National Tsing Hua University Calculus(I) Final Exam

Instrutor:Hsuan-Yi Liao

1. Let
$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$
   (a) Prove that the sequence$(x_n)_{n=1}^{\infty}$ converges
   (b) Prove that $0 < \lim_{n\to\infty} x_n < 3$
2. Let f be a function define on$(c-1, c+1)$ where $c \in \mathbb{R}$. Supposethat f is differentiable at c. Prove that f is continuous at c.
3. Let $f(x)$ be a polynomial of degree n with zeros
$$a_1 < a_2 < \cdots < a_n$$
   Let $g(x)$ be a polynomial of degree at most $n-1$. Suppose that $A_1, \cdots, A_n \in \mathbb{R}$ are numbers such that
$$\frac{g(x)}{f(x)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_n}{x-a_n} \quad \forall x \in \mathbb{R} \setminus \{a_1, \cdots, a_n\}$$
   (a) Prove that $A_j = \dfrac{g(x)}{f'(x)}$
   (b) Assume, in addition, that $A_k A_{k+1} > 0$ for $k = 1, 2, \cdots, n-1$. Prove that $g(x)$ is of degree $n-1$, and it has $n-1$ distinct real zeros.

4. Suppose $f \in C^1[0,1]$, $f(x) \in [0,1]$, and $|f'(x)| < 1$ for all $x \in [0,1]$

   (a) Prove that there exist a constant M, $0 \leq M < 1$, such that$|f'(x)| \leq M$for all $x \in [0,1]$

   (b) Let M be a number such that $|f'(x)| \leq M$ for all $x \in [0,1]$. Prove that $|f(x) - f(y)| \leq M\,|x-y|$

   (c) Let $x_0 \in [0,1]$. Since $f([0,1]) \subset [0,1]$, we can define a sequence $(x_n)_{n=1}^{\infty}$ in [0, 1] by interation:

$$x_1 = f(x_0), x_2 = f(x_1), \cdots, x_n = f(x_{n-1})$$

   Prove that the sequence $(x_n)_{n=1}^{\infty}$ is convergent, and

$$f(\lim_{n\to\infty}) = \lim_{n\to\infty} x_n$$

5. (a) Prove that $x^6 - \frac{x^5}{5} + 1 > 0$ for all $x \in [0,1]$.

   (b) Suppose a continuous function $f \in C[0,1]$ satisfies the following condition

   (i) $\int_0^1 f(x) \cdot x^{10}\,dx = 5;$

   (ii) $\int_0^1 f(x) \cdot (x^6 - \frac{x^5}{5} + 1)\,dx = -6$

   Prove that there exists $c \in [0,1]$ such that $f(c) = 0$

6. Calculate the following for $x > 0$

   (a) $\frac{d}{dx}\left(\int_{x^x}^{3} \frac{dt}{\sqrt{2t+5}}\right)$

   (b) $\frac{d}{dx}\left(\int_{3x}^{\frac{1}{x}} \cos 2t\,dt\right)$

7. Evaluate the integrals.

(a) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{\sqrt{1-e^{-x}}}\,dx$

(b) $\displaystyle\int_{e^2}^{1} x\ln(\sqrt{x})\,dx$

(c) $\displaystyle\int_{0}^{\frac{\pi}{6}} \sin^3 3x\,dx$

(d) $\displaystyle\int_{0}^{1} \frac{x^5}{(x-2)^2}\,dx$

(e) $\displaystyle\int_{-\infty}^{0} xe^{x}\,dx$