National Tsing Hua University Calculus(I)

Midterm Exam

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1. Let $(a_n)_{n=1}^{\infty}$ be a sequence, and $L \in \mathbb{R}$
   (a) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$
   (b) Is there a divergent sequence $(b_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} |b_n| = L$? Prove your answer.(Hint:Consider two cases $L = 0$ and $L \neq 0$)

2. Let $(b_n)_{n=1}^{\infty}$ be a convergent sequence such that $\lim_{n\to\infty} b_n \neq 0$. Use the definition of limits to prove that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim\limits_{n\to\infty} b_n}$$

3. Prove the convergence of the sequence:
   (a) Prove that the sequence $\left(\frac{\sin n + \cos n}{\sqrt{n}}\right)_{n=1}^{\infty}$ convergent.
   (b) Prove that the sequence $\left(\frac{n^4 - 3n^2 + n + 2}{n^3 - 7n}\right)_{n=1}^{\infty}$ divergent.
   (c) Prove that the sequence $\left(\cos \frac{n\pi}{3}\right)_{n=1}^{\infty}$ convergent.

4. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} a_n = \alpha$. Define the average sequence by

$$\sigma_n = \frac{a_1 + \cdots + a_n}{n}$$

(a) Prove that for any $\epsilon > 0$, there exists N_ϵ such that for all $n > N_\epsilon$, the following inequality holds:

$$\frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n - N_\epsilon)(\alpha - \epsilon)}{n} < \sigma_n < \frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n - N_\epsilon)(\alpha + \epsilon)}{n}$$

(b) Prove that the sequence $(\sigma_n)_{n=1}^{\infty}$ is also convergent, and $\lim\limits_{n\to\infty} \sigma_n = \alpha$

5. Evaluate the limits:

(a) $\lim\limits_{x\to -1} |x|(x^4 - 3)$

(b) $\lim\limits_{x\to 1} \dfrac{x^{10} - 1}{x^3 - 1}$

(c) $\lim\limits_{x\to 2^+} f(x)$ if $f(x) = \begin{cases} 2x - 1, & x \leq 2 \\ x^2 - x, & x > 2 \end{cases}$

(d) $\lim\limits_{x\to \pi} \dfrac{\sin x}{x - \pi}$

(e) $\lim\limits_{x\to 0} \dfrac{\sin 7x - \sin 5x}{\sin x}$

(f) $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

6. Let $f(x)$ be a real valued function which is continuous in $(-\infty, \infty)$. Suppose that

$$\lim_{n\to\infty} \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} = 1$$

Prove that $f(x)$ has infinitely many zeros in the half line$(0, \infty)$.

7. Let

$$f(x) = \begin{cases} \dfrac{\sin(x - \frac{\pi}{2}\cos x)}{x^2+1}, & |x| > 2\pi \\ -\dfrac{x^2}{4\pi^2(4\pi^2+1)}, & |x| \le 2\pi \end{cases}$$

   (a) Prove that f is continuous on $(-\infty, \infty)$.

   (b) Prove that f is bounded on $(-\infty, \infty)$, i.e. there exists $M > 0$ such that $|f(x)| \le M$ for any $x \in (-\infty, \infty)$

8. Let $f(x)$ be differentiable function.

   (a) State the definition of f'(x), the derivative of $f/$

   (b) Suppose $f(x) = x^3$. Use the definition of derivative to prove $f'(x) = 3x^2$