The Pólya–Szegő Conjecture on Polygons: A Numerical Approach

By

LOGAN REED

A thesis submitted to the

Graduate School–Camden

Rutgers, The State University of New Jersey

In partial fulfullment of the requirements

For the degree of Master of Science

Graduate Program in Mathematical Sciences

Written under the direction of

Siqi Fu

And approved by

Dr. One

Dr. Two

Dr. Three

Dr. Four

Camden, New Jersey

May 2023

TEST PAGE

THESIS ABSTRACT

TODO SOMETHING ABOUT THE ABSTRACT THAT IS ABOUT THIS LONG OR SO

by LOGAN REED

Thesis Director:

Siqi Fu

The topic that I chose to explore for this thesis is a study of the eigenvalues of the Dirichlet Laplacian on a two dimensional domain and . . .

Acknowledgment

I would be remiss if I did not take a moment to express appreciation for all of the people who have helped me through this process.

Contents

Chapter 1

Introduction

1.1 Physical Motivation

Physical drums consist of a rigid shell with a membrane which produces sound when hit. A similar thing can be "created" in a pure mathematical setting by studing specific partial differential equations over a closed region. Specifically, the frequencies of the drum membrane corresponds to the eigenvalues of the Dirichlet Laplacian. This construction makes it possible to "hear" drums where the shape of the drumhead is any closed and simple curve. To do this, we begin with the shape of our drumhead, which is a region of the real plane bounded by piecewise smooth curves. Since we wish to emulate the physical properties of a drum, we want to define some system that models the vibration of the drum membrane which produces the sound. This is done using the wave equation over our boundary, with the boundary condition that the function is 0 on the boundary. The Dirichlet Laplacian allows us to model the physical vibration of the drum, and we can calculate its eigenvalues to find the fundamental frequency and overtones. Thus, by modeling the vibration of the drumhead using the Dirichlet Laplacian and studying its properties, we can produce "sound" via the eigenvalues.

In 1877, Lord Rayleigh conjectured the following [7]

> If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.

In 1923, Faber published a proof which was followed by an independent proof by Krahn in 1925 [4].

Theorem 1.1.1 (Faber-Krahn)**.** *Let c be a positive number and B the ball with volume c. Then,*

$$\lambda_1(B) = \min \left\{ \lambda_1(\Omega), \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c \right\}.$$

From the Faber-Krahn inequality, we know that for any drumhead with a given area, the circle is the one with the lowest tone. In 1951, Polya and Szego conjectured that a similar statement holds for drumheads with a polygonal shape [5]. This conjecture has been shown to be true for 3 and 4 sided polygons, but remains unproven for any other number of sides.

There are two main hurdles that are halting progress on this conjecture. The first is that the tools that were used to prove both Lord Rayleighs conjecture as well as the small cases for the Polya-Szego conjecture are not available when the number of sides is greater than four. The main tool that is used is called Steiner Symmetrization, and when there are more than four sides this symmetrization method creates additional sides at each step.

The purpose of this paper is to show a specific method for running numerical approximations to suggest that this conjecture is indeed true. This is done using a method based on fundamental solutions [3]. Specifically, we consider all functions that satisfy the Laplace Equation and then solve for the linear coefficients using the boundary conditions. Once this is done we use gradient descent to find the polygon with the minimum first eigenvalue, which is equivalent to the first fundamental tone of the drum.

Consider a homogeneous elastic drumhead, or membrane, stretched over a rigid frame. We will represent the frame as a domain $\Omega \subset \mathbb{R}^2$. Take the function $u(x, y, t)$ to be the vertical displacement of the membrane from its resting position. Then for any disk $D \subset \Omega$, Newton's second law of motion states that

$$\int_{\partial D} T \frac{\partial u}{\partial \mathbf{n}} \, dS = \int_D \rho u_{tt} \, dA$$

where T is the constant tension, ρ is the density constant, and $\mathbf{n}$ is the outward normal of the boundary. By the divergence theorem, we have

$$\int_D T \Delta u \, dA = \int_D \rho u_{tt} \, dA$$

where Δ is the Laplace operator. From this we can get the wave equation on Ω

$$u_{tt} = c^2 \Delta u$$

where we define u to be 0 on the boundary and where $c = \sqrt{T/\rho}$. We can solve this wave equation using $u(x, y, t) = T(t)V(x, y)$ which gives us

$$\frac{T''}{c^2 T} = \frac{\Delta V}{V} = -\lambda$$

and finally we have reduced our problem to the Dirichlet Laplacian

$$\Delta V = -\lambda V$$

where V on the boundary is zero.

NOTE: The best reference I could find is Logan's Applied Partial Differential Equations. I could also use Ryans paper

In the next section, we will start from the Dirichlet Laplacian and introduce the conjectures in a formal setting.

1.2 Polya-Szego's Conjecture

1. Introduce Rigorous Definitions from 1.1.2 Henrot
2. Dirichlet Laplacian eigenvalues prereqs
3. Faber Krahn
4. Polya-Szego Conjecture [6]

1.3 Known Results

1. All Explicit Cases

2. Tools for n=3 and n=4

1.4 Numerical Analysis Tools

Chapter 2

Background

2.1 Notations and Definitions

2.2 Measure Theory

Definition 2.2.1. *A Measure space is a pair (X, A) where X is a non-empty set and A is a σ-algebra of subsets of X. That is, A satisfies the following*

1. $\emptyset \in A$
2. *For countably many* $A_j \in A$, $\cup A_j \in A$
3. *If* $B \in A$, *then* $X - B \in A$.

TODO Integration and maybe supports

2.3 Function Spaces

Definition 2.3.1. *A complex linear space $\mathbb{H}$ is called a normed linear space if there exists a map $||\cdot|| : \mathbb{H} \to \mathbb{R}^+$ such that for any $x, y \in \mathbb{H}$ and $\lambda \in \mathbb{C}$,*

1. $||\lambda x|| = |\lambda|||x||$
2. $||x + y|| \leq ||x|| + ||y||$
3. $||x|| \geq 0$, *and* $||x|| = 0$ *if and only if* $x = 0$

Definition 2.3.2. *A complex linear space $\mathbb{H}$ is called an inner product space with inner product $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ if for any $x, y, z \in \mathbb{H}$ and $\lambda \in \mathbb{C}$,*

1. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
4. $\langle x, x \rangle \geq 0$, *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

Definition 2.3.3. *A Hilbert space is a complete inner product space*

2.4 PDEs

Definition 2.4.1 (Laplacian)**.**

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}.$$

Definition 2.4.2. *The Dirichlet Laplacian is the Laplace Operator subject to Dirichlet boundary conditions. That is, we call u a solution to the Dirichlet Laplacian if u is a solution to*

$$\begin{cases} \Delta u = \lambda u & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases}.$$

2.5 Calculus of Variations

2.6 Tools

Definition 2.6.1 (Schwarz Rearrangement)**.** *For any measurable set ω in $\mathbb{R}^N$, we denote by ω^* the ball of same volume as ω. If u is a non-negative measurable function defined on a measurable set Ω and vanishing on its boundary $\partial\Omega$, we denote by $\Omega(c) = \{x \in \Omega \,|\, u(x) \geq c\}$ its level sets. The Schwarz rearrangement of u is the function u^* defined on Ω^* by*

$$u^*(x) = \sup\{c/x \in \Omega(c)^*\}.$$

Without loss of generality, we fix the hyperplane of symmetry to be $x_N = 0$. Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a measurable set. We denote by Ω' the projection of Ω on $\mathbb{R}^{N-1}$, and for every $x' \in \mathbb{R}^{N-1}$ we denote by $\Omega(x')$ the projection of Ω with $\{x'\} \times \mathbb{R}$.

Definition 2.6.2 (Steiner Symmetrization)**.** *Let $\Omega \subset \mathbb{R}^N$ be measurable. Then the set*

$$\Omega^* := \left\{ x = (x', x_N) : -\frac{1}{2}|\Omega(x')| < x_N < \frac{1}{2}|\Omega(x')|, x' \in \Omega' \right\}$$

is the Steiner symmetrization of Ω with respect to the hyperplane $x_N = 0$.

Theorem 2.6.1. *Let Ω be a measurable set and u be a non-negative measurable function defined on Ω and vanishing on its boundary $\partial\Omega$. Let ϕ be any measurable function defined on $\mathbb{R}^+$ with values in $\mathbb{R}$, then*

$$\int_\Omega \phi(u(x))\,\mathrm{d}x = \int_{\Omega^*} \phi(u^*(x))\,\mathrm{d}x.$$

Theorem 2.6.2 (Pólya's Inequality)**.** *Let Ω be an open set and u a non-negative function belonging to the Sobolev space $H_0^1(\Omega)$. Then $u^* \in H_0^1(\Omega^*)$ and*

$$\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x \geq \int_{\Omega^*} |\nabla u^*(x)|^2\,\mathrm{d}x.$$

TODO Steiner Symmetrization

Chapter 3

Eigenvalues of the Dirichlet Laplacian

3.1 Definition

Definition 3.1.1 (Rayleigh Quotient)**.** *For an operator L, we define the Rayleigh quotient to be*

$$R_L[v] := \frac{\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \,\mathrm{d}x + \int_\Omega a_0(x) v^2(x) \,\mathrm{d}x}{\int_\Omega v(x)^2 \,\mathrm{d}x}.$$

This is used to express the first eigenvalue of the Dirichlet Laplacian in the following way

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2 \,\mathrm{d}x}{\int_\Omega v(x)^2 \,\mathrm{d}x}.$$

Theorem 3.1.1. *Let Ω be a bounded open set. We assume that $\lambda_k'(\Omega)$ is simple. Then, the functions $t \to \lambda_k(t), t \to u_t \in L^2(\mathbb{R}^N)$ are differentiable at $t = 0$ with*

$$\lambda_k'(0) := -\int_\Omega \mathrm{div}(|\nabla u|^2 V) \,\mathrm{d}x.$$

If, moreover, Ω is of class C^2 or if Ω is convex, then

$$\lambda_k'(0) := -\int_\Omega \left(\frac{\partial u}{\partial n} \right)^2 V.n \,\mathrm{d}\sigma$$

and the derivative u' of u_t is the solution of

$$\begin{cases} -\Delta u' = \lambda_k u' + \lambda_k' u & \text{in} \Omega \\ u' = -\frac{\partial u}{\partial n} V.n & \text{on} \partial\Omega \\ \int_\Omega u u' \,\mathrm{d}\sigma = 0. \end{cases}$$

Theorem 3.1.2. *Each eigenvalue is real, $\lambda_1 \leq \lambda_2 \leq \dots$*

Proof. pg 335 evans □

NOTE: pg 336 Evans has an alternative method of defining eigenvalues of elliptic operators

3.2 Known Results

1. invariant under translations rotations

2. homothety

3. continuous

Theorem 3.2.1 (Faber-Krahn). *Let c be a positive number and B the ball with volume c. Then,*

$$\lambda_1(B) = \min\left\{\lambda_1(\Omega), \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c\right\}.$$

Proof. This proof is a straightforward application of Schwarz rearrangement (2.6.1) [5]. Let Ω be a bounded open set of measure c and $\Omega^* = B$ be the ball of the same volume. Let u_1 be a en eigenfunction with eigenvalue $\lambda_1(\Omega)$ and u_1^* its Schwarz rearrangement. Using 2.6.1 we have

$$\int_{\Omega^*} u_1^*(x)^2 \,\mathrm{d}x = \int_{\Omega} u_1(x)^2 \,\mathrm{d}x.$$

Further, using 2.6.2 we have

$$\int_{\Omega^*} |\nabla u_1^*(x)|^2 \,\mathrm{d}x \leq \int_{\Omega} |\nabla u_1(x)|^2 \,\mathrm{d}x.$$

Using Rayleigh quotients (3.1.1) we get the following

$$\lambda_1(\Omega^*) \leq \frac{\int_\Omega |\nabla u_1^*(x)|^2 \,\mathrm{d}x}{\int_\Omega u_1^*(x)^2 \,\mathrm{d}x}.$$

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u_1(x)|^2 \,\mathrm{d}x}{\int_\Omega u_1(x)^2 \,\mathrm{d}x}.$$

Using the previous two statements yields the desired results. □

3.3 Polygons

Note P_N is the class of plane polygons with at most N edges.

Theorem 3.3.1. *Let $a > 0$ and $N \in \mathbb{N}$ be fixed. Then the problem*

$$\min \{\lambda_1(\Omega), \Omega \in P_N, |\Omega| = a\}$$

has a solution.

Proof. 47 henrot □

Theorem 3.3.2. *Let $M \in \mathbb{N}$ and Ω be a polygon with M edges. Then Ω cannot be a (local) minimum for $|\Omega|\lambda_1(\Omega)$ in the class P_{M+1}.*

Theorem 3.3.3 (Pólya)**.** *The equilateral triangle has the least first eigenvalue among all triangles of given area. The square has the least first eigenvalue among all quadrilaterals of given area.*

Proof. pg 50 henrot □

TODO: Explain Issue with proof method for cases N ¿= 5

Theorem 3.3.4. *For $n \geq 3$ the regular polygon with n sides is an extreme point for the first eigenvalue of the Dirichlet Laplace operator among polygons with n sides and a fixed area.*

Proof. pg 56 of bogosel paper □

Chapter 4

A Numerical Approach

4.1 Overview

1. Method of fundamental solutions using hankel functions
2. Additional requirements for the matrix for the next part
3. Calculate derivative of first eigenvalue using boundary method
4. Apply gradient descent to find minimum

Theorem 4.1.1. *Laplace's Equation is invariant under rotations.*

4.2 Method of Fundamental Solutions

We will use the method of fundamental solutions (MFS) to compute the eigenvalues of a given polygon. The following construction is based on a similar method by Alves and Antunes [1]

Our goal is to numerically solve the Helmholtz equation with Dirichlet boundaries

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We will consider the group of functions which satisfy $-\Delta u = \lambda u$ that are of the form

$$u = a_1\phi_1^\lambda + \ldots + a_N\phi_N^\lambda,$$

where $\phi_i^\lambda, i = 1, 2, \ldots, M$ are fundamental radial solutions of $-\Delta u = \lambda u$ with singularities laying outside of Ω. Let (y_i) be the singularities of ϕ_i^λ outside of Ω.

To find the coefficients $a_1, \ldots, a_N$ we impose the Dirichlet boundary condition on a discretization of $\partial\Omega$. Let (x_i) be a discretization of $\partial\Omega$, and let $x_{N+1} \in \Omega$. This leads to a system of equations

$$\begin{cases} u(x_i) = 0 & \text{if } 1 \leq i \leq N \\ u(x_i) = 1 & \text{if } i = N+1 \end{cases}$$

Note that the equation when $i = N+1$ is used to guarantee that $u(x) \not\equiv 0$ [2].

Obviously we are interested when the system has non-trivial solutions. This occurs when the matrix $A_\lambda = (\phi_i^\lambda(x_j))_{i,j=1}^N$ is singular. As this shows the existence of an eigenfunction, we can find eigenvalues using the determinate of the matrix A_λ. Specifically, we can find the eigenvalues of Ω on some interval I by locating the values $\lambda \in I$ where $\det A_\lambda = 0$. Once we have found an eigenvalue, we can solve the system to find a corresponding eigenfunction.

To apply MFS to our specific problem, we need to find suitable radial functions as well as (x_i) and (y_i).

First, we will find suitable radial functions. Let $\phi := x(r)$ be a radial function in polar coordinates. Then Helmholtz's equation becomes

$$-x'' - \frac{1}{r}x' = \lambda x$$
$$r^2x'' + rx' + r^2\lambda x = 0.$$

Substituting in $s = \sqrt{\lambda}r$ we have

$$s^2y'' + sy' + s^2y = 0,$$

where $y(s) = x(r)$. Note this is a specific case of Bessel's differential equation. Thus, our radial fundamental solutions can be Bessel functions of order 0. We choose to use the Hankel function of the first kind with order 0 as it is the most efficient computationally.

Definition 4.2.1. *We define the Bessel function of the first kind with order* 0 *in the following way*

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin \tau)\, \mathrm{d}\tau.$$

We define the Bessel function of the second kind with order 0 *in the following way*

$$Y_0(x) = \frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi} \cos(x \cos \tau)\left(e + \ln\left(2x \sin^2 \tau\right)\right)\, \mathrm{d}\tau.$$

Finally, we define the Hankel function (of the first kind) with order 0 *as*

$$H_0(x) = J_0(x) + iY_0(x).$$

Thus our fundamental solutions will be of the form $\phi_i^\lambda = H_0(\sqrt{\lambda}|x - y_i|)$.

TODO: Should I go into how to choose $(x_i), (y_i)$ or refer to various papers and summarize?

4.3 Optimization

In the previous section we outlined MFS, a method to calculate the eigenvalues of our equation. In this section we will outline a method to calculate the derivative of the eigenvalue and use gradient descent to find extremum.

From 3.1.1, the derivative of an eigenvalue is given by

$$\lambda_k'(0) := -\int_\Omega \left(\frac{\partial u}{\partial n}\right)^2 V.n \, \mathrm{d}\sigma.$$

As we are taking the derivative with respect to the domain, we will begin by defining vector fields that allow us to write the derivative with respect to geometric parameters. We will find particular vector fields V which allow us to compute the derivative with respect to the coordinates of the vertices. Fix a vertex and label it v_0. Next, label the remaining vertices $v_1, v_2, \ldots, v_{N-1}$ going around counterclockwise. That is, for a vertex v_i the adjacent vertices should be v_{i-1}, v_{i+1} modulo N. Finally, take (x_{2i-1}, x_{2i}) to be the coordinates of v_i.

TODO: Outline Perturbation and give derivative equations. NOTE: $n_{j,j+1} = (n^1_{j,j+1}, n^2_{j,j+1})$

$$\frac{d\lambda_1}{dx_{2i-1}} = -\int_{v_i v_{i-1}} P_{i-1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^1_{i-1,i} \, \mathrm{d}\sigma - \int_{v_i v_{i+1}} P_{i+1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^1_{i+1,i} \, \mathrm{d}\sigma.$$

$$\frac{d\lambda_1}{dx_{2i}} = -\int_{v_i v_{i-1}} P_{i-1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^2_{i-1,i} \, \mathrm{d}\sigma - \int_{v_i v_{i+1}} P_{i+1,i} \left(\frac{\partial u}{\partial n}\right)^2 n^2_{i+1,i} \, \mathrm{d}\sigma.$$

Bibliography

[1] Antunes Alves. "The Method of Fundamental Solutions applied to the calculation of eigenfrequencies and eigenmodes of 2D simply connected shapes". In: *Tech Science Press CMC* 2 (Dec. 2005), pp. 251–265.

[2] Antunes Alves. "The method of fundamental solutions applied to the calculation of eigensolutions for 2D plates". In: *International Journal for Numerical Methods in Engineering* 77 (Jan. 2009), pp. 177–194.

[3] Beniamin Bogosel. *Faber-Krahn inequality for polygons - numerical study.* 2015. URL: `http://www.cmap.polytechnique.fr/~beniamin.bogosel/faber_krahn_polygons.html`.

[4] Daniel Daners. "Krahn's proof of the Rayleigh conjecture revisited". In: *Archiv der Mathematik* (2011), pp. 187–199.

[5] Antoine Henrot. *Extremum Problems for Eigenvalues of Elliptic Operators.* Birkhauser, 2006.

[6] G. Polya and G. Szego. *Isoperimetric inequalities in mathematical physics.* Princeton University Press, 1951.

[7] Lord Rayleigh. *The Theory of Sound.* 1st edition. Macmillan, 1887.