The Pólya–Szegő Conjecture on Polygons: A Numerical Approach

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THESIS ABSTRACT

TODO SOMETHING ABOUT THE ABSTRACT THAT IS ABOUT THIS LONG OR SO

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The topic that I chose to explore for this thesis is a study of the eigenvalues of the Dirichlet Laplacian on a two dimensional domain and ...

Acknowledgment

I would be remiss if I did not take a moment to express appreciation for all of the people who have helped me through this process.

Contents

Chapter 1

Introduction

1.1 Physical Motivation

Physical drums consist of a rigid shell with a membrane which produces sound when hit. A similar thing can be "created" in a pure mathematical setting by studing specific partial differential equations over a closed region. Specifically, the frequencies of the drum membrane corresponds to the eigenvalues of the Dirichlet Laplacian. This construction makes it possible to "hear" drums where the shape of the drumhead is any closed and simple curve. To do this, we begin with the shape of our drumhead, which is a region of the real plane bounded by piecewise smooth curves. Since we wish to emulate the physical properties of a drum, we want to define some system that models the vibration of the drum membrane which produces the sound. This is done using the wave equation over our boundary, with the boundary condition that the function is 0 on the boundary. The Dirichlet Laplacian allows us to model the physical vibration of the drum, and we can calculate its eigenvalues to find the fundamental frequency and overtones. Thus, by modeling the vibration of the drumhead using the Dirichlet Laplacian and studying its properties, we can produce "sound" via the eigenvalues.

In 1877, Lord Rayleigh conjectured the following [5]

> If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.

In 1923, Faber published a proof which was followed by an independent proof by Krahn in 1925 [2].

Theorem 1.1.1 (Faber-Krahn). *Let c be a positive number and B the ball with volume c. Then,*

$$\lambda_1(B) = \min\left\{\lambda_1(\Omega), \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c\right\}.$$

From the Faber-Krahn inequality, we know that for any drumhead with a given area, the circle is the one with the lowest tone. In 1951, Polya and Szego conjectured that a similar statement holds for drumheads with a polygonal shape [3]. This conjecture has been shown to be true for 3 and 4 sided polygons, but remains unproven for any other number of sides.

There are two main hurdles that are halting progress on this conjecture. The first is that the tools that were used to prove both Lord Rayleighs conjecture as well as the small cases for the Polya-Szego conjecture are not available when the number of sides is greater than four. The main tool that is used is called Steiner Symmetrization, and when there are more than four sides this symmetrization method creates additional sides at each step.

The purpose of this paper is to show a specific method for running numerical approximations to suggest that this conjecture is indeed true. This is done using a method based on fundamental solutions [1]. Specifically, we consider all functions that satisfy the Laplace Equation and then solve for the linear coefficients using the boundary conditions. Once this is done we use gradient descent to find the polygon with the minimum first eigenvalue, which is equivalent to the first fundamental tone of the drum.

Consider a homogeneous elastic drumhead, or membrane, stretched over a rigid frame. We will represent the frame as a domain $\Omega \subset \mathbb{R}^2$. Take the function $u(x, y, t)$ to be the vertical displacement of the membrane from its resting position. Then for any disk $D \subset \Omega$, Newton's second law of motion states that

$$\int_{\partial D} T \frac{\partial u}{\partial \mathbf{n}} \, dS = \int_D \rho u_{tt} \, dA$$

where T is the constant tension, ρ is the density constant, and $\mathbf{n}$ is the outward normal of the boundary. By the divergence theorem, we have

$$\int_D T \Delta u \, dA = \int_D \rho u_{tt} \, dA$$

where Δ is the Laplace operator. From this we can get the wave equation on Ω

$$u_{tt} = c^2 \Delta u$$

where we define u to be 0 on the boundary and where $c = \sqrt{T/\rho}$. We can solve this wave equation using $u(x, y, t) = T(t)V(x, y)$ which gives us

$$\frac{T''}{c^2 T} = \frac{\Delta V}{V} = -\lambda$$

and finally we have reduced our problem to the Dirichlet Laplacian

$$\Delta V = -\lambda V$$

where V on the boundary is zero.

NOTE: The best reference I could find is Logan's Applied Partial Differential Equations. I could also use Ryans paper

In the next section, we will start from the Dirichlet Laplacian and introduce the conjectures in a formal setting.

1.2 Polya-Szego's Conjecture

1. Introduce Rigorous Definitions from 1.1.2 Henrot

2. Dirichlet Laplacian eigenvalues prereqs

3. Faber Krahn

4. Polya-Szego Conjecture [4]

1.3 Known Results

1. All Explicit Cases

2. Tools for n=3 and n=4

1.4 Numerical Analysis Tools

Chapter 2

Background

2.1 Notations and Definitions

2.2 Function Spaces

Definition 2.2.1. *A complex linear space* $\mathbb{H}$ *is called a normed linear space if there exists a map* $||\cdot|| : \mathbb{H} \to \mathbb{R}^+$ *such that for any* $x, y \in \mathbb{H}$ *and* $\lambda \in \mathbb{C}$*,*

1. $||\lambda x|| = |\lambda| ||x||$
2. $||x + y|| \leq ||x|| + ||y||$
3. $||x|| \geq 0$*, and* $||x|| = 0$ *if and only if* $x = 0$

Definition 2.2.2. *A complex linear space* $\mathbb{H}$ *is called an inner product space with inner product* $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ *if for any* $x, y, z \in \mathbb{H}$ *and* $\lambda \in \mathbb{C}$*,*

1. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
4. $\langle x, x \rangle \geq 0$*, and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$*.*

Definition 2.2.3. *A Hilbert space is a complete inner product space*

2.3 PDEs

Definition 2.3.1 (Laplacian)**.**

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}.$$

Definition 2.3.2. *The Dirichlet Laplacian is the Laplace Operator subject to Dirichlet boundary conditions. That is, we call u a solution to the Dirichlet Laplacian if u is a solution to*

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

2.4 Calculus of Variations

2.5 Tools

Definition 2.5.1 (Schwarz Rearrangement)**.** *For any measurable set* ω *in* $\mathbb{R}^N$*, we denote by* ω^* *the ball of same volume as* ω*. If* u *is a non-negative measurable function defined on a measurable set* Ω *and vanishing on its boundary* $\partial\Omega$*, we denote by* $\Omega(c) = \{x \in \Omega \,|\, u(x) \geq c\}$ *its level sets. The Schwarz rearrangement of* u *is the function* u^* *defined on* Ω^* *by*

$$u^*(x) = \sup\{c \,|\, x \in \Omega(c)^*\}.$$

Theorem 2.5.1. *Let* Ω *be a measurable set and* u *be a non-negative measurable function defined on* Ω *and vanishing on its boundary* $\partial\Omega$*. Let* ϕ *be any measurable function defined on* $\mathbb{R}^+$ *with values in* $\mathbb{R}$*, then*

$$\int_\Omega \phi(u(x))\,\mathrm{d}x = \int_{\Omega^*} \phi(u^*(x))\,\mathrm{d}x.$$

Theorem 2.5.2 (Pólya's Inequality)**.** *Let* Ω *be an open set and* u *a non-negative function belonging to the Sobolev space* $H_0^1(\Omega)$*. Then* $u^* \in H_0^1(\Omega^*)$ *and*

$$\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x \geq \int_{\Omega^*} |\nabla u^*(x)|^2\,\mathrm{d}x.$$

TODO Steiner Symmetrization

Chapter 3

Eigenvalues of Dirichlet Laplacian

3.1 Definition

Definition 3.1.1 (Rayleigh Quotient)**.** *For an operator L, we define the Rayleigh quotient to be*

$$R_L[v] := \frac{\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x + \int_\Omega a_0(x) v^2(x) \, \mathrm{d}x}{\int_\Omega v(x)^2 \, \mathrm{d}x}.$$

This is used to express the first eigenvalue of the Dirichlet Laplacian in the following way

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2 \, \mathrm{d}x}{\int_\Omega v(x)^2 \, \mathrm{d}x}.$$

Theorem 3.1.1. *Let Ω be a bounded open set. We assume that $\lambda_k'(\Omega)$ is simple. Then, the functions $t \to \lambda_k(t), t \to u_t \in L^2(\mathbb{R}^N)$ are differentiable at $t = 0$ with*

$$\lambda_k'(0) := -\int_\Omega \operatorname{div}(|\nabla u|^2 V) \, \mathrm{d}x.$$

If, moreover, Ω is of class C^2 or if Ω is convex, then

$$\lambda_k'(0) := -\int_\Omega \left(\frac{\partial u}{\partial n} \right)^2 V.n \, \mathrm{d}\sigma$$

and the derivative u' of u_t is the solution of

$$\begin{cases} -\Delta u' = \lambda_k u' + \lambda_k' u & \text{in} \Omega \\ \quad u' = -\frac{\partial u}{\partial n} V.n & \text{on} \partial\Omega \\ \int_\Omega u u' \, \mathrm{d}\sigma = 0. \end{cases}$$

3.2 Known Results

1. invariant under translations rotations
2. homothety
3. continuous

Theorem 3.2.1 (Faber-Krahn)**.** *Let c be a positive number and B the ball with volume c. Then,*

$$\lambda_1(B) = \min\left\{\lambda_1(\Omega), \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c\right\}.$$

Proof. This proof is a straightforward application of Schwarz rearrangement (2.5.1) [3]. Let Ω be a bounded open set of measure c and $\Omega^* = B$ be the ball of the same volume. Let u_1 be a en eigenfunction with eigenvalue $\lambda_1(\Omega)$ and u_1^* its Schwarz rearrangement. Using 2.5.1 we have

$$\int_{\Omega^*} u_1^*(x)^2 \,\mathrm{d}x = \int_{\Omega} u_1(x)^2 \,\mathrm{d}x.$$

Further, using 2.5.2 we have

$$\int_{\Omega^*} |\nabla u_1^*(x)|^2 \,\mathrm{d}x \leq \int_{\Omega} |\nabla u_1(x)|^2 \,\mathrm{d}x.$$

Using Rayleigh quotients (3.1.1) we get the following

$$\lambda_1(\Omega^*) \leq \frac{\int_\Omega |\nabla u_1^*(x)|^2 \,\mathrm{d}x}{\int_\Omega u_1^*(x)^2 \,\mathrm{d}x}.$$

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u_1(x)|^2 \,\mathrm{d}x}{\int_\Omega u_1(x)^2 \,\mathrm{d}x}.$$

Using the previous two statements yields the desired results. □

3.3 Polygons

Note P_N is the class of plane polygons with at most N edges.

Theorem 3.3.1. *Let $a > 0$ and $N \in \mathbb{N}$ be fixed. Then the problem*

$$\min \{\lambda_1(\Omega), \Omega \in P_N, |\Omega| = a\}$$

has a solution.

Proof. 47 henrot □

Theorem 3.3.2. *Let $M \in \mathbb{N}$ and Ω be a polygon with M edges. Then Ω cannot be a (local) minimum for $|\Omega|\lambda_1(\Omega)$ in the class P_{M+1}.*

Theorem 3.3.3 (Pólya)**.** *The equilateral triangle has the least first eigenvalue among all triangles of given area. The square has the least first eigenvalue among all quadrilaterals of given area.*

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