UNIT-2

Laplace Transformation

Let $f(t)$ be function defined for all +ve values of t than its laplace transform is denoted by

unknown

$$\boxed{L[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

$$L[1] = \int_0^\infty e^{-st} \cdot 1\, dt \Rightarrow \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$\Rightarrow -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s}[0-1] = \frac{1}{s}$$

$f(t)$ should be well defined.

Let c be any ~~an Constant~~ integers

$$\boxed{L[c] = \frac{c}{s}}$$

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot dt$$

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Formula

~~Standard Formula of Differential eq^n~~

$$= -\left[\frac{e^{-(s-a)t}}{(s-a)}\right]_0^{\infty} \Rightarrow \frac{-1}{(s-a)}\left[e^{-\infty} - e^{-0}\right]$$

$$= \frac{-1}{s-a}[0-1] = \frac{1}{s-a}$$

$$\boxed{L[e^{at}] = \frac{1}{s-a}}$$

~~Sinat = e~~

$$\sin h\, at = \frac{e^{at} - e^{-at}}{2}$$

$$L[\sin at] = L\left[\frac{e^{iat} - e^{-iat}}{2i}\right]$$

$$= \frac{1}{2i}\{L[e^{iat}] - L[e^{-iat}]\}$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ia-s+ia}{s^2+a^2}\right] = \frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right]$$

$$\boxed{L[\sin at] = \frac{a}{s^2+a^2}}$$

$$\boxed{L[t^n] = \frac{n!}{s^{n+1}} \;;\; n = 0,1,2,3}$$

$$\boxed{L[\cos at] = \frac{s}{s^2+a^2}}$$

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$$\boxed{L[\cosh at] = \frac{s}{s^2-a^2}}$$

$$\boxed{L[\sinh at] = \frac{a}{s^2-a^2}}$$

Q find laplace of $f(t) = \frac{t}{K}$; when $0<t<k$

1 ; when $t>k$

W.K.T Laplace transformation of a func. of t i.e. $f(t)$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt.$$

$$= \int_0^k e^{-st} \cdot \frac{t}{k}\,dt + \int_k^\infty e^{-st} \cdot 1\,dt$$

$$= \frac{1}{K}\left[\left(t\frac{e^{-st}}{-s}\right)_0^k - \int_0^k \frac{e^{-st}}{-s}\,dt\right] + \left[\frac{e^{-st}}{-s}\right]_k^\infty$$

$$= \frac{1}{K}\left[K\frac{e^{-Ks}}{-s} - \left[\frac{e^{-st}}{s^2}\right]_0^k\right] + \left[\frac{e^{-\infty}}{-s} - \frac{e^{-sK}}{-s}\right]$$

$$= \frac{1}{k}\left[k\frac{e^{-ks}}{-s} - \left[\frac{e^{-ks}}{s^2} - \frac{e^{-0}}{s^2}\right]\right] + \left[0 + \frac{e^{-ks}}{s}\right]$$

$$= -\frac{e^{-ks}}{ks^2} + \frac{1}{ks^2}$$

$$= \frac{1-e^{-ks}}{ks^2} \text{ Ans}$$

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H-W

Q $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ t & 1 \le t < 2 \\ t^2 & 2 \le t < \infty \end{cases}$

H-W

Q $f(t) = \begin{cases} t-1 & 1 < t \le 2 \\ 3-t & 2 < t < 3 \end{cases}$

Q $L[1 + \sin 2t]$

$= L[1] + L[\sin 2t]$

$= \frac{1}{s} + \frac{2}{s^2+4}$

Property:

$$\boxed{L[af_1(t) + bf_2(t)] = aL[f_1(t)] + bL[f_2(t)]}$$

* First shifting Theorem :-

If $L[f(t)] = f(s)$

then, $\boxed{L[e^{at} f(t)] = f(s-a)}$

- $L[e^{at} \cdot t^n] = \frac{n!}{(s-a)^{n+1}}$; $n = 0, 1, 2, 3$
- $L[e^{at} \cdot \sin bt] = \frac{b}{(s-a)^2 + b^2}$
- $L[e^{at} \cdot \cos bt] = \frac{s-a}{(s-a)^2 + b^2}$

~~L [sinh b~~

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- $L[e^{at} \sinh bt] = \frac{b}{(s-a)^2 - b^2}$
- $L[e^{at} \cosh bt] = \frac{s-a}{(s-a)^2 - b^2}$

Q Find Laplace of 2 sin 2t cos t

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$+ \qquad + \qquad +$$

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$$

$$A = 2t, \; B = t$$

$$2\sin 2t \cos t = \sin(3t) + \sin(t)$$

$$L[2\sin t \cos t] = L[\sin 3t] + L[\sin t]$$

$$\Rightarrow \frac{3}{s^2+9} + \frac{1}{s^2+1}$$

Q ~~$t + t^2 + t^3$~~ 4 cosh 2t sin 4t

$$4\left[\frac{e^{2t}+e^{-2t}}{2}\right]\sin 4t$$

$$2\,L[e^{2t}\sin 4t + e^{-2t}\sin 4t]$$

$$2\left[\frac{4}{(s-2)^2+16} + \frac{4}{(s+2)^2+16}\right]$$

Q $L[\sin 2t \sin 3t]$

Sol $L\left[\frac{e^{i2t}-e^{-i2t}}{2i}[\sin 3t]\right]$

$$\Rightarrow \frac{1}{2i}[e^{i2t}\cdot\sin 3t - e^{-i2t}\sin 3t]$$

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Q $f(t) = 1 \quad 0 \le t < 1$

$\qquad t \quad 1 < t < 2$

$\qquad t^2 \quad 2 \le t < \infty$

Sol

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$= \underbrace{\int_0^1 e^{-st} \cdot 1 \, dt}_{I} + \underbrace{\int_1^2 e^{-st} \cdot t}_{II} + \underbrace{\int_2^\infty e^{-st} \cdot t^2}_{III}$$

$$I = \cancel{\left[\frac{e^{-st}}{-s}\right]_0^1} \int_0^1 e^{-st} \cdot 1 \, dt = \left[\frac{e^{-st}}{-s}\right]_0^1$$

$$= -\frac{1}{s}\left[e^{-1} - e^0\right]$$

$$-\frac{1}{s}\left\{\frac{1}{e} - 1\right]$$

$$= \frac{e-1}{se}$$

$$II = \int_1^2 e^{-st} \cdot t = \left[t \frac{e^{-st}}{s}\right]_1^2 - \int 1 \cdot \frac{e^{-st}}{-s} \cdot dt$$

$$= \cancel{\frac{t}{s}[e^{-2}} \left[2\frac{e^{-2}}{-s} - \frac{e^{-1}}{-s}\right] - \left[\frac{e^{-st}}{s^2}\right]_1^2$$

$$II = 2\frac{e^{-2}}{-s} + \frac{e^{-1}}{s} - \frac{e^{-2}}{s^2} + \frac{e^{-1}}{s^2}$$

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Q Laplace of ~~cos2t~~ $L[\cos^2 t]$

$$L\left[\frac{1+\cos 2t}{2}\right]$$ $2\cos^2\theta - 1 =$

L

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\frac{1}{2} L[1] + \frac{1}{2} L[\cos 2t]$$

$$\frac{1}{2} \times \frac{1}{s} + \frac{1}{2} \frac{s}{s^2+4}$$ Ans

Q $L[t^3 e^{-2t}]$

Using First Shifting Theorem.

$$L[e^{-2t} t^3] \cdot \quad L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

$$L[e^{-2t} t^3] = \frac{6}{(s+2)^4}$$ by F.S.T

Q $L\{t \sin at\}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{t \sin at\} = (-1)^1 \frac{d}{ds} F(s)$

$= (-1)\cdot \frac{d}{ds} \frac{a}{s^2+a^2} \Rightarrow -a \cdot \frac{d}{ds} \frac{1}{s^2+a^2}$

$= \frac{2as}{(s^2+a^2)^2}$

Q $t^2 e^t \sin 4t$

$L\{\sin 4t\} = \frac{4}{s^2+16}$

$L\{e^t \sin 4t\} = \frac{4}{(s-1)^2+16}$

$L\{t^2 e^t \sin 4t\} = (-1)^2 \frac{d^2}{ds^2} \left\{ \frac{4}{(s-1)^2+16} \right\}$

$= 4 \frac{d^2}{ds^2} \frac{1}{(s-1)^2+16}$

$= 4 \frac{d}{ds} \frac{-2(s-1)\cancel{(s-1)^2+16)\times 0}}{[(s-1)^2+16]^2}$ $0 + 1 \cdot 2(s-1)$

$\frac{d}{ds}\cancel{2(s-1)}$

$\frac{d}{ds}$

$= -8 \frac{d}{ds} \frac{(s-1)}{[(s-1)^2+16]^2}$

Property :- Laplace Transform of $t^n f(t)$

If $L[f(t)] = f(s)$ $\quad (n>0)$

then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

Q $f(t) = t e^{-t} \sin 2t$

Right to left.

$L[\sin 2t] = \frac{2}{s^2+4}$

$L[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+4}$

$L[t e^{-t}\sin 2t] = (-1)^1 \cdot \frac{d}{ds} \cdot \left\{\frac{2}{(s+1)^2+4}\right\}$

$= -2 \cdot \frac{d}{ds} \frac{1}{s^2+2s+5}$

$= \frac{-2(-2)(s+1)}{((s+1)^2+4)^2} = \frac{4(s+1)}{[(s+1)^2+4]^2}$

$$\Rightarrow \frac{1}{2i}\left[\frac{3}{(s-i2)^2+9} - \frac{3}{(s+i2)^2+9}\right]$$

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Q $e^{-t}\cos^2 t$

sol

$$e^{-t}\frac{(1+\cos 2t)}{2} = \frac{1}{2}\left[e^{-t} + e^{-t}\cos 2t\right]$$

$$\frac{1}{2}\left[\frac{1}{(s+1)} + \frac{s+1}{(s+1)^2+4}\right]$$

Q $\cos at \cdot \sinh at$

sol

$$\frac{e^{at}-e^{-at}}{2}\cdot \cos at \Rightarrow \frac{1}{2}\left[e^{at}\cdot\cos at - e^{-at}\cdot\cos at\right]$$

$$\frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right]$$

Propety: Change of Scale

if $L[f(t)] \longrightarrow f(s)$ then,

$$L[f(at)] \rightarrow \frac{1}{a} f\left(\frac{s}{a}\right)$$

Q if $L[\cos^2 t) = \dfrac{s^2+2}{s(s^2+4)}$

find $L(\cos^2 at)$

$$\Rightarrow \frac{1}{a}\left\{\frac{\left(\frac{s}{a}\right)^2+2}{\left(\frac{s}{a}\right)\left[\left(\frac{s}{a}\right)^2+4\right]}\right\}$$

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UNIT – 2
LAPLACE
Continue

Q $L\left[\frac{\cos at - \cos bt}{t}\right] = L\left[\frac{1}{t}\{\cos at - \cos bt\}\right]$

W.K.T $= L[\cos at] - L[\cos bt]$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left[\frac{1}{t}(\cos at - \cos bt)\right] = \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$s^2+a^2 = t$

$2s \cdot ds = dt$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$\frac{1}{\infty} = 0$

$$= \frac{1}{2}\log\left[\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left\{\log 1 - \log \frac{s^2+a^2}{s^2+b^2}\right\}$$

$$= -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

Property

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$\frac{1}{t} f(t) = ?$

** If $L[f(t)] = f(s)$ then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty f(s)\,ds$

Q $L\left[\frac{\sin 2t}{t}\right] = L\left[\frac{1}{t} \cdot \sin 2t\right]$

$L[\sin 2t] = \frac{2}{s^2+4}$

i.e. $L\left[\frac{1}{t} \sin 2t\right] = 2\int_s^\infty \frac{1}{s^2+2^2}\,ds$

$= 2 \cdot \frac{1}{2}\left[\tan^{-1}\left[\frac{s}{2}\right]\right]_s^\infty$

$=$

$\tan^{-1}\infty - \tan^{-1}\frac{s}{2}$

$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$

$= \cot^{-1}\frac{s}{2}$

$( )^{-2-1}$ $( )^{-3}$ -2

$$= -8 \times \left\{ \frac{[(s-1)^2+16]^2 - 4(s-1)^2[(s-1)^2+16]}{\{[(s-1)^2+16]\}^4} \right\}$$

$$\frac{-8 \times \cancel{(s-1)^2+16}}{((s-1)^2+16)^3}\left[(s-1)^2+16 - 4(s-1)^2\right]$$

$$= \frac{24}{\{(s-1)^2+16\}^3}\left[\cancel{3}(s-1)^2 + 16/3\right]$$

Q $L[t^2 e^{2t} \sin t]$

Sol $L\{\sin t\} = \frac{1}{s^2+1}$

$L\{e^{2t}\sin t\} = \frac{1}{(s-2)^2+1}$

$L[t^2 e^{2t} \sin t] = \frac{d^2}{ds^2}\left[\frac{1}{(s-2)^2+1}\right]$

$x \to \infty$

$\frac{x}{e^x}$ $\frac{\infty}{e^\infty}$ $\frac{1}{e^\infty}$ $\frac{\infty}{\infty}$

$\frac{1}{\infty} = 0$

Q. $\frac{1}{t}[1-\cos at]$

$$L\left[\frac{1}{t}(1-\cos at)\right]$$

$$= L\left[\frac{1}{t}\cdot 1\right] - L\left[\frac{1}{t}\cos at\right]$$

$$\Rightarrow L[1] = \frac{1}{s}.$$

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$= L\left[\frac{1}{t}\right] = \int_s^\infty \frac{1}{s} \quad = [\log s]_s^\infty$$

$$L\left[\frac{1}{t}\cos at\right] = \int_s^\infty \frac{s}{s^2+a^2} \quad \cancel{= \frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty}$$

$$= \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$$

$$\Rightarrow [\log s]_s^\infty - \frac{1}{2}[\log(s^2+a^2)]_s^\infty$$

$$\left[\frac{2\log s}{2} - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+a^2$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+a^2}\right]_s^\infty$$

Q 4 (iii) $\frac{d^4y}{dx^4} - y = x\sin x$

$D^2 = -1$

$(D^4 - 1^4)y = x\sin x$

$D^4 - 1^4$,

$(D^2-1^2)(D^2+1^2)$

$\downarrow \quad \downarrow$

$1, -1 \quad -i, i$

$\pm i$

$m^4 - 1 = 0$

$m^4 = 1$

$m = 1, 1, 1, 1$

$CF = e^x + e^{-x} + e^0(\cos x + \sin x)$

Q Laplace of $\frac{1-\cos t}{t^2}$

$$L\left[\frac{1-\cos t}{t^2}\right] = L\left[\frac{1}{t}\left[\frac{1}{t}(1-\cos t)\right]\right]$$

$= \frac{d}{t}$ w r t

$$L\left[\frac{1}{t}(1-\cos t)\right] = L[1] - L[\cos t]$$
$$= \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left[\frac{1}{t}(1-\cos t)\right] = \int_s^\infty \frac{1}{s} - \frac{s}{s^2+1}$$

$$" = [\log s]_s^\infty - \frac{1}{2}\log[s^2+1]_s^\infty$$

$$" = \left[\frac{2\log s}{2} - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$" = \left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$'' = \frac{1}{2}\left[\log \frac{s^2}{s^2+1}\right]_s^\infty$$

$$'' = \frac{1}{2}\left[\log \frac{1}{1+\frac{1}{s^2}}\right]_s^\infty$$

$$'' = \frac{1}{2}\left[\log 1 - \log \frac{s^2}{s^2+1}\right]$$

$$\log\left[\frac{1}{t}(1-\cos t)\right] = -\frac{1}{2}\log\frac{s^2}{s^2+1} = \frac{1}{2}\log\frac{s^2+1}{s^2}$$

$$L\left[\frac{1}{t}\left[\frac{1}{t}(1-\cos t)\right]\right]$$

$$= \frac{1}{2}\int_s^\infty \log\frac{s^2+1}{s^2}$$

$s \cdot \log s^2+1 - \int \frac{1}{s^2+1} \cdot s$

$\left[s\log(s^2+1)\right]_s^\infty - \left[\frac{1}{2}\log(s^2+1)\right]$

$$\frac{1}{2}\left[\int_s^\infty 1\cdot\log s^2+1 - \underbrace{\int_s^\infty 1\cdot\log s^2}\right]$$

$[s\cdot\log s^2]_s^\infty - \int \frac{1\cdot s}{s^2}$

$\log s$

$$-\left[s\log(s^2+1)\right]_s^\infty + \frac{1}{2}\left[\log(s^2+1)\right]_s^\infty$$

$$+\left[s\log(s^2+1)\right]_s^\infty - \frac{1}{2}\left[\log(s^2+1)\right]_s^\infty - \left[s\log s^2\right]_s^\infty + \left[\frac{2}{2}\log s\right]_s^\infty$$

$$+\left[s\{\log(s^2+1) - \log s^2\}\right]_s^\infty + \frac{1}{2}\left[\log(s^2+1) - \log(s^2+1)\right]_s^\infty$$

$$\left[s\left\{\log\frac{s^2+1}{s^2}\right\}\right]_s^\infty + \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty$$

$$s\left[s\log\frac{1+\frac{1}{s^2}}{1}\right]_s^\infty + \frac{1}{2}\left[\log\frac{1}{1+\frac{1}{s^2}}\right]_s^\infty$$

Property :-

If $L[f(t)] = f(s)$ then,

$$L[e^{at} f(t)] = f(s-a)$$

$$L[t^n \{f(t)\}] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$L\left[\frac{1}{t} f(t)\right] = \int_s^\infty f(s)\, ds$$

☆ then,

$$\boxed{L\left[\int_0^t f(t)\, dt\right] = \frac{1}{s} F(s)}$$

Q $f(t) = \int_0^t \frac{\sin at}{t}\, dt$ Laplace

$t \to 0$ mean it can be ∞

$$\text{w.r.t} = L[\sin at] = \frac{a}{s^2+a^2}$$

$$L\left[\frac{1}{t} \sin at\right] = \int_s^\infty \cancel{\sin at}\, \frac{a}{s^2+a^2} \cdot ds$$

$$= \left[\tan^{-1}\left[\frac{s}{a}\right]\right]_s^\infty$$

$$= \cot^{-1}\left[\frac{s}{a}\right]$$

By first shifting theorem]

$$L\left[e^{-4t}\frac{1}{t}\sin 3t\right] = \cot^{-1}\left[\frac{(s+4)}{3}\right]$$

Q $\frac{1}{t}\left[e^{-at} - e^{-bt}\right]$

$$L\left[e^{-at} - e^{-bt}\right] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\left[\frac{1}{t}e^{-at}\cdot 1 - \frac{1}{t}e^{-bt}\cdot 1\right]$$

$$L[1] = \frac{1}{s}$$

$$L\left[\frac{1}{t}\cdot 1\right] = \int_s^{\infty}\frac{1}{s} = \left[\log s\right]_s^{\infty}$$

$$L\left[\frac{1}{t}\left[\frac{1}{s+a} - \frac{1}{s-b}\right]\right]$$

$$= \int_s^{\infty}\frac{1}{s+a} - \int_s^{\infty}\frac{1}{s+b}$$

$$= \left[\log(s+a)\right]_s^{\infty} - \left[\log(s+b)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^{\infty} \qquad \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)_s^{\infty}$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right) \qquad \log\left[\frac{1+\frac{a}{s}\infty}{1+\frac{b}{s}\infty}\right]$$

$$= \log\frac{s+b}{s+a} \qquad \log[1] = 0$$

$$\left[\infty \times 0 - s \cdot \log \frac{s^2+1}{s^2}\right] + \frac{1}{2}\left[0 - \log \frac{s^2}{s^2+1}\right]$$

$$- s \log \frac{s^2+1}{s^2} \mp \frac{1}{2} \log \frac{s^2}{s^2+1}$$

$$- s\log \frac{s^2+1}{s^2} + \frac{1}{2} \log \frac{s^2+1}{s^2}$$

$$\log \left[\frac{s^2+1}{s^2}\right] \left\{\frac{1}{2} - s\right\}$$

$$\left(\frac{1-2s}{2}\right) \left[\log \frac{s^2+1}{s^2}\right]$$

Q $L\left[\frac{1}{t} e^{-4t} \sin 3t\right]$

Sol $L\{\sin 3t\} = \frac{3}{s^2+9}$

$$L\left[\frac{1}{t} \sin 3t\right] = \int_s^\infty \frac{3}{s^2+3^2} \, ds$$

$$\frac{3 \times 1}{3}\left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

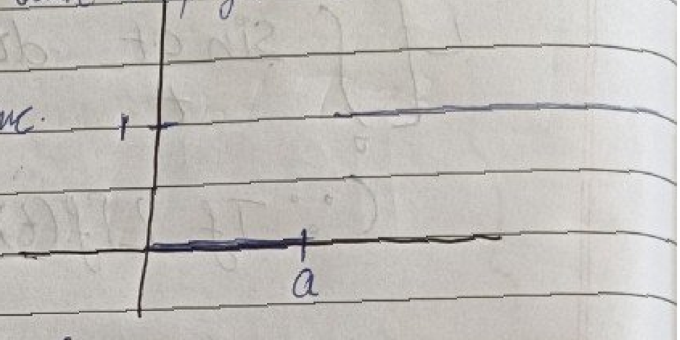
$$\frac{\pi}{2} - \tan^{-1} \frac{s}{3}$$

$$= \cot^{-1} \frac{s}{3}$$

Unit Step function
(U.S.f)

Q $u(t-a) = \begin{cases} 0 ; & t<a \\ 1 ; & t \geq a \end{cases}$

~~If unit step funct~~ unit step function is ats



Laplace of unit step func.

* $L[u(t-a)] =$

$$\int_0^a e^{-st} \, 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt \qquad \left\{ L[f(t)] = \int_0^\infty e^{-st} f(t) dt \right.$$

$$= \int_a^\infty e^{-st} \cdot 1 \, dt = -\left[\frac{e^{-st}}{s}\right]_a^\infty$$

First shifting
↓
$L[e^{at} f(t)] = f(s-a)$

$$= -\left[\frac{e^{-\infty}}{s} - \frac{e^{-as}}{s}\right] = \frac{e^{-as}}{s}$$

Second Shifting Theorem

If : $L[f(t)] = f(s)$

then

$$\boxed{L[\underbrace{f(t-a) \cdot u(t-a)}_{\text{these func. shoulb be same}}] = e^{-as} \cdot F(s)}$$

Q $f(t) = \begin{cases} t-1 & , 1<t<2 \\ 3-t & , 2 \leq t < 3 \end{cases}$; and find its Laplace transform

given : unit step func is defined at both 1 & 2

$$\underset{\downarrow f(t)}{(t-1)} \cdot [u(t-1) - u(t-2)] + \underset{\downarrow f(t)}{(3-t)} \cdot [u(t-2) - u(t-3)]$$

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~~Now, $L\left[\int_0^t \cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{s}\cot^{-1}\left(\frac{s}{a}\right)$~~

Now,

$$L\left[\int_0^t \frac{\sin at}{t} \cdot dt\right] = \frac{1}{s}\cot^{-1}\left(\frac{s}{a}\right)$$

$\left(\because \text{If } L[f(t)] = f(s), \text{ Then } L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s)\right)$

Q. Evaluate $L\int_0^t t \cdot e^{3t}\sin t\,dt$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[t\sin t] = (-1)\frac{d}{ds}\frac{1}{(s^2+1)}$$

$\left\{\begin{array}{l}\text{If } L[f(t) = f(s)] \text{ Then} \\ L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)\end{array}\right\}$

$$= +\frac{2s}{(s^2+1)^2}$$

$$L[e^{-3t} \cdot t\sin t] = \frac{2(s+3)}{((s+3)^2+1)^2} \quad f(s)$$

$$L\left[\int_0^t t \cdot e^{-3t}\sin t\,dt\right] = \frac{1}{s} \times \frac{2(s+3)}{((s+3)^2+1)^2}$$

~~Now, $L\left[\int_0^t \cot^{-1}\left[\frac{s}{a}\right]\right] = \frac{1}{s}\cot^{-1}\left(\frac{s}{a}\right)$~~

Now,

$$L\left[\int_0^t \frac{\sin at}{t}\,dt\right] = \frac{1}{s}\cot^{-1}\left(\frac{s}{a}\right)$$

$\left(\because \text{If } L[f(t)] = f(s), \text{Then } L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s)\right)$

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$$= + \frac{1\cdot 2s}{(s^2+1)^2}$$

$$L[e^{-3t}\cdot t\sin t] = \frac{2(s+3)}{((s+3)^2+1)^2}$$

$f(s)$

$$L\left[\int_0^t t\cdot e^{-3t}\sin t\,dt\right] = \frac{1}{s}\times\frac{2(s+3)}{((s+3)^2+1)^2}$$

$= (t-1)\cdot u(t-1) - (t-1)\, u(t-2) + (3-t)\, u(t-2) - (3-t)\, u(t-3)$

$= (t-1)\, u(t-1) + [-t+1+3-t]\cdot u(t-2) + (t-3)\, u(t-3)$

$= \underbrace{(t-1)\cdot u(t-1)}_{\text{usf at 1}} - 2\underbrace{(t-2)\, u(t-2)}_{\text{usf at 2}} + \underbrace{(t-3)\cdot u(t-3)}_{\text{usf at 3}}$

Now, applying second shifting theorem (ASST)

$L[(t-1)\cdot u(t-1)] - 2L[(t-2)\cdot u(t-2)] + L[(t-3)\cdot u(t-2)]$

$L[f(t)] = e^{-s} L[t] - 2e^{-2s}\cdot L[t] + e^{-3s}\cdot L[t]$

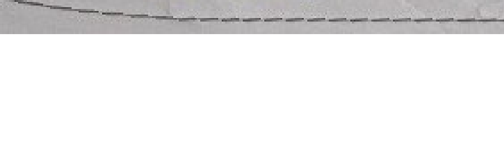
$= e^{-s}\cdot \frac{1}{s^2} - 2e^{-2s}\cdot \frac{1}{s^2} + e^{-3s}\cdot \frac{1}{s^2}$

Periodic function

Let $f(t)$ be periodic function with period T then

* $$\boxed{L[f(t)] = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}}$$

Periodic ↓



$\sinh x = \frac{e^x - e^{-x}}{2}$

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Q $f(t) = \left(\frac{2t}{3}\right)$; $0 \le t \le 3$

Solve laplace of this periodic func.

Sol given func of $f(t)$ has period $T = 3$

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

$$= \frac{1}{1-e^{-3s}} \times \frac{2}{3} \int_0^3 e^{-st} \cdot t \cdot dt$$

$$= \frac{2}{3}\left[\frac{1}{1-e^{-3s}}\right] \int_0^3 \underset{I}{t} \cdot \underset{II}{e^{-st}} \cdot dt$$

$$= \text{,,} \left[t \cdot \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s}\right]$$

$$\text{,,} \left[\frac{t \cdot e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3} \times \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{e^{-0}}{s^2}\right]$$

$$= \frac{2}{3}\left(\frac{1}{1-e^{-3s}}\right)\left[\frac{1 - e^{-3s} - 3se^{-3s}}{s^2}\right]$$

$m = a \pm \sqrt{b}$ $\quad m = a \pm \sqrt{b}$

classmate

$e^{ax}(C_1 \cosh \sqrt{b}x + C_2 \sinh \sqrt{b}x) = \quad C_1 e^{a+\sqrt{b}} + C_2 e^{a-\sqrt{b}} \quad \cosh = e^{x}$

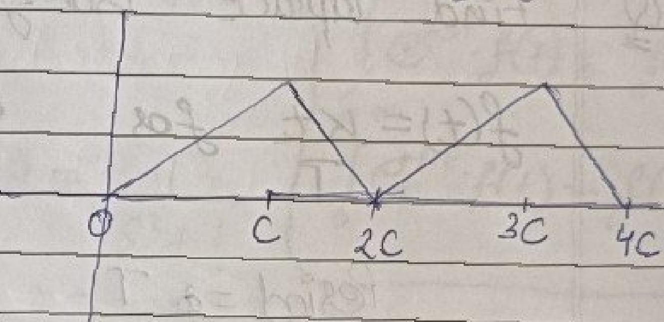
transformation $\quad C_1 e^{a} \cdot e^{\sqrt{b}} + C_2 e^{a} \cdot \frac{1}{e^{\sqrt{b}}}$

Q find the laplace transformation of triangular wave func.

$$f(t) = \begin{cases} t & ; 0 \le t < c \\ 2c - t & ; c \le t \le 2c \end{cases}$$

since $f(t)$ is periodic
it has period $2c$ $\quad$ if $c = 1$, t, $2-t$

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$



$$= \frac{1}{1 - e^{-2cs}} \left[\int_0^c e^{-st} \cdot t + \int_c^{2c} e^{-st}(2c - t) \right] dt$$

$$= \frac{1}{1 - e^{-2cs}}$$ Continue later

Q

$$f(t) = \begin{cases} t & , 0 \le t \le \pi \\ \pi - t & \pi \le t \le 2\pi \end{cases}$$

Period $= 2$, w.r.t, L.T. of $f(t)$ is

$$f(t) = \frac{1}{1 - e^{-2s}} \left[\int_0^{\pi} e^{-st} \cdot t + \int_{\pi}^{2\pi} e^{-st} \cdot (\pi - t) \right]$$

$$\left[t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\pi} + \left[(\pi - t) \cdot \frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_{\pi}^{2\pi}$$

$$= [\tan^{-1} s]_s^{\infty}$$

$$= \pi/2 - \tan^{-1} s$$

$$= \cot^{-1} s$$

Now $\int_0^{\infty} \frac{\sin t}{t} \cdot dt = \int_0^{\infty} e^{-0t} \frac{\sin t}{t} \cdot dt = \cot^{-1}(0) = \frac{\pi}{2}$

Q Evalute $\int_0^{\infty} e^{-t} \frac{\sin t}{t} \cdot dt$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left[\frac{1}{t}\sin t\right] = \int_s^{\infty} \frac{1}{s^2+1}$$

$$= [\tan^{-1} s]_s^{\infty} = \cot^{-1} s$$

$$\int_s^{\infty} e^{-t} \frac{\sin t}{t} \cdot dt = \cot^{-1}(1) = \pi/4$$

Q $\int_0^{\infty} t \cdot e^{-4t} \cdot \sin t \cdot dt$

$$L[\sin t \cdot dt] = \frac{1}{s^2+1}$$

$$L\{t \cdot \sin t\} = -1 \cdot \frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= +1 \frac{2s}{(s^2+1)^2}$$

W.R Evaluate of integrals

$\int_0 e^{st} f(t)$

We can evaluate number of integrals having lower limit 0 & upper limit ∞ by the help of laplace transform

Q Evaluate $\int_0^\infty t e^{-3t} \sin t \cdot dt$.

$$= \int_0^\infty e^{-3t} \cdot t \sin t \cdot dt$$

{ With help of laplace
$\int_0^\infty e^{-st} \cdot f(t) = f(s)$
$\int_0^\infty e^{kt} \cdot f(t) = f(k)$ }

W.R.T. $L[\sin t] = \frac{1}{s^2+1}$

$$L[t \cdot \sin t] = (-1)^1 \cdot \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2s}{(s^2+1)^2}$$

Now $\left[\int_0^\infty e^{-3t} t \sin t \cdot dt\right]$

~~s=3 Replace~~ { Replacing s by 3 }

$$= \frac{2 \times 3}{(3^2+1)^2} = \frac{3}{50}$$

Q Evalute $\int_0^\infty \frac{\sin t}{t} \cdot dt$

$L[\sin t] = \frac{1}{s^2+1}$

$$L\left[\frac{1}{t} \sin t\right] = \int_s^\infty \frac{1}{s^2+1} \cdot ds$$

$$\frac{1}{1-e^{-2\pi s}}\left[-\frac{\pi e^{-s\pi}}{s}-\frac{e^{-\pi s}}{s^2}+\frac{1}{s^2}\right]+\left[\frac{\pi e^{-2\pi s}}{s}+\frac{e^{-2\pi s}}{s^2}-\frac{e^{-\pi s}}{s^2}\right]$$

$$\frac{1}{1-e^{-2\pi s}}\left[\frac{1}{s^2}-\frac{2e^{-\pi s}}{s^2}+\frac{e^{-2\pi s}}{s^2}-\frac{\pi e^{-s\pi}}{s}+\frac{\pi e^{-2s\pi}}{s}\right]$$

Q Find laplace transformation of

$f(t)=\frac{Kt}{T}$ for $0<t<T$; $f(t+T)=f(t)$.

it means func. is period of T

Period $= T$

$$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}\,\frac{Kt}{T}$$

$$\frac{K}{T(1-e^{-sT})}\int_0^T e^{-st}\cdot t$$

$$\frac{K}{T(1-e^{-sT})}\left[\frac{t\cdot e^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^T$$

$$\frac{K}{T(1-e^{-sT})}\left[-\frac{T\cdot e^{-sT}}{s}-\frac{e^{-sT}}{s^2}+\frac{1}{s^2}\right]$$

$$\frac{K}{T(1-e^{-sT})}\left\{\frac{1-e^{-sT}}{s^2}-\frac{Te^{-sT}}{s}\right\}$$

Inverse Laplace

- $L[f(t)] = f(s)$ | $L^{-1}[f(s)] = f(t)$
- $L[e^{at}] = \frac{1}{s-a}$ | $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
- $L[\sin at] = \frac{a}{s^2+a^2}$ | $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$ OR $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$
- $L[\cos at] = \frac{s}{s^2+a^2}$ | $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
- $L[\sinh at] = \frac{a}{s^2-a^2}$ | $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$
- $L[\cosh at] = \frac{s}{s^2-a^2}$ | $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

explanation at end * $L[t^n] = \frac{n!}{s^{n+1}} = \frac{\overline{|n+1}}{s^{n+1}}$ | $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

— By First shifting theorem

- $L[e^{at}f(t)] = f(s-a)$ | $L^{-1}[f(s-a)] = e^{at}f(t)$

⇒ $L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$ ⇒ $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$

⇒ $L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}$ ⇒ $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

⇒ $L[e^{at}\sinh bt] = \frac{b}{(s-a)^2-b^2}$ ⇒ $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b}e^{at}\sinh bt$

Convolution Theorem

* If $L[f_1(t)] = f_1(S)$

$L[f_2(t)] = f_2(S)$

then, $L\left[\int_0^t f_1(x) \cdot f_2(t-x)\,dx\right] = F_1(S) \cdot F_2(S)$

Q Find laplace transformation of

$$\int_0^t e^x \sin(t-x) \cdot dx$$

Sol

Comparing the given problem with the Convolution theorem

$$L\left[\int_0^t f_1(x) f_2(t-x) \cdot dx\right] = f_1(S) \cdot f_2(S)$$

where $f_1(S) = L[f_1(t)]$
$f_2(S) = L[f_2(t)]$

i.e.

$f_1(x) = e^x \Rightarrow f_1(t) = e^t$

& $f_2(t-x) = \sin(t-x)$ $\quad f_2(t) = \sin t$

Now $L[f_1(t)] = L[e^t] = \dfrac{1}{S-1}$

$L[f_2(t)] = L[\sin t] = \dfrac{1}{S^2+1}$

i.e.

$$L\left[\int_0^t e^x \sin(t-x)\,dx\right] = \frac{1}{S-1} \cdot \frac{1}{S^2+1}$$

$$\int_0^\infty e^{-4t} \cdot t \sin t \cdot dt = \frac{2\times -4}{(16+1)^2} = -\frac{8}{289}$$

Q $$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \cdot dt$$

$$L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{(s+b)}$$

$$L\left\{\frac{1}{t}(e^{-ab} - e^{-bt})\right\} = \int_s^\infty \frac{1}{s+a} - \frac{1}{s+b}$$

$$= \log\frac{(s+b)}{(s+a)}$$

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \cdot dt = \log\left(\frac{b}{a}\right).$$

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$$-6L^{-1}\left[\frac{s/16}{s^2+9/16}\right]$$

$$=3e^{3/2}-\frac{1}{4}\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4t}{3}$$

$$+\frac{8}{16}\times\frac{4}{3}\sin\frac{3}{4}t-\frac{6}{16}\cos\frac{3t}{4}$$

$$=3e^{3/2}-\frac{1}{4}\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4}{3}t+\frac{2}{3}\sin\frac{3t}{4}-\frac{3}{8}\cos\frac{3t}{4}$$

Q $L^{-1}\left[\frac{1}{s^n}\right]$ exist only when n is

Q $L^{-1}\left[\frac{2s-5}{4s^2+25}+\frac{4s-18}{9-s^2}\right]$

$$=2L^{-1}\left[\frac{s}{4s^2+25}\right]-5L^{-1}\left[\frac{1}{4s^2+25}\right]-4L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$+18L^{-1}\left[\frac{1}{s^2-9}\right]$$

$$=\frac{2}{4}L^{-1}\left[\frac{s}{s^2+25/4}\right]-\frac{5}{4}L^{-1}\left[\frac{1}{s^2+25/4}\right]-4L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$+18L^{-1}\left[\frac{1}{s^2-9}\right]$$

$$=\frac{2}{4}\cos\frac{5}{2}t-\frac{5}{4}\times\frac{2}{5}\sin\frac{5}{2}t-4\cosh 3t+18\times\frac{1}{3}\sinh 3t$$

Q $L^{-1}\left[\frac{s+2}{(s+2)^2-25}\right]$

By first shifting theorem

$e^{-2t}\cosh 5t$

Q $\frac{s-1}{(s-1)^2+4} = e^{t}\cos 2t$ Ans

$\downarrow$

$e^{t}\cdot L^{-1}\left[\frac{s}{s^2+2^2}\right] = e^{t}\cos 2t$

Q $L^{-1}\left[\frac{s^2+2s+6}{s^3}\right]$

Sol

$L^{-1}\left[\frac{s^2}{s^3}\right]+2L^{-1}\left[\frac{s}{s^3}\right]+6L^{-1}\left[\frac{1}{s^3}\right]$

$= 1+2t+\frac{6t^2}{2!} \Rightarrow 1+2t+3t^2$

Q $L^{-1}\left[\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}\right]$

$L^{-1}\left[\frac{6}{2s-3}\right]-L^{-1}\left[\frac{3+4s}{9s^2-16}\right]+L^{-1}\left[\frac{8-6s}{16s^2+9}\right]$

$= L^{-1}\left[\frac{3}{s-3/2}\right]-L^{-1}\left[\frac{3}{9s^2-16}\right]-4L^{-1}\left[\frac{s}{9s^2-16}\right]+L^{-1}\left[\frac{8}{16s^2+9}\right]$

$-6L^{-1}\left[\frac{s}{16s^2+9}\right]$

$= 3\cdot e^{3/2} - L^{-1}\left[\frac{3/9}{s^2-16/9}\right]-4L^{-1}\left[\frac{s/9}{s^2-16/9}\right]+8L^{-1}\left[\frac{1/16}{s^2+9/16}\right]$

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s) \Rightarrow L^{-1}[\,] = t^n f(t)$

Similarly all other formulae

$\Rightarrow \cdot\ L[e^{at}\cosh bt] = \frac{s-a}{(s-a)^2-b^2}$	$\Rightarrow L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at}\cosh bt.$
Same explanation at * $\Rightarrow \cdot\ L[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$ (Last Page)	$\Rightarrow L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{1}{(n-1)!}e^{at}t^{n-1}$

— Convolution

$\cdot\ L\left[\int_0^t f_1(x) f_2(t-x)\,dx\right] = f_1(s)\cdot f_2(s)$	$L^{-1}[f_1(s)\cdot f_2(s)] = \int_0^t f_1(x)\cdot f_2(t-x)\,dx$
$\cdot\ L\left[\frac{1}{t} f(t)\right] = \int_s^\infty f(s)\,ds$	$L^{-1}\left[\int_s^\infty f(s)\,ds\right] = \frac{1}{t} f(t)$
$\cdot\ L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)$	$= L^{-1}\left[\frac{1}{s} f(s)\right] = \int_0^t f(t)\,dt$

$L\left[\frac{\cancel{f}}{t}\right]$

Q $\frac{1}{s^2-9} = \frac{1}{s^2-3^2} = \frac{3}{3(s^2-3^2)}$

$L^{-1}\left[\frac{1}{s^2-9}\right] = \frac{1}{3} L^{-1}\left[\frac{3}{s^2-3^2}\right] = \frac{1}{3}\sinh 3t.$

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$= \frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin\frac{5}{2}t - 4\cosh 3t + 6\sinh 3t$

Q $L^{-1}\left[\frac{1}{4s} + \frac{16}{1-s^2}\right]$

$\frac{1}{4}L^{-1}\left[\frac{1}{s}\right] - 16\, L^{-1}\left[\frac{1}{s^2-1}\right]$

$\frac{1}{4} - 16 \times \frac{1}{1} \times \sinh t$

☆ $\boxed{L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(t) \cdot dt}$

Q $\frac{3(s^2-2)^2}{2s^5}$

Q $L^{-1}\left[\frac{1}{s(s+a)}\right]$

$\int_0^t L^{-1}[f(s)]$

w.k.T

$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

Now

$= \int_0^t L^{-1}[F(s)]dt$

Now $L^{-1}\left[\frac{1}{s(s+a)}\right]$

$= \int_0^t e^{-at}dt$

$= \left[\frac{e^{-at}}{-a}\right]_0^t = \frac{e^{-at}}{-a} + \frac{1}{a}$

Q $\dfrac{s^2+3}{s(s^2+9)} = \dfrac{s^2}{s(s^2+9)} + \dfrac{3}{s(s^2+9)}$

$= \dfrac{s}{s^2+9} + \dfrac{1}{s}\left(\dfrac{3}{s^2+9}\right)$

i.e.

$L^{-1}\left[\dfrac{s}{s^2+9}\right] + L^{-1}\left[\dfrac{1}{s} \cdot \dfrac{3}{s^2+9}\right]$

$= \cos 3t + \int_0^t \sin 3t$ $\qquad L^{-1}\left[\dfrac{3}{s^2+9}\right] = \sin 3t$

$= \cos 3t - \left[\dfrac{\cos 3t}{3}\right]_0^t$ $\qquad \int_0^t L^{-1}[f(s)]$

$= \cos 3t - \dfrac{\cos 3t}{3} + 1$

$= \dfrac{2}{3}\cos 3t + \dfrac{1}{3}$

Q $L^{-1}\left[\dfrac{1}{s(s^2+a^2)}\right]$

$L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at$

$\dfrac{1}{a}\int_0^t \sin at = \dfrac{1}{a}\left[\dfrac{\cos at}{a}\right]_0^t$

$= \dfrac{1}{a^2}\cos at - \dfrac{\cos at}{a^2}$

$= [1 - \cos at]\dfrac{1}{a^2}$

Q $\dfrac{1}{S(S^2-16)}$

Sol

$$L^{-1}\left[\frac{1}{S(S^2-4^2)}\right] = L^{-1}\left[\frac{1}{S^2-4^2}\right] = \frac{1}{4}\sinh 4t$$

$$\frac{1}{4}\int_0^t \sinh 4t$$

$$= \frac{1}{4}\left[\frac{\cosh 4t}{4}\right]_0^t$$

$$= \frac{1}{4}\frac{\cosh 4t}{4} - \frac{1}{16}$$

* First Shifting Theorem:

If $L[f(t)] = f(s)$, Then

$$L[e^{at} \cdot f(t)] = f(s-a)$$

For inverse laplace.

If $L^{-1}[f(s)] = f(t)$ then $L^{-1}[f(s-a)] = e^{at} L^{-1}[f(s)]$

Q $\dfrac{1}{(S+2)^5} \Rightarrow L^{-1}\left[\dfrac{1}{(S+2)^5}\right] = e^{-2t} L^{-1}\left[\dfrac{1}{S^5}\right]$

$$e^{-2t}\frac{t^4}{4!}$$

Since $L^{-1}\left[\dfrac{1}{s^{n+1}}\right] = \dfrac{t^n}{n!}$

Or

$L^{-1}\left[\dfrac{1}{S^n}\right] = \dfrac{t^{n-1}}{(n-1)!}$

Q $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

$$L^{-1}[f(s)] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right] \quad \text{——— (1)}$$

$$L^{-1}[\log(s+1) - \log(s-1)]$$

$$L^{-1}[\log(s+1)] - L^{-1}[\log(s-1)]$$

using — (1)

$$\Rightarrow -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log(s+1)\right] + \frac{1}{t} L^{-1}\left[\frac{d}{ds}\log(s-1)\right]$$

$$\Rightarrow -\frac{1}{t} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{t} L^{-1}\left[\frac{1}{s-1}\right]$$

$$\Rightarrow -\frac{1}{t}\cdot e^{-t} + \frac{1}{t} e^{t}$$

$$\Rightarrow \frac{1}{t}\left[e^{t} - e^{-t}\right]$$

Q $F(s) = \log\left(\frac{s+a}{s+b}\right)$ Find its ~~ILP~~ ILT

$$L^{-1}[\log(s+a)] - L^{-1}(s+b]$$

using — (1)

$$-\frac{1}{t} L^{-1}\left[\frac{1}{s+a}\right] + \frac{1}{t} L^{-1}\left[\frac{1}{s+b}\right]$$

Q $\frac{1}{2(s-1)^2+32}$

$\frac{1}{2} L^{-1}\left[\frac{1}{(s-1)^2+4^2}\right]$

$= \frac{1}{2} e^{t} \cdot \frac{1}{4} \sin 4t = \frac{1}{8} e^{t} \sin 4t$

Theorem :

If $L[f(t)] = f(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

or

If $L[f(t)] = f(s)$ then, $L[t f(t)] = (-1) \frac{d}{ds} F(s)$

I.L.T

If $L^{-1}[f(s)] = f(t)$ then,

$\Rightarrow L^{-1}\left[-\frac{d}{ds} F(s)\right] = t f(t)$

$\Rightarrow -L^{-1}\left[\frac{d}{ds} F(s)\right] = t \cdot L^{-1}[f(s)]$

$\Rightarrow \boxed{L^{-1}[f(s)] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]}$ Can be used for log cosec etc, etc...

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$(S+2)^2$
$S^2+4S+4+9$

$\frac{S}{(S+2)^2+9}$

$\frac{(S+2)-2}{(S+2)^2+3^2}$

Q $\frac{S}{S^2+4S+13}$

$\frac{S}{S^2+4S+4+9}$

$\Rightarrow \frac{S}{(S+2)^2+9} \Rightarrow \frac{(S+2)-2}{(S+2)^2+9}$

$\Rightarrow \frac{S+2}{(S+2)^2+9} - \frac{2}{(S+2)^2+9}$

$= L^{-1}\left[\frac{S+2}{(S+2)^2+9}\right] - 2L^{-1}\left[\frac{1}{(S+2)^2+3^2}\right]$

By using first shifting theorem $\longrightarrow$

$= e^{-2t}\cdot\cos 3t - 2\times e^{-2t}\cdot\frac{1}{3}\sin 3t$

Q $\frac{S+8}{S^2+4S+5}$

$\Rightarrow \frac{S+8}{S^2+4S+4+1} \Rightarrow \frac{S+8}{(S+2)^2+1^2}$

$\Rightarrow \frac{S+2}{(S+2)^2+1^2} + \frac{6}{(S+2)^2+1^2}$

$\Rightarrow L^{-1}\left[\frac{S+2}{(S+2)^2+1^2}\right] + 6L^{-1}\left[\frac{1}{(S+2)^2+1^2}\right]$

$e^{-2t}\cdot\cos t + 6\cdot\frac{1}{1}\times e^{-2t}\cdot\sin t.$

If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \bar{f}(s)\, ds$

Inverse Laplace theorem

If

$$L^{-1}[\bar{f}(s)] = f(t) \text{ then } L^{-1}\left[\int_s^\infty \bar{f}(s) ds\right] = \frac{1}{t} f(t)$$

i.e. $$L^{-1}\left[\int_s^\infty \bar{f}(s) ds\right] = \frac{1}{t} L^{-1}[\bar{f}(s)]$$

$$\Rightarrow L^{-1}[\bar{f}(s)] = t \cdot L^{-1}\left[\int_s^\infty \bar{f}(s) ds\right]$$

Q Obtain inverse lape of

$$L^{-1}\left[\frac{2s}{(s^2+1)^2}\right]$$

$$L^{-1}[\bar{f}(s)] = t \cdot L^{-1}\left[\int_s^\infty \bar{f}(s) ds\right]$$

$$L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = t \cdot L^{-1}\left[\int_s^\infty \frac{2s}{(s^2+1)^2} \cdot ds\right]$$

$$t \cdot L^{-1}\left[\int_s^\infty \frac{1}{t^2} \cdot dt\right]$$ let $s^2+1 = t$

$2s\, ds = dt$

Q Find the func. whose L.T is

$$\log\left(1+\frac{1}{s}\right)$$

$\frac{s+1}{s}$ $\frac{m}{n}$

$m-n$

$$L^{-1}\left[\log\left(1+\frac{1}{s}\right)\right] = F(s)$$

?

$$= L^{-1}[\log(s+1)] - L^{-1}[\log s]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{t} L^{-1}\left[\frac{1}{s}\right]$$

$$= -\frac{1}{t} e^{-t} + \frac{1}{t}$$

Q $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{1}{s}\right)\right]$$

$\frac{1}{1+x^2} = \frac{d}{ds}\tan^{-1}x$

$\frac{1}{1+\left(\frac{1}{s}\right)^2} \times \frac{-1}{s^2}$

$$= -\frac{1}{t} L^{-1}\left[\frac{-\cancel{s^2}}{s^2+1} \times \frac{1}{\cancel{s^2}}\right]$$

$$= +\frac{1}{t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{t} \cdot 1 \times \sin t$$

$$= \frac{1}{t} e^{-bt} - \frac{1}{t} e^{-at}$$

$$= \frac{1}{t} [e^{-bt} - e^{-at}]$$

Q ILT of $\log\left[\frac{S^2-1}{S^2}\right]$

$$\log(S^2-1) - \log S^2$$

$$L^{-1}[\log(S^2-1)] - L^{-1}[\log S^2]$$

using theorem

$$-\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log(S^2-1)\right] + \frac{1}{t} L^{-1}\left[\frac{d}{ds}\log S^2\right]$$

$$-\frac{1}{t} L^{-1}\left[\frac{1}{S^2-1} \times 2S\right] + \frac{1}{t} L^{-1}\left[\frac{2S}{S^2}\right]$$

$$L^{-1}\left[\frac{1}{S}\right] = 1$$

$$-\frac{1}{t} \cdot 2 \cdot \cosh t + \frac{2}{t}$$

$$\Rightarrow \frac{2}{t}[1 - \cosh t]$$

Partial fraction

Q $\frac{1}{s^2-7s+12} = \frac{1}{s^2-4s-3s+12}$

$= \frac{1}{s(s-4)-3(s-4)} = \frac{1}{(s-3)(s-4)}$

$\frac{1}{(s-3)(s-4)} = \frac{A}{(s-3)} + \frac{B}{s-4}$

$1 = A(s-4) + B(s-3)$

$A+B=0 \quad — (I) \longrightarrow A=-B$

$-4A+3B=1 \quad — (II)$

Putting A in eq (II)

$4B-3B=1$

$\boxed{B=1}$

$\boxed{A=-1}$

i.e.

$\frac{1}{s^2-7s+12} = -\frac{1}{s-3} + \frac{1}{s-4}$

i.e.

$L^{-1}\left[\frac{1}{s^2-7s+12}\right] = -L^{-1}\left[\frac{1}{s-3}\right] + L^{-1}\left[\frac{1}{s-4}\right]$

$= -e^{3t} + e^{4t}$ Ans

Ex-6.6 → Q4

eg :- 22, 23

Q $\frac{5S+3}{(S-1)(S^2+2S+5)} = \frac{A}{S-1} + \frac{BS+C}{S^2+2S+5}$

$\Rightarrow 5S+3 = A(S^2+2S+5) + (BS+C)(S-1)$

$5S+3 = AS^2+2AS+5A + BS^2 - BS + CS - C$

Comparing

$A+B=0$

$2A-B+C=5$ $\quad \boxed{A=-B}$

$5A-C=3$

$2A+A+C=5 \Rightarrow 3A+C=5$

$5A-C=3$

$8A=8$

$\boxed{A=1}$

$\boxed{B=-1}$ $\quad \boxed{C=2}$

$= \frac{1}{S-1} + \frac{2-S}{S^2+2S+5}$

$\frac{1}{S-1} - \frac{S-2}{S^2+2S+5}$

$L^{-1}\left[\frac{1}{S-1}\right] - L^{-1}\left[\frac{S-2}{S^2+2S+5}\right]$

$\quad\text{"}\quad - L^{-1}\left[\frac{S-2}{(S+1)^2+4}\right]$

Apply first shifting

$\quad\text{"}\quad - L^{-1}\left[\frac{S+1-3}{(S+1)^2+4}\right]$

$L^{-1}\left[\frac{1}{S-1}\right] - L^{-1}\left[\frac{S+1}{(S+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(S+1)^2+4}\right]$

$e^{t} - e^{-t}\cdot \cos 2t + 3e^{-t}\cdot\frac{1}{2}\sin 2t$

Q Find $L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right]$

$$\frac{s+4}{s(s-1)(s^2+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$s+4 = A(s-1)(s^2+1) + Bs(s^2+4) + (Cs+D)(s)(s-1)$$

$s=1$

$5 = 0 + B\times 5$

$B=1$

$A+B+C=0$

$-A-C+D=0$

$+4A+4B-D=0$

$-4A=4$ — (IV)

$A=-1$

$B=1$

$C=0$

$D=-1$

$s=2$

$6 = 5A + 16B + (2C+D)2$

$6 = -5 + 16 + 4C + 2D$

$-5 = 4C + 2D$

i.e.

$$\frac{s+4}{s(s-1)(s^2+4)} = \frac{-1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

i.e.

$$L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right] = -L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= -1 + e^{t} - \frac{1}{2}\sin 2t$$

I·L·T Convolution theorem

If $L^{-1}[f_1(s)] = f_1(t)$ & $L^{-1}[f_2(s)] = f_2(t)$

then,

$$L^{-1}[f_1(s) \cdot f_2(s)] = \int_0^t f_1(x) \cdot f_2(t-x)\, dx$$

Q using C·T evaluate

$$L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$$

$$L^{-1}\left[\frac{1}{(s^2+4)} \cdot \frac{s}{(s^2+4)}\right]$$

$$F_1(s) = \frac{1}{s^2+4} \Rightarrow L^{-1}[F_1(s)] = L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{2}\sin 2t = f_1(t)$$

$$f_2(s) = \frac{s}{s^2+4} \Rightarrow L^{-1}[f_2(s)] = L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= \cos 2t = f_2(t)$$

i.e.

- $f_1(t) = \frac{1}{2}\sin 2t \Rightarrow f_1(x) = \frac{1}{2}\sin 2x$

- $f_2(t) = \cos 2t \Rightarrow f_2(t-x) = \cos 2(t-x)$

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Q. $L^{-1}\left[\frac{1}{(S+1)(S^2+2S+2)}\right]$

$$\Rightarrow \frac{1}{(S+1)(S^2+2S+2)} = \frac{A}{S+1} + \frac{B\cancel{S}S+C}{S^2+2S+2}$$

$$1 = A(S^2+2S+2) + (B\cancel{S}S+C)(S+1)$$

$$AS^2+2AS+2A+BS^2+BS+CS+C$$

$A+B=0 \Rightarrow \boxed{A=-B}$

$2A+B+C=0$

$C-B=0$

$\boxed{C=B}$

$2A+C=1$

$-2B+B=1$

$\boxed{B=-1}$

$\boxed{A=1}$

$\boxed{C=-1}$

Q. $$\frac{1}{S^4+4} = \frac{1}{(S^2+2)^2-(2S)^2}$$

$$= \frac{1}{(S^2+2-2S)(S^2+2+2S)}$$

Now partial fraction.

$\frac{1}{S+2}$ S^4 $(S^2)^2$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B)-\cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

$$L^{-1}\left[\frac{1}{(S+1)(S^2+2S+2)}\right]$$

$$\Rightarrow \frac{1}{(S+1)(S^2+2S+2)} = \frac{A}{S+1} + \frac{BS+C}{S^2+2S+2}$$

$$1 = A(S^2+2S+2) + (BS+C)(S+1)$$

$$AS^2+2AS+2A+BS^2+BS+CS+C$$

$$A+B=0 \Rightarrow \boxed{A=-B}$$

$$2A+B+C=0$$

$$C-B=0$$

$$\boxed{C=B}$$

$$2A+C=1$$

$$-2B+B=1$$

$$\boxed{B=-1}$$

$$\boxed{A=1}$$

$$\boxed{C=-1}$$

$\frac{1}{S+2}$ $\quad S^4 \quad (S^2)^2$

Q. $$\frac{1}{S^4+4} = \frac{1}{(S^2+2)^2-(2S)^2}$$

$$= \frac{1}{(S^2+2-2S)(S^2+2+2S)}$$

Now partial fraction.

$$F_2(s) = \frac{s}{s^2+b^2} = L^{-1}[F(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos bt = f_2(t)$$

$$f_1(t) = \cos at \Rightarrow f_1(x) = \cos ax$$

$$f_2(t) = \cos bt \Rightarrow f_2(t-x) = \cos b(t-x)$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] \Rightarrow \int_0^t f_1(x) \cdot f_2(t-x)$$

$$= \int_0^t \cos ax \cdot \cos b(t-x)$$

$$= \frac{1}{2}\int_0^t [\cos(ax+bt-bx) + \cos(ax-bt+bx)]\,dx$$

$$= \frac{1}{2}\int_0^t [\cos(bt+(a-b)x) + \cos(\cancel{b}(a+b)x - bt)]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(bt+(a-b)x)}{a-b} + \frac{\sin[(a+b)x-bt]}{a+b}\right]_0^t$$

$$\frac{1}{2}\left[\frac{\sin(bt+at-bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b}\right.$$

$$\left. - \frac{\sin(bt)}{a-b} - \frac{\sin(-bt)}{a+b}\right]$$

$$F_2(s) = \frac{s}{s^2+b^2} = L^{-1}\{F(s)\} = L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos bt = f_2(t)$$

$$f_1(t) = \cos at \Rightarrow f_1(x) = \cos ax$$

$$f_2(t) = \cos bt \Rightarrow f_2(t-x) = \cos b(t-x)$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] \Rightarrow \int_0^t f_1(x) \cdot f_2(t-x)$$

$$= \int_0^t \cos ax \cdot \cos b(t-x)$$

$$= \frac{1}{2}\int_0^t \left[\cos(ax+bt-bx) + \cos(ax-bt+bx)\right] dx$$

$$= \frac{1}{2}\int_0^t \left[\cos(bt+(a-b)x) + \cos((a+b)x-bt)\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(bt+(a-b)x)}{a-b} + \frac{\sin\left[(a+b)x-bt\right]}{a+b}\right]_0^t$$

$$\frac{1}{2}\left[\frac{\sin(bt+at-bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b}\right.$$

$$\left. - \cancel{\cos a} + \frac{\sin(bt)}{a-b} + \frac{\sin(-bt)}{a+b}\right]$$

Now, $L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \int_0^t \frac{1}{2}\sin 2x \cdot \cos 2(t-x)\, dx$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^t \left[\sin(2x+2t-x) + \sin(2x-2t+2x)\right] dx$$

$$= \frac{1}{4}\int_0^t \left[\sin 2t + \sin(4x-2t)\right] dx$$

$$= \frac{1}{4}\left[x\sin 2t - \frac{\cos(4x-2t)}{4}\right]_0^t$$

$$= \frac{1}{4}\left[t\sin 2t - \frac{\cos 2t}{4} - 0 + \frac{\cos(-2t)}{4}\right]$$

$$= \frac{1}{4}\left[t\sin 2t - \frac{\cos 2t}{4} + \frac{\cos 2t}{4}\right]$$

$$= \frac{t\sin 2t}{4}$$

Using Convolution Theorem Evalute (UCTE) Evalute:-

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] : a \neq b$$

$$L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}\right] = L^{-1}\left[F_1(s)\cdot F_2(s)\right]$$

$$F_1(s) = \frac{s}{s^2+a^2} = L^{-1}[F(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$= f_1(t)$$

Eg: ~~34~~ eg:- 34, 35, 37 $\frac{t^n}{a-}$

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$$\frac{1}{2}\left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

Q $$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right]$$

$$F_1(s) = \frac{1}{s} \quad = L^{-1}\left[\frac{1}{s}\right] = 1 = f_1(t)$$

$$F_2(s) = \frac{1}{s^2+a^2} \quad = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f_2(t)$$

$$f_1(t) = 1 \Rightarrow f(x) = 1$$

$$f_2(t) = \frac{1}{a}\sin at \Rightarrow f(t-x) = \frac{1}{a}\sin a(t-x)$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t 1\cdot\frac{1}{a}\sin a(t-x)\cdot dx$$

$$= \frac{1}{a^2}\left[\frac{-\cos a(t-x)}{-}\right]_0^t$$

$$= \frac{1}{a^2}\left[\cos a(t-t) - \cos at\right]$$

$$= \frac{1}{a^2}\left[1-\cos at\right] \text{ Ans}$$

Sol^n Of Diff. Equation

$y = f(x)$

derivative w.r.t x is denoted by

$\frac{dy}{dx}$ or y'

L.T. for derivative

$y = f(t)$

$L[f(t)] = \bar{f}(s)$

$y = f(x)$

$\boxed{L[y] = \bar{y} = L[f(x)]}$

* $L[y'] = s\bar{y} - y(0)$

$L[y''] = s^2\bar{y} - sy(0) - y'(0)$

$L[y'''] = s^3\bar{y} - s^2 y(0) - sy'(0) - y''(0)$

$L[y^n] = s^n\bar{y} - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - s^0 y^{n-1}(0)$

Apply convolution Theorem Solve following Diff. Equation (A C T S F D E)

$y'' + y = \sin 3t \quad ; y(0) = 0, \; y'(0) = 0$

Sol $\Rightarrow L[y''] + L[y] = L[\sin 3t]$

$= s^2\bar{y} - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{0} + \bar{y} = \frac{3}{s^2+3^2}$

$(s^2+1)\bar{y} = \frac{3}{s^2+3^2}$

$$\bar{y} = \frac{3}{(s^2+3^2)\cdot(s^2+1)}$$

$$L[y] = \frac{3}{(s^2+3^2)(s^2+1)}$$

$$y = L^{-1}\left[\frac{3}{s^2+3^2} \cdot \frac{1}{s^2+1}\right]$$

Now apply convolution.

Q $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$

where $y=1$, $\frac{dy}{dt} = 2$, $\frac{d^2y}{dt^2} = 2$ at $t=0$

D.E.I.G.I.T.F.O (Diff. eq in given transfer form)

$y''' + 2y'' - y' - 2y = 0$

$\Rightarrow L[y'''] + 2L[y''] - L[y'] - 2[L[y]] = 0$

$\Rightarrow s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0) + 2\{s^2\bar{y}(s) - sy(0) - y'(0)\}$

$- \{s\bar{y}(s) - y(0)\} - 2L[y] = 0$

Given: $y(0) = 1$, $y'(0) = 2$, $y''(0) = 2$

$s^3\bar{y} - s^2 - 2s - 2 + 2s^2\bar{y} - 2s - 4 - s\bar{y} + 1 - 2L[y] = 0$

$s^3L[y] - s^2 - 2s - 2 + 2s^2L[y] - 2s - 4 - sL[y] + 1 - 2L[y] = 0$

$L[y]\{s^3 + 2s^2 - s - 2\} = \cancel{s^2+2}\ s^2 + 4s + 5$

~~$y''+y = \sin 3t$~~

Q $y''+25y = 10\cos 5t$

$y(0)=2,\ y'(0)=0$

$L[y'']+25L[y] = 10\cos 5t$

$s^2\bar{y} - sy(0) - y'(0) + 25\bar{y} = L[10\cos 5t]$

$\bar{y}(s^2+25) - 2s = 10\ \frac{s}{s^2+25}$

$\bar{y}(s^2+25) = \frac{10s}{s^2+25} + 2s$

$\bar{y}(s^2+25) = \frac{50 - 2s^2 - 50s}{s^2+25}$

$\bar{y}(s^2+25) = \frac{10s + 2s^3 + 50s}{s^2+25}$

$\bar{y} = \frac{+2s^3+60s}{(s^2+25)^2}$

$L[y] = \frac{+2s(s^2+30)}{(s^2+25)^2}$

$y = +2L^{-1}\left[\frac{s}{s^2+25} \cdot \frac{s^2+30}{s^2+25}\right]$

F_1

$$\frac{s^2+4s+5}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$s^2+4s+5 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$

$s = 1$

$10 = A(2)(3)$

$$\boxed{A = \frac{10}{6} = \frac{5}{3}}$$

$s = -1$

$1-4+5$

$2 = B(-2)(+1)$

$$\boxed{B = -1}$$

$s = -2$

$1 = C(-3)(-1)$

$$\boxed{C = \frac{1}{3}}$$

$$\bar{y} = \frac{5}{3(s-1)} - \frac{1}{s+1} + \frac{1}{3(s+2)}$$

taking inverse laplace transform

$$y = \frac{5}{3} L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3} L^{-1}\left[\frac{1}{s+2}\right]$$

$$y = \frac{5}{3} e^{t} - e^{-t} + \frac{1}{3} e^{-2t}$$

$$\Rightarrow L[y] = \frac{s^2+4s+5}{s^3+2s^2-s-2}$$

$$y = L^{-1}\left[\frac{s^2+4s+5}{s^3+2s^2-s-2}\right]$$

$$\bar{y} = \frac{s^2+4s+5}{s^3+2s^2-s-2}$$

$s=1$ satisfies eqn $s^3+2s^2-s-2=0$

i.e.

$(s-1)$ will be one factor of the polynomial

$\cancel{s^3+}\ s^3+2s^2-s-2$

$$\begin{array}{r|l} & s^2+3s+2 \\ \hline s-1 & s^3+2s^2-s-2 \\ & \underline{s^3-s^2} \\ & \quad -\ \ + \\ & \quad 3s^2-s \\ & \quad \underline{3s^2-3s} \\ & \quad\quad -\ \ + \\ & \quad\quad 2s-2 \\ & \quad\quad \underline{2s-2} \\ & \quad\quad\ \ \times \end{array}$$

i.e. $s^3+2s^2-s-2 = (s^2+3s+2)(s-1)$

$$= (s-1)(s+1)(s+2)$$

i.e. $$\bar{y} = \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

$$\bar{y} = 10\frac{s}{s^2+25}\cdot\frac{1}{s^2+5^2} + \frac{2s}{s^2+25}$$

Taking I - Laplace both side

$$y = 2L^{-1}\left[\frac{s}{s^2+5^2}\right] + 10L^{-1}\left[\frac{1}{s^2+5^2}\cdot\frac{s}{s^2+5^2}\right]$$

$$y = 2\cos 5t + 10L^{-1}\left[\frac{1}{s^2+5^2}\cdot\frac{s}{s^2+5^2}\right] \quad \text{(A)}$$

Let $f_1(s) = \frac{1}{s^2+5^2}$; $L^{-1}[f_1(s)] = \frac{1}{5}\sin 5t = f_1(t)$

$f_2(s) = \frac{s}{s^2+5^2}$; $L^{-1}[f_2(s)] = \cos 5t = f_2(t)$

If $f_1(t) = \frac{1}{5}\sin 5t$ then $f_1(x) = \frac{1}{5}\sin 5x$

$f_2(t) = \cos 5t$ Then $f_2(t-x) = \cos 5(t-x)$

A.C.T

$$L^{-1}\left[\frac{1}{s^2+5^2}\cdot\frac{s}{s^2+5^2}\right] = \int_0^t \frac{1}{5}\sin 5x \cdot \cos 5(t-x)\,dx$$

$$= \frac{1}{5}\times\frac{1}{2}\int_0^t \left[\sin(5x+5t-5x) + \sin(5x-5t+5x)\right]$$

$$= \frac{1}{10}\int_0^t \left[\sin 5t + \sin(10x-5t)\right]dx$$

$$= \frac{1}{10}\left[x\sin 5t - \frac{\cos(10x-5t)}{10}\right]_0^t$$

$$= \frac{1}{10}\left[t\sin 5t - \frac{\cos 5t}{10} + \frac{\cos 5t}{10} \cdot 0\right]$$

$$= \frac{t\sin 5t}{10}$$

Put in eq A

$$y = 2\cos 5t + \frac{10}{10} t\sin 5t$$

$$= 2\cos 5t + t\sin 5t$$