1 Definitions

$$
\begin{aligned}
\text{Let } m &= \left\lceil \left(\frac{N}{93}\right)^{\frac{1}{d}} \right\rceil, \text{ base of the counter} \\
MSR &= \text{ most signifcant digit region} \\
\mathcal{C}_0 &= \text{starting value of counter} \\
d &= \lceil \log_m(\mathcal{C}_0) \rceil, \text{ number of digits per row} \\
\mathcal{DR} &= \left\lceil \frac{d}{3} \right\rceil, \text{ digit regions per row} \\
\mathcal{C}_f &= m^d - 1, \text{ final value of the counter} \\
\mathcal{C}_\Delta &= \mathcal{C}_f - \mathcal{C}_0, \text{ number of rows} \\
l &= \lceil \log m \rceil + 2, \text{ bits needed to represent each digit plus 2 for MSR and MSD} \\
\mathcal{DR}_{height} &= 3 \cdot (l + 30), \text{ height of a row} \\
h &= c_\Delta \cdot \mathcal{DR}_{height}, \text{ height of constuction without roof tiles}
\end{aligned}
$$

In order to determine the number of digits per row, given some value k, let $R = k \mod 6$. There are 3 distinct cases to consider.

- if $0 \leq R \leq 1$, then $\{\ d \in \mathbb{N} \mid\ 3 \text{ divides } d\ \}$
  MSR $\leftarrow$ 3 digits
- if $2 \leq R \leq 3$, then $\{\ d \in \mathbb{N} \mid\ 3 \text{ divides } d-1\ \}$
  MSR $\leftarrow$ 1 digit
- if $4 \leq R \leq 5$, then $\{\ d \in \mathbb{N} \mid\ 3 \text{ divides } d-2\ \}$
  MSR $\leftarrow$ 2 digits

2 Determining the starting value $\mathcal{C}_0$

...therefore, let $d = \left\lfloor \frac{k}{2} \right\rfloor$, $m = \left\lceil \left(\frac{N}{93}\right)^{\frac{1}{d}} \right\rceil$, $l = \lceil \log m \rceil + 2$, $\mathcal{C}_0 = m^d - \left\lfloor \frac{N-3l-76}{3l+90} \right\rfloor$, where d is the number of digits per row of the counter, m is the base of the counter, l is the number of bits needed to encode each digit in binary plus 2 for indicating whether a digit is in the MSR and is the MSD in that region, and $\mathcal{C}_0$ is the start of the counter in decimal.

In general, the height of a digit region is $3l + 90$. There are two cases when this is false, in the first digit region the height is $3l + 91$, and the last digit region has a height of $3l + 75$. To ensure the rectangle ends with height N, let h be the height of the construction before any filler/roof tiles are added. If we define $\mathcal{C}_\Delta$ as the number of `Counter` unit rows, then $h = n(3l + 90) + (3l + 91) + (3l + 75)$, which we will re-write as $n(3l + 90) + 3l + 76$. So then the maximum height of the counter is $m^d(3l + 90) + 3l + 76$. Since our goal is to end with with a rectangle of height N, we need to pick a base such that the counter can increment so many times that when it stops, it is at least N.

Lemma 1. *$N \leq m^d(3l + 90) + 3l + 76$.*

Proof.

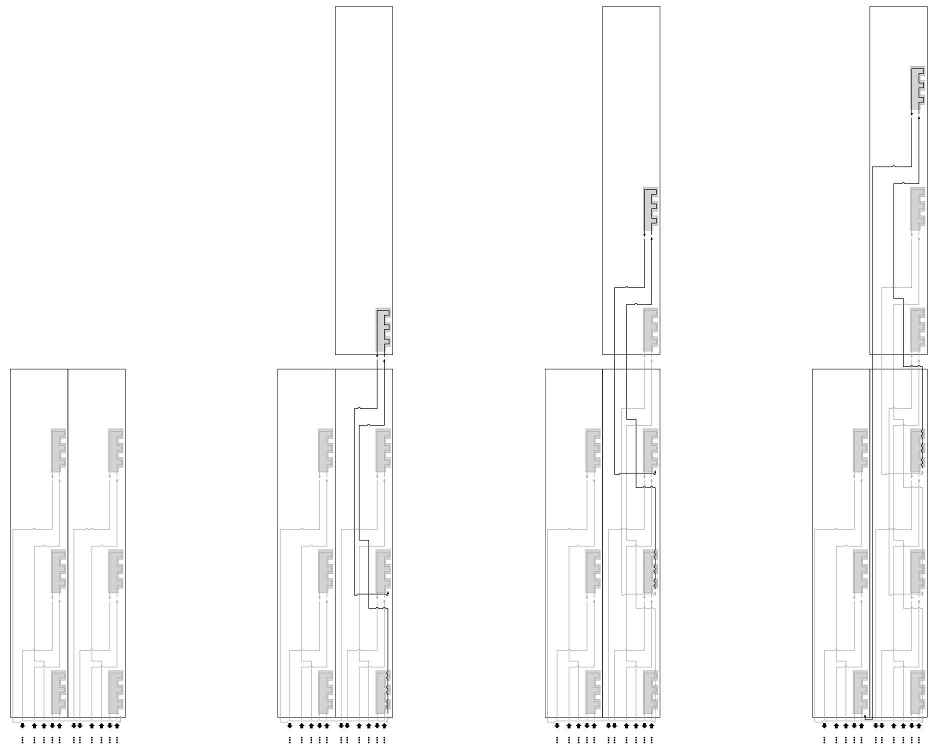
$$
\begin{aligned}
N = 93\left(\frac{N}{93}\right) = 93\left(\left(\frac{N}{93}\right)^{\frac{1}{d}}\right)^{d} \leq 93\left\lceil\left(\frac{N}{93}\right)^{\frac{1}{d}}\right\rceil^{d} \\
= 93m^d \leq 3lm^d + 90m^d \leq 3lm^d + 90m^d + 3l + 76 \\
= m^d(3l + 90) + 3l + 76
\end{aligned}
$$

□

2.1 Filling in the gaps

This means that the number of `Counter` unit rows $\mathcal{C}_\Delta$ is $m^d - \mathcal{C}_0$, where we have defined $\mathcal{C}_0$ as the starting value of the counter. To choose the best starting value, we find the value for $\mathcal{C}_\Delta$ that gets h as close to N without exceeding N. It follows from the equation $h = \mathcal{C}_\Delta(3l + 90) + 3l + 76$, that $\mathcal{C}_\Delta = \left\lfloor \frac{N-3l-76}{3l+90} \right\rfloor$. Thus, $\mathcal{C}_0 = m^d - \left\lfloor \frac{N-3l-76}{3l+90} \right\rfloor$. As a result of each digit requiring a width of 2 tiles, if k is odd, one additional tile column must be added. The number of filler tiles needed for the width is $k \mod 2$, and the number of filler tiles for the height is $N - h$, which is $N - 3l - 76 \mod 3l + 90$.

3 General counter



(a) A "clean" counter row, before any reading has started.

(b) Reading digit 1, writing digit 1 in the next row, and returning to read digit 2 of the current row.

(c) Reading digit 2, writing digit 2 in the next row, and returning to read digit 3 of the current row.

(d) Reading digit 3, writing digit 3 in the next row, and returning to read digit 4 of the current row.

Figure 1: Progression of the counter as it reads the 3 least significant digits in one value, and writes the corresponding digits in the next row/value.