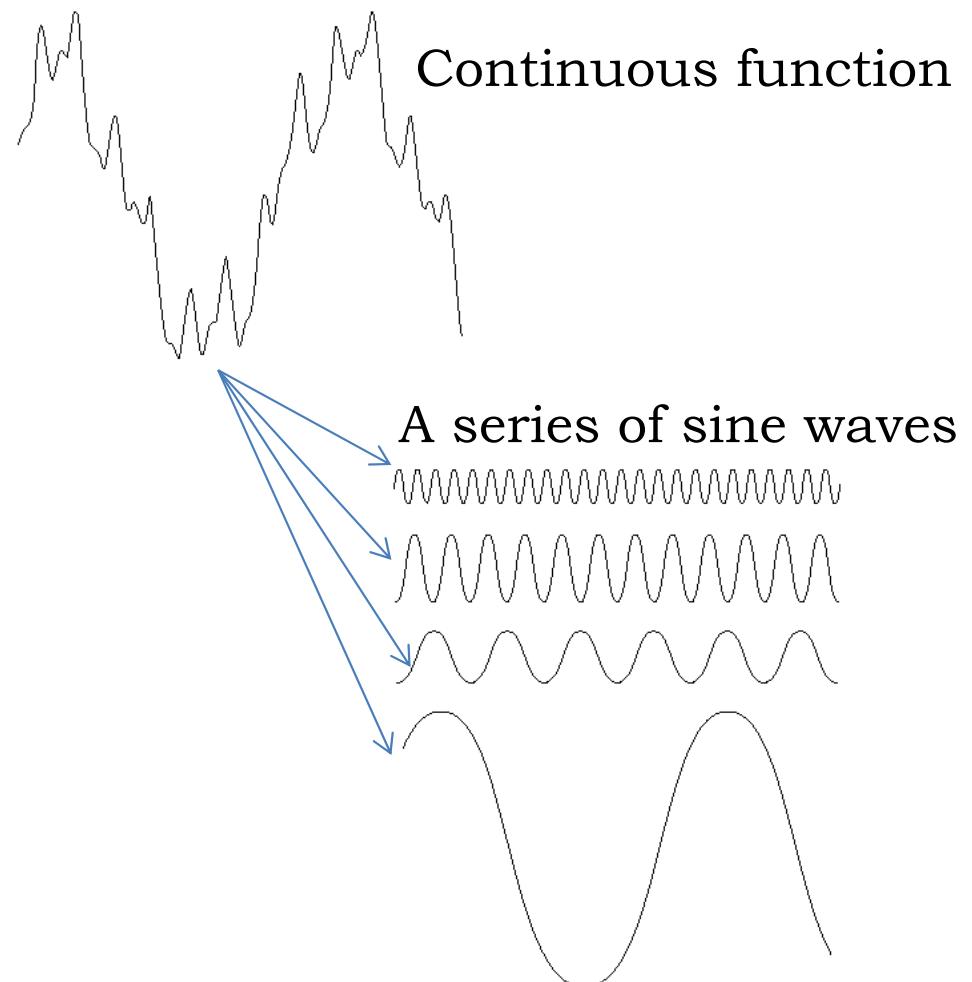


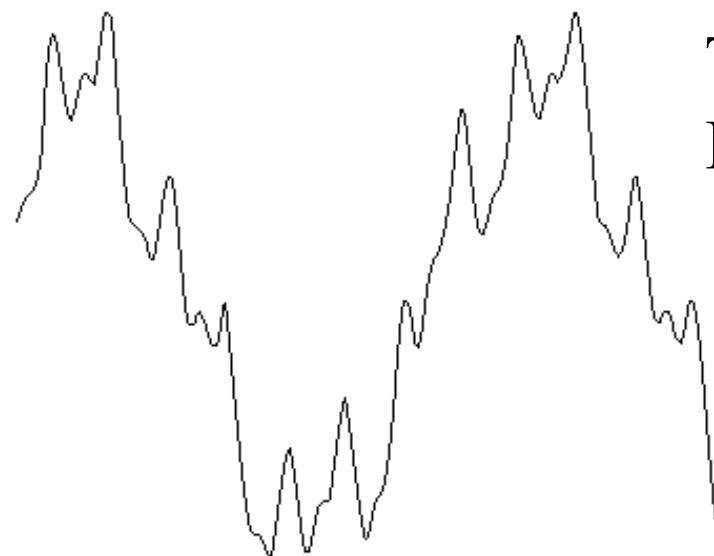
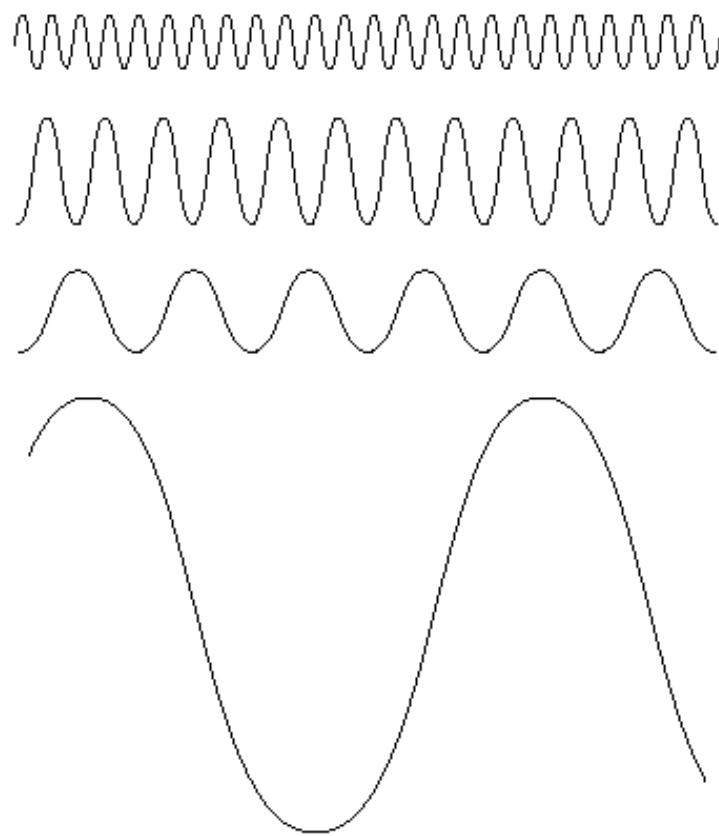
Image Enhancement in the Frequency Domain

Fourier Transform

http://en.wikipedia.org/wiki/Fourier_transform

- A continuous function can be transformed to a series of sine waves
- This gives us a way to describe a function in terms of its frequencies

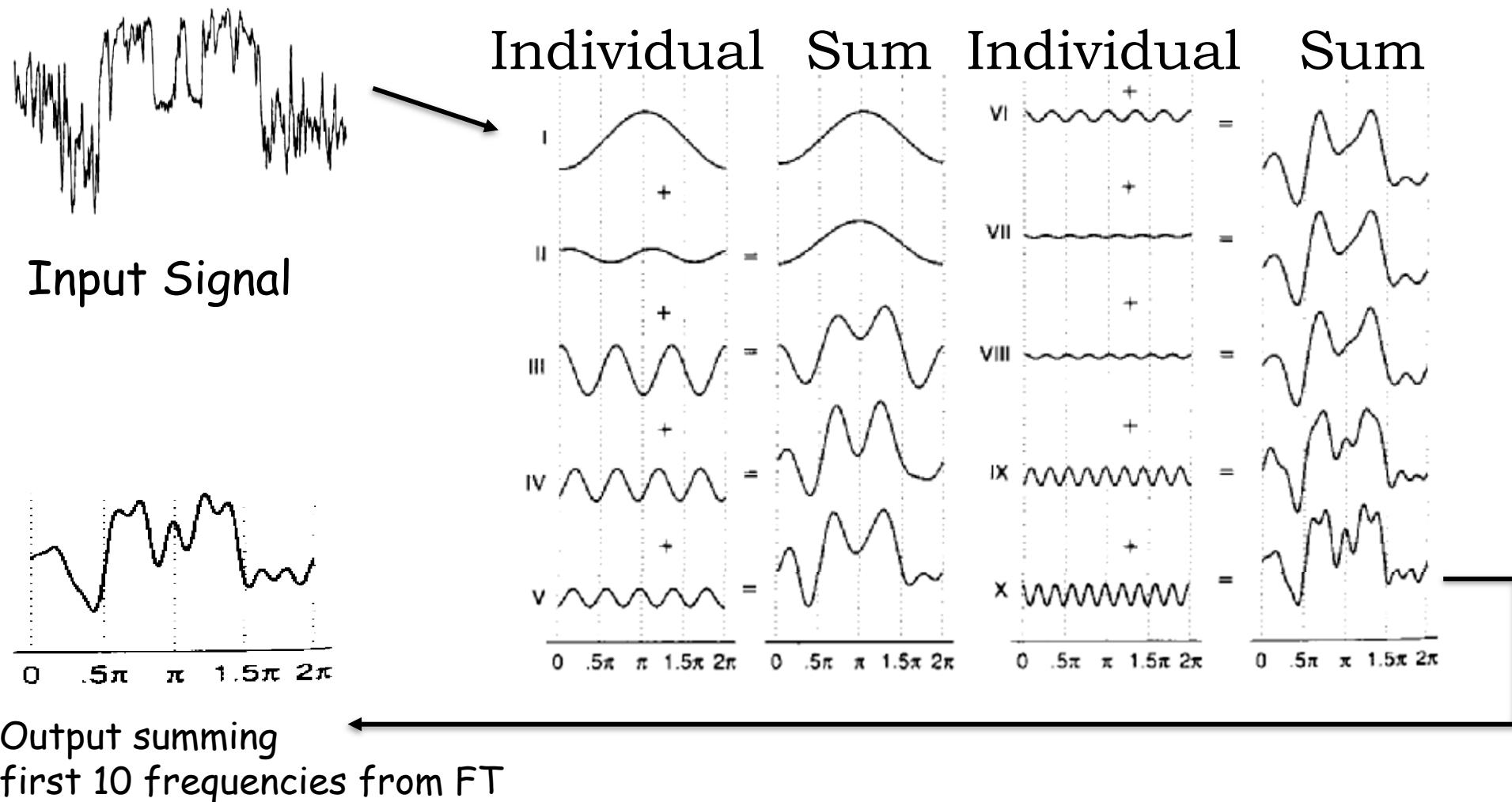




The function, $f(x)$
Input Signal

FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Sum of Frequencies



Fourier Transform

http://en.wikipedia.org/wiki/Fourier_transform

- One dimensional Fourier transform (based on the indefinite integral defined below) may have infinite frequency u .

$$F(u) = \int_{-\infty}^{\infty} f(x)[\cos 2\pi u x - i \sin 2\pi u x] dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u)[\cos 2\pi u x + i \sin 2\pi u x] du$$

Inverse Fourier Transform

Sine wave: http://en.wikipedia.org/wiki/Sine_wave

Fourier Transform

http://en.wikipedia.org/wiki/Fourier_transform

- One dimensional Fourier transform (based on the indefinite integral defined below) may have infinite frequency u .

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi ux}dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u)e^{i2\pi ux}du$$

Inverse Fourier Transform

Sine wave: http://en.wikipedia.org/wiki/Sine_wave

Euler's formula: https://en.wikipedia.org/wiki/Euler%27s_formula

Discrete Fourier Transform (DFT)

The discrete Fourier transformation (definite summation) of the sampled $f(x)$ is:

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi x u / N}$$

for $u = 0, 1, 2, 3, \dots, N-1$

Inverse:

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{j2\pi x u / N}$$

Exponential function: http://en.wikipedia.org/wiki/Exponential_function

Sometimes this is written with the (1/N) switched.

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-j2\pi x u / N}$$

for $u = 0, 1, 2, 3, \dots, N-1$

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) e^{j2\pi x u / N}$$

2D DFT

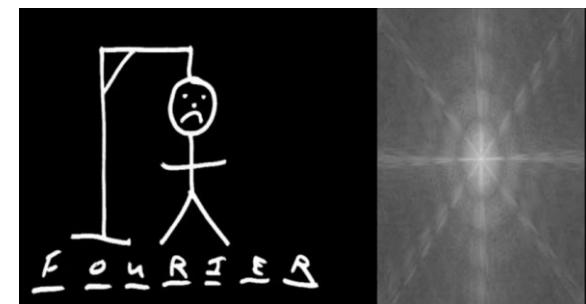
- In the two-variable case, the DFT pair is:

$$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

Live Fourier Transform Demo:

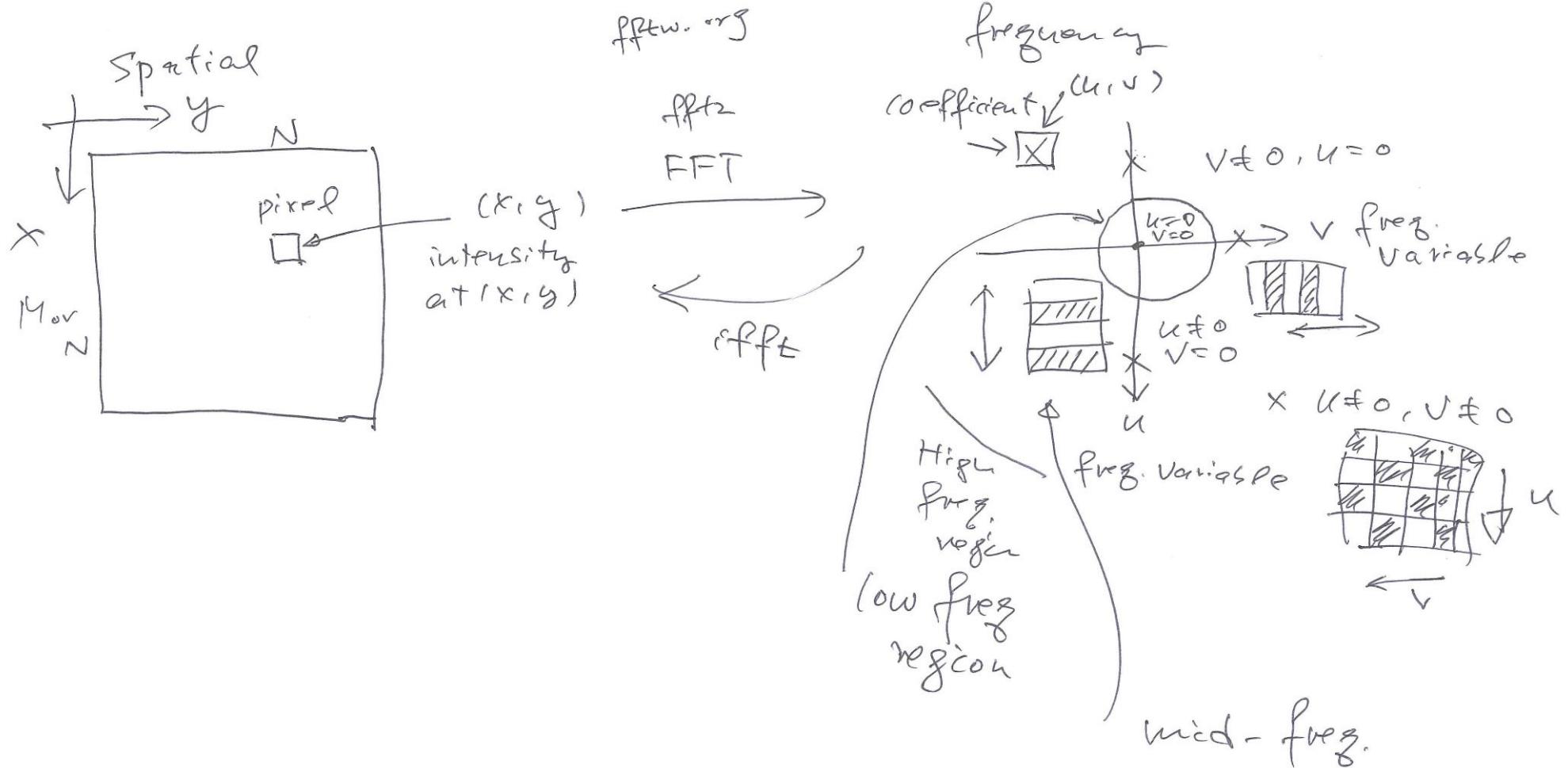
<https://www.youtube.com/watch?v=qalZxK9Y1Tw>



A Faster Fast Fourier Transform:

<http://spectrum.ieee.org/computing/software/a-faster-fast-fourier-transform>

Spatial and Frequency Representations



DFT example

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j \frac{2\pi u x}{N}}$$

If $N=3$, 3 samples

$f(x=0)$

$f(x=1)$

$f(x=2)$

$N-1 = 3-1 = 2$

$\text{O}(N^2)$

$$F(u=0) = \frac{1}{3} \sum_{x=0}^2 f(x) e^{-j \frac{2\pi u x}{N}}$$

$$= \frac{1}{3} \left[f(x=0) e^0 + f(x=1) e^{-j \frac{2\pi u}{3}} + f(x=2) e^{-j \frac{4\pi u}{3}} \right]$$

\downarrow intensity value at $x=0$

$$= \frac{1}{3} (f(x=0) + f(x=1) + f(x=2)) = \underline{\text{average intensity}}$$

$$e^{jx} = \cos x + j \sin x$$

Imaginary

e^{jx}

$\cos x$

$\sin x$

Real

 $|e^{jx}| = 1$

Spatial

$x=0$	$x=1$	$x=2$
$f(x=0)$	$f(x=1)$	$f(x=2)$
$u=0$	$u=1$	$u=2$
$F(u=0)$	$F(u=1)$	$F(u=2)$

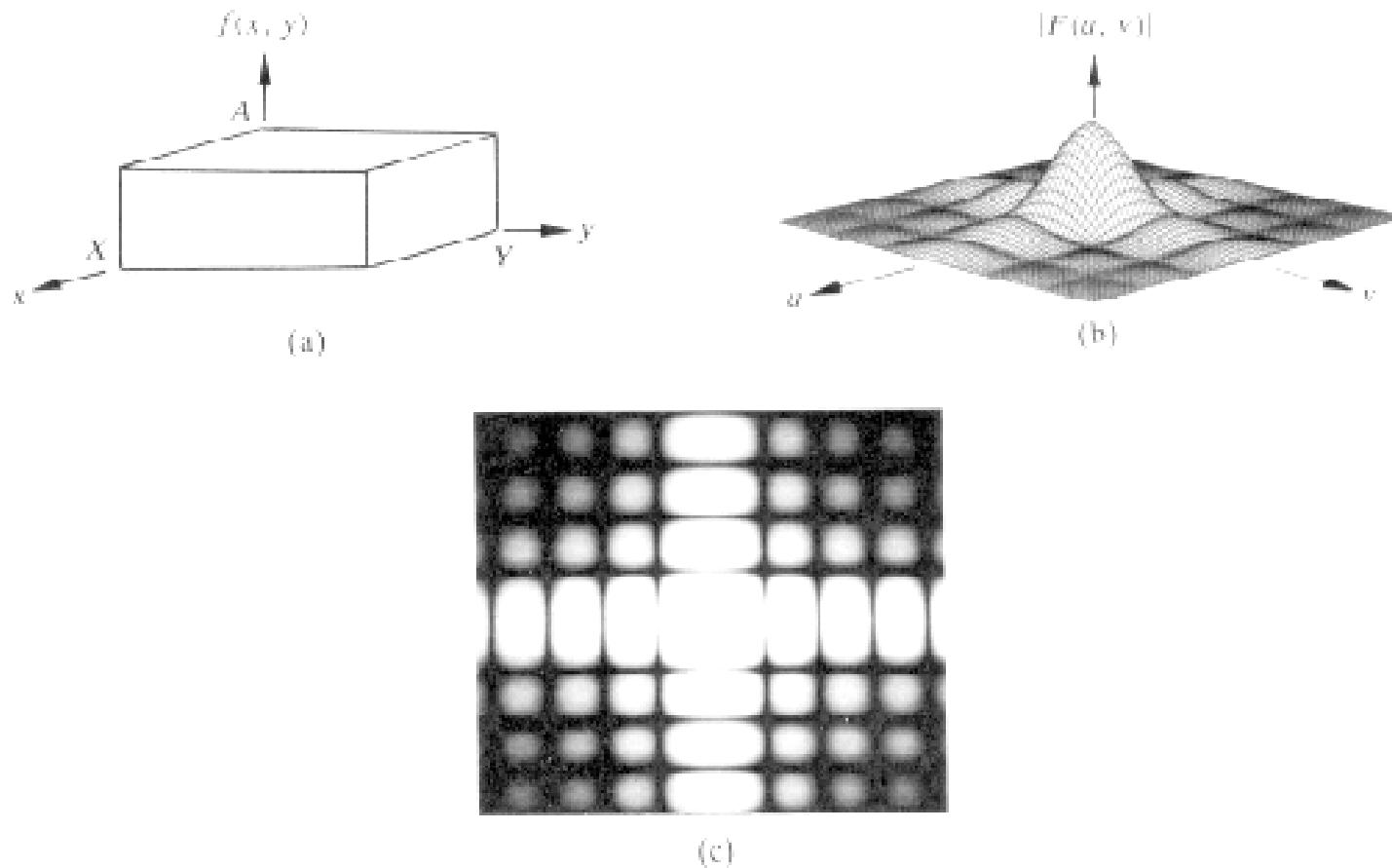
Freq.

FFT

IFFT

fftw.org

Example



- We show $|F(u,v)|$ and $\log(1 + |F(u,v)|)$
- Note, these values have also been shifted such that $F(0,0)$ is at $F(N/2, N/2)$

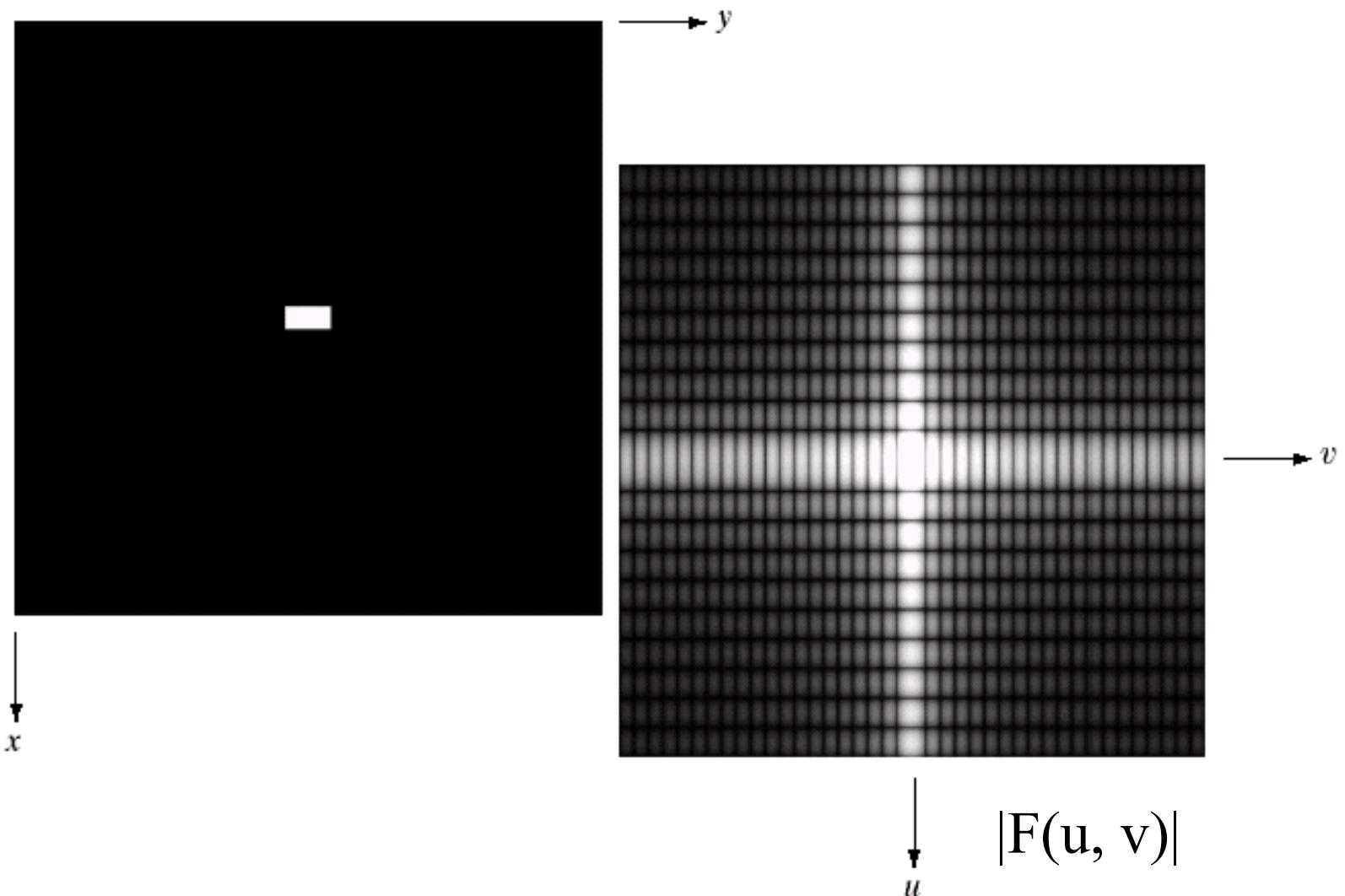
$$f(x, y)$$

a b

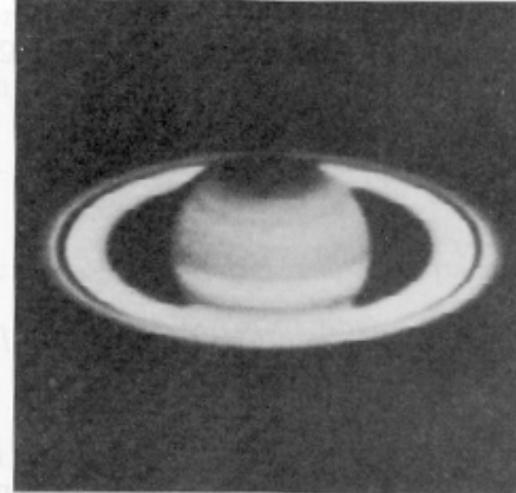
FIGURE 4.3

(a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels.

(b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

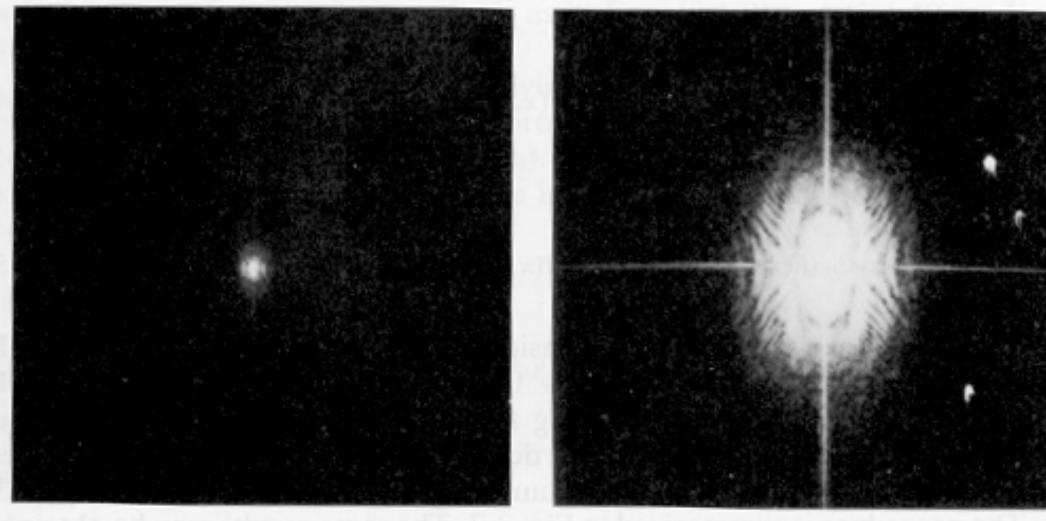


More Examples



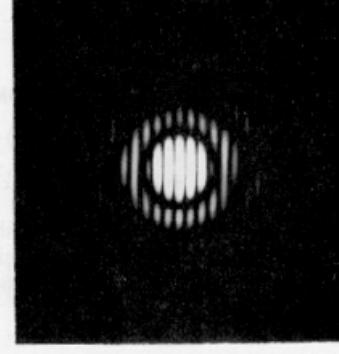
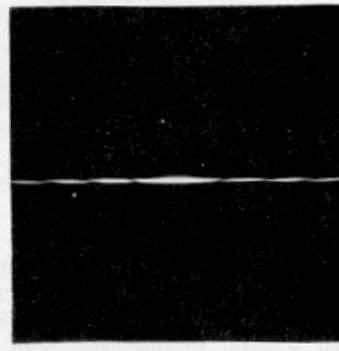
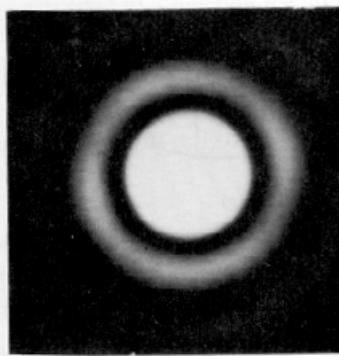
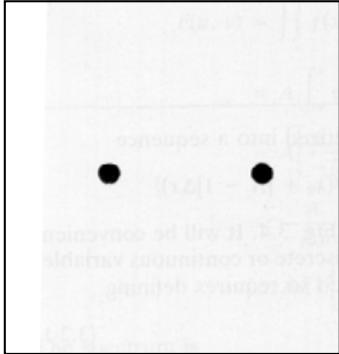
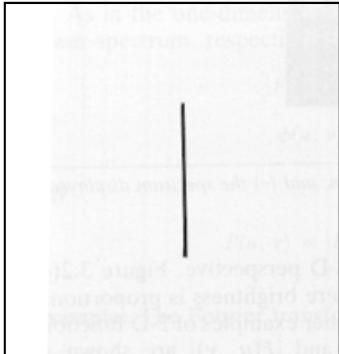
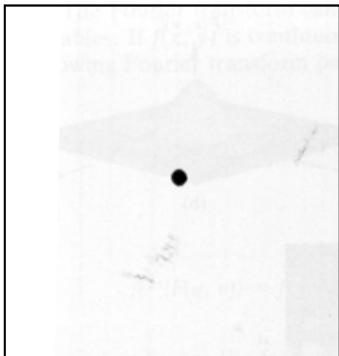
$$f(x, y)$$

$$|F(u, v)|$$



$$\log(1+|F(u,v)|)$$

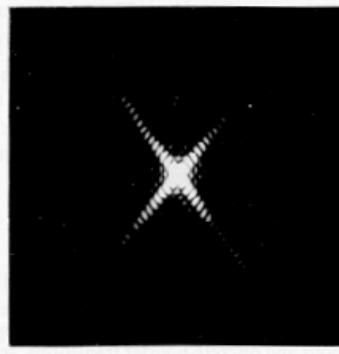
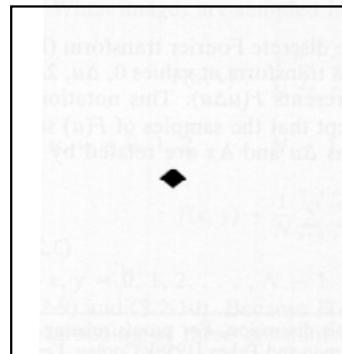
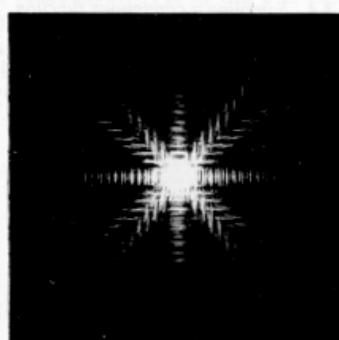
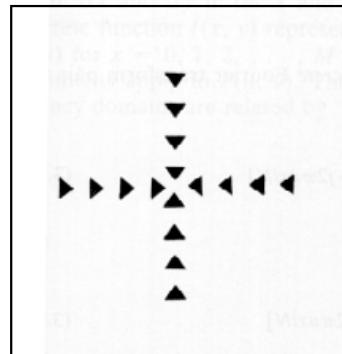
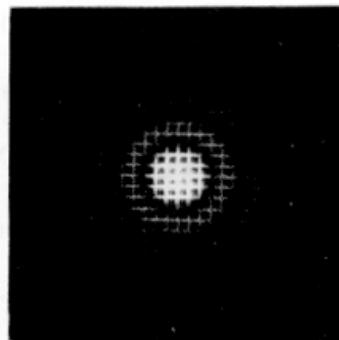
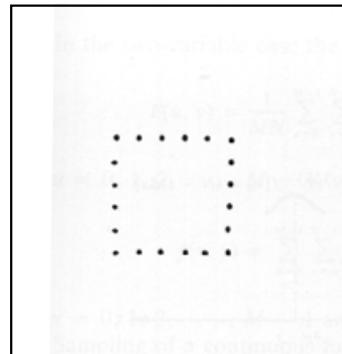
More Examples



$f(x, y)$

$|F(u, v)|$

More Examples



$f(x, y)$

$|F(u, v)|$

Image Mean or Average Value

- Consider the mean image intensity

$$\bar{f}(x, y) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y)$$

- Consider $F(0,0)$

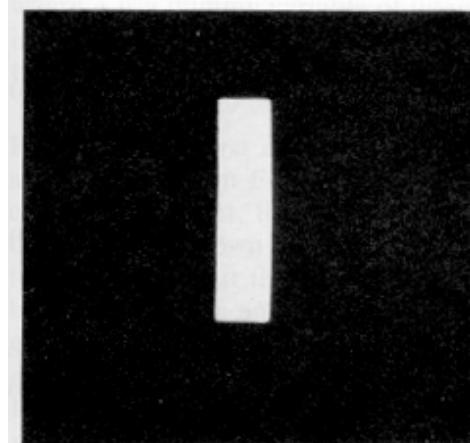
$$F(0,0) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) e^{(0)}$$

Rotation

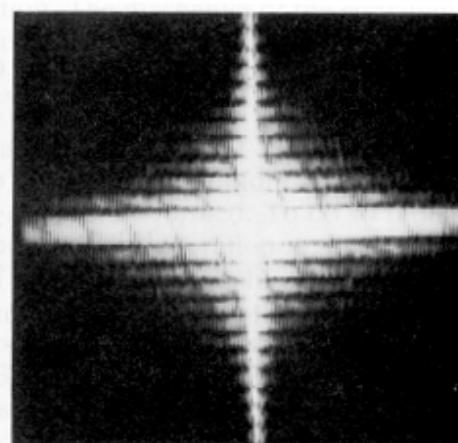
- Introduce polar coordinates,
 $r = \text{radius, theta } \theta/\phi$ $\phi = \text{angle}$
 $x = r \cos\theta$ $y = r \sin\theta$ $u = r \cos\phi$ $v = r \sin\phi$
- Then, using the polar coordinates, we have
 $f(x,y) = f(r, \theta)$ and $F(u,v) = F(r, \phi)$
- Then $f(r, \theta + \theta_0) \Leftrightarrow F(r, \phi + \theta_0)$

Rotation: Example

$$f(x,y) = f(r, \theta) \text{ and } F(u,v) = F(r, \phi)$$

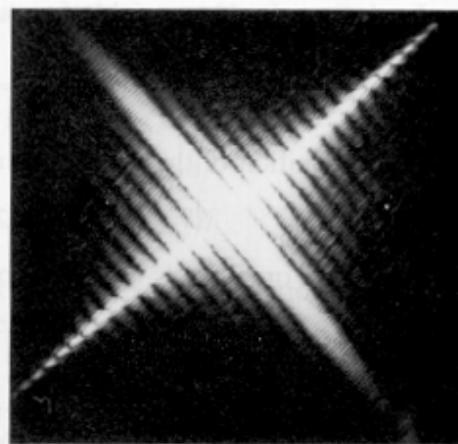
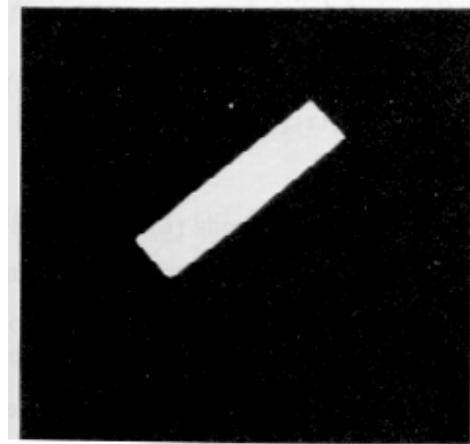


(a)



(b)

$f(x, y)$



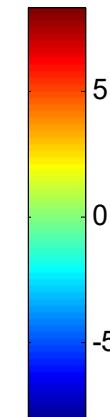
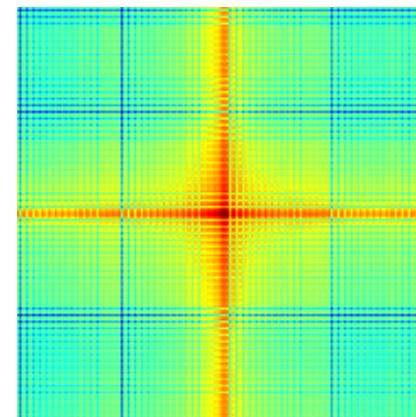
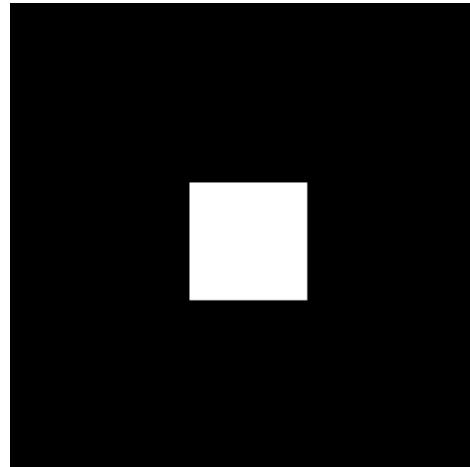
$|F(u, v)|$

$$f(r, \theta + \theta_0) \Leftrightarrow F(r, \phi + \theta_0)$$

Translation properties of FT

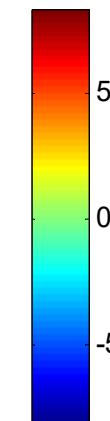
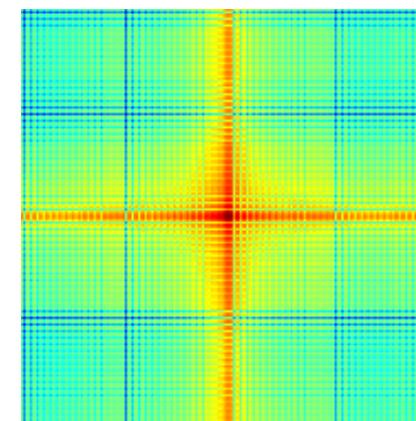
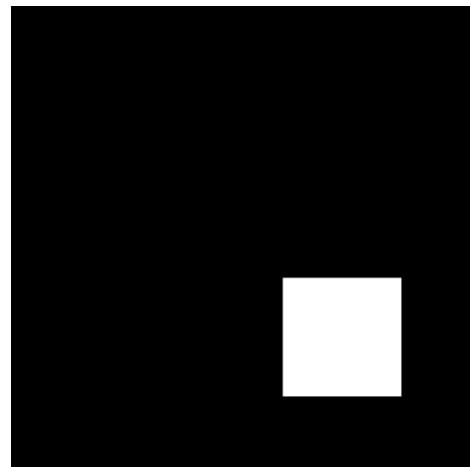
$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{-j2\pi(ux_0/M + vy_0/N)}$$

$f(x, y)$



$|F(u, v)|$

$f(x - x_0, y - y_0)$



$|F(u, v)|$

Magnitude of
the exponential
function is one

Filtering in Frequency Domain

Filtering in Frequency Domain

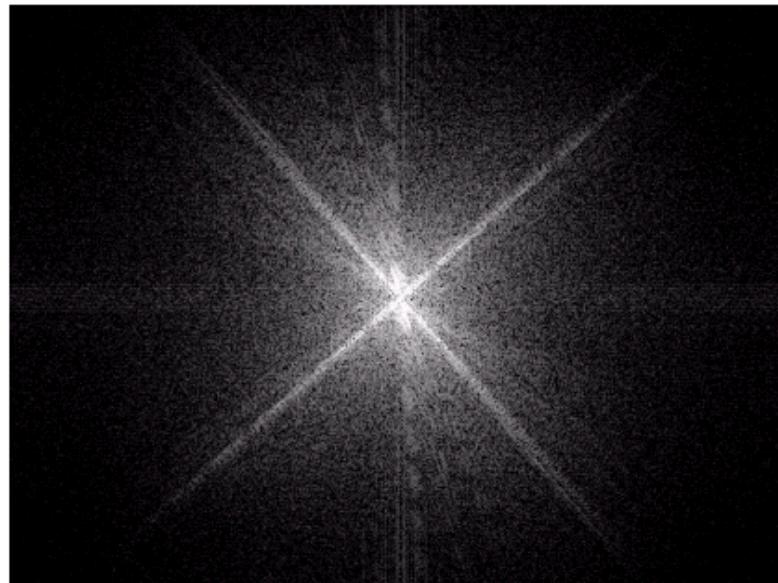
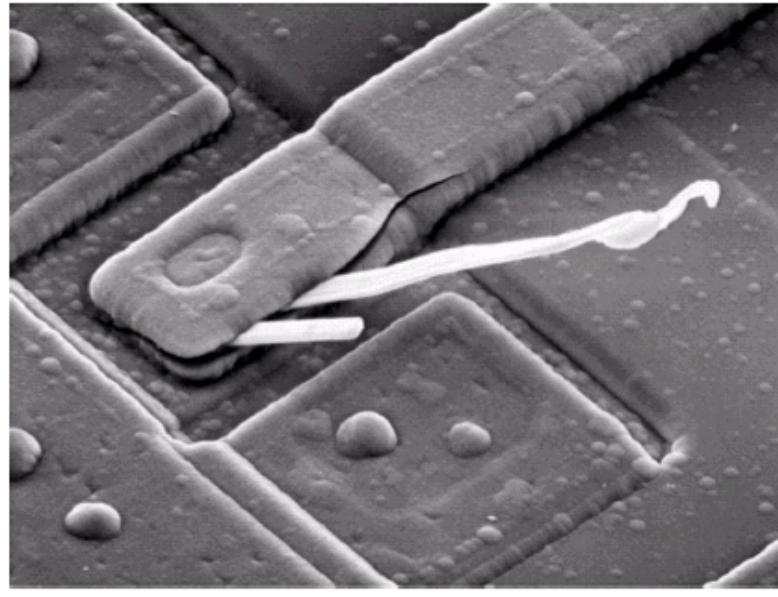
- One image = two representations

$$f(x, y)$$

*Spatial
(a collection
of pixels)*

$$|F(u, v)|$$

*Frequency
(rate of change of
intensity values or
grey level)*



a
b

FIGURE 4.4
(a) SEM image of a damaged integrated circuit.
(b) Fourier spectrum of (a).
(Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Strong edges at $\pm 45^\circ$ of the spectrum.

Filtering in Frequency Domain

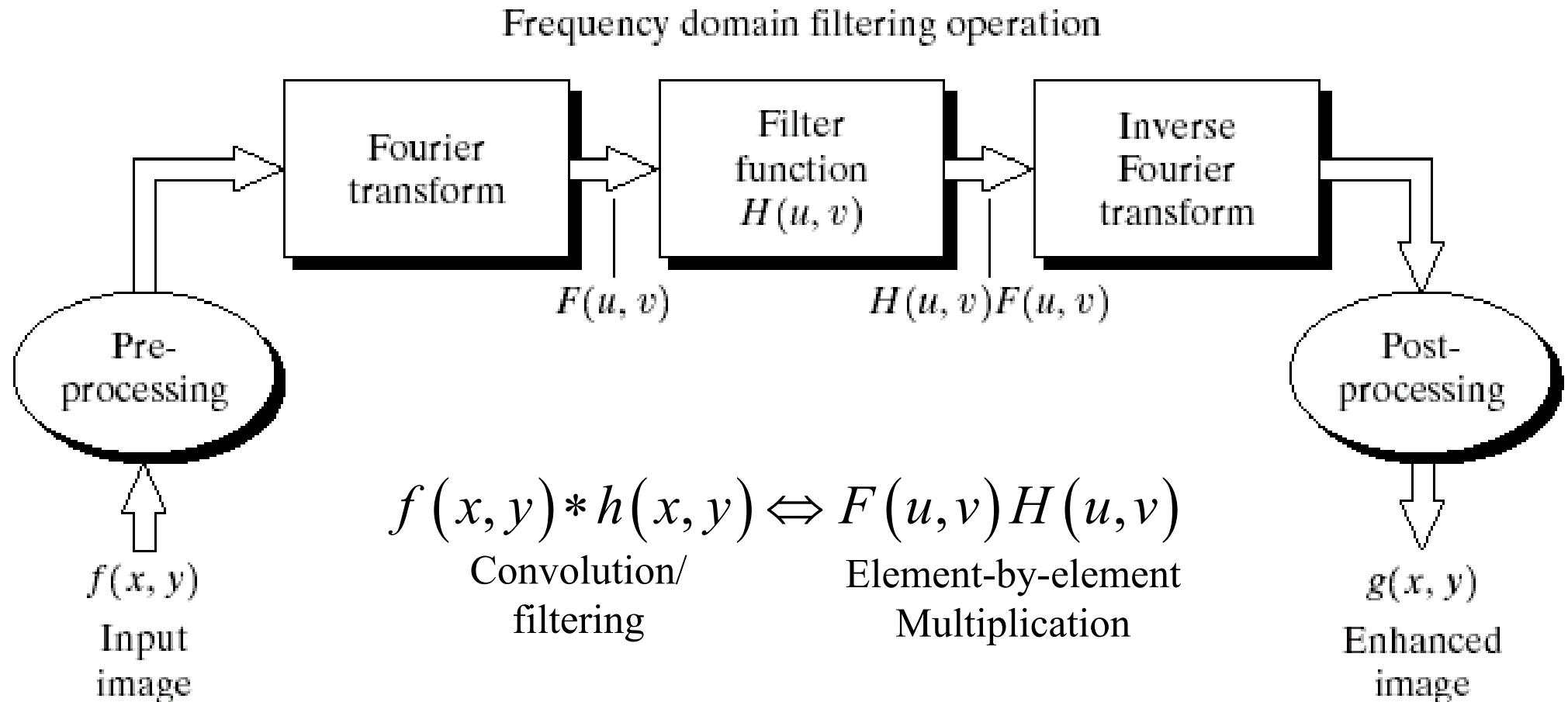


FIGURE 4.5 Basic steps for filtering in the frequency domain. $g(x, y) = f(x, y) * h(x, y)$

$$G(u, v) = F(u, v)H(u, v)$$

$$g(x, y) = \mathcal{I}^{-1}\{G(u, v)\}$$

Filtering in Frequency Domain

- Filtering Frequencies
 - Low Pass, High Pass, . . .
- Filter will take the form
 - $G(u,v) = H(u,v) F(u,v)$ (*element-by-element multiplication*)
 - $F(u,v)$ is DFT of $f(x,y)$, DFT = Discrete Fourier Transform
 - $H(u,v)$ is a filter, and it attenuates or selects frequencies
 - $G(u,v)$ is the filtered version of $F(u,v)$

Example 1: Notch filter

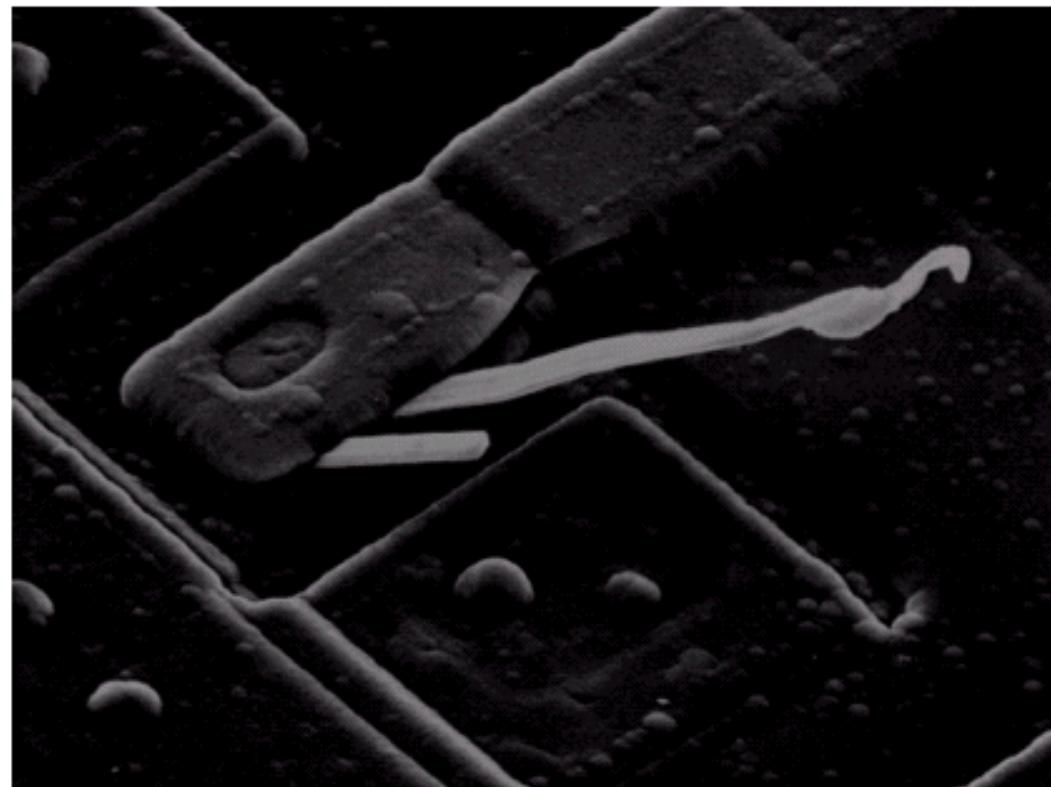
$$H(u, v) = \begin{cases} 0 & \text{if } (u, v) = (0, 0) \\ 1 & \text{otherwise} \end{cases}$$

Note: The origin can also locate at $(M/2, N/2)$ for a $M \times N$ image matrix

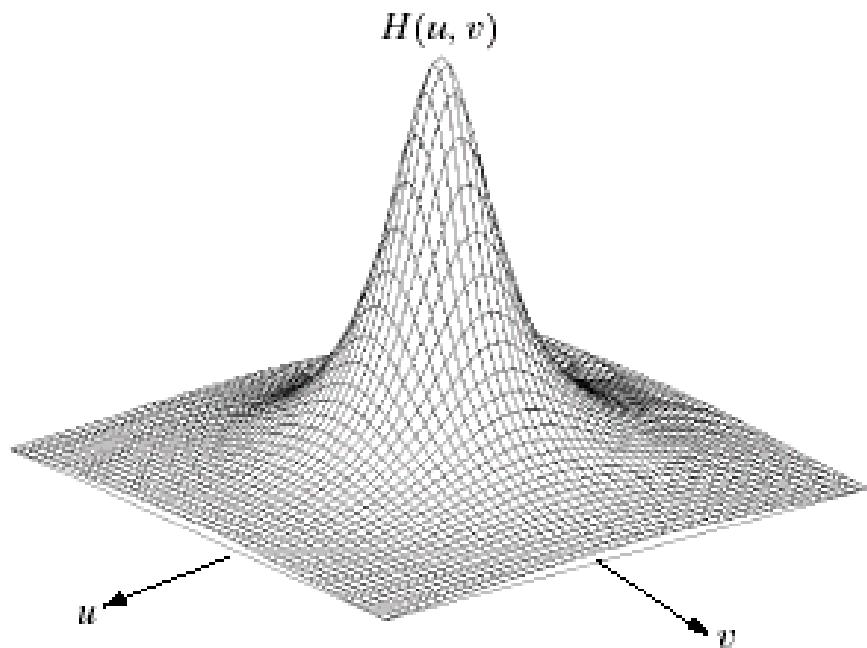
FIGURE 4.6

Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.

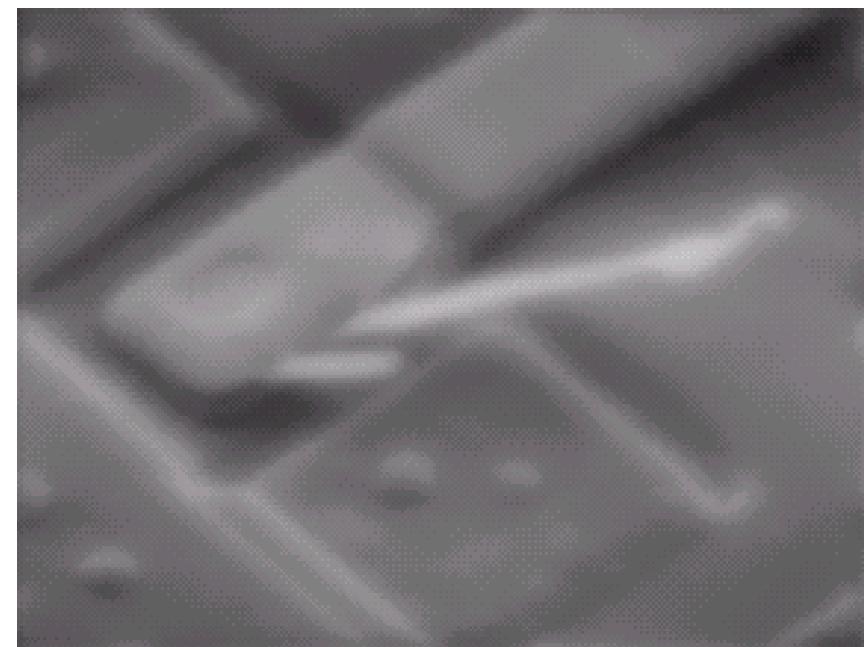
Filtered image



Example 2: Lowpass filter

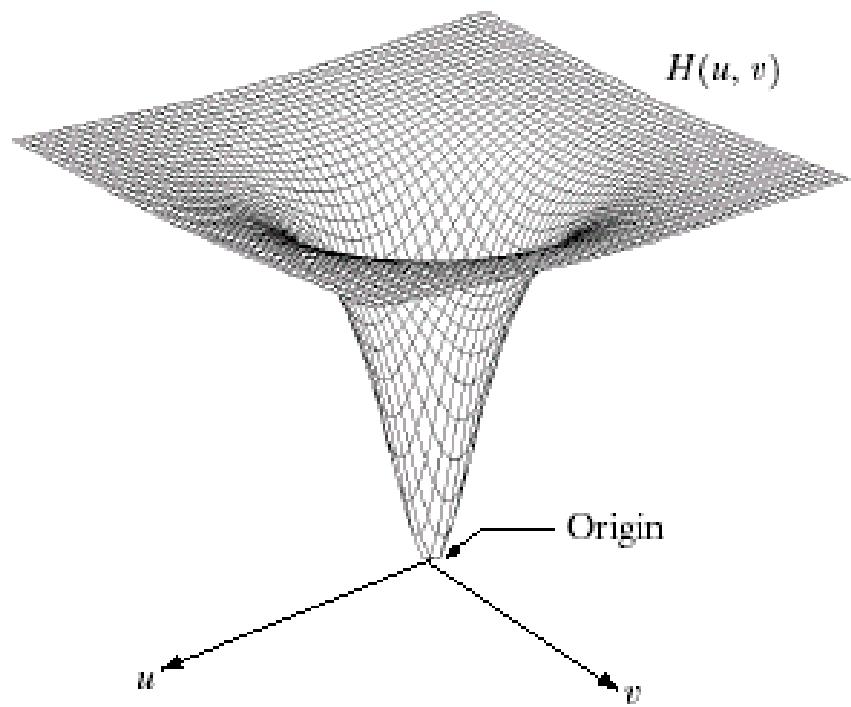


Filter
Circularly symmetric



Filtered image

Example 3: Highpass filter



Filter
Circularly symmetric



Filtered image

Smoothing Frequency-Domain Filters

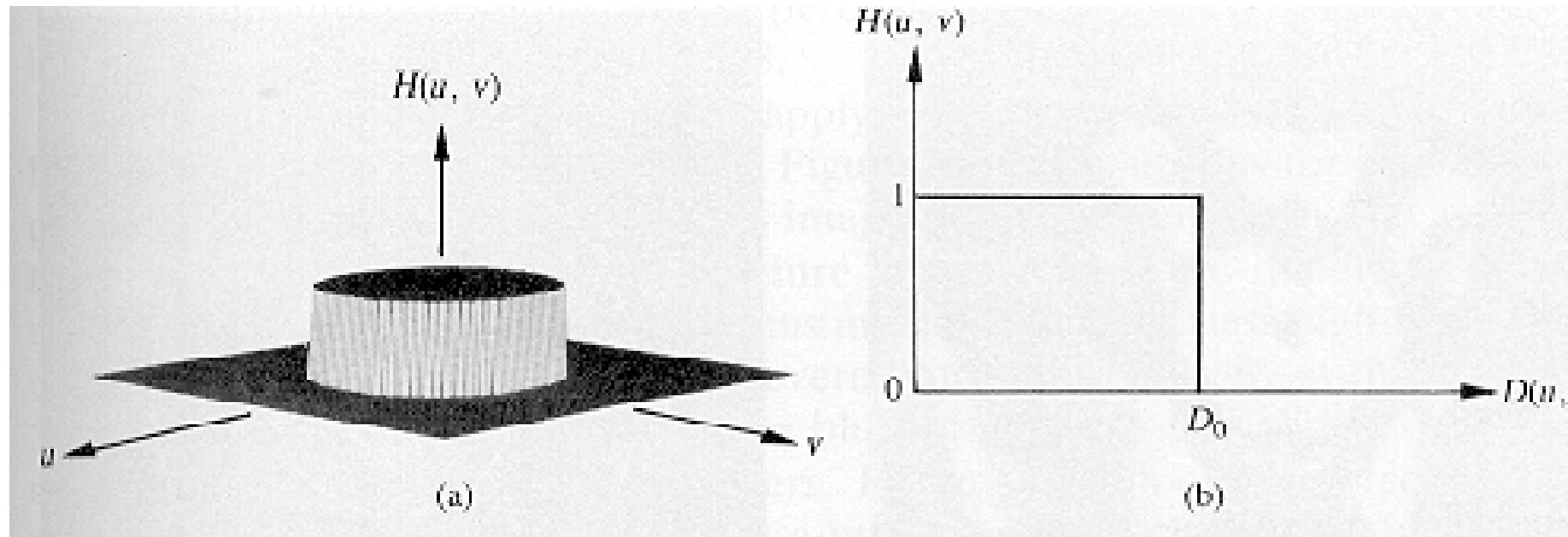
Ideal Lowpass Filter (ILPF)

- Ideal Lowpass Filter

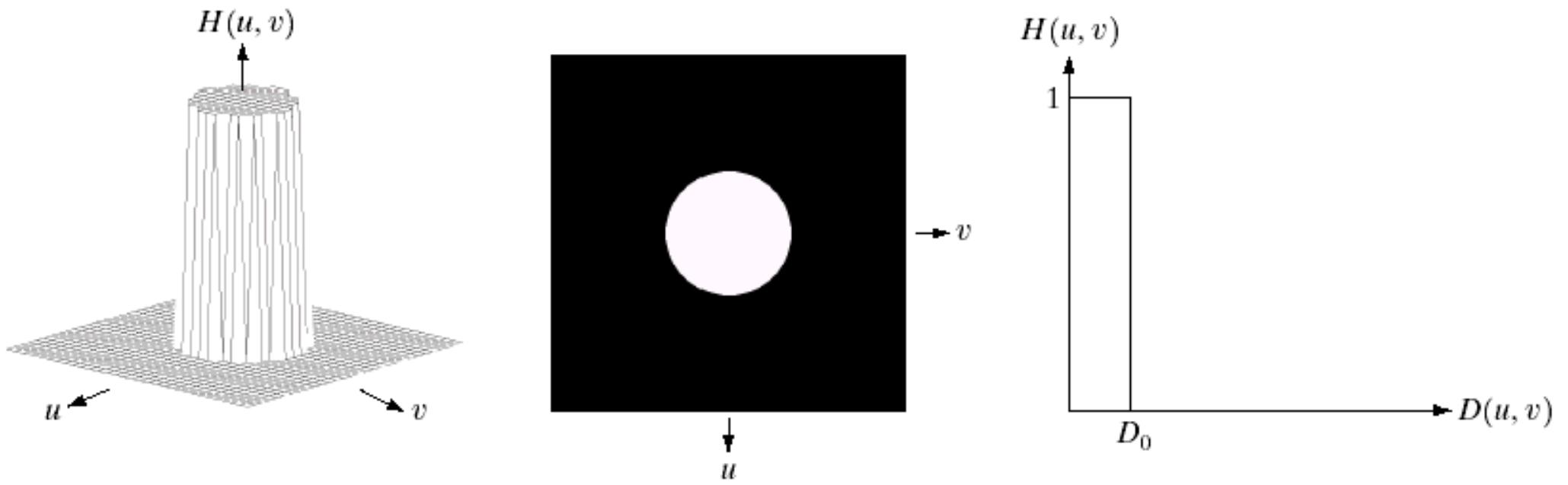
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

- $D(u,v)$ is the distance from point (u,v) to the origin of the frequency plane; $D(u,v) = (u^2 + v^2)^{1/2}$
- D_0 is a non-negative quantity

Graphical Representation of the Ideal Lowpass Filter



Graphical Representation of the Ideal Lowpass Filter



a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Image Power

- We can compute the amount of power a filter “encloses” from the total Power P_T

$$P_T = \sum_u \sum_v P(u, v)$$

$$P(u, v) = R(u, v)^2 + I(u, v)^2$$

(Real and Imaginary Components of DFT)

Image Power

- A filter, centered at the origin of the frequency plane with radius r , enclosed β percent of power

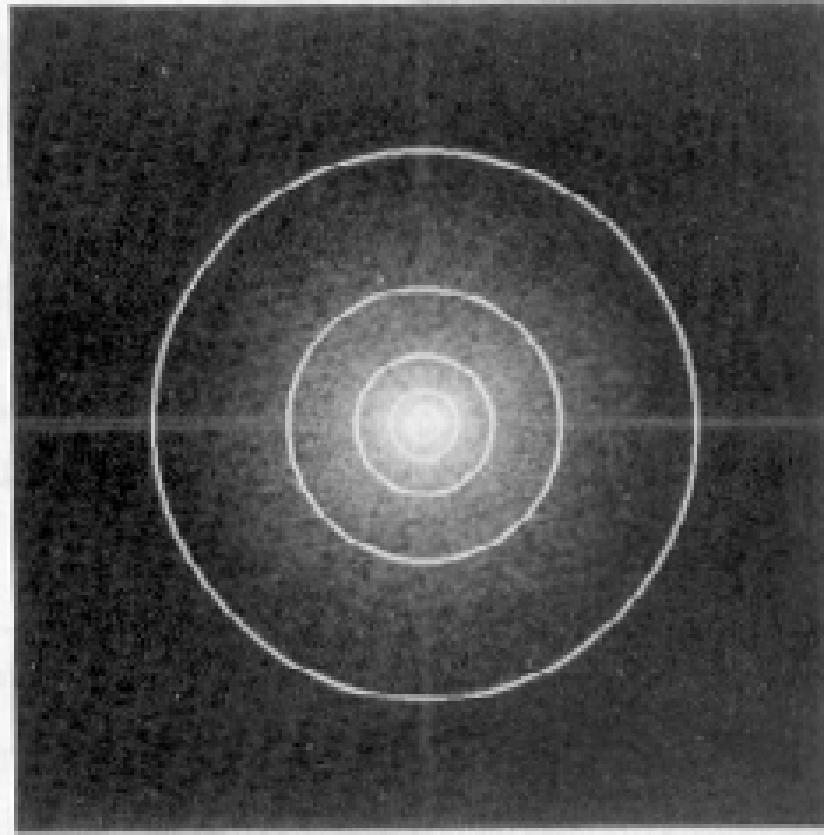
$$\beta = \sum_u \sum_v \frac{P(u, v)}{P_T} \times 100\%$$

- where (u, v) are summed over the coordinates that lie within the circle or on its boundary

Example



(a)



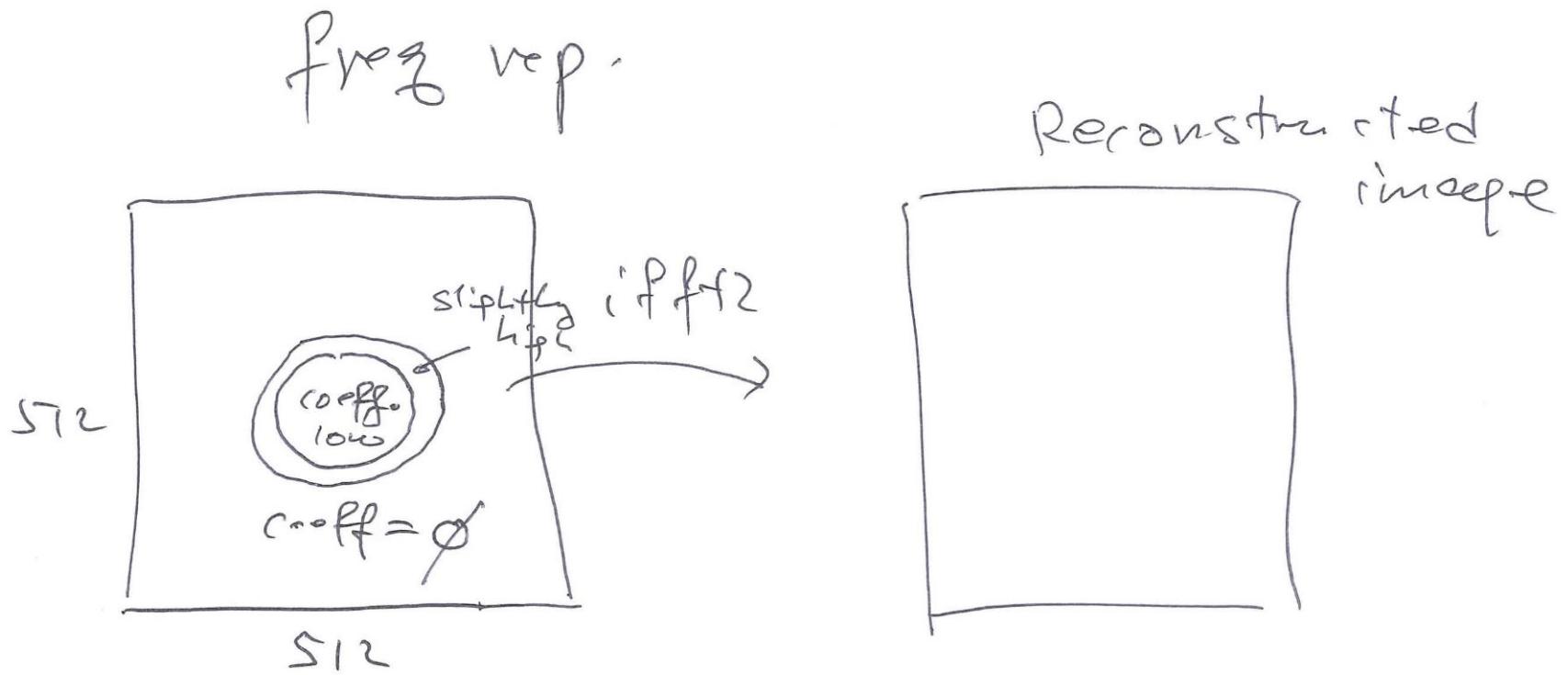
(b)

- A 512x512 image
- Filters with radii = 8,18,43,78,153 pixels
- Power = 90%, 93%, 95%, 99%, 99.5% respectively

A Faster Fast Fourier Transform: “Natural signals often have relatively few frequency components of significance.”

<http://spectrum.ieee.org/computing/software/a-faster-fast-fourier-transform>

Example

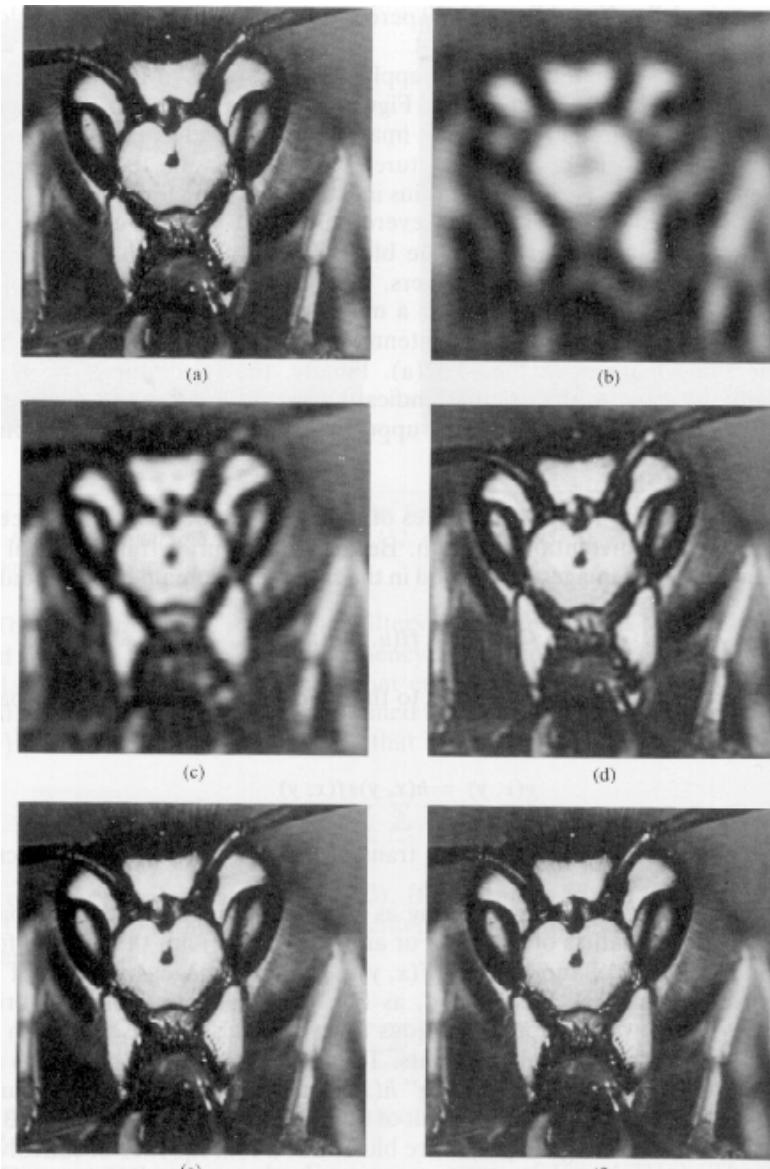


$$512 \times 512 = 262144 \text{ coefficients}$$

$$\pi((53)^2) = 73542 \text{ (28\%)}$$

Filtering with Ideal filter

Original



8 (Filter Radius)

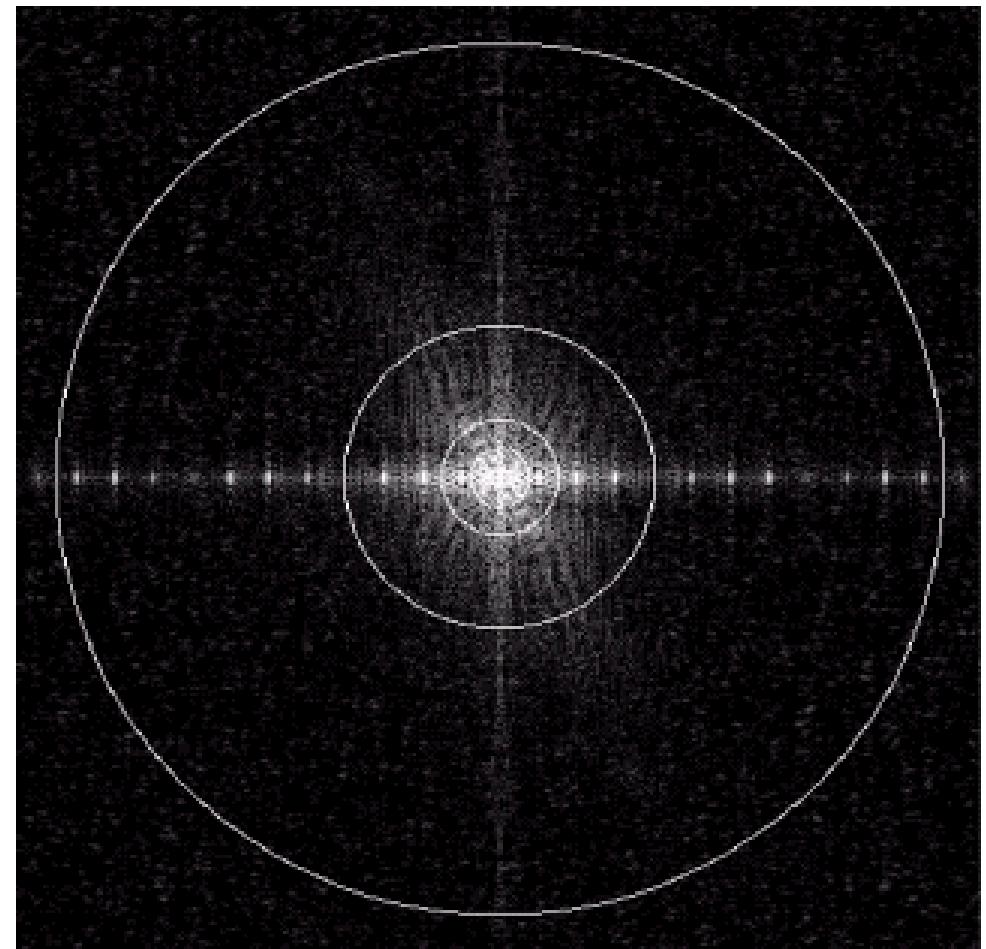
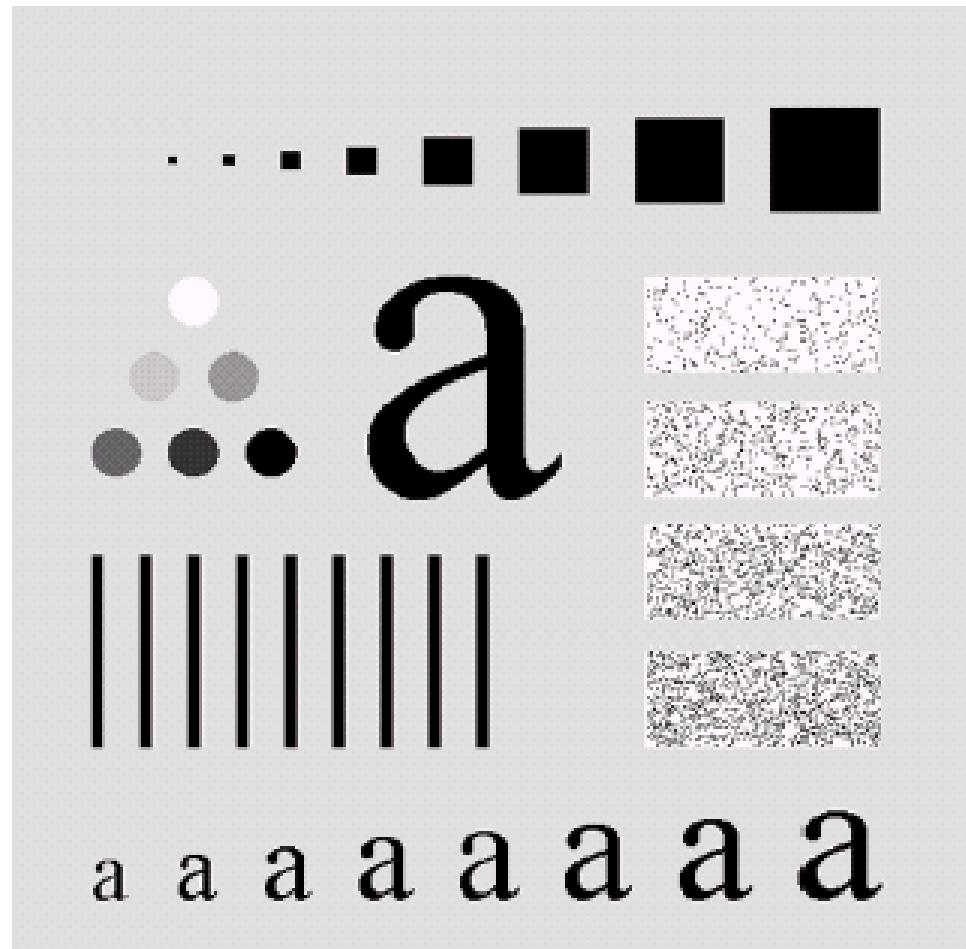
43

18

78

153

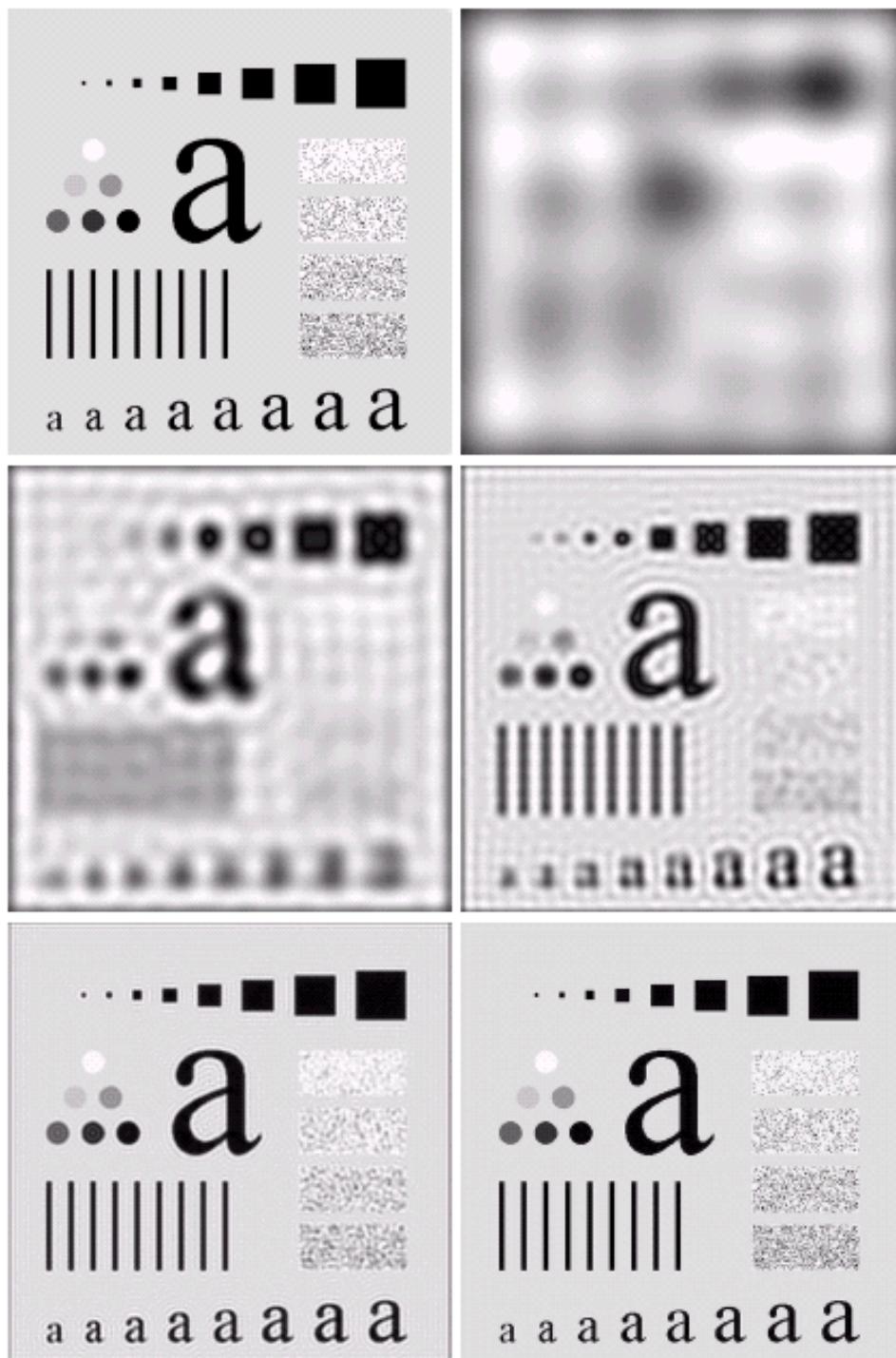
Another example



a b

FIGURE 4.11 (a) An image of size 500×500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Filtering with Ideal filter



a b
c d
e f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

Ideal Filter

- The sharp cut off of the ideal filter in the frequency domain can result in ripples appearing in the image. It is called ringing artifact.
- We can design a filter with a smooth cut off.

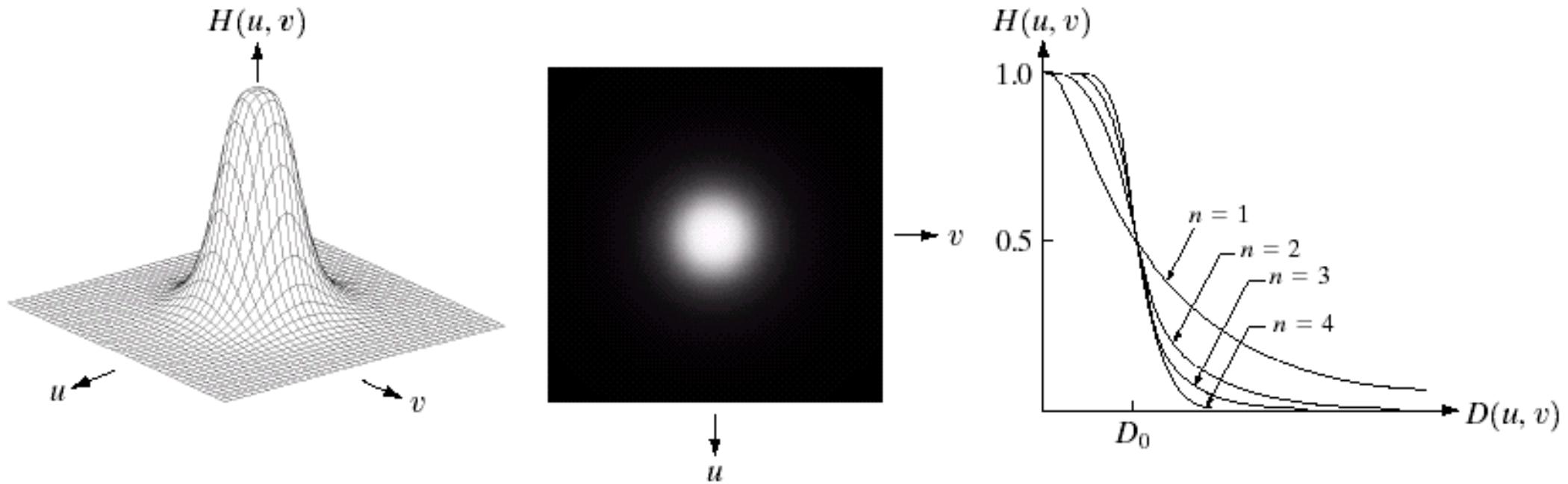
http://en.wikipedia.org/wiki/Ringing_artifacts

Butterworth Lowpass Filter (BLPF)

- This filter does not have a sharp discontinuity
 - Instead it has a smooth transition
- A Butterworth filter of order n and cutoff frequency locus at a distance D_0 has the form

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$

- where $D(u, v)$ is the distance from the center of the frequency plane, and D_0 is a positive quantity

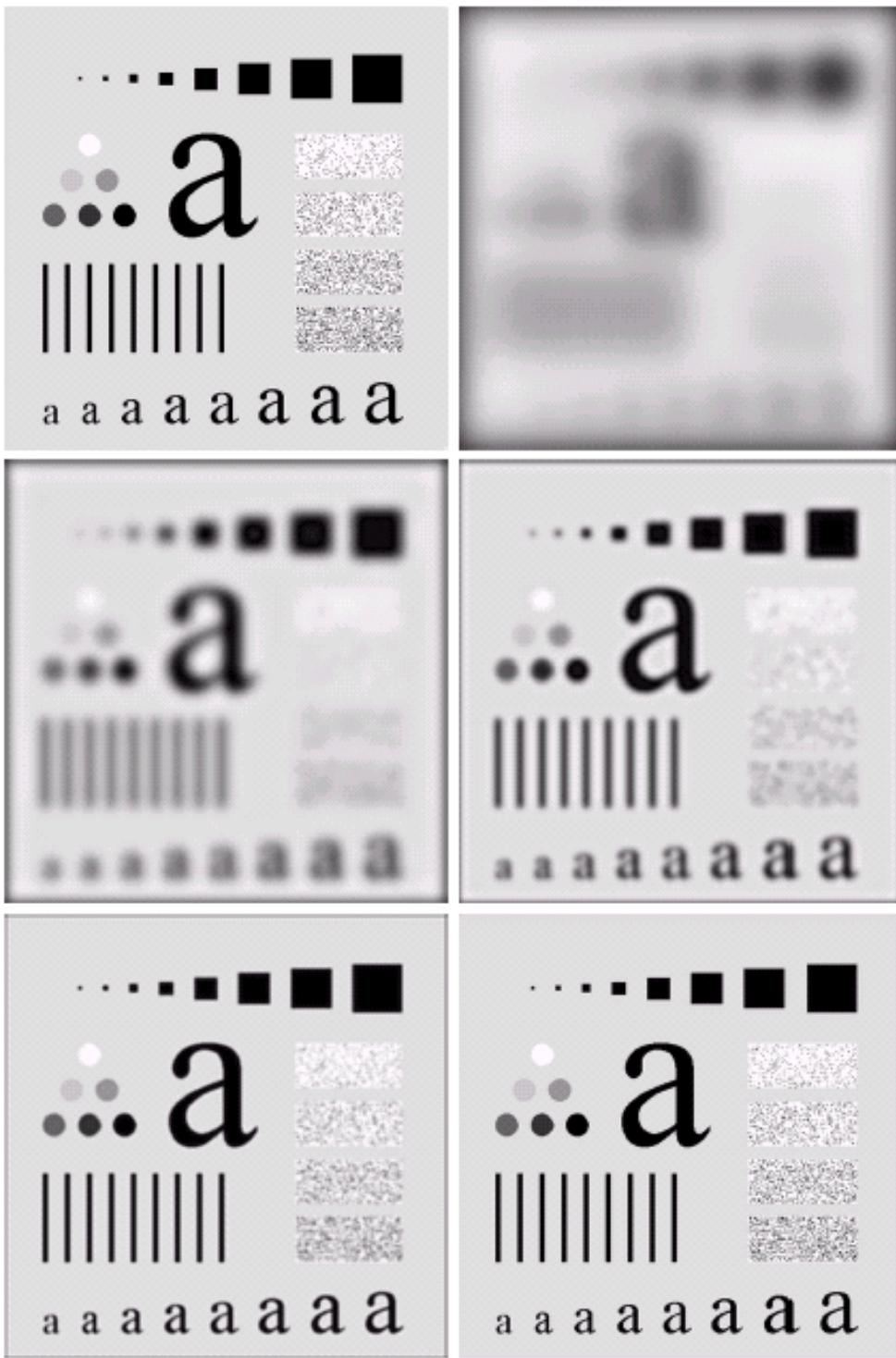


a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

1. The BLPF transfer function does not have a sharp discontinuity that sets up a clear cutoff between passed and filtered frequencies.
2. No ringing artefact visible when $n = 1$. Very little artefact appears when $n \leq 2$ (hardly visible).

Example
(BLPFs)
Order = 2



a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Slight Variation

- For these type filters
 - We often want to define a cutoff frequency
 - For which $H(u,v)$ is decreased to a certain fraction of its maximum value
 - For example: Let $H(u,v) = 0.5$ when $D(u,v) = D_0$ and α is 1

$$H(u,v) = \frac{1}{1 + (\alpha)[D(u,v)/D_0]^{2n}}$$

Relationship between filter shape and parameters

$$H(D) = \frac{1}{1 + \alpha \left(\frac{|D|}{D_0}\right)^{2n}}$$

$$\left. \frac{dH}{dD} \right|_{D=D_0} = \frac{-2n\alpha}{D_0 (1+\alpha)^2}$$

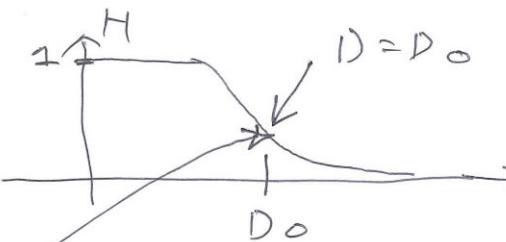
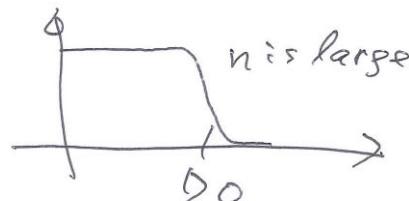
when $\alpha = 1$

$$\boxed{\left. \frac{dH}{dD} \right|_{D=D_0} = \frac{-n}{2D_0}}$$

$$n \uparrow \Rightarrow \left. \frac{dH}{dD} \right|_{D=D_0} \uparrow$$

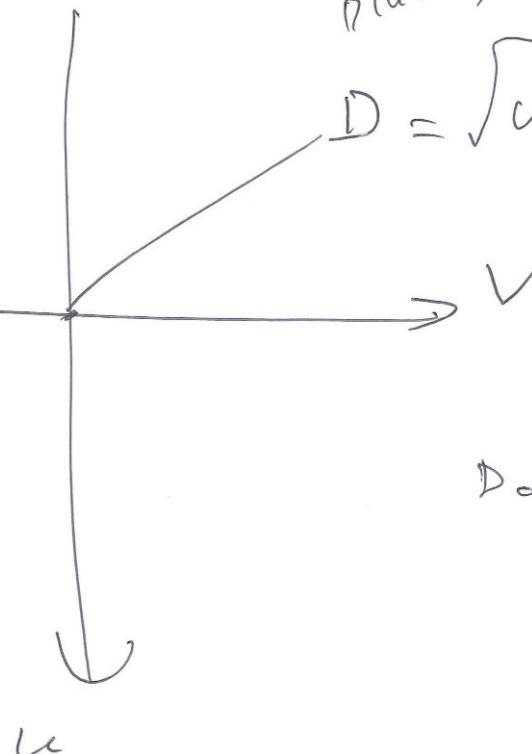
more negative

\Rightarrow more steep



$$\eta(u, v)$$

$$D = \sqrt{u^2 + v^2}$$



$D_0 = \text{cutoff}$

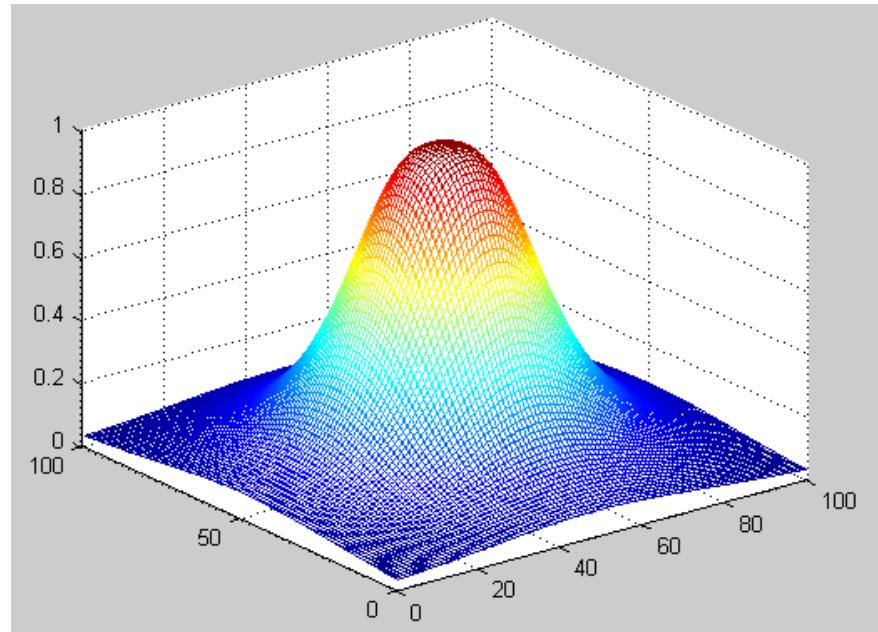
Slight Variation

- Another example is to have $H(u,v)$ be $1/\sqrt{2}$ ($=0.707$) of the maximum value of $H(u,v)$ when $D(u,v) = D_0$ and α is $\sqrt{2} - 1$ ($=0.414$).

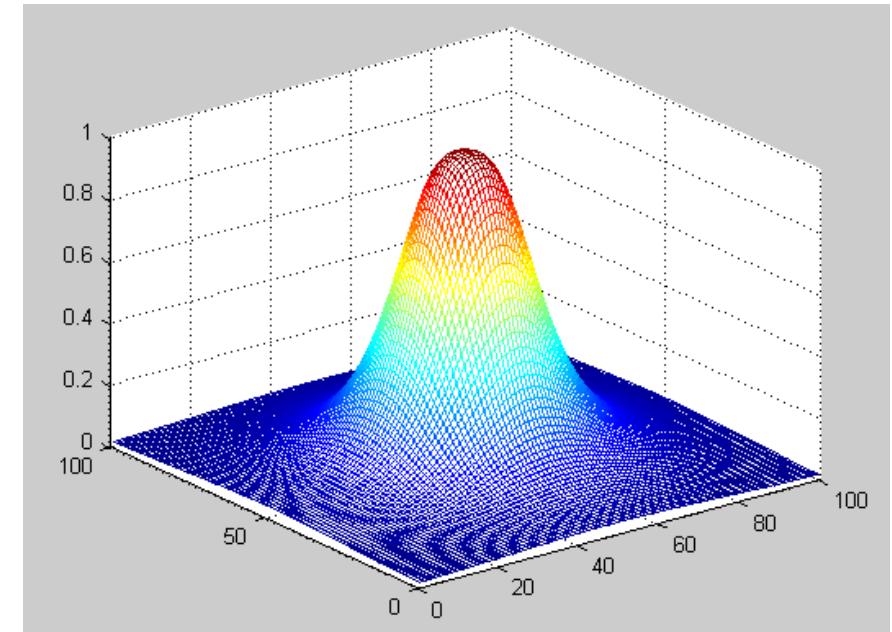
$$H(u,v) = \frac{1}{1 + (\sqrt{2} - 1)[D(u,v)/D_0]^{2n}}$$

$$H(u,v) = \frac{1}{1 + 0.414[D(u,v)/D_0]^{2n}}$$

Examples

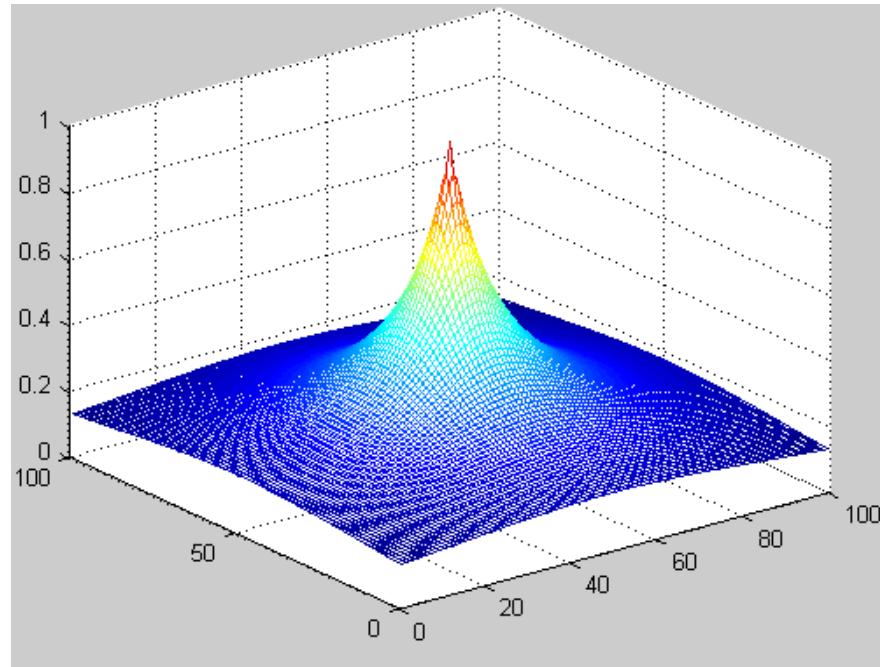


100×100
 $D_0 = 20$
 $\alpha = 0.414$
 $n = 3$

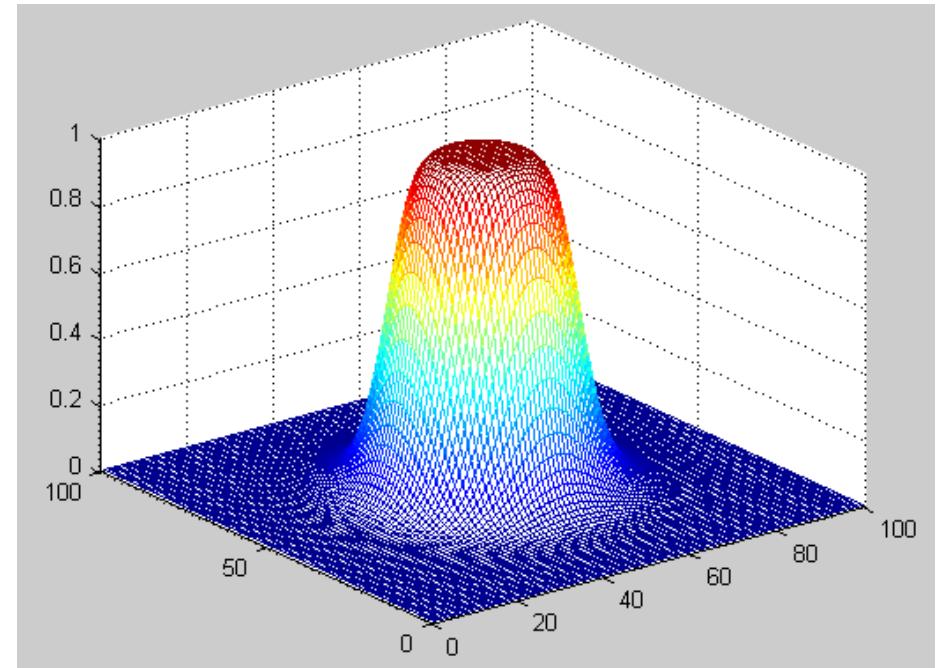


100×100
 $D_0 = 20$
 $\alpha = 1$
 $n = 3$

Examples

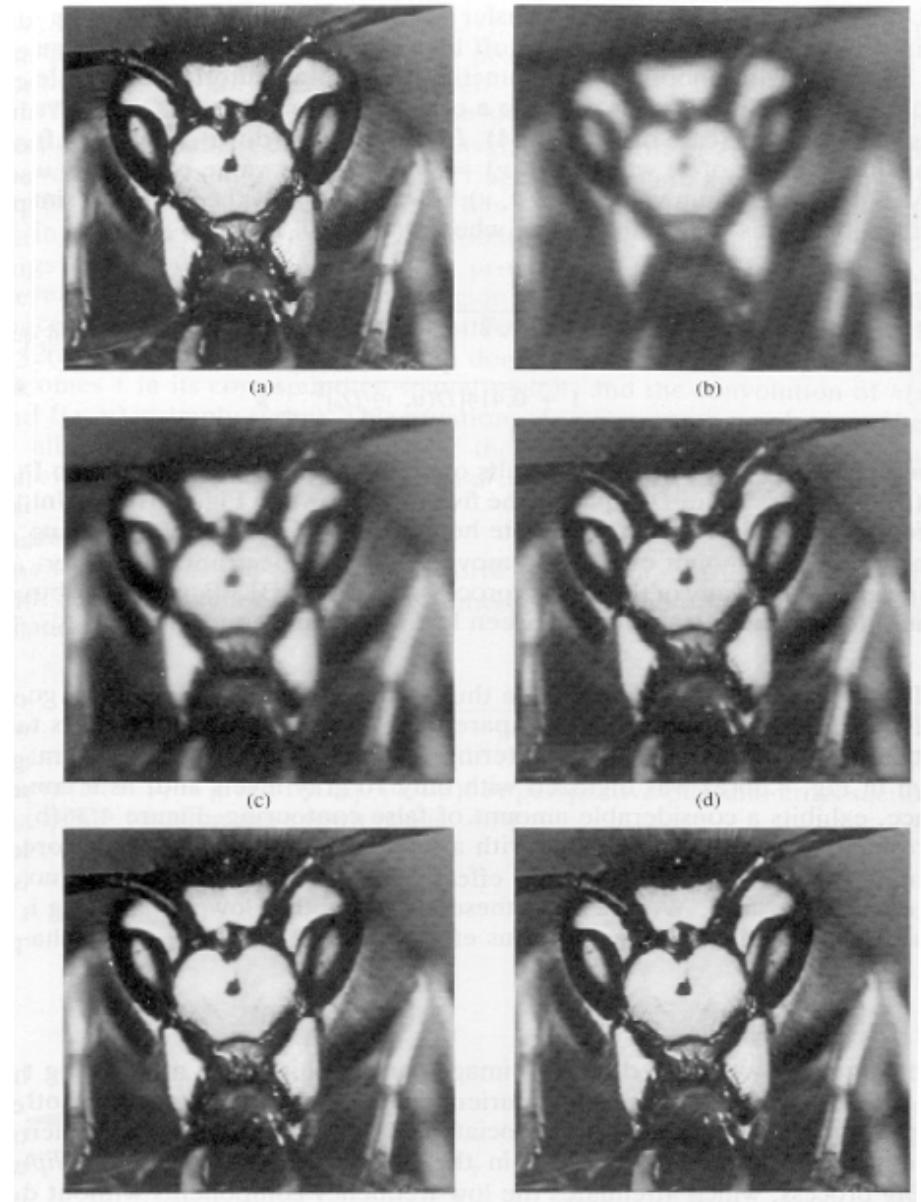


100×100
 $D_0 = 20$
 $\alpha = 1$
 $n = 1$



100×100
 $D_0 = 20$
 $\alpha = 1$
 $n = 7$

Example



Butterworth filters
with different
cut-offs.

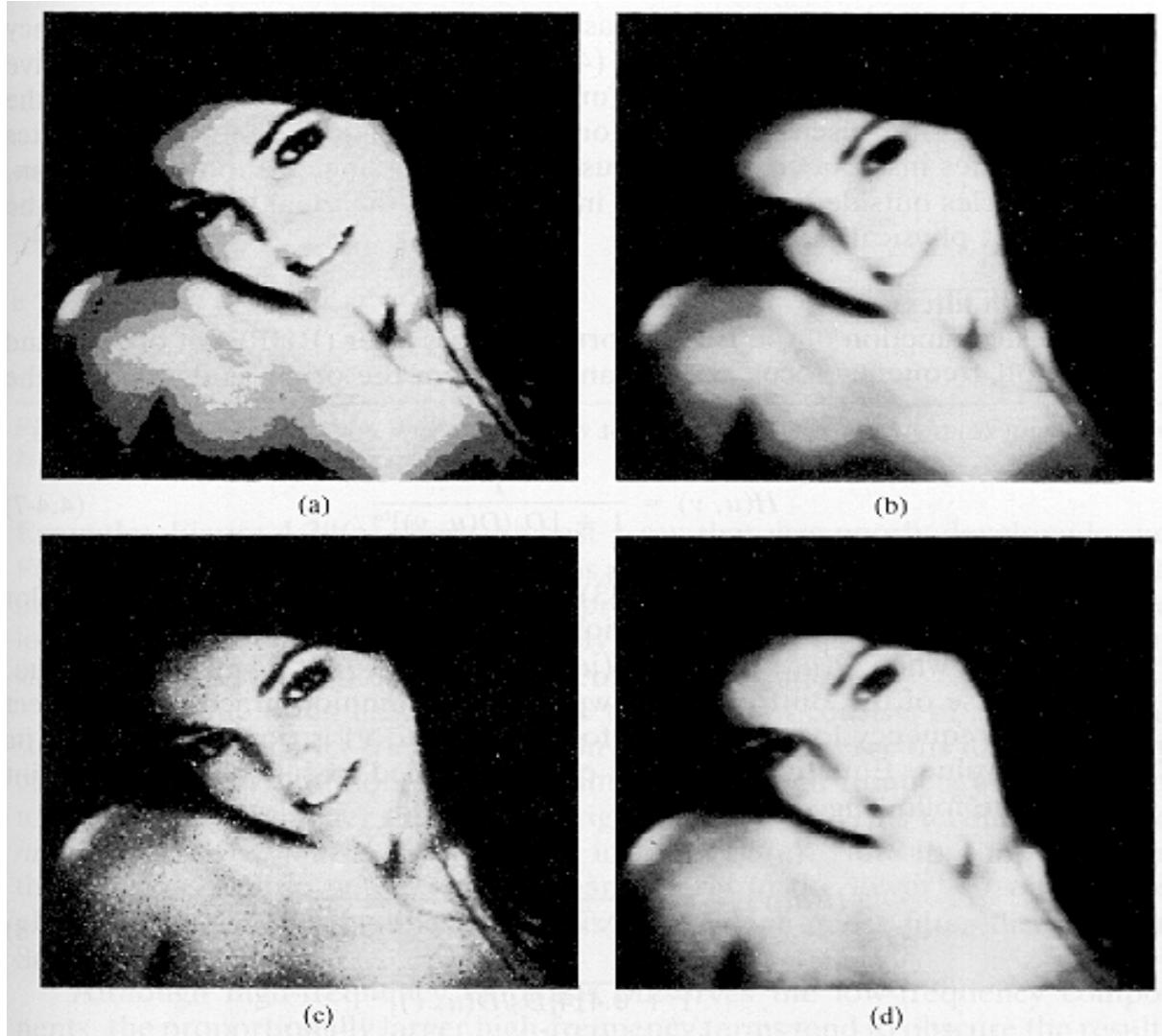
This is smoother,
although it is letting
more high-frequencies
Through.

Example

Quantization
Noise

Random
Noise

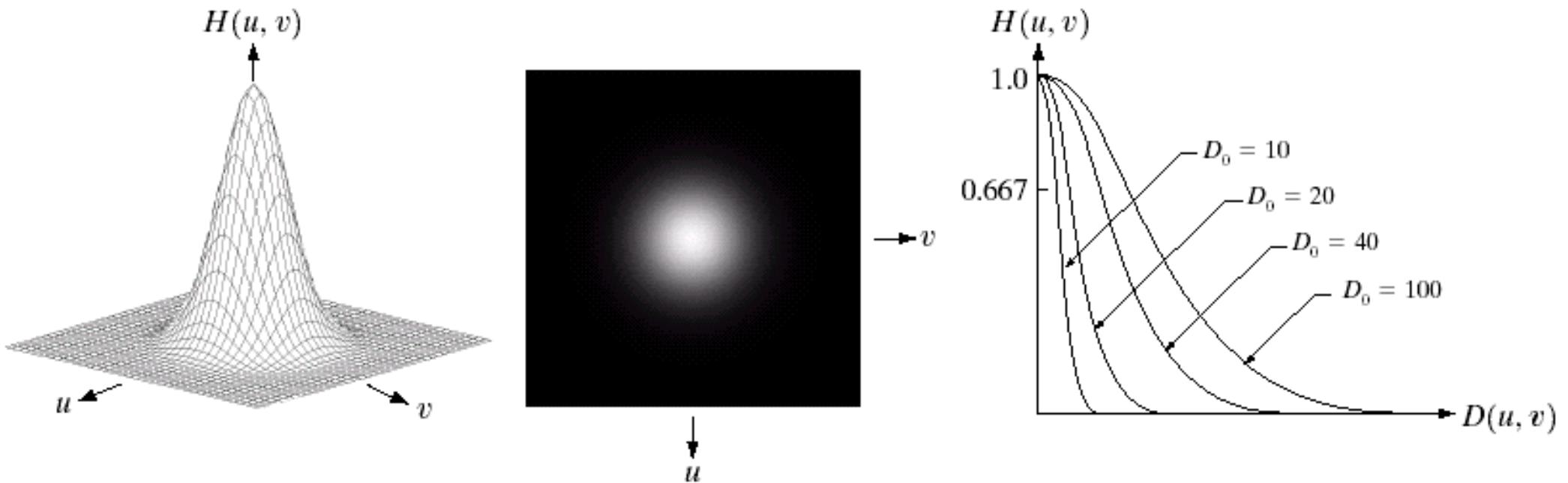
Butterworth Filters



Gaussian Lowpass Filter (GLPF)

- This filter is one of the commonly used filters.

$$H(u, v) = e^{-D^2(u, v)/2D_0^2}$$

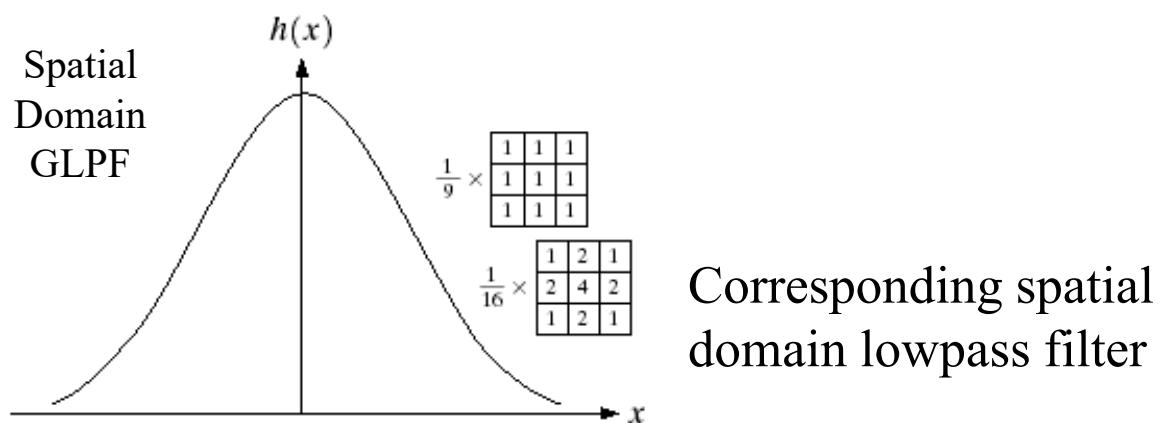
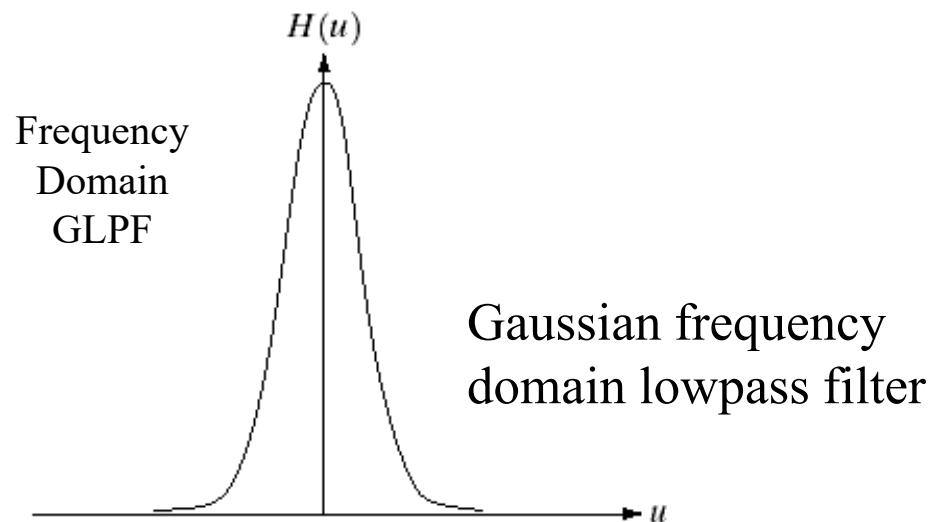


a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0 .

Gaussian Lowpass Filter (GLPF)

$$H(u, v) = e^{-D^2(u, v)/2D_0^2} \Leftrightarrow h(x, y) = \sqrt{2\pi} D_0 e^{-2\pi^2 D_0^2 D^2(x, y)}$$

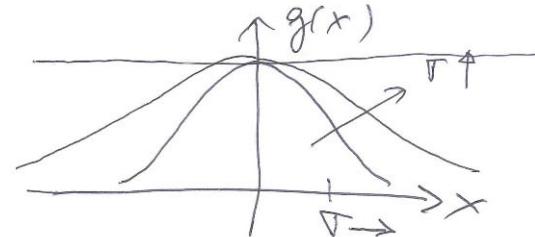


Relationship between D_0 and Standard Deviation

$$h(x, y) = \sqrt{2\pi} D_0 e^{-\frac{2\pi^2 D_0^2}{\Gamma^2} \text{distance}}$$

↑
cutoff
freq.

$$g(x, y) = \sqrt{2\pi} D_0 e^{-\frac{(x^2 + y^2)}{2\sigma^2}}$$



$$\sqrt{2\pi^2 D_0^2} = \sqrt{\frac{x^2 + y^2}{2\sigma^2}}$$

$$\dots | g(x=-1) | g(x=0) | g(x=1) | \dots$$

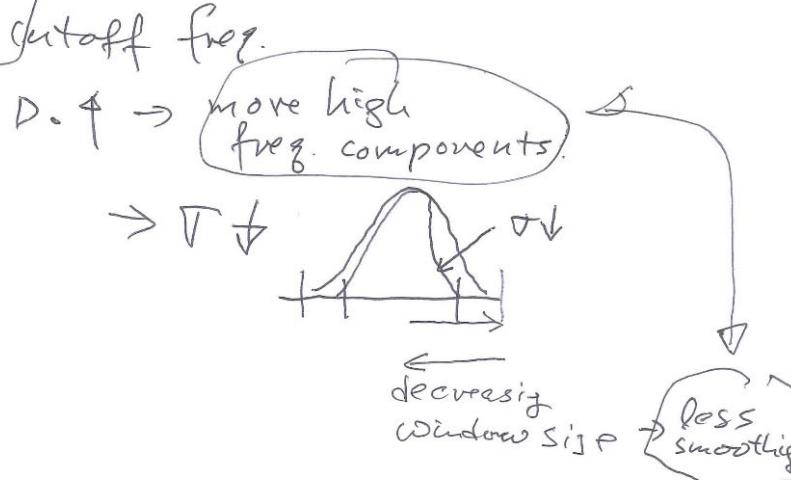
imfilter(\square, g)

$$D_0 = \frac{1}{2\pi\sigma}$$

or

$$\sigma = \frac{1}{2\pi D_0}$$

↑
cutoff freq.



Example (GLPFs)

- No ringing artefact.

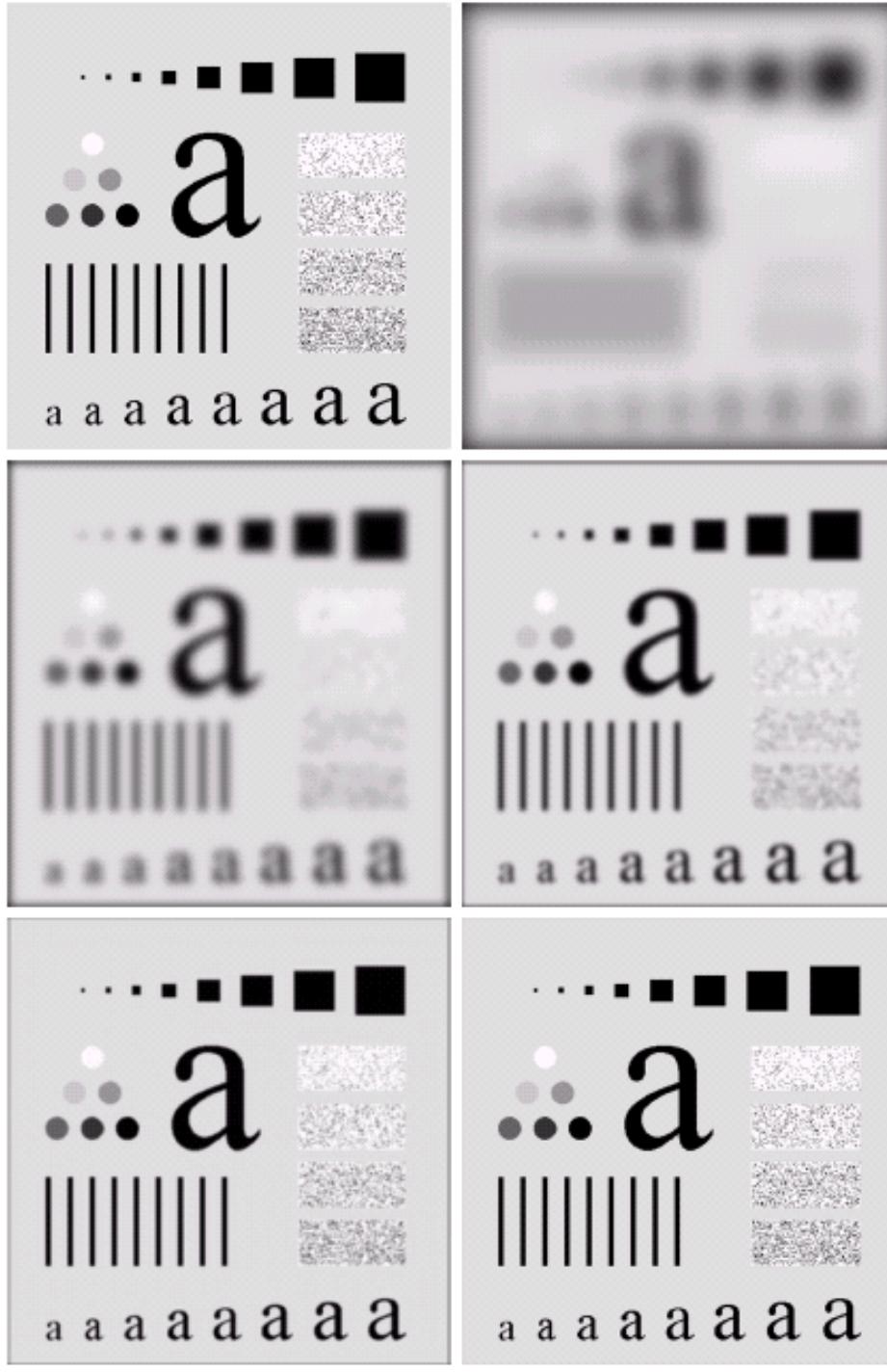


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

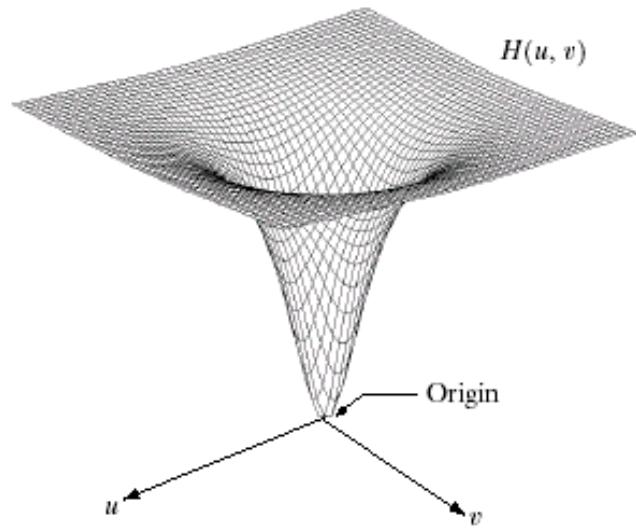
Sharpening Frequency-Domain Filters

Concepts

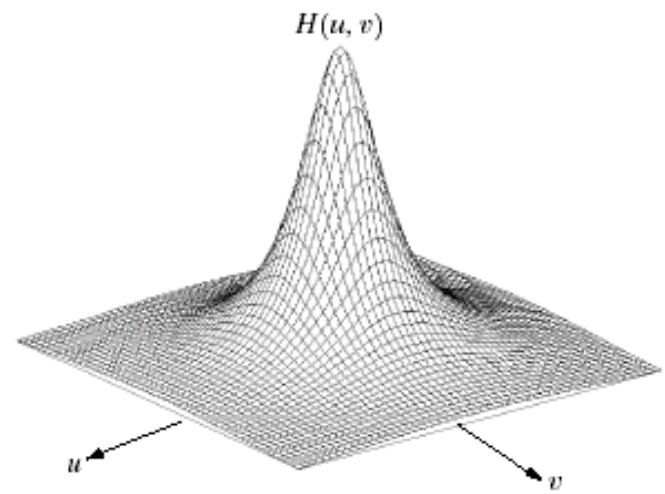
1. Because edges and other abrupt changes in grey levels are associated with high-frequency components, image sharpening can be achieved by using the high-pass frequency filtering process.
2. The high-pass filter attenuates (suppresses) the low-frequency components without disturbing high-frequency information in the frequency domain.
3. Relation between low-pass and high-pass filters

$$H_{hp}(u, v) = 1 - H_{lp}(u, v)$$

(u, v) represents position variables in the frequency domain (frequency variables).



= 1 -



$$H_{hp}(u, v) = 1 - H_{lp}(u, v)$$

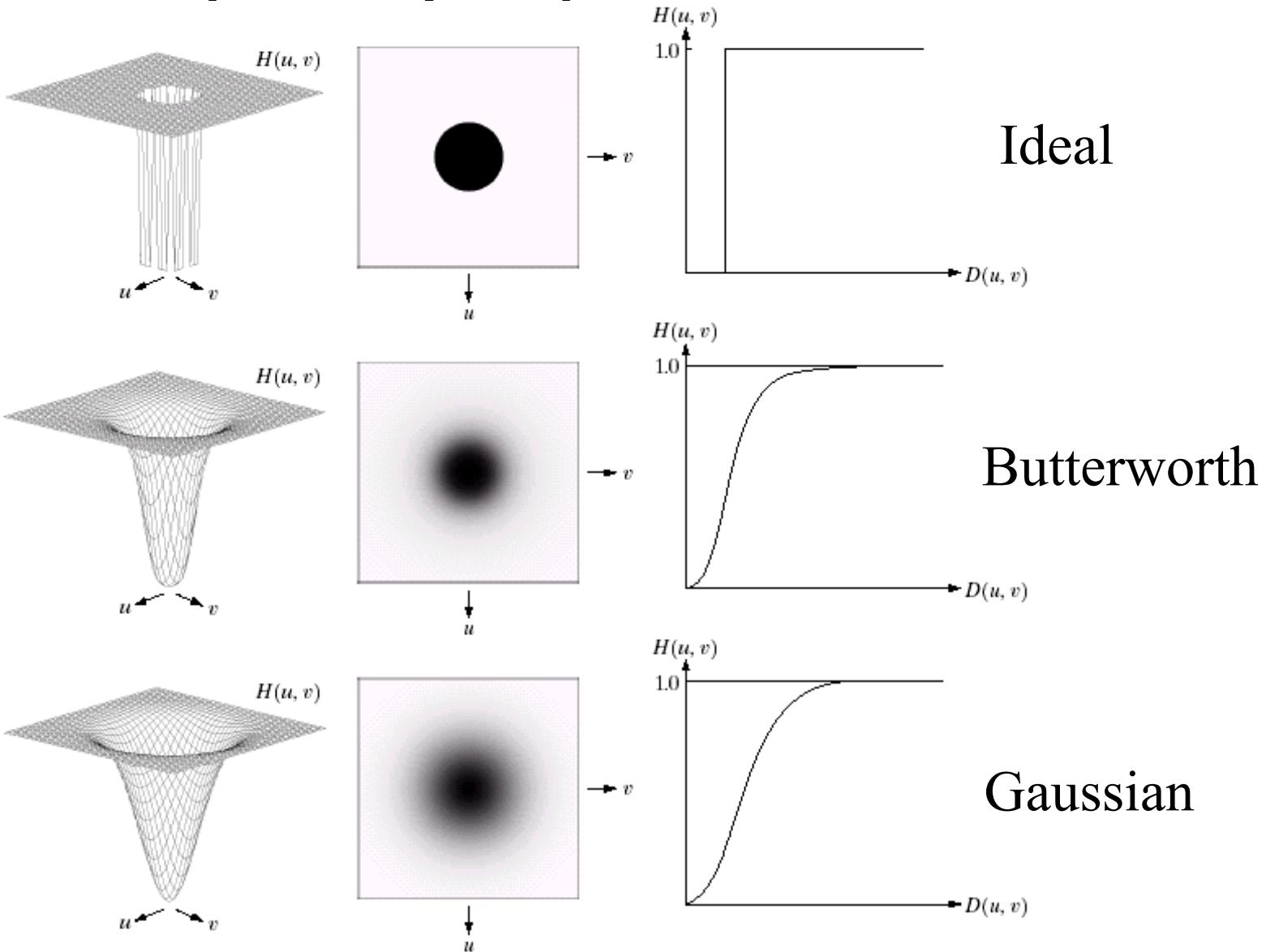
Ideal High Pass Filter (IHPF)

- Ideal Filter

$$- H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

- where $D(u,v)$ is the distance from point (u,v) to the origin of the frequency plane; $D(u,v) = (u^2 + v^2)^{1/2}$
- D_0 is a non-negative quantity, cutoff frequency

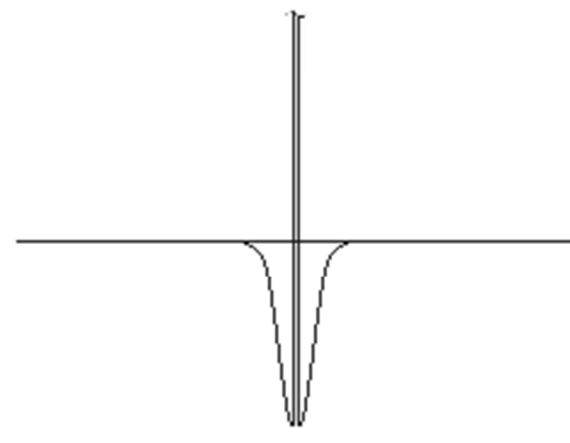
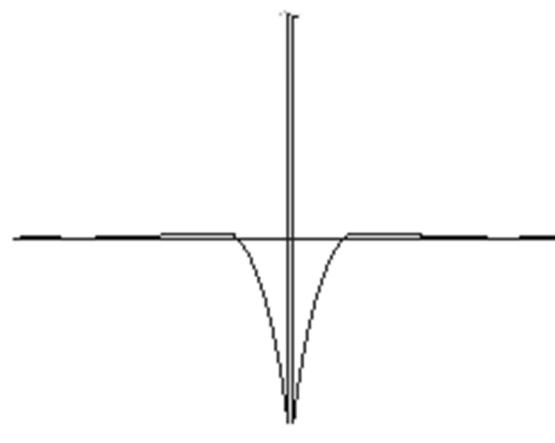
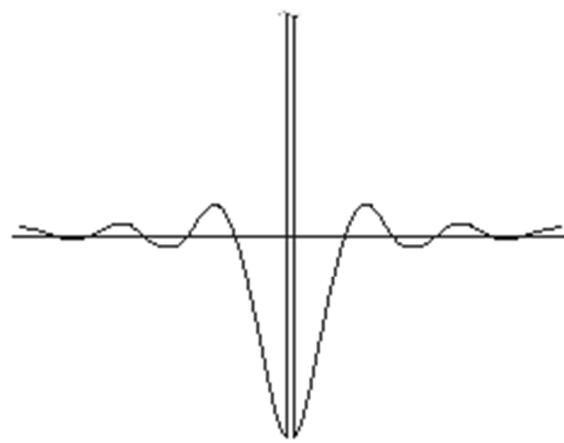
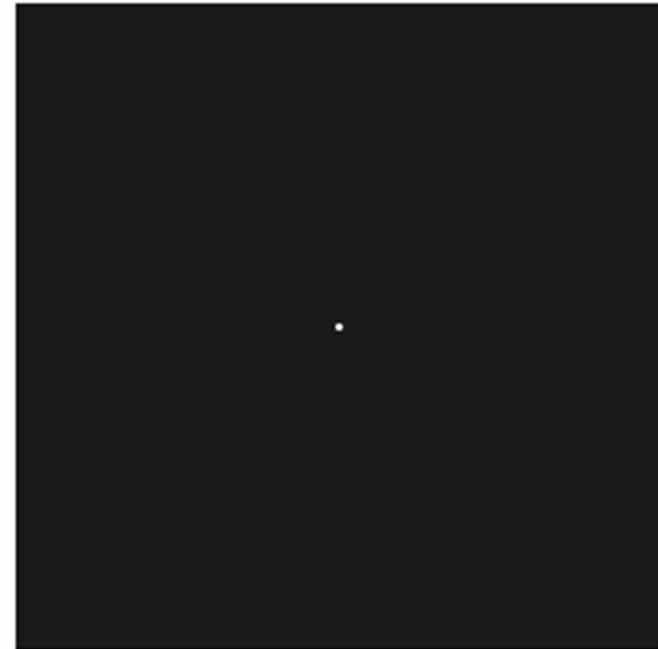
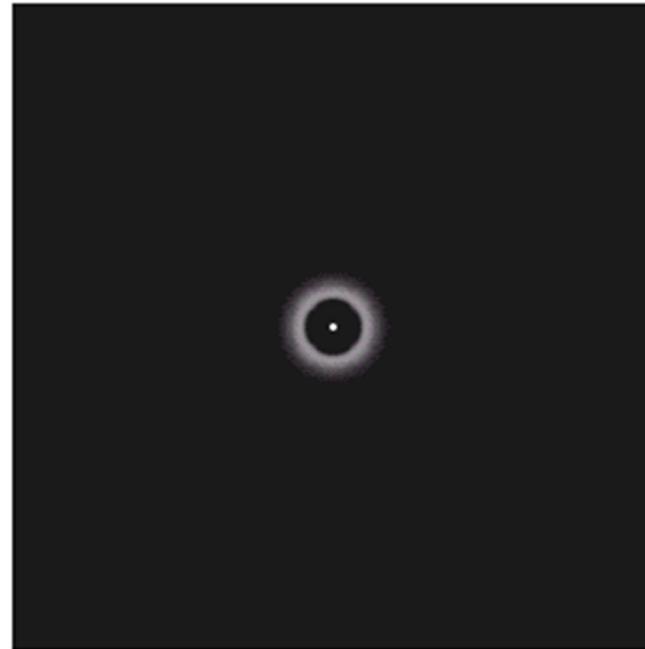
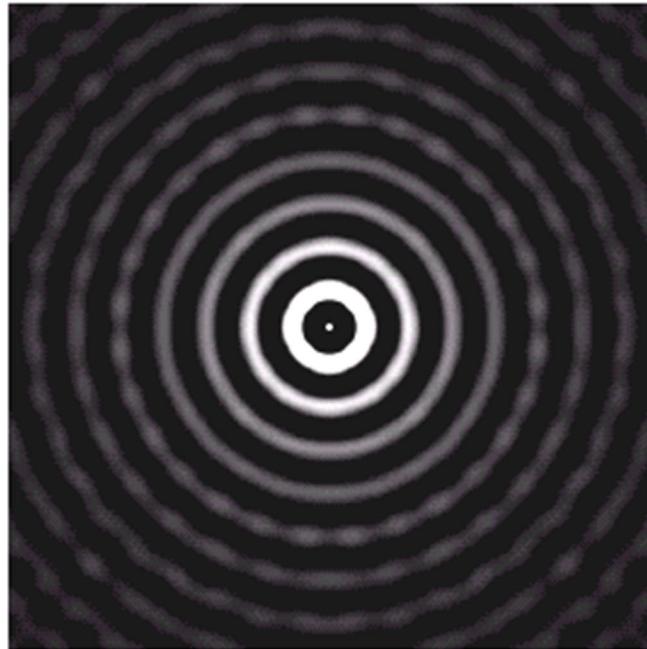
Frequency representations



a	b	c
d	e	f
g	h	i

FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Spatial representations

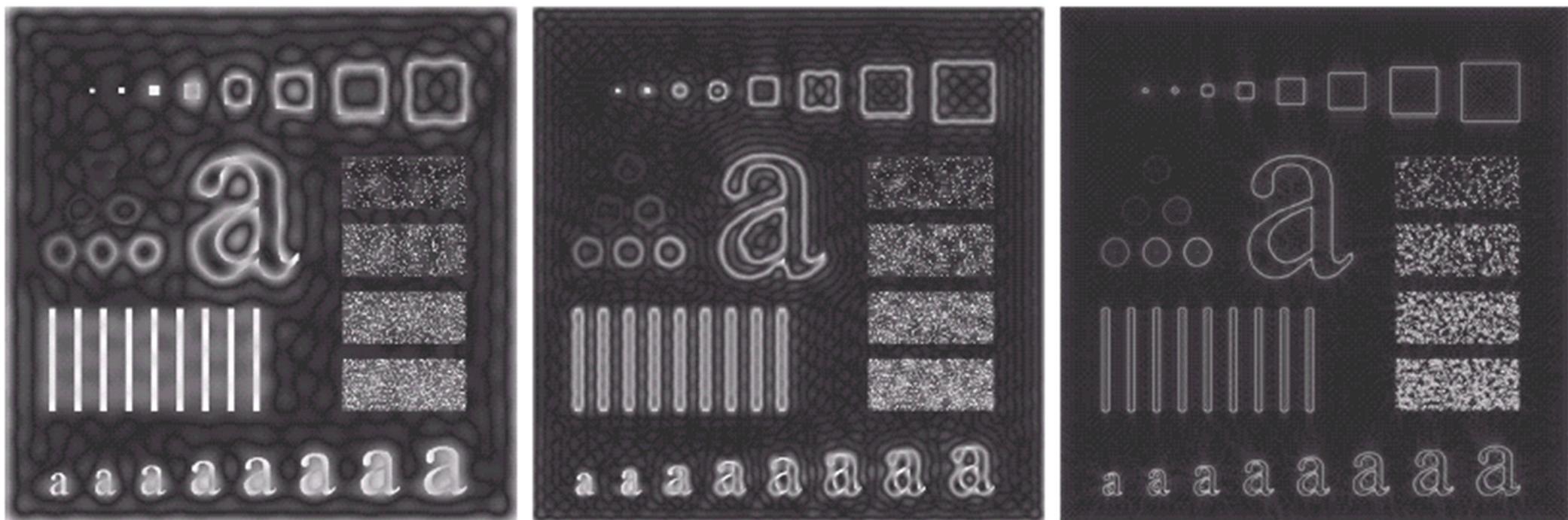


a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Ideal High Pass Filter

- We can expect IHPFs to have the same ringing properties as ILPFs.



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15, 30,$ and $80,$ respectively. Problems with ringing are quite evident in (a) and (b).

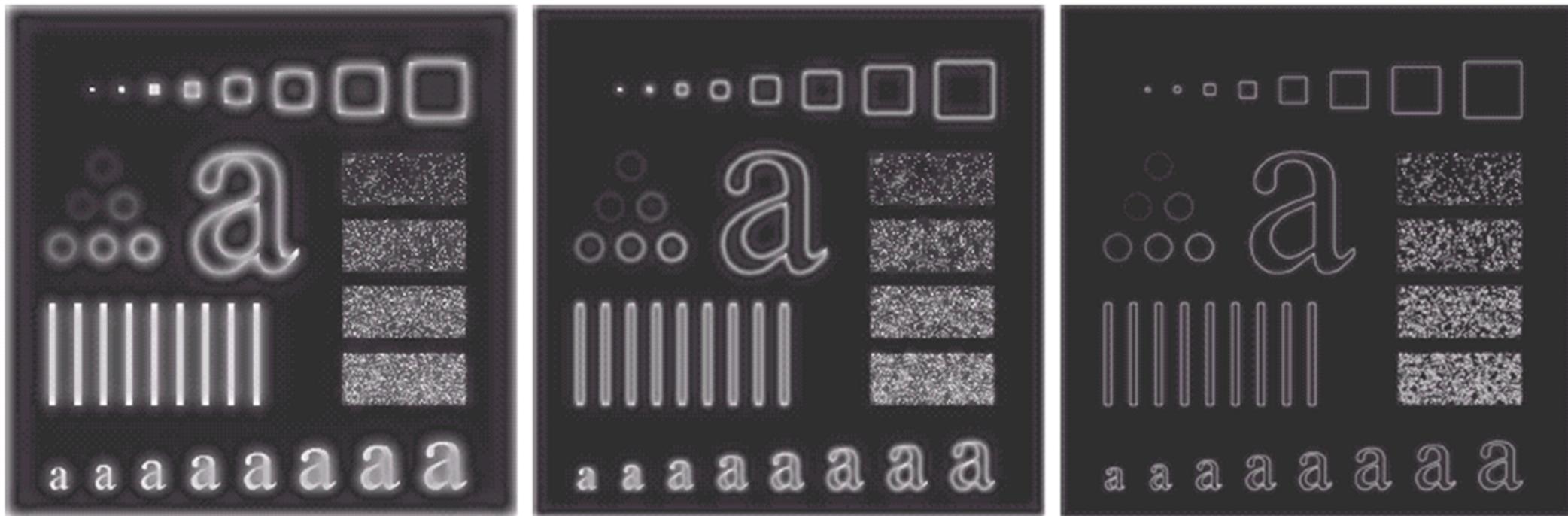
High Pass Butterworth Filter

$$H(u, v) = \frac{1}{1 + [D_0 / D(u, v)]^{2n}}$$

$$H(u, v) = \frac{1}{1 + \alpha[D_0 / D(u, v)]^{2n}}$$

High-Pass Butterworth Filter

- We can expect Butterworth highpass filters to behave smoother than IHPFs.
- The transition into higher values of cutoff frequencies is much smoother with the BHPF.



a b c

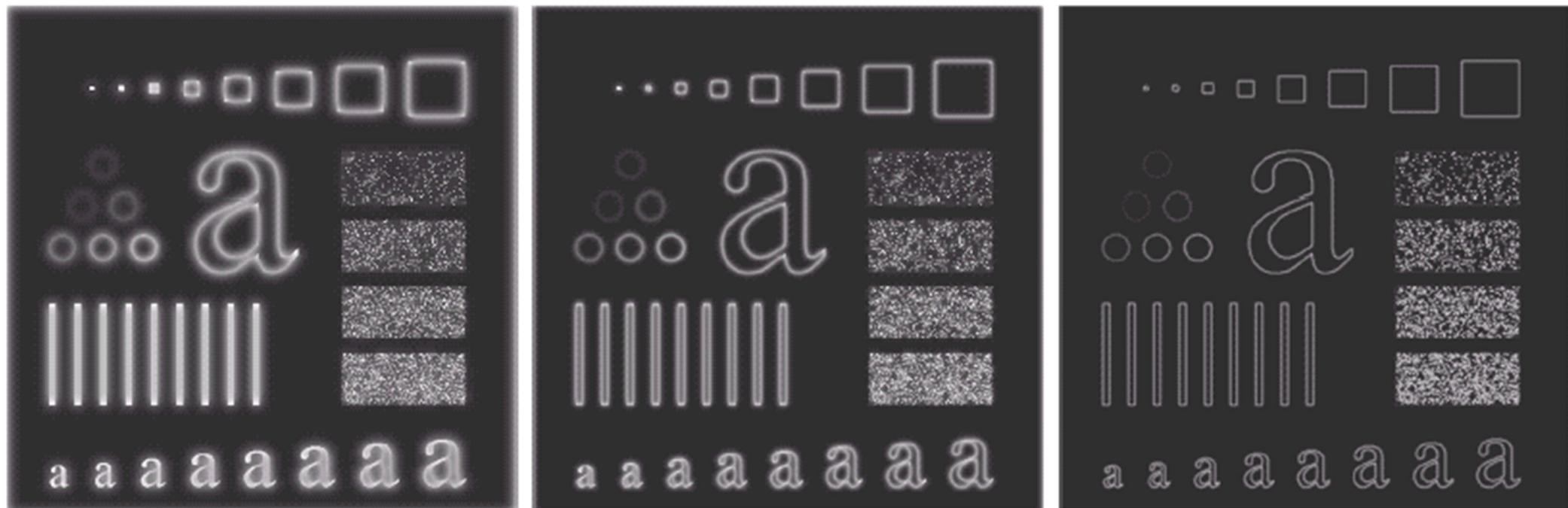
FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an IHPF.

High Pass Gaussian Filter

$$H(u, v) = 1 - e^{-D^2(u, v)/2D_0^2}$$

- It is possible to construct highpass Gaussian filters as the difference of lowpass Gaussian filters.
- Even the filtering of the smaller objects and thin bars is clearer with the Gaussian filter.

High-Pass Gaussian Filter



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

High-Frequency Emphasis

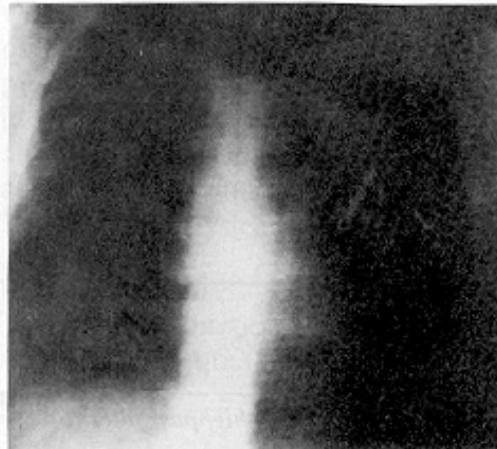
- Add a constant to the filter
- $0 \leq a \leq 1, b > a$

$$H(u, v) = a + b \frac{1}{1 + [D_0 / D(u, v)]^{2n}}$$

- Preserves low frequencies
- Amplifies high-frequencies
- (This technique is often used in conjunction with a histogram equalization)
- Analogous to the “high-boost” filter

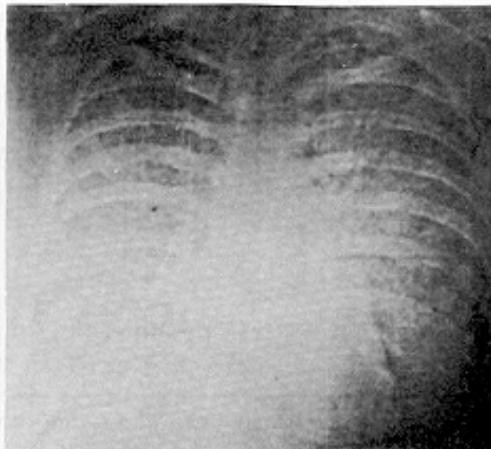
Examples

Original



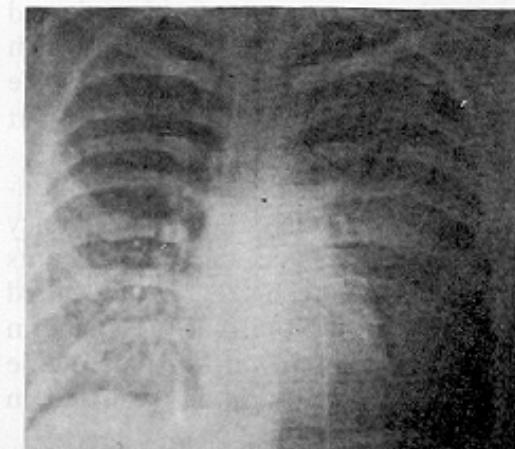
(a)

Butterworth



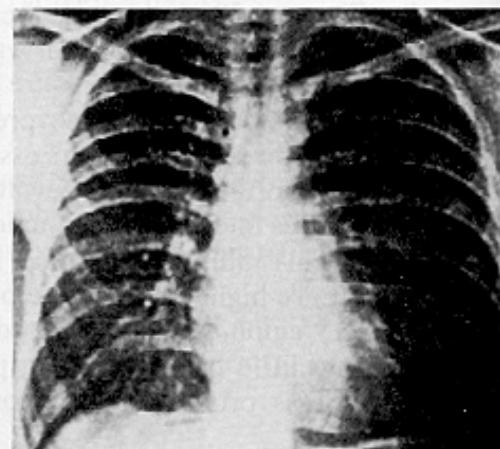
(b)

High-frequency
emphasis
 $a=0.5, b = 2.0$



(c)

High-frequency
emphasis +
histogram equalization



(d)

More example: http://www.ee.oulu.fi/research/imag/courses/dkk/exercises/2008/matlab/2/e2_h_emphasis.html