The Mathematics of Music Theory

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A conjecture of Polya and Szego states that among the polygons with the same number of sides and area, the regular polygon minimizes the first eigenvalue of the Dirichlet Laplacian. This conjecture has only been proven in the cases of three and four sides. In this paper introduce a computational approach to checking the Polya-Szego conjecture for five or more sides.

[16%] Assignment Outline

Introduction

Consider the eigenvalue solutions for the Laplace operator with Dirichlet boundary conditions for any open, bounded set $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u = \lambda u \\ \quad u = 0 \end{cases} \tag{1}$$

Due to Rellich's compacness lemma TODO TODO, the spectrum of the Dirichlet Laplacian consists only of discrete eigenvalues

$$0 < \lambda_1(\Omega) \leq \lambda_1(\Omega) \leq \lambda_1(\Omega) \leq ... \to +\infty$$

which can be ordered by their multiplicity [Hen06]. The first eigenvalue λ_1, which is also called the fundamental tone, is of particular importance. TODO Explain why we care about this (physics stuff) There are few polygons whose spectrum can be explicitely calculated. These polygons are equilateral triangles, hemi-equilateral triangles, and isosceles-right triangles [McC11].

Bibliography

References

[1] Antoine Henrot. *Extremum Problems for Eigenvalues of Elliptic Operators*. Birkhauser, 2006.

[2] B. McCartin. "Laplacian Eigenstructure of the Equilateral Triangle". In: *Hikari* (2011), pp. 8–28.