1 Definitions

1.1 Misc

$$\text{Let } m = \left\lceil \left(\frac{N}{102}\right)^{\frac{1}{d}} \right\rceil, \text{ base of the counter}$$

$$\begin{aligned}
MSR &= \text{ most signifcant digit region}\\
s &= \text{starting value of counter}\\
d &= \lceil \log_m s \rceil = \left\lfloor \frac{k}{2} \right\rfloor, \text{ number of digits per row}\\
\mathcal{C}_f &= m^d, \text{ final value of the counter}\\
\mathcal{C}_\Delta &= \mathcal{C}_f - s, \text{ number of rows/ times to count}\\
l &= \lceil \log m \rceil + 2, \text{ bits needed to encode each digit in binary, plus 2 for MSR and MSD}
\end{aligned}$$

1.2 Determining the starting value

...therefore, let $d = \lfloor \frac{k}{2} \rfloor$, $m = \left\lceil \left(\frac{N}{102}\right)^{\frac{1}{d}} \right\rceil$, $l = \lceil \log m \rceil + 2$, $s = m^d - \left\lfloor \frac{N-12l-94}{12l+90} \right\rfloor$, where d is the number of digits per row of the counter, m is the base of the counter, l is the number of bits needed to encode each digit in binary plus 2 for indicating whether a digit is in the MSR and is the MSD in that region, and s is the start of the counter in decimal.

In general, the height of a digit region is $12l + 90$. There is one exception, being the seed row, which has a height of $12l + 94$. Let h be the height of the construction before any filler/roof tiles are added. If we define n as the number of `Counter` unit rows, then $h = n(12l + 90) + (12l + 94)$. So then the maximum height of the counter is $m^d(12l + 90) + 12l + 94$. Since our goal is to end with a rectangle of height N, we need to pick a base such that the counter can increment so many times that when it stops, it is at least N.

Lemma 1. *$N \leq m^d(12l + 90) + 12l + 94$.*

Proof.

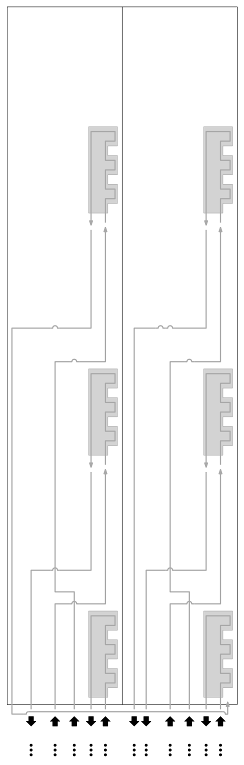
$$\begin{aligned}
N = 102\left(\frac{N}{102}\right) &= 102\left(\left(\frac{N}{102}\right)^{\frac{1}{d}}\right)^d \leq 102\left\lceil\left(\frac{N}{102}\right)^{\frac{1}{d}}\right\rceil^d\\
&= 102m^d \leq 12lm^d + 90m^d \leq 12lm^d + 90m^d + 12l + 94\\
&= m^d(12l + 90) - 12l + 94
\end{aligned}$$

□

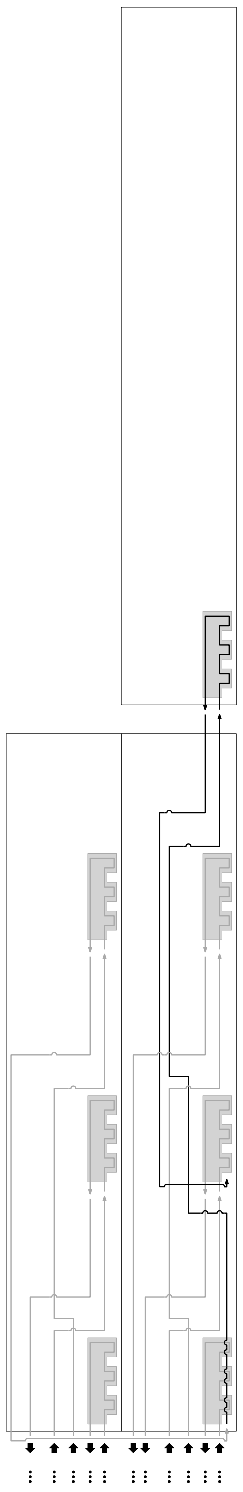
1.3 Filling in the gaps

...this means that the number of `Counter` unit rows n is $m^d - s$, where we have defined s as the starting value of the counter. To choose the best starting value, we find the value for n that gets h as close to N without exceeding N. It follows from the equation $h = n(12l + 90) + 12l + 94$, that $n = \left\lfloor \frac{N-12l-94}{12l+90} \right\rfloor$. Thus, $s = m^d - \left\lfloor \frac{N-12l-94}{12l+90} \right\rfloor$. As a result of each digit requiring a width of 2 tiles, if k is odd, one additional tile column must be added. The number of filler tiles needed for the width is $k \mod 2$, and the number of filler tiles for the height is $N - 12l - 94 \mod 12l + 90$.

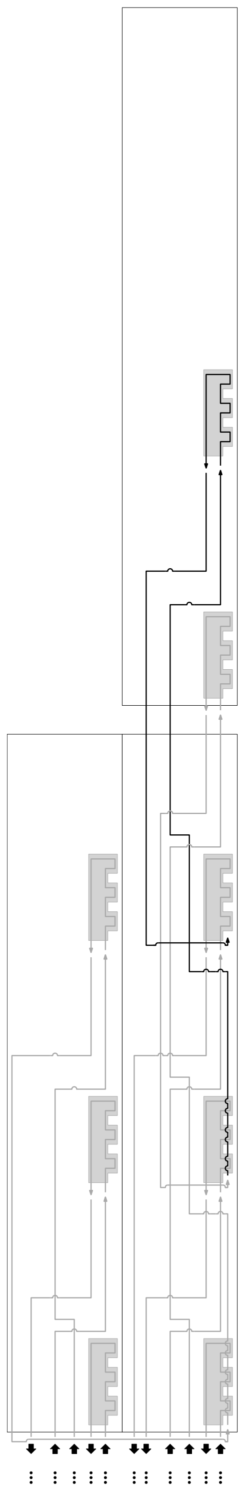
2 General counter



(a) A "clean" counter row, before any reading has started.



(b) Read digit 1 in the current row, write digit 1 in the next row.


(c) Read digit 2 in the current row, write digit 2 in the next row.

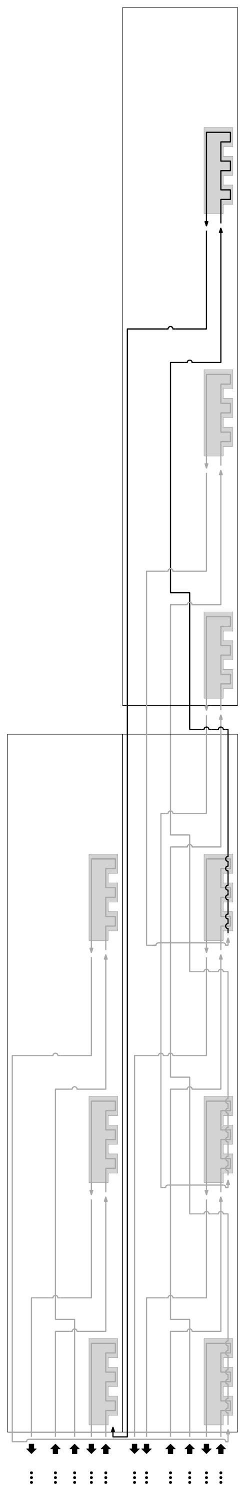
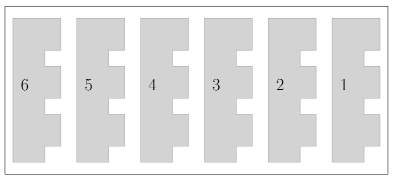

(d) Read digit 3 in the current row, write digit 3 in the next row.

Figure 1: This illustrates how a counter reads and writes a digit region, in a general sense. The counter starts in the rightmost digit region by reading the bottommost digit within that region. After reading digit 1 in the current row, the corresponding digit region in the next row be started in the next row. The counter writes the first digit in the next row, and then returns to the second digit in the current digit region. Once all the digits in the current digit region are read and written into the next row, the counter can then do one of the following: continue reading digits by moving on to the next digit region, cross back all the way to the right of the rectangle and start reading the next row, or halt.

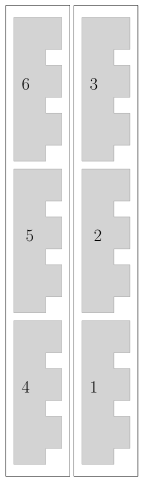
2.1 Digit region explanation (in progress)

Each logical row of the counter is made up of $\lceil \frac{d}{3} \rceil$ "digit regions". A digit region is a group of 1-3 digits, stacked vertically on top of one another. Within a digit region, the digits are sorted in order of significance, thus the top digit is the most signifcant digit, the middle digit is second most significant and the bottommost digit is the least signifcant.

The leftmost digit region is most signifcant and the rightmost is the least signifcant. The counter reads the least signifcant digit (1) in digit region 1, and continues in the current row until it detects the final digit, in the most signifcant digit region (MSR).



(a) Digits in a typical counter



(b) Digits in two digit regions, stacked vertically, minimizing the width.

Contrary to a typical counter, each counter row has an approximate height of 3 digits $\approx 12l$. The digits are stacked up to 3 before increasing the width.

2.2 Detecting the edges

The counter must detect if a digit is in the MSR and if it's in the MSR, whether or not it is the most signifcant digit. To do this, all digits are encoded with two additional bits on the least signifcant end. If bit 0 is 1, the reader tiles know they could be reading the most signifcant digit (MSD) or in case 2, the second most signifcant digit. If bit 1 is 1, the digit currently being read is the MSD, otherwise the digit is digit 1 in case 2.

bit_1	bit_0	Meaning
0	0	digit is not in MSR
0	1	digit is in the MSR but is not the MSD
1	0	
1	1	digit is in the MSR and is MSD

2.3 Tile set

When describing a special case, i.e. "digit x – case y", whatever follows will only apply to the MSR (due to each case only affecting the MSR.)

2.3.1 Line Gadgets

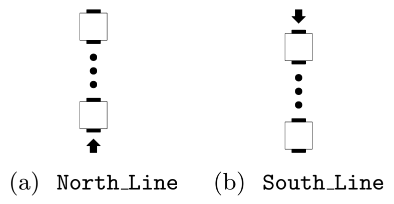
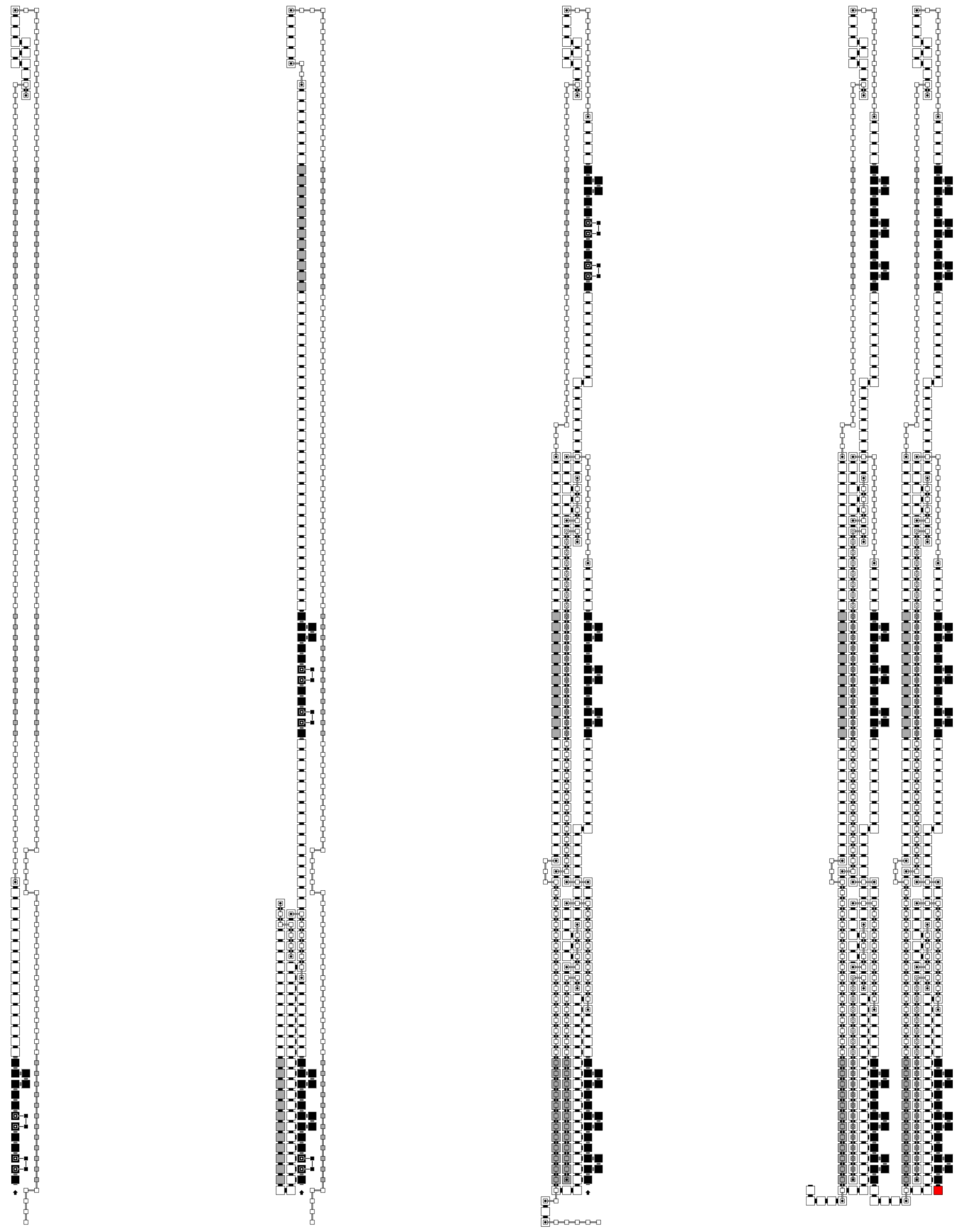

(a) `North_Line` (b) `South_Line`

Figure 3: `Line` gadgets

We will use the notation `NorthN_Line` and `SouthN_Line` where `N` corresponds to the length of a specific line gadget.

2.3.2 Initial Value (updated to assemble right to left like the other gadgets)

We begin by encoding s with the Seed unit. It has $\lceil \frac{d}{3} \rceil$ digit regions. Each digit region has three digits, except for the most significant digit region (MSR) which has $d \mod 3$ if $d \mod 3 \neq 0$, otherwise it has 3 digits.



(a) MSR case 1

(b) MSR case 2

(c) MSR case 3

(d) General digit regions

Figure 4: These figures show an example construction of the initial value, with all the possible MSR to the left. Of the three possible MSRs, of course only one would occur in a real assembly.

Note that we use i as the index of a digit in s and j as the index of a bit in a encoded digit.

- Create Seed($\langle \texttt{CounterWrite}, 1, \texttt{seed}, 0, 0\rangle$)

The idea here is to repeat these steps starting from $i = 0$ and repeating until i is the index of the first digit in the MSR. These steps build general non-MSR digit regions shown in Figure 4d.

- Start:
- Digit: for each $j = 0, \ldots, l-1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$:
    create Counter_Write($\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle$, $\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1\rangle$)
    from the general gadget shown in Figure 12a.
  - if $j = 1$:
    create Counter_Write($\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle$, $\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1\rangle$)
    from the general gadget shown in Figure 12a.
  - if $1 < j < l-1$:
    create Counter_Write($\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle$, $\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l-1$: create Counter_Write($\langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j\rangle$, $\langle \texttt{DigitTop}, 1, \texttt{seed}, i\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
- Digit_Top: the following statements create the gadget shown in Figure 14a.
  - Create North_Line5($\langle \texttt{DigitTop}, 1, \texttt{seed}, i\rangle$, $\langle \texttt{DigitTopA}, 1, \texttt{seed}, i\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create Topper($\langle \texttt{DigitTopA}, 1, \texttt{seed}, i\rangle$, $\langle \texttt{DigitTopB}, 1, \texttt{seed}, i\rangle$)
    from the micro-gadget shown in Figure 13a.
  - Create South_Line4l($\langle \texttt{DigitTopB}, 1, \texttt{seed}, i\rangle$, $\langle \texttt{ReturnPath}, 1, \texttt{seed}, i\rangle$)
    from the micro-gadget shown in Figure 3b.
- Create Return_Path($\langle \texttt{ReturnPath}, 1, \texttt{seed}, i\rangle$, $\langle \texttt{NextRead}, 1, \texttt{seed}, i\rangle$)
  (single tile).
- $i \leftarrow i + 1$
- Create Next_Read($\langle \texttt{NextRead}, 1, \texttt{seed}, i-1\rangle$, $\langle \texttt{SecondWarp}, 2, \texttt{seed}, i\rangle$)
  (single tile).
- Create Second_Warp($\langle \texttt{SecondWarp}, 2, \texttt{seed}, i\rangle$, $\langle \texttt{PostWarp}, 2, \texttt{seed}, i\rangle$) (single tile).
- Create Post_Warp($\langle \texttt{PostWarp}, 2, \texttt{seed}, i\rangle$, $\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, 0\rangle$)
  from the general gadget show in Figure 11c.
- Digit: for each $j = 0, \ldots, l-1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$:
    create Counter_Write($\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle$, $\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle$)
    from the general gadget shown in Figure 12a.
  - if $j = 1$:
    create Counter_Write($\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j\rangle$, $\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1\rangle$)
    from the general gadget shown in Figure 12a.

  - if $1 < j < l - 1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 2,$ `seed`$, i, j + 1\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l - 1$: create `Counter_Write`($\langle$`CounterWrite`$, 2,$ `seed`$, i, j\rangle$, $\langle$`DigitTop`$, 2,$ `seed`$, i\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.

- `Digit_Top`: the following statements create the gadget shown in Figure 14a.
  - Create `North_Line5`($\langle$`DigitTop`$, 2,$ `seed`$, i\rangle$, $\langle$`DigitTopA`$, 2,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create `Topper`($\langle$`DigitTopA`$, 2,$ `seed`$, i\rangle$, $\langle$`DigitTopB`$, 2,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 13a.
  - Create `South_Line4`l($\langle$`DigitTopB`$, 2,$ `seed`$, i\rangle$, $\langle$`ReturnPath`$, 2,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.
- Create `Return_Path`($\langle$`ReturnPath`$, 2,$ `seed`$, i\rangle$, $\langle$`NextRead`$, 2,$ `seed`$, i\rangle$)
  from the gadget in Figure 15d.
- $i \leftarrow i + 1$
- Create `Next_Read`($\langle$`NextRead`$, 2,$ `seed`$, i - 1\rangle$, $\langle$`FirstWarp`$, 3,$ `seed`$, i\rangle$)
  from the general gadget shown in Figure 16e.
- Create `First_Warp`($\langle$`FirstWarp`$, 3,$ `seed`$, i\rangle$, $\langle$`WarpBridge`$, 3,$ `seed`$, i\rangle$)
- Create `Warp_Bridge`($\langle$`WarpBridge`$, 3,$ `seed`$, i\rangle$, $\langle$`SecondWarp`$, 3,$ `seed`$, i\rangle$)
  from the general gadget shown in Figure 9a.
- Create `Second_Warp`($\langle$`SecondWarp`$, 3,$ `seed`$, i\rangle$, $\langle$`PostWarp`$, 3,$ `seed`$, i\rangle$)
- Create `Post_Warp`($\langle$`PostWarp`$, 3,$ `seed`$, i\rangle$, $\langle$`CounterWrite`$, 3,$ `seed`$, i, 0\rangle$)
  from the general gadget shown in Figure 11c.
- `Digit`: for each $j = 0, \ldots, l - 1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$:
    create `Counter_Write`($\langle$`CounterWrite`$, 3,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 3,$ `seed`$, i, j + 1\rangle$)
    from the general gadget shown in Figure 12a.
  - if $j = 1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 3,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 3,$ `seed`$, i, j + 1\rangle$)
    from the general gadget shown in Figure 12a.
  - if $1 < j < l - 1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 3,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 3,$ `seed`$, i, j + 1\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l - 1$: create `Counter_Write`($\langle$`CounterWrite`$, 3,$ `seed`$, i, j\rangle$, $\langle$`DigitTop`$, 3,$ `seed`$, i\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
- `Digit_Top`: the following statements create the gadget shown in Figure 14a.
  - Create `North_Line5`($\langle$`DigitTop`$, 3,$ `seed`$, i\rangle$, $\langle$`DigitTopA`$, 3,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create `Topper`($\langle$`DigitTopA`$, 3,$ `seed`$, i\rangle$, $\langle$`DigitTopB`$, 3,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 13a.

  - Create `South_Line4`l($\langle$`DigitTopB`$, 3,$ `seed`$, i\rangle$, $\langle$`ReturnPath`$, 3,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.

- Create `Return_Path`($\langle$`ReturnPath`$, 3,$ `seed`$, i\rangle$, $\langle$`NextRead`$, 3,$ `seed`$, i, \rangle$)
  from the gadget in Figure 15g.

- $i \leftarrow i + 1$

- Create `Next_Read`($\langle$`NextRead`$, 3,$ `seed`$, i - 1\rangle$, $\langle$`CounterWrite`$, 1,$ `seed`$, i\rangle$)
  from the general gadget shown in Figure 16i.

- if i is not an index in the MSR, go to `start`, else go to MSR.

MSR

Case 1 – if $d - i = 1$ to create the assembly shown in 4a.

- `Digit`: for each $j = 0, \ldots, l - 1$ and each b in $bin(C_0[i])[j]$:

  - if $j = 0$:
    create `Counter_Write`($\langle$`CounterWrite`$, 1,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 1,$ `seed`$, i, j + 1\rangle$)
    from the general gadget shown in Figure 12b.
  - if $j = 1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 1,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 1,$ `seed`$, i, j + 1\rangle$)
    from the general gadget shown in Figure 12b.
  - if $1 < j < l - 1$:
    create `Counter_Write`($\langle$`CounterWrite`$, 1,$ `seed`$, i, j\rangle$, $\langle$`CounterWrite`$, 1,$ `seed`$, i, j + 1\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l - 1$: create `Counter_Write`($\langle$`CounterWrite`$, 1,$ `seed`$, i, j\rangle$, $\langle$`DigitTop`$, 1,$ `seed`$, i\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.

- `Digit_Top`: the following statements create the gadget shown in Figure 14d

  - Create `North_Line4`l($\langle$`DigitTop`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopA`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create `North_Line4`($\langle$`DigitTopA`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopB`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create `Topper`($\langle$`DigitTopB`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopC`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 13a.
  - Create `South_Line4`l($\langle$`DigitTopC`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopD`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.
  - Create `South_Line30`($\langle$`DigitTopD`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopE`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.
  - Create `South_Line4`l($\langle$`DigitTopE`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopF`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.
  - Create `South_Line14`($\langle$`DigitTopF`$, 1,$ `seed`$, i\rangle$, $\langle$`DigitTopG`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.
  - Create `South_Line17`($\langle$`DigitTopG`$, 1,$ `seed`$, i\rangle$, $\langle$`ReturnPath`$, 1,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.

- Create Return_Path($\langle$ReturnPath$, 1, $seed$, i\rangle$, $\langle$NextRead$, 1, $seed$, i\rangle$)
  from the general gadget shown in Figure 15m
- Create Next_Read($\langle$NextRead$, 1, $seed$, i\rangle$, $\langle$Cross_Next_Row, increment$\rangle$)
  from the micro-gadget shown in Figure 16k.

Case 2 – if $d - i = 2$ to create the assembly shown in 4b.

- Digit: for each $j = 0, \ldots, l-1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$:
    create Counter_Write($\langle$CounterWrite$, 2, $seed$, i, j\rangle$, $\langle$CounterWrite$, 2, $seed$, i, j+1\rangle$)
    from the general gadget shown in Figure 12b.
  - if $j = 1$:
    create Counter_Write($\langle$CounterWrite$, 2, $seed$, i, j\rangle$, $\langle$CounterWrite$, 2, $seed$, i, j+1\rangle$)
    from the general gadget shown in Figure 12a.
  - if $1 < j < l-1$:
    create Counter_Write($\langle$CounterWrite$, 2, $seed$, i, j\rangle$, $\langle$CounterWrite$, 2, $seed$, i, j+1\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l-1$: create Counter_Write($\langle$CounterWrite$, 2, $seed$, i, j\rangle$, $\langle$DigitTop$, 2, $seed$, i\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
- Digit_Top: the following statements create the gadget shown in Figure 14g
  - Create Topper($\langle$DigitTop$, 1, $seed$, i\rangle$, $\langle$DigitTopA$, 1, $seed$, i\rangle$)
    from the micro-gadget shown in Figure 13b
  - Create South_Line4l($\langle$DigitTopA$, 1, op, msr\rangle$, $\langle$ReturnPath$, 1, op, msr\rangle$)
    from the micro-gadget shown in Figure 3b
- Create Return_Path($\langle$ReturnPath$, 1, $seed$, i\rangle$, $\langle$NextRead$, 1, $seed$, i\rangle$)
  (single tile)
- $i \leftarrow i + 1$
- Create Next_Read($\langle$NextRead$, 1, $seed$, i-1\rangle$, $\langle$SecondWarp$, 2, $seed$, i\rangle$)
  (single tile)
- Create Second_Warp($\langle$SecondWarp$, 2, $seed$, i\rangle$, $\langle$PostWarp$, 2, $seed$, i\rangle$)
  (single tile)
- Create Post_Warp($\langle$PostWarp$, 2, $seed$, i\rangle$, $\langle$CounterWrite$, 2, $seed$, i, 0\rangle$)
  from the general gadget show in Figure 11l.
- Digit: for each $j = 0, \ldots, l-1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$:
    create Counter_Write($\langle$CounterWrite$, 2, $seed$, i, j\rangle$, $\langle$CounterWrite$, 2, $seed$, i, j+1\rangle$)
    from the general gadget shown in Figure 12b.
  - if $j = 1$:
    create Counter_Write($\langle$CounterWrite$, 2, $seed$, i, j\rangle$, $\langle$CounterWrite$, 2, $seed$, i, j+1\rangle$)
    from the general gadget shown in Figure 12b.
  - if $1 < j < l-1$:
    create Counter_Write($\langle$CounterWrite$, 2, $seed$, i, j\rangle$, $\langle$CounterWrite$, 2, $seed$, i, j+1\rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.

  - if $j = l - 1$: create $\texttt{Counter\_Write}(\ \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j \rangle, \langle \texttt{DigitTop}, 2, \texttt{seed}, i \rangle\ )$ from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.

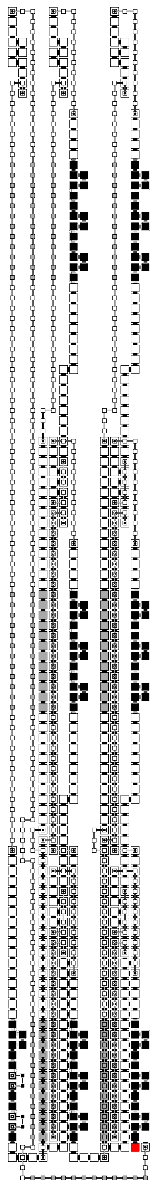
- `Digit_Top`: the following statements create the gadget shown in Figure 14j
  - Create $\texttt{North\_Line4}l(\ \langle \texttt{DigitTop}, 2, \texttt{seed}, i \rangle, \langle \texttt{DigitTopA}, 2, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\ \langle \texttt{DigitTopA}, 2, \texttt{seed}, i \rangle, \langle \texttt{DigitTopB}, 2, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 13c.
  - Create $\texttt{South\_Line4}l(\ \langle \texttt{DigitTopB}, 2, \texttt{seed}, i \rangle, \langle \texttt{DigitTopC}, 2, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 3b.
  - Create $\texttt{South\_Line30}(\ \langle \texttt{DigitTopC}, 2, \texttt{seed}, i \rangle, \langle \texttt{ReturnPath}, 2, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 3b.
- Create $\texttt{Return\_Path}(\ \langle \texttt{ReturnPath}, 2, \texttt{seed}, i \rangle, \langle \texttt{NextRead}, 2, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 15m.
- Create $\texttt{Next\_Read}(\ \langle \texttt{NextRead}, 2, \texttt{seed} \rangle, \langle \texttt{Cross\_Next\_Row}, \texttt{increment} \rangle\ )$ from the micro-gadget shown in Figure 16k.

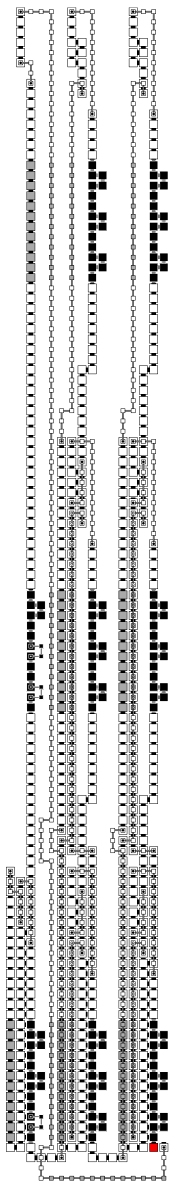
Case 3 – if $d - i = 3$ to create the assembly shown in 4c.

- `Digit`: for each $j = 0, \ldots, l-1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$:
    create $\texttt{Counter\_Write}(\ \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j \rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1 \rangle\ )$ from the general gadget shown in Figure 12a.
  - if $j = 1$:
    create $\texttt{Counter\_Write}(\ \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j \rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1 \rangle\ )$ from the general gadget shown in Figure 12a.
  - if $1 < j < l - 1$:
    create $\texttt{Counter\_Write}(\ \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j \rangle, \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j+1 \rangle\ )$ from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l - 1$: create $\texttt{Counter\_Write}(\ \langle \texttt{CounterWrite}, 1, \texttt{seed}, i, j \rangle, \langle \texttt{DigitTop}, 1, \texttt{seed}, i \rangle\ )$ from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
- `Digit_Top`: the following statements create the gadget shown in Figure 14a.
  - Create $\texttt{North\_Line5}(\ \langle \texttt{DigitTop}, 1, \texttt{seed}, i \rangle, \langle \texttt{DigitTopA}, 1, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper}(\ \langle \texttt{DigitTopA}, 1, \texttt{seed}, i \rangle, \langle \texttt{DigitTopB}, 1, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 13a.
  - Create $\texttt{South\_Line4}l(\ \langle \texttt{DigitTopB}, 1, \texttt{seed}, i \rangle, \langle \texttt{ReturnPath}, 1, \texttt{seed}, i \rangle\ )$ from the micro-gadget shown in Figure 3b.
- $i \leftarrow i + 1$
- Create $\texttt{Return\_Path}(\ \langle \texttt{ReturnPath}, 1, \texttt{seed}, i-1 \rangle, \langle \texttt{SecondWarp}, 2, \texttt{seed}, i \rangle\ )$ (single tile).
- Create $\texttt{Second\_Warp}(\ \langle \texttt{SecondWarp}, 2, \texttt{seed}, i \rangle, \langle \texttt{PostWarp}, 2, \texttt{seed}, i \rangle\ )$ (single tile).
- Create $\texttt{Post\_Warp}(\ \langle \texttt{PostWarp}, 2, \texttt{seed}, i \rangle, \langle \texttt{CounterWrite}, 2, \texttt{seed}, i, 0 \rangle\ )$ from the general gadget show in Figure 11c.

- `Digit`: for each $j = 0, \ldots, l-1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$
    create `Counter_Write`($\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j \rangle$, $\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1 \rangle$)
    from the general gadget shown in Figure 12a.
  - if $j = 1$:
    create `Counter_Write`($\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j \rangle$, $\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1 \rangle$)
    from the general gadget shown in Figure 12a.
  - if $1 < j < l-1$:
    create `Counter_Write`($\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j \rangle$, $\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j+1 \rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l-1$: create `Counter_Write`($\langle \texttt{CounterWrite}, 2, \texttt{seed}, i, j \rangle$, $\langle \texttt{DigitTop}, 2, \texttt{seed}, i \rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
- `Digit_Top`: the following statements create the gadget shown in Figure 14a.
  - Create `North_Line5`($\langle \texttt{DigitTop}, 2, \texttt{seed}, i \rangle$, $\langle \texttt{DigitTopA}, 2, \texttt{seed}, i \rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create `Topper`($\langle \texttt{DigitTopA}, 2, \texttt{seed}, i \rangle$, $\langle \texttt{DigitTopB}, 2, \texttt{seed}, i \rangle$)
    from the micro-gadget shown in Figure 13a.
  - Create `South_Line4`l($\langle \texttt{DigitTopB}, 2, \texttt{seed}, i \rangle$, $\langle \texttt{ReturnPath}, 2, \texttt{seed}, i \rangle$)
    from the micro-gadget shown in Figure 3b.
- Create `Return_Path`($\langle \texttt{ReturnPath}, 2, \texttt{seed}, i \rangle$, $\langle \texttt{NextRead}, 2, \texttt{seed}, i \rangle$)
  from the gadget in Figure 15d.
- $i \leftarrow i + 1$
- Create `Next_Read`($\langle \texttt{NextRead}, 2, \texttt{seed}, i-1 \rangle$, $\langle \texttt{FirstWarp}, 3, \texttt{seed}, i \rangle$)
  from the general gadget shown in Figure 16e.
- Create `First_Warp`($\langle \texttt{FirstWarp}, 3, \texttt{seed}, i \rangle$, $\langle \texttt{WarpBridge}, 3, \texttt{seed}, i \rangle$)
- Create `Warp_Bridge`($\langle \texttt{WarpBridge}, 3, \texttt{seed}, i \rangle$, $\langle \texttt{SecondWarp}, 3, \texttt{seed}, i \rangle$)
  from the general gadget shown in Figure 9a.
- Create `Second_Warp`($\langle \texttt{SecondWarp}, 3, \texttt{seed}, i \rangle$, $\langle \texttt{PostWarp}, 3, \texttt{seed}, i \rangle$)
- Create `Post_Warp`($\langle \texttt{PostWarp}, 3, \texttt{seed}, i \rangle$, $\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, 0 \rangle$)
  from the general gadget shown in Figure 11c.
- `Digit`: for each $j = 0, \ldots, l-1$ and each b in $bin(C_0[i])[j]$:
  - if $j = 0$:
    create `Counter_Write`($\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j \rangle$, $\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1 \rangle$)
    from the general gadget shown in Figure 12b.
  - if $j = 1$:
    create `Counter_Write`($\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j \rangle$, $\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1 \rangle$)
    from the general gadget shown in Figure 12b.
  - if $1 < j < l-1$:
    create `Counter_Write`($\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j \rangle$, $\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j+1 \rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.
  - if $j = l-1$: create `Counter_Write`($\langle \texttt{CounterWrite}, 3, \texttt{seed}, i, j \rangle$, $\langle \texttt{DigitTop}, 3, \texttt{seed}, i \rangle$)
    from the general gadget shown in Figure 12a if $b = 0$ or Figure 12b if $b = 1$.

- `Digit_Top`: the following statements create the gadget shown in Figure 14a.
  - Create `North_Line5`($\langle$`DigitTop`$, 3,$ `seed`$, i\rangle$, $\langle$`DigitTopA`$, 3,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create `Topper`($\langle$`DigitTopA`$, 3,$ `seed`$, i\rangle$, $\langle$`DigitTopB`$, 3,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 13a.
  - Create `South_Line4`l($\langle$`DigitTopB`$, 3,$ `seed`$, i\rangle$, $\langle$`ReturnPath`$, 3,$ `seed`$, i\rangle$)
    from the micro-gadget shown in Figure 3b.
- Create `Return_Path`($\langle$`ReturnPath`$, 3,$ `seed`$, i\rangle$, $\langle$`NextRead`$, 3,$ `increment`$,$ `msr`$,$ `msd`$\rangle$)
  from the gadget in Figure 15g.


(a) Initial value case 1


(b) Initial value case 2

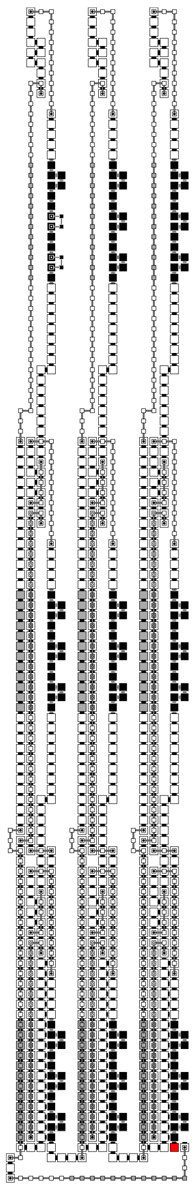

(c) Initial value case 3

Figure 5: These figures show a full example of an initial value, with both general regions and the MSRs together, instead of separated as shown above.

2.4 Counter Unit

2.4.1 Digit readers

- For each $i = 1, 2, 3$, $j = 0, \ldots, l-2$, $u \in \{0,1\}^j$, and $op \in \{\texttt{increment}, \texttt{copy}\}$:
  - if $j = 0$: create $\texttt{Counter\_Read}(\ \langle \texttt{CounterRead}, i, \lambda, op \rangle\ ,$
    $\langle \texttt{CounterRead}, i, 0, op \rangle\ ,$
    $\langle \texttt{CounterRead}, i, 1, op \rangle\ )$

from the general gadget in Figure 6.

  – else: create `Counter_Read`($\langle$`CounterRead`$, i, u, op\rangle$,
  $\langle$`CounterRead`$, i, 0u, op\rangle$,
  $\langle$`CounterRead`$, i, 1u, op\rangle$)
  from the general gadget in Figure 6.

- For each $i = 1, 2, 3$ and each $u \in \{0, 1\}^{l-1}$:
  - Create `Counter_Read`($\langle$`CounterRead`$, i, u,$ `copy`$\rangle$, $\langle$`PreWarp`$, i, 0u,$ `copy`$\rangle$, $\langle$`PreWarp`$, i, 1u,$ `copy`$\rangle$) from the general gadget in Figure 6.

Since the counter must only increment the current value if the result will be less than m, the `Counter_Read` gadgets that have both an `increment` signal and input size of $l-2$ must first right shift the bits 2 spots, and then for each possible value after reading one more bit, check whether that value is less than $m-1$. Basically, if the next bit read is a 0, we check if the current value + 1 is less than m. If the next bit read is a 1, we check if current value + $2^{\log(m)-1}$ + 1 is less than m. For both cases, if the counter can increment the current value, then the `Counter_Read` gadgets output the incremented value to the `Pre_Warp` gadgets and output a `copy` signal. Otherwise, if the counter is unable to increment the value, it outputs signal in which the bits of the digit is all zeroes and it will propagate the `increment` signal to the next digit.

- For each $i = 1, 2, 3$ and each $u \in \{0, 1\}^{l-1}$:

Algorithm 1 Incrementing and halting

```
function ReadMSB
    if $u$ ends with "11" then
        $out0 \leftarrow \langle \texttt{halt} \rangle$.
        $out1 \leftarrow \langle \texttt{halt} \rangle$.
    else
        $guess0 \leftarrow 0u >> 2$.
        $guess1 \leftarrow 1u >> 2$.
        if $convertToDecimal(guess0) + 1 < m - 1$ then
            $out0 \leftarrow \langle \texttt{PreWarp}, i, convertToBinary(convertToDecimal(guess0) + 1) + u[1] + [0], \texttt{copy} \rangle$.
        else
            $out0 \leftarrow \langle \texttt{PreWarp}, i, repeat(\text{"0"}, m) + u[1] + u[0], \texttt{increment} \rangle$.
        if $convertToDecimal(guess1) + 1 < m - 1$ then
            $out1 \leftarrow \langle \texttt{PreWarp}, i, convertToBinary(convertToDecimal(guess1) + 1) + u[1] + [0], \texttt{copy} \rangle$.
        else
            $out1 \leftarrow \langle \texttt{PreWarp}, i, repeat(\text{"0"}, m) + u[1] + u[0], \texttt{increment} \rangle$.
```

Create `Counter_Read`($\langle$`CounterRead`$, i, u,$ `increment`$\rangle$, $out0, out1$)
from the general gadget in Figure 6.

Let $guess0 = 0u >> 2$, $guess1 = 1u >> 2$

if $dec(guess0) + 1 < m - 1$

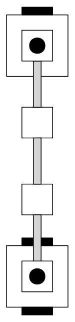
then $R0 = bin(dec(guess0) + 1) + u[1] + u[0],$ `copy` else $R0 = \{0\}^l, \texttt{increment}$

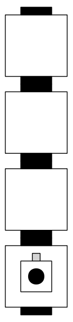
if $dec(guess1) + 1 < m - 1$

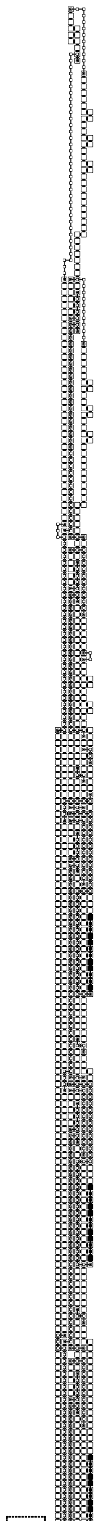
then $R1 = bin(dec(guess1) + 1) + u[1] + u[0], \texttt{copy}$

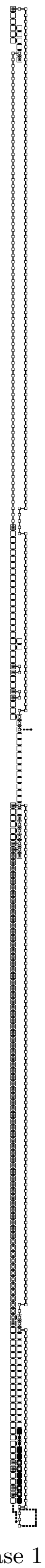
else $R1 = \{0\}^l, \texttt{increment}$

- Create `Counter_Read`($\langle \texttt{CounterRead}, i, u, \texttt{increment} \rangle$, $\langle \texttt{PreWarp}, i, R0 \rangle$, $\langle \texttt{PreWarp}, i, R1 \rangle$) from the general gadget in Figure 6.


(a) `Counter_Read_0`


(b) `Counter_Read_1`


(c) Digits 1, 2, & 3 - general overview


(d) Digit 1 - case 1 overview

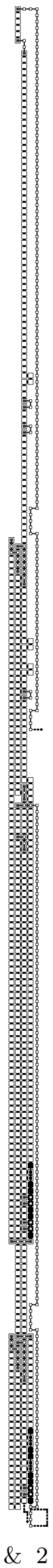

(e) Digits 1 & 2 - case 2 overview

Figure 6: The `Counter_Read` gadgets.

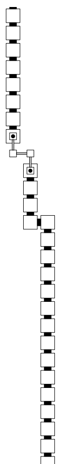
2.4.2 Warping

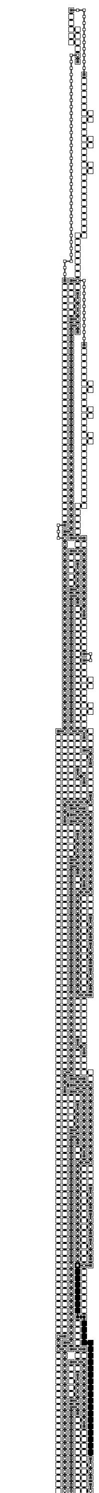
We now explain the `Warp_Unit`. A warp unit generally consists of the following 5 gadgets: `Pre_Warp`, `First_Warp`, `Warp_Bridge`, `Second_Warp`, `Post_Warp`. The job of these 5 gadgets is to transport the value read by the `Counter_Read` all the way to the digit region in the next row, so that the `Counter_Write` gadgets can write the next value in the correct locations. The `First_Warp` and `Second_Warp` gadgets are single tile gadgets that have north and south glues with identical labels. This allows the gadgets to continuously assemble until stopped by earlier parts of the assembly. These single tile gadgets also have one additional glue that will allow the next piece in the warp unit to assemble, however the assembly will also block this side of the tile all the way until the gadget can no longer continue assembling in the north direction.

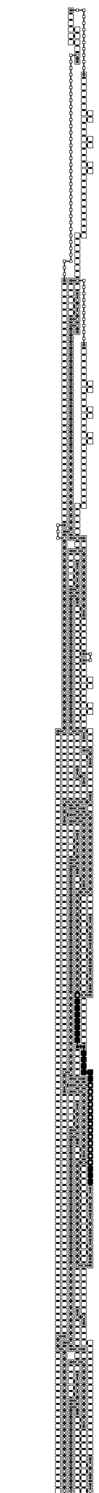
- `Pre_Warp`: These gadgets use the bits read from the `Counter_Read` gadgets to translate them into a signal used to tell the counter whether to begin reading another digit in the current row, or cut across the rectangle and begin reading the first digit in the next row. This signal is used from the `Pre_Warp` gadgets through the `Digit_Top` gadgets are attached after writing the current digit.

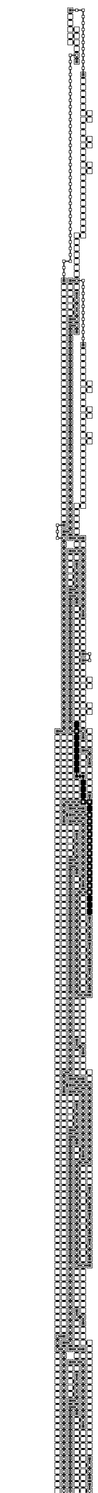
  For each $i = 1, 2, 3, u \in \{0, 1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}\}$:

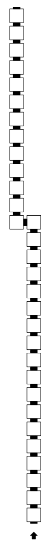
  - if u ends with 00: create `Pre_Warp`($\langle \texttt{PreWarp}, i, u, op \rangle$, $\langle \texttt{FirstWarp}, i, u, op \rangle$)
    from the general gadget in Figure 7a.
  - if u ends with 01: create `Pre_Warp`($\langle \texttt{PreWarp}, 1, u, op \rangle$, $\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr} \rangle$)
    from the general gadget in Figure 7e.
  - if u ends with 11: create `Pre_Warp`($\langle \texttt{PreWarp}, i, u, op \rangle$, $\langle \texttt{FirstWarp}, i, u, op, \texttt{msr}, \texttt{msd} \rangle$) from the general gadget in Figure 7g if $i = 1$ (case 1), or Figure 7i if $i = 2$ (case 2), or Figure 7a if $i = 3$ (case 3).

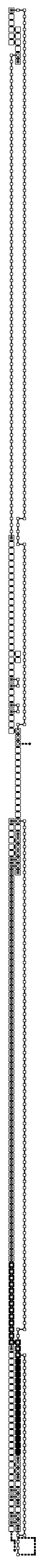

(a) Digits 1, 2, & 3 - general

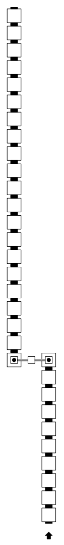

(b) Digit 1 - general overview

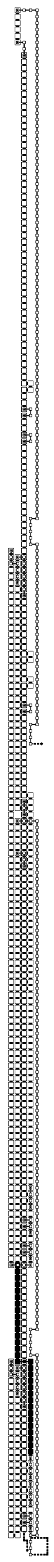

(c) Digit 2 - general overview

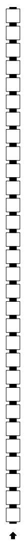

(d) Digit 3 - general overview


(e) Digit 1 - case 1


(f) Digit 1 - case 1 overview


(g) Digit 1 - case 2


(h) Digit 1 - case 2 overview


(i) Digit 2 - case 2

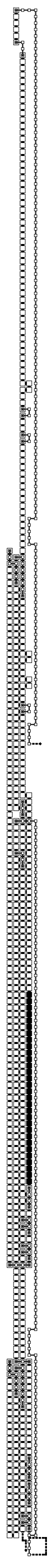

(j) Digit 2 - case 2 overview

Figure 7: The `Pre_Warp` gadgets.

- `First_Warp`: The idea of the `First_Warp` gadget is to transport the information read by the `Counter_Read` gadgets, usually across a distance $O(\log m)$. We do this using a single tile that assembles an infinite line in the north direction, and has one unique glue either in the east direction or west direction. This unique glue will at some point later in the assembly, that is determined by earlier parts of the assembly, no longer be blocked. When this occurs, it can finally attach to the `Warp_Bridge` gadget (except in a few special cases). This process signifies the "waking up" of the `First_Warp` gadgets. When this gadget wakes up, it must also be blocked in the north direction, which prevents a truly infinite line from assembling. The geometry required for this process is guarenteed to be in place by earlier-assembled `Digit_Top` gadgets.

  For each $u \in \{0,1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}\}$:

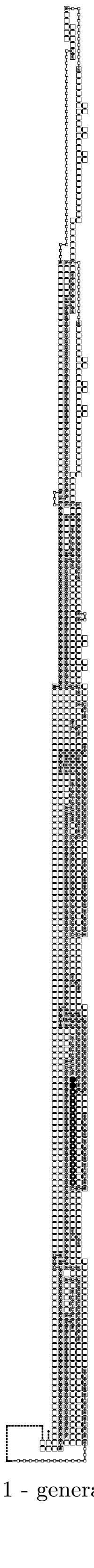
  - For each $i = 1, 2, 3$: create `First_Warp`($\langle \texttt{FirstWarp}, i, u, op \rangle$,
    $\langle \texttt{FirstWarp}, i, u, op \rangle$,
    $\langle \texttt{WarpBridge}, i, u, op \rangle$)
    from the single tile gadget, shown in Figure 8a if $i = 1$ or Figure 8b if $i = 2$, otherwise from Figure 8c if $i = 3$.

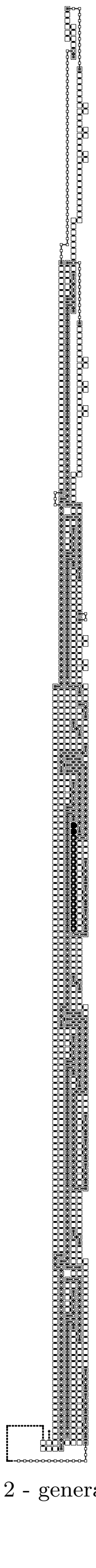
  - Create `First_Warp`($\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr} \rangle$,
    $\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr} \rangle$,
    $\langle \texttt{PostWarp}, 1, u, op, \texttt{msr} \rangle$)
    from the single tile gadget shown in Figure 8f.

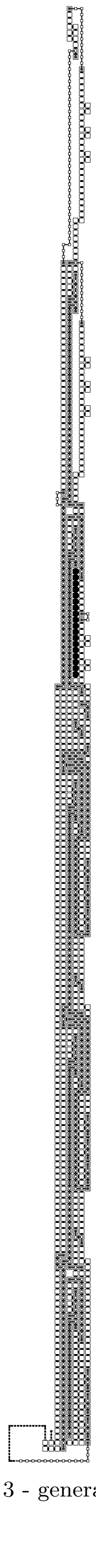
  - Create `First_Warp`($\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{FirstWarp}, 1, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{PostWarp}, 1, u, op, \texttt{msr}, \texttt{msd} \rangle$)
    from the single tile gadget shown in Figure 8e.

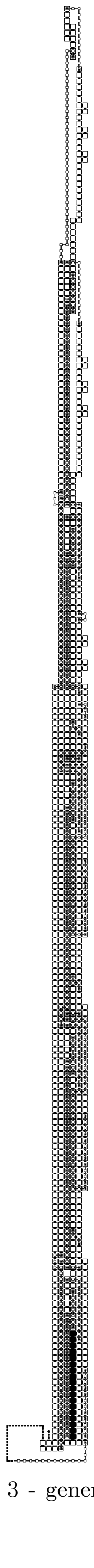
  - Create `First_Warp`($\langle \texttt{FirstWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{FirstWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{WarpBridge}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$)
    from the single tile gadget shown in Figure 8g.

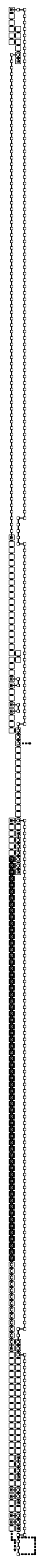
  - Create `First_Warp`($\langle \texttt{FirstWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{FirstWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{WarpBridge}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$)
    from the single tile gadget shown in Figure 8h.

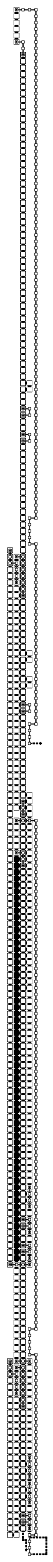

(a) Digit 1 - general overview

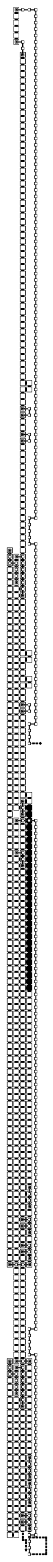

(b) Digit 2 - general overview


(c) Digit 3 - general overview


(d) Digit 3 - general (seed) overview


(e) Digit 1 - case 1 overview


(f) Digit 1 - case 2 overview


(g) Digit 2 - case 2 overview

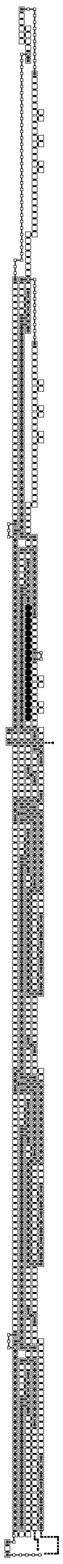

(h) Digit 3 - case 3 overview

Figure 8: The `First_Warp` gadget overviews.

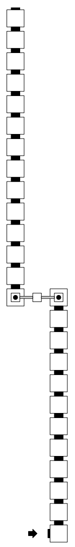
- `Warp_Bridge`: a `Warp_Bridge` gadget is a gadget which assembles when the `First_Warp` gadget makes it to its final destination. The goal of the `Warp_Bridge` is to assemble a path from the end of the `First_Warp` gadgets to the start of the `Second_Warp` gadgets. For digit 1 in cases 1 and 2, the `Warp_Bridge` is omitted from the `Warp_Unit`.

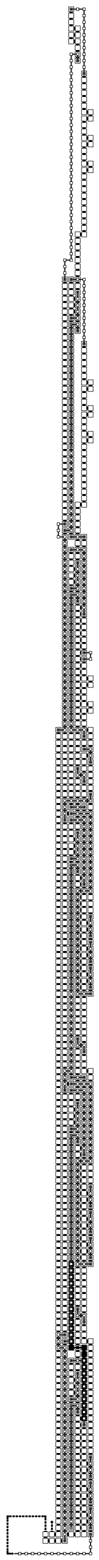
  For each $u \in \{0,1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}\}$:

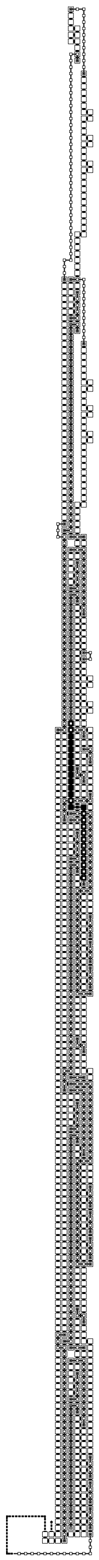
  - For each $i = 1, 2, 3$: create `Warp_Bridge`($\langle \texttt{WarpBridge}, i, u, op \rangle$, $\langle \texttt{SecondWarp}, i, u, op \rangle$) from the general gadget in Figure 9a.

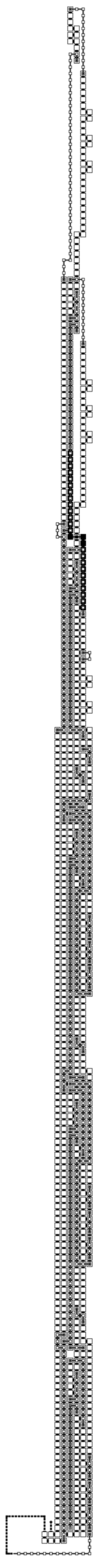
  - Create `Warp_Bridge`($\langle \texttt{WarpBridge}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{SecondWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$) from the general gadget in Figure 9g.

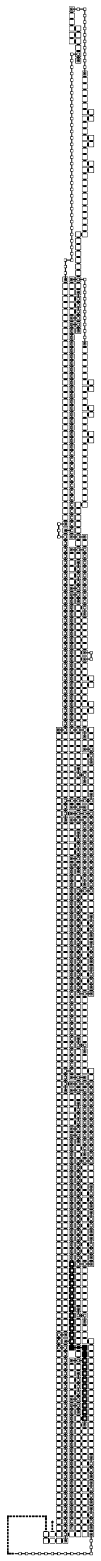
  - Create `Warp_Bridge`($\langle \texttt{WarpBridge}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{SecondWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$) from the general gadget in Figure 9a.

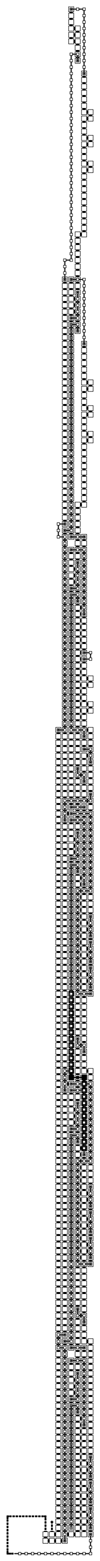

(a) General

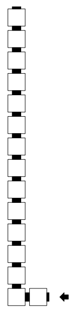

(b) Digit 1 - general overview


(c) Digit 2 - general overview


(d) Digit 3 - general overview


(e) Digit 2 - general (seed) overview


(f) Digit 3 - general (seed) overview


(g) Digit 2

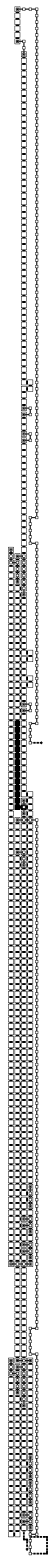

(h) Digit 2 - case 2 overview

Figure 9: The Warp_Bridge gadgets.

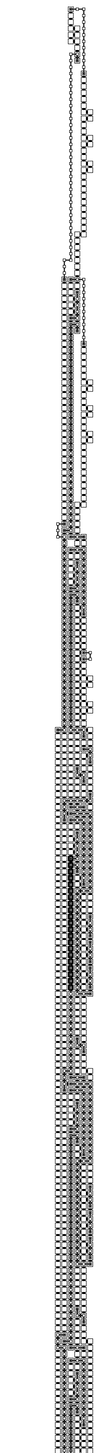
- `Second_Warp`: Similar to the `First_Warp` gadgets, the idea of the `Second_Warp` gadget is to also to transport the information read by the `Counter_Read` gadgets. We do this using a single tile that assembles an infinite line in the north direction, and has one unique glue either in the east direction or up direction. This unique glue will at some point later in the assembly, that is determined by earlier parts of the assembly, no longer be blocked. When this occurs, it can finally attach to the `Post_Warp` gadget. This process signifies the "waking up" of the `Second_Warp` gadgets. When this gadget wakes up, it must also be blocked in the north direction, which prevent a truly infinite line from assembling. The geometry required for this process is guarenteed to be in place by earlier-assembled `Digit_Top` gadgets.

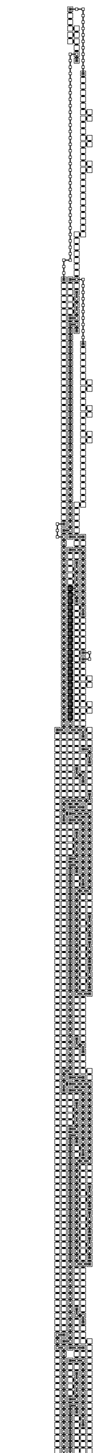
  For each $u \in \{0,1\}^l$, and each $op \in \{\texttt{increment}, \texttt{copy}\}$:

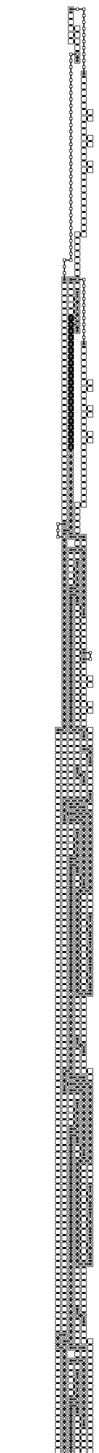
  - For each $i = 1, 2, 3$:
    Create `Second_Warp`($\langle \texttt{SecondWarp}, i, u, op \rangle$, $\langle \texttt{SecondWarp}, i, u, op \rangle$, $\langle \texttt{PostWarp}, i, u, op \rangle$)
    from the single tile gadget, shown in Figure 10a if $i = 1$ or Figure 10b if $i = 2$, otherwise from Figure 10c if $i = 3$.

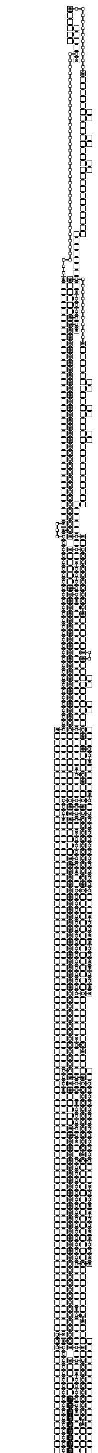
  - Create `Second_Warp`($\langle \texttt{SecondWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{SecondWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{PostWarp}, 2, u, op, \texttt{msr}, \texttt{msd} \rangle$)
    from the single tile gadget shown in Figure 10f.

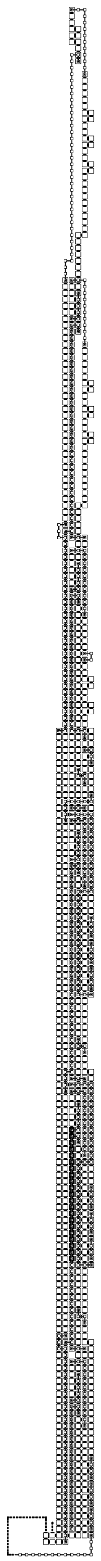
  - Create `Second_Warp`($\langle \texttt{SecondWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{SecondWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$,
    $\langle \texttt{PostWarp}, 3, u, op, \texttt{msr}, \texttt{msd} \rangle$)
    from the single tile gadget shown in Figure 10h.

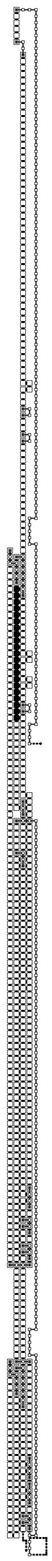

(a) Digit 1 - general overview

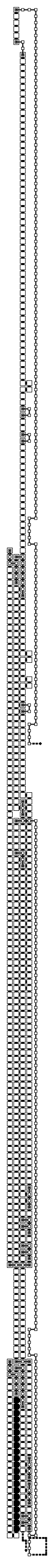

(b) Digit 2 - general overview


(c) Digit 3 - general overview


(d) Digit 2 - general (seed) overview


(e) Digit 3 - general (seed) overview


(f) Digit 2 - case 2 overview


(g) Digit 2 - case 2 (seed) overview

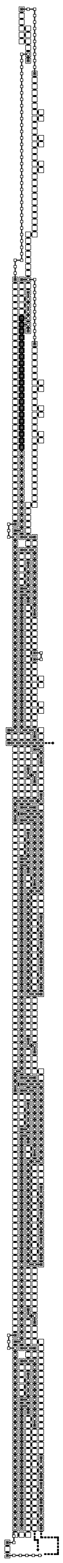
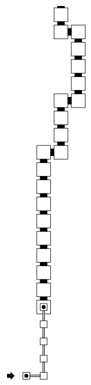
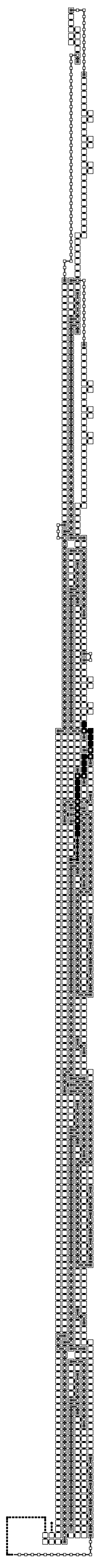
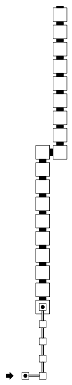

(h) Digit 3 - case 3 overview

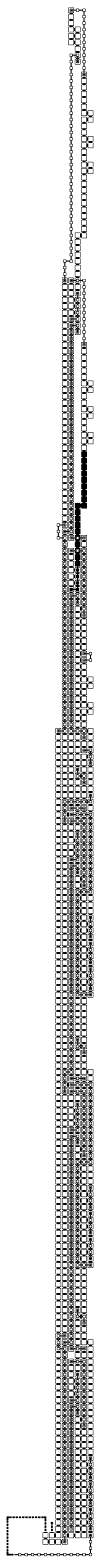
Figure 10: The `Second_Warp` gadget overviews.

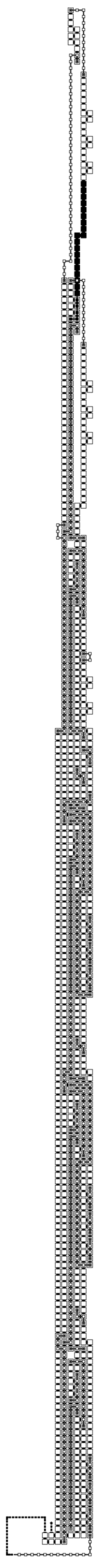
- `Post_Warp`: The `Post_Warp` gadget is final gadget to assemble in the `Warp_Unit`. The idea of this gadget is to assemble from wherever the last warping tiles "wake up", forming a path that ends where the next digit needs to start being written.
  - For each $i = 1, 2, 3$: create
    `Post_Warp`($\langle \texttt{PostWarp}, i, u, op \rangle$, $\langle \texttt{CounterWrite}, i, u, op \rangle$)
    from the general gadget shown in Figure 11a if $i = 1$, or Figure 11c if $i = 2$ or $i = 3$.
  - Create `Post_Warp`($\langle \texttt{PostWarp}, 1, u, op, \texttt{msr} \rangle$, $\langle \texttt{CounterWrite}, 1, u, op, \texttt{msr} \rangle$)
    from the general gadget in Figure 11j.
  - For each $i = 1, 2, 3$: create
    `Post_Warp`($\langle \texttt{PostWarp}, i, u, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{CounterWrite}, i, u, op, \texttt{msr}, \texttt{msd} \rangle$)
    from the general gadget shown in Figure 11h if $i = 1$, or Figure 11l if $i = 2$, or Figure 11c if $i = 3$.

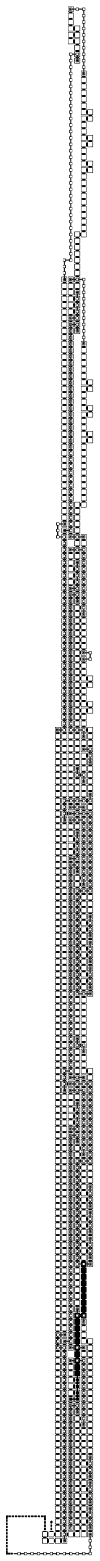

(a) Digit 1 - general

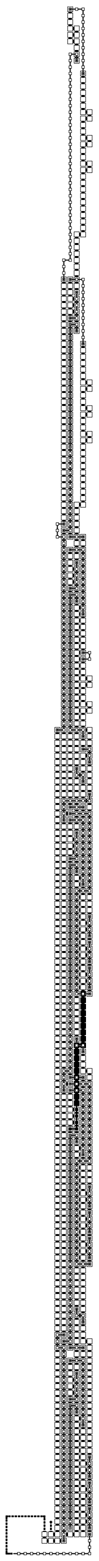

(b) Digit 1 - general overview. The black tiles in this figure is the gadget shown in subfigure a.

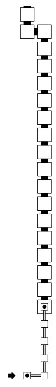

(c) Digit 2 & 3 - general


(d) Digit 2 - general overview

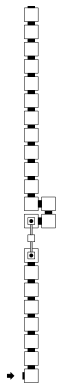

(e) Digit 3 - general overview


(f) Digit 2 - general (seed) overview

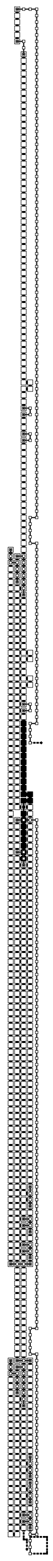

(g) Digit 3 - general (seed) overview


(h) Digit 1 - case 1

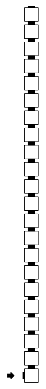
(i) Digit 1 - case 2 overview

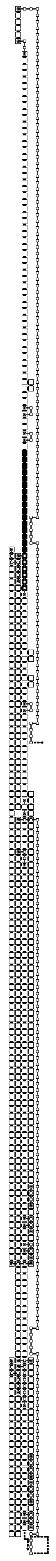


(j) Digit 1 - case 2



(k) Digit 1 - case 2 overview


(l) Digit 2 - case 2


(m) Digit 2 - case 2 overview

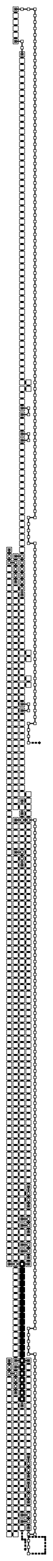
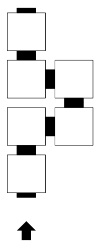
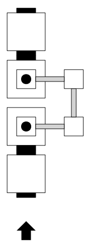

(n) Digit 2 - case 2 (seed) overview

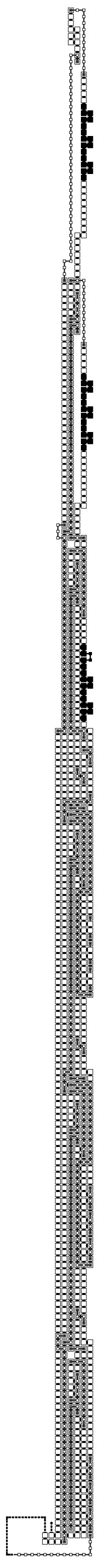
Figure 11: The Post_Warp gadgets.

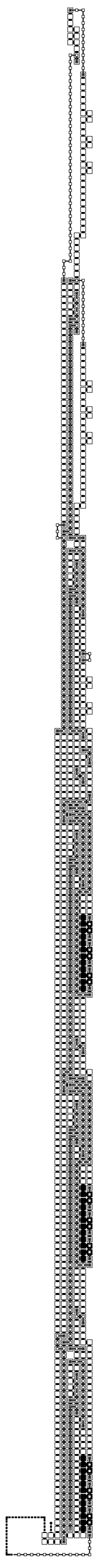
2.4.3 Counter write

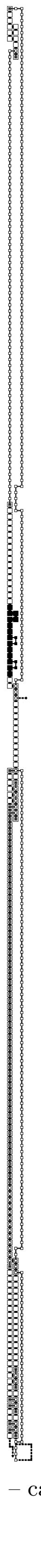
- For each $i = 1, 2, 3$, $j = l - 1, \ldots, 1$, $u \in \{0, 1\}^j$, and each $op \in \{\texttt{increment}, \texttt{copy}\}$:
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, u0, op \rangle$, $\langle \texttt{CounterWrite}, i, u, op \rangle$) from the general gadget in Figure 12a
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, u1, op \rangle$, $\langle \texttt{CounterWrite}, i, u, op \rangle$) from the general gadget in Figure 12b
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, 1, u0, op, \texttt{msr} \rangle$, $\langle \texttt{CounterWrite}, 1, u, op, \texttt{msr} \rangle$) from the general gadget in Figure 12a
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, 1, u1, op, \texttt{msr} \rangle$, $\langle \texttt{CounterWrite}, 1, u, op, \texttt{msr} \rangle$) from the general gadget in Figure 12b
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, u0, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{CounterWrite}, i, u, op, \texttt{msr}, \texttt{msd} \rangle$) from the general gadget in Figure 12a
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, u1, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{CounterWrite}, i, u, op, \texttt{msr}, \texttt{msd} \rangle$) from the general gadget in Figure 12b
- For each $i = 1, 2, 3$ and each $op \in \{\texttt{increment}, \texttt{copy}\}$:
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, 0, op \rangle$, $\langle \texttt{DigitTop}, i, op \rangle$) from the general gadget in Figure 12a
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, 1, op \rangle$, $\langle \texttt{DigitTop}, i, op \rangle$) from the general gadget in Figure 12b
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, 1, 0, op, \texttt{msr} \rangle$, $\langle \texttt{DigitTop}, 1, op, \texttt{msr} \rangle$) from the general gadget in Figure 12a
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, 1, 1, op, \texttt{msr} \rangle$, $\langle \texttt{DigitTop}, 1, op, \texttt{msr} \rangle$) from the general gadget in Figure 12b
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, 0, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{DigitTop}, i, op, \texttt{msr}, \texttt{msd} \rangle$) from the general gadget in Figure 12a
  - Create `Counter_Write`($\langle \texttt{CounterWrite}, i, 1, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{DigitTop}, i, op, \texttt{msr}, \texttt{msd} \rangle$) from the general gadget in Figure 12b

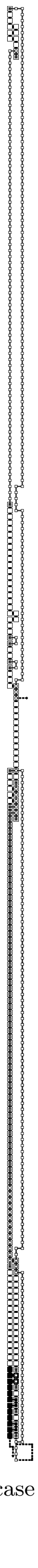

(a) `Counter_Write_0`

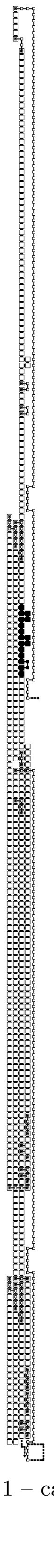

(b) `Counter_Write_1`

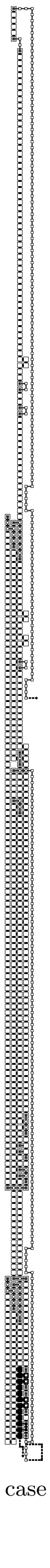

(c) Digits 1, 2, & 3 - general overview

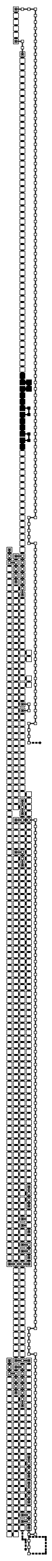

(d) Digits 1, 2, & 3 - general (seed) overview


(e) Digit 1 – case 1


(f) Digit 1 – case 1 (seed)


(g) Digit 1 – case 2


(h) Digit 1 – case 2 (seed)


(i) Digit 2 – case 2

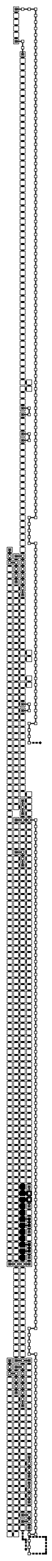
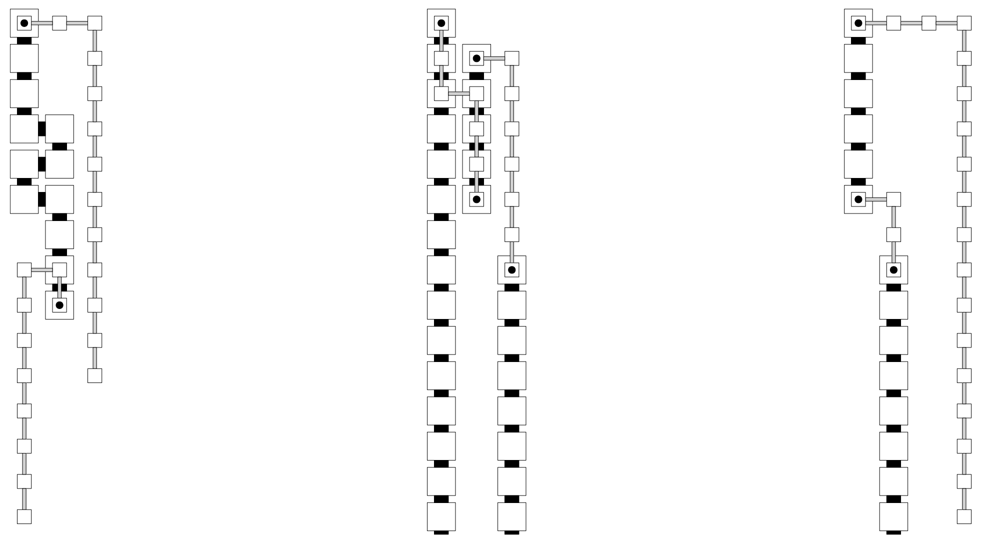

(j) Digit 2 – case 2 (seed)

Figure 12: The `Counter_Write` gadgets

2.4.4 Digit tops

The Digit_Top gadgets have special geometry designed so that First_Warp and Second_Warp tiles are allowed to "wake up", and complete their warping journey. Each digit has some type of Digit_Top gadget, however, depending on the digit region and index of a specific digit, the exact digit top will differ.



(a) General topper (b) Case 1 – topper (c) Case 2 – topper

Figure 13: Topper micro-gadgets

For each $op \in \{\texttt{increment}, \texttt{copy}\}$

- Digit 1 (general): the following statements create the gadget shown in Figure 14a
  - Create North_Line5($\langle \texttt{DigitTop}, 1, op \rangle$, $\langle \texttt{DigitTopA}, 1, op \rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create Topper($\langle \texttt{DigitTopA}, 1, op \rangle$, $\langle \texttt{DigitTopB}, 1, op \rangle$)
    from the micro-gadget shown in Figure 13a.
  - Create South_Line4l($\langle \texttt{DigitTopB}, 1, op \rangle$, $\langle \texttt{Return\_Path}, 1, op \rangle$)
    from the micro-gadget shown in Figure 3b.

- Digit 1 (MSR): the following statements create the gadget shown in Figure 14g
  - Create Topper($\langle \texttt{DigitTop}, 1, op, \texttt{msr} \rangle$, $\langle \texttt{DigitTopA}, 1, op, \texttt{msr} \rangle$)
    from the micro-gadget shown in Figure 13b.
  - Create South_Line4l($\langle \texttt{DigitTopA}, 1, op, \texttt{msr} \rangle$, $\langle \texttt{Return\_Path}, 1, op, \texttt{msr} \rangle$)
    from the micro-gadget shown in Figure 3b.

- Digit 1 (MSD): the following statements create the gadget shown in Figure 14d

- Create North_Line4l($\langle$DigitTop$, 1, op,$ msr, msd$\rangle$, $\langle$DigitTopA$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 3a.
- Create North_Line4($\langle$DigitTopA$, 1, op,$ msr, msd$\rangle$, $\langle$DigitTopB$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 3a.
- Create Topper($\langle$DigitTopB$, 1, op,$ msr, msd$\rangle$, $\langle$DigitTopC$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 13a.
- Create South_Line4l($\langle$DigitTopC$, 1, op,$ msr, msd$\rangle$, $\langle$DigitTopD$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 3b.
- Create South_Line30($\langle$DigitTopD$, 1, op,$ msr, msd$\rangle$, $\langle$DigitTopE$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 3b.
- Create South_Line4l($\langle$DigitTopE$, 1, op,$ msr, msd$\rangle$, $\langle$DigitTopF$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 3b.
- Create South_Line14($\langle$DigitTopF$, 1, op,$ msr, msd$\rangle$, $\langle$DigitTopG$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 3b.
- Create South_Line17($\langle$DigitTopG$, 1, op,$ msr, msd$\rangle$, $\langle$Return_Path$, 1, op,$ msr, msd$\rangle$)
  from the micro-gadget shown in Figure 3b.

- Digit 2 (general): the following statements create the gadget shown in Figure 14a
  - Create North_Line5($\langle$DigitTop$, 2, op\rangle$, $\langle$DigitTopA2$, op\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create Topper($\langle$DigitTopA2$, op\rangle$, $\langle$DigitTopB2$, op\rangle$)
    from the micro-gadget shown in Figure 13a.
  - Create South_Line4l($\langle$DigitTopB2$, op\rangle$, $\langle$Return_Path$, 2, op\rangle$)
    from the micro-gadget shown in Figure 3b.

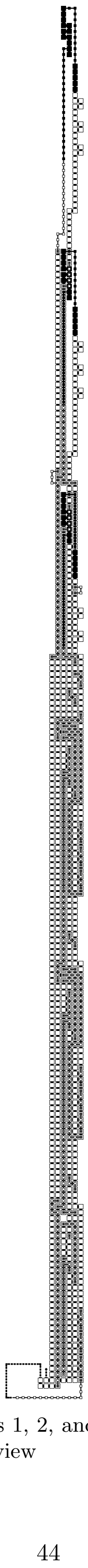
- Digit 2 (MSD): the following statements create the gadget shown in Figure 14j
  - Create North_Line4l($\langle$DigitTop$, 2, op,$ msr, msd$\rangle$, $\langle$DigitTopA$, 2, op,$ msr, msd$\rangle$)
    from the micro-gadget shown in Figure 3a.
  - Create Topper($\langle$DigitTopA$, 2, op,$ msr, msd$\rangle$, $\langle$DigitTopB$, 2, op,$ msr, msd$\rangle$)
    from the micro-gadget shown in Figure 13c.
  - Create South_Line4l($\langle$DigitTopB$, 2, op,$ msr, msd$\rangle$, $\langle$DigitTopC$, 2, op,$ msr, msd$\rangle$)
    from the micro-gadget shown in Figure 3b.
  - Create South_Line30($\langle$DigitTopC$, 2, op,$ msr, msd$\rangle$, $\langle$Return_Path$, 2, op,$ msr, msd$\rangle$)
    from the micro-gadget shown in Figure 3b.

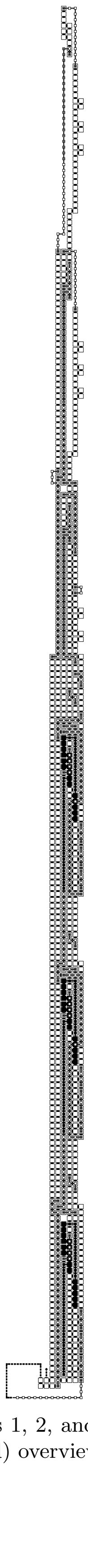
- Digit 3 (general): the following statements create the gadget from Figure 14a
  - Create North_Line5($\langle$DigitTop$, 3, op\rangle$, $\langle$DigitTopA$, 3, op\rangle$)
    from the micro-gadget shown in Figure 3a.

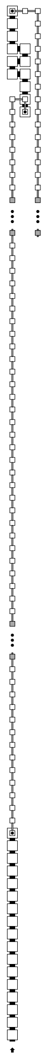
  - Create $\texttt{Topper(}\ \langle\texttt{DigitTopA}, 3, op\rangle\ , \langle\texttt{DigitTopB}, 3, op\rangle\ )$
    from the micro-gadget shown in Figure 13a.
  - Create $\texttt{South\_Line4}l(\ \langle\texttt{DigitTopB}, 3, op\rangle\ , \langle\texttt{Return\_Path}, 3, op\rangle\ )$
    from the micro-gadget shown in Figure 3b.

- Digit 3 (MSD): the following statements create the gadget from Figure 14a
  - Create $\texttt{North\_Line5(}\ \langle\texttt{DigitTop}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ , \langle\texttt{DigitTopA}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ )$
    from the micro-gadget shown in Figure 3a.
  - Create $\texttt{Topper(}\ \langle\texttt{DigitTopA}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ , \langle\texttt{DigitTopB}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ )$
    from the micro-gadget shown in Figure 13a.
  - Create $\texttt{South\_Line4}l(\ \langle\texttt{DigitTopB}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ , \langle\texttt{Return\_Path}, 3, op, \texttt{msr}, \texttt{msd}\rangle\ )$
    from the micro-gadget shown in Figure 3b.

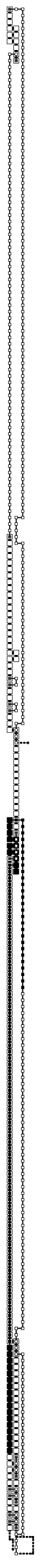
(a) Digit top general

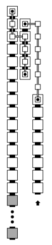

(b) Digits 1, 2, and 3 - general overview

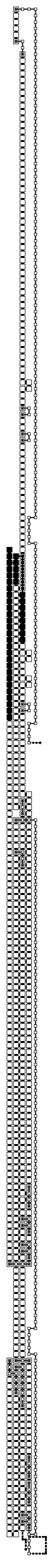

(c) Digits 1, 2, and 3 - general (seed) overview

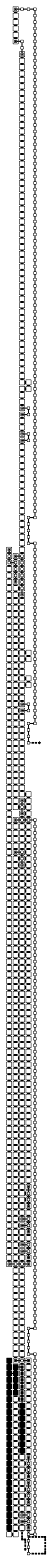

(d) Digit 1 case 1

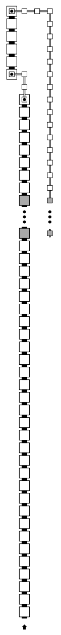
(e) Digit 1 case 1 - general overview

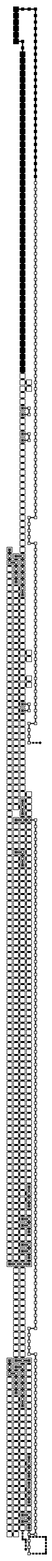

(f) Digit 1 case 1 - general (seed) overview


(g) Digit 1 case 2


(h) Digit 1 case 2 - general overview


(i) Digit 1 case 2 - general (seed) overview


(j) Digit 2 case 2


(k) Digit 2 case 2 - general overview

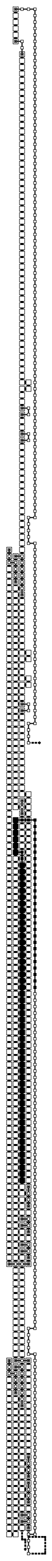

(l) Digit 2 case 2 - general (seed) overview

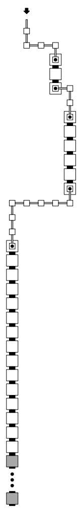
Figure 14: The Digit_Top gadgets.

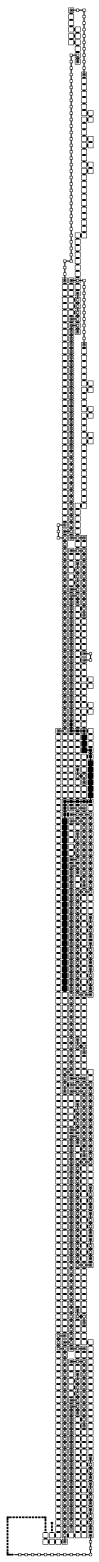
2.4.5 Return paths

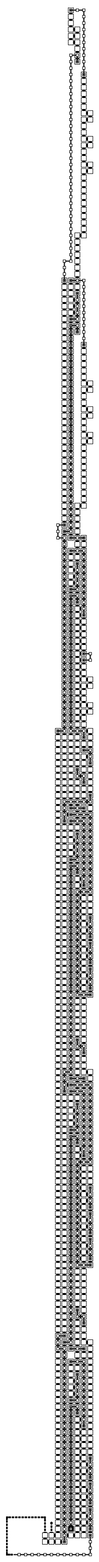
After a `Digit_Top` has assembled, we know that the geometry the allows for the `Warp_Unit` to work has been placed, therfore the counter is free to return back near where it started reading the previous digit. The next gadget that assembles is the `Return_Path` gadget. The basic idea of this gadget is simply to provide a path from the end of `Digit_Top` to some general area closer to where the most recently read digit is located. Once the `Return_Path` has completely assembled, it output's a glue for the `Next_Read` gadgets to determine where the counter needs to assemble next.

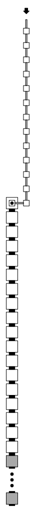
For each $op \in \{\texttt{increment}, \texttt{copy}\}$:

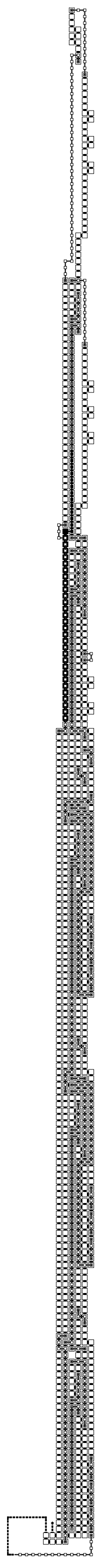
- Create `Return_Path`($\langle \texttt{ReturnPath}, 1, op \rangle$, $\langle \texttt{NextRead}, 1, op \rangle$)
  from the general gadget shown in Figure 15a.
- Create `Return_Path`($\langle \texttt{ReturnPath}, 1, op, \texttt{msr} \rangle$, $\langle \texttt{NextRead}, 1, op, \texttt{msr} \rangle$)
  from the general gadget shown in Figure 15j
- Create `Return_Path`($\langle \texttt{ReturnPath}, 1, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{NextRead}, 1, op, \texttt{msr}, \texttt{msd} \rangle$)
  from the general gadget shown in Figure 15m.
- Create `Return_Path`($\langle \texttt{ReturnPath}, 2, op \rangle$, $\langle \texttt{NextRead}, 2, op \rangle$)
  from the general gadget shown in Figure 15d.
- Create `Return_Path`($\langle \texttt{ReturnPath}, 2, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{NextRead}, 2, op, \texttt{msr}, \texttt{msd} \rangle$)
  from the general gadget shown in Figure 15m.
- Create `Return_Path`($\langle \texttt{ReturnPath}, 3, op \rangle$, $\langle \texttt{NextRead}, 3, op \rangle$)
  from the general gadget shown in Figure 15g.
- Create `Return_Path`($\langle \texttt{ReturnPath}, 3, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{NextRead}, 3, op, \texttt{msr}, \texttt{msd} \rangle$)
  from the general gadget shown in Figure 15g.

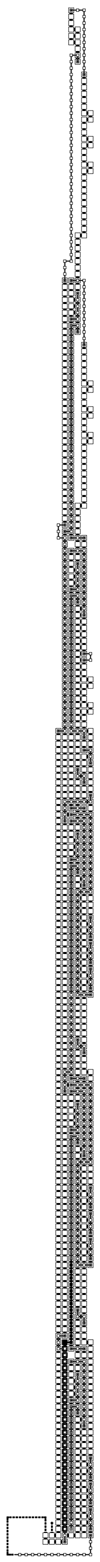

(a) Digit 1 - general

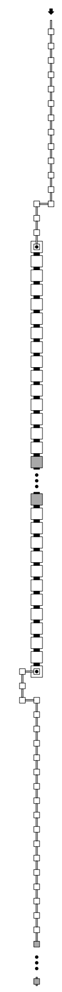

(b) Digit 1 - general overview

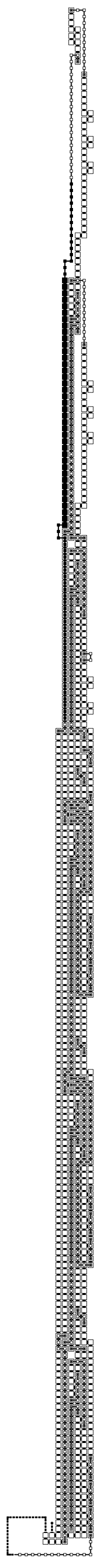

(c) Digit 1 - general (seed) overview

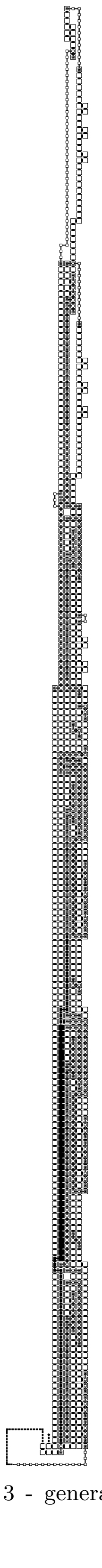

(d) Digit 2 - general

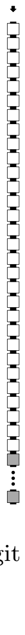

(e) Digit 2 - general overview

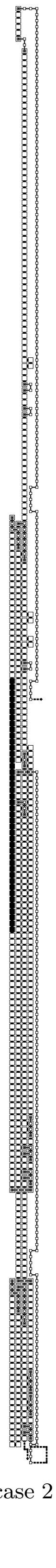

(f) Digit 2 - general (seed) overview

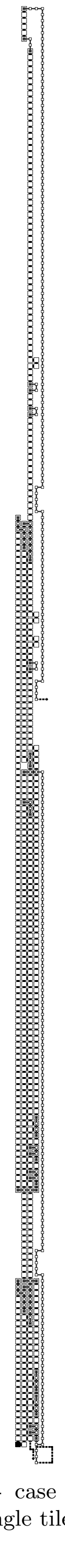

(g) Digit 3 - general

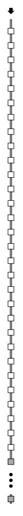

(h) Digit 3 - general overview

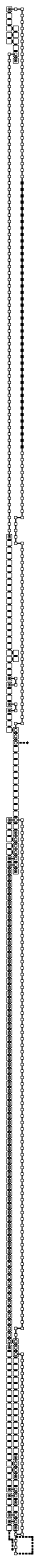

(i) Digit 3 - general (seed) overview

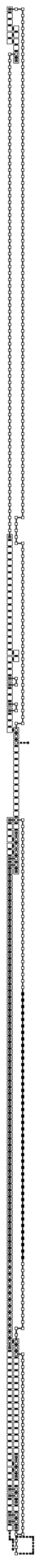

(j) Digit 1 - case 2


(k) Digit 1 - case 2 overview


(l) Digit 1 - case 2 (seed) overview (single tile)


(m) Digit 1 - case 1, Digit 2 case 2


(n) Digit 1 - case 1 overview


(o) Digit 1 - case 1 (seed) overview

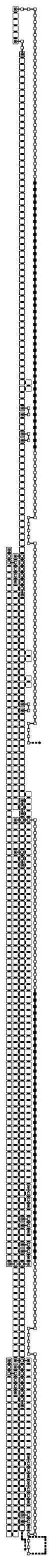

(p) Digit 2 - case 2 overview

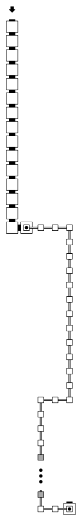
Figure 15: The `Return_Path` gadgets.

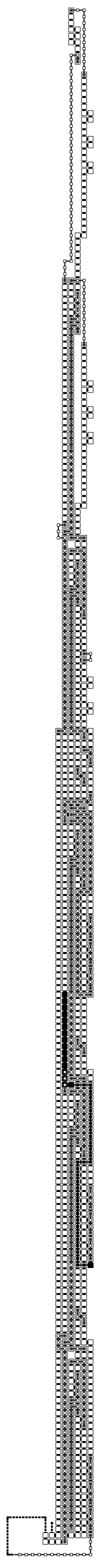
2.4.6 The Next Read gadgets

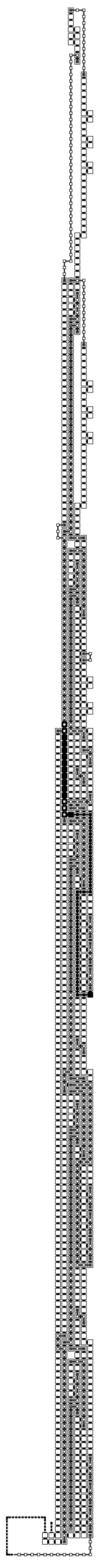
Once the `Return_Path` gadget has assembled, the counter has a few options to choose from. The gadget that controls what happens after a digit is written is the `Next_Read` gadget. If there is a `msr` and `msd` signal in the input, this gadget knows that the most significant digit was just read and will output a glue for the `Cross_Next_Row` gadget to assemble so that the counter crosses back to the right and begins reading the first digit in the next row. Otherwise (for regular digits – not in the MSR), this gadget will assemble the second half of the return path, terminating at the next digit in the current row. When this happens, the gadget increments the digit index (unless it is already digit 3, in which case it resets to 1) and outputs an empty `Counter_Read` signal to force the counter to begin reading the next digit.

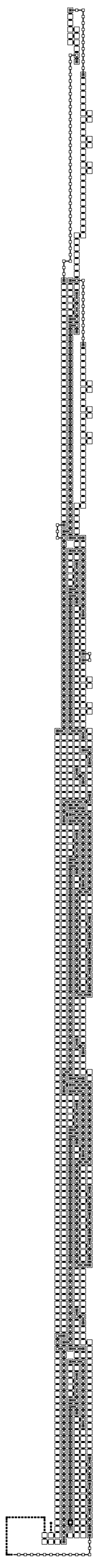
For each $op \in \{\texttt{increment}, \texttt{copy}\}$:

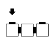
- Create `Next_Read`($\langle \texttt{NextRead}, 1, op \rangle$, $\langle \texttt{CounterRead}, 2, \lambda, op \rangle$)
  from the gadget shown in Figure 16a.
- Create `Next_Read`($\langle \texttt{NextRead}, 1, op, \texttt{msr} \rangle$, $\langle \texttt{CounterRead}, 2, \lambda, op \rangle$)
  from the gadget shown in Figure 16o.
- Create `Next_Read`($\langle \texttt{NextRead}, 1, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{CrossNextRow}, op \rangle$)
  from the gadget shown in Figure 16k.
- Create `Next_Read`($\langle \texttt{NextRead}, 2, op \rangle$, $\langle \texttt{CounterRead}, 3, \lambda, op \rangle$)
  from the gadget shown in Figure 16a.
- Create `Next_Read`($\langle \texttt{NextRead}, 2, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{CrossNextRow}, op \rangle$)
  from the gadget shown in Figure 16k.
- Create `Next_Read`($\langle \texttt{NextRead}, 3, op \rangle$, $\langle \texttt{CounterRead}, 1, \lambda, op \rangle$)
  from the gadget shown in Figure 16g.
- Create `Next_Read`($\langle \texttt{NextRead}, 3, op, \texttt{msr}, \texttt{msd} \rangle$, $\langle \texttt{CrossNextRow}, op \rangle$)
  from the gadget shown in Figure 16q.

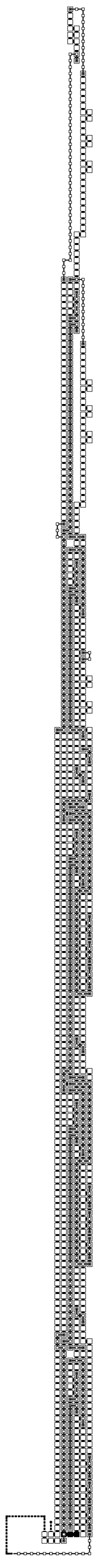

(a) Digit 1 & 2 - general

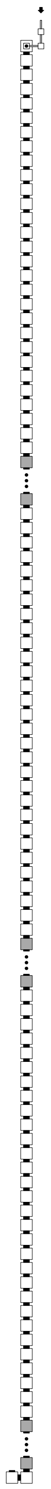

(b) Digit 1 - general overview

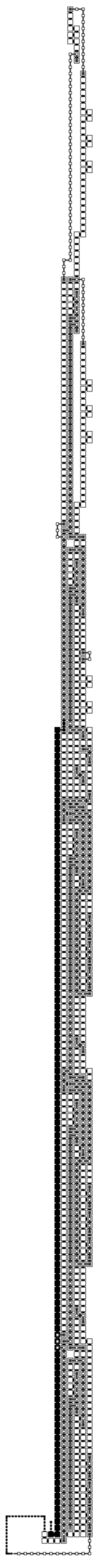

(c) Digit 2 - general overview

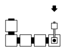

(d) Digit 1 - general (seed) overview

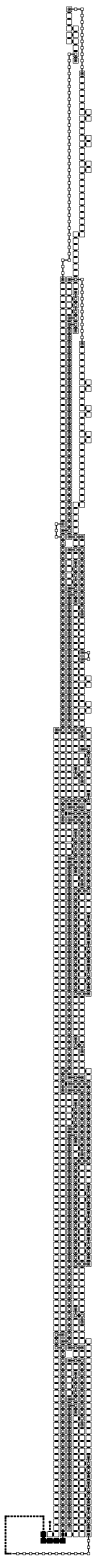

(e) Digit 2 - seed

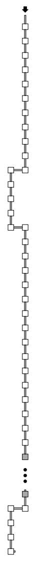

(f) Digit 2 - general (seed) overview

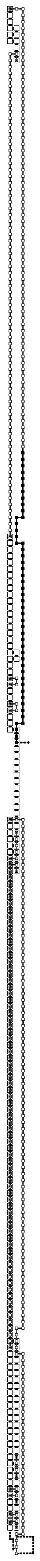

(g) Digit 3 - general

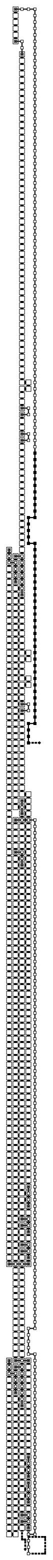

(h) Digit 3 - general overview

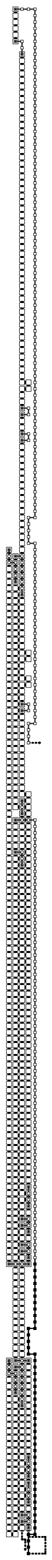

(i) Digit 3 - seed

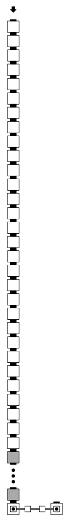

(j) Digit 3 - general (seed) overview

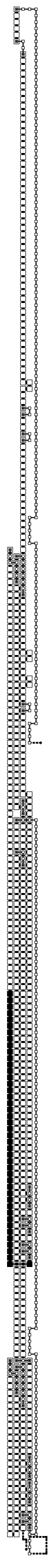

(k) Digit 1 - case 1, Digit 2 - case 2

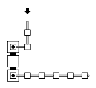

(l) Digit 1 - case 1 overview

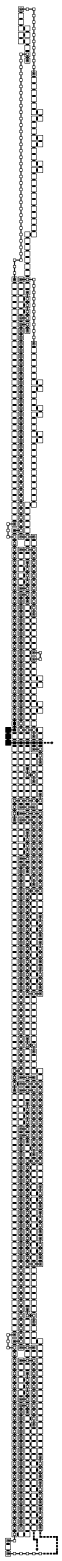

(m) Digit 2 - case 2 overview


(n) Digit 2 - case 2 (seed) overview


(o) Digit 1 - case 2


(p) Digit 1 - case 2 overview


(q) Digit 3 - case 3


(r) Digit 3 - case 3 overview

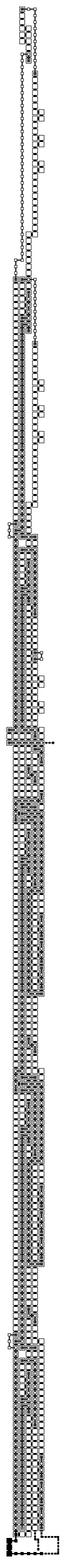

(s) Digit 3 - case 3 (seed) overview

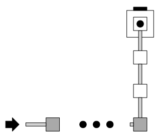
Figure 16: The Next_Read gadgets.

2.4.7 Cross over gadget

The idea this gadget is to assemble after reading the MSD, routing the counter back to the start of the next row, in position for the counter to begin reading the first digit. The number of tiles shaded in darker grey is $6 \cdot \left\lfloor \frac{d}{3} \right\rfloor$.

For each $op \in \{\texttt{increment}, \texttt{copy}\}$:

- Create `Cross_Next_Row`($\langle \texttt{CrossNextRow}, op \rangle$, $\langle \texttt{CounterRead}, 1, \lambda, op \rangle$) from the gadget shown in Figure 17a.


(a) General

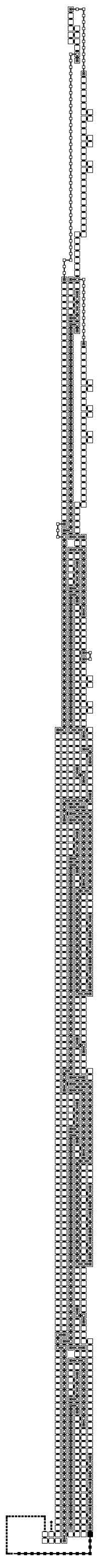
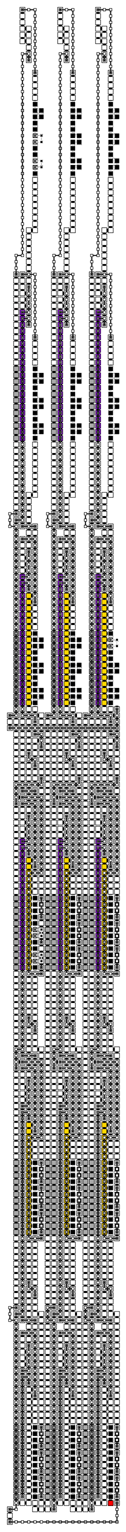
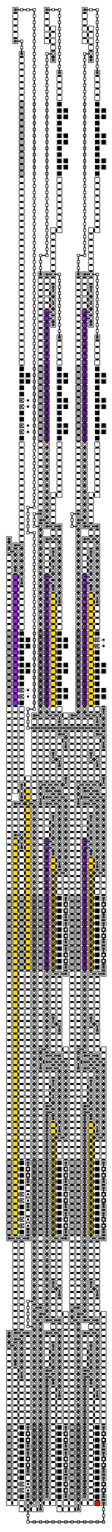
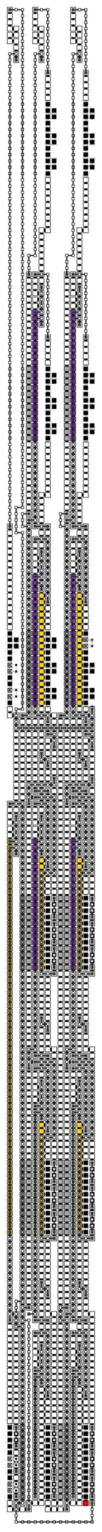

(b) General overview

Figure 17: The `Cross_Next_Row` gadget.

2.5 Overviews


(a) Full overview case 3


(b) Full overview case 2


(c) Full overview case 1