Linear Algebra I Midterm Exam

January 13, 2026

1. Determine if each of the following subsets of $\mathbb{R}^4$ is a subspace, and prove your answers.
   (a) $\{(a^2, b^2, a^2, b^2)|a, b \in \mathbb{R}\}$
   (b) $\{((a^3, b^3, a^3, b^3)|a, b \in \mathbb{R}\}$
2. Let k and n be positive integers with $k < n$. Let $\{v_1, \cdots, v_n\}$ be a set of n linearly independent vectors in an F-vector space V. Show that if $u \in V$ is a vector such that $\{u, v_1, \cdots, v_k\}$ and $\{u, v_{k+1}, \cdots, v_n\}$ are both linearly dependent, then $u = 0$.
3. Let V be an F-vector space. Let S and T be finite subsets of V with $|S| \leq |T|$. Show that if T is linearly independent and $T \subseteq \text{span}(S)$, then T is a basis for $\text{span}(S)$.
4. Let V be a finite dimensional F-vector space, and let U be a subspace of V. Show that there exists a subspace W of V such that $U \subsetneq W \subsetneq V$ if and only if $\dim U \leq \dim V - 2$
5. Let V and W be F-vector spaces. Let $T : V \to W$ be a linear transformation, and let $v_1, \cdots, v_n \in V$. Show that $\{T(v_1), \cdots, T(v_n)\}$ spans $R(T)$ if and only if $N(T)+\text{span}\{v_1, \cdots, v_n\} = V$.
6. Let $T : P_n(F) \to P_n(F)$ be a linear transformation, and let A be the matrix representation of T relative to the ordered basis $\{1, x, \cdots, x^n\}$. Show that A is upper triangular if and only if $\deg T(f) \leq \deg f$ for all $f \in P_n(F)$.