The Pólya–Szegő Conjecture on Polygons: A Numerical Approach

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TEST PAGE

THESIS ABSTRACT

The Pólya–Szegő Conjecture on Polygons: A Numerical Approach

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The famous Faber-Krahn inequality states that the ball is the minimizer for the first eigenvalue of the Dirichlet Laplacian. In 1950, Pólya and Szegő conjectured a similar result over the class of plane polygons. They conjectured that for all polygons with N vertices and a fixed area, the regular polygon minimizes the first Dirichlet eigenvalue. While it has been proven for the cases $N = 3, 4$, this problem remains open for all $N \geq 5$ despite its apparent simplicity. In this paper we construct a numerical method to approximate the polygon which minimizes the first Dirichlet eigenvalue, which we compare to Pólya's conjecture. We will also prove various properties of the Dirichlet eigenvalue.

Acknowledgment

Write Achknowledgement

I would be remiss if I did not take a moment to express appreciation for all of the people who have helped me through this process.

Contents

Chapter 1

Introduction

1.1 Physical Motivation

Physical drums consist of a rigid shell with a membrane which produces sound when hit. A similar thing can be "created" in a pure mathematical setting by studing specific partial differential equations over a closed region. Specifically, the frequencies of the drum membrane corresponds to the eigenvalues of the Dirichlet Laplacian. This construction makes it possible to "hear" drums where the shape of the drumhead is any closed and simple curve. To do this, we begin with the shape of our drumhead, which is a region of the real plane bounded by piecewise smooth curves. Since we wish to emulate the physical properties of a drum, we want to define some system that models the vibration of the drum membrane which produces the sound. This is done using the wave equation over our boundary, with the boundary condition that the function is 0 on the boundary. The Dirichlet Laplacian allows us to model the physical vibration of the drum, and we can calculate its eigenvalues to find the fundamental frequency and overtones. Thus, by modeling the vibration of the drumhead using the Dirichlet Laplacian and studying its properties, we can produce "sound" via the eigenvalues. Let us show this construction in more detail.

Consider a homogeneous elastic drumhead, or membrane, stretched over a rigid frame. We will represent the frame as a domain $\Omega \subset \mathbb{R}^2$. Take the function $u(x, y, t)$ to be the vertical displacement of the membrane from its resting position. Then for any disk $D \subset \Omega$, Newton's second law of motion states that

$$\int_{\partial D} T \frac{\partial u}{\partial \mathbf{n}} \, dS = \int_D \rho u_{tt} \, dA$$

where T is the constant tension, ρ is the density constant, and $\mathbf{n}$ is the outward normal of the boundary. By the divergence theorem, we have

$$\int_D T \Delta u \, dA = \int_D \rho u_{tt} \, dA$$

where Δ is the Laplace operator. From this we can get the wave equation on Ω

$$u_{tt} = c^2 \Delta u$$

where we define u to be 0 on the boundary and where $c = \sqrt{T/\rho}$. We can solve this wave equation using $u(x, y, t) = T(t)V(x, y)$ which gives us

$$\frac{T''}{c^2 T} = \frac{\Delta V}{V} = -\lambda$$

and finally we have reduced our problem to the Dirichlet Laplacian

$$\Delta V = -\lambda V$$

where V on the boundary is zero.

Add History and problems part

Use ppts for structure

mention isoperimetric and add it

Explain Issue with proof method for cases N ¿= 5

1.2 Background

Talk about symmetrizations and maybe move the specific eigenvalue examples here

Should have Disk, rect, Tri, Pólya showing two cases and the rest being open

. The isoperimetric problem is as follows: Among all plane figures with a given perimeter L, which one encloses the greatest area A? Equivalently: among all plane figures with a given area, which one has the least perimeter? The importance of the isoperimetric problem can hardly be understated. One can find it in Vergil's version of the legend of Dido as well as one of the fundamental problems in the classical calculus of variations.

The isoperimetric problem has been studied in various forms since the days of ancient greece. As far back as 200BC, a greek mathematician named Zenodorus showed that a circle has greater area than any polygon with the same perimeter[10]. Zenodorus also proved that for regular n-polygons with the same perimeter L, its area A_n increases as $n \to \infty$, and that $A_n \leq \frac{L^2}{4\pi}$

In 1877, Lord Rayleigh conjectured the following [9]

> If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.

Note that this is a specific example of an isoperimetric problem. More importantly, it is the original motivating conjecture which will eventually lead us to the Pólya-Szego conjecture.

In 1923, Faber published a proof which was followed by an independent proof by Krahn in 1925 [4].

Theorem 1.2.1 (Faber-Krahn). *Let c be a positive number and B the ball with* *volume c. Then,*

$$\lambda_1(B) = \min \left\{ \lambda_1(\Omega), \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c \right\}.$$

In terms of our physical motivation, this theorem states that for any drumhead with a given area, the circle is the one with the lowest tone. In 1951, Polya and Szego conjectured a similar statement about regular polygons[8]

Paste Polya conjecture here

This conjecture can be viewed as analogous to the Faber-Krahn Theorem, as regular polygons are in some sense the roundest polygons for a given number of sides. However, while this conjecture is simple to state and simple to understand, it has been largely left unsolved for 70 years. A major contributor to the difficulty is that there are few polygons whose spectrum can be explicitly calculated. These polygons are equilateral triangles, hemi-equilateral triangles, and isosceles-right triangles [7].

The following figure contains every polygon whose spectrum we can calculate explicitely,

Copy this file from seminar class folder

Although our ability to explicitely calculate the fundamental eigenvalue is very limited, there has been some success in proving the conjecture. Polya himself proved his conjecture in the cases of $N = 3$ and $N = 4$ [6].

Theorem 1.2.2 (Polya)**.** *The equilateral triangle has the least first eigenvalue among all triangles of given area. The square has the least first eigenvalue among all quadrilaterals of given area.*

Should I put the explicit eigenvalues examples here or elsewhere?

Unfortunately, the method Pólya used to prove these two cases utilizes the Steiner symmetrization. This causes issues as applying Steiner symmetrization to polynomials with $N \geq 5$ will always increase the number of vertices, which means we cannot produce a sequence of polynomials with vertices N that fixes the area and minimizes the perimeter.

From all of this, it seems as though the lack of a proof comes from the difficulty of constructing the minimizing sequence and not from the validity of the statement. To study this, we will come up with an algorithm that approximates the minimizing polygon. This will give us more concrete data to justify (or contradict) the Pólya-Szego conjecture.

1.3 Examples

Build from Henrot pg 10

We begin with a simple one dimensional case. Let $\Omega = (0, L)$. Solving the differential equation

$$\begin{cases} -u'' = \lambda u & x \in \Omega \\ u(0) = u(L) = 0 \end{cases},$$

we find that the only non-trivial solutions are

$$\lambda_n = \frac{n^2\pi^2}{L^2},\ u_n = \sin\left(\frac{n\pi x}{L}\right),\ n \geq 1.$$

$$\Omega = \{(x, y) : 0 < x < a, 0 < y < b, a, b \in \mathbb{R}\}.$$

Finish Rect and add Tri and maybe disk

Should I include a section on the numerical stuff?

Chapter 2

Preliminaries

2.1 Measure Theory

Definition 2.1.1. *A Measure space is a pair (X, A) where X is a non-empty set and A is a σ-algebra of subsets of X. That is, A satisfies the following*

1. $\emptyset \in A$
2. *For countably many $A_j \in A$, $\cup A_j \in A$*
3. *If $B \in A$, then $X - B \in A$.*

Elements in A are called *measurable sets*. A *measure* μ on (X, A) is a non-negative function $\mu : A \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\cup A_j) = \Sigma\mu(A_j)$ for any countably many, mutually disjoint $A_j \in A$. μ is said to be *finite* if $\mu(X) < \infty$, and μ is σ-finite if X is a countable union of sets in A with finite measures. A property is said to hold *almost everywhere* on a set A if it holds on A save a subset with zero measure.

A function $f : X \to [-\infty, \infty]$ is *measurable* if $\{x \in X : f(x) < \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$. A *simple function* is a function of the form

$$f = \sum_{j=1}^{m} a_j \chi_{A_j},$$

where χ_S is the characteristic function on the set S, $a_j \in \mathbb{R}$, and $A_j \in A$.

Theorem 2.1.1 (Simple Function Approximation Theorem)**.** *Let $f : X \to [-\infty, \infty]$ be a measurable function. If f is non-negative, then there exists an increasing sequence of simple functions ϕ_j such that $0 \leq \phi_j \leq f$ and $\lim_{j\to\infty} \phi_j(x) = f(x)$. If f is bounded, then there exists a sequence of simple functions ϕ_j such that $\phi_j \to f$ uniformly on X.*

Proof.

Write Proof: Fu Notes and Evans Appendix

□

We will assume our measure spaces (X, A, μ) are complete. That is, if $B \subset N$ and $\mu(N) = 0$ then $B \in A$. The *integration* with respect to μ is defined in the following way. We first define the integral for non-negative simple functions. Let $\phi = \sum_{j=1}^{m} a_j \chi_{A_j} \geq 0$, and define

$$\int_X \phi \,\mathrm{d}\mu = \sum_{j=1}^{m} a_j \mu(A_j)$$

where we use the convention that $0 \cdot \infty = 0$. We define the integral for non-negative measurable functions as

$$\int_X f \,\mathrm{d}\mu = \sup\left\{\int_X \phi \,\mathrm{d}\mu ; 0 \leq \phi \leq f, \phi \text{ simple}\right\}.$$

For a measurable function $f : X \to [-\infty, \infty]$ we write $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, and we define

$$\int_X f \,\mathrm{d}\mu = \int_X f^+ \,\mathrm{d}\mu - \int_X f^- \,\mathrm{d}\mu$$

given at least one of the integrals on the right hand side is finite. When f is complex-valued we define the integral by integrating the real and complex parts seperately. When $\int_X |f| \,\mathrm{d}\mu < \infty$ we say f is *integrable*.

Definition 2.1.2. *If $0 < p < \infty$ and if f is a complex measurable function on X, define*

$$||f||_p = \left\{\int_X |f|^p \,\mathrm{d}\mu\right\}^{\frac{1}{p}}$$

and let $L^p(\mu)$ consist of all f for which $||f||_p < \infty$.

For $f : X \to \mathbb{C}$, the *support* of f is defined as $\operatorname{supp}(f) = \overline{\{x \in X; f(x) \neq 0\}}$. Denote by $C_c(X)$ the family of continuous functions on X with compact support and by $C_0(X)$ the family of continuous functions that vanish at infinity.

2.2 Functional Analysis

Definition 2.2.1. *A complex linear space* $\mathbb{H}$ *is called a normed linear space if there exists a map* $||\cdot|| : \mathbb{H} \to \mathbb{R}^+$ *such that for any* $x, y \in \mathbb{H}$ *and* $\lambda \in \mathbb{C}$*,*

1. $||\lambda x|| = |\lambda| ||x||$
2. $||x + y|| \leq ||x|| + ||y||$
3. $||x|| \geq 0$*, and* $||x|| = 0$ *if and only if* $x = 0$

We call this map a norm

Definition 2.2.2. *A Banach space* X *is a complete, normed linear space.*

Definition 2.2.3. *We say* X *is seperable if* X *contains a countable dense subset.*

Definition 2.2.4. *A complex linear space* $\mathbb{H}$ *is called an inner product space with inner product* $\langle\cdot,\cdot\rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ *if for any* $x, y, z \in \mathbb{H}$ *and* $\lambda \in \mathbb{C}$*,*

1. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
4. $\langle x, x\rangle \geq 0$*, and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$*.*

For an inner product $\langle,\rangle$, the associated *norm* is $||u|| := \langle u, u\rangle^{\frac{1}{2}}$ for $u \in \mathbb{H}$. One could verify both of the definitions of a *norm* via the Cauchy-Schwarz inequality. We say that two elements $u, v \in \mathbb{H}$ are *orthogonal* if $\langle u, v\rangle = 0$. A *countable basis* $\{w_k\}_{k=1}^{\infty} \subset \mathbb{H}$ is *orthonormal* if the elements are pairwize orthogonal and the norm of each element is one.

Definition 2.2.5. *A Hilbert space* $\mathbb{H}$ *is a Banach space endowed with an inner product which generates the norm.*

For the remainder of this paper all Hilbert spaces will be assumed to be seperable.

Let X, Y be real Banach spaces.

Definition 2.2.6. *A mapping $A : X \to Y$ is a linear operator provided*

$$A(au + bv) = aAu + bAv$$

for all $u, v \in X$ and $a, b \in \mathbb{R}$.

Definition 2.2.7. *A linear operator $A : X \to Y$ is bounded if*

$$||A|| := \sup \{||Au||; ||u|| \leq 1\} \leq \infty.$$

Definition 2.2.8. *A linear operator $A : X \to Y$ is closed if whenever $u_k \to u$ in X and $Au_k \to v$ in Y, then $Au = v$*

Definition 2.2.9. *Let $T \in L(\mathbb{H})$ be a non-negative, self-adjoint operator. The form defined by*

$$Q(f, g) = \left(T^{1/2} f, T^{1/2} g\right)$$

with $Dom(Q) = Dom(T^{1/2})$ is called the sesquilinear form associated with T.

Let $f \in C^\infty(\Omega)$. Then using integration by parts we have

$$(\partial^\alpha f, \phi) = \left(f, (-1)^{|\alpha|} \partial^\alpha \phi\right)$$

for any $\phi \in C_c^\infty$. We denote $L_{loc}(\Omega)$ to be the space of Lebesgue measurable functions that are integrable over a compact subset of Ω.

Definition 2.2.10. *Let $f, g \in L_{loc}(\Omega)$. We say that $g = \partial^\alpha f$ in the sense of distribution if $\left(f, (-1)^{|\alpha|} \partial^\alpha \phi\right) = (g, \phi)$ for any $\phi \in C_c^\infty(\Omega)$.*

Rewrite below using Fu's notes

The following is primarily from Henrot[6]. Let Ω be a bounded open set in $\mathbb{R}^N$. We denote by $L^2(\Omega)$ the Hilbert space of square summable functions defined on Ω and by $H^1(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives (in the sense of distrubutions) are in $L^2(\Omega)$. When $H^1(\Omega)$ is endowed with the scalar product

$$(u, v)_{H^1} := \int_\Omega u(x)v(x)\,\mathrm{d}x + \int_\Omega \nabla u(x)\nabla v(x)\,\mathrm{d}x$$

and the corresponding norm

$$||u||_{H^1} := \left(\int_\Omega u(x)^2\,\mathrm{d}x + \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

$H^1(\Omega)$ becomes a Hilbert space. In the case of Dirichlet boundary conditions, we will use the subspace $H_0^1(\Omega)$ which is defined as the closure of C^∞ functions compactly supported in Ω for the norm $||\cdot||_{H^1}$. This is also a Hilbert space.

We will occasionally need to work with spaces $L^p, p \geq 1$ instead of L^2. In this case we define the Sobolev spaces in the same way and denote them $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively. These are Banach spaces.

2.3 PDEs

We begin by defining the Laplacian Operator

Definition 2.3.1 (Laplacian)**.**

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2},$$

where derivatives are to be understood in the sense of distrubutions.

The following two results will be used in the definition of the Dirichlet Laplacian.

Theorem 2.3.1. *Let $T \in L(\mathbb{H})$ be non-negative and self-adjoint. Then*

1. *$f \in Dom(T)$ if and only if $f \in Dom(Q)$ and there exists a vector $g \in \mathbb{H}$ such that $Q(f, \phi) = (g, \phi)$ for all $\phi \in Dom(Q)$. In this case $g = Tf$.*

2. *$Dom(T)$ is dense in $Dom(Q)$ in $|| \cdot ||_1 - norm$.*

Source/proof

Theorem 2.3.2. *Let Q be a non-negative sesquilinear form with dense domain $Dom(Q) in \mathbb{H}$. The following are equivalent*

1. *Q is closed.*

2. *The associated quadratic form $\widetilde{Q}$ is lower semi-continuous.*

3. *There exists a unique non-negative, self-adjoint operator T such that $Dom(T^{\frac{1}{2}}) = Dom(Q)$ and $Q(f, g) = \left(T^{\frac{1}{2}} f, T^{\frac{1}{2}} g\right)$.*

Rewrite proof from Dr. Fu's lecture notes or find reference.

Definition 2.3.2. *Let Ω be an open set in $\mathbb{R}^N$, and let $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N})$ be the gradient operator. Let*

$$Q_\Omega^D(u, v) = (\nabla u, \nabla v)\,,\ Dom(Q_\Omega^D) = W_0^1(\Omega).$$

Since $C_c^\infty(\infty) \subset W_0^1(\Omega) \subset L^2(\Omega)$ *and* C_c^∞ *is dense in* $L^2(\Omega)$*, the sesquilinear form* Q_Ω^D *is densely defined on* $L^2(\Omega)$*. Also, since* $W_0^1(\Omega)$ *is complete,* Q_Ω^D *is closed. Then, by (2.3.2),* Q_Ω^D *uniquely determines a densely defined self adjoint operator* $\Delta : L^2(\Omega) \to L^2(\Omega)$ *such that*

$$Q_\Omega^D(u,v) = \left(\Delta^{\frac{1}{2}}u, \Delta^{\frac{1}{2}}v\right), \; Dom(\Delta^{\frac{1}{2}}) = Dom(Q^D) = W_0^1(\Omega).$$

The operator Δ *thus defined is called the Dirichlet Laplacian, and we will denote it by* Δ_Ω^D*.*

In order for this definition to be useful, we will show this definition is equivalent to the standard definition of the Dirichlet Laplacian we used in the introduction. By (2.3.1), $f \in \mathrm{Dom}(\Delta^D)$ if and only if $f \in W_0^1(\Omega)$ and there exists some $g \in L^2(\Omega)$ such that

$$(g,\phi) = Q^D(f,\phi) = (\nabla f, \nabla \phi) = \left(-\nabla^2 f, \phi\right)$$

for all $\phi \in C_0^\infty(\Omega)$. Thus $-\nabla^2 f = g \in L^2(\Omega)$ in the sense of distribution. Therefore we have

$$\Delta^D = -\nabla^2 = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}, \; \mathrm{Dom}(\Delta^D) = W_0^1(\Omega) \cap \left\{f \in L^2(\Omega) : \nabla^2 f \in L^2(\Omega)\right\}.$$

Since $f \in W_0^1$ implies $f = 0$ on the boundary of Ω, we have shown this definition is consistent with the classical definition.

Next, lets state a general property of the Laplacian that will be used constantly.

Theorem 2.3.3. *The Laplacian is invariant under orthogonal transformations.*

Proof. Let $f \in C_c^2(\mathbb{R}^N)$ and let A be an orthogonal $n \times n$ matrix over $\mathbb{R}$. Also, let $x = (x_1, x_2, \ldots, x_N)$. Since A is orthogonal, $\sum_{j=1}^N a_{ij}a_{kj} = \delta_{ik}$ where δ_{ik} is the Kronecker Delta function. So we have

$$(f \circ A)(x) = f\left(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i\right).$$

Take $z_i = g_i(x_1, x_2, \ldots, x_N) = \sum_{k=1}^{N} a_{ik}x_k$. From a direct application of the chain rule we obtain

$$\frac{d}{dx_j}(f \circ A)(x) = \sum_{k=1}^{N} a_{kj} \cdot (\partial_k f)(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i).$$

Further, by taking $\partial_k f$ in place of f, we obtain

$$\frac{d^2}{dx_j^2}(f \circ A)(x) = \sum_{k=1}^{N} a_{kj} \sum_{\ell=1}^{N} a_{\ell j}(\partial_\ell \partial_k f)(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i).$$

With all of these pieces in place, we have the following

$$\begin{aligned}
\Delta(f \circ A)(x) &= \sum_{j=1}^{N} \frac{d^2}{dx_j^2}(f \circ A)(x) \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} a_{kj} \sum_{\ell=1}^{N} a_{\ell j}(\partial_\ell \partial_k f)(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i) \\
&= \sum_{k,\ell=1}^{N} \left(\sum_{j=1}^{N} a_{kj}a_{\ell j}\right)(\partial_\ell \partial_k f)(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i) \\
&= \sum_{k,\ell=1}^{N} \delta_{k,\ell}(\partial_\ell \partial_k f)(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i) \\
&= \sum_{k=1}^{N} (\partial_k^2 f)(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i) \\
&= (\Delta f)(\sum_{i=1}^{N} a_{1i}x_i, \ldots, \sum_{i=1}^{N} a_{di}x_i) \\
&= (\Delta f)(Ax) \\
&= ((\Delta f) \circ A)(x).
\end{aligned}$$

Hence $\Delta(f \circ A)(x) = ((\Delta f) \circ A)(x)$ and so f is invariant under orthogonal transformations.

$\square$

We will use this result thoughout the rest of the proofs without explicit reference, especially when using translations and rotations.

2.4 Tools

Definition 2.4.1 (Schwarz Rearrangement)**.** *For any measurable set ω in $\mathbb{R}^N$, we denote by ω^* the ball of same volume as ω. If u is a non-negative measurable function defined on a measurable set Ω and vanishing on its boundary $\partial\Omega$, we denote by $\Omega(c) = \{x \in \Omega \,|\, u(x) \geq c\}$ its level sets. The Schwarz rearrangement of u is the function u^* defined on Ω^* by*

$$u^*(x) = \sup\{c/x \in \Omega(c)^*\}.$$

Without loss of generality, we fix the hyperplane of symmetry to be $x_N = 0$. Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a measurable set. We denote by Ω' the projection of Ω on $\mathbb{R}^{N-1}$, and for every $x' \in \mathbb{R}^{N-1}$ we denote by $\Omega(x')$ the projection of Ω with $\{x'\} \times \mathbb{R}$.

Definition 2.4.2 (Steiner Symmetrization)**.** *Let $\Omega \subset \mathbb{R}^N$ be measurable. Then the set*

$$\Omega^* := \left\{ x = (x', x_N) : -\frac{1}{2}|\Omega(x')| < x_N < \frac{1}{2}|\Omega(x')|, x' \in \Omega' \right\}$$

is the Steiner symmetrization of Ω with respect to the hyperplane $x_N = 0$.

Theorem 2.4.1. *Let Ω be a measurable set and u be a non-negative measurable function defined on Ω and vanishing on its boundary $\partial\Omega$. Let ϕ be any measurable function defined on $\mathbb{R}^+$ with values in $\mathbb{R}$, then*

$$\int_\Omega \phi(u(x))\,\mathrm{d}x = \int_{\Omega^*} \phi(u^*(x))\,\mathrm{d}x.$$

Theorem 2.4.2 (Pólya's Inequality)**.** *Let Ω be an open set and u a non-negative function belonging to the Sobolev space $H_0^1(\Omega)$. Then $u^* \in H_0^1(\Omega^*)$ and*

$$\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x \geq \int_{\Omega^*} |\nabla u^*(x)|^2\,\mathrm{d}x.$$

Chapter 3

Eigenvalues of the Dirichlet Laplacian

3.1 Definition

Definition 3.1.1 (Rayleigh Quotient)**.** *For an operator L, we define the Rayleigh quotient to be*

$$R_L[v] := \frac{\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x + \int_\Omega a_0(x) v^2(x) \, \mathrm{d}x}{\int_\Omega v(x)^2 \, \mathrm{d}x}.$$

This is used to express the first eigenvalue of the Dirichlet Laplacian in the following way

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2 \, \mathrm{d}x}{\int_\Omega v(x)^2 \, \mathrm{d}x}.$$

Theorem 3.1.1. *Let Ω be a bounded open set. We assume that $\lambda_k'(\Omega)$ is simple. Then, the functions $t \to \lambda_k(t), t \to u_t \in L^2(\mathbb{R}^N)$ are differentiable at $t = 0$ with*

$$\lambda_k'(0) := -\int_\Omega \operatorname{div}(|\nabla u|^2 V) \, \mathrm{d}x.$$

If, moreover, Ω is of class C^2 or if Ω is convex, then

$$\lambda_k'(0) := -\int_\Omega \left(\frac{\partial u}{\partial n} \right)^2 V.n \, \mathrm{d}\sigma$$

and the derivative u' of u_t is the solution of

$$\begin{cases} -\Delta u' = \lambda_k u' + \lambda_k' u & \text{in} \Omega \\ \quad u' = -\frac{\partial u}{\partial n} V.n & \text{on} \partial\Omega \\ \int_\Omega u u' \, \mathrm{d}\sigma = 0. \end{cases}$$

Theorem 3.1.2. *Each eigenvalue is real. Furthermore, if we repeat each eigenvalue according to its (finite) multiplicity, we have $\Sigma = \{\lambda_k\}_{k=1}^\infty$ where Σ is the set of eigenvalues, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$, and $\lambda_k \to \infty$ as $k \to \infty$.*

Proof.

Pg 335 Evans. Ask Dr. Fu about elementary proof for first statement

□

3.2 Known Results

Rework Sections for this chapter. Known Results isn't descriptive

Let $k > 0$ and H_k be a homothety of origin α and ratio k. That is, $H_k(x) := kx$. For a function u defined on Ω, we define the function $H_k u$ on $H_k(\Omega)$ by $H_k u(x) := u(x/k)$. Since $H_k \circ \Delta = k^2 \Delta \circ H_k$, we have

$$\lambda_n(H_k(\Omega)) = \frac{\lambda_n(\Omega)}{k^2}.$$

Using these basic properties, we can construct a correspondence between two minimization problems [6].

Theorem 3.2.1. *The minimization problems* $\min \{\lambda_n(\Omega); |\Omega| = c\}$ *as well as* $\min \left\{|\Omega|^{2/N} \lambda_n(\Omega)\right\}$ *are equivalent. That is, there exists a bijective correspondence between the solutions of these two problems.*

Further, as the functional $\Omega \mapsto |\Omega|^{2/N} \lambda_n(\Omega)$ is invariant under homothety, we can construct the coorespondence explicitely as follows. Every solution of $\min \{\lambda_n(\Omega); |\Omega| = c\}$ is a solution of $\min \left\{|\Omega|^{2/N} \lambda_n(\Omega)\right\}$. In the other direction, if Ω is a solution of $\min \left\{|\Omega|^{2/N} \lambda_n(\Omega)\right\}$ with volume c', then for $k = \frac{c}{c'}^{1/N}$ the homothety $H_k(\Omega)$ is a solution of $\min \{\lambda_n(\Omega); |\Omega| = c\}$.

Theorem 3.2.2 (Faber-Krahn)**.** *Let c be a positive number and B the ball with volume c. Then,*

$$\lambda_1(B) = \min \left\{\lambda_1(\Omega), \Omega \text{ open subset of } \mathbb{R}^N, |\Omega| = c\right\}.$$

Proof. This proof is a straightforward application of Schwarz rearrangement (2.4.1) [6]. Let Ω be a bounded open set of measure c and $\Omega^* = B$ be the ball of the same volume. Let u_1 be a en eigenfunction with eigenvalue $\lambda_1(\Omega)$ and u_1^* its Schwarz rearrangement. Using 2.4.1 we have

$$\int_{\Omega^*} u_1^*(x)^2 \, \mathrm{d}x = \int_{\Omega} u_1(x)^2 \, \mathrm{d}x.$$

Further, using 2.4.2 we have

$$\int_{\Omega^*} |\nabla u_1^*(x)|^2 \, \mathrm{d}x \leq \int_{\Omega} |\nabla u_1(x)|^2 \, \mathrm{d}x.$$

Using Rayleigh quotients (3.1.1) we get the following

$$\lambda_1(\Omega^*) \leq \frac{\int_{\Omega} |\nabla u_1^*(x)|^2 \, \mathrm{d}x}{\int_{\Omega} u_1^*(x)^2 \, \mathrm{d}x}.$$

$$\lambda_1(\Omega) = \frac{\int_{\Omega} |\nabla u_1(x)|^2 \, \mathrm{d}x}{\int_{\Omega} u_1(x)^2 \, \mathrm{d}x}.$$

Using the previous two statements yields the desired results. $\square$

3.3 Continuity

To prove the existence of minimizers (or maximizers) for eigenvalues and for functions of eigenvalues we will need continuity of eigenvalues with respect to the variable. While the classical case is when the eigenvalues depend on the coefficients of the operator, our eigenvalues will depend on the domain. We will study the domain-continuity of the eigenvalues with a concept called γ-convergence.

We will denote by $A_L^D(\Omega)$ the linear operator defined by

$$A_L^D : L^2(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega),$$

$$f \mapsto u \text{ solution of Dirichlet Laplacian..}$$

I need to add essentially all of 2.3.3 in Henrots book pg 28

Look into sources for proofs

3.4 Polygons

Note P_N is the class of plane polygons with at most N edges.

Theorem 3.4.1. *Let $M \in \mathbb{N}$ and Ω be a polygon with M edges. Then Ω cannot be a (local) minimum for $|\Omega|\lambda_1(\Omega)$ in the class P_{M+1}.*

Note that by local we mean for the Hausdorff distance. So, for any $\varepsilon > 0$ we can find a polygon Ω_ε with $M+1$ edges and $d_H(\Omega, \Omega_\varepsilon) < \varepsilon$ such that $|\Omega_\varepsilon|\lambda_1(\Omega_\varepsilon) < |\Omega|\lambda_1(\Omega)$.

Proof. Take x_0 to be a vertex of Ω with an angle $\alpha < \pi$. Without loss of generality we can assume that x_0 is the origin. We want to show that by removing a cap of size ε from the domain we can decrease $|\Omega|\lambda_1(\Omega)$. We denote by η the normalized inward bisector, $C_\varepsilon = \{x \in \Omega; x.\eta \leq \varepsilon\}$ which we call the cap, $\Omega_\varepsilon = \Omega - C_\varepsilon$ which is the polygon obtained by removing the cap. Also, let $B_\varepsilon = \{x \in \Omega; \varepsilon < x.\eta \leq 2\varepsilon\}$, $C_{2\varepsilon} = C_\varepsilon \cup B_\varepsilon$, and $\Omega_{2\varepsilon} = \Omega_\varepsilon - B_\varepsilon = \Omega - C_{2\varepsilon}$. Let u_1 be the first normalized eigenfunction of Ω. It is well known that u_1 has a gradient which vanishes at the corner [6]. Specifically, we have

$$\lim_{x \to 0, x \in \Omega} |\nabla u_1(x)| = 0.$$

Let $\beta > 0$ be a sufficiently small number (specified at the end) Then using the fact that u_1 has a gradient which vanishes at the corner alongside the mean value theorem, we can choose ε such that for all $x \in C_{2\varepsilon}$, $|u_1(x)| \leq \beta|x|$. In particular, we have

$$\int_{C_{2\varepsilon}} |u_1(x)|^2 \,\mathrm{d}x \leq \beta^2 \int_{C_{2\varepsilon}} |x|^2 \,\mathrm{d}x = \frac{8}{3} \tan\frac{\alpha}{2} \left(3 + \tan^2\frac{\alpha}{2}\right) \beta^2 \varepsilon^4.$$

Define c_1 such that the right hand side of the previous equation is equal to

$c_1\beta^2\varepsilon^4$. Next, let χ_ε be a C^1 cut-off function such that

$$\begin{cases} \chi_\varepsilon(x) = 1 & \text{if } x \in \Omega_{2\varepsilon} \\ 0 \leq \chi_\varepsilon(x) \leq 1 & \text{if } x \in B_\varepsilon \\ \chi_\varepsilon(x) = 0 & \text{if } x \in C_\varepsilon \end{cases}$$

and the function $u_\varepsilon^1 := \chi_\varepsilon u_1$ which belongs to $H_0^1(\Omega_\varepsilon)$. According to the definition of λ_1 using the Rayleigh coefficient, we have

$$\lambda_1(\Omega_\varepsilon) \leq \frac{\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^1|^2 \,\mathrm{d}x}{\int_{\Omega_\varepsilon} (u_\varepsilon^1)^2 \,\mathrm{d}x}.$$

Next, we have the following

$$\int_{\Omega_\varepsilon} (u_\varepsilon^1)^2 \,\mathrm{d}x \geq \int_{\Omega_{2\varepsilon}} u_1^2 \,\mathrm{d}x = 1 - \int_{C_{2\varepsilon}} u_1^2 \,\mathrm{d}x \geq 1 - c_1\beta^2\varepsilon^4.$$

Also, we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^1|^2 \,\mathrm{d}x \leq \int_\Omega |\nabla u_1|^2 \,\mathrm{d}x + \int_{B_\varepsilon} |\nabla_\varepsilon|^2 u_1^2 \,\mathrm{d}x.$$

Due to the construction of a cut-off function, there exists a constant c_2 such that $|\nabla \chi_\varepsilon|^2 \leq \frac{c_2}{\varepsilon^2}$ and thus

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^1|^2 \,\mathrm{d}x \leq \lambda_1 + c_1 c_2 \beta^2 \varepsilon^2.$$

Using these inequalities, we can use the definition of λ_1 to get

$$\lambda_1(\Omega_\varepsilon) \leq \frac{\lambda_1 + \beta^2\varepsilon^2 c_1 c_2}{1 - c_1\beta^2\varepsilon^4}.$$

Also, using $|\Omega_\varepsilon| = |\Omega| - |C_{2\varepsilon}| = |\Omega| - 4\varepsilon^2 \tan(\alpha/2) + o(\varepsilon^2)$ we obtain

$$|\Omega_\varepsilon|\lambda_1(\Omega_\varepsilon) \leq |\Omega|\lambda_1 + \varepsilon^2 \left(\beta^2 c_1 c_2 |\Omega| - 4\lambda_1 \tan(\alpha/2)\right) + o(\varepsilon^2).$$

Thus, for sufficiently small ε, once $\beta^2 < \frac{4\lambda_1 \tan(\alpha/2)}{c_1 c_2 |\Omega|}$ we have $|\Omega_\varepsilon|\lambda_1(\Omega_\varepsilon) < |\Omega|\lambda_1$. □

Theorem 3.4.2. *Let $a > 0$ and $N \in \mathbb{N}$ be fixed. Then the problem*

$$\min \{\lambda_1(\Omega), \Omega \in P_N, |\Omega| = a\}$$

has a solution.

Proof. We will use the direct method of calculus of variations. Let Ω_n be a minimizing sequence in P_N for λ_1. We will begin by showing the diameter $D(\Omega_n)$ is bounded. Assume that this is not the case.

I'd like to go over this section and rewrite it.

Then, since the area must be fixed, we can choose some length going to infinity but with a width, for example at its basis $A_n B_n$ going to zero. Let us now construct another minimizing sequence $\widetilde{\Omega_n}$ by cutting the pick at its basis. Let $\widetilde{\Omega_n}$ be the polygon we obtain by replacing our choice by the segment $A_n B_n$. Obviously $|\widetilde{\Omega_n}| \leq |\Omega_n|$, so if we prove that $\lambda_1(\widetilde{\Omega_n}) - \lambda_1(\Omega_n) \to 0$, it will show that $\lambda_1(\widetilde{\Omega_n})$ is also a minimizing sequence for the product $|\Omega|\lambda_1(\Omega)$. Since the number of possible picks is bounded by $N/2$, this will prove that we can consider a minimizing sequence with bounded diameter.

We denote by $\eta_n = A_n B_n$ the width of the basis of the choice ($\eta_n \to 0$) and $\omega_n = \Omega_n \cap B(\frac{A_n+B_n}{2}, 3\eta_n)$. Let χ_n be a cut-off function which satisfies:

1. $\chi_n = 1$ outside $B(\frac{A_n+B_n}{2}, 3\eta_n)$,
2. $\chi_n = 0$ on the segment $A_n B_n$,
3. χ_n is C^1 on $\overline{\widetilde{\Omega_n}}$,
4. $\exists C > 0$ (independent of n) such that $|\nabla \chi_n| \leq \frac{C}{\eta_n}$.

Let u_n be the normalized first eigenfunction of Ω_n. By construction $\chi_n u_n \in H_0^1(\widetilde{\Omega_n})$ as so it is admissible in the min formula that defines λ_1.

Now, for any C^1 function v we have

$$|\nabla(vu_n)|^2 = |u_n \nabla v + v \nabla u_n|^2 = u_n^2 |\nabla v|^2 + \nabla u_n \nabla (u_n v^2)$$

or

$$|\nabla(vu_n)|^2 = u_n^2 |\nabla v|^2 + \operatorname{div}(u_n v^2 \nabla u_n) + \lambda_1(\Omega_n) u_n^2 v^2.$$

Replacing v by χ_n and integrating on $\widetilde{\Omega_n}$ yields

$$\int_{\overline{\Omega_n}} |\nabla(\chi_n u_n)|^2 = \int_{\overline{\Omega_n}} u_n^2 |\nabla \chi_n|^2 + \lambda_1(\Omega_n) \int_{\overline{\Omega_n}} \chi_n^2 u_n^2.$$

Then, the variational definition of $\lambda_1(\widetilde{\Omega_n})$ is

$$\lambda_1(\widetilde{\Omega}_n) \leq \lambda_1(\Omega_n) + \frac{\int_{\overline{\Omega}_n} u_n^2 |\nabla \chi_n|^2}{\int_{\overline{\Omega}_n} \chi_n^2 u_n^2}.$$

Now, using $|\nabla \chi_n| = 0$ outside $B(\frac{A_n+B_n}{2}, 3\eta_n)$, $|\nabla \chi_n|^2 \leq \frac{C}{\eta_n^2}$ in ω_n and $\int_{\overline{\Omega}_n} \chi_n^2 u_n^2 \geq \frac{1}{2}$, we obtain

$$\lambda_1(\widetilde{\Omega}_n) \leq \lambda_1(\Omega_n) + \frac{2C}{\eta_n^2} \int_{\omega_n} u_n^2 \leq \lambda_1(\Omega_n) + C' \sup_{\omega_n} u_n^2.$$

However, since $\sup_{\omega_n} u_n^2 \to 0$ the result has been proven.

All of the above is for bounded domain, should I move it to a lemma?

Since λ_1 is invariant by translation, we can assume that all of the domains $\widetilde{\Omega}_n$ are included in a fixed ball B.

Should I handle the below property introduction by adding a citation to the sentence, adding the theorem in a previous section, or add theorem and proof?

By compactness of the Hausdorff convergence, there exists an open set Ω and a subsequence $\widetilde{\Omega}_{n_k}$ which converge to Ω for the Hausdorff distance. Moreover, since the vertices $A_n^j, j = 1, 2, \ldots, M, M \leq N$ of $\widetilde{\Omega}_n$ stay in B, we can also assume up to a subsequence $\widetilde{\Omega}_{n_k}$ that each $A_{n_k}^j$ converges to some point A^j in

B. So, using Hausdorff convergence, Ω must be a polygon with vertices A^j. Further, since any polygon in P_N has at most $N/3$ holes, we can apply the Sverak Theorem to show $\lambda_1(\widetilde{\Omega}_{n_k})$ converges to $\lambda_1(\Omega)$. Recall that minimizing $\lambda_1(\Omega)$ under an area constraint is equivalent to minimizing the product $|\Omega|\lambda_1(\Omega)$ without constraint. Then, as a consequence of (3.4.1), Ω must have exactly N edges. □

Theorem 3.4.3 (Pólya)**.** *The equilateral triangle has the least first eigenvalue among all triangles of given area. The square has the least first eigenvalue among all quadrilaterals of given area.*

Proof. This proof is analogous to the Faber-Krahn Theorem, but uses Steiner Symmetrization instead of Schwarz rearrangement. We note that as the Steiner symmetrization shares the properties 2.4.1 and 2.4.2, we know that any Steiner symmetrization will not increase the first eigenvalue.

We will construct a sequence of Steiner symmetrizations that makes a triangular domain converge to an equilateral triangle. Let a_n, h_n, and A_n be the base, height, and one of the base's incident angles of the triangle T_n that we obtain at step n. Then we have

$$\frac{h_n}{a_{n+1}} = \frac{h_{n+1}}{a_n} = \sin A_n.$$

Denote the ratio $x_n = \frac{h_n}{a_n}$. Then we have

$$x_{n+1} = \frac{\sin^2 A_n}{x_n} = \frac{\sin^2(tan^{-1}(2x_n))}{x_n} = \frac{4x_n}{1+4x_n^2}.$$

Thus we have constructed the sequence $x_{n+1} = \frac{4x_n}{1+4x_n^2}$. This will converge to the fixed point of $f(x) = \frac{4x}{1+4x^2}$, which is $\frac{\sqrt{3}}{2}$.

$$\frac{4x}{1+4x^2} = x$$
$$x\left(4x^2 - 3\right) = 0$$

and so $x = \frac{\sqrt{3}}{2}$ is the fixed point of f.

One can use elementary geometry to find that for an equilateral triangle with side length a, the height h is $\frac{\sqrt{3}}{2}a$. So $\frac{h}{a} = \frac{\sqrt{3}}{2}$, and thus our sequence converges to the value characteristic of equilateral triangles. Moreover, by Sverak's Theorem, the sequence of triangles γ-converges to the equilateral triangle which we will denote by T_e. Then, for an initial triangle domain T, we have shown

$$\lambda_1(T_e) = \lim \lambda_1(T_n) \leq \lambda_1(T).$$

For quadrilaterals we can use a more elementary proof. One can show that by choosing a sequence of three Steiner symmetrizations you can transform any quadrilateral into a rectangle [6].

Use Example eigenvalue def to show square minimizes eigen

. $\square$

Theorem 3.4.4. *For $n \geq 3$ the regular polygon with n sides is an extreme point for the first eigenvalue of the Dirichlet Laplace operator among polygons with n sides and a fixed area.*

Proof. By 3.2.1, our problem is equivalent (up to homothety) to solving the problem

$$\min_{P \in P_n} \lambda_1(P) + |P|.$$

Note that the first eigenfunction u_1 when P is a regular polygon is a H^2 function

[5]. Then for this case the shape derivative of $G(P) = \lambda_1(P) + |P|$ is

$$\frac{dG}{dV}(P) = -\int_{\partial P}\left(\frac{\partial u_1}{\partial n}\right)^2 V.n\,\mathrm{d}\sigma + \int_{\partial P} V.n\,\mathrm{d}\sigma.$$

Let P be the regular polygon which minimizes $\lambda_1(P)+|P|$, and l its side length. Without loss of generality we will assume P is centered at the origin. Also, let r be the inradius of P. Consider the vector field $V = |x|/r$, and note that $V.n = 1$. Since V will preserve the regularity of P, we can use the shape derivative to show

$$\int_{\partial P}\left(\frac{\partial u_1}{\partial n}\right)^2 \mathrm{d}\sigma = \int_{\partial P}\mathrm{d}\sigma = nl.$$

As we will see when we build the numeric method, all relevant perturbations can be described by the perturbations of the n vertices. Further, each perturbation of a vertex v_i can be written as a linear combination of perturbations of the type $v_i + cE_{i-1,i}$ where $E_{i-1,i}$ is the edge incident to v_{i-1}, v_i. From this, we can write the derivative of G as

$$\frac{dG}{dV}(P) = -\int_0^l \left(\partial_n u_1(p(t/l))^2\right)\frac{l-t}{l}V.n\,\mathrm{d}t + \int_0^l \frac{l-t}{l}V.n\,\mathrm{d}t,$$

where the parameterization of the side $E_{i,i+1}$ is $p(s) = (1-s)v_i + sv_{i+1}$ and n is the normal vector to $E_{i,i+1}$. Since $V.n$ is constant we have

$$\frac{dG}{dV}(P) = V.n\left(-\int_0^l \left(\partial_n u_1(p(t/l))^2\right)\frac{l-t}{l}\,\mathrm{d}t + \int_0^l \frac{l-t}{l}\,\mathrm{d}t\right).$$

Since the first eigenfunction and the regular polygon share the same symmetries, we have

$$\int_0^l \left(\partial_n u_1(p(t/l))^2\right)\frac{l-t}{l}\,\mathrm{d}t = \int_0^l \left(\partial_n u_1(p(t/l))^2\right)\frac{t}{l}\,\mathrm{d}t = \frac{\int_0^l(\partial_n u_1(p(t/l))^2)\,\mathrm{d}t}{2} = \frac{l}{2},$$

where we've used the change of variables $t \mapsto l-t$. Thus for these types of perturbations V, $\frac{dG}{dV}(P) = 0$. Since the shape derivative is linear and every perturbation of the vertices of the polygon can be represented by a linear combination of these perturbations, $\frac{dG}{dV}(P) = 0$ for all vertex perturbations V. Therefore the regular polygon P is a critical point for G. $\square$

Chapter 4

A Numerical Approach

4.1 Overview

Copy this section from Final Essay from last year

Talk about why we're allowed to minimize sum of area and eigenvalue instead of forcing the area constraint

4.2 Method of Fundamental Solutions

We will use the method of fundamental solutions (MFS) to compute the eigenvalues of a given polygon. The following construction is based on a similar method by Alves and Antunes [1]

Our goal is to numerically solve the Helmholtz equation with Dirichlet boundaries

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We will consider the group of functions which satisfy $-\Delta u = \lambda u$ that are of the form

$$u = a_1\phi_1^\lambda + \ldots + a_N\phi_N^\lambda,$$

where $\phi_i^\lambda, i = 1, 2, \ldots, M$ are fundamental radial solutions of $-\Delta u = \lambda u$ with singularities laying outside of Ω. Let (y_i) be the singularities of ϕ_i^λ outside of Ω.

To find the coefficients $a_1, \ldots, a_N$ we impose the Dirichlet boundary condition on a discretization of $\partial\Omega$. Let (x_i) be a discretization of $\partial\Omega$, and let $x_{N+1} \in \Omega$. This leads to a system of equations

$$\begin{cases} u(x_i) = 0 & \text{if } 1 \leq i \leq N \\ u(x_i) = 1 & \text{if } i = N+1 \end{cases}$$

Note that the equation when $i = N+1$ is used to guarantee that $u(x) \not\equiv 0$ [2].

Obviously we are interested when the system has non-trivial solutions. This occurs when the matrix $A_\lambda = (\phi_i^\lambda(x_j))_{i,j=1}^N$ is singular. As this shows the existence of an eigenfunction, we can find eigenvalues using the determinate of the matrix A_λ. Specifically, we can find the eigenvalues of Ω on some interval I by locating the values $\lambda \in I$ where $\det A_\lambda = 0$. Once we have found an eigenvalue, we can solve the system to find a corresponding eigenfunction.

To apply MFS to our specific problem, we need to find suitable radial functions as well as (x_i) and (y_i).

First, we will find suitable radial functions. Let $\phi := x(r)$ be a radial function in polar coordinates. Then Helmholtz's equation becomes

$$-x'' - \frac{1}{r}x' = \lambda x$$
$$r^2 x'' + rx' + r^2 \lambda x = 0.$$

Substituting in $s = \sqrt{\lambda} r$ we have

$$s^2 y'' + sy' + s^2 y = 0,$$

where $y(s) = x(r)$. Note this is a specific case of Bessel's differential equation. Thus, our radial fundamental solutions can be Bessel functions of order 0. We choose to use the Hankel function of the first kind with order 0 as it is the most efficient computationally.

Definition 4.2.1. *We define the Bessel function of the first kind with order* 0 *in the following way*

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin \tau) \, \mathrm{d}\tau.$$

We define the Bessel function of the second kind with order 0 *in the following way*

$$Y_0(x) = \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} \cos(x \cos \tau) \left(e + \ln\left(2x \sin^2 \tau\right) \right) \mathrm{d}\tau.$$

Finally, we define the Hankel function (of the first kind) with order 0 *as*

$$H_0(x) = J_0(x) + iY_0(x).$$

Thus our fundamental solutions will be of the form $\phi_i^\lambda = H_0(\sqrt{\lambda}|x - y_i|)$.

Write overview of how to choose $(x_i), (y_i)$ or refer to various papers and summarize?

4.3 Optimization

In the previous section we outlined the method of fundamental solutions, a method to calculate the eigenvalues of our equation. In this section we will outline a method to calculate the derivative of the eigenvalue and use gradient descent to find extremum.

From 3.1.1, the derivative of an eigenvalue is given by

$$\lambda_k'(0) := -\int_\Omega \left(\frac{\partial u}{\partial n}\right)^2 V.n \, \mathrm{d}\sigma.$$

As we are taking the derivative with respect to the domain, we will begin by defining vector fields that allow us to write the derivative with respect to geometric parameters. We will find particular vector fields V which allow us to compute the derivative with respect to the coordinates of the vertices. Fix a vertex and label it v_0. Next, label the remaining vertices $v_1, v_2, \ldots, v_{N-1}$ going around the polygon counterclockwise. That is, for a vertex v_i the adjacent vertices should be v_{i-1}, v_{i+1} modulo N. Finally, take (x_i, y_i) to be the coordinates of v_i.

To find the derivative of λ_1 with respect to x_i we make a perturbation of v_i with $(1, 0)$. This induces a perturbation of the adjacent edges of the boundary, which we will denote as $E_{i-1,i}$ and $E_{i,i+1}$. For our particular case V will have the following form on the boundary

$$\begin{cases} L_{i-1,i}(x,y) & (x,y) \in E_{i-1,i} \\ L_{i,i+1}(x,y) & (x,y) \in E_{i,i+1} \\ 0 & \text{otherwise} \end{cases}$$

where $L_{j,k} : E_{j,k} \to [0,1]$ is the following affine function

$$L_{j,k}(x,y) = \begin{cases} (x_k - x_j)^{-1}(x - x_i) & \text{if } x_i \neq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Denote the outer normal of the edge $E_{i,i+1}$ by $n_{j,j+1} = (n^1_{j,j+1}, n^2_{j,j+1})$. Then we can rewrite the derivative of the fundamental eigenvalue as

$$\frac{d\lambda_1}{dx_i} = -\int_{E_{i-1,i}} L_{i-1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^1_{i-1,i}\, \mathrm{d}\sigma - \int_{E_{i,i+1}} L_{i+1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^1_{i,i+1}\, \mathrm{d}\sigma.$$

Likewise, we can find the derivative with respect to the y value as

$$\frac{d\lambda_1}{dx_{2i}} = -\int_{E_{i-1,i}} L_{i-1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^2_{i-1,i}\, \mathrm{d}\sigma - \int_{E_{i,i+1}} L_{i+1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^2_{i,i+1}\, \mathrm{d}\sigma.$$

Following these computations, we have all of the pieces needed to optimize the fundamental eigenvalue using gradient descent.

4.4 Results

Due to the construction of our algorithm, the computational cost will increase rapidly as the number of vertices N increases. Specifically, by defining the vector field using a combination of vector fields which are affine transformations of a singular vertex we guarantee the need to calculate the derivative of the eigenvalue for each vertex. This necessitates that we focus on the cases $N = 5, 6, \ldots, M$ where M is chosen exclusively based on computational restrictions. Obviously the algorithm could be parallelized to largely mitigate this problem, but that is outside of the purpose of this paper.

Should I include a basic derivation of the runtime? That is more of a CS thing, so I'm not sure that it is appropriate to add.

In this paper we will show results for polygons with $N = 3, 4, \ldots, 23$ vertices. We have included the cases $N = 3, 4$ as these are the only known results we can compare to an explicit minimization of our problem. In order to demonstrate "similarity" of the final domain to the regular polygon we give the standard deviation of the side lengths as well as the interior angles. Let $|P|$ be the number of vertices of P, σ_E be the standard deviation of the edge lengths, and σ_A the standard deviation of the interior angles. Then we have the following table:

$\lvert P\rvert$	σ_E	σ_A
3	$2e-4$	$1e-4$
4	$3e-4$	$6e-4$
5	$7e-4$	$2e-4$
6	$8e-3$	$5e-3$
7	$3e-3$	$2e-3$
8	$1e-3$	$3e-4$
9	$7e-3$	$6e-3$
10	$5e-4$	$7e-4$
11	$1e-2$	$6e-3$
12	$2e-2$	$1e-2$

$\lvert P\rvert$	σ_E	σ_A
13	$2e-2$	$1e-2$
14	$1e-2$	$8e-3$
15	$1e-2$	$1e-2$
16	$1e-2$	$1e-2$
17	$2e-2$	$2e-2$
18	$2e-2$	$1e-2$
19	$7e-3$	$1e-2$
20	$1e-2$	$2e-2$
21	$1e-2$	$2e-2$
22	$2e-2$	$1e-2$
23	$2e-2$	$2e-2$

It is worth taking a moment to analyze these results. For the method of gradient descent, we can guarantee the result converges to the minimization solution for sufficiently small step sizes[3]. The difficulty in choosing these step sizes is that decreasing the step size will necesarily increase the number of iterations needed to get within a specific distance to the solution. Since the total number of iterations is constant, the difference in standard deviation values simply demonstrates that the algorithm will take longer to converge. We can obtain specific values for the standard deviations by a combination of increasing the number of iterations and decreasing the step size.

Finally, it may seem odd that we have not included the eigenvalue for some fixed area. This is due to us using (3.2.1) to include the constraint inside of our functional. As a consequence of this, each polygon may end up with a different area which makes comparing their eigenvalues uninteresting. However, there are two ways of modifying the algorithm to produce the eigenvalue for some fixed area. The first option is to choose a different vector field, one which we can parameterize using some geometric property which will also maintain the area of the polygon. The other option is to use (3.2.1), specifically the explicit correspondence, to find the eigenvalue for some fixed area.

should I add eigenvalues for area=1? I haven't coded it yet but it seems straightforward

I also have graphs of the functional as a function of num iterations. I can include some of them, though they are not very interesting

I also have Images of the polygons that were the result of the computations for the table above. I don't know what they could be used for, but they look pretty nice.

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