Linear Algebra I Final Exam

January 13, 2026

1. Let A be a square matrix whose entries are all integers. Show that A has an inverse matrix whose entries are all integers if and only if $\det A = \pm 1$

2. Let V and W be F-vector spaces, $n \in \mathbb{N}$, and $f : V^n \to W$ be a multilinear alternating function. Show that if $v_1, \cdots, v_n$ are linearly dependent, then $f(v_1, \cdots, v_n) = 0$

3. Let $W_1, W_2 \subseteq \mathbb{R}^3$ be two planes through the origin. For $i = 1, 2$, let $T_i : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto W_i, and let A_i be the matrix representation of T_i relative to the standard basis for $\mathbb{R}^3$

   (a) Determine which of the following four sentences is correct, and give a geometric explanation of how you know.

   (A) $\text{rank}(A_1, A_2) = 2$ if and only if W_1 and W_2 coincide.

   (B) $\text{rank}(A_1, A_2) = 2$ if and only if W_1 and W_2 are perpendicular.

   (C) $\text{rank}(A_1, A_2) = 1$ if and only if W_1 and W_2 coincide.

   (D) $\text{rank}(A_1, A_2) = 1$ if and only if W_1 and W_2 are perpendicular.

   Your explanation should include the geometric meaning of $\text{rank}(A_1, A_2)$. You may include pictures in your explanation.

   (b) Explain why any basis $\{u, v\}$ for W_1 can ve extended to a basis $\{u, v, w\}$ for $\mathbb{R}^3$ such that the matrix representation of T_1 relative to $\{u, v, w\}$ is

   $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   Your explanation should include how to choose w and why your choice makes the matrix of T_1 the above form.

   (c) Write down the definition of what it means for two square matrices to be similar. Then prove that A_1 and A_2 are similar. You should clearly state any theorems that you use in your proof. You may also use the result of (b) even if you did not explain (b).

4. Let $c \in \mathbb{R}$ and let $T : P_3(\mathbb{R}) \to \mathbb{R}^4$ be the linear transformation that sends any $f \in P_3(\mathbb{R})$ to the vector

$$(f(0) + f'(-1)\ ,\ \ cf(0),\ \ 2f(0) + cf''(-1) + f'''(1),\ \ 2f'(-1) + f''(-1) + cf'''(1)) \in \mathbb{R}^4$$

   (a) If T is invertible and $T^{-1}((-1,0,1,0)) = x + x^2$, determine the value of c.

   (b) Find all $c \in \mathbb{R}$ such that T is NOT invertible.

5. Find the dimension and a basis of the $\mathbb{R}$-vector space

$$\{f \in P_5(\mathbb{R}) | f'''(1) = 6f(-1), f''''(1) = 24f(-1)\}$$

6. Let $A \in M_{m\times n}(\mathbb{R})$. Show that the system of linear equations $Ax = b$ has infinitely many solutions for all $b \in \mathbb{R}^m$ if and only if $m < n$ and the row vectors of A are linearly independent.