The Pólya–Szegő Conjecture on Polygons: A Numerical Approach

By

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THESIS ABSTRACT

TODO SOMETHING ABOUT THE ABSTRACT THAT IS ABOUT THIS LONG OR SO

by LOGAN REED

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The topic that I chose to explore for this thesis is a study of the eigenvalues of the Dirichlet Laplacian on a two dimensional domain and ...

Acknowledgment

I would be remiss if I did not take a moment to express appreciation for all of the people who have helped me through this process.

Contents

Introduction

This is the intro.

Chapter 1

Introduction

1.1 Physical Motivation

Consider a homogeneous elastic drumhead, or membrane, stretched over a rigid frame. We will represent the frame as a domain $\Omega \subset \mathbb{R}^2$. Take the function $u(x, y, t)$ to be the vertical displacement of the membrane from its resting position. Then for any disk $D \subset \Omega$, Newton's second law of motion states that

$$\int_{\partial D} T \frac{\partial u}{\partial \mathbf{n}} \, dS = \int_D \rho u_{tt} \, dA$$

where T is the constant tension, ρ is the density constant, and $\mathbf{n}$ is the outward normal of the boundary. By the divergence theorem, we have

$$\int_D T \Delta u \, dA = \int_D \rho u_{tt} \, dA$$

where Δ is the Laplace operator. From this we can get the wave equation on Ω

$$u_{tt} = c^2 \Delta u$$

where we define u to be 0 on the boundary and where $c = \sqrt{T/\rho}$. We can solve this wave equation using $u(x, y, t) = T(t)V(x, y)$ which gives us

$$\frac{T''}{c^2 T} = \frac{\Delta V}{V} = -\lambda$$

and finally we have reduced our problem to the Dirichlet Laplacian

$$\Delta V = -\lambda V$$

where V on the boundary is zero.

NOTE: The best reference I could find is Logan's Applied Partial Differential Equations. I could also use Ryans paper

In the next section, we will start from the Dirichlet Laplacian and introduce the conjectures in a formal setting.

1.2 Polya-Szego's Conjecture

1. Introduce Rigorous Definitions from 1.1.2 Henrot

2. Dirichlet Laplacian eigenvalues prereqs

3. Faber Krahn

4. Polya-Szego Conjecture [1]

1.3 Known Results

1. All Explicit Cases

2. Tools for n=3 and n=4

1.4 Numerical Analysis Tools

Chapter 2

Background

2.1 Notations and Definitions

2.2 Function Spaces

Definition 2.2.1. *A complex linear space $\mathbb{H}$ is called a normed linear space if there exists a map $||\cdot|| : \mathbb{H} \to \mathbb{R}^+$ such that for any $x, y \in \mathbb{H}$ and $\lambda \in \mathbb{C}$,*

1. $||\lambda x|| = |\lambda| ||x||$
2. $||x + y|| \leq ||x|| + ||y||$
3. $||x|| \geq 0$, *and* $||x|| = 0$ *if and only if* $x = 0$

Definition 2.2.2. *A complex linear space $\mathbb{H}$ is called an inner product space with inner product $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ if for any $x, y, z \in \mathbb{H}$ and $\lambda \in \mathbb{C}$,*

1. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
4. $\langle x, x \rangle \geq 0$, *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

Definition 2.2.3. *A Hilbert space is a complete inner product space*

2.3 PDEs

Definition 2.3.1. *Laplacian*

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}.$$

Definition 2.3.2. *The Dirichlet Laplacian is the Laplace Operator subject to Dirichlet boundary conditions*

2.4 Calculus of Variations

Chapter 3

Eigenvalues of Dirichlet Laplacian

3.1 Definition

Theorem 3.1.1. *Let Ω be a bounded open set. We assume that $\lambda'_k(\Omega)$ is simple. Then, the functions $t \to \lambda_k(t), t \to u_t \in L^2(\mathbb{R}^N)$ are differentiable at $t = 0$ with*

$$\lambda'_k(0) := -\int_\Omega \mathrm{div}(|\nabla u|^2 V)\,\mathrm{d}x.$$

If, moreover, Ω is of class C^2 or if Ω is convex, then

$$\lambda'_k(0) := -\int_\Omega \left(\frac{\partial u}{\partial n}\right)^2 V.n\,\mathrm{d}\sigma$$

and the derivative u' of u_t is the solution of

$$\begin{cases} -\Delta u' = \lambda_k u' + \lambda'_k u & \text{in}\Omega \\ u' = -\frac{\partial u}{\partial n} V.n & \text{on}\partial\Omega \\ \int_\Omega u u'\,\mathrm{d}\sigma = 0. \end{cases}$$

3.2 Known Results

1. invariant under translations rotations

2. homothety

3. continuous

Theorem 3.2.1 (Faber-Krahn)**.** *Let c be a positive number and B the ball with volume c. Then,*

$$\lambda_1(B) = \min \left\{ \lambda_1(\Omega), \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c \right\}.$$

Proof. page 46 henrot □

3.3 Polygons

Note P_N is the class of plane polygons with at most N edges.

Theorem 3.3.1. *Let $a > 0$ and $N \in \mathbb{N}$ be fixed. Then the problem*

$$\min \{\lambda_1(\Omega), \Omega \in P_N, |\Omega| = a\}$$

has a solution.

Proof. 47 henrot □

Theorem 3.3.2. *Let $M \in \mathbb{N}$ and Ω be a polygon with M edges. Then Ω cannot be a (local) minimum for $|\Omega|\lambda_1(\Omega)$ in the class P_{M+1}.*

Theorem 3.3.3 (Pólya)**.** *The equilateral triangle has the least first eigenvalue among all triangles of given area. The square has the least first eigenvalue among all quadrilaterals of given area.*

3.4 Tools

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Bibliography

[1] G. Polya and G. Szego. *Isoperimetric inequalities in mathematical physics.* Princeton University Press, 1951.