Part I

Upper bound

1 Notation for gadgets and figures

$$\text{Let } m = \left\lceil \left(\frac{N}{102}\right)^{\frac{1}{d}} \right\rceil, \text{ base of the counter}$$

$$\begin{aligned}
\text{MSR} &= \text{ most signifcant digit region} \\
s &= \text{starting value of counter} \\
d &= \lceil \log_m s \rceil = \left\lfloor \frac{k}{2} \right\rfloor, \text{ number of digits per row} \\
m^d &= \text{ final value of the counter} \\
n &= m^d - s, \text{ number of rows/ times to count} \\
l &= \lceil \log m \rceil + 2, \text{ bits needed to encode each digit in binary, plus 2 for MSR and MSD}
\end{aligned}$$

2 Parameters for the counter

...therefore, let $d = \lfloor \frac{k}{2} \rfloor$, $m = \left\lceil \left(\frac{N}{102}\right)^{\frac{1}{d}} \right\rceil$, $l = \lceil \log m \rceil + 2$, $s = m^d - \left\lfloor \frac{N-12l-94}{12l+90} \right\rfloor$, where d is the number of digits per row of the counter, m is the base of the counter, l is the number of bits needed to encode each digit in binary plus 2 for indicating whether a digit is in the MSR and is the MSD in that region, and s is the starting value of the counter in decimal. If d is a multiple of 3, let $g = \frac{d}{3} - 1$ otherwise let $g = \lfloor \frac{d}{3} \rfloor$, where g is the number of general digit regions.

Let h be the height of the construction without any additional "roof" tiles. By inspection...we see that the height of the construction without additional tiles is $12l + 94$ for the `Seed` unit, plus $12l + 90$ for each incrementation of the counter, adding a `Counter` unit row each time. If we define n as the number of `Counter` unit rows, then $h = n(12l + 90) + (12l + 94)$. So then the maximum height of the counter is $m^d(12l + 90) + 12l + 94$. Since our goal is to end with a rectangle of height N, we need to pick a base such that the counter can increment so many times that when it stops, it is at least N.

Lemma 1. *$N \leq m^d(12l + 90) + 12l + 94$.*

Proof.

$$\begin{aligned}
N = 102\left(\frac{N}{102}\right) &= 102\left(\left(\frac{N}{102}\right)^{\frac{1}{d}}\right)^d \leq 102\left\lceil \left(\frac{N}{102}\right)^{\frac{1}{d}} \right\rceil^d \\
&= 102m^d \leq 12lm^d + 90m^d \leq 12lm^d + 90m^d + 12l + 94 \\
&= m^d(12l + 90) + 12l + 94
\end{aligned}$$

□

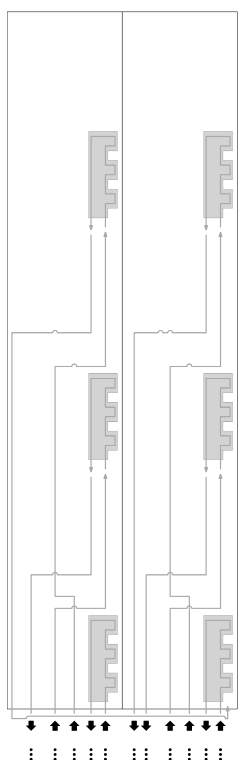
And the mininum height is $24l + 184$.

One row of the counter might result in a final assembly that will not be tall enough but ... says that having all possible rows of the counter might result in a final assembly that is too tall. Therefore, we must start the counter at an appropriate value to get the correct height of the final assembly.

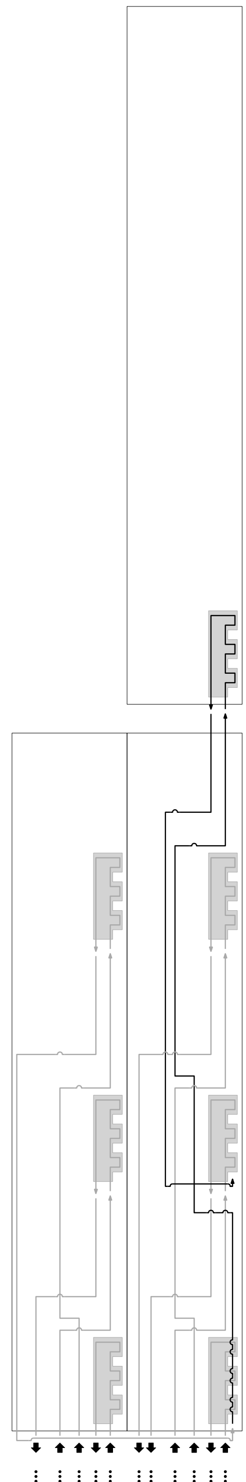
The counter can start at any whole number less than m^d and ends when it reaches 0 by rolling over $m^d - 1$. This means that the number of Counter unit rows n, is $m^d - s$, where we have defined s as the starting value of the counter. To choose the best starting value, we find the value for n that gets h close to N without exceeding N. It follows from the equation $h = n(12l + 90) + 12l + 94$, that $n = \left\lfloor \frac{N-12l-94}{12l+90} \right\rfloor$. Thus, $s = m^d - \left\lfloor \frac{N-12-94}{12l+90} \right\rfloor$.

If k is an even number, no filler tiles are needed. We can assemble any rectangle that has a width k, provided that k is some even number ≥ 6. If k is any odd number ≥ 6, we use one filler tile, which adds a column to the right and increases the width of the rectangle by 1. ...as a result of each digit requiring a width of 2 tiles, if k is odd, one additional tile column must be added. The number of filler tiles needed for the width is $k \mod 2$, and the number of filler tiles for the height is $N - 12l - 90 \mod 12l + 90$.

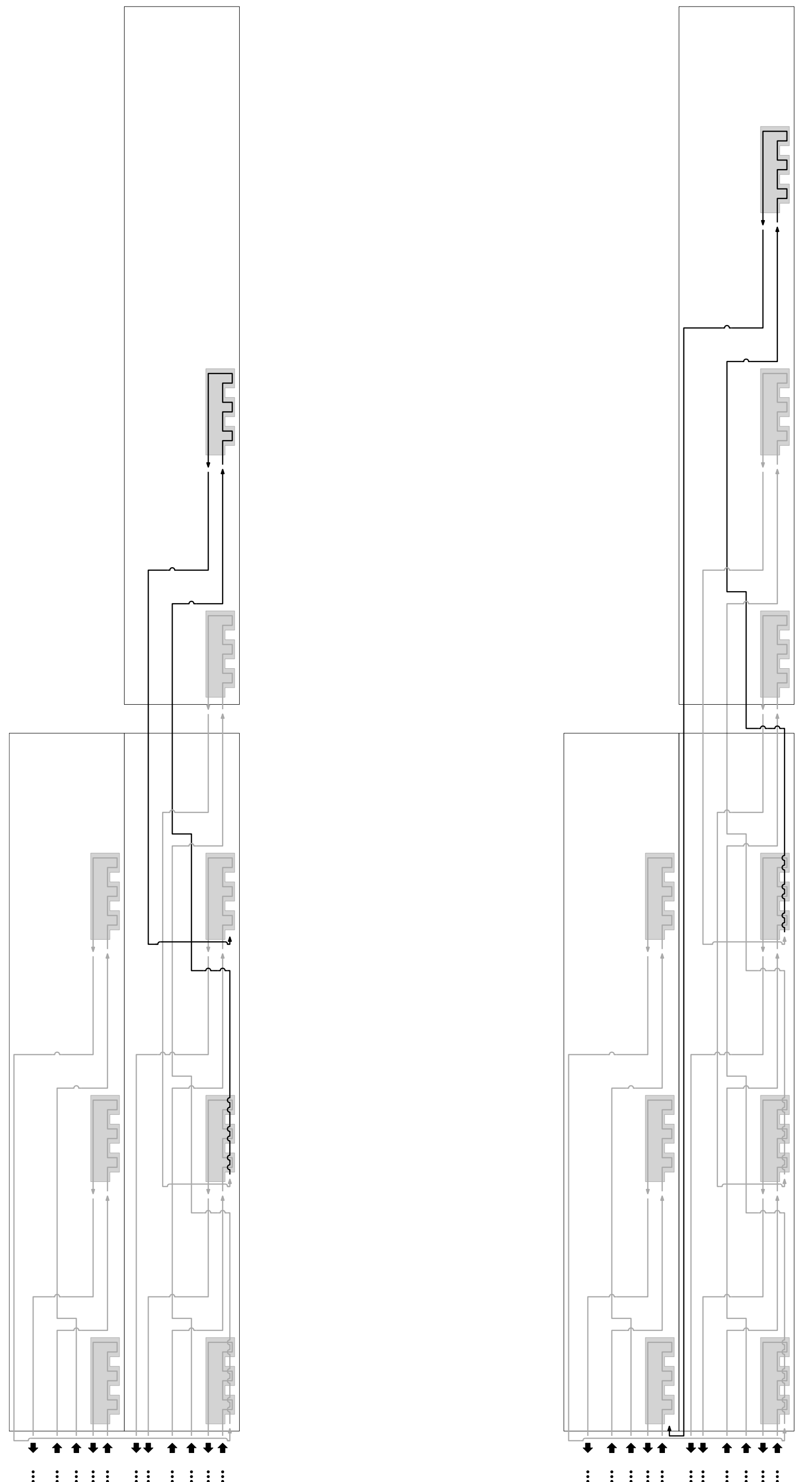
3 Counting with digit regions



(a) A "clean" counter row, before any reading has started.



(b) Read digit 1 in the current row, write digit 1 in the next row.



(c) Read digit 2 in the current row, write digit 2 in the next row.

(d) Read digit 3 in the current row, write digit 3 in the next row.

Figure 1: This illustrates how a counter reads and writes a digit region, in a general sense. The counter starts in the rightmost digit region by reading the bottommost digit within that region. After reading digit 1 in the current row, the corresponding digit region in the next row be started in the next row. The counter writes the first digit in the next row, and then returns to the second digit in the current digit region. Once all the digits in the current digit region are read and written into the next row, the counter can then do one of the following: continue reading digits by moving on to the next digit region, cross back all the way to the right of the rectangle and start reading the next row, or halt.

3.1 Digit regions

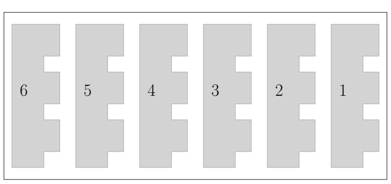
Each logical row of the counter is made up of $\lceil \frac{d}{3} \rceil$ "digit regions". A digit region is a group of 1-3 digits, stacked vertically on top of one another. Within a digit region, the digits are sorted in order of significance, thus the top digit is the most signifcant digit, the middle digit is second most significant and the bottommost digit is the least signifcant. The leftmost digit region is most signifcant and the rightmost is the least signifcant.

The counter reads digit 1 in the rightmost digit region, then writes digit $1'$ in the digit region directly north in the next row. After writing this first digit, the counter returns back to the current row and reads digit 2. Upon reading digit 2 in the current row, it then writes digit $2'$ in the next row and returns to the current row. Finally, in the current row, it reads digit 3, the last digit remaining, and writes digit $3'$ in the next row. Now that all three digits in this digit region have been evaluated, the counter will repeat this process for all remaining digit regions in the current row.

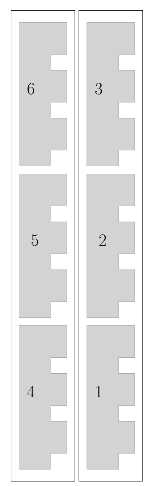
In the last digit region, which we will refer to as the MSR (most significant region), the region may or may not contain all 3 digits like the earlier regions. Because of this, there are three cases we must account for: case 1 if the MSR only has one digit; case 2 if the MSR only has two digits; or case 3 if the MSR has three digits.

In case 1, the MSR adds an additional 2 tiles to the overall width of the rectangle. In case 2, the MSR adds an additional 4 tiles to the overall width of the rectangle. And, similar to all regular digit regions with 3 digits, in case 3 the MSR adds an additional 6 tiles to the overall width of the rectangle. As a result of these cases, we're able to utilize these different cases to fine tune the width of the rectangle such that we only need one filler tile in the event that k is odd.

Figure 2: Digits in a typical counter vs. digits separated into digit regions.



(a) Digits in a typical counter.



(b) Digits in two digit regions, stacked vertically. Digits 1-3 are in digit region 1, and digits 4-6 are in digit region 2.

Contrary to a typical counter, each counter row has an approximate height of 3 digits $\approx 12l$. The digits are stacked up to 3, once a digit region is full, we add another digit region to the left, and repeat.

3.2 Detecting the edges

The counter must detect if a digit is in the MSR and if it's in the MSR, whether or not it is the most signifcant digit. To do this, all digits are encoded with two additional bits on the least signifcant end. If bit 0 is 1, the reader tiles know they could be reading the most signifcant digit (MSD) or in case 2, the second most signifcant digit. If bit 1 is 1, the digit currently being read is the MSD, otherwise the digit is digit 1 in case 2.

bit_1	bit_0	Meaning
0	0	digit is not in MSR
0	1	digit is in the MSR but is not the MSD
1	0	
1	1	digit is in the MSR and is MSD

4 Tile set

When describing a special case, i.e. "digit x – case y", whatever follows will only apply to the MSR (due to each case only affecting the MSR.)

4.1 Gadgets

4.1.1 Line Gadgets

(a) `North_Line` (b) `South_Line`

Figure 3: `Line` gadgets

We will use the notation `NorthN_Line` and `SouthN_Line` where `N` corresponds to the length of a specific line gadget.

4.1.2 `Counter_Read`

A group of `Counter_Read` gadgets read over a series of bit-bumps protruding into their row from the preceding row. After a group of `Counter_Read` gadgets read a bit pattern, a `Counter_Read` gadget with a output length of l produces a new bit pattern which it outputs to the `Pre_Warp` gadgets to begin the process of writing it in the row above it that encodes a copy or increment of the current digit. If the input signal for a `Counter_Read` gadget is `copy`, it will always propagate a `copy` signal. However, if the input signal for a `Counter_Read` gadget is `increment`, the output will always depend on the bits of the digit that gets read. First, if the value of the bits is less than $m-1$, the digit can be incremented so the output of the `Counter_Read` is the value read + 1, and a `copy` signal is started. Otherwise, the value of the bits is $m-1$ so the digit is reset to 0, but the `Counter_Read` gadget needs to check the two least signicant bits that it read to determine whether it should output a `increment` or `halt` signal. There are 3 possible values for these bits, 00, 01, and 11. If the bits are 00 or 01, an `increment` signal will be output. If the bits are 11, a `halt` signal will be output, as these bits indicate this digit is the most significant digit in the most significant digit region. By passing a `halt` signal to the `Pre_Warp` gadget, this signal continues through the rest of the `Warp_Unit` gadgets and into a special `Counter_Write` gadget, which will allow for the `Roof_Unit` to take over and complete the rectangle.

- For each $i = 1, 2, 3$, $j = 0, \ldots, l-2$, $u \in \{0,1\}^j$, and each $op \in \{$`increment`, `copy`$\}$:

- Create $\texttt{Counter\_Read}(\langle \texttt{CounterRead}, i, \lambda, op\rangle,$
  $\langle \texttt{CounterRead}, i, 0, op\rangle,$
  $\langle \texttt{CounterRead}, i, 1, op\rangle\ )$
  from the general gadget in Figure 4.

Note that 9 tiles are added for a single `Counter_Read` gadget. In this step,

$$
\begin{aligned}
\sum_{j=0}^{l-2} 9 \cdot 2 \cdot 3 \cdot 2^j &= 27(2^l - 2) \\
&= 27\left(2^{\lceil \log m \rceil + 2} - 2\right) \\
&= 27\left(4 \cdot 2^{\lceil \log m \rceil} - 2\right) \\
&= 108\left(2^{\lceil \log m \rceil} - \frac{1}{2}\right) \\
&\leq 108\left(2 \cdot 2^{\log m} - \frac{1}{2}\right) \\
&= 108\left(2m - \frac{1}{2}\right) \\
&= 216m - 81 \\
&= O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)
\end{aligned}
$$

tiles were created in this step.

- For each $i = 1, 2, 3$ and each $u \in \{0, 1\}^{l-1}$:
  - Create $\texttt{Counter\_Read}(\langle \texttt{CounterRead}, i, u, \texttt{copy}\rangle, \langle \texttt{PreWarp}, i, 0u, \texttt{copy}\rangle, \langle \texttt{PreWarp}, i, 1u, \texttt{copy}\rangle\ )$ from the general gadget in Figure 4.

Since the counter must only increment the current value if the result will be less than m, the `Counter_Read` gadgets that have both an `increment` signal and input size of $l - 2$ must first right shift the bits 2 spots, and then for each possible value after reading one more bit, check whether that value is less than $m - 1$. Basically, if the next bit read is a 0, we check if the current value + 1 is less than m. If the next bit read is a 1, we check if current value + $2^{\log(m)-1}$ + 1 is less than m. For both cases, if the counter can increment the current value, then the `Counter_Read` gadgets output the incremented value and a `copy` signal to the `Pre_Warp` gadgets. Otherwise, if the counter is unable to increment the value, it will output a signal in which the bits of the digit are all zeroes and the `increment` signal is propagated to the next digit.

Algorithm 1 Incrementing and halting

function READMOSTSIGNICANTBIT( )
  $guess0 \leftarrow 0u >> 2$.
  $guess1 \leftarrow 1u >> 2$.
  if $convertToDecimal(guess0) + 1 \leq m - 1$ **then**
    $out0 \leftarrow \langle \texttt{PreWarp}, i, convertToBinary(convertToDecimal(guess0) + 1) + u[1] + [0], \texttt{copy} \rangle$.
  else
    if u ends with "11" **then**
      $out0 \leftarrow \langle \texttt{PreWarp}, i, repeat(\text{"0"}, m) + 11, \texttt{halt} \rangle$.
    else
      $out0 \leftarrow \langle \texttt{PreWarp}, i, repeat(\text{"0"}, m) + u[1] + u[0], \texttt{increment} \rangle$.
  if $convertToDecimal(guess1) + 1 \leq m - 1$ **then**
    $out1 \leftarrow \langle \texttt{PreWarp}, i, convertToBinary(convertToDecimal(guess1) + 1) + u[1] + [0], \texttt{copy} \rangle$.
  else
    if u ends with "11" **then**
      $out1 \leftarrow \langle \texttt{PreWarp}, i, repeat(\text{"0"}, m) + 11, \texttt{halt} \rangle$.
    else
      $out1 \leftarrow \langle \texttt{PreWarp}, i, repeat(\text{"0"}, m) + u[1] + u[0], \texttt{increment} \rangle$.

– Create $\texttt{Counter\_Read}(\langle \texttt{CounterRead}, i, u, \texttt{increment} \rangle, out0, out1\ )$ from the general gadget in Figure 4.

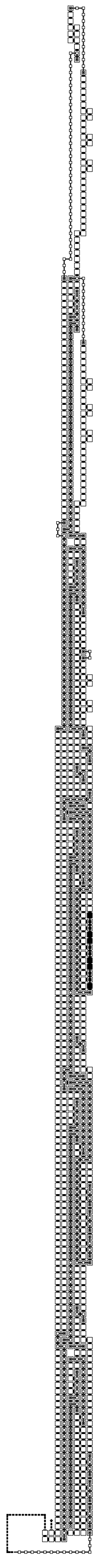
Note $\{0,1\}^{l-1} = \{0,1\}^{\lceil \log m \rceil + 1}$, so $\left|\{0,1\}^{\lceil \log m \rceil + 1}\right| = 2^{\lceil \log m \rceil} + 2^1 \leq 4 \cdot 2^{\log m} = 4m$. So in this step $O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

(a) `Counter_Read_0`

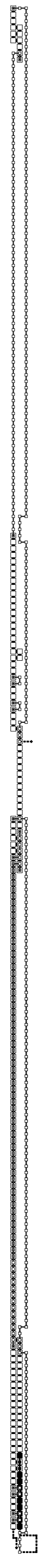
(b) `Counter_Read_1`

(c) Digit 1 - general overview.

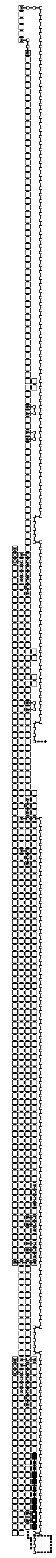
(d) Digit 2 - general overview.



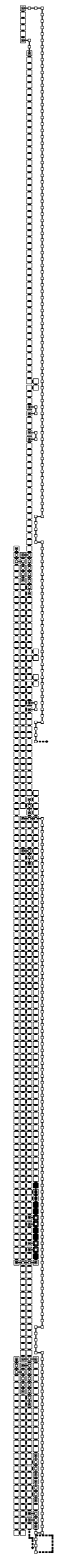
(e) Digit 3 - general overview.



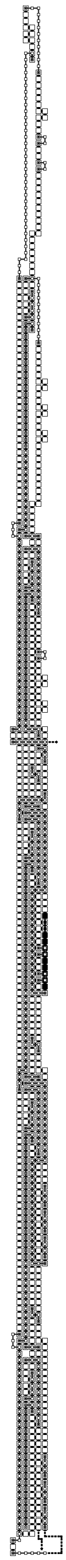
(f) Digit 1 - case 1 overview.



(g) Digit 1 - case 2 overview.



(h) Digit 2 - case 2 overview.



(i) Digit 3 - case 3 overview.

Figure 4: The Counter_Read gadgets.

4.1.3 Warp_Unit

A `Warp_Unit` generally consists of the following 5 gadgets: `Pre_Warp`, `First_Warp`, `Warp_Bridge`, `Second_Warp`, and `Post_Warp`. The job of these 5 gadgets is to transport the value read by the `Counter_Read` all the way to the digit region in the next row, so that the `Counter_Write` gadgets can write the next value in the correct locations.

- `Pre_Warp`: These gadgets decode the least significant two bits passed in from the read `Counter_Read` gadgets into a special signal that indicates whether a gadget is currently in the MSR, and if it is, whether it is also the MSD. Here is how the signal is decoded: if the least significant bit is 1, add a marker to indicate the current gadgets are in the MSR; if the second least significant bit is also 1, add a marker on the glues to indicate that the gadget is also part of the MSD.

  For each $i = 1, 2, 3, u \in \{0,1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}, \texttt{halt}\}$:

  - if u ends with 00: create $\texttt{Pre\_Warp}(\langle\texttt{PreWarp}, i, u, op\rangle, \langle\texttt{FirstWarp}, i, u, op\rangle\ )$ from the general gadget in Figure 5a.
  - if u ends with 01: $\texttt{Pre\_Warp}(\langle\texttt{PreWarp}, 1, u, op\rangle, \langle\texttt{FirstWarp}, 1, u, op, \texttt{msr}\rangle\ )$ from the general gadget in Figure 5e.
  - if u ends with 11: create $\texttt{Pre\_Warp}(\langle\texttt{PreWarp}, i, u, op\rangle, \langle\texttt{FirstWarp}, i, u, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the general gadget in Figure 5g if $i = 1$ (case 1), or Figure 5i if $i = 2$ (case 2), or Figure 5a if $i = 3$ (case 3).

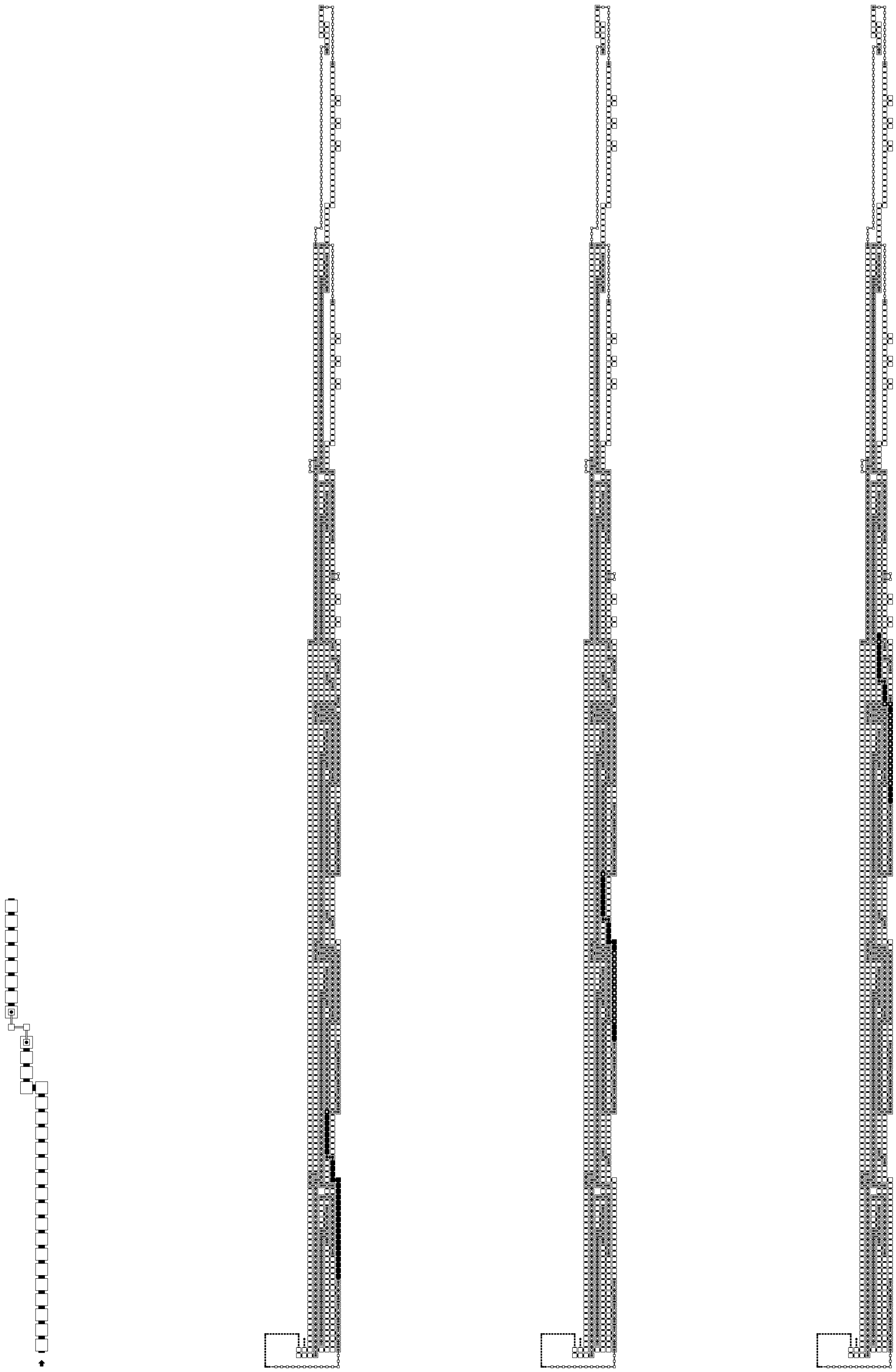
  In this step, for digits 1-3 in the general case, $9 \cdot 2^l \cdot 34 = 306 \cdot 2^l = 306 \cdot 2^{\lceil \log m \rceil + 2} = 1224 \cdot 2^{\lceil \log m \rceil} \leq 2448 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

  For digit 1 in case 1, $3 \cdot 2^l \cdot 31 = 93 \cdot 2^l = 93 \cdot 2^{\lceil \log m \rceil + 2} = 372 \cdot 2^{\lceil \log m \rceil} \leq 744 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

  For digit 1 in case 2, $3 \cdot 2^l \cdot 34 = 102 \cdot 2^l = 102 \cdot 2^{\lceil \log m \rceil + 2} = 408 \cdot 2^{\lceil \log m \rceil} \leq 816 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

  For digit 2 in case 2, $3 \cdot 2^l \cdot 30 = 90 \cdot 2^l = 90 \cdot 2^{\lceil \log m \rceil + 2} = 360 \cdot 2^{\lceil \log m \rceil} \leq 720 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

  For digit 3 in case 3, $3 \cdot 2^l \cdot 34 = 102 \cdot 2^l = 102 \cdot 2^{\lceil \log m \rceil + 2} = 408 \cdot 2^{\lceil \log m \rceil} \leq 816 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

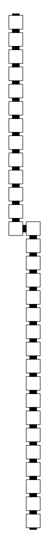


(a) Digits 1, 2, & 3 - general, Digit 3 - case 3. There are 34 tiles in this gadget.

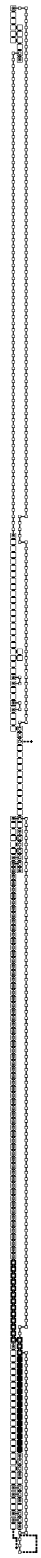
(b) Digit 1 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

(c) Digit 2 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

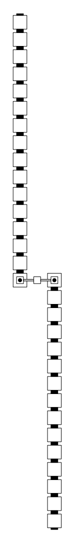
(d) Digit 3 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



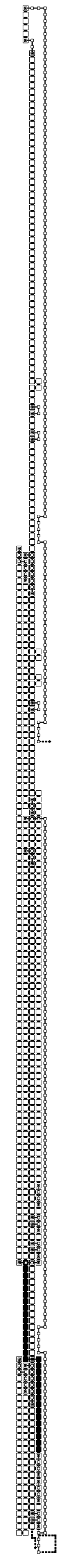
(e) Digit 1 - case 1. There are 31 tiles in this gadget.



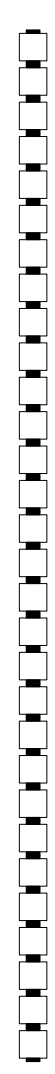
(f) Digit 1 - case 1 overview. The black tiles in this figure correspond to the gadget shown in subfigure e.



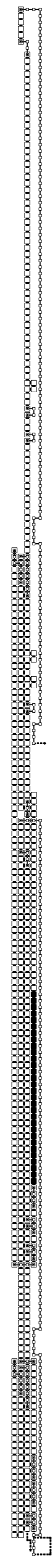
(g) Digit 1 - case 2. There are 34 tiles in this gadget.



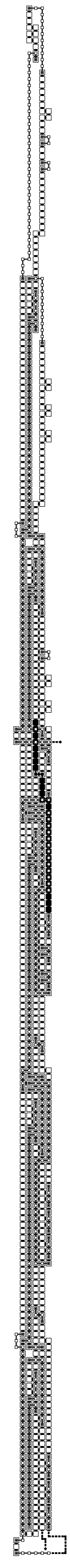
(h) Digit 1 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure g.



(i) Digit 2 - case 2. There are 30 tiles in this gadget.



(j) Digit 2 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure i.



(k) Digit 3 - case 3 overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

Figure 5: The `Pre_Warp` gadgets.

- `First_Warp`:

  The idea of the `First_Warp` gadget is to transport the information read by the `Counter_Read` gadgets, usually across a large distance. We do this using a single special tile that has three glues. Both the north and south glues have the same labels, allowing this tile to assemble into a line indefinitely. The third glue is the special glue; this glue can be either on the east or west side of the tile, and has the output label. This unique glue will at some point later in the assembly, in a spot determined by earlier pieces of the assembly, no longer be blocked from the direction of it special glue. When this occurs, it can finally attach to the `Warp_Bridge` gadget (except in a few special cases where the `Warp_Bridge` gadget is skipped). This process signifies the "waking up" of the `First_Warp` gadgets. When this gadget wakes up, it must also be blocked in the north direction. By blocking the north glue, the line stops, which prevents a truly infinite line from assembling. The earlier pieces of the assembly that guarentee this process to work are known as the `Digit_Top` gadgets.

  For each $i = 1, 2, 3$, $u \in \{0,1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}, \texttt{halt}\}$:

  – Create $\texttt{First\_Warp}(\langle \texttt{FirstWarp}, i, u, op \rangle,$
  $\langle \texttt{FirstWarp}, i, u, op \rangle,$
  $\langle \texttt{WarpBridge}, i, u, op \rangle\ )$
  from the single tile gadget, shown in Figure 6a if $i = 1$ or Figure 6b if $i = 2$, otherwise from Figure 6c if $i = 3$.

  – Create $\texttt{First\_Warp}(\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr} \rangle,$
  $\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr} \rangle,$
  $\langle \texttt{PostWarp}, 1, u, op, \texttt{msr} \rangle\ )$
  from the single tile gadget shown in Figure 6f.

  – Create $\texttt{First\_Warp}(\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr}, \texttt{msd} \rangle,$
  $\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr}, \texttt{msd} \rangle,$
  $\langle \texttt{PostWarp}, 1, u, op, \texttt{msr}, \texttt{msd} \rangle\ )$
  from the single tile gadget shown in Figure 6e.

  – Create $\texttt{First\_Warp}(\langle \texttt{FirstWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle,$
  $\langle \texttt{FirstWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle,$
  $\langle \texttt{WarpBridge}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle\ )$
  from the single tile gadget shown in Figure 6g.

  – Create $\texttt{First\_Warp}(\langle \texttt{FirstWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle,$
  $\langle \texttt{FirstWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle,$
  $\langle \texttt{WarpBridge}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle\ )$
  from the single tile gadget shown in Figure 6h.

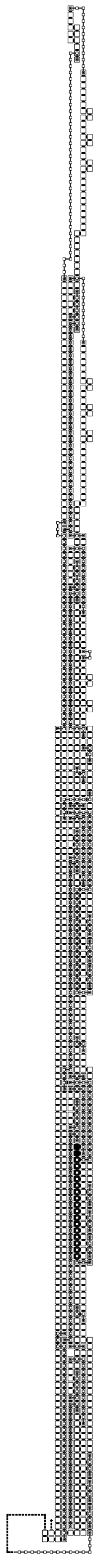
In this step, for digits 1-3 in the general case, $9 \cdot 2^l = 9 \cdot 2^{\lceil \log m \rceil + 2} = 36 \cdot 2^{\lceil \log m \rceil} \leq 72 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

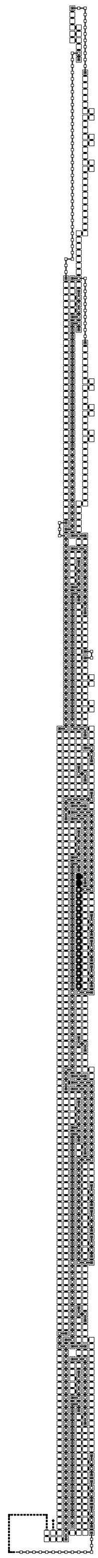
For digit 1 in case 1, $3 \cdot 2^l = 3 \cdot 2^{\lceil \log m \rceil + 2} = 12 \cdot 2^{\lceil \log m \rceil} \leq 24 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

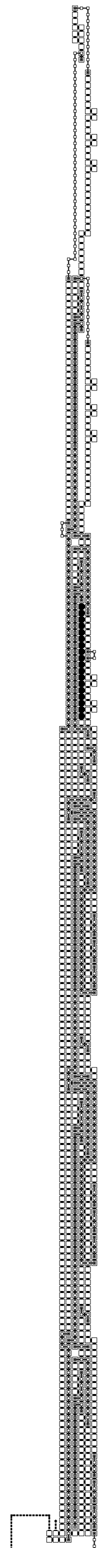
For digit 1 in case 2, $3 \cdot 2^l = 3 \cdot 2^{\lceil \log m \rceil + 2} = 12 \cdot 2^{\lceil \log m \rceil} \leq 24 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

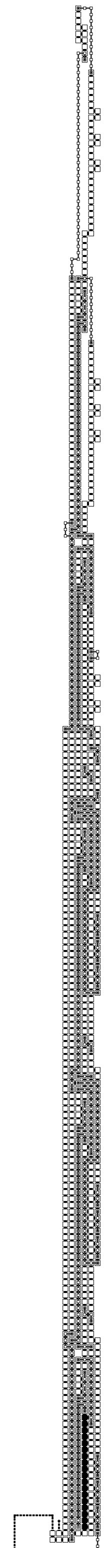
For digit 2 in case 2, $3 \cdot 2^l = 3 \cdot 2^{\lceil \log m \rceil + 2} = 12 \cdot 2^{\lceil \log m \rceil} \leq 24 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

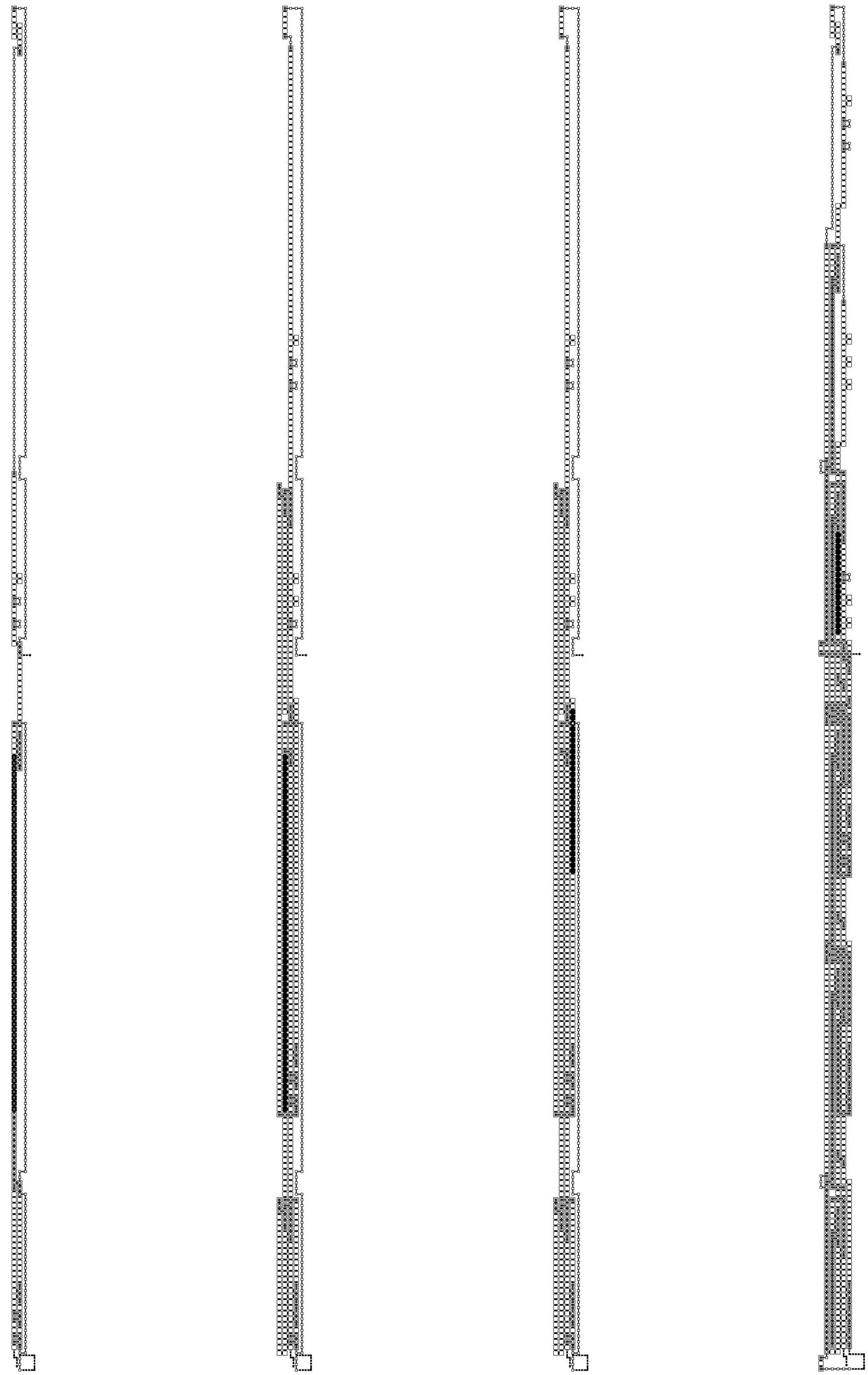
For digit 3 in case 3, $3 \cdot 2^l = 3 \cdot 2^{\lceil \log m \rceil + 2} = 12 \cdot 2^{\lceil \log m \rceil} \leq 24 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.


(a) Digit 1 - general overview.


(b) Digit 2 - general overview.


(c) Digit 3 - general overview.


(d) Digit 3 - general (seed) overview.



(e) Digit 1 - case 1 overview. (f) Digit 1 - case 2 overview. (g) Digit 2 - case 2 overview. (h) Digit 3 - case 3 overview.

Figure 6: The First_Warp gadget overviews.

- `Warp_Bridge`:
  The idea of the `Warp_Bridge` gadget is to assemble after the `First_Warp` gadget makes it to its final destination. Its objective is to assemble a path from the end of the `First_Warp` gadgets to the start of the `Second_Warp` gadgets. For digit 1 in cases 1 and 2, the `Warp_Bridge` is omitted from the `Warp_Unit`.

  For each $i = 1, 2, 3, u \in \{0, 1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}, \texttt{halt}\}$:

  – Create `Warp_Bridge`($\langle\texttt{WarpBridge}, i, u, op\rangle, \langle\texttt{SecondWarp}, i, u, op\rangle$) from the general gadget in Figure 7a.

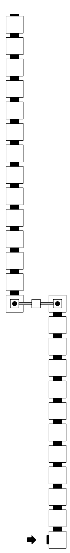
  – Create `Warp_Bridge`($\langle\texttt{WarpBridge}, 2, u, op, \texttt{msr}, \texttt{msd}\rangle, \langle\texttt{SecondWarp}, 2, u, op, \texttt{msr}, \texttt{msd}\rangle$) from the general gadget in Figure 7g.

  – Create `Warp_Bridge`($\langle\texttt{WarpBridge}, 3, u, op, \texttt{msr}, \texttt{msd}\rangle, \langle\texttt{SecondWarp}, 3, u, op, \texttt{msr}, \texttt{msd}\rangle$) from the general gadget in Figure 7a.

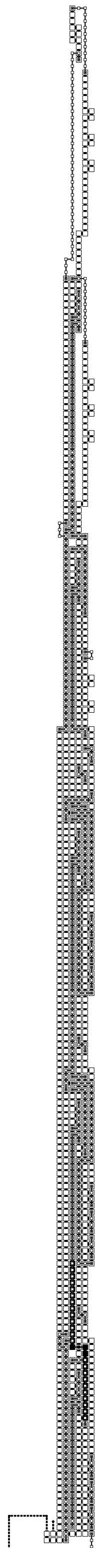
  In this step, for digits 1-3 in the general case, $9 \cdot 2^l \cdot 29 = 261 \cdot 2^l = 261 \cdot 2^{\lceil \log m \rceil + 2} = 1044 \cdot 2^{\lceil \log m \rceil} \leq 2088 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

  For digit 2 in case 2, $3 \cdot 2^l \cdot 15 = 45 \cdot 2^l = 45 \cdot 2^{\lceil \log m \rceil + 2} = 180 \cdot 2^{\lceil \log m \rceil} \leq 360 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

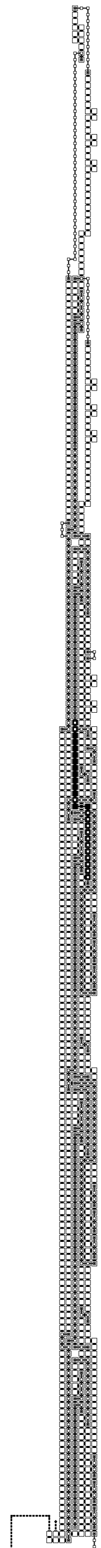
  For digit 3 in case 3, $3 \cdot 2^l \cdot 29 = 87 \cdot 2^l = 87 \cdot 2^{\lceil \log m \rceil + 2} = 348 \cdot 2^{\lceil \log m \rceil} \leq 696 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.



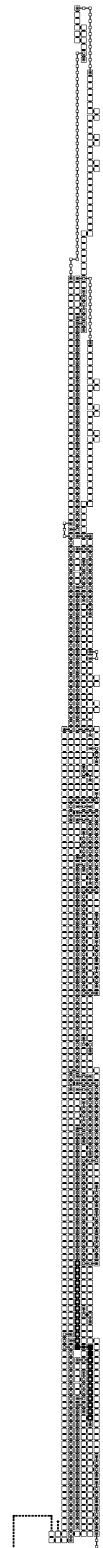
(a) Digits 1, 2, & 3 general, digit 3 - case 3. There are 29 tiles in this gadget.



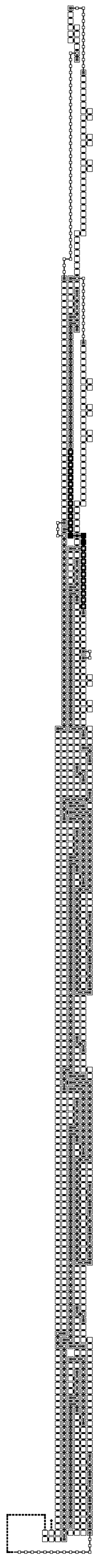
(b) Digit 1 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



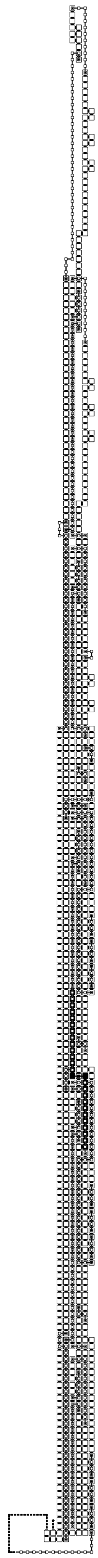
(c) Digit 2 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



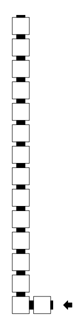
(d) Digit 2 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



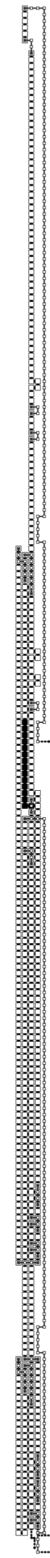
(e) Digit 3 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



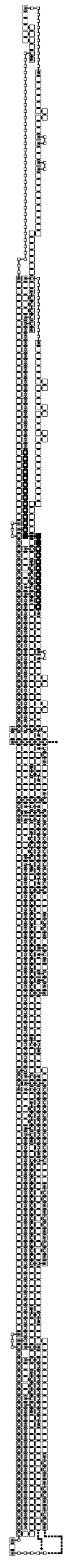
(f) Digit 3 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



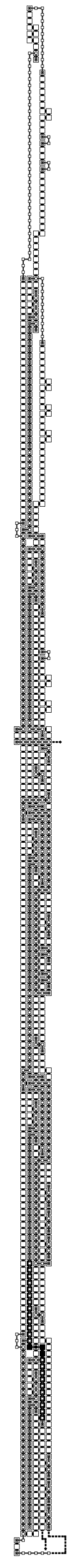
(g) Digit 2 - case 2. There are 15 tiles in this gadget.



(h) Digit 2 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure g.



(i) Digit 3 - case 3 overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



(j) Digit 3 - case 3 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

Figure 7: The `Warp_Bridge` gadgets.

- `Second_Warp`:

  Similar to the `First_Warp` gadget, the idea of this gadget is also transport the information read by the `Counter_Read` gadgets across a large distance. We do this using a single special tile that has three glues. Both the north and south glues have the same labels, allowing this tile to assemble into a line indefinitely. The third glue is also the special glue; this glue can be either on the east or up side of the tile, and has the output label. This unique glue will at some point later in the assembly, in a spot determined by earlier pieces of the assembly, no longer be blocked from the direction of it special glue. When this occurs, it can finally attach to the `Post_Warp` gadget. This process signifies the "waking up" of the `Second_Warp` gadgets. When this gadget wakes up, it must also be blocked in the north direction. By blocking the north glue, the line stops, which prevents a truly infinite line from assembling. The earlier pieces of the assembly that guarentee this process to work are known as the `Digit_Top` gadgets.

  For each $i = 1, 2, 3, u \in \{0,1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}, \texttt{halt}\}$:

  - Create $\texttt{Second\_Warp}(\langle \texttt{SecondWarp}, i, u, op\rangle, \langle \texttt{SecondWarp}, i, u, op\rangle, \langle \texttt{PostWarp}, i, u, op\rangle\ )$ from the single tile gadget, shown in Figure 8a if $i = 1$ or Figure 8b if $i = 2$, otherwise from Figure 8c if $i = 3$.

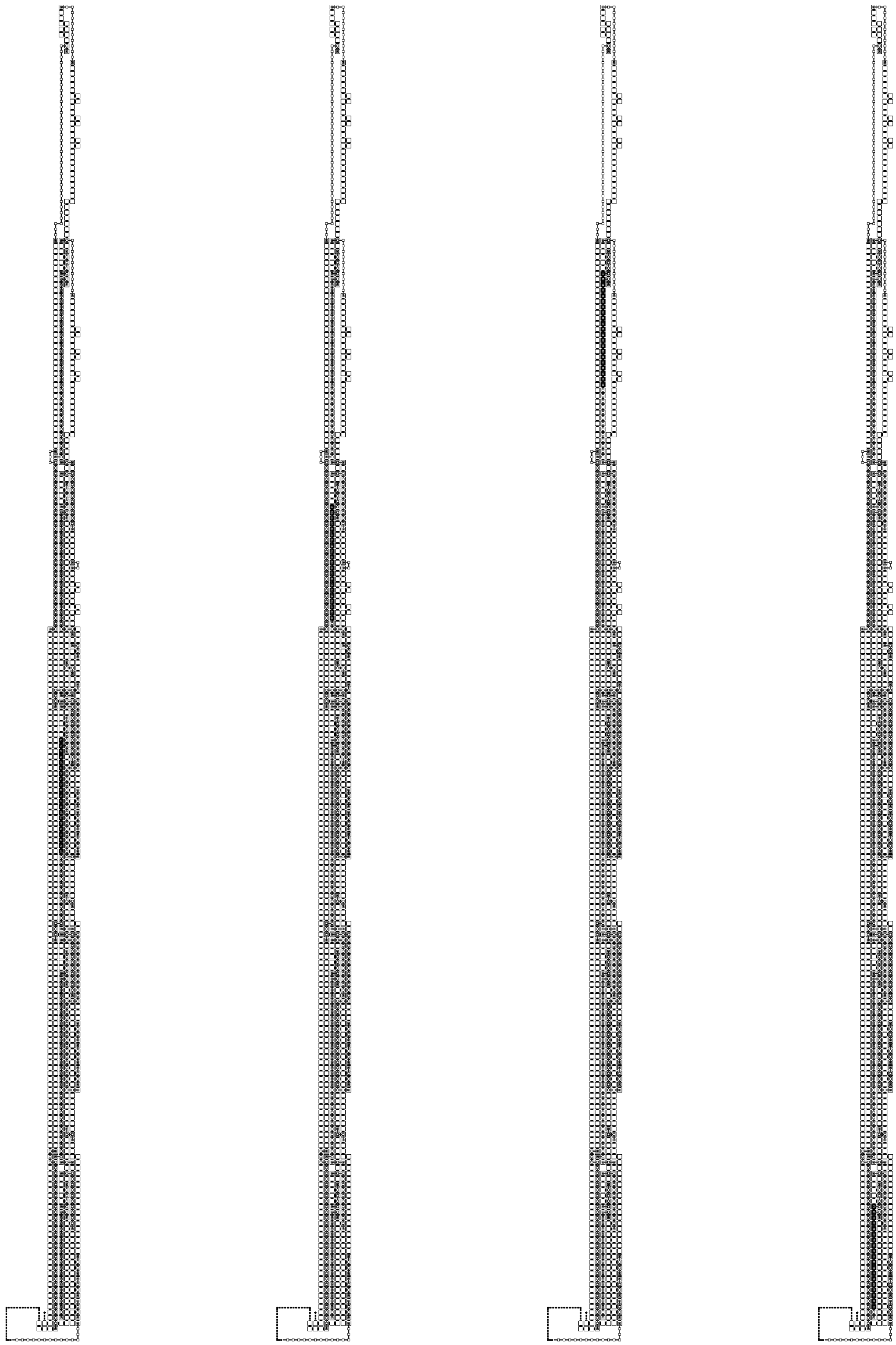
  - Create $\texttt{Second\_Warp}(\langle \texttt{SecondWarp}, 2, u, op, \texttt{msr}, \texttt{msd}\rangle,$
    $\langle \texttt{SecondWarp}, 2, u, op, \texttt{msr}, \texttt{msd}\rangle,$
    $\langle \texttt{PostWarp}, 2, u, op, \texttt{msr}, \texttt{msd}\rangle\ )$
    from the single tile gadget shown in Figure 8f.

  - Create $\texttt{Second\_Warp}(\langle \texttt{SecondWarp}, 3, u, op, \texttt{msr}, \texttt{msd}\rangle,$
    $\langle \texttt{SecondWarp}, 3, u, op, \texttt{msr}, \texttt{msd}\rangle,$
    $\langle \texttt{PostWarp}, 3, u, op, \texttt{msr}, \texttt{msd}\rangle\ )$
    from the single tile gadget shown in Figure 8h.

  In this step, for digits 1-3 in the general case, $9 \cdot 2^l = 9 \cdot 2^{\lceil \log m\rceil + 2} = 36 \cdot 2^{\lceil \log m\rceil} \leq 72 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2}\rfloor}}\right)$ tiles were created.

  For digit 2 in case 2, $3 \cdot 2^l = 3 \cdot 2^{\lceil \log m\rceil + 2} = 12 \cdot 2^{\lceil \log m\rceil} \leq 24 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2}\rfloor}}\right)$ tiles were created.

  For digit 3 in case 3, $3 \cdot 2^l = 3 \cdot 2^{\lceil \log m\rceil + 2} = 12 \cdot 2^{\lceil \log m\rceil} \leq 24 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2}\rfloor}}\right)$ tiles were created.

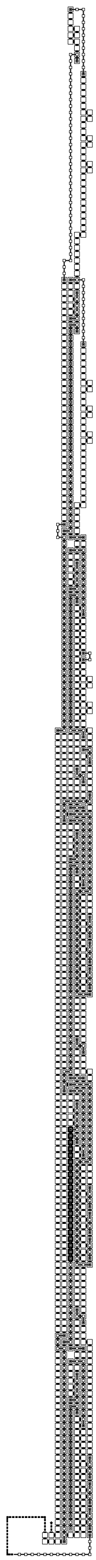


(a) Digit 1 - general overview.

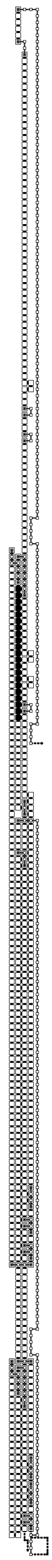
(b) Digit 2 - general overview.

(c) Digit 3 - general overview.

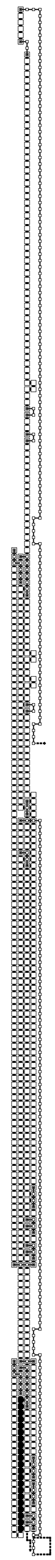
(d) Digit 2 - general (seed) overview.



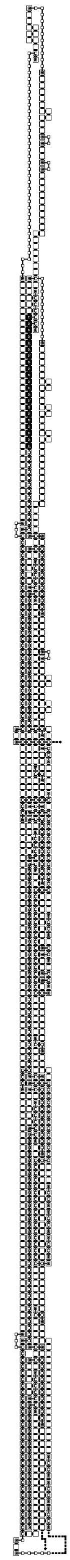
(e) Digit 3 - general (seed) overview.



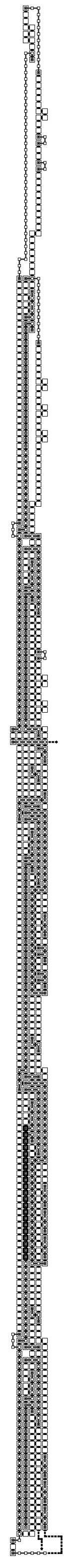
(f) Digit 2 - case 2 overview.



(g) Digit 2 - case 2 (seed) overview.



(h) Digit 3 - case 3 overview.



(i) Digit 3 - case 3 (seed) overview.

Figure 8: The Second_Warp gadget overviews.

- `Post_Warp`:

  The `Post_Warp` gadget is the final gadget to assemble in the `Warp_Unit`. The idea of this gadget is to assemble from wherever the last warping tiles "wake up", forming a path that ends where the next digit needs to start being written. Since it is the last gadget to assemble in the `Warp_Unit`, this gadget outputs a signal allowing the `Counter_Write` gadgets to assemble to digit that was being passed along.

  For each $i = 1, 2, 3, u \in \{0, 1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}, \texttt{halt}\}$:

  – Create $\texttt{Post\_Warp}(\langle \texttt{PostWarp}, i, u, op\rangle, \langle \texttt{CounterWrite}, i, u, op\rangle\ )$
    from the general gadget shown in Figure 9a if $i = 1$, or Figure 9c if $i = 2$ or $i = 3$.

  – Create $\texttt{Post\_Warp}(\langle \texttt{PostWarp}, 1, u, op, \texttt{msr}\rangle, \langle \texttt{CounterWrite}, 1, u, op, \texttt{msr}\rangle\ )$
    from the general gadget in Figure 9j.

  – For each $i = 1, 2, 3$: create
    $\texttt{Post\_Warp}(\langle \texttt{PostWarp}, i, u, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{CounterWrite}, i, u, op, \texttt{msr}, \texttt{msd}\rangle\ )$
    from the general gadget shown in Figure 9h if $i = 1$, or Figure 9l if $i = 2$, or Figure 9c if $i = 3$.

In this step, for digit 1 in the general case, $3 \cdot 2^l \cdot 27 = 81 \cdot 2^l = 81 \cdot 2^{\lceil \log m \rceil + 2} = 324 \cdot 2^{\lceil \log m \rceil} \leq 648 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

For digits 2 & 3 in the general case, $6 \cdot 2^l \cdot 25 = 150 \cdot 2^l = 150 \cdot 2^{\lceil \log m \rceil + 2} = 600 \cdot 2^{\lceil \log m \rceil} \leq 1200 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

For digit 1 in case 1, $3 \cdot 2^l \cdot 25 = 75 \cdot 2^l = 75 \cdot 2^{\lceil \log m \rceil + 2} = 300 \cdot 2^{\lceil \log m \rceil} \leq 600 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

For digit 1 in case 2, $3 \cdot 2^l \cdot 26 = 78 \cdot 2^l = 78 \cdot 2^{\lceil \log m \rceil + 2} = 312 \cdot 2^{\lceil \log m \rceil} \leq 624 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

For digit 2 in case 2, $3 \cdot 2^l \cdot 22 = 66 \cdot 2^l = 66 \cdot 2^{\lceil \log m \rceil + 2} = 264 \cdot 2^{\lceil \log m \rceil} \leq 528 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

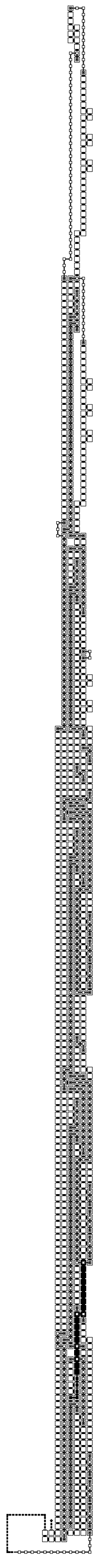
For digit 3 in case 3, $3 \cdot 2^l \cdot 25 = 75 \cdot 2^l = 75 \cdot 2^{\lceil \log m \rceil + 2} = 300 \cdot 2^{\lceil \log m \rceil} \leq 600 \cdot 2^{\log m} = O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)$ tiles were created.

(a) Digit 1 - general. There are 27 tiles in this gadget.

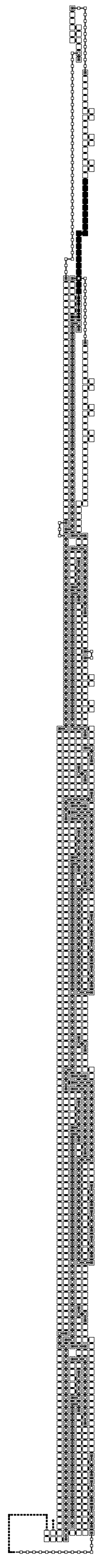
(b) Digit 1 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

(c) Digit 2 & 3 - general, digit 3 - case 3. There are 25 tiles in this gadget.

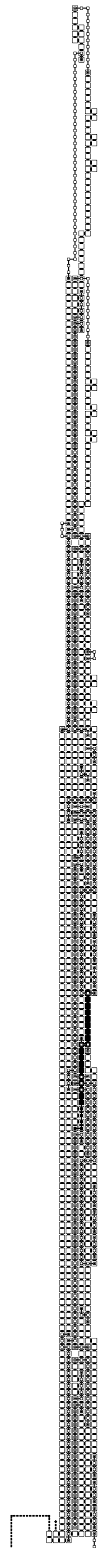
(d) Digit 2 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure c.



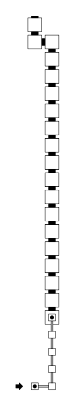
(e) Digit 2 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure c.



(f) Digit 3 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure c.

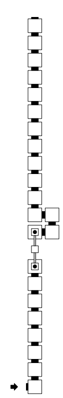


(g) Digit 3 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure c.

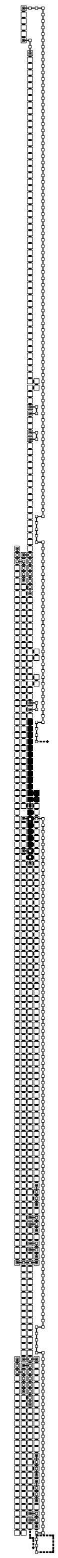


(h) Digit 1 - case 1. There are 25 tiles in this gadget.

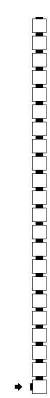
(i) Digit 1 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure h.



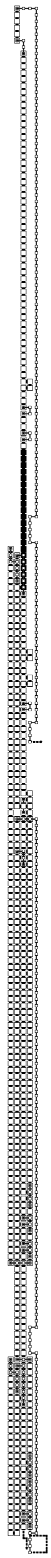
(j) Digit 1 - case 2. There are 26 tiles in this gadget.



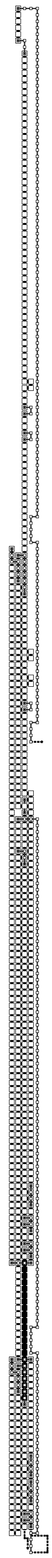
(k) Digit 1 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure j.



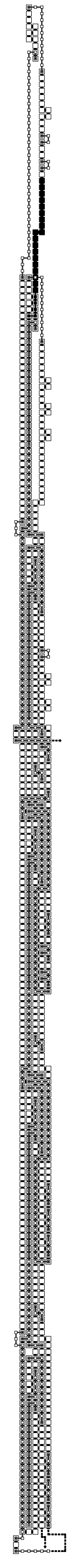
(l) Digit 2 - case 2. There are 22 tiles in this gadget.



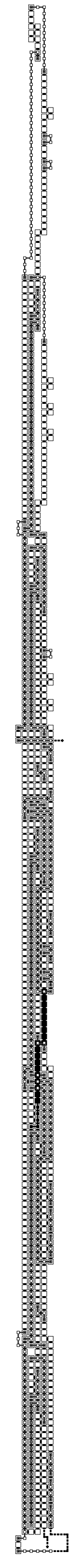
(m) Digit 2 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure l.



(n) Digit 2 - case 2 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure l.



(o) Digit 3 - case 3 overview. The black tiles in this figure correspond to the gadget shown in subfigure c.



(p) Digit 3 - case 3 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure c.

Figure 9: The `Post_Warp` gadgets.

4.1.4 `Counter_Write`

- For each $i = 1, 2, 3$, $j = 1, \ldots, l-1$, $u \in \{0,1\}^j$, and each $op \in \{\texttt{increment}, \texttt{copy}, \texttt{halt}\}$:
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, u0, op\rangle, \langle \texttt{CounterWrite}, i, u, op\rangle\ )$ from the general gadget in Figure 10a.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, u1, op\rangle, \langle \texttt{CounterWrite}, i, u, op\rangle\ )$ from the general gadget in Figure 10b.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, u0, op, \texttt{msr}\rangle, \langle \texttt{CounterWrite}, 1, u, op, \texttt{msr}\rangle\ )$ from the general gadget in Figure 10a.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, u1, op, \texttt{msr}\rangle, \langle \texttt{CounterWrite}, 1, u, op, \texttt{msr}\rangle\ )$ from the general gadget in Figure 10b.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, u0, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{CounterWrite}, i, u, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the general gadget in Figure 10a.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, u1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{CounterWrite}, i, u, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the general gadget in Figure 10b.

  In this step,

$$
\begin{aligned}
\sum_{j=0}^{l-1} 3 \cdot 2^j \cdot 3 \cdot 6 \cdot 7 &= 378\big(2^l - 2\big) \\
&= 378\left(2^{\lceil \log m \rceil + 2} - 2\right) \\
&= 378\left(4 \cdot 2^{\lceil \log m \rceil} - 2\right) \\
&= 1512\left(2^{\lceil \log m \rceil} - \frac{1}{2}\right) \\
&\leq 1512\left(2 \cdot 2^{\log m} - \frac{1}{2}\right) \\
&= 1512\left(2m - \frac{1}{2}\right) \\
&= 3024m - 756 \\
&= O(m) = O\left(N^{\frac{1}{\lfloor \frac{k}{2} \rfloor}}\right)
\end{aligned}
$$

  tiles were created.

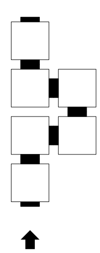
- For each $i = 1, 2, 3$ and each $op \in \{\texttt{increment}, \texttt{copy}\}$:
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, 0, op\rangle, \langle \texttt{DigitTop}, i, op\rangle\ )$ from the general gadget in Figure 10a.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, 1, op\rangle, \langle \texttt{DigitTop}, i, op\rangle\ )$ from the general gadget in Figure 10b.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, 0, op, \texttt{msr}\rangle, \langle \texttt{DigitTop}, 1, op, \texttt{msr}\rangle\ )$ from the general gadget in Figure 10a.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, 1, op, \texttt{msr}\rangle, \langle \texttt{DigitTop}, 1, op, \texttt{msr}\rangle\ )$ from the general gadget in Figure 10b.

  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, 0, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTop}, i, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the general gadget in Figure 10a.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTop}, i, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the general gadget in Figure 10b.

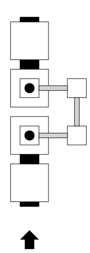
  In this step, $9 \cdot 6 \cdot 7 = 378 = O(1)$ tiles were created.

- For each $i = 1, 2, 3$:
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, 0, \texttt{halt}, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{RoofScaffolding}, i, \texttt{msr}, \texttt{msd}\rangle\ )$ from the general gadget in Figure 10a.
  - Create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, i, 1, \texttt{halt}, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{RoofScaffolding}, i, \texttt{msr}, \texttt{msd}\rangle\ )$ from the general gadget in Figure 10b.

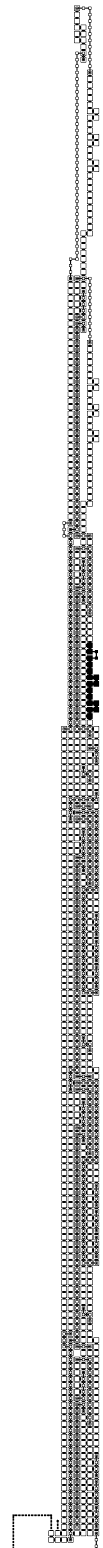
  In this step, $3 \cdot 2 \cdot 7 = 42 = O(1)$ tiles were created.



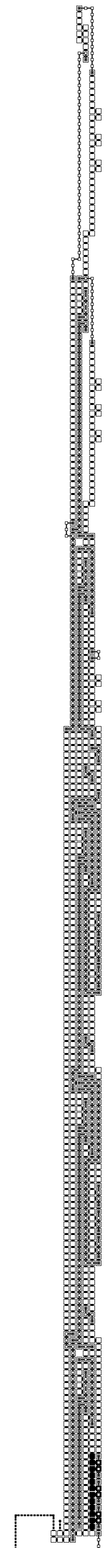
(a) `Counter_Write_0`. There are 6 tiles in this gadget.



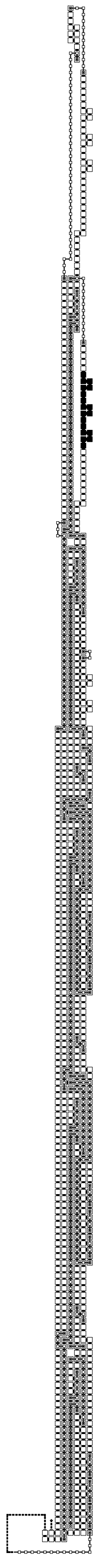
(b) `Counter_Write_1`. There are 8 tiles in this gadget.



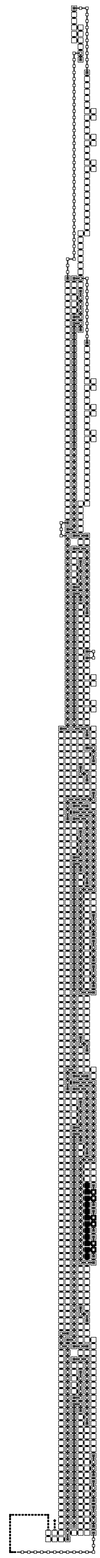
(c) Digit 1 - general overview.



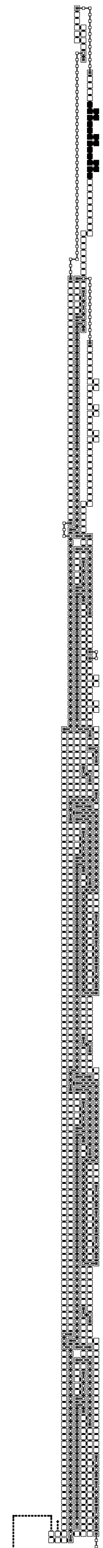
(d) Digit 1 - general (seed) overview.



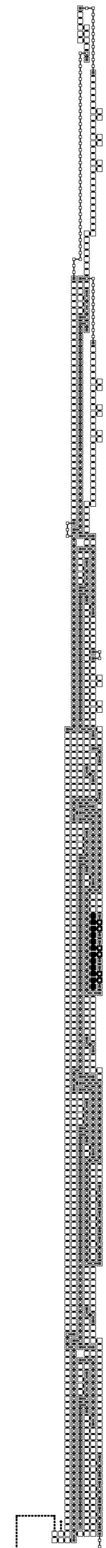
(e) Digit 2 - general overview.



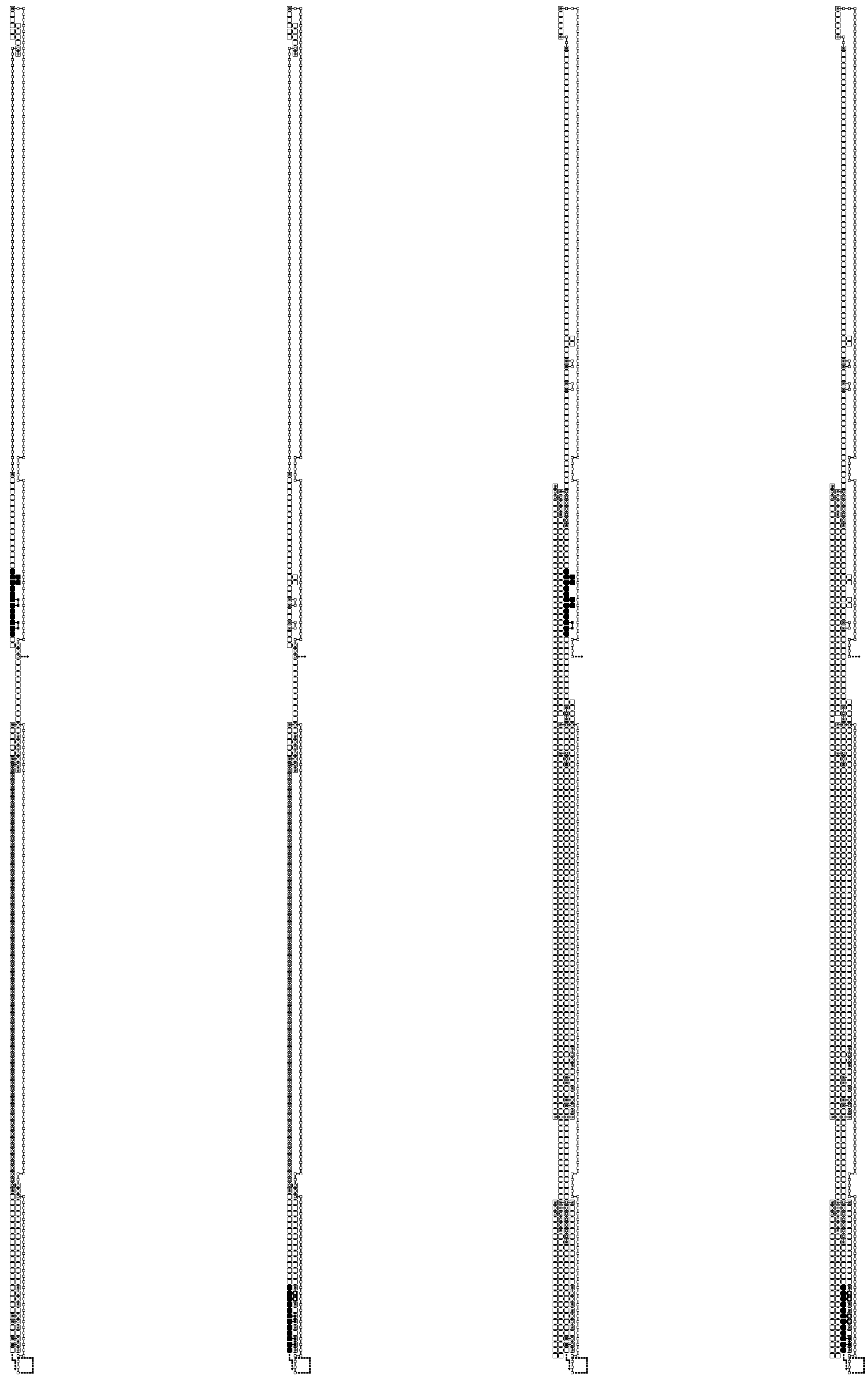
(f) Digit 2 - general (seed) overview.



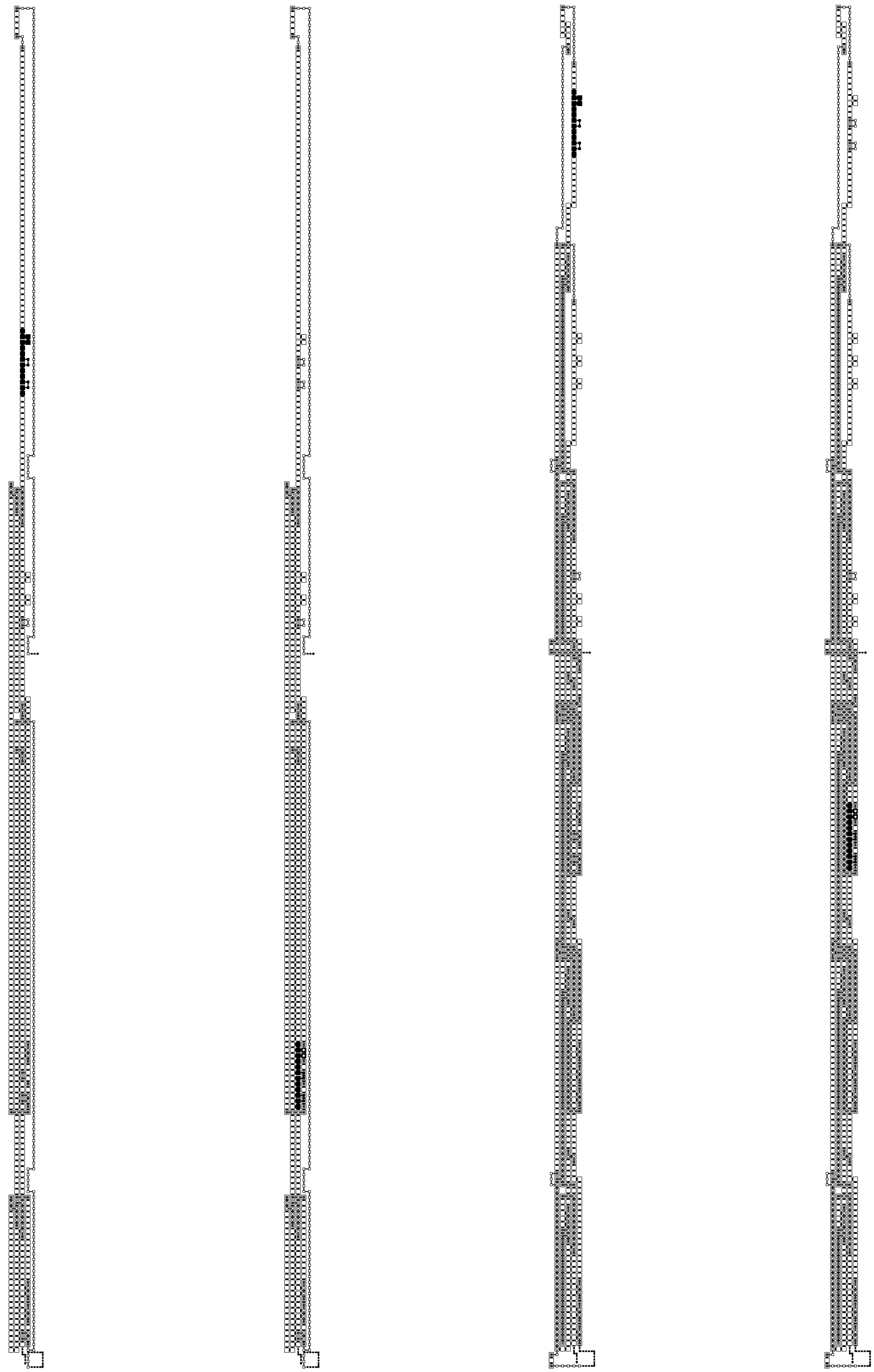
(g) Digit 3 - general overview.



(h) Digit 3 - general (seed) overview.



(i) Digit 1 - case 1. (j) Digit 1 - case 1 (seed). (k) Digit 1 - case 2. (l) Digit 1 - case 2 (seed).

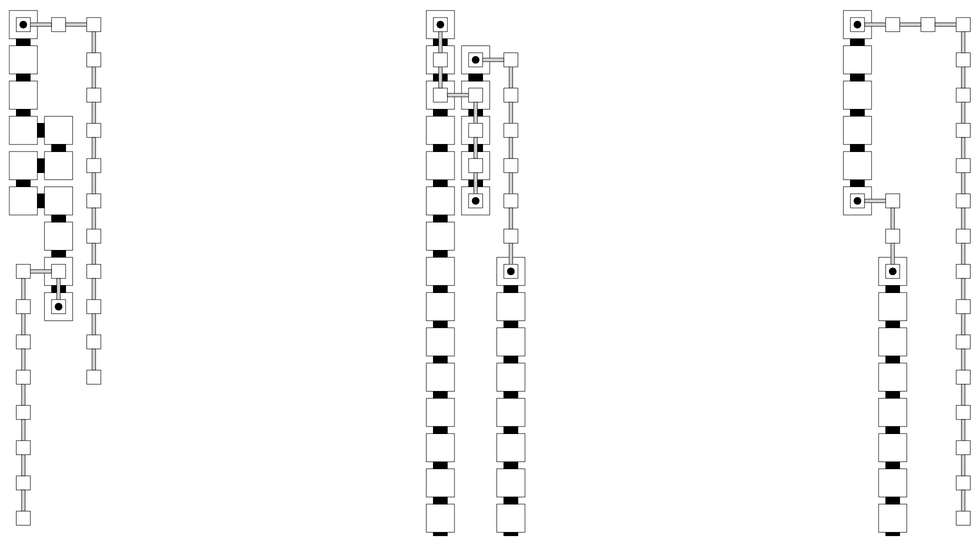


(m) Digit 2 - case 2. (n) Digit 2 - case 2 (seed). (o) Digit 3 - case 3. (p) Digit 3 - case 3 (seed).

Figure 10: The Counter_Write gadgets.

4.1.5 `Digit_Top`

The `Digit_Top` gadgets have special geometry designed so that `First_Warp` and `Second_Warp` tiles are allowed to "wake up", and complete their warping journey. Each digit has some type of `Digit_Top` gadget, however, depending on the digit region and index of a specific digit, the exact digit top will differ.



(a) General topper (b) Case 1 – topper (c) Case 2 – topper

Figure 11: Topper micro-gadgets

For each $op \in \{\texttt{increment}, \texttt{copy}\}$

- Digit 1 (general): the following statements create the gadget shown in Figure 12a.
  - Create $\texttt{North\_Line5}(\langle\texttt{DigitTop}, 1, op\rangle, \langle\texttt{DigitTopA}, 1, op\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle\texttt{DigitTopA}, 1, op\rangle, \langle\texttt{DigitTopB}, 1, op\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle\texttt{DigitTopB}, 1, op\rangle, \langle\texttt{Return\_Path}, 1, op\rangle\ )$ from the micro-gadget shown in Figure 3b.

  In this step, $2 \cdot (40 + 4l) = 80 + 8l = 80 + 8 \cdot (\lceil \log m \rceil + 2) \leq 80 + 8 \cdot (\log m + 3) = 104 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Digit 1 (MSR): the following statements create the gadget shown in Figure 12k.
  - Create $\texttt{Topper}(\langle\texttt{DigitTop}, 1, op, \texttt{msr}\rangle, \langle\texttt{DigitTopA}, 1, op, \texttt{msr}\rangle\ )$ from the micro-gadget shown in Figure 11b.
  - Create $\texttt{South\_Line4}l(\langle\texttt{DigitTopA}, 1, op, \texttt{msr}\rangle, \langle\texttt{Return\_Path}, 1, op, \texttt{msr}\rangle\ )$ from the micro-gadget shown in Figure 3b.

  In this step, $2 \cdot (43 + 4l) = 86 + 8l = 86 + 8 \cdot (\lceil \log m \rceil + 2) \leq 86 + 8 \cdot (\log m + 3) = 110 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Digit 1 (MSD): the following statements create the gadget shown in Figure 12h.
  - Create $\texttt{North\_Line4}l(\langle \texttt{DigitTop}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopA}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{North\_Line4}(\langle \texttt{DigitTopA}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopB}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopB}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopC}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopC}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopD}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line30}(\langle \texttt{DigitTopD}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopE}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopE}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopF}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line14}(\langle \texttt{DigitTopF}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopG}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line17}(\langle \texttt{DigitTopG}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{Return\_Path}, 1, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.

  In this step, $2\cdot(100 + 12l) = 200+24l = 200+24\cdot(\lceil \log m\rceil + 2) \leq 200+24\cdot(\log m + 3) = 272+24\cdot\log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Digit 2 (general): the following statements create the gadget shown in Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 2, op\rangle, \langle \texttt{DigitTopA2}, op\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA2}, op\rangle, \langle \texttt{DigitTopB2}, op\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB2}, op\rangle, \langle \texttt{Return\_Path}, 2, op\rangle\ )$ from the micro-gadget shown in Figure 3b.

  In this step $2 \cdot (40 + 4l) = 80 + 8l = 80 + 8 \cdot (\lceil \log m\rceil + 2) \leq 80 + 8 \cdot (\log m + 3) = 104 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Digit 2 (MSD): the following statements create the gadget shown in Figure 12n.
  - Create $\texttt{North\_Line4}l(\langle \texttt{DigitTop}, 2, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopA}, 2, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 2, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopB}, 2, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 11c.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 2, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopC}, 2, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line30}(\langle \texttt{DigitTopC}, 2, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{Return\_Path}, 2, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.

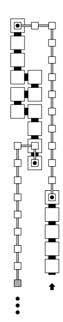
  In this step, $2\cdot(58 + 8l) = 116+16l = 116+16\cdot(\lceil \log m\rceil + 2) \leq 116+16\cdot(\log m + 3) = 164+16\cdot\log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Digit 3 (general): the following statements create the gadget from Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 3, op\rangle, \langle \texttt{DigitTopA}, 3, op\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 3, op\rangle, \langle \texttt{DigitTopB}, 3, op\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 3, op\rangle, \langle \texttt{Return\_Path}, 3, op\rangle\ )$ from the micro-gadget shown in Figure 3b.

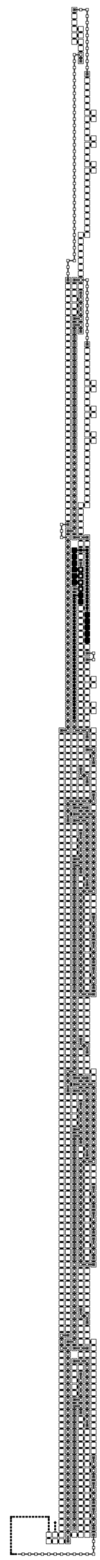
  In this step, $2 \cdot (40 + 4l) = 80 + 8l = 80 + 8 \cdot (\lceil \log m \rceil + 2) \leq 80 + 8 \cdot (\log m + 3) = 104 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Digit 3 (MSD): the following statements create the gadget from Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 3, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopA}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 3, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{DigitTopB}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 3, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{Return\_Path}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ )$ from the micro-gadget shown in Figure 3b.

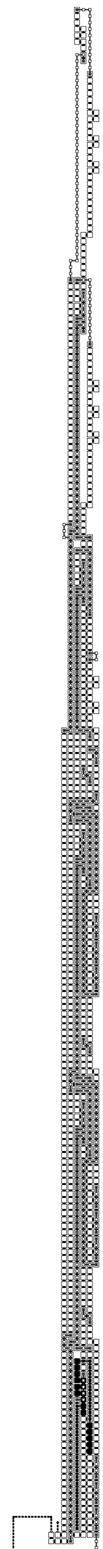
  In this step, $2 \cdot (40 + 4l) = 80 + 8l = 80 + 8 \cdot (\lceil \log m \rceil + 2) \leq 80 + 8 \cdot (\log m + 3) = 104 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.



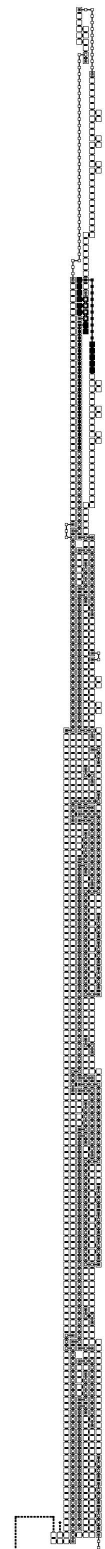
(a) Digits 1, 2, & 3 - general. Digit 3 - case 3. There are $40 + 4l$ tiles in this gadget.



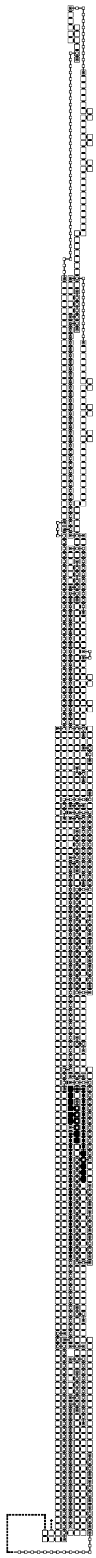
(b) Digit 1 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



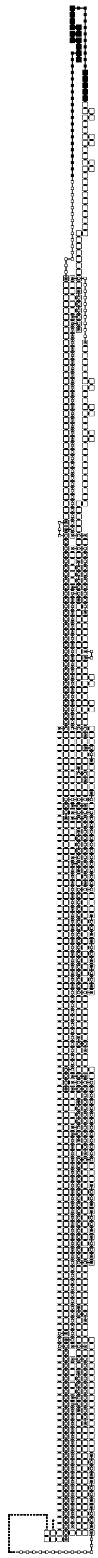
(c) Digit 1 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



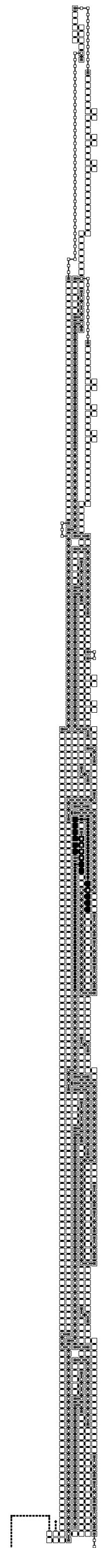
(d) Digit 2 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



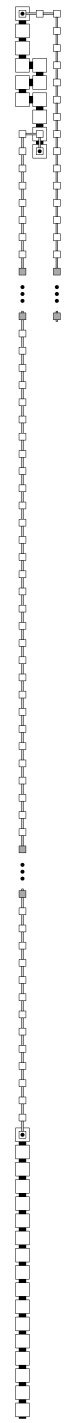
(e) Digit 2 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



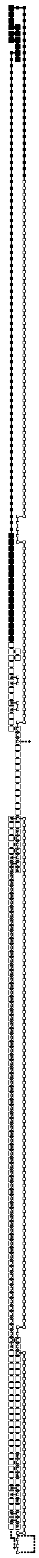
(f) Digit 3 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



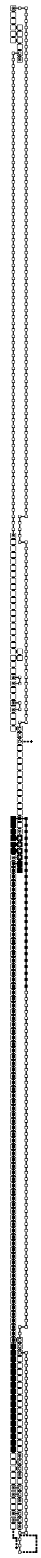
(g) Digit 3 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



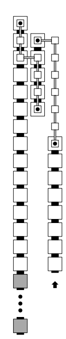
(h) Digit 1 - case 1. There are $100+12l$ tiles in this gadget.



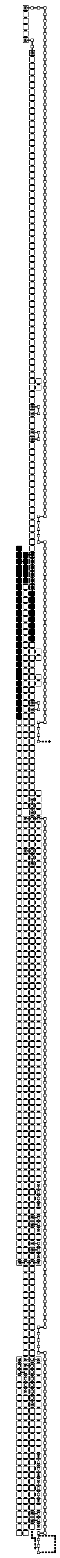
(i) Digit 1 - case 1 overview. The black tiles in this figure correspond to the gadget shown in subfigure h.



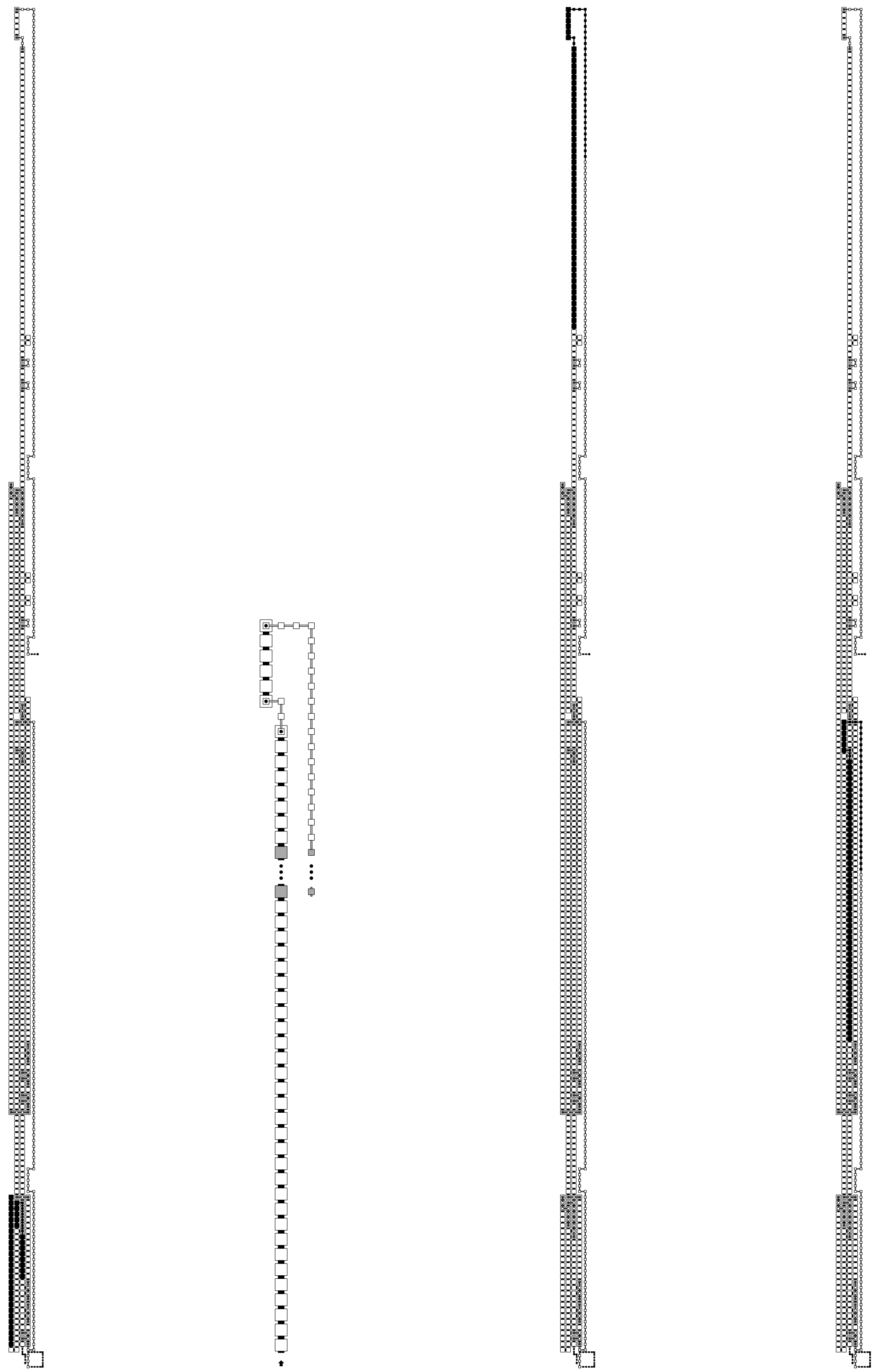
(j) Digit 1 - case 1 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure h.



(k) Digit 1 - case 2. There are $43 + 4l$ tiles in this gadget.



(l) Digit 1 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure k.

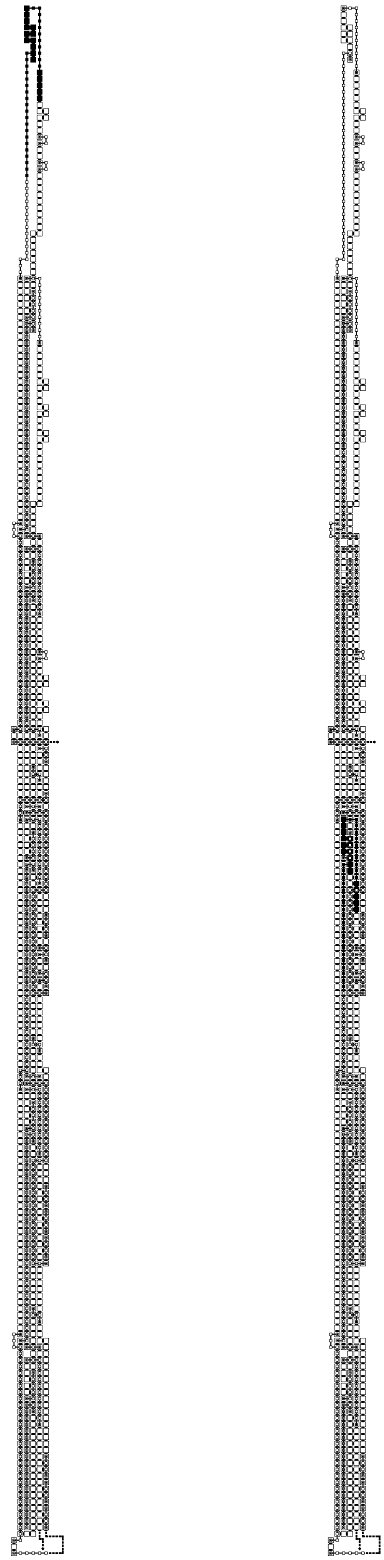


(m) Digit 1 - case 2 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure k.

(n) Digit 2 - case 2. There are $58 + 8l$ tiles in this gadget.

(o) Digit 2 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure n.

(p) Digit 2 - case 2 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure n.



(q) Digit 3 - case 3 overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

(r) Digit 3 - case 3 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

Figure 12: The `Digit_Top` gadgets.

4.1.6 `Return_Path`

After a `Digit_Top` has assembled, we know that the geometry that allows for the `Warp_Unit` to work has been placed, therefore the counter is free to return back near where it started reading the previous digit. The next gadget that assembles is the `Return_Path` gadget. The basic idea of this gadget is simply to provide a path from the end of `Digit_Top` to some general area closer to where the most recently read digit is located. Once the `Return_Path` has completely assembled, it output's a glue for the `Next_Read` gadgets to determine where the counter needs to assemble next.

For each $op \in \{\texttt{increment}, \texttt{copy}\}$:

- Create `Return_Path`($\langle \texttt{ReturnPath}, 1, op \rangle, \langle \texttt{NextRead}, 1, op \rangle$)
  from the general gadget shown in Figure 13a. In this step, $2 \cdot (42 + 4l) = 84 + 8l = 84 + 8 \cdot (\lceil \log m \rceil + 2) \leq 84 + 8 \cdot (\log m + 3) = 108 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create `Return_Path`($\langle \texttt{ReturnPath}, 1, op, \texttt{msr} \rangle, \langle \texttt{NextRead}, 1, op, \texttt{msr} \rangle$)
  from the general gadget shown in Figure 13j. In this step, $2 \cdot (30 + 4l) = 60 + 8l = 60 + 8 \cdot (\lceil \log m \rceil + 2) \leq 60 + 8 \cdot (\log m + 3) = 84 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

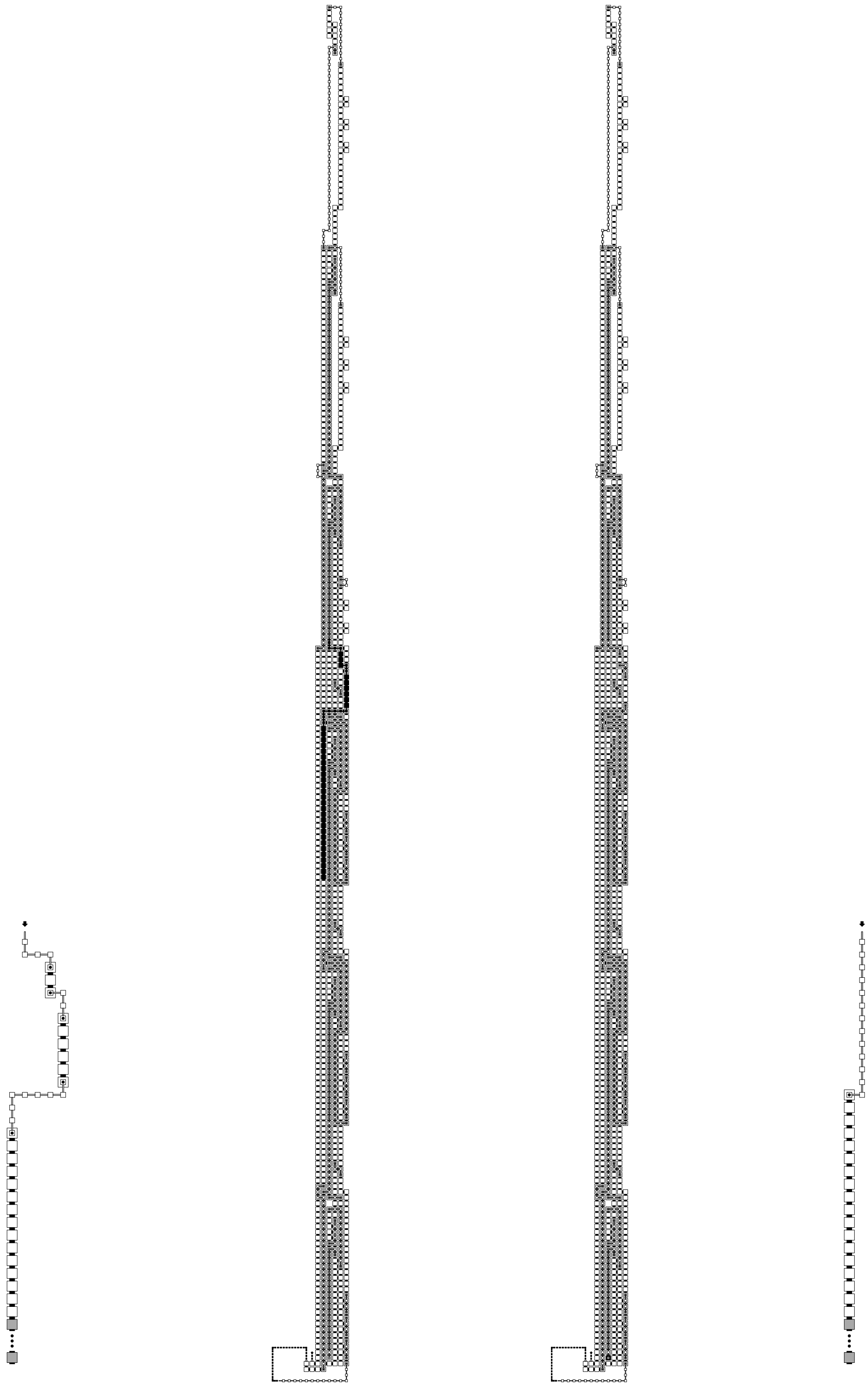
- Create `Return_Path`($\langle \texttt{ReturnPath}, 1, op, \texttt{msr}, \texttt{msd} \rangle, \langle \texttt{NextRead}, 1, op, \texttt{msr}, \texttt{msd} \rangle$)
  from the general gadget shown in Figure 13m. In this step, $2 \cdot (30 + 4l) = 60 + 8l = 60 + 8 \cdot (\lceil \log m \rceil + 2) \leq 60 + 8 \cdot (\log m + 3) = 84 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create `Return_Path`($\langle \texttt{ReturnPath}, 2, op \rangle, \langle \texttt{NextRead}, 2, op \rangle$)
  from the general gadget shown in Figure 13d. In this step, $2 \cdot (32 + 4l) = 64 + 8l = 64 + 8 \cdot (\lceil \log m \rceil + 2) \leq 64 + 8 \cdot (\log m + 3) = 88 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create `Return_Path`($\langle \texttt{ReturnPath}, 2, op, \texttt{msr}, \texttt{msd} \rangle, \langle \texttt{NextRead}, 2, op, \texttt{msr}, \texttt{msd} \rangle$)
  from the general gadget shown in Figure 13m. In this step, $30 + 4l$ tiles were created. In this step, $2 \cdot (30 + 4l) = 60 + 8l = 60 + 8 \cdot (\lceil \log m \rceil + 2) \leq 60 + 8 \cdot (\log m + 3) = 84 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create `Return_Path`($\langle \texttt{ReturnPath}, 3, op \rangle, \langle \texttt{NextRead}, 3, op \rangle$)
  from the general gadget shown in Figure 13g. In this step, $2 \cdot (65 + 8l) = 130 + 16l = 130 + 16 \cdot (\lceil \log m \rceil + 2) \leq 130 + 16 \cdot (\log m + 3) = 178 + 16 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create `Return_Path`($\langle \texttt{ReturnPath}, 3, op, \texttt{msr}, \texttt{msd} \rangle, \langle \texttt{NextRead}, 3, op, \texttt{msr}, \texttt{msd} \rangle$)
  from the general gadget shown in Figure 13g. In this step, $2 \cdot (65 + 8l) = 130 + 16l = 130 + 16 \cdot (\lceil \log m \rceil + 2) \leq 130 + 16 \cdot (\log m + 3) = 178 + 16 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

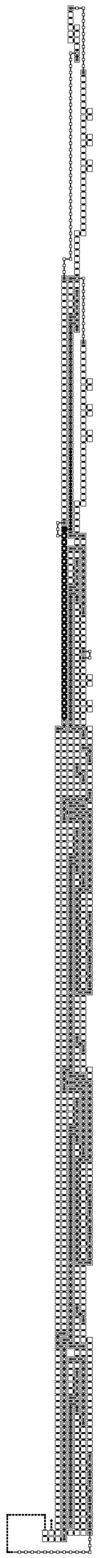


(a) Digit 1 - general. There are $42 + 4l$ tiles in this gadget.

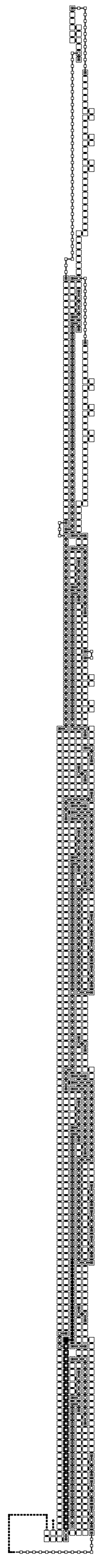
(b) Digit 1 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.

(c) Digit 1 - general (seed) overview. The black tile in this figure is a single tile gadget used only in the initial value.

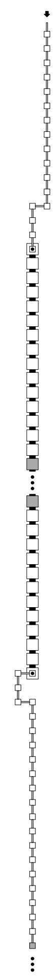
(d) Digit 2 - general. There are $32 + 4l$ tiles in this gadget.



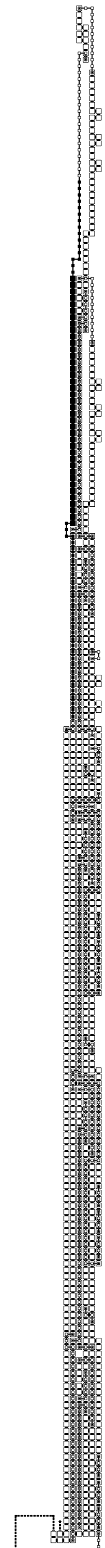
(e) Digit 2 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure d.



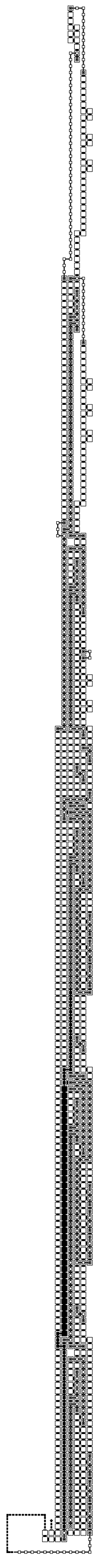
(f) Digit 2 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure d.



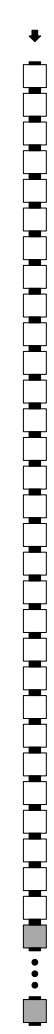
(g) Digit 3 - general. There are $65 + 8l$ tiles in this gadget.



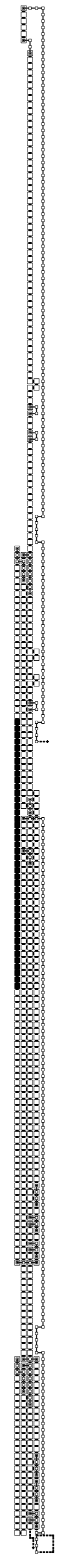
(h) Digit 3 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure g.



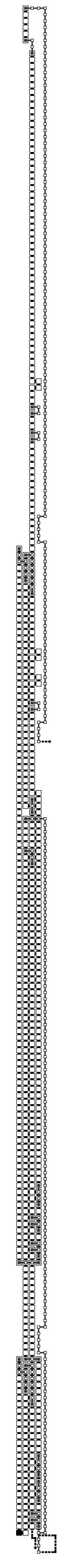
(i) Digit 3 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure g.



(j) Digit 1 - case 2. There are $30 + 4l$ tiles in this gadget.

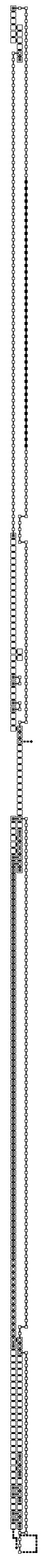


(k) Digit 1 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure j.

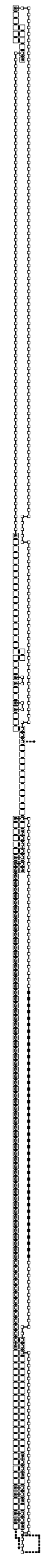


(l) Digit 1 - case 2 (seed) overview. The black tile in this figure is a single tile gadget used only in the initial value.

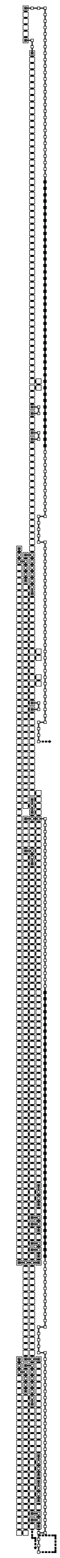
(m) Digit 1 - case 1, Digit 2 case 2. There are $30 + 4l$ tiles in this gadget.



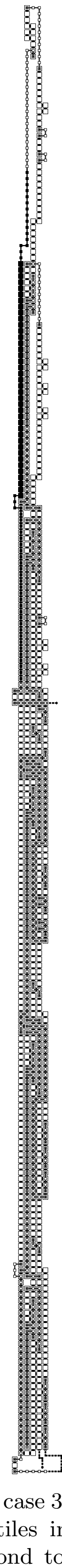
(n) Digit 1 - case 1 overview. The black tiles in this figure correspond to the gadget shown in subfigure m.



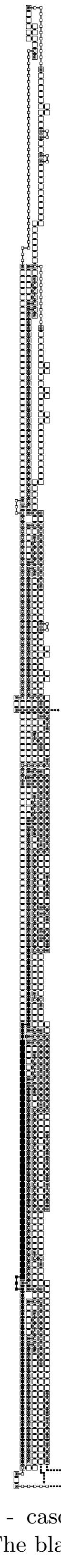
(o) Digit 1 - case 1 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure m.



(p) Digit 2 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure m.



(q) Digit 3 - case 3 overview. The black tiles in this figure correspond to the gadget shown in subfigure g.



(r) Digit 3 - case 3 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure g.

Figure 13: The Return_Path gadgets.

4.1.7 `Next_Read`

Once the `Return_Path` gadget has assembled, the counter has a few options to choose from. The gadget that controls what happens after a digit is written is the `Next_Read` gadget. If there is a `msr` and `msd` signal in the input, this gadget knows that the most significant digit was just read and will output a glue for the `Cross_Next_Row` gadget to assemble so that the counter crosses back to the right and begins reading the first digit in the next row. Otherwise (for regular digits not in the MSR), this gadget will assemble the second half of the return path, terminating at the next digit in the current row. When this happens, the gadget increments the digit index (unless it is already digit 3, in which case it resets to 1) and outputs an empty `Counter_Read` signal to force the counter to begin reading the next digit.

For each $op \in \{\texttt{increment}, \texttt{copy}\}$:

- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 1, op\rangle, \langle \texttt{CounterRead}, 2, \lambda, op\rangle\ )$ from the gadget shown in Figure 14a. In this step, $2 \cdot (41 + 4l) = 82 + 8l = 82 + 8 \cdot (\lceil \log m \rceil + 2) \leq 82 + 8 \cdot (\log m + 3) = 106 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 1, op, \texttt{msr}\rangle, \langle \texttt{CounterRead}, 2, \lambda, op\rangle\ )$ from the gadget shown in Figure 14o. In this step, $2 \cdot (36 + 4l) = 72 + 8l = 72 + 8 \cdot (\lceil \log m \rceil + 2) \leq 72 + 8 \cdot (\log m + 3) = 96 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

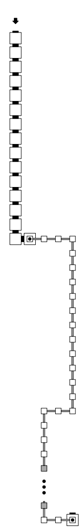
- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 1, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{CrossNextRow}, op\rangle\ )$ from the gadget shown in Figure 14k. In this step, $2 \cdot (37 + 4l) = 74 + 8l = 74 + 8 \cdot (\lceil \log m \rceil + 2) \leq 74 + 8 \cdot (\log m + 3) = 98 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 2, op\rangle, \langle \texttt{CounterRead}, 3, \lambda, op\rangle\ )$ from the gadget shown in Figure 14a. In this step, $2 \cdot (41 + 4l) = 82 + 8l = 82 + 8 \cdot (\lceil \log m \rceil + 2) \leq 82 + 8 \cdot (\log m + 3) = 106 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

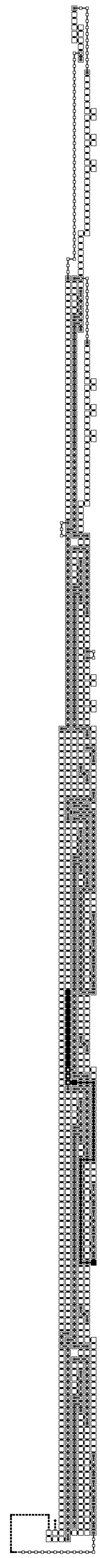
- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 2, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{CrossNextRow}, op\rangle\ )$ from the gadget shown in Figure 14k. In this step $37+4l$ tiles were created. In this step, $2 \cdot (37 + 4l) = 74+8l = 74+8 \cdot (\lceil \log m \rceil + 2) \leq 74+8 \cdot (\log m + 3) = 98+8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 3, op\rangle, \langle \texttt{CounterRead}, 1, \lambda, op\rangle\ )$ from the gadget shown in Figure 14e. In this step $94+12l$ tiles were created. In this step, $2 \cdot (94 + 12l) = 188+24l = 188+24 \cdot (\lceil \log m \rceil + 2) \leq 188+24 \cdot (\log m + 3) = 260+24 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

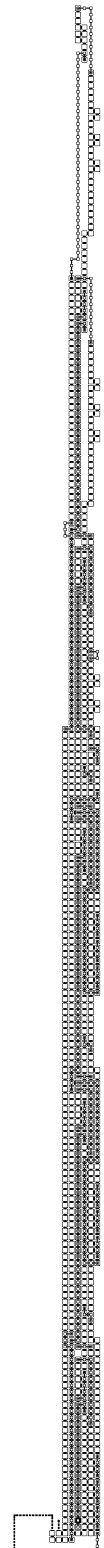
- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 3, op, \texttt{msr}, \texttt{msd}\rangle, \langle \texttt{CrossNextRow}, op\rangle\ )$ from the gadget shown in Figure 14q. In this step, $12 = O(1)$ tiles were created.



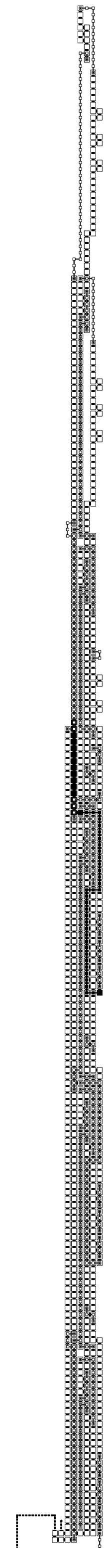
(a) Digit 1 & 2 - general. There are $41+4l$ tiles in this gadget.



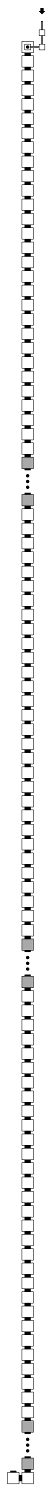
(b) Digit 1 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



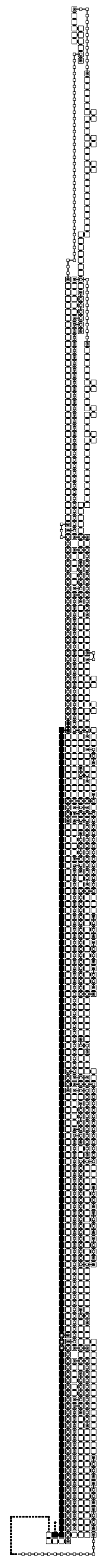
(c) Digit 1 - general (seed) overview. The black tile in this figure is a single tile gadget used only in the initial value.



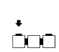
(d) Digit 2 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



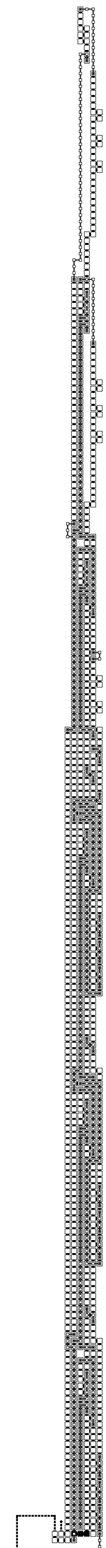
(e) Digit 3 - general. There are $94+12l$ tiles in this gadget.



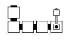
(f) Digit 3 - general overview. The black tiles in this figure correspond to the gadget shown in subfigure e.



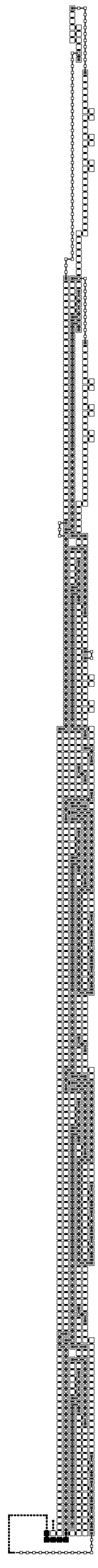
(g) Digit 2 - seed. There are 3 tiles in this gadget.



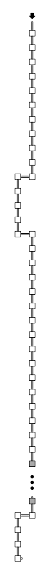
(h) Digit 2 - seed overview. The black tiles in this figure correspond to the gadget shown in subfigure g.



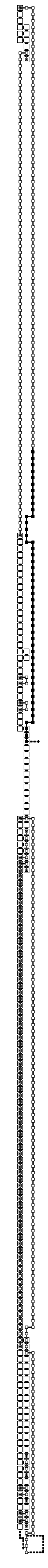
(i) Digit 3 - seed. There are 7 tiles in this gadget.



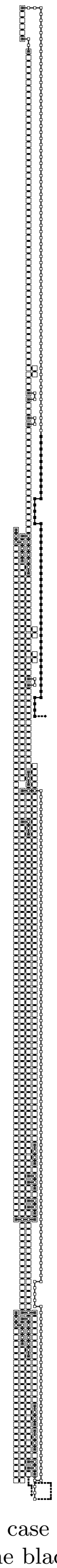
(j) Digit 3 - general (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure i.



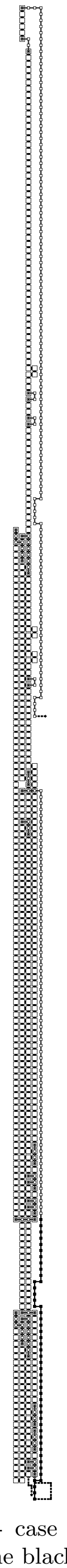
(k) Digit 1 - case 1, Digit 2 - case 2. There are $37 + 4l$ tiles in this gadget.



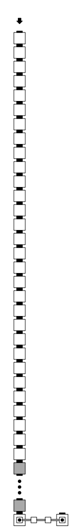
(l) Digit 1 - case 1 overview. The black tiles in this figure correspond to the gadget shown in subfigure k.



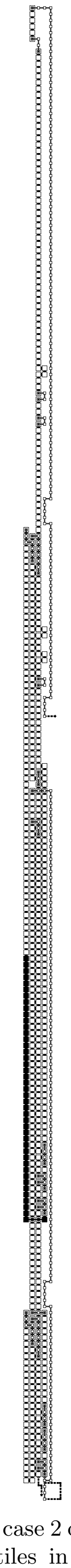
(m) Digit 2 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure k.



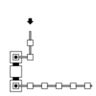
(n) Digit 2 - case 2 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure k.



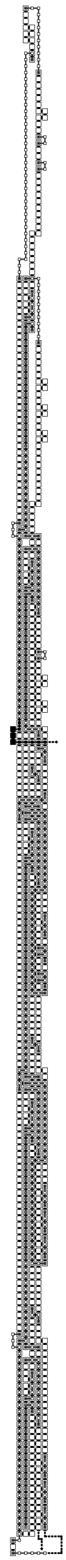
(o) Digit 1 - case 2. There are $36 + 4l$ tiles in this gadget.



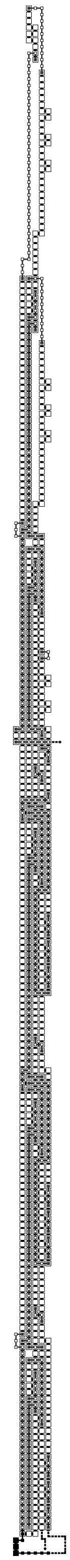
(p) Digit 1 - case 2 overview. The black tiles in this figure correspond to the gadget shown in subfigure o.



(q) Digit 3 - case 3. There are 12 tiles in this gadget.



(r) Digit 3 - case 3 overview. The black tiles in this figure correspond to the gadget shown in subfigure q.



(s) Digit 3 - case 3 (seed) overview. The black tiles in this figure correspond to the gadget shown in subfigure q.

Figure 14: The `Next_Read` gadgets.

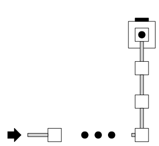
4.1.8 `Cross_Next_Row`

The idea this gadget is to assemble after reading the MSD, routing the counter back to the start of the next row, in position for the counter to begin reading the first digit.

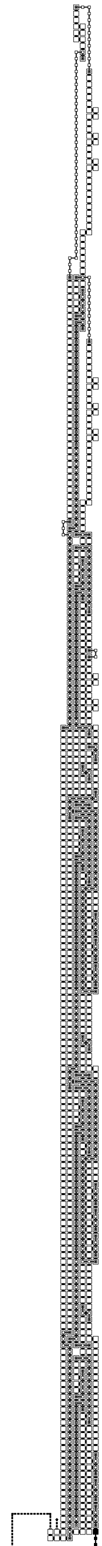
For each $op \in \{\texttt{increment}, \texttt{copy}\}$:

- Create `Cross_Next_Row`($\langle \texttt{CrossNextRow}, op\rangle, \langle \texttt{CounterRead}, 1, \lambda, op\rangle$)
  from the gadget shown in Figure 15a.

In this step, $6g + 4 \leq 6\frac{d}{3} + 4 = 2d + 4 = O(d) = O(k) = O(\log N)$ tiles were created.



(a) General. There are $6g+4$ tiles in this gadget.



(b) General overview. The black tiles in this figure is the gadget shown in subfigure a.

Figure 15: The `Cross_Next_Row` gadget.

4.1.9 `Roof_Unit`

The `Roof_Unit` gadgets consist of two parts, the scaffolding and the shingles. The scaffolding is responsible for assembling a path from the top of the final most significant digit and adding some height last row. Since the most significant digit is positioned at a different height within a digit region depending on how many digits are in the MSR, the `Roof_Scaffolding` gadget's height must take this into consideration. If the MSR has 3 digits, the height is 30 tiles. If the MSR has 2 digits, the height is $60 + 4l$ tiles. Otherwise, if the MSR has 1 digit the height is $90 + 8l$ tiles.

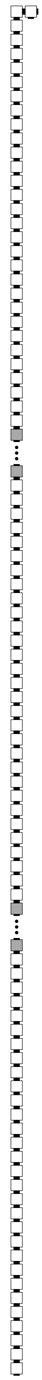
- If case 1: create `Roof_Scaffolding`($\langle$`RoofScaffolding`$, 1,$ `halt`, `msr`, `msd`$\rangle$, $\langle$`RoofShingle`$, 0\rangle$) from the gadget shown in Figure 16a. In this step, $90 + 8l + 1 = 91 + 8l = 91 + 8 \cdot (\lceil \log m \rceil + 2) \leq 91 + 8 \cdot (\log m + 3) = 115 + 8 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

- If case 2: create `Roof_Scaffolding`($\langle$`RoofScaffolding`$, 2,$ `halt`, `msr`, `msd`$\rangle$, $\langle$`RoofShingle`$, 0\rangle$) from the gadget shown in Figure 16b. In this step, $60 + 4l + 3\ 63 + 4l = 63 + 4 \cdot (\lceil \log m \rceil + 2) \leq 63 + 4 \cdot (\log m + 3) = 75 + 4 \cdot \log m = O(\log m) = O\left(\frac{\log N}{k}\right) = O(\log N)$ tiles were created.

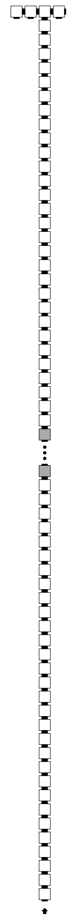
- If case 3: create `Roof_Scaffolding`($\langle$`RoofScaffolding`$, 3,$ `halt`, `msr`, `msd`$\rangle$, $\langle$`RoofShingle`$, 0\rangle$) from the gadget shown in Figure 16c. In this step, $30 + 5 = 35 = O(1)$ tiles were created.

- For each $i = 0, \ldots, g$, create `Roof_Shingle`($\langle$`RoofShingle`$, i\rangle$, $\langle$`RoofShingle`$, i + 1\rangle$) from the gadget shown in Figure 16d. In this step, $6g \leq 6\frac{d}{3} = 2d = O(d) = O(k) = O(\log N)$ tiles were created.

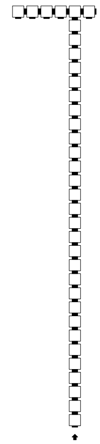
- If k is odd, we add one additional tile to the rightmost shingle, with a west glue labeled $\left\langle \texttt{RoofShingle}, \left\lceil \frac{d}{3} \right\rceil \right\rangle$. In this step $1 = O(1)$ tile was created.



(a) Case 1 - roof unit scaffolding. There are $91 + 8l$ tiles in this gadget.

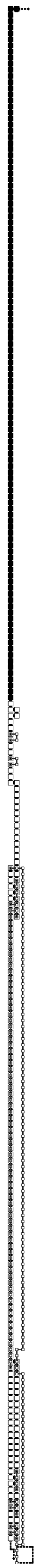


(b) Case 2 - roof unit scaffolding. There are $63 + 4l$ tiles in this gadget.

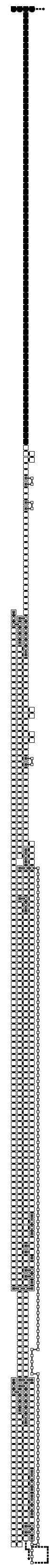


(c) Case 3 - roof unit scaffolding. There are 35 tiles in this gadget.

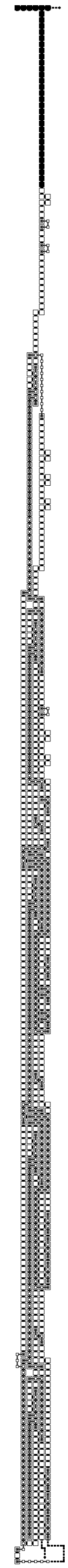
(d) Right shingles. There are 6 tiles in this gadget.



(e) Case 1 - overview. The black tiles in this figure correspond to the gadget shown in subfigure a.



(f) Case 2 - overview. The black tiles in this figure correspond to the gadget shown in subfigure b.

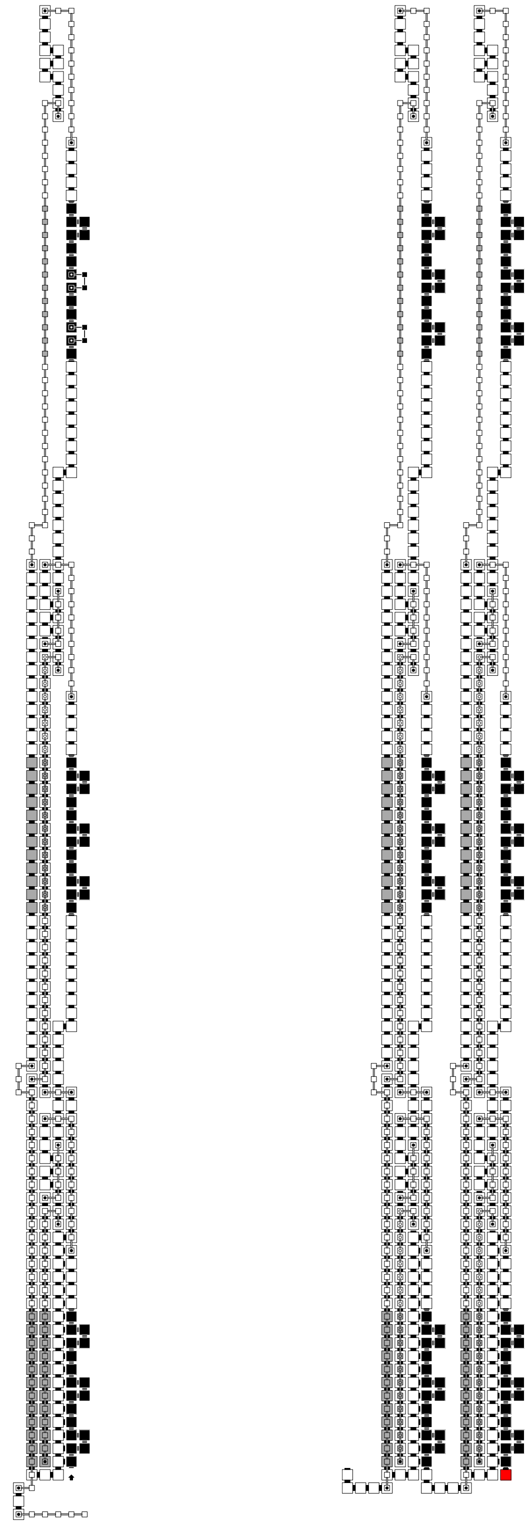


(g) Case 3 - overview. The black tiles in this figure correspond to the gadget shown in subfigure c.

Figure 16: The `Roof_Unit` gadgets.

4.2 Initial value

We begin by encoding s with the Seed unit. It has $\lceil \frac{d}{3} \rceil$ digit regions. Each digit region has three digits, except for the most significant digit region (MSR) which has $d \mod 3$ if $d \mod 3 \neq 0$, otherwise it has 3 digits.



(a) MSR case 1

(b) MSR case 2

(c) MSR case 3

(d) General digit regions

Figure 17: These figures show an example construction of the initial value, with all the possible MSR to the left. Of the three possible MSRs, of course only one would occur in a real assembly.

Note that we use $bin(s[i])[j]$, where i is the index of a digit in s and j is the index of a bit within the encoded digit.

- Create $\texttt{Seed}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, 0, 0\rangle\ )$
  TODO: include special seed that has 4 additional tiles to stop fillers if k is odd?

We repeat these steps starting from $i = 0$ and repeating until i is the index of the first digit in the MSR. These steps build general non-MSR digit regions shown in Figure 17d.

4.2.1 General

For $i = 0, \ldots, 3g$:

- $\texttt{Start}$.
- $\texttt{Digit}$, for each $j = 0, \ldots, l-1$, where $b = bin(s[i])[j]$:
  - if $j = 0$:
    create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1\rangle\ )$
    from the general gadget shown in Figure 10a.
  - if $j = 1$:
    create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1\rangle\ )$
    from the general gadget shown in Figure 10a.
  - if $1 < j < l-1$:
    create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1\rangle\ )$
    from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.
  - if $j = l-1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{DigitTop}, 1, \texttt{seed}, i\rangle\ )$
    from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

  In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m \rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.
- $\texttt{Digit\_Top}$, the following statements create the gadget shown in Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 1, \texttt{seed}, i\rangle, \langle \texttt{DigitTopA}, 1, \texttt{seed}, i\rangle\ )$
    from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 1, \texttt{seed}, i\rangle, \langle \texttt{DigitTopB}, 1, \texttt{seed}, i\rangle\ )$
    from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 1, \texttt{seed}, i\rangle, \langle \texttt{ReturnPath}, 1, \texttt{seed}, i\rangle\ )$
    from the micro-gadget shown in Figure 3b.

  In this step, $40 + 4l = 40 + 4 \cdot (\lceil \log m \rceil + 2) \leq 40 + 4 \cdot (\log m + 3) = 52 + 4 \cdot \log m$ tiles were created.
- Create $\texttt{Return\_Path}(\langle \texttt{ReturnPath}, 1, \texttt{seed}, i\rangle, \langle \texttt{NextRead}, 1, \texttt{seed}, i\rangle\ )$
  (single tile). In this step, 1 tile was created.
- $i \leftarrow i + 1$.
- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 1, \texttt{seed}, i-1\rangle, \langle \texttt{SecondWarp}, 2, \texttt{seed}, i\rangle\ )$
  (single tile). In this step, 1 tile was created.
- Create $\texttt{Second\_Warp}(\langle \texttt{SecondWarp}, 2, \texttt{seed}, i\rangle, \langle \texttt{PostWarp}, 2, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.

- Create $\texttt{Post\_Warp}(\langle \texttt{PostWarp}, 2, \texttt{seed}, i\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, 0\rangle\ )$ from the general gadget show in Figure 9c. In this step, 25 tiles were created.
- $\texttt{Digit}$: for each $j = 0, \ldots, l-1$, where b is $bin(s[i])[j]$:
  - if $j = 0$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a.
  - if $j = 1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a.
  - if $1 < j < l-1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.
  - if $j = l-1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{DigitTop}, 2, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

  In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m \rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.
- $\texttt{Digit\_Top}$: the following statements create the gadget shown in Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 2, \texttt{seed}, i\rangle, \langle \texttt{DigitTopA}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 2, \texttt{seed}, i\rangle, \langle \texttt{DigitTopB}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 2, \texttt{seed}, i\rangle, \langle \texttt{ReturnPath}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3b.

  In this step, $40 + 4l = 40 + 4 \cdot (\lceil \log m \rceil + 2) \leq 40 + 4 \cdot (\log m + 3) = 52 + 4 \cdot \log m$ tiles were created.
- Create $\texttt{Return\_Path}(\langle \texttt{ReturnPath}, 2, \texttt{seed}, i\rangle, \langle \texttt{NextRead}, 2, \texttt{seed}, i\rangle\ )$ from the gadget in Figure 13d. In this step, $32 + 4l$ tiles were created.
- $i \leftarrow i + 1$.
- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 2, \texttt{seed}, i-1\rangle, \langle \texttt{FirstWarp}, 3, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 14g. In this step, 3 tiles were created.
- Create $\texttt{First\_Warp}(\langle \texttt{FirstWarp}, 3, \texttt{seed}, i\rangle, \langle \texttt{WarpBridge}, 3, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.
- Create $\texttt{Warp\_Bridge}(\langle \texttt{WarpBridge}, 3, \texttt{seed}, i\rangle, \langle \texttt{SecondWarp}, 3, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 7a. In this step, 29 tiles were created.
- Create $\texttt{Second\_Warp}(\langle \texttt{SecondWarp}, 3, \texttt{seed}, i\rangle, \langle \texttt{PostWarp}, 3, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.
- Create $\texttt{Post\_Warp}(\langle \texttt{PostWarp}, 3, \texttt{seed}, i\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, 0\rangle\ )$ from the general gadget shown in Figure 9c. In this step, 25 tiles were created.
- $\texttt{Digit}$: for each $j = 0, \ldots, l-1$, where $b = bin(s[i])[j]$:
  - if $j = 0$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a.

– if $j = 1$:
create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1\rangle\ )$
from the general gadget shown in Figure 10a.

– if $1 < j < l - 1$:
create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1\rangle\ )$
from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

– if $j = l - 1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{DigitTop}, 3, \texttt{seed}, i\rangle\ )$
from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m \rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.

- $\texttt{Digit\_Top}$, the following statements create the gadget shown in Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 3, \texttt{seed}, i\rangle, \langle \texttt{DigitTopA}, 3, \texttt{seed}, i\rangle\ )$
    from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 3, \texttt{seed}, i\rangle, \langle \texttt{DigitTopB}, 3, \texttt{seed}, i\rangle\ )$
    from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 3, \texttt{seed}, i\rangle, \langle \texttt{ReturnPath}, 3, \texttt{seed}, i\rangle\ )$
    from the micro-gadget shown in Figure 3b.

  In this step, $40 + 4l = 40 + 4 \cdot (\lceil \log m \rceil + 2) \leq 40 + 4 \cdot (\log m + 3) = 52 + 4 \cdot \log m$ tiles were created.

- Create $\texttt{Return\_Path}(\langle \texttt{ReturnPath}, 3, \texttt{seed}, i\rangle, \langle \texttt{NextRead}, 3, \texttt{seed}, i, \rangle\ )$
  from the gadget in Figure 13g. In this step, $65 + 8l$ tiles were created.

- $i \leftarrow i + 1$.

- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 3, \texttt{seed}, i-1\rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, 0\rangle\ )$
  from the general gadget shown in Figure 14i. In this step, 7 tiles were created.

- if i is not an index in the MSR, go to `start`, else break.

In this step, $\sum_{i=0}^{3g} 311 + 48l = (3g + 1) + (48l + 311) = O(dl)$ tiles were created. This means that

$$
\begin{aligned}
dl &= O(d \log m) \\
&= O\left(d \log \left\lceil \left(\frac{N}{102} \right)^{\frac{1}{d}} \right\rceil \right) \\
&= O\left(d \log \left(2 \left(\frac{N}{102} \right)^{\frac{1}{d}} \right) \right) \\
&= O\left(d \log \left(\frac{N}{102} \right)^{\frac{1}{d}} + d \log 2 \right) \\
&= O\left(\log N + \left\lfloor \frac{k}{2} \right\rfloor \right) \\
&= O(\log N)
\end{aligned}
$$

tiles were created in this step.

4.2.2 Case 1

If $d-i=1$, create the assembly shown in 17a.

- `Digit`, for each $j=0,\ldots,l-1$, where $b=bin(s[i])[j]$:
  - if $j=0$:
    create $\texttt{Counter\_Write}(\langle\texttt{CounterWrite},1,\texttt{seed},i,j\rangle,\langle\texttt{CounterWrite},1,\texttt{seed},i,j+1\rangle\ )$
    from the general gadget shown in Figure 10b.
  - if $j=1$:
    create $\texttt{Counter\_Write}(\langle\texttt{CounterWrite},1,\texttt{seed},i,j\rangle,\langle\texttt{CounterWrite},1,\texttt{seed},i,j+1\rangle\ )$
    from the general gadget shown in Figure 10b.
  - if $1<j<l-1$:
    create $\texttt{Counter\_Write}(\langle\texttt{CounterWrite},1,\texttt{seed},i,j\rangle,\langle\texttt{CounterWrite},1,\texttt{seed},i,j+1\rangle\ )$
    from the general gadget shown in Figure 10a if $b=0$ or Figure 10b if $b=1$.
  - if $j=l-1$: create $\texttt{Counter\_Write}(\langle\texttt{CounterWrite},1,\texttt{seed},i,j\rangle,\langle\texttt{DigitTop},1,\texttt{seed},i\rangle\ )$
    from the general gadget shown in Figure 10a if $b=0$ or Figure 10b if $b=1$.

  In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1}8=8l=8\cdot(\lceil\log m\rceil+2)\leq 8\cdot(\log m+3)=8\cdot\log m+24$ tiles were created.

- `Digit_Top`, the following statements create the gadget shown in Figure 12h.
  - Create $\texttt{North\_Line4}l(\langle\texttt{DigitTop},1,\texttt{seed},i\rangle,\langle\texttt{DigitTopA},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 3a.
  - Create $\texttt{North\_Line4}(\langle\texttt{DigitTopA},1,\texttt{seed},i\rangle,\langle\texttt{DigitTopB},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle\texttt{DigitTopB},1,\texttt{seed},i\rangle,\langle\texttt{DigitTopC},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle\texttt{DigitTopC},1,\texttt{seed},i\rangle,\langle\texttt{DigitTopD},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line30}(\langle\texttt{DigitTopD},1,\texttt{seed},i\rangle,\langle\texttt{DigitTopE},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line4}l(\langle\texttt{DigitTopE},1,\texttt{seed},i\rangle,\langle\texttt{DigitTopF},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line14}(\langle\texttt{DigitTopF},1,\texttt{seed},i\rangle,\langle\texttt{DigitTopG},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line17}(\langle\texttt{DigitTopG},1,\texttt{seed},i\rangle,\langle\texttt{ReturnPath},1,\texttt{seed},i\rangle\ )$
    from the micro-gadget shown in Figure 3b.

  In this step, $100+12l=100+12\cdot(\lceil\log m\rceil+2)\leq 100+12\cdot(\log m+3)=136+36\cdot\log m$ tiles were created.

- Create $\texttt{Return\_Path}(\langle\texttt{ReturnPath},1,\texttt{seed},i\rangle,\langle\texttt{NextRead},1,\texttt{seed},i\rangle\ )$
  from the general gadget shown in Figure 13m. In this step, $30+4l$ tiles were created.

- Create $\texttt{Next\_Read}(\langle\texttt{NextRead},1,\texttt{seed},i\rangle,\langle\texttt{Cross\_Next\_Row},\texttt{increment}\rangle\ )$
  from the general gadget shown in Figure 14k. In this step, $37+4l$ tiles were created.

This means that, $8l+(100+12l)+(30+4l)+(37+4l)=167+28l=167+28\cdot(\lceil\log m\rceil+2)\leq 167+28\cdot(\log m+3)=O(\log m)$ tiles were created in this step.

4.2.3 Case 2

If $d - i = 2$, create the assembly shown in 17b.

- `Digit`, for each $j = 0, \ldots, l-1$, where $b = bin(s[i])[j]$:
  - if $j = 0$:
    create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, j+1\rangle$)
    from the general gadget shown in Figure 10b.
  - if $j = 1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, j+1\rangle$)
    from the general gadget shown in Figure 10a.
  - if $1 < j < l-1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, j+1\rangle$)
    from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.
  - if $j = l-1$: create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`DigitTop`$, 2,$ `seed`$, i\rangle$)
    from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

  In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m \rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.

- `Digit_Top`, the following statements create the gadget shown in Figure 12k.
  - Create `Topper`($\langle$`DigitTop`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopA`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 11b
  - Create `South_Line4`l($\langle$`DigitTopA`$, 1, op,$ `msr`$\rangle$, $\langle$`ReturnPath`$, 1, op,$ `msr`$\rangle$)
    from the micro-gadget shown in Figure 3b

  In this step, $43 + 4l = 43 + 4 \cdot (\lceil \log m \rceil + 2) \leq 43 + 4 \cdot (\log m + 3) = 55 + 4 \cdot \log m$ tiles were created.

- Create `Return_Path`($\langle$`ReturnPath`$, 1,$ `seed`$, i\rangle$, $\langle$`NextRead`$, 1,$ `seed`$, i\rangle$)
  (single tile). In this step, 1 tile was created.

- $i \leftarrow i + 1$

- Create `Next_Read`($\langle$`NextRead`$, 1,$ `seed`$, i-1\rangle$, $\langle$`SecondWarp`$, 2,$ `seed`$, i\rangle$)
  (single tile). In this step, 1 tile was created.

- Create `Second_Warp`($\langle$`SecondWarp`$, 2,$ `seed`$, i\rangle$, $\langle$`PostWarp`$, 2,$ `seed`$, i\rangle$)
  (single tile). In this step, 1 tile was created.

- Create `Post_Warp`($\langle$`PostWarp`$, 2,$ `seed`$, i\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, 0\rangle$)
  from the general gadget show in Figure 9l. In this step, 22 tiles were created.

- `Digit`, for each $j = 0, \ldots, l-1$, where $b = bin(s[i])[j]$:
  - if $j = 0$:
    create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, j+1\rangle$)
    from the general gadget shown in Figure 10b.
  - if $j = 1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, j+1\rangle$)
    from the general gadget shown in Figure 10b.
  - if $1 < j < l-1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, j+1\rangle$)
    from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

- if $j = l - 1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{DigitTop}, 2, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m\rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.

- $\texttt{Digit\_Top}$, the following statements create the gadget shown in Figure 12n.
  - Create $\texttt{North\_Line4}l(\langle \texttt{DigitTop}, 2, \texttt{seed}, i\rangle, \langle \texttt{DigitTopA}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 2, \texttt{seed}, i\rangle, \langle \texttt{DigitTopB}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 11c.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 2, \texttt{seed}, i\rangle, \langle \texttt{DigitTopC}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line30}(\langle \texttt{DigitTopC}, 2, \texttt{seed}, i\rangle, \langle \texttt{ReturnPath}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3b.

  In this step, $58 + 8l = 58 + 8 \cdot (\lceil \log m\rceil + 2) \leq 58 + 8 \cdot (\log m + 3) = 82 + 8 \cdot \log m$ tiles were created.

- Create $\texttt{Return\_Path}(\langle \texttt{ReturnPath}, 2, \texttt{seed}, i\rangle, \langle \texttt{NextRead}, 2, \texttt{seed}, i\rangle\ )$ from the gadget shown in Figure 13m. In this step, $30 + 4l$ tiles were created.
- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 2, \texttt{seed}\rangle, \langle \texttt{Cross\_Next\_Row}, \texttt{increment}\rangle\ )$ from the gadget shown in Figure 14k. In this step, $37 + 4l$ tiles were created.

This means that, $8l + (43 + 4l) + 1 + 1 + 1 + 22 + 8l + (58 + 8l) + (30 + 41) + (37 + 4l) = 193 + 36l = 193 + 36 \cdot (\lceil \log m\rceil + 2) \leq 193 + 36 \cdot (\log m + 3) = O(\log m)$ tiles were created in this step.

4.2.4 Case 3

If $d - i = 3$, create the assembly shown in 17c.

- $\texttt{Digit}$, for each $j = 0, \ldots, l - 1$, where $b = bin(s[i])[j]$:
  - if $j = 0$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j + 1\rangle\ )$ from the general gadget shown in Figure 10a.
  - if $j = 1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j + 1\rangle\ )$ from the general gadget shown in Figure 10a.
  - if $1 < j < l - 1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j + 1\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.
  - if $j = l - 1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle, \langle \texttt{DigitTop}, 1, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

  In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m\rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.

- $\texttt{Digit\_Top}$, the following statements create the gadget shown in Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 1, \texttt{seed}, i\rangle, \langle \texttt{DigitTopA}, 1, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3a.

- Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 1, \texttt{seed}, i\rangle, \langle \texttt{DigitTopB}, 1, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 11a.
- Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 1, \texttt{seed}, i\rangle, \langle \texttt{ReturnPath}, 1, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3b.

In this step, $40 + 4l = 40 + 4 \cdot (\lceil \log m \rceil + 2) \leq 40 + 4 \cdot (\log m + 3) = 52 + 4 \cdot \log m$ tiles were created.

- Create $\texttt{Return\_Path}(\langle \texttt{ReturnPath}, 1, \texttt{seed}, i-1\rangle, \langle \texttt{NextRead}, 1, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.
- $i \leftarrow i + 1$.
- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 1, \texttt{seed}, i-1\rangle, \langle \texttt{SecondWarp}, 2, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.
- Create $\texttt{Second\_Warp}(\langle \texttt{SecondWarp}, 2, \texttt{seed}, i\rangle, \langle \texttt{PostWarp}, 2, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.
- Create $\texttt{Post\_Warp}(\langle \texttt{PostWarp}, 2, \texttt{seed}, i\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, 0\rangle\ )$ from the general gadget show in Figure 9c. In this step, 25 tiles were created.
- $\texttt{Digit}$, for each $j = 0, \ldots, l-1$, where $b = bin(s[i])[j]$:
  - if $j = 0$ create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a.
  - if $j = 1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a.
  - if $1 < j < l-1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.
  - if $j = l-1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle, \langle \texttt{DigitTop}, 2, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m \rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.

- $\texttt{Digit\_Top}$, the following statements create the gadget shown in Figure 12a.
  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 2, \texttt{seed}, i\rangle, \langle \texttt{DigitTopA}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 2, \texttt{seed}, i\rangle, \langle \texttt{DigitTopB}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 2, \texttt{seed}, i\rangle, \langle \texttt{ReturnPath}, 2, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3b.

In this step, $40 + 4l = 40 + 4 \cdot (\lceil \log m \rceil + 2) \leq 40 + 4 \cdot (\log m + 3) = 52 + 4 \cdot \log m$ tiles were created.

- Create $\texttt{Return\_Path}(\langle \texttt{ReturnPath}, 2, \texttt{seed}, i\rangle, \langle \texttt{NextRead}, 2, \texttt{seed}, i\rangle\ )$ from the gadget in Figure 13d. In this step, $32 + 4l$ tiles were created.
- $i \leftarrow i + 1$

- Create $\texttt{Next\_Read}(\langle \texttt{NextRead}, 2, \texttt{seed}, i-1\rangle, \langle \texttt{FirstWarp}, 3, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 14g. In this step, 3 tiles were created.

- Create $\texttt{First\_Warp}(\langle \texttt{FirstWarp}, 3, \texttt{seed}, i\rangle, \langle \texttt{WarpBridge}, 3, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.

- Create $\texttt{Warp\_Bridge}(\langle \texttt{WarpBridge}, 3, \texttt{seed}, i\rangle, \langle \texttt{SecondWarp}, 3, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 7a. In this step, 29 tiles were created.

- Create $\texttt{Second\_Warp}(\langle \texttt{SecondWarp}, 3, \texttt{seed}, i\rangle, \langle \texttt{PostWarp}, 3, \texttt{seed}, i\rangle\ )$ (single tile). In this step, 1 tile was created.

- Create $\texttt{Post\_Warp}(\langle \texttt{PostWarp}, 3, \texttt{seed}, i\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, 0\rangle\ )$ from the general gadget shown in Figure 9c. In this step, 25 tiles were created.

- $\texttt{Digit}$, for each $j = 0, \ldots, l-1$, where $b = bin(s[i])[j]$:

  - if $j = 0$:
    create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10b.
  - if $j = 1$:
    create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10b.
  - if $1 < j < l-1$:
    create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.
  - if $j = l-1$: create $\texttt{Counter\_Write}(\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j\rangle, \langle \texttt{DigitTop}, 3, \texttt{seed}, i\rangle\ )$ from the general gadget shown in Figure 10a if $b = 0$ or Figure 10b if $b = 1$.

  In this step, assuming the maximum of 8 tiles are used for each bit b, then $\sum_{j=0}^{l-1} 8 = 8l = 8 \cdot (\lceil \log m \rceil + 2) \leq 8 \cdot (\log m + 3) = 8 \cdot \log m + 24$ tiles were created.

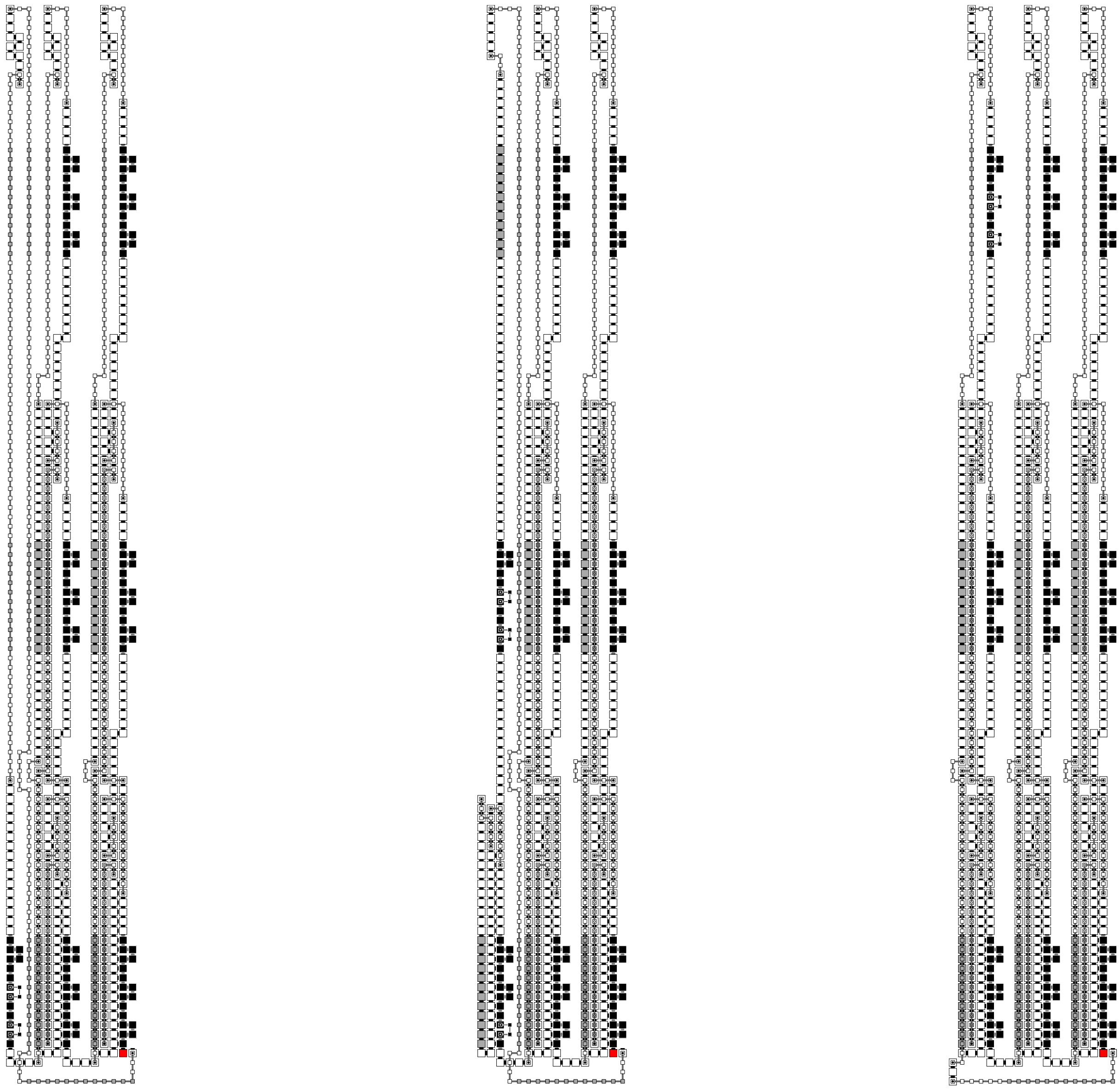
- $\texttt{Digit\_Top}$, the following statements create the gadget shown in Figure 12a.

  - Create $\texttt{North\_Line5}(\langle \texttt{DigitTop}, 3, \texttt{seed}, i\rangle, \langle \texttt{DigitTopA}, 3, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\langle \texttt{DigitTopA}, 3, \texttt{seed}, i\rangle, \langle \texttt{DigitTopB}, 3, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 11a.
  - Create $\texttt{South\_Line4}l(\langle \texttt{DigitTopB}, 3, \texttt{seed}, i\rangle, \langle \texttt{ReturnPath}, 3, \texttt{seed}, i\rangle\ )$ from the micro-gadget shown in Figure 3b.

  In this step, $40 + 4l = 40 + 4 \cdot (\lceil \log m \rceil + 2) \leq 40 + 4 \cdot (\log m + 3) = 52 + 4 \cdot \log m$ tiles were created.

- Create $\texttt{Return\_Path}(\langle \texttt{ReturnPath}, 3, \texttt{seed}, i\rangle, \langle \texttt{NextRead}, 3, \texttt{increment}, \texttt{msr}, \texttt{msd}\rangle\ )$ from the gadget in Figure 13g. In this step, $65 + 8l$ tiles were created.

This means that $8l+(40+4l)+1+1+1+25+8l+(40+4l)+(32+4l)+3+1+29+1+25+8l+(40+4l)+(65+8l) = 304 + 48l = 304 + 48 \cdot (\lceil \log m \rceil + 2) \leq 304 + 48 \cdot (\log m + 3) = O(\log m)$ tiles were created in this step.



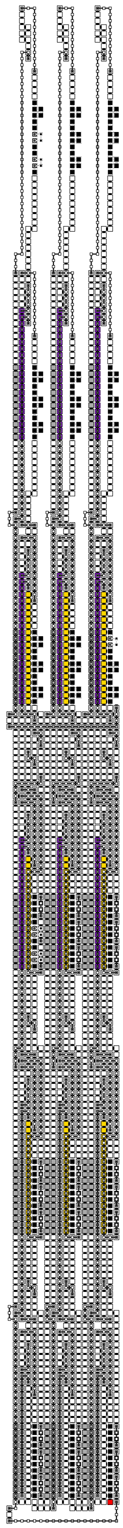
(a) Initial value case 1

(b) Initial value case 2

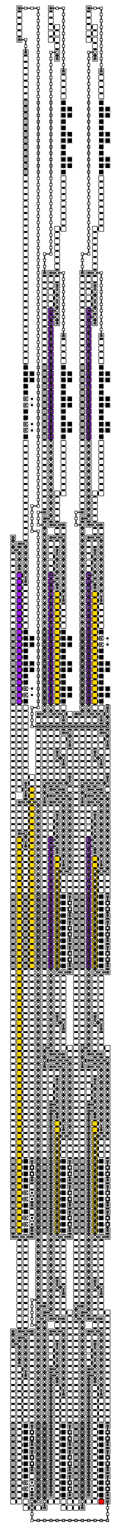
(c) Initial value case 3

Figure 18: These figures show a full example of an initial value, with both general regions and the MSRs together, instead of separated as shown above.

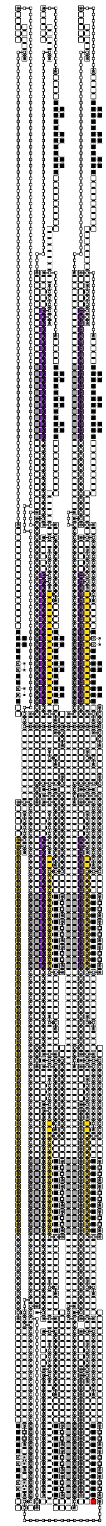
4.3 Overviews



(a) Full overview case 3



(b) Full overview case 2



(c) Full overview case 1