The Mathematics of Music Theory

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In this paper, we will summarize a few connections between mathematics and music theory. We will look at the circle of fifths and it's connection to modular arithmetic. This is paired with a summary of the mathematics of chord construction. Specifically, we will focus on the ratio's between different notes in a chord and how these ratios are used to name the chords. We also study how one could define when two drums have the same shape, and we connect connect this concept to Euclid's proof of SAS criteria for triangle congruence. We follow this with how one could "hear" the area of a drum. We also give a brief introduction to the theory of Fourier series and its applications.

Intervals and Chords

In music theory we divide an octave into twelve equidistant notes and give each a name [Gol65]. Given two of these notes, we can find the ratio between their frequencies. We call these ratios the "size" of the interval between the two notes. The sound produced from three or more notes played simultaneously is called a chord, and we can categorize it using the interval's of the underlying notes. The most common chords are triads, which consist of a root note and notes which are a third and a fifth above the root note. The triads are named after the intervals that they contain, and we will list the two most common cases. If the chord consists of a major third and a perfect fifth we call the chord major. Similarly, if the chord consists of a minor third and a perfect fifth we call the chord minor. For example, the chord consisting of the notes C,E,G is a major triad and the chord consisting of the notes A,C,E is a minor triad. Larger chords can be made by adding additional notes on top of triads. For example, the chord created by combining a major triad with a note that is a minor seventh above the root is called a dominant seventh chord [Gol65]. This chord is fundamental in most styles of jazz. One of the reasons we name chords based on the intervals they consist of is that it tells us how consonant or dissonant the chord sounds. Because of the way humans process sound, notes whose interval ratios are simple fractions sound consonant together. A perfect fifth has a interval ratio of 3:2, so it will be consonant with the root. Likewise, the minor and major thirds have interval ratios of 6:5 and 5:4 respectively, so they will also be consonant to the root. The dominant seventh chord is a great example of a chord that sounds dissonant. This is caused by the third and seventh note being a tritone apart, which has the interval ratio of 64:45.

The Circle of Fifths

The circle of fifths is a sequence of chord tones, each followed by its dominant, which loops through all twelve tones [Gol65]. Grouping the notes in this way makes it easy to write chord progressions and key changes, among many other relationships. For example, if you follow the circle of fifths backwards you get a progression of IV to I chord changes. These chord changes show up in practically every song you hear on the radio, and is the building block of musical harmony [Gol65]. Another place that the circle of fifths shows up is with sheet music. There is a simple process that connects key signatures and the circle of fifths. We start in the key of C Major, which has no accidentals. Following the circle of fifths, we add one sharp for each step we take in the clockwise direction. For example, suppose we are writing a song in the key of E Major. Since E is four clockwise steps from C, we know we need four sharps. Similarly, if we are trying to read a piece of music with one sharp we can use the circle of fifths to quickly determine that we are in the key of G Major. It is worth noting that this connection also works for minor keys. This comes from the fact that every major key has a relative minor. The relative minor of a major key is obtained by taking the sixth note as the root instead of the first note. This concept of the circle of fifths is strikingly similar to modular arithmetic. Modular arithmetic is a system of arithmetic that groups numbers together based on their remainder with respect to some fixed natural number [Ros12]. If we represent notes by the number of semitones they are from C, we can study them using modular arithmetic. Suppose we look at the integers modulo 12, then we can represent each note as one of the equivalence classes. This representation of music theory inside of modular arithmetic allows us to use mathematical methods to answer questions about music. One such question is “are there other intervals we could choose to cycle through every note?” An example of an interval that doesn’t work is the minor third, which is 3 semitones. Starting from C, we get the sequence C,D#,F#,A,C. So we end up with a cycle that contains four notes instead of the twelve notes that we wanted. Fortunately, we know from modular arithmetic that we can cycle through every number if we continuously add a number that is relatively prime to the base we are in [Ros12]. One can check that the numbers less than 12 which are relatively prime to 12 are 1, 5, 7, and 11. One represents a semitone, which gives us the sequence C, C#, . . . , B, C. Similarly, eleven gives us the same sequence as one but in reverse order. This only leaves five and seven. Five semitones is a perfect forth, and seven semitones is a perfect fifth. Obviously, the sequence of notes we get from these intervals is the circle of fifths. Thus, there are no other interval circles that contain every note.

The Shape of Drums

Physical drums consist of a rigid shell with a membrane which produces sound when hit. A similar thing can be “created” in a pure mathematical setting by studing specific partial differential equations over a closed region. Specifically, the frequencies of the drum membrane corresponds to the eigenvalues of the Dirichlet Laplacian. This construction makes it possible to “hear” drums where the shape of the drumhead is any closed and simple curve. With all of these abstractions from the physical definition of a drum, one may wonder how we determine if two drums have a similar shape. Because of the vast number of possible shapes for any given drum, we will study a simple case and expand our results to the general case. Suppose we have two drums, both of which have triangular drumheads. We wish to give some criteria for the two shapes to be similar. One possible criteria was given by Euclid in his book, the Elements. In the introduction to his online translation of Euclid’s Elements, Dr. David Joyce states “Euclid’s Elements form one of the most beautiful and influential works of science in the history of humankind. Its beauty lies in its logical development of geometry and other branches of mathematics.” [Joy98] During Euclid’s “logical development of geometry” he lays out many possible criteria to determine if two triangles are similar. The one we will use, called the Side-Angle-Side (SAS) criteria, is proved in Book One, Proposition Four [Joy98]. The proposition states "If two triangles have two sides equal to two sides respectively, and if the angles contained by those sides are also equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides."[Joy98] In other words, we can guarantee that two triangles have the same shape so long as two of the sides of one triangle match two sides of the other and the angle between these two sides are also the same on both triangles. In his proof of the SAS criteria, Euclid uses the method of superposition [Joy98]. Essentially, this method consists of using some combination of rotation, translation, and reflection to overlay one shape on top of the other. If the shapes perfectly match one another, then they must be the same shape. This method of superposition can be generalized to general curves using an algebraic structure called the group of isometries [Tot02]. This structure contains all of the possible reflections, rotations, and translations that we can preform without changing the shape or size of our theoretical drum. Thus, fundamentally, we can determine whether two drums are the same shape by lining them up and seeing if they match.

How to Hear a Drum

In the last section, we introduced a way of constructing a "drum" inside of a pure math setting. In this section we will go into more detail and describe how this construction works. We begin with the shape of our drumhead, which is a region of the real plane bounded by piecewise smooth curves. Since we wish to emulate the physical properties of a drum, we want to define some system that models the vibration of the drum membrane which produces the sound. This is done using the wave equation over our boundary, with the boundary condition that the function is 0 on the boundary. This boundary condition comes from the fact that the membrane is fixed to the drum shell around the edge of the drumhead, and thus does not vibrate freely. We call this system of the differential equation with the boundary conditions the Dirichlet Laplacian. The Dirichlet Laplacian allows us to model the physical vibration of the drum, and we can calculate its eigenvalues to find the fundamental frequency and overtones. Thus, by modeling the vibration of the drumhead using the Dirichlet Laplacian and studying its properties, we can produce “sound” via the eigenvalues.

Fourier Series

Throughout this paper we have studied functions that behave cyclically. Specifically, both sound waves and the vibrations of a drum membrane are cyclic in nature. This cyclic nature can be very powerful if we have tools that can utilize it, but the common tools of polynomial functions do not have this capability. This causes a problem, as the amazing utility of Taylor expansions and similar polynomial tools are not well suited to this area. Ideally, we would want the ability to decompose complicated cyclic functions into distinct simple components in the same way Taylor series decomposes complicated functions into the relatively simple polynomials. This is what Fourier did when he introduced the concept of Fourier series. Fourier series decomposes periodic functions into a (possibly infinite) sum of sines and cosines. This decomposition is analogous to Taylor series, but we replace the monomial terms with sines and cosines. Thus we have the same concerns of calculating the coefficients and determining convergence. Using Fourier series to approximate periodic functions gives us massive improvements to using polynomials, but we also have a few downsides to account for. The most important improvement is that Fourier series maintains the same level of approximation on any cycle of the function. This is a large improvement over using Taylor series approximations, which becomes less accurate the further away from the center you go. However, calculating the coefficients requires the computation of an integral over the period of the function [Rud87]. Despite the potential drawback's of Fourier analysis, it has been applied to a plethora of physics and engineering domains with great success. Specifically, it is used in areas such as image processing, audio processing, and even discretely in the JPEG compression algorithm.

References

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