x - (1 + x^2)Y$$

be the numerator of the minimal polynomial of $G(x)$. Then $P(x, G(x)) = 0$, and so $P(tF(0; t), t) = 0$ by replacing x by $tF(0; t)$. The latter holds because $G(x)$ has valuation 1, which guarantees that it has a right-inverse, and because the composition of series is associative, which implies that every right-inverse is also left-inverse. Therefore, $F(0; t)$ is algebraic and a multiple of its minimal polynomial is

$$P(tY, t) = -t(1 - Y + t^2Y^2).$$

The expression for $F(0; t)$ and $F(x; t)$ from before can be recovered from that. Moreover, the coefficients of $F(0; t)$ can be computed from (5) using *Lagrange inversion*. For the convenience of the reader we state it here [27].

Proposition 1. Let $H(x)$ be a power series in x whose valuation is 1, and let $G(x)$ be its compositional inverse, that is, the series satisfying $G(H(x)) = x$. Then

$$[x^n]G^k(x) = \frac{k}{n}[x^{-k}]H^{-n}(x), \quad n \neq 0.$$

Applying Lagrange inversion to $x/(1 + x^2)$ we find

$$f(0; n) = [t^{n+1}]tF(0; t) = \frac{1}{n+1}[t^{-1}](t + t^{-1})^{n+1} = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{2}, \\ \frac{1}{n/2+1} \binom{n}{n/2} & \text{else.} \end{cases} \quad (6)$$

6 Invariant method

The symmetry of the kernel polynomial the orbit-sum method is based on is not the only symmetry that can be exploited to solve the kernel equation. There is also a symmetry that appears in $K(x, t)$ having a (non-trivial) rational multiple in $\mathbb{Q}(x) + \mathbb{Q}(t)$ such as

$$-\frac{K(x, t)}{xt} = x + \bar{x} - \frac{1}{t}. \quad (7)$$

Formulated in a more algebraic way, the subfields of the function field of $K(x, t)$ generated by x and t , respectively, have a non-trivial intersection. It is generated by any of t and $x + \bar{x}$.

Now how is this useful? We can eliminate all terms involving x from the right-hand side of equation (7) by subtracting the kernel equation and adding

$$K(x, t) \frac{F(x; t)}{xtF(0; t)} = \frac{1}{tF(0; t)} - \bar{x}.$$

Multiplying the resulting equation by $tF(0; t)$ gives

$$K(x, t)(-\bar{x}F(0; t) - tF(0; t)F(x; t) + \bar{x}F(x; t)) = 1 - F(0; t) + t^2F(0; t)^2$$

The valuations of $K(x; t)$ and $1 - F(0; t) + t^2F(0; t)^2$ are zero with respect to both x and t , and hence so are the valuations of the coefficient of $K(x, t)$ on the left-hand side of the equation. Its right-hand side is therefore a multiple of $K(x, t)$ in $\mathbb{Q}[[x, t]]$ that does not depend on x . Consider now a monomial order on monomials in x and t with $x < t$. The smallest term in $K(x, t)$ is x , so the smallest term on the left-hand side of the equation is a multiple of x . But the right-hand side does not involve x . Thus it is zero and

$$1 - F(0; t) + t^2F(0; t)^2 = 0.$$

7 Continued fractions

The algebraic equation a generating function satisfies often has a combinatorial interpretation. The equation

$$F(0; t) = 1 + t^2F(0; t)^2, \quad (8)$$

reflects that any walk in \mathbb{N} that starts and ends in 0 is either of length 0, or a walk that consists of a right step, followed by an excursion which does not visit 0, possibly of length 0, followed by a left step, possibly continued by another excursion. This decomposition of a lattice walk is referred to as the “first passage” decomposition. It is distinguished from the “arch” decomposition. The latter decomposes a path into a sequence of triples consisting of a right step, a path which does not visit 0 and a left step. It gives rise to the following equivalent equation

$$F(0; t) = \frac{1}{1 - t^2F(0; t)}.$$

Applied recursively it implies a representation of $F(0; t)$ as the *continued fraction*

$$F(0; t) = \cfrac{1}{1 - \cfrac{t^2}{1 - \cfrac{t^2}{1 - \dots}}}. \quad (9)$$

The finite continued fractions (F_k) defined by

$$F_0 = 1, \quad \text{and} \quad F_k = \frac{1}{1 - t^2F_{k-1}}, \quad k \geq 1,$$

are called the *convergents* of the continued fraction 9. Of course the convergents (F_k) converge to $F(0; t)$ in the sense that

$$\lim_{k \rightarrow \infty} \text{val}(F(0; t) - F_k) = 0.$$

The convergents have a combinatorial meaning too: F_k is the generating function of excursions on $\{0, 1, \dots, k\}$ that start at 0 and take their steps from $\{-1, 1\}$. This is clearly true for F_0 , and since any excursion on $\{0, 1, \dots, k\}$ is a sequence of walks that start and end with a right and left step, respectively, and whose other steps form a walk on $\{0, \dots, k-1\}$, it is easily seen that it also holds for $k \geq 1$. The question how to compute the coefficients of the series expansion of F_k efficiently relates the enumeration of lattice walks to the theory of continued fractions and orthogonal polynomials.

8 Reflection principle

It is natural to ask whether also the other expressions we have found for $F(x; t)$ and its evaluation give more insight into the combinatorial problem, apart from the fact that they represent the generating function of the counting sequence. It is obvious how to interpret the right-hand side of

$$F(x; t) = \frac{1 - \bar{x}x_0}{1 - t(x + \bar{x})}$$

combinatorially having in mind that $F(0; t) = x_0(t)/t$ is the generating function of walks in \mathbb{N} that start and end in 0 and $\frac{1}{1-t(x+\bar{x})}$ represents the generating function of walks in \mathbb{Z} that start at 0. It is the generating function of walks in \mathbb{Z} that start at 0 minus the generating function of the subset of those walks which do not lie in \mathbb{N} . It might be less obvious how to interpret the right-hand side of

$$F(x; t) = [x^{\geq 0}] \frac{1 - \bar{x}^2}{1 - t(x + \bar{x})} \quad (10)$$

in this respect. The right-hand side of the equation reflects the existence of a bijection between walks in \mathbb{Z} that start at 0, end in \mathbb{N} but do not lie in \mathbb{N} and walks that start at -2 and end in \mathbb{N} . It is defined by reflecting the initial part of a walk up to its last visit of -1 at -1 . See the figure below for an illustration. As the bijection neither affects the length nor the end point of a walk, this provides another argument why identity (10) holds.

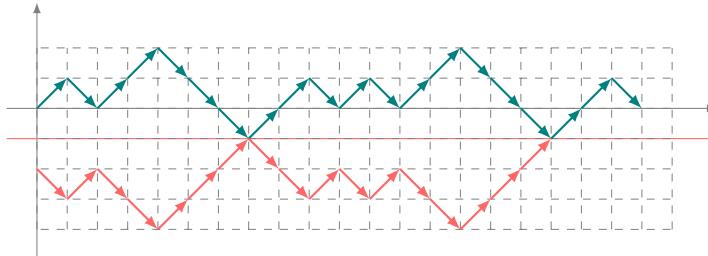


Figure 2: A lattice walk in \mathbb{Z}^2 that starts at $(0,0)$, consists of steps $(1,1)$ or $(1,-1)$, and crosses the x -axis, and its reflection along the line $y = -1$.

This combinatorial argument is referred to as the *reflection principle*, and the variant of the kernel method which leads to the expression of $F(x; t)$ as the non-negative part of a rational function is therefore sometimes referred to as the algebraic variant of the reflection principle.

The reflection principle also allows one to directly determine the number $f(i; n)$ of walks in \mathbb{N} that start at 0, end in i and have length n . It is

$$f(i; n) = \binom{n}{\frac{n-i}{2}} - \binom{n}{\frac{n-i-2}{2}} = \frac{n-i}{n+i+2} \binom{n}{\frac{n-i}{2}},$$

since the binomial coefficient $\binom{n}{\frac{n-i}{2}}$ counts the number of walks in \mathbb{Z} that start at 0, end at i and have length n .

9 Combinatorial factorization

In Section 3 we met a factorization of the kernel polynomial $K(x, t)$ to which we now give a combinatorial meaning. Let us slightly rewrite it in the form

$$K(x, t) = x(1 - \bar{x}x_0)tx_1(1 - xx_1^{-1}),$$

and point out that $K(x, t)$ is the product of x and series s_- , s_0 and s_+ such that

$$s_-^{\pm 1} \in \mathbb{Q}[\bar{x}][[t]], \quad s_+^{\pm 1} \in \mathbb{Q}[x][[t]], \quad \text{and} \quad s_0^{\pm 1} \in \mathbb{Q}[[t]].$$

Furthermore,

$$[x^0]s_- = 1 = [x^0]s_+.$$

The latter makes this decomposition unique. This has the following consequence. Any walk on \mathbb{Z} that starts at 0 and takes its steps from $\{-1, 1\}$ can be written as the concatenation of three walks w_- , w_0 and w_+ that have the following property: w_- is a walk that starts at 0 and does not end at a positive integer, w_0 is a walk in \mathbb{N} that starts and ends at 0, and w_+ is a walk in \mathbb{N} that starts at 0. Requiring that w_0 is of maximal length makes this decomposition unique. This translates into a factorization of the generating function of walks on \mathbb{Z}

$$\frac{1}{1 - t(x + \bar{x})} = F_-(x; t)F(0; t)F_+(x; t),$$

where again

$$F_-^{\pm 1} \in \mathbb{Q}[\bar{x}][[t]], \quad F_+^{\pm 1} \in \mathbb{Q}[x][[t]] \quad \text{and} \quad F_0 \in \mathbb{Q}[[t]]$$

and

$$[x^0]F_- = 1 = [x^0]F_+.$$

It now follows that

$$F_-(x; t) = \frac{1}{1 - \bar{x}x_0}, \quad F(0; t) = \frac{1}{tx_1} \quad \text{and} \quad F_+(x; t) = \frac{1}{1 - xx_1^{-1}}.$$

Furthermore, since a walk w on \mathbb{Z} is a walk on \mathbb{N} if and only if w_- is of length zero, we find that

$$F(x; t) = F(0; t)F_+(x; t) = \frac{1}{t(x_1 - x)}.$$

10 Cycle lemma

We have seen how Lagrange inversion implies that

$$f(0; n) = \frac{1}{n+1} [t^{-1}] (t + t^{-1})^{n+1}.$$

Since $[t^{-1}] (t + t^{-1})^{n+1}$ is the number of walks in \mathbb{Z} which start at 0, take their steps from $\{1, -1\}$ and end in -1 this identity begs for a combinatorial explanation. The cycle lemma provides such an explanation. There are different versions of the cycle lemma, the one we present here is due to Spitzer and also known as Spitzer's lemma. For details and proofs we refer to [37] from which the following lemma and proposition are taken.

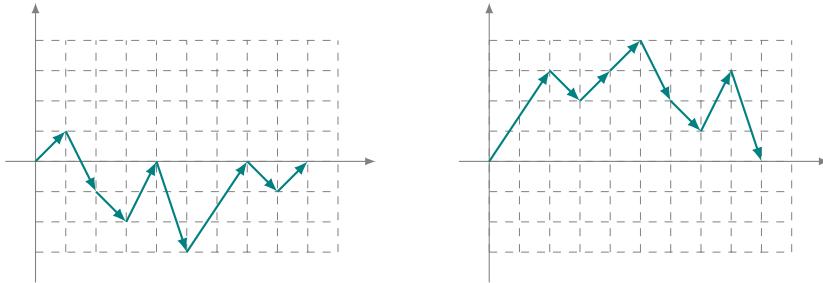
Lemma 1. Let a_1, \dots, a_N be a sequence of real numbers such that $a_1 + \dots + a_N = 0$ and no partial sum of consecutive a_i 's read cyclically vanishes. Then there is a unique cyclic permutation $a_i, a_{i+1}, \dots, a_N, a_1, \dots, a_{i-1}$ with the property that for $j = 1, \dots, N$ the sum of its first j terms is non-negative.

A consequence of the cycle lemma is the following proposition.

Proposition 2. Let r and s be two positive integers which are relative prime. Then the number of paths in \mathbb{N}^2 that start at $(0,0)$, consist of steps $(1,0)$ and $(0,1)$, end at (r,s) and stay weakly below the line $ry = sx$ is $\frac{1}{r+s} \binom{r+s}{s}$.

Proof. Let \mathfrak{P} be the set of lattice walks that start at $(0,0)$ and end at (r,s) and let \mathfrak{G} be the group generated by the permutation $(1,2,\dots,r+s)$. Clearly, \mathfrak{G} acts on \mathfrak{P} by permuting the steps of a walk cyclically. Since \mathfrak{P} has cardinality $\binom{r+s}{s}$ and since each orbit has cardinality $r+s$ the set \mathfrak{P} partitions into $\frac{1}{r+s} \binom{r+s}{s}$ orbits. We finish the proof of the proposition by observing that each orbit contains exactly one walk that lies weakly below the line $ry = sx$. To see this note that we can encode a walk by a sequence of numbers by replacing a step $(1,0)$ by s and a step $(0,1)$ by $-r$. Since r and s are relatively prime the cycle lemma applies. \square

Proposition 2 provides a combinatorial explanation for equation 6 as there is a bijection between walks of length $2n$ in \mathbb{N} that start at 0, take their steps from $\{-1, 1\}$ and end at 0 and walks of length $2n+1$ in \mathbb{N}^2 that start at $(0,0)$, consist of steps $(1,0)$ or $(0,1)$, end at $(r,s) = (n, n+1)$ and stay weakly below the line $ry = sx$.



11 Guess and prove

Equation (8) shows that the generating function $F(0;t)$ of excursions is algebraic and its minimal polynomial over $\mathbb{Q}(t)$ is $P_0(t,Y) = Y^2 - Y/t^2 + 1/t^2$. Since $F(x;t)$ is a rational function in $F(0;t)$ over $\mathbb{Q}(x,t)$ closure properties of algebraic functions [35] imply that $F(x;t)$ is algebraic too. These closure properties are effective, so we could use them to determine the minimal polynomial of $F(x;t)$ given the minimal polynomial of $F(0;t)$. Although in this particular example this would be particularly simple we will proceed differently: we will *guess* the minimal polynomial of $F(x;t)$ and give an argument for its correctness which is independent of the discussion from before. By *guessing* we mean to make an ansatz

$$P(x,t,Y) = \sum_{i,j,k=0}^d p_{ijk} x^i t^j Y^k$$

with undetermined coefficients p_{ijk} for (the numerator of) the minimal polynomial of $F(x; t)$. By computing a truncation of $F(x; t)$ up to some order, plugging it into $P(x, t, Y)$ and setting the coefficients of the monomials in x and t equal to zero results in a linear system for the p_{ijk} 's. In general, there are two situations that can arise: either the linear system has a non-trivial solution which gives rise to an annihilating polynomial for the truncation of $F(x; t)$, or its only solution is zero. If its only solution is zero, then either because the ansatz was not appropriate, that is, the d was too small, or because $F(x; t)$ is not algebraic. For this example we already know that the generating function $F(x; t)$ is algebraic. Indeed, computing $F(x; t)$ up to order 8 with respect to t and making an ansatz for $P(x, t, Y)$ with $d = 2$ we find that the solution space for its coefficients has dimension 1 and one of its solutions corresponds to

$$P(x, t, Y) = 1 - (1 - 2xt)Y - xt(1 - t(x + \bar{x}))Y^2.$$

Of course, having a non-trivial guess for the minimal polynomial does in general not prove the algebraicity of $F(x; t)$. It only means that we have found a polynomial that annihilates $F(x; t)$ up to a given order. But having a guess for the minimal polynomial one can increase confidence by checking whether it also annihilates truncations of $F(x; t)$ at higher orders. Having a reasonable candidate for the minimal polynomial, the next step is to verify its correctness. For this particular example it is again particularly easy. Since $P(x, t, Y)$ has degree 2 with respect to Y , we can simply compute its roots and check whether one of them is an element of $\mathbb{Q}[x][[t]]$ that solves equation (4). If there is one, then $P(x, t, Y)$ is indeed the minimal polynomial of $F(x; t)$, since $F(x; t)$ is the only solution of the kernel equation in $\mathbb{Q}[x][[t]]$.

The general situation is more complicated though the underlying principle is the same. The difficulty comes from the fact that the roots of the minimal polynomial may not be given explicitly. But this is actually not necessary. The existence of a root $F_{\text{cand}}(x; t)$ in $\mathbb{Q}[x][[t]]$ can be derived from the Newton-Puiseux algorithm [18]. To prove that $P(x, t, Y)$ is the minimal polynomial of $F(x; t)$ it is sufficient to show that $F(x; t)$ equals $F_{\text{cand}}(x; t)$. Using closure properties of algebraic functions and Gröbner bases (or resultants) we find that

$$A(Y) = Y(1 - 4t^2 - t^2Y^2)$$

is an annihilating polynomial for

$$x(1 - t(x + \bar{x}))F_{\text{cand}}(x; t) - x + tF_{\text{cand}}(0; t). \quad (11)$$

By computing a truncation of $F_{\text{cand}}(x; t)$ up to some order and plugging the corresponding truncation of expression (11) into the second factor of $A(Y)$ we find that it is not a root of $1 - 4t^2 - t^2Y^2$. Since it is a root of $A(Y)$, it is necessarily a root of Y , and therefore identically zero. Consequently, $F_{\text{cand}}(x; t)$ is a solution of the kernel equation and hence equal to $F(x; t)$ since there is only one solution in $\mathbb{Q}[x][[t]]$, and $P(x, t, Y)$ is indeed the minimal polynomial of $F(x; t)$.

We now provide some details on the computation of $A(Y)$. By construction $P(x, t, Y)$ is the minimal polynomial of $F_{\text{cand}}(x; t)$, an annihilating polynomial for $F_{\text{cand}}(0; t)$ is $P(0, t, Y)$, and the minimal polynomial of a polynomial $p(x, t) \in \mathbb{Q}[x, t]$

is $p(x, t) - Y$. So we know the annihilating polynomial for the building blocks expression (11) is made of. To find an annihilating polynomial for the full expression we need to effectively perform the closure properties plus and times. This is easily done by using Gröbner bases and noting that if $p_1(Y)$ and $p_2(Y)$ are annihilating polynomials for y_1 and y_2 then the generators of the elimination ideals

$$\langle p_1(y_1), p_2(y_2), y_1 + y_2 - Y \rangle \cap \mathbb{Q}[Y] \quad \text{and} \quad \langle p_1(y_1), p_2(y_2), y_1 y_2 - Y \rangle \cap \mathbb{Q}[Y]$$

are annihilating polynomials for $y_1 + y_2$ and $y_1 y_2$, respectively. Note that y_1 and y_2 are treated as variables.

Having determined the minimal polynomial of $F(x; t)$ it is now natural to ask why it is advantageous to have it at all. In general there is no closed form for its roots and hence for the generating function. But the minimal polynomial together with the first terms of $F(x; t)$, sufficiently many to distinguish it from the other roots, allows one to represent $F(x; t)$ by a finite amount of data. The class of algebraic functions has many closure properties and we have seen before how some of them can be performed effectively. Apart from doing exact computations these representations also admit many other useful operations such as the fast computation of their coefficients, the derivation of closed form expressions, the computation of their asymptotics, and numerical evaluation [44]. Before we elaborate on the computation of coefficients we introduce the notion of D-finiteness.

12 Differential algebra

Definition 3. A series $f \in \mathbb{Q}[[t_1, \dots, t_n]]$ is called *D-finite*, if its derivatives with respect to each of its variables form a finite-dimensional vector space over $\mathbb{Q}(t_1, \dots, t_n)$, that is, if for each $t \in \{t_1, \dots, t_n\}$ there are $d \in \mathbb{N}$ and $p_0, \dots, p_d \in \mathbb{Q}[t_1, \dots, t_n]$, not all zero, such that

$$p_0 \frac{\partial^d}{\partial t^d} f + p_1 \frac{\partial^{d-1}}{\partial t^{d-1}} f + \dots + p_d f = 0.$$

It is straight-forward to see that a series $f(t) = \sum_{n=0}^{\infty} f_n t^n$ is D-finite if and only if its coefficient sequence (f_n) satisfies a linear recurrence with polynomial coefficients, i.e. if and only if there are $r \in \mathbb{N}$ and polynomials $q_0, \dots, q_r \in \mathbb{Q}[t]$, not all zero, such that

$$q_0 f_{n+r} + q_1 f_{n+r-1} + \dots + q_r f_n = 0 \quad \text{for all } n \in \mathbb{N}.$$

Consequently, a D-finite function can be encoded by a finite amount of data: a recurrence relation for its coefficient sequence and finitely many of its initial terms.

Any algebraic function is D-finite, and we will now explain how to determine a differential equation for $F(x; t)$, and see how it translates into a recurrence relation for its coefficient sequence. There is more than one way of doing so. For instance, we could use that $F(x; t)$ equals $\frac{1-\bar{x}x_0}{1-t(x+\bar{x})}$, find differential equations for the building blocks of the expression and then use closure properties of D-finite functions to construct a differential equation for $F(x; t)$ from them. We could

also use that $F(x; t)$ is the non-negative part of a rational function and construct a differential equation for $F(x; t)$ using linear algebra as suggested in [38] or creative telescoping as proposed in [8]. Or we guess a differential equation and verify its correctness similarly as we did before for its minimal polynomial. We will do none of that, but explain how to derive a differential equation directly from its minimal polynomial by unrolling the classical proof that an algebraic function is D-finite. For simplicity we show it for $F_0 \equiv F(0; t)$. Note that $\mathbb{Q}(x, t)[F_0]$ is a finite-dimensional vector space whose dimension equals the degree of the minimal polynomial of F_0 and that the extended Euclidean algorithm implies that $\mathbb{Q}(x, t)[F_0]$ is also a field which is closed under taking derivatives. Recall that $P_0(t, Y) = 1 - Y + t^2 Y^2$ is the minimal polynomial of F_0 . By differentiating $P_0(t, F_0) = 0$ with respect to t we find that

$$\frac{\partial P_0}{\partial t}(t, F_0) + \frac{\partial P_0}{\partial Y}(t, F_0) \frac{\partial F_0}{\partial t} = 0. \quad (12)$$

Since $\deg_Y \frac{\partial P_0}{\partial Y} < \deg_Y P_0$, and because P_0 is irreducible, there are $u, v \in \mathbb{Q}(t)[Y]$ such that

$$u \frac{\partial P_0}{\partial Y} + v P_0 = 1, \quad \text{and hence} \quad u(t, F_0) \frac{\partial P_0}{\partial Y}(t, F_0) = 1.$$

Multiplying equation (12) by $u(t, F_0)$ and rearranging terms gives

$$\frac{\partial F_0}{\partial t} = -u(t, F_0) \frac{\partial P_0}{\partial t}(t, F_0). \quad (13)$$

Since $\mathbb{Q}(t)[F_0]$ is a 2-dimensional vector space there is a non-trivial linear relation between $1, F_0$ and $\frac{\partial F_0}{\partial t}$ over $\mathbb{Q}(t)$ that gives rise to a non-trivial linear differential relation for F_0 :

$$t(1 - 4t^2) \frac{\partial F_0}{\partial t} + 2(1 - 2t^2)F_0 = 2.$$

By differentiating the equation with respect to t we can get rid of the inhomogeneous part:

$$t(1 - 4t^2) \frac{\partial^2 F_0}{\partial t^2} + (3 - 16t^2) \frac{\partial F_0}{\partial t} - 8tF_0 = 0. \quad (14)$$

Hence, F_0 is D-finite as stated before. From the differential equation one can deduce the following recurrence relation

$$(4+n)f(0; n+2) - 4(1+n)f(0; n) = 0 \quad (15)$$

for the coefficients of F_0 . It has essentially order 1, so the coefficient sequence of F_0 is essentially hypergeometric [39]. In case one does not care whether the recurrence for the coefficients of F_0 is linear or not one can as well apply $[t^n]$ to $F_0 = 1 + t^2 F_0^2$ and find that

$$f(0; n) = \begin{cases} 1, & \text{if } n = 0, \\ \sum_{k=0}^{n-2} f(0; k)f(0; n-2-k) & \text{else.} \end{cases} \quad (16)$$

The latter recurrence has an evident combinatorial meaning which is not surprising since this is the case for the minimal polynomial of F_0 it is derived from. Given an excursion e of length $n \geq 2$ we can write it as the concatenation of excursions e_1 and e_2 such that e_1 visits 0 at its beginning and end only. If e_1 has

length k then removing its first and last step induces a bijection between excursions of this kind and excursions of length $k - 2$. Since the set of excursions of length n partitions according to the first time the starting point is visited again we have $f(0; n) = \sum_{k=2}^n f(0; k-2)f(0; n-k)$. Together with $f(0; 0) = 1$ and $f(0; 1) = 0$ this gives the recurrence in (16). The advantage of the linear recurrence over the non-linear one is the number of operations needed to compute $f(0; n)$. The linear recurrence requires $O(n)$ many arithmetic operations, when used in a naive way, while the other requires $O(n^2)$ many. The computation of $f(0; n)$ for fixed and large n can be done even faster when not caring about the previous terms. We refer the interested reader to [47, 23, 9].

The reflection principle implies a simple formula for $g(n) := f(0; 2n)$. It can also be deduced from the recurrence relation it satisfies:

$$g(n) = \prod_{k=0}^{n-1} \frac{4(1+2k)}{4+2k} = \frac{4^n \frac{(2n)!}{2^n n!}}{2^n (n+1)!} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n}.$$

We point out that there are algorithms for finding closed form solutions of linear recurrences such as polynomial and rational, hypergeometric and d'Alembertian solutions. We refer to [34] for more information on that. The differential equation (14) we derived for F_0 is not only useful for encoding the generating function, performing exact computations, or deriving linear recurrences for its coefficient sequence. It also allows one to decide whether F_0 has some closed form representation and to find it in case there is one. The differential operator L that corresponds to the differential equation (14) is

$$L = t(1-4t^2) \frac{\partial^2}{\partial t^2} + (3-16t^2) \frac{\partial}{\partial t} - 8t.$$

It is the least common left multiple of the operators

$$L_1 = t \frac{\partial}{\partial t} + 2 \quad \text{and} \quad L_2 = t(2t-1)(2t+1) \frac{\partial}{\partial t} + 2(2t^2-1),$$

and its solution space is the direct sum of the solution spaces of L_1 and L_2 . The latter are easily determined since L_1 and L_2 both have order 1. We just note that the solutions of a differential equation of the form $q_0 f' + q_1 f = 0$, with $q_0, q_1 \in \mathbb{Q}(t)$, are constant multiples of $\exp(-\int \frac{q_1}{q_0} dt)$. The solution space of L_1 is therefore generated by $f_1 = 1/t^2$, and the solution space of L_2 is easily seen to be generated by $f_2 = \frac{\sqrt{1-4t^2}}{t^2}$ by performing the partial fraction decomposition

$$\frac{2(2t^2-1)}{t(2t-1)(2t+1)} = \frac{2}{t} + \frac{1}{1-2t} - \frac{1}{1+2t}$$

and integrating each summand separately. Both, f_1 and f_2 , have a singularity at 0, a pole of order 2. It can be eliminated by linearly combining f_1 and f_2 to $f := (1 - \sqrt{1-4t^2})/t^2$. By comparing $f(0) = 2$ with $f(0; 0) = 1$ we find that

$$F(0; t) = \frac{1 - \sqrt{1-4t^2}}{2t^2}.$$

13 Asymptotic methods

Often one is not interested in an exact formula for a combinatorial sequence but in an *asymptotic* one. There can be different reasons for that. It could be that

there is no simple exact formula, or that there is one, yet it is difficult to evaluate. There are different ways an asymptotic formula can be derived. For instance, using Stirling's formula,

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \rightarrow \infty,$$

one can find for

$$g(n) = \frac{1}{n+1} \binom{2n}{n}$$

the following asymptotic approximation

$$g(n) \sim \frac{1}{\sqrt{\pi}} 4^n n^{-\frac{3}{2}}, \quad n \rightarrow \infty. \quad (17)$$

This asymptotic formula consists of several parts: 4^n and $n^{-\frac{3}{2}}$ are called the exponential and polynomial growth, respectively, and $1/\sqrt{\pi}$ is referred to as the growth constant.

We next explain how to derive information about the asymptotics of the sequence $(g(n))$ from the linear recurrence it satisfies. It is based on the notion of generalized series solutions.

Definition 4. A *generalized series* is a \mathbb{C} -linear combination of formal objects of the form

$$\begin{aligned} & \Gamma(x)^{u/v} \phi^x \exp(s_1 x^{1/v} + s_2 x^{2/v} + \cdots + s_{v-1} x^{(v-1)/v}) \\ & \times x^\alpha ((c_{0,0} + c_{0,1} x^{-1/v} + c_{0,2} x^{-2/v} + \dots) \\ & + (c_{1,0} + c_{1,1} x^{-1/v} + c_{1,2} x^{-2/v} + \dots) \log(x) \\ & + \dots \\ & + (c_{m,0} + c_{m,1} x^{-1/v} + c_{m,2} x^{-2/v} + \dots) \log(x)^m), \end{aligned} \quad (18)$$

where $u \in \mathbb{Z}$, $v \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{N}$, and ϕ , α , s_1, \dots, s_{v-1} , $c_{i,j} \in \mathbb{C}$, and its parts behave as follows with respect to the shift operation:

$$\Gamma(x+i) = x^i \Gamma(x), \quad \phi^{x+i} = \phi^i \phi^x, \quad (x+i)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} i^\alpha x^{\alpha-n},$$

$$\exp(s_l(x+i)^{l/v}) = \exp(s_l x^{l/v}) \sum_{n=0}^{\infty} \frac{1}{n!} \left(s_l \sum_{k=1}^{\infty} \binom{l/v}{k} i^k x^{l/v-k} \right)^n$$

and

$$\log(x+i) = \log(x) - \sum_{n=1}^{\infty} \frac{(-i)^n}{n} x^{-n}.$$

A generalized series which solves a given linear recurrence is called a generalized series solution. It can be shown that if the recurrence has order r , then there are exactly as many linearly independent generalized series solutions, all of which are of the above form. Their terms can be computed term by term. We refer to [31] for an implementation of an algorithm in Mathematica. The recurrence for $(g(n))$,

$$(4+2n)g(n+1) - 4(1+2n)g(n) = 0,$$

has, up to multiples, only one generalized series solution. Its first terms are

$$4^x x^{-3/2} \left(1 - \frac{9}{8x} + \frac{145}{128x^2} - \frac{1155}{1024x^3} + \frac{36939}{32768x^4} + O(x^{-5}) \right).$$

Although a priori only correct in an algebraic sense, any sequence that solves a linear recurrence has an asymptotic expansion which is a linear combination of generalized series solutions [48, 3, 4]. Hence, there is a $c \in \mathbb{C}$ such that

$$g(n) = c 4^n n^{-3/2} \left(1 - \frac{9}{8n} + \frac{145}{128n^2} - \frac{1155}{1024n^3} + \frac{36939}{32768n^4} + O(n^{-5}) \right), \quad n \rightarrow \infty.$$

The exponential and polynomial growth of $g(n)$ is therefore 4^n and $n^{-3/2}$, respectively. In general, it is not known how to determine the coefficients of the linear combination. However, numerical approximations can be found by comparing $g(n)$ with

$$4^n n^{-3/2} \left(1 - \frac{9}{8n} + \frac{145}{128n^2} - \frac{1155}{1024n^3} + \frac{36939}{32768n^4} \right)$$

for different and large values of n [32]. Based thereon it is often possible to guess their exact values [33].

14 Some references

The methods which have been presented here are not new. They have appeared in many other peoples' works. In the following section we list the references where they can be found.

The method presented in Section 2 dates back at least to [36], where it is presented as the solution to Exercise 2.2.1.-4. Having found considerable attention since then [14, 20, 42, 7, 10], it is meanwhile referred to as the classical kernel method.

The approach of Section 3 which is based on a particular factorization of the kernel polynomial has not appeared before in this formal / algebraic setting. However, the usefulness of such factorizations is known in the context of Riemann-Hilbert boundary problems [24, Section 2.2.3], [29].

The orbit-sum method presented in Section 4 was introduced in [15]. Its generalization was initiated in [6] and then continued in [21] based on the Newton-Puiseux algorithm [18].

It is well-known that compositional inverses and Lagrange inversion relate to the enumeration of lattice walks [26]. Yet, to the best of our knowledge, it has not been applied before as shown in Section 5.

Inspired by Tutte's work on the enumeration of maps [46], the invariant method sketched in 6 was introduced and applied to the enumeration of lattice walks in [2, 12, 13]. Related algorithmic questions have received considerable attention in [22, 17, 16, 19] and [5].

The connection of lattice walks with continued fractions and orthogonal polynomials sketched in Section 7 is classical, and so are the reflection principle and the cycle lemma illustrated in 8 and 10. We refer to [25], [37, Sec 10.9 - 10.11] and [28], [37, Sec 10.3, 10.18] and [45], [37, Sec 10.4], respectively, and the references therein for further information.

The combinatorial factorization presented in Section 9 is taken and adapted from [26].

The paradigm of guess and prove illustrated in Section 11 has ever been popular in science [43]. However, only since the advent of high performing computers it has given rise to powerful effective methods. The availability of nowadays hard- and software makes the class of D-finite functions introduced in 12 accessible to practical computations. We refer to [34] for extensive information on that.

There are methods we have not discussed here though they do fall within the framework of this text. We have not presented them because they cannot be applied to equation 4. Among these methods are the half-orbit-sum method, the obstinate kernel method and the iterated kernel method. They were introduced and applied in [15, 1] and [11] and [40, 41], respectively.

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