Numerical Estimations of the Dirichlet Laplacian on Polygons

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Abstract

A conjecture of Polya and Szego states that among the polygons with the same number of sides and area, the regular polygon minimizes the first eigenvalue of the Dirichlet Laplacian. This conjecture has only been proven in the cases of three and four sides. In this paper introduce a computational approach to checking the Polya-Szego conjecture for five or more sides.

Introduction

Physical drums consist of a rigid shell with a membrane which produces sound when hit. A similar thing can be "created" in a pure mathematical setting by studing specific partial differential equations over a closed region. Specifically, the frequencies of the drum membrane corresponds to the eigenvalues of the Dirichlet Laplacian. This construction makes it possible to "hear" drums where the shape of the drumhead is any closed and simple curve. To do this, we begin with the shape of our drumhead, which is a region of the real plane bounded by piecewise smooth curves. Since we wish to emulate the physical properties of a drum, we want to define some system that models the vibration of the drum membrane which produces the sound. This is done using the wave equation over our boundary, with the boundary condition that the function is 0 on the boundary. This boundary condition comes from the fact that the membrane is fixed to the drum shell around the edge of the drumhead, and thus does not vibrate freely. We call this system of the differential equation with the boundary conditions the Dirichlet Laplacian. The Dirichlet Laplacian allows us to model the physical vibration of the drum, and we can calculate its eigenvalues to find the fundamental frequency and overtones. Thus, by modeling the vibration of the drumhead using the Dirichlet Laplacian and studying its properties, we can produce "sound" via the eigenvalues.

In 1877, Lord Rayleigh conjectured the following [8]

> If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.

This conjecture was left unsolved for a very long time. In 1923, Faber published a proof which was followed by an independent proof by Krahn in 1925 [2]. From the so called Faber-Krahn inequality, we know that for any drumhead with a given area, the circle is the one with the lowest tone.

In 1951, Polya and Szego conjectured that a similar statement holds for drumheads with a polygonal shape [3]. This conjecture has been shown to be true for 3 and 4 sided polygons, but remains unproven for any other number of sides.

There are two main hurdles that are halting progress on this conjecture. The first is that the tools that were used to prove both Lord Rayleighs conjecture as well as the small cases for the Polya-Szego conjecture are not available when the number of sides is greater than four. The main tool that is used is called Steiner Symmetrization, and when there are more than four sides this symmetrization method creates additional sides at each step.

The purpose of this paper is to show a specific method for running numerical approximations to suggest that this conjecture is indeed true. This is done using a method based on fundamental solutions [1]. Specifically, we consider all functions that satisfy the Laplace Equation and then solve for the linear coefficients using the boundary conditions. Once this is done we use gradient descent to find the polygon with the minimum first eigenvalue, which is equivalent to the first fundamental tone of the drum.

Background

Physical Definition

Consider a homogeneous elastic drumhead, or membrane, stretched over a rigid frame. We will represent the frame as a domain Ω in $\mathbb{R}^2$. Take the function $u(x,y,t)$ to be the vertical displacement of the membrane from its resting position. Then for any disk $D \subset \Omega$, Newton's second law of motion states that

$$\int_{\partial D} T \frac{\partial u}{\partial \mathbf{n}} \, dS = \int_D \rho u_{tt} \, dA$$

where T is the constant tension, ρ is the density constant, and $\mathbf{n}$ is the outward normal of the boundary [9]. By the divergence theorem, we have

$$\int_D T \Delta u \, dA = \int_D \rho u_{tt} \, dA$$

where Δ is the Laplace operator. From this we can get the wave equation on Ω

$$u_{tt} = c^2 \Delta u$$

where we define u to be 0 on the boundary and where $c = \sqrt{T/\rho}$. We can solve this wave equation using $u(x,y,t) = T(t)V(x,y)$ which gives us

$$\frac{T''}{c^2 T} = \frac{\Delta V}{V} = -\lambda$$

and finally we have reduced our problem to the Dirichlet Laplacian

$$\Delta V = -\lambda V$$

where V on the boundary is zero [9]. In the next section, we will start from the Dirichlet Laplacian and introduce the conjectures in a formal setting.

Polya-Szego's Conjecture

Consider the eigenvalue solutions for the Laplace operator with Dirichlet boundary conditions for any open, bounded set $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u = \lambda u & \text{in}\, \Omega \\ \quad u = 0 & \text{on}\, \partial\Omega \end{cases} \tag{1}$$

Due to Rellich's compactness lemma, the spectrum of the Dirichlet Laplacian consists only of discrete eigenvalues

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq ... \to +\infty$$

which can be ordered by their multiplicity [3]. The first eigenvalue λ_1, which is also called the fundamental tone, is of particular importance. One noteworthy property of the first eigenvalue is it's connection with the fundamental pitch of a drum with the same shape, as we saw in the previous section.

In 1877, Rayleigh conjectured that for all domains with a fixed area, the fundamental tone is minimized by the disc [8]. This was eventually proven in any euclidean space by Faber and also independently by Krahn [3].

Theorem 0.1 (Faber-Krahn)**.** *Let c be a positive number and B the ball of* *volume c. Then,*

$$\lambda_1(B) = min\{_1(\Omega), \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c\}$$

In 1951, Polya and Szego conjectured a similar statement about regular polygons [7]. This conjecture feels like it follows from the Faber-Krahn Theorem, as regular polygons are in some sense the roundest polygons for a given number of sides. However, while this conjecture is simple to state and simple to understand, it has been largely left unsolved for 70 years.

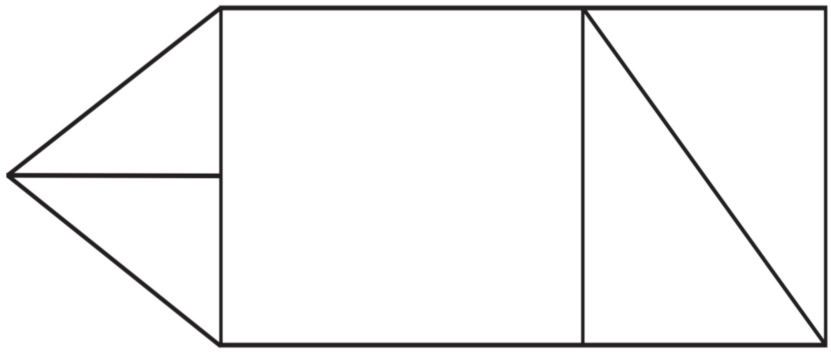
Conjecture 0.2 (Polya-Szego)**.** *Let P_n be the set of simple polygons with n sides. The unique solution to the minimization problem*

$$\min_{P \in P_n, |P| = \pi} \lambda_1(P)$$

is the regular polygon with n sides and area π

There are few polygons whose spectrum can be explicitly calculated. These polygons are equilateral triangles, hemi-equilateral triangles, and isosceles-right triangles [5].

The following figure contains every polygon whose spectrum we can calculate explicitely.



Although our ability to explicitely calculate shapes is very limited, there has been some success in proving the conjecture. Polya himself proved his conjecture in the cases of $N = 3$ or $N = 4$ [6].

Theorem 0.3 (Polya)**.** *The equilateral triangle has the least first eigenvalue among all triangles of given area. The square has the least first eigenvalue among all quadrilaterals of given area.*

This proof relies on a technique called Steiner Symmetrization, which does not work on polygons with five or greater sides [9]. All other cases remain unsolved. In the remaining parts of this paper, we will introduce a method to numerically estimate the polygon which minimizes the first eigenvalue for an arbitrary number of sides.

Numerical Estimation

We will introduce a method that has the following two properties

1. Provide a algorithmic process which is precise and fast

2. Find the derivative of λ_1 with respect to the vertices of the polygon

This will be done based on the method of fundamental solutions, which will produce a system of equations that depends on the vertices of the polygon. From this system of equations, we will find the derivative of the first eigenvalue based on the vertices of the polygon. This will allow us to use gradient descent on this system, which will eventually leave us with a polygon that has a locally minimal first eigenvalue.

Method of Fundamental Solutions

We begin by considering the functions which already satisfy the equation

$$-\Delta u = \lambda u \text{ in } \Omega$$

One way to do this is to consider the functions

$$u = \alpha_1 \phi_1^\lambda + ... + \alpha_N \phi_N^\lambda$$

where ϕ_i^λ, $i = 1...M$ are fundamental radially symmetric solutions of

$$-\Delta \phi = \lambda \phi$$

For our purposes, we choose these solutions to be of the form

$$\phi_i^\lambda(x) = H_0(\sqrt{\lambda}|x - y_i|)$$

where H_0 is the Hankel function of order 0, $\{y_i\}$ is the set of singularities of the functions ϕ_i^λ outside of the boundary Ω [1]. The coefficients $\alpha_1, \alpha_2 ... \alpha_N$ are found using the Dirichlet boundary conditions on a discretization of $\partial\Omega$ which we will denote as $\{x_i\}$. From this set up, we have constructed a system of equations

$$u = \alpha_1 \phi_1^\lambda(x_i) + ... + \alpha_N \phi_N^\lambda(x_i) = 0, \; i = 1, 2, ..., N$$

From this system of equations, we can define a matrix for a given λ

$$A_\lambda = \{\phi_j \{x_i\}^\lambda\}_{i,j=1}^N$$

Since we want non-trivial solutions to the system of equations, our goal is to find values for λ which causes A_λ to be singular. In other words, we wish to find values that make the matrix non-invertible and so we wish to find values where the determinant is 0. Formally, for some interval I, we wish to locate the values $\lambda \in I$ where

$$\det A_\lambda = 0$$

However, this creates a problem. Since the matrix is singular, the matrix does not have a unique solution. To combat this, we introduce another equation to some interior point such that

$$\Sigma \alpha_i \phi_i^\lambda$$

does not vanish [1]. This will allow us to find a unique solution of A_λ.

Gradient Descent

In the last section, we found a transformation that gives us information about the eigenvalues depending only on the vertices of the polygon. In this section we will compute the derivative of the first eigenvalue with respect to the vertices of the polygon. This is done by perturbing the shape Ω by some vector field V. From this we get the derivative

$$\frac{d\lambda}{dV} = -\int_{\partial\Omega} (\frac{\partial u}{\partial \mathbf{n}})^2 V \mathbf{n} \, d\sigma$$

where $\mathbf{n}$ is the outer unit normal field to $\partial\Omega$ [4]. However, as we wish to compute the derivative with respects to vertices of the polygon, we will consider particular vector fields V. Specifically, we will look at perturbations of one vertex. This will in turn perturb the segments connected to the vertex. Let $A_{i-1}A_i, A_iA_{i+1}$ be the segments of the polygon connected to the vertex. Then for this specific perturbation, V has the following form on the border of the polygon

$$\begin{cases} \mathbb{I}_{i-1,i}(x)(1,0) & x \in [A_{i-1}A_i] \\ \mathbb{I}_{i,i+1}(x)(1,0) & x \in [A_iA_{i+1}] \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $\mathbb{I}_{i,j} : A_iA_j \to [0,1]$ is an affine function such that

$$\mathbb{I}_{i,j}(A_i) = 0$$

$$\mathbb{I}_{i,j}(A_j) = 1$$

We will denote the outer normal of the segment A_iA_{i+1} by $\mathbf{n}_{i,i+1}$. Finally, we can write the derivative of the first eigenvalue with respect to the vertices of the polygon in the following way

$$\frac{d\lambda_1}{dx_{2i-1}} = -\int_{A_{i-1}A_i} \mathbb{I}_{i-1,i}(\frac{\partial u}{\partial \mathbf{n}})^2 \mathbf{n}_{i-1,i}\, d\sigma$$

$$-\int_{A_iA_{i+1}} \mathbb{I}_{i,i+1}(\frac{\partial u}{\partial \mathbf{n}})^2 \mathbf{n}_{i,i+1}\, d\sigma$$

and we also have

$$\frac{d\lambda_1}{dx_{2i}} = -\int_{A_{i-1}A_i} \mathbb{I}_{i-1,i}(\frac{\partial u}{\partial \mathbf{n}})^2 \mathbf{n}_{i-1,i}\, d\sigma$$

$$-\int_{A_iA_{i+1}} \mathbb{I}_{i,i+1}(\frac{\partial u}{\partial \mathbf{n}})^2 \mathbf{n}_{i,i+1}\, d\sigma$$

Once we have computed everything we have introduced, the last step is to find the minimal value using gradient descent. Following this method one can see that for polygons with $N \in [5, 15]$ number of sides we quickly converge to the regular polygon [1].

Conclusion

The shapes that minimize the value of the first eigenvalue for the Dirichlet Laplacian have been of interest since Lord Rayleigh's Conjecture. This was extended to polynomials by Polya and Szego, and was proven for specific cases by Polya. Because of the difficulty of proving this conjecture directly, we have introduced a numerical approximation method for checking the conjecture for polygons that have 5 or more sides.

To further this area of study, one could study the rate at which the gradient descent approaches a critical point. One could also investigate improvements of the process we took to set up the gradient descent. The major downside of this approach is that the runtime grows quickly as the number of edges increases. It would be of great interest to use computational methods to compute the necessary values for each vertex in parallel.

References

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