Introduction to non-linear filtering

Sharpening and Enhancement

Original

Processed

PDE and variational approaches for stable sharpening and denoising.

http://guygilboa.eew.technion.ac.il/researches/sharpening-and-enhancement/

ANISOTROPIC DIFFUSION USING POWER WATERSHEDS

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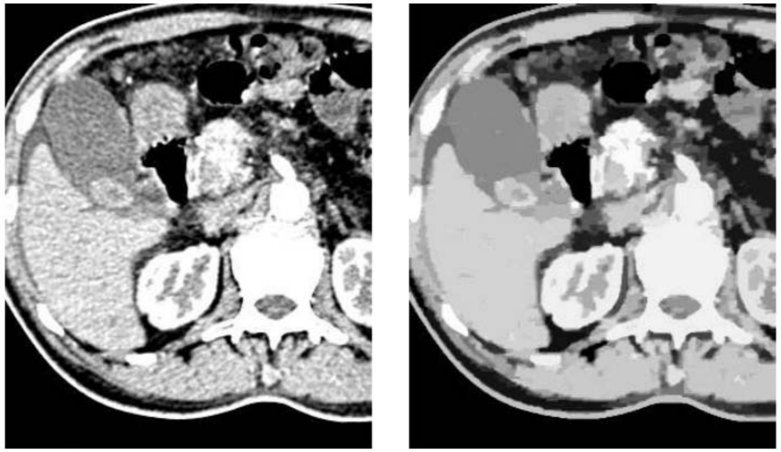
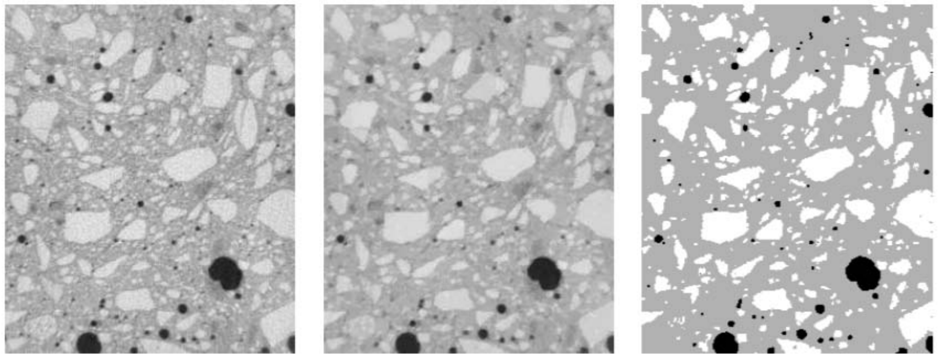


(a) Original image (b) PW result

Fig. 3. Filtering of a liver image by power watershed. Here noise and small vessels are both removed, leading to a result which may be used as a first step before segmentation.



(a) Original image (b) PW result (6 iter.) (c) Segmentation

Fig. 4. The segmentation of concrete images in three classes (bubbles in black, stones in light grey) is useful for the study of the material's mechanical properties. The segmentation is obtained by two thresholds of the filtered image by PW.

https://hal-enpc.archives-ouvertes.fr/hal-00744091/document

Introduction to non-linear filtering

References:

(1) P. Perona and J. Malik, *Scale-space and edge detection using anisotropic diffusion*, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 12, No. 7, pp. 629-639, 1990.

(2) Joachim Weickert, *A review of nonlinear diffusion filtering*, Scale-Space Theory in Computer Vision, Lecture Notes in Computer Science, Vol. 1252, Springer, Berlin, pp. 3-28, 1997.

(3) G. Gerig, *Nonlinear Anisotropic Filtering of MRI Data*, IEEE Transactions on Medical Imaging, Vol. 11, No. 2, pp. 221-232, 1992.

Roadmap

1. Filtering based on Gaussian low-pass filter
2. Concepts of diffusion
3. Linear diffusion filtering
4. Non-linear diffusion filtering
5. Non-linear anisotropic diffusion filtering
   (Edge enhancing anisotropic diffusion filtering)

Filtering based on Gaussian low-pass filter

1. Low-pass Gaussian filtering of a 2D image

$$u(\vec{x},t)=\begin{cases} I(\vec{x}) & (t=0) \\ \left(G_{\sqrt{2t}} * I\right)(\vec{x}) & (t>0) \end{cases}$$

where $\vec{x}=(x,y)=$ Position vector

$I(\vec{x})=$ Initial image

$u(\vec{x},t)=$ Filtered image

$*=$ Convolution operator

$$G_\sigma(\vec{x})=\frac{1}{2\pi\sigma^2}\cdot\exp\left(\frac{-|\vec{x}|^2}{2\sigma^2}\right)$$

Filtering based on Gaussian low-pass filter

1. Low-pass Gaussian filtering of a 2D image

$$u(\vec{x},t)=\begin{cases} I(\vec{x}) & (t=0) \\ (G_{\sqrt{2t}} * I)(\vec{x}) & (t>0) \end{cases}$$

$u = G_{\sqrt{2t}} * I = \frac{1}{4\pi t} e^{\frac{-(x^2+y^2)}{4t}} * I(x,y)$

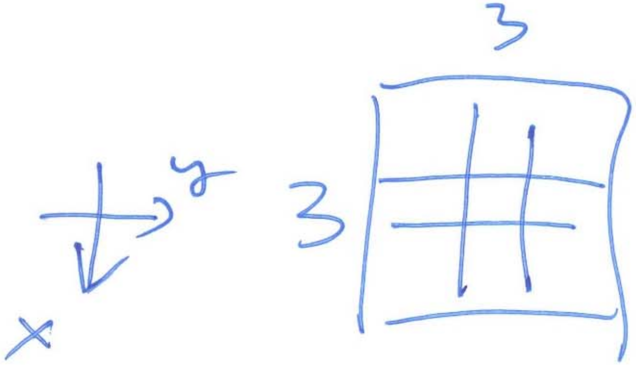
where $\vec{x}=(x,y)=$ Position vector

$I(\vec{x})=$ Initial image

$u(\vec{x},t)=$ Filtered image

$*=$ Convolution operator

$$G_\sigma(\vec{x})=\frac{1}{2\pi\sigma^2}\exp\left(\frac{-|\vec{x}|^2}{2\sigma^2}\right)$$

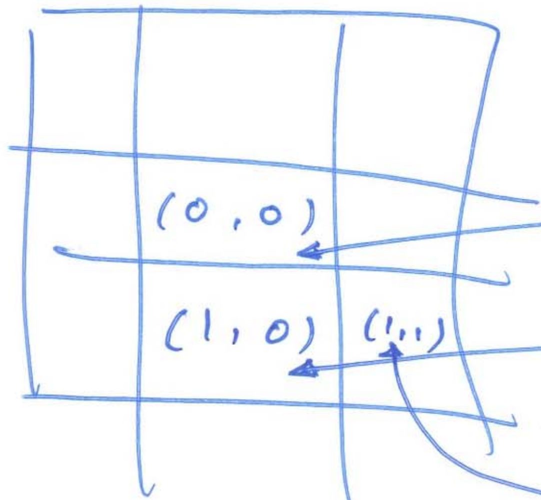


$$G_\sigma(\vec{x}) = \frac{1}{2\pi\sigma^2} e^{-\frac{|\vec{x}|^2}{2\sigma^2}}$$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

imfilter(I , w)

filter 3x3, 5x5



$$G_{\sigma=1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{2\pi}$$

$$G_{\sigma=1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2\pi} e^{-\frac{1}{2}}$$

$$G_{\sigma=1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2\pi} e^{-\frac{2}{2}}$$

$$= \frac{1}{2\pi} e^{-1}$$

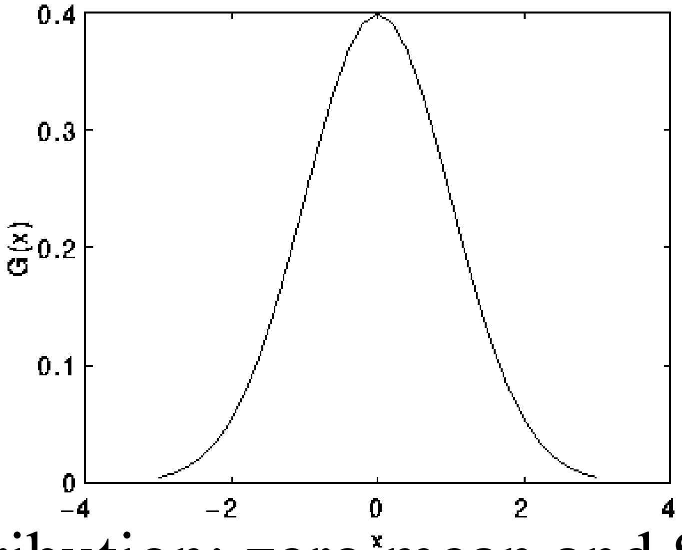
Filtering based on Gaussian low-pass filter

2. Gaussian filtering – it filters or can smooth an initial image *I* by filtering (convolving) the image with a Gaussian filter.

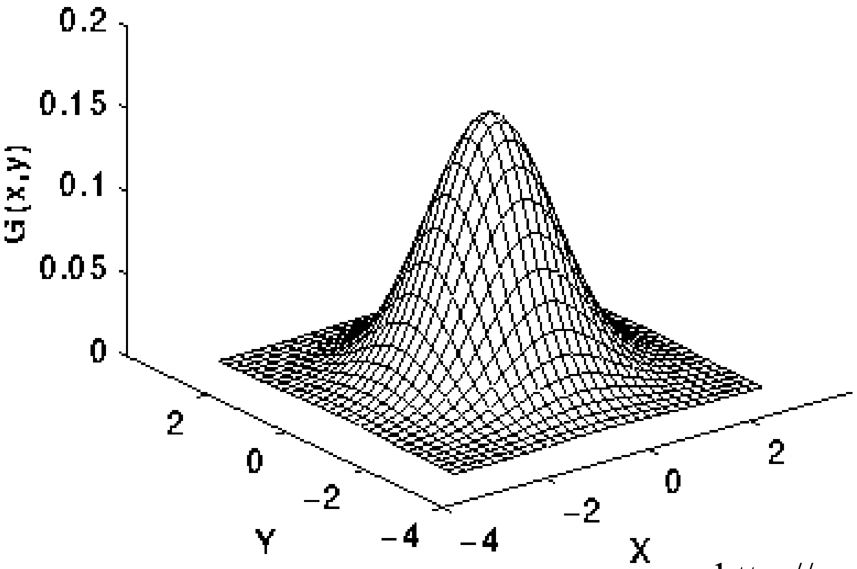
3. Filtering (convolution) represents the weighted sum of local intensity values, in which the weights are determined by a Gaussian distribution with higher values in the center and lower values in the tails.

Filtering based on Gaussian low-pass filter

4. 1D Gaussian distribution: zero mean and SD = 1

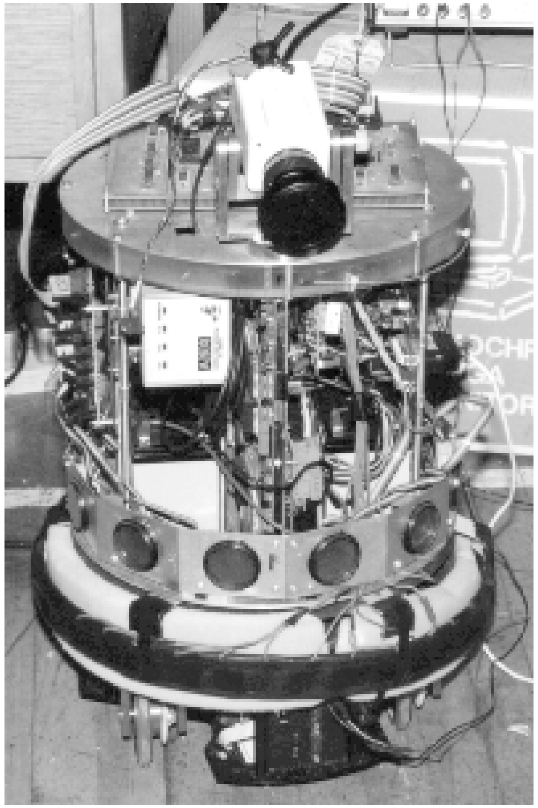


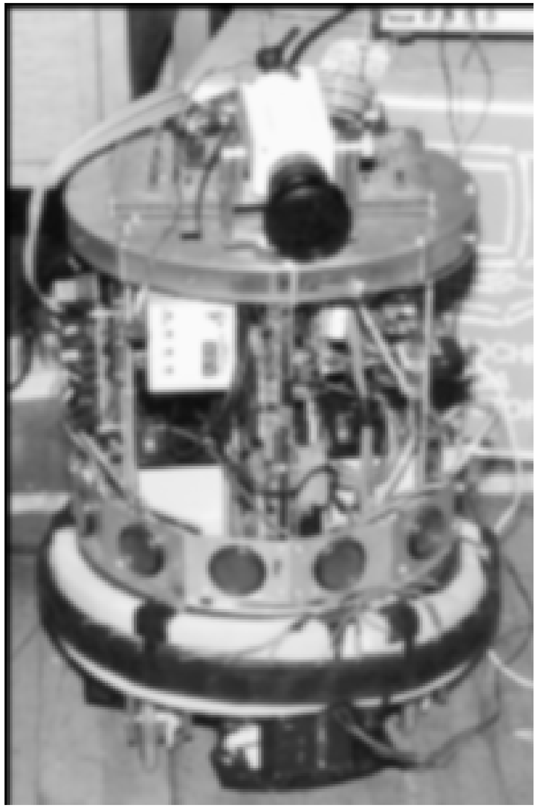
5. 2D Gaussian distribution: zero mean and SD = 1



http://www.dai.ed.ac.uk/HIPR2/gsmooth.htm

Filtering based on Gaussian low-pass filter



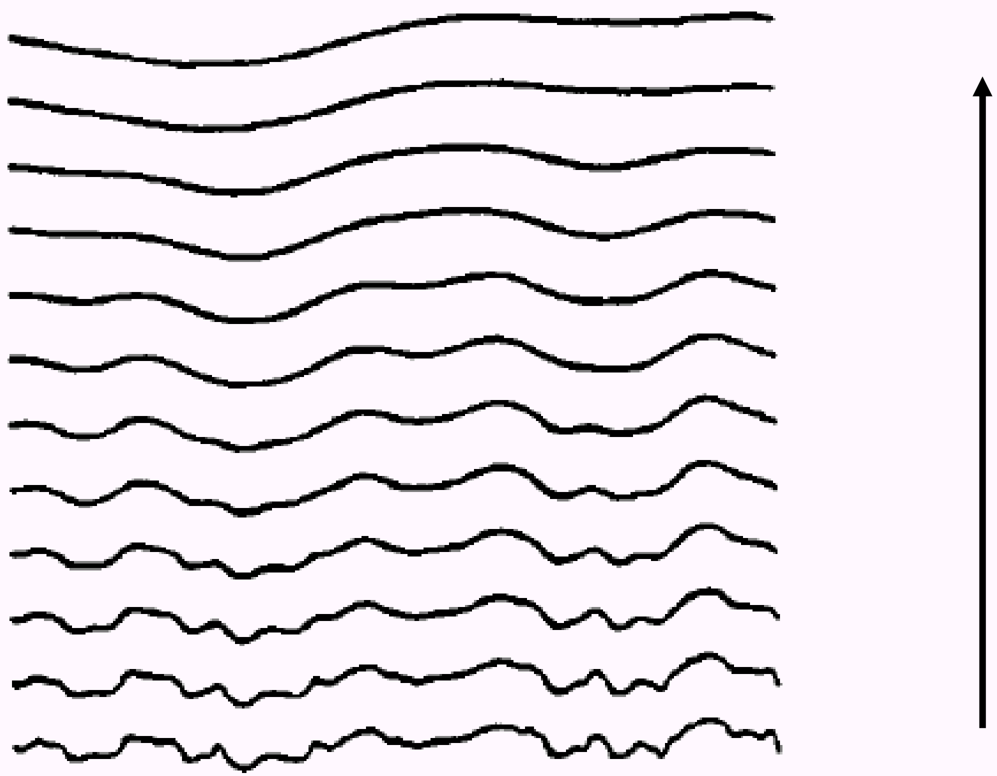


$\sigma = 1$ (5X5 kernel size)

$\sigma = 2$ (9X9 kernel size)

Gaussian smoothing: http://www.dai.ed.ac.uk/HIPR2/gsmooth.htm

Filtering based on Gaussian low-pass filter



The value of t (variance) is increasing.

Fig. 1. A family of 1-D signals $I(x, t)$ obtained by convolving the original one (bottom) with Gaussian kernels whose variance increases from bottom to top (adapted from Witkin [21]).

A family of smoother image intensity profiles is formed by filtering (convolving) the original image with a Gaussian filter of increasing variance.

Perona and Malik

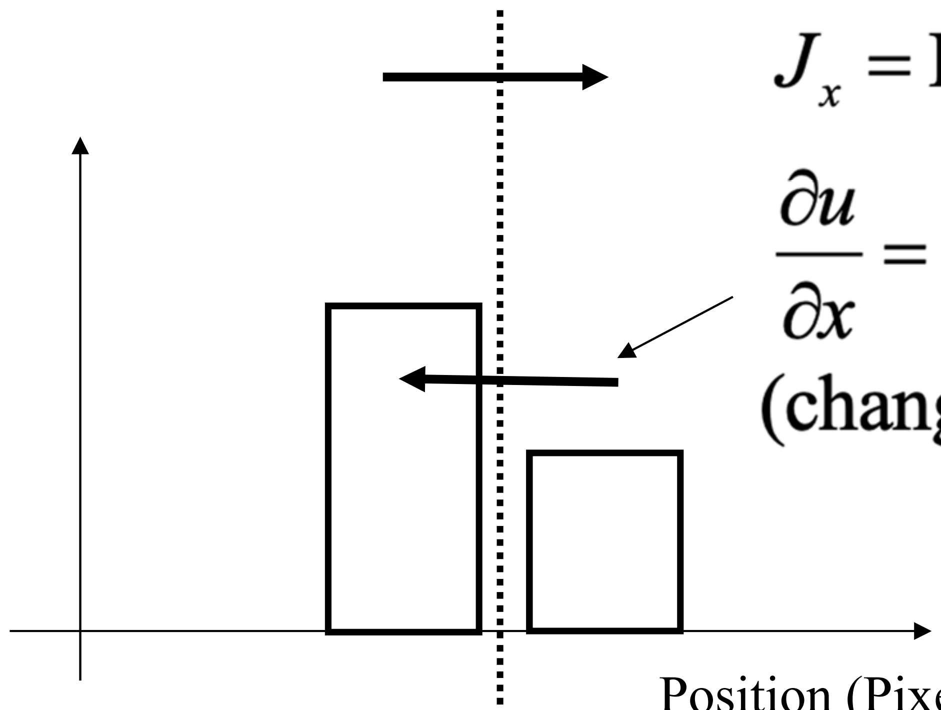
Concepts of diffusion

1. The first law of diffusion – mass flux (diffusion) is proportional to the concentration gradient (change in concentration)

$$\vec{J} = -D \cdot \nabla u$$

$$\left(J_x, J_y\right)^{\mathrm{T}} = -D \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)^{\mathrm{T}}$$

Concentration, u
(Intensity)



$$J_x = \mathrm{Flux} = -\frac{\partial \mathrm{u}}{\partial \mathrm{x}}$$

$$\frac{\partial u}{\partial x} = \text{gradient}$$

(change in concentration)

Position (Pixel position)

Animation: https://en.wikipedia.org/wiki/Molecular_diffusion

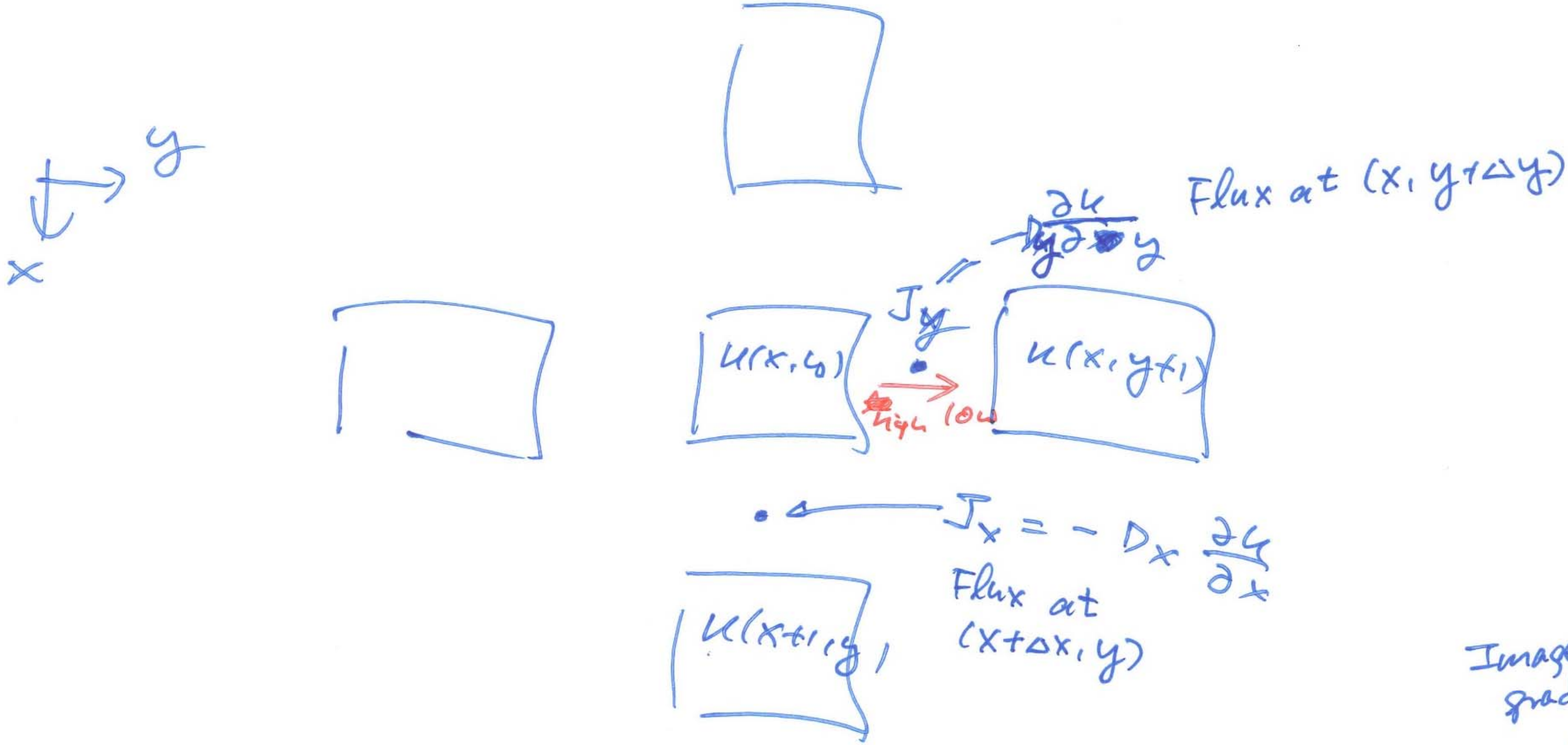


Image gradient

$$\therefore \quad J_x = -D_x \frac{\partial u}{\partial x}$$

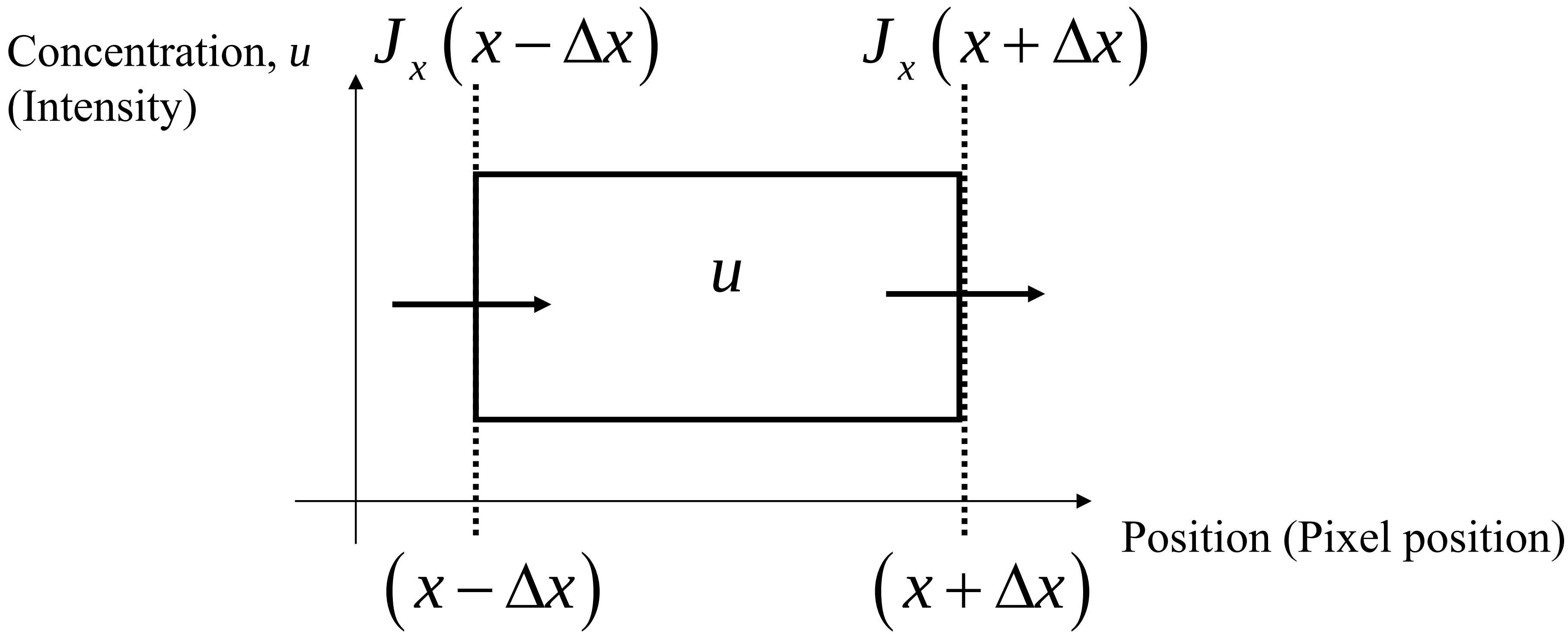
$$J_y = -D_y \frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \vec{J} = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = -\begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

Concepts of diffusion

2. The second law of diffusion – the rate of accumulation of concentration within a volume is proportional to the change of local concentration gradient (continuity equation).

For example, for 1D $$\frac{\partial u}{\partial t} = -\frac{\partial J_x}{\partial x}$$

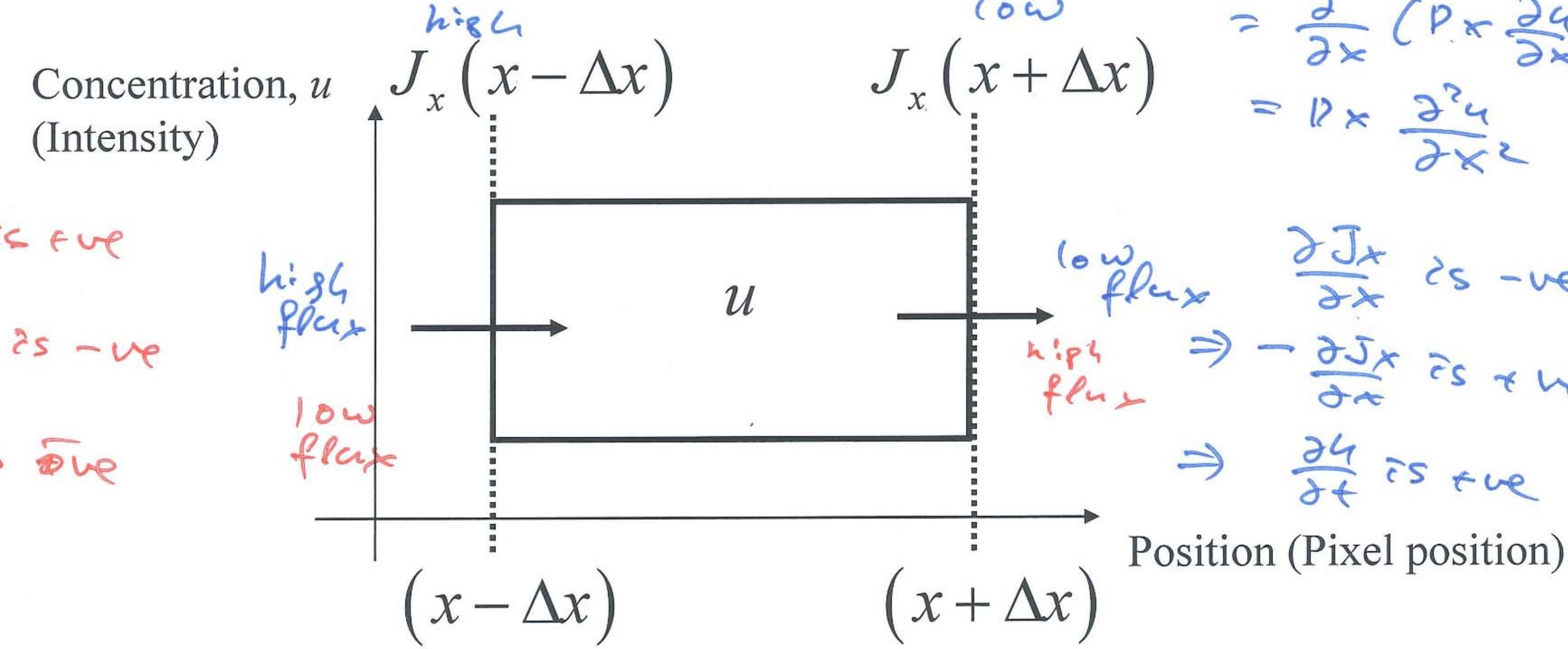


Concepts of diffusion

2. The second law of diffusion – the rate of accumulation of concentration within a volume is proportional to the change of local concentration gradient (continuity equation).

2nd law ↓ 1st law ↓

For example, for 1D $$\frac{\partial u}{\partial t} = -\frac{\partial J_x}{\partial x} = -\frac{\partial}{\partial x}\left(-D_x \frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(D_x \frac{\partial u}{\partial x}\right) = D_x \frac{\partial^2 u}{\partial x^2}$$



$\frac{\partial J_x}{\partial x}$ is +ve

$\Rightarrow -\frac{\partial J_x}{\partial x}$ is −ve

$\Rightarrow \frac{\partial u}{\partial t}$ is −ve

$\frac{\partial J_x}{\partial x}$ is −ve

$\Rightarrow -\frac{\partial J_x}{\partial x}$ is +ve

$\Rightarrow \frac{\partial u}{\partial t}$ is +ve

Concepts of diffusion

3. Equation of diffusion (Heat equation)

E.g., for 1D $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$ where $D = \text{diffusivity}$

or Conductance

or Diffusion constant

E.g., for 2D $\dfrac{\partial u}{\partial t} = \text{div}\left(D \ \nabla u\right)$

where $D = \text{diffusivity tensor}$

$$\text{div}\left(\vec{x}\right) = \nabla \cdot \vec{x}$$

Concepts of diffusion

3. Equation of diffusion (Heat equation)

E.g., for 1D $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$ where $D =$ diffusivity

or Conductance

or Diffusion constant

E.g., for 2D $\dfrac{\partial u}{\partial t} = \operatorname{div}\big(\underset{2\times 2}{D}\ \underset{2\times 1}{\nabla u}\big)$

where $D =$ diffusivity tensor

$\operatorname{div}(\vec{x}) = \nabla \overset{\text{dot}}{\cdot} \vec{x} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \vec{x}$

$D_x = D_y = 1$

for 2D, $\dfrac{\partial u}{\partial t} = \nabla \cdot \left(\underset{2\times 2}{\begin{pmatrix} D_x & 0 \\ 0 & D_y \end{pmatrix}} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \right)$

$\dfrac{\partial u}{\partial t} = \nabla \cdot \begin{pmatrix} D_x \frac{\partial u}{\partial x} \\ D_y \frac{\partial u}{\partial y} \end{pmatrix} = \dfrac{\partial}{\partial x}\left(D_x \dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(D_y \dfrac{\partial u}{\partial y}\right)$

Concepts of diffusion

4. Linear diffusion filtering: $D = I$ (Identity matrix), it means heat (intensity) diffuses according to

$$\frac{\partial u}{\partial t} = \operatorname{div}(\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Initial condition

$$u(\vec{x}, t = 0) = I(\vec{x})$$

(Remark: The low-pass Gaussian filtering can be viewed as the solution of the heat equation.)

Concepts of diffusion

4. Linear diffusion filtering: $D = I$, it means heat (intensity) diffuses according to

$$\left(-\frac{1}{t} + \frac{x^2+y^2}{4t^2}\right)u$$

$$\frac{\partial}{\partial t}\left(\overbrace{G_{\sqrt{2t}} * I}^{u}\right) = \frac{\partial u}{\partial t} = \operatorname{div}(\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Initial condition

$$u(\vec{x}, t=0) = I(\vec{x})$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = \boxed{\left(\frac{-1}{t} + \frac{x^2+y^2}{4t^2}\right)u}$$

(Remark: The low-pass Gaussian filtering can be viewed as the solution of the heat equation.)

Effect of low-pass Gaussian filtering

& Effect of linear diffusion filtering

are the same.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(G_{\sqrt{2t}} * I \right)\Big|_{t>0}$$

$$\frac{\partial u}{\partial t} = \left(\frac{-1}{t} + \frac{x^2+y^2}{2t^2} \right) \underbrace{G_{\sqrt{2t}} * I}_{\text{smooth}}$$

$$\frac{\partial^2}{\partial x^2}\left(G_{\sqrt{2t}} * I \right) = \left(\frac{-1}{2t} + \frac{x^2}{4t^2} \right) \underbrace{G_{\sqrt{2t}} * I}$$

$$\frac{\partial^2}{\partial y^2}\left(G_{\sqrt{2t}} * I \right) = \left(\frac{-1}{2t} + \frac{y^2}{4t^2} \right) G_{\sqrt{2t}} * I$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(-\frac{1}{t} + \frac{x^2+y^2}{2t^2} \right) \underbrace{G_{\sqrt{2t}} * I}$$

$$\therefore \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$u = \underbrace{G_{\sqrt{2t}} * I}$$ low-pass Gaussian filtering

Linear diffusion filtering

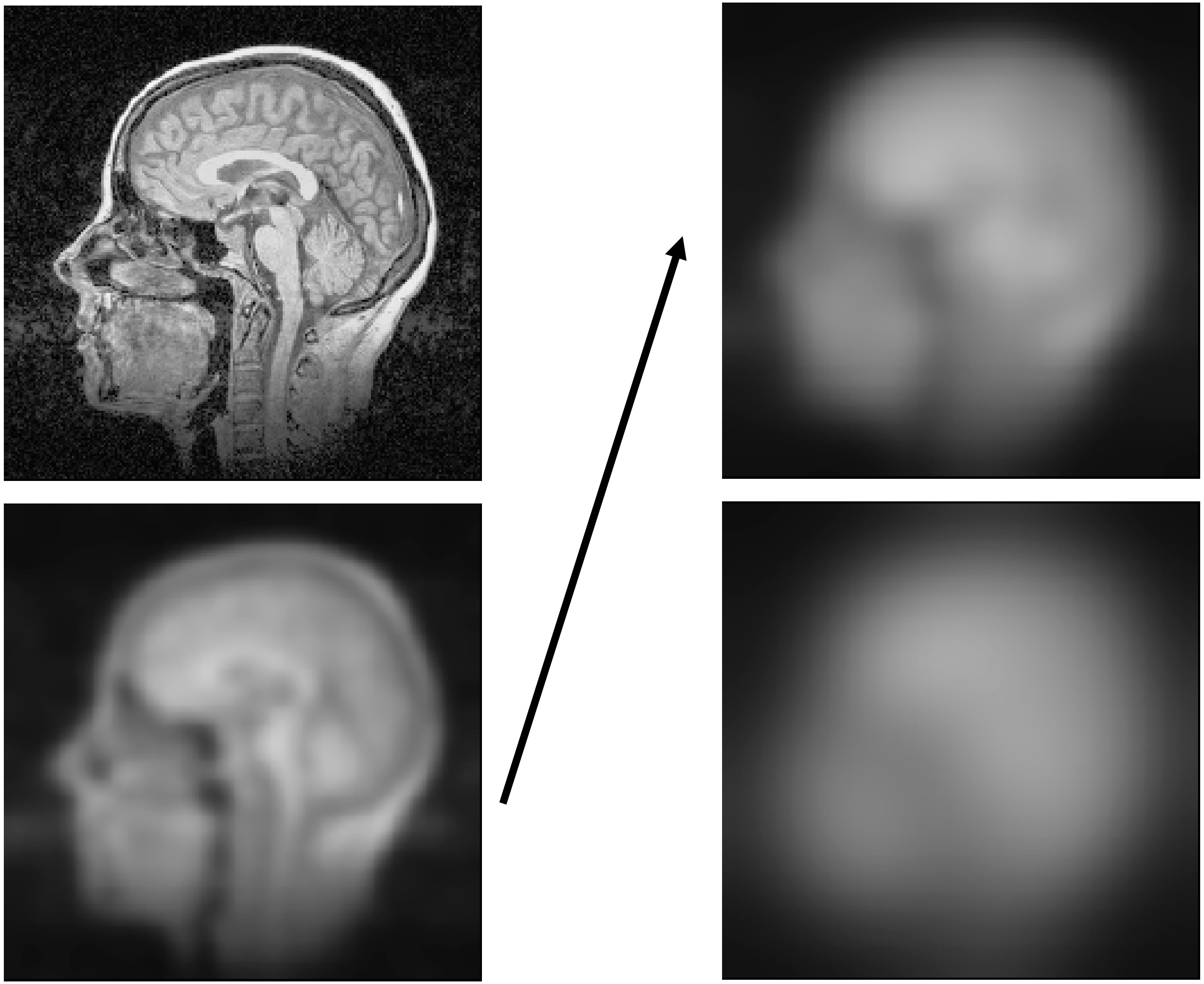
1. Assume that $D = I$ Identity Matrix
2. Heat equation (2D)

$$\frac{\partial u}{\partial t} = \Delta u$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Initial condition $\quad u\left(\vec{x}, t = 0\right) = I\left(\vec{x}\right)$

Linear diffusion filtering



Linear diffusion examples: t=0, 12.5, 50, 200.

Weickert

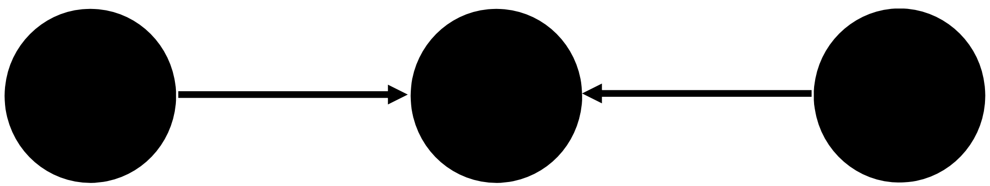
Linear diffusion filtering

3. Implementation. For example, in 1D

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition $\quad u(x, t=0) = I(x)$

$$u(x, t+\Delta t) = u(x,t) + \frac{\Delta t}{(\Delta x)^2}\big(u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)\big)$$



$u(x-\Delta x, t) \qquad u(x,t) \qquad u(x+\Delta x, t)$

Linear diffusion filtering

3. Implementation. For example, in 1D

unknown   Forward diff. approximation   known

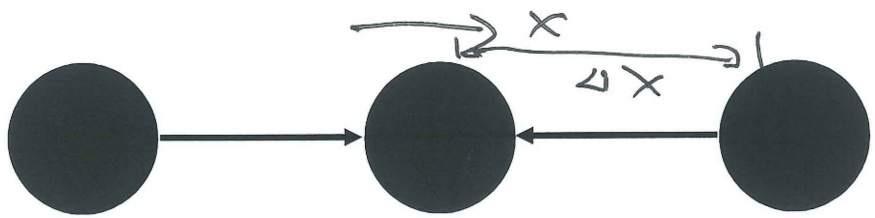
$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} \doteq \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \doteq \frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{(\Delta x)^2}$$

known   central diff. approximation

Δt = time step = 0.001, 0.01, 0.1

Initial condition $\quad u(x,t=0)=I(x)$

$$u(x,t+\Delta t)=u(x,t)+\frac{\Delta t}{(\Delta x)^2}\Big(u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)\Big)$$



$u(x-\Delta x,t) \qquad u(x,t) \qquad u(x+\Delta x,t)$

Linear diffusion filtering

4. Gaussian smoothing / linear diffusion filtering reduces noise, but also blurs important features such as edges and thus makes them harder to identify (see Fig. 1 on next page).

5. Gaussian smoothing / linear diffusion filtering dislocates edges when moving from finer to coarser scales (see Fig. 2 on next page).

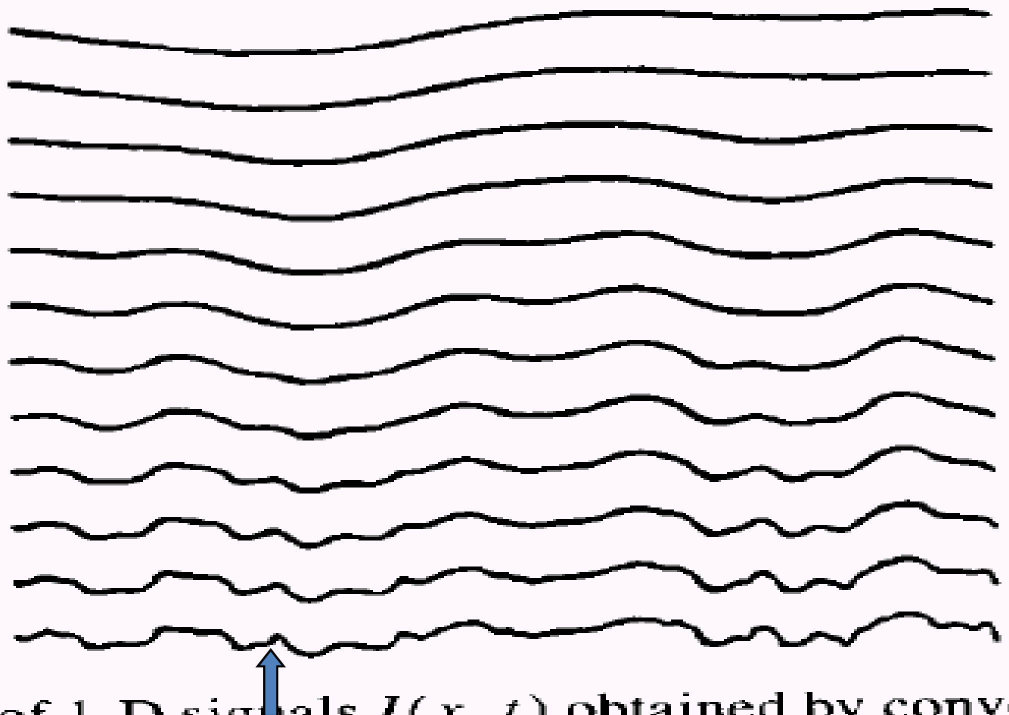


Fig. 1. A family of 1-D signals $I(x, t)$ obtained by convolving the original one (bottom) with Gaussian kernels whose variance increases from bottom to top (adapted from Witkin [21]).

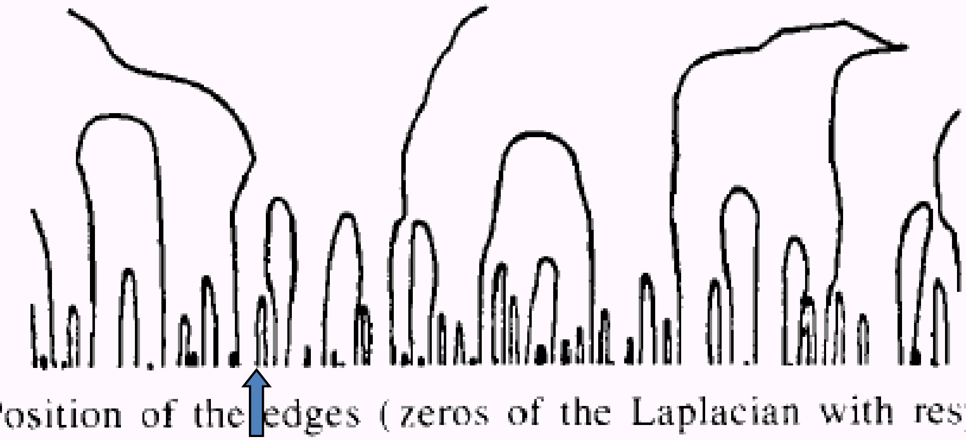


Fig. 2. Position of the edges (zeros of the Laplacian with respect to x) through the linear scale space of Fig. 1 (adapted from Witkin [21]).

Non-linear diffusion filtering

1. We want to encourage smoothing within a region in preference to smoothing across the boundaries.
2. This could be achieved by setting the diffusivity to be 0 on the boundary and to be non-zero in the interior of the region.

E.g., for 2D $\frac{\partial u}{\partial t} = \mathrm{div}\left(D \nabla u\right)$

where $D = g\left(\left|\nabla u\right|\right)$

$$g\left(\left|\nabla u\right|\right) = \exp\left(-\left(\frac{\left|\nabla u\right|}{K}\right)^2\right) \quad \text{or} \quad g\left(\left|\nabla u\right|\right) = \frac{1}{1+\left(\frac{\left|\nabla u\right|}{K}\right)^2}$$

Non-linear diffusion filtering

3. Implementation. For example, in 2D

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g \nabla u\right)$$

Initial condition $\quad u\left(\vec{x}, t=0\right) = I\left(\vec{x}\right)$

$$u\left(x, y, t+\Delta t\right) = u\left(x, y, t\right)$$

$$+\Delta t\left[\left(\Phi_N - \Phi_S\right) + \left(\Phi_E - \Phi_W\right)\right]$$

Non-linear diffusion filtering

3. Implementation. For example, in 2D

$$\frac{u(x,y,t+\Delta t)-u(x,y,t)}{\Delta t} = \frac{\partial u}{\partial t} = \operatorname{div}(g\nabla u)$$

PDE

magnitude

$|\nabla u|$

Initial condition $u(\vec{x}, t=0) = I(\vec{x})$

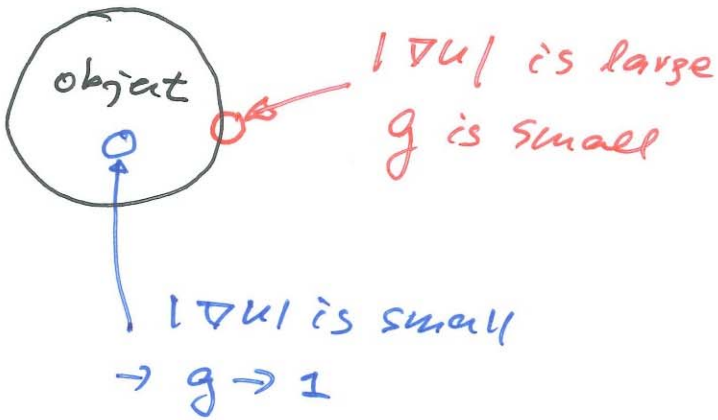
vector

$$u(x,y,t+\Delta t) = u(x,y,t) + \Delta t \overbrace{\left[\left(\Phi_N - \Phi_S \right) + \left(\Phi_E - \Phi_W \right) \right]}^{\operatorname{div}(g\nabla u)}$$

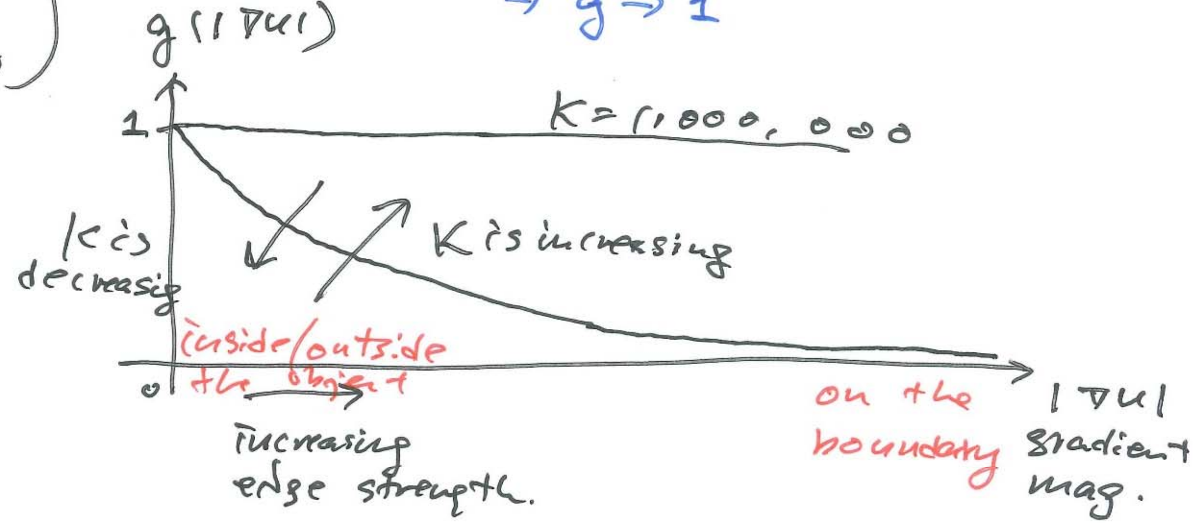
North

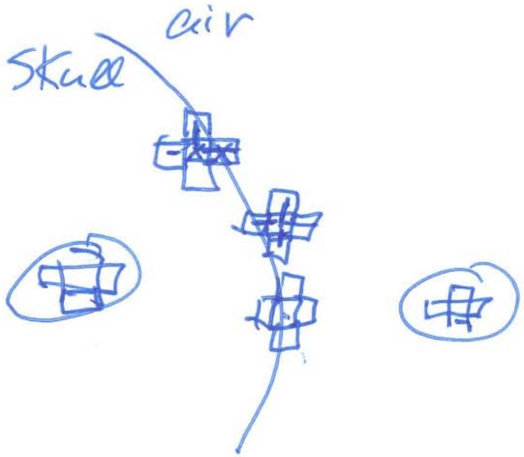
$$g(\underbrace{|\nabla u|}_{\text{image gradient}}) = \frac{1}{1 + \frac{|\nabla u|^2}{k^2}}$$

mag.



$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$



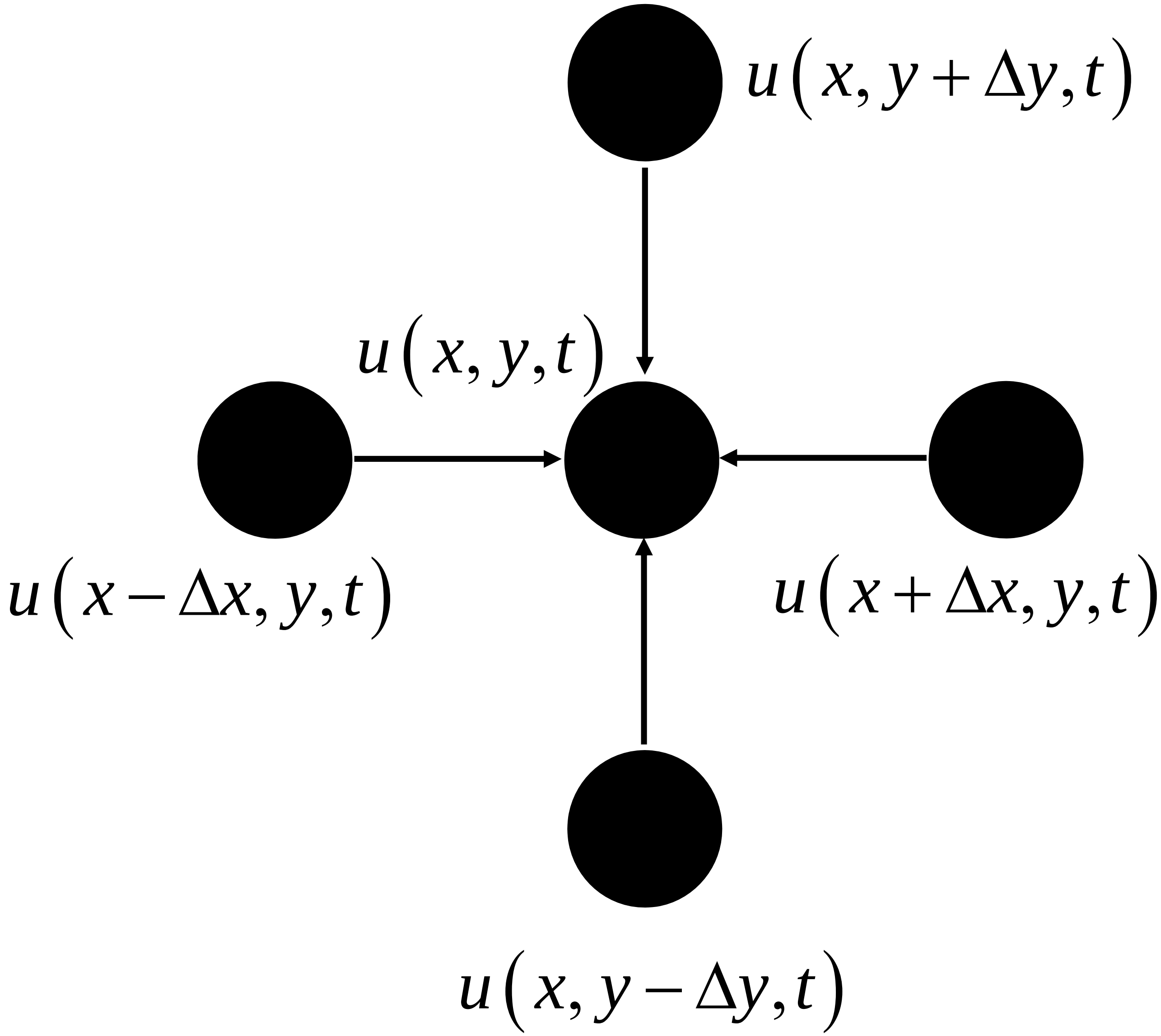


$$\mathrm{div} = \underset{\substack{\uparrow \\ \text{dot}}}{\nabla \cdot}$$

$$\frac{\partial u}{\partial t} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} g \frac{\partial u}{\partial x} \\ g \frac{\partial u}{\partial y} \end{pmatrix}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(g \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(g \frac{\partial u}{\partial y}\right)$$

Non-linear diffusion filtering



Non-linear diffusion filtering

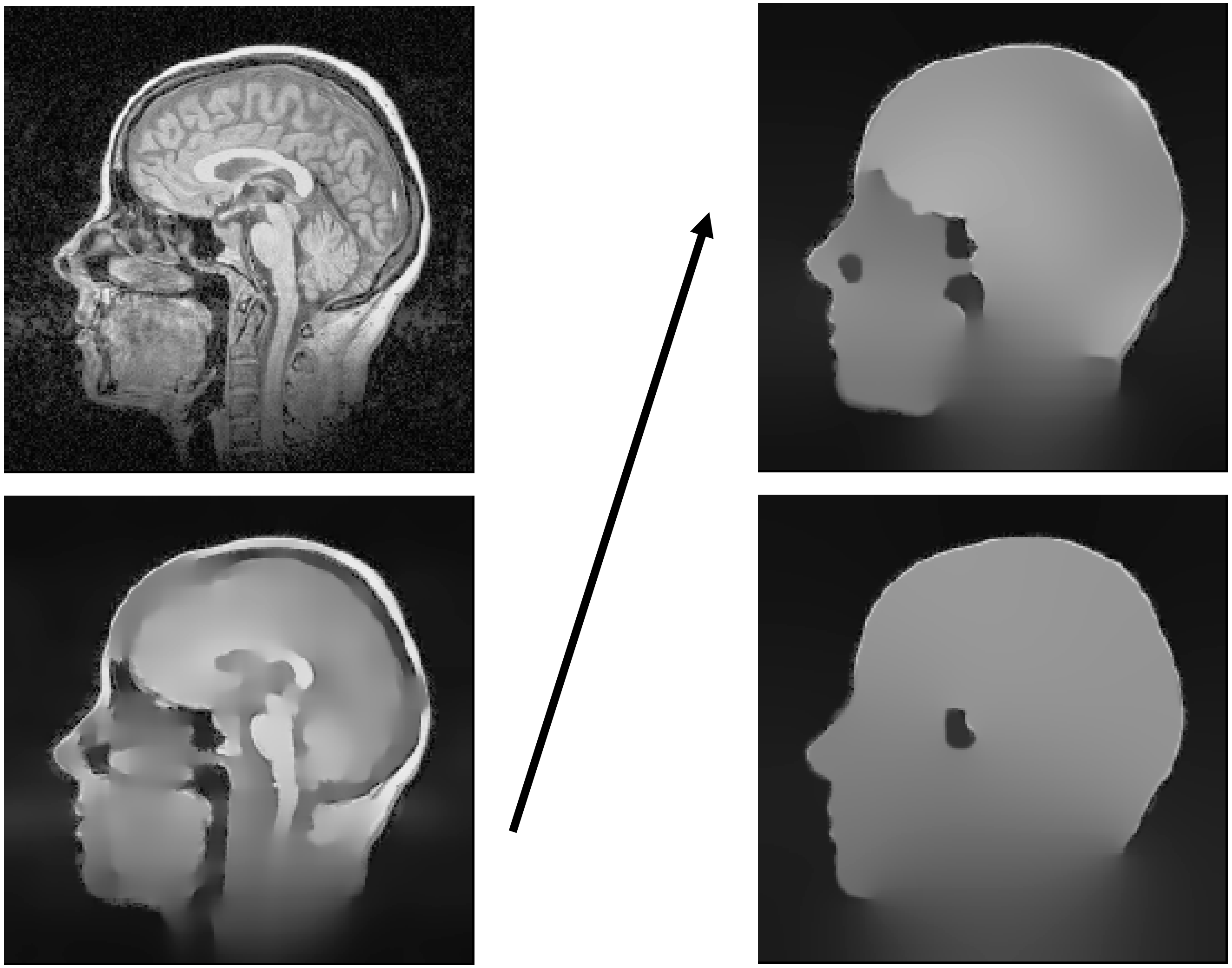
$$\Phi_N = g\left(\left|\nabla u\left(x, y+\frac{\Delta y}{2}, t\right)\right|\right)\left(\frac{u(x, y+\Delta y, t)-u(x, y, t)}{\Delta y^2}\right)$$

$$\Phi_S = g\left(\left|\nabla u\left(x, y-\frac{\Delta y}{2}, t\right)\right|\right)\left(\frac{u(x, y, t)-u(x, y-\Delta y, t)}{\Delta y^2}\right)$$

$$\Phi_E = g\left(\left|\nabla u\left(x+\frac{\Delta x}{2}, y, t\right)\right|\right)\left(\frac{u(x+\Delta x, y, t)-u(x, y, t)}{\Delta x^2}\right)$$

$$\Phi_W = g\left(\left|\nabla u\left(x-\frac{\Delta x}{2}, y, t\right)\right|\right)\left(\frac{u(x, y, t)-u(x-\Delta x, y, t)}{\Delta x^2}\right)$$

Non-linear diffusion filtering



Non-linear diffusion examples: K=3, t=0, 40, 400, 1500.

Weickert

Linear diffusion

Non-linear diffusion

$$\frac{\partial u}{\partial t} = \Delta u$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(g \ \nabla u\right)$$

Perona and Malik

(a)

(a)

(b)

Non-linear anisotropic diffusion filtering

1. For isotropic diffusion, the diffusion direction is always parallel to the gradient vector ∇u.
2. Let $\vec{v} \perp \nabla u$ be a flux perpendicular to the direction of gradient ∇u. For example,

If $\vec{v} = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right)^{\mathrm{T}}$ Tangent vector

then $\frac{\partial u}{\partial t} = \operatorname{div}\left(\vec{v}\right) = 0$ No smoothing/filtering along edge

3. Diffusion favors the gradient direction.
   a. Interior region: linear filter
   b. Boundary region: no diffusion if gradient function $g\left(\left|\nabla u\right|\right)$ is used. Problem: no smoothing along edge.

More details: https://en.wikipedia.org/wiki/Anisotropic_diffusion

Non-linear anisotropic diffusion filtering

1. For isotropic diffusion, the diffusion direction is always parallel to the gradient vector ∇u.
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$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

If $$\vec{v} = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x} \right)^T$$ Tangent vector

then $$\frac{\partial u}{\partial t} = \operatorname{div}\left(\vec{v}\right) = 0$$ No smoothing/filtering along edge

3. Diffusion favors the gradient direction.
   a. Interior region: linear filter
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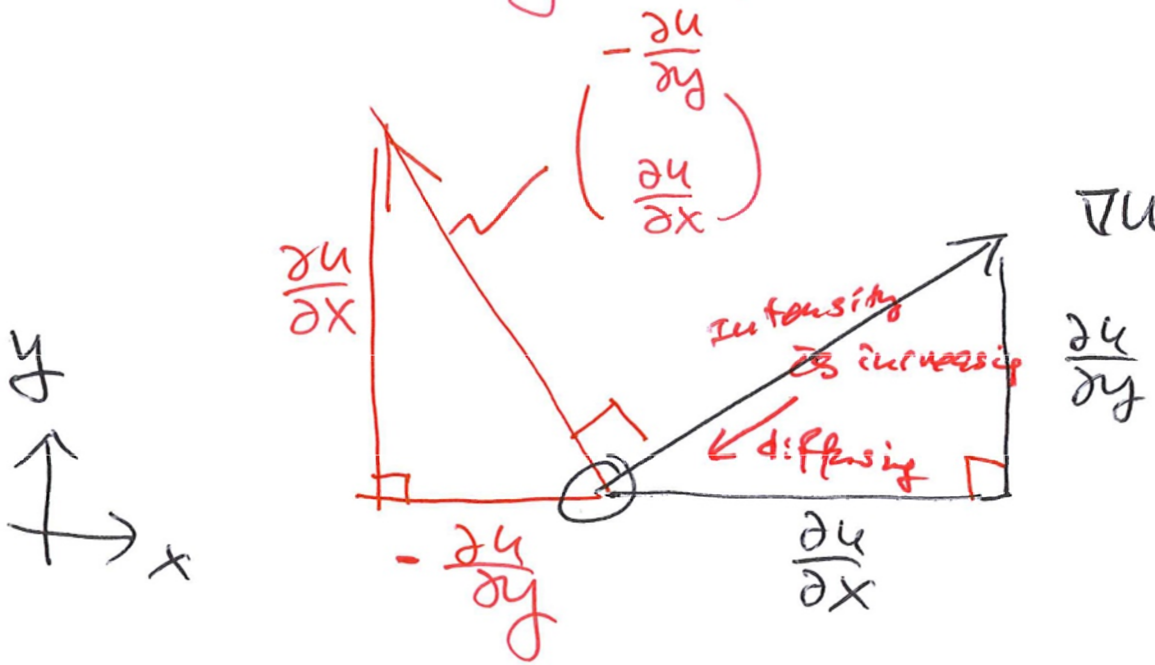
More details: https://en.wikipedia.org/wiki/Anisotropic_diffusion

Tangent of ∇u



$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)^T$$

$$\frac{\partial u}{\partial t} = \mathrm{div}(\vec{v}), \quad \vec{v} = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x} \right)^T$$

$$= \nabla \cdot \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x} \right)^T$$

$$= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y}$$

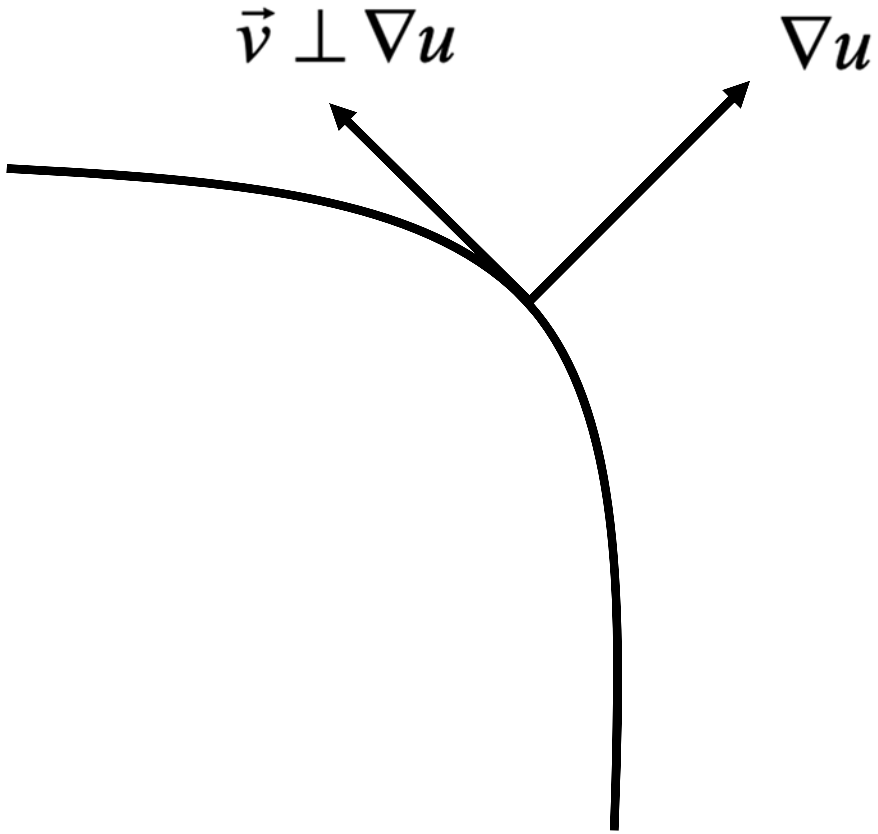
$$= 0$$

$$\nabla \cdot = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

Non-linear anisotropic diffusion filtering

Zero flux and hence
no diffusion
(along the negative
tangent vector)

Maximum flux for
diffusion
(along the negative
gradient vector)



Non-linear anisotropic diffusion filtering

Zero flux and hence
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Maximum flux for
diffusion
(along the negative
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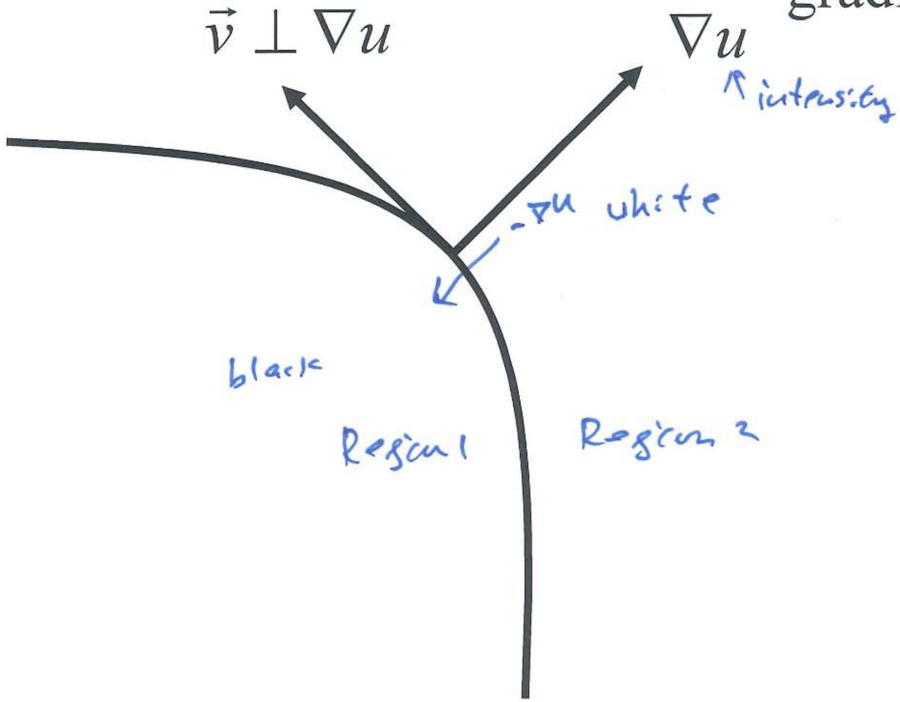


$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

Edge-enhancing anisotropic diffusion

4. Smoothing along edge can be achieved by diffusing along the negative gradient and negative tangent vectors of a Gaussian intensity smoothed image u_σ.
5. Definitions:

$$u_\sigma = G_\sigma * u$$

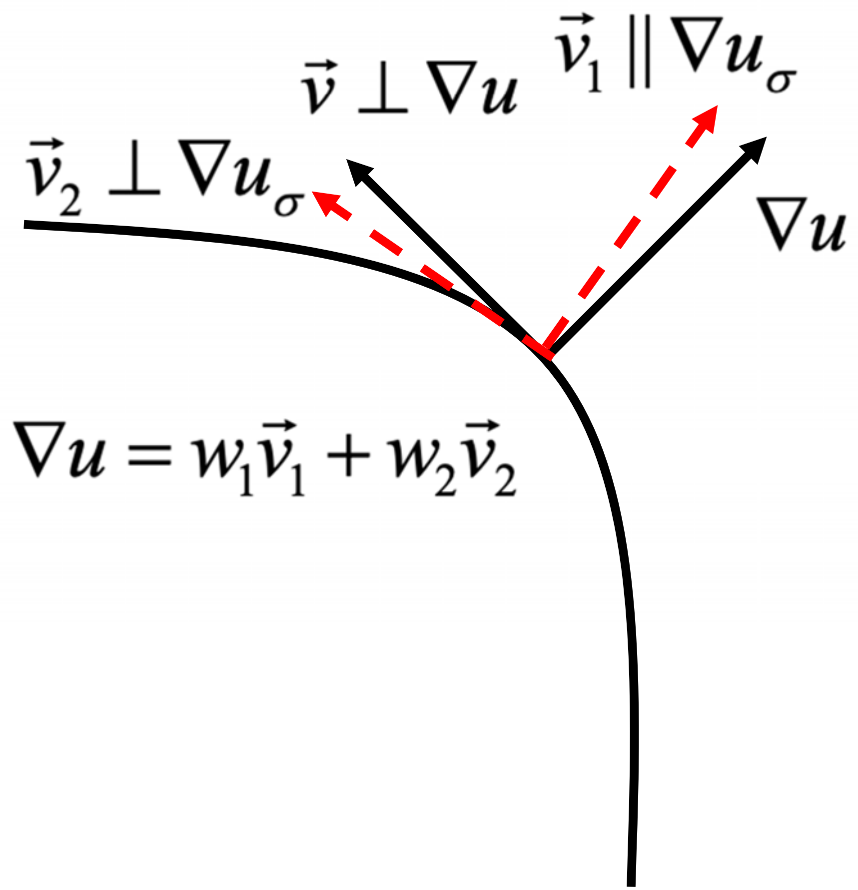
$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{\frac{-x^2-y^2}{2\sigma^2}}$$

Gradient vector of the smoothed image $\vec{v}_1 \parallel \nabla u_\sigma$

Tangent vector of the smoothed image $\vec{v}_2 \perp \nabla u_\sigma$

Edge-enhancing anisotropic diffusion

6. In general, ∇u_σ will not be parallel to ∇u.



$$\lambda_1 = g\left(\left|\nabla u_\sigma\right|\right)$$

$$\lambda_2 = 1$$

7. If we assign a relatively large weight to $\vec{v}_2$ and a very small weight to $\vec{v}_1$, then smoothing along the edge is achieved.

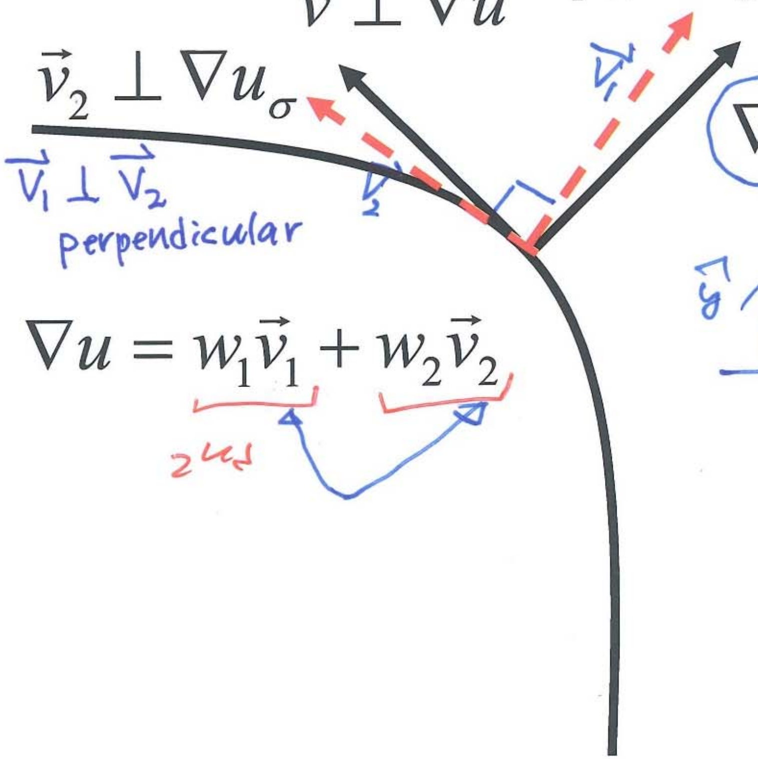
Edge-enhancing anisotropic diffusion

6. In general, ∇u_σ will not be parallel to ∇u.

grad. of new image → u_σ = smooth version of u with σ

large σ

small σ



$$\nabla u = \underbrace{w_1 \vec{v}_1}_{} + \underbrace{w_2 \vec{v}_2}_{}$$

2 w's

$$\lambda_1 = g\left(\left|\nabla u_\sigma\right|\right) \quad \text{for } \vec{v}_1$$

$$\lambda_2 = 1 \quad \text{for } \vec{v}_2$$

$\hat{y}$ y

x $\hat{x}$

$$\nabla u = c_1 \hat{x} + c_2 \hat{y}$$

$$\nabla u = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

7. If we assign a relatively large weight to $\vec{v}_2$ and a very small weight to $\vec{v}_1$, then smoothing along the edge is achieved.

Edge-enhancing anisotropic diffusion

8. Changes have to be made in the heat equation

Original $$\frac{\partial u}{\partial t} = \operatorname{div}\left(D \cdot \nabla u\right)$$ Heat equation

$$\downarrow$$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\overleftrightarrow{D} \cdot \nabla u\right)$$

where $\overleftrightarrow{D}$ represents a matrix with eigenvectors $\vec{v}_1$ and $\vec{v}_2$, and with eigenvalues λ_1 and λ_2.

$$\overleftrightarrow{D} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}^{-1}$$

$$\vec{v}_1 \parallel \nabla u_\sigma \,, \vec{v}_2 \perp \nabla u_\sigma \,, \lambda_1 = g\left(\left|\nabla u_\sigma\right|\right) \text{ and } \lambda_2 = 1$$

Notes on Matrix, eigenvectors and eigenvalues

http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

Matrix multiplication $\overleftrightarrow{A} \cdot \vec{v} = \lambda \vec{v}$ Linear transformation

$$\left(\overleftrightarrow{A} - \lambda \overleftrightarrow{I} \right) \cdot \vec{v} = \vec{0}$$

Let the eigenvectors be $\vec{v}_1$ and $\vec{v}_2$

Let the eigenvalues be λ_1 and λ_2

$$\overleftrightarrow{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}^{-1}$$

eig

2×2

identity matrix

$$a = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|a - \lambda \overleftrightarrow{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 - 1$$

$$= 4 - 4\lambda + \lambda^2 - 1$$

$$= 3 - 4\lambda + \lambda^2$$

eigenvalues

$$\lambda_1 = 1 \quad , \quad \lambda_2 = 3$$

$$\hat{e}_1 = \begin{pmatrix} -0.7071 \\ 0.7071 \end{pmatrix} \quad , \quad \hat{e}_2 = \begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix}$$

unit vector

$$\sqrt{(0.7071)^2 + (0.7071)^2} = 1$$

inv

$$a = \begin{pmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{pmatrix}^{-1}$$

$$a = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad (\hat{e}_1 \ \hat{e}_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\hat{e}_1 \ \hat{e}_2)^{-1}$$

Notes on Matrix, eigenvectors and eigenvalues

http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

2x2

$$A\ \vec{v} = \lambda \vec{v}$$

$$D\ \vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\left(A - \lambda I\right)\ \vec{v} = \vec{0}$$

Let the eigenvectors be $\vec{v}_1$ and $\vec{v}_2$

Let the eigenvalues be λ_1 and λ_2

2x2

$$D = A = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}^{-1}$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\overleftrightarrow{D} \cdot \nabla u\right)$$

$$\downarrow$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\overleftrightarrow{D} \cdot \left(w_1 \vec{v}_1 + w_2 \vec{v}_2\right)\right)$$

$$\downarrow$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(w_1 \overleftrightarrow{D} \cdot \vec{v}_1 + w_2 \overleftrightarrow{D} \cdot \vec{v}_2\right)$$ Matrix multiplication

$$\downarrow$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(w_1 \lambda_1 \vec{v}_1 + w_2 \lambda_2 \vec{v}_2\right)$$ Linear transformation

$$\downarrow$$

At the boundary, $\lambda_1 \approx 0$

Therefore, $\dfrac{\partial u}{\partial t} = w_2 \lambda_2 \mathrm{div}\left(\vec{v}_2\right)$

This represents diffusion along edge.

original

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\overleftrightarrow{D}\ \nabla u\right)$$

version 2

$$\downarrow$$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\overleftrightarrow{D}\ \left(w_1\vec{v}_1 + w_2\vec{v}_2\right)\right)$$

$$\downarrow$$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(w_1\overleftrightarrow{D}\ \vec{v}_1 + w_2\overleftrightarrow{D}\ \vec{v}_2\right)$$

$$\downarrow$$

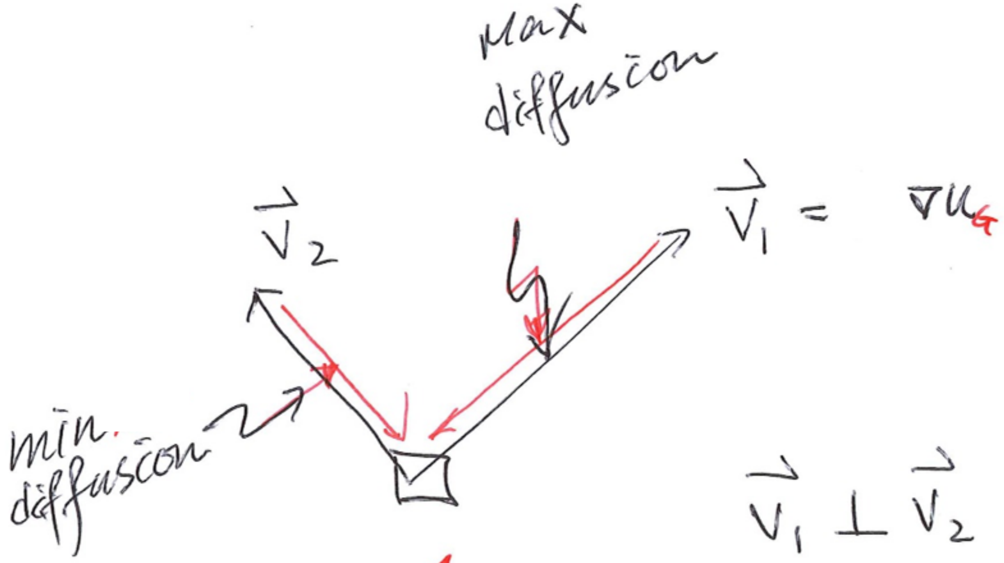
$$\frac{\partial u}{\partial t} = \operatorname{div}\left(w_1\lambda_1\vec{v}_1 + w_2\lambda_2\vec{v}_2\right)$$

$$\downarrow$$

At the boundary, $\lambda_1 \approx 0$ $\lambda_1 = g(|\nabla u|)$

Therefore, $\dfrac{\partial u}{\partial t} = w_2\lambda_2 \operatorname{div}\left(\vec{v}_2\right)$

This represents diffusion along edge.



$\vec{v}_1 \perp \vec{v}_2$

orthogonal

perpendicular

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(w_1 \lambda_1 \vec{v}_1 + w_2 \overset{1}{\lambda_2} \vec{v}_2\right)$$

controlled by g based on gradient

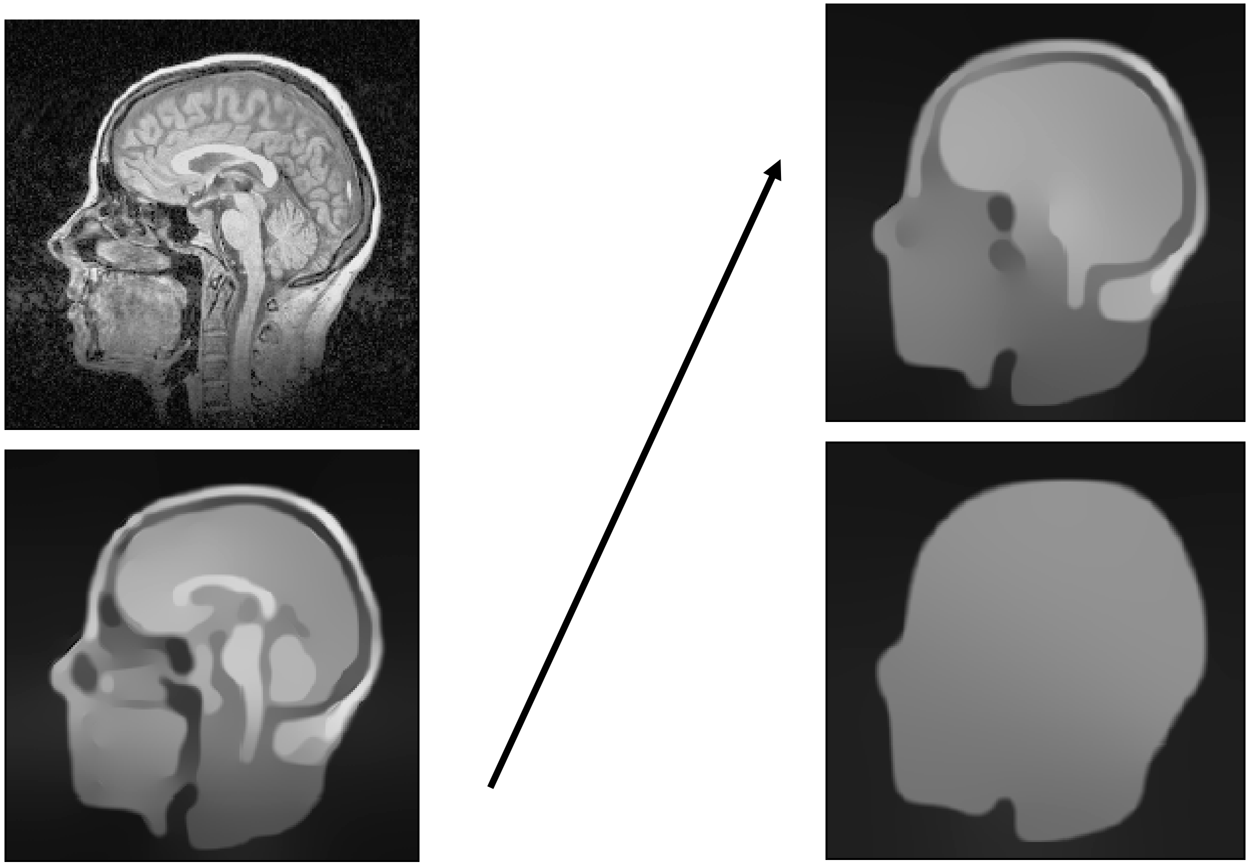
$$= \operatorname{div}\left(w_1 \overleftrightarrow{D} \vec{v}_1 + w_2 \overleftrightarrow{D} \vec{v}_2\right)$$

$$= \operatorname{div}\left(\overleftrightarrow{D}\left(w_1 \vec{v}_1 + w_2 \vec{v}_2\right)\right)$$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\underset{2\times 2}{\overleftrightarrow{D}} \; \underset{2\times 1}{\nabla u}\right)$$

Edge-enhancing anisotropic diffusion

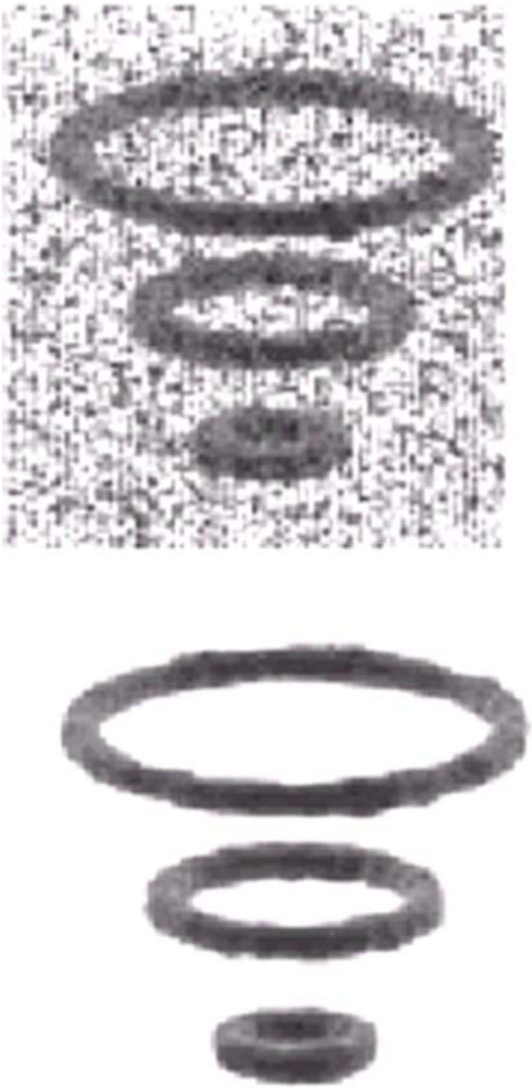
9. The new model behaves anisotropic.
10. When $\sigma \to 0$, the new model switches back to the isotropic diffusion method.



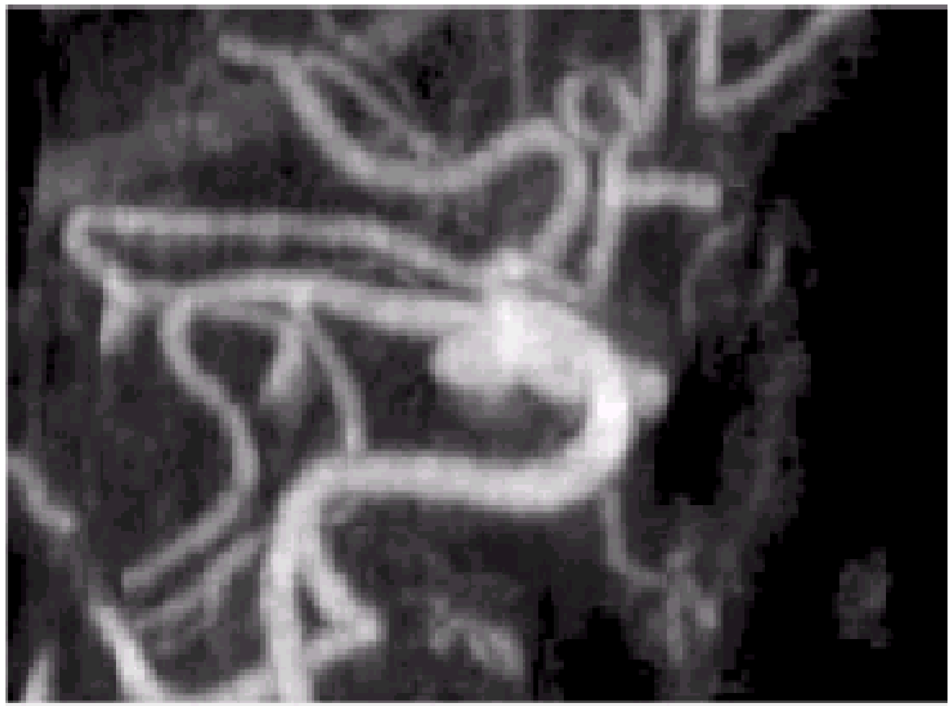
Nonlinear anisotropic diffusion examples: K=3, t=0, 250, 875, 3000.

Weickert

Edge-enhancing anisotropic diffusion



Enhancement of tubular structures
(Karl Krissian, et. al.)



Enhancement of vessels
(Karl Krissian, et. al.)

Coherence-enhancing anisotropic diffusion

Fig. 4. Coherence-enhancing anisotropic diffusion of a fingerprint image. (a) LEFT: Original image, $\Omega = (0, 256)^2$. (b) RIGHT: Filtered, $\sigma = 0.5$, $\rho = 4$, $t = 20$. From [74].

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Coherence-enhancing anisotropic diffusion

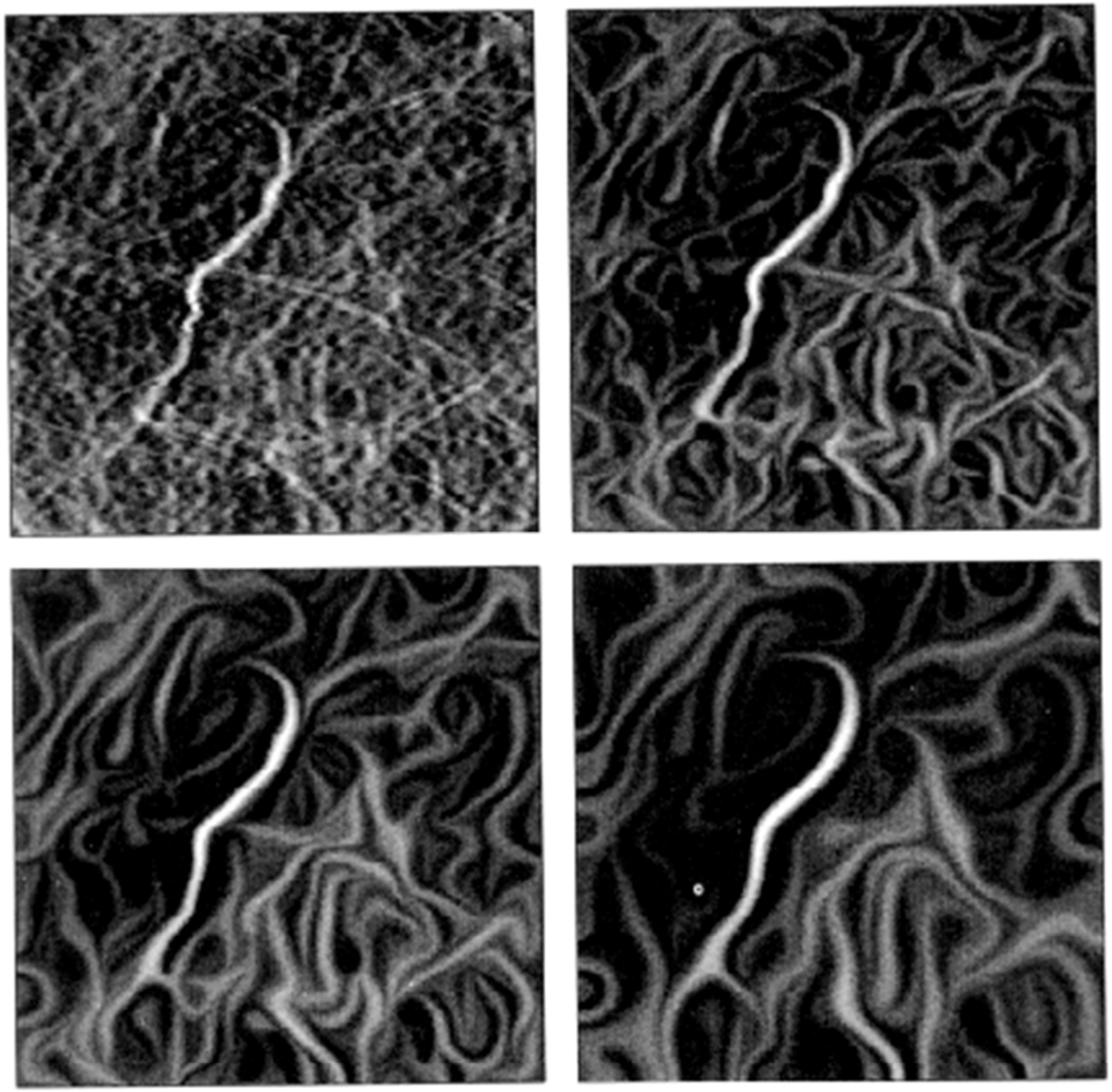

Figure 5.12: Scale-space behaviour of coherence-enhancing diffusion ($\sigma = 0.5$, $\rho = 2$). (a) TOP LEFT: Original fabric image, $\Omega = (0, 257)^2$. (b) TOP RIGHT: $t = 20$. (c) BOTTOM LEFT: $t = 120$. (d) BOTTOM RIGHT: $t = 640$.

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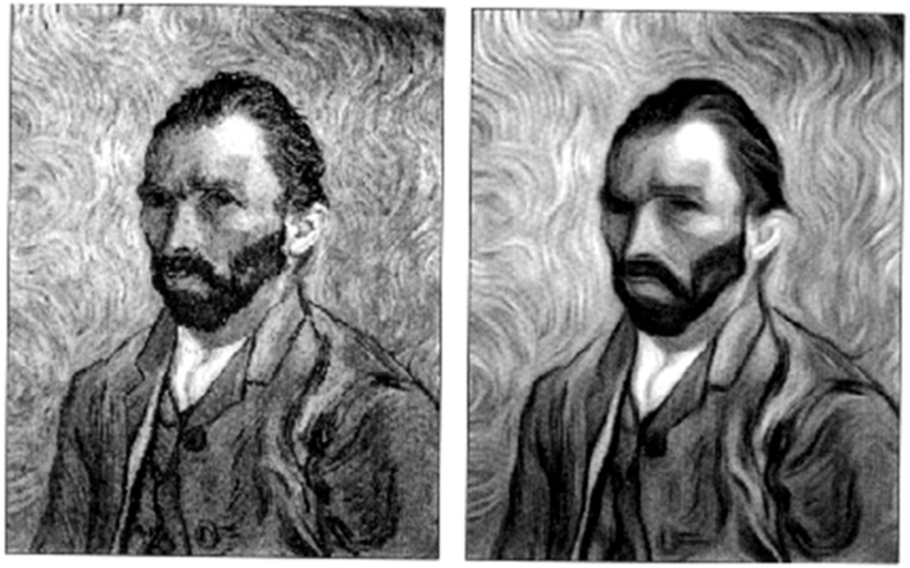

Figure 5.14: Image restoration using coherence-enhancing anisotropic diffusion. (a) LEFT: "Selfportrait" by van Gogh (Saint-Rémy, 1889, Paris, Museé d'Orsay). $\Omega = (0, 215) \times (0, 275)$. (b) RIGHT: Filtered, $\sigma = 0.5$, $\rho = 4$, $t = 6$.

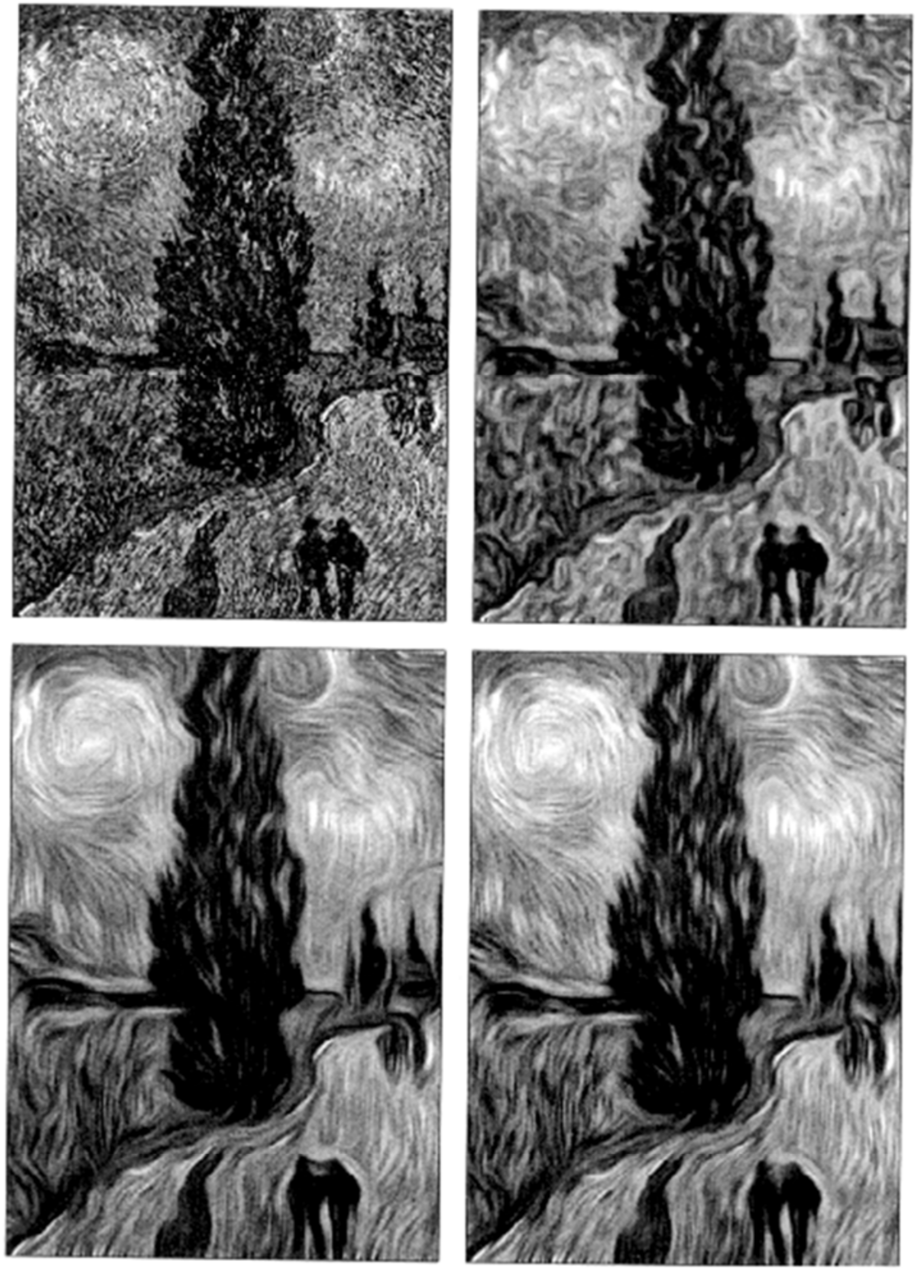

Figure 5.15: Impact of the integration scale on coherence-enhancing anisotropic diffusion ($\sigma = 0.5$, $t = 8$). (a) TOP LEFT: "Road with Cypress and Star" by van Gogh (Auvers-sur-Oise, 1890; Otterlo, Rijksmuseum Kröller-Müller), $\Omega = (0, 203) \times (0, 290)$. (b) TOP RIGHT: Filtered with $\rho = 1$. (c) BOTTOM LEFT: $\rho = 4$. (d) BOTTOM RIGHT: $\rho = 6$.

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