1 Definitions

$$
\begin{aligned}
\text{Let } m &= \left\lceil \left(\frac{N}{93}\right)^{\frac{1}{d}} \right\rceil, \text{ base of the counter} \\
MSR &= \text{ most signifcant digit region} \\
\mathcal{C}_0 &= \text{starting value of counter} \\
d &= \lceil \log_m \mathcal{C}_0 \rceil = \left\lfloor \frac{k}{2} \right\rfloor, \text{ number of digits per row} \\
\mathcal{C}_f &= m^d, \text{ final value of the counter} \\
\mathcal{C}_\Delta &= \mathcal{C}_f - \mathcal{C}_0, \text{ number of rows/ times to count} \\
l &= \lceil \log m \rceil + 2, \text{ bits needed to encode each digit in binary, plus 2 for MSR and MSD}
\end{aligned}
$$

In order to determine the number of digits per row, given some value k, let $R = k \mod 6$. There are 3 distinct cases to consider.

- if $0 \le R \le 1$, then $\{\ d \in \mathbb{N} \mid \text{ 3 divides } d\ \}$
  MSR $\leftarrow$ 3 digits
- if $2 \le R \le 3$, then $\{\ d \in \mathbb{N} \mid \text{ 3 divides } d-1\ \}$
  MSR $\leftarrow$ 1 digit
- if $4 \le R \le 5$, then $\{\ d \in \mathbb{N} \mid \text{ 3 divides } d-2\ \}$
  MSR $\leftarrow$ 2 digits

2 Determining the starting value $\mathcal{C}_0$

...therefore, let $d = \lfloor \frac{k}{2} \rfloor$, $m = \left\lceil \left(\frac{N}{93}\right)^{\frac{1}{d}} \right\rceil$, $l = \lceil \log m \rceil + 2$, $\mathcal{C}_0 = m^d - \left\lfloor \frac{N-3l-76}{3l+90} \right\rfloor$, where d is the number of digits per row of the counter, m is the base of the counter, l is the number of bits needed to encode each digit in binary plus 2 for indicating whether a digit is in the MSR and is the MSD in that region, and $\mathcal{C}_0$ is the start of the counter in decimal.

In general, the height of a digit region is $3l+90$. There are two cases when the height is different, namely in the first and last digit regions, where the height is $3l+91$ and $3l+75$, respectively. Let h be the height of the construction before any filler/roof tiles are added. If we define $\mathcal{C}_\Delta$ as the number of `Counter` unit rows, then $h = \mathcal{C}_\Delta(3l+90) + (3l+91) + (3l+75)$, simplifying to $\mathcal{C}_\Delta(3l+90) + 3l + 76$. So then the maximum height of the counter is $m^d(3l+90) + 3l + 76$. Since our goal is to end with a rectangle of height N, we need to pick a base such that the counter can increment so many times that when it stops, it is at least N.

Lemma 1. *$N \le m^d(3l+90) + 3l + 76$.*

Proof.

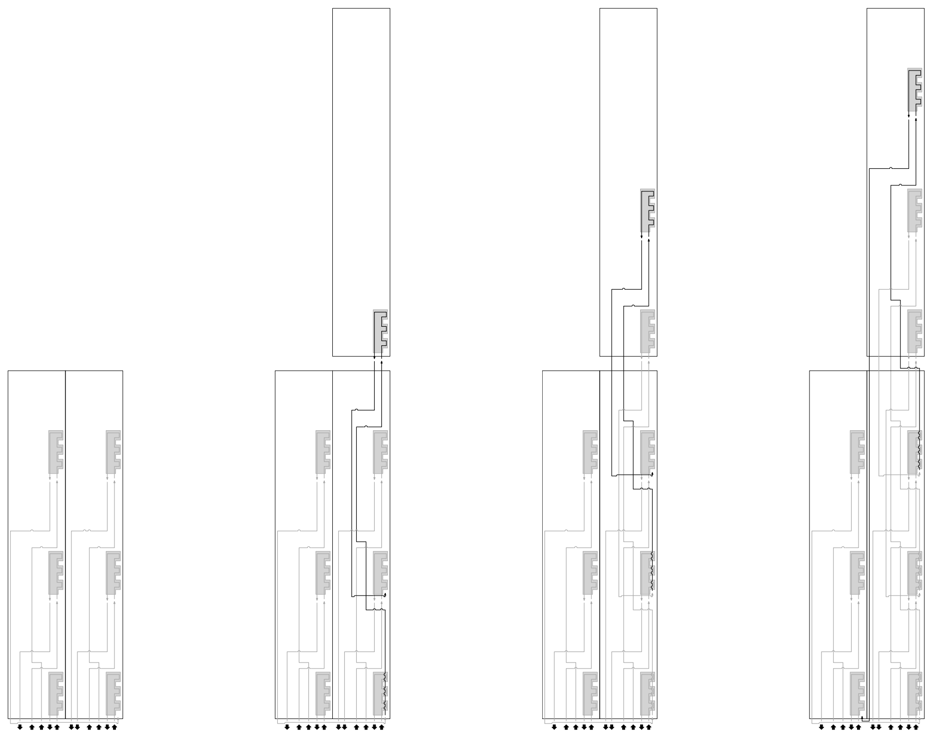
$$
\begin{aligned}
N = 93\left(\frac{N}{93}\right) &= 93\left(\left(\frac{N}{93}\right)^{\frac{1}{d}}\right)^d \le 93\left\lceil \left(\frac{N}{93}\right)^{\frac{1}{d}} \right\rceil^d \\
&= 93m^d \le 3lm^d + 90m^d \le 3lm^d + 90m^d + 3l + 76 \\
&= m^d(3l+90) + 3l + 76
\end{aligned}
$$

□

2.1 Filling in the gaps

...this means that the number of `Counter` unit rows $\mathcal{C}_\Delta$ is $m^d - \mathcal{C}_0$, where we have defined $\mathcal{C}_0$ as the starting value of the counter. To choose the best starting value, we find the value for $\mathcal{C}_\Delta$ that gets h as close to N without exceeding N. It follows from the equation $h = \mathcal{C}_\Delta(3l + 90) + 3l + 76$, that $\mathcal{C}_\Delta = \left\lfloor \frac{N-3l-76}{3l+90} \right\rfloor$. Thus, $\mathcal{C}_0 = m^d - \left\lfloor \frac{N-3l-76}{3l+90} \right\rfloor$. As a result of each digit requiring a width of 2 tiles, if k is odd, one additional tile column must be added. The number of filler tiles needed for the width is $k \mod 2$, and the number of filler tiles for the height is $N - 3l - 76 \mod 3l + 90$.

3 General counter



(a) A "clean" counter row, before any reading has started.

(b) Digit 1 in row i is read, and row $i + 1$ is started. After this digit is written, the counter returns to row i, and is read to read digit 2.

(c) Reading digit 2, writing digit 2 in the next row, and returning to read digit 3 of the current row.

(d) Reading digit 3, writing digit 3 in the next row, and returning to read digit 4 of the current row.

Figure 1: These illustrate how the counter reads and writes a digit region, in a general sense. The counter starts in the rightmost digit region, and reads the bottomost digit within that region. After reading digit 1, the corresponding digit region will be started in row $i + 1$. The counter writes the first digit, and then returns to digit 2 in row i. Once all the digits in the current digit region are read and written into row $i+1$, the counter can then move to the next digit region in row i, begin reading row $i + 1$, or halt, depending on what signals are encoded in the geometry of the current row.