Part 5

Multi-Period Discounting

So far this book has mainly worked in a framework that is motivated by the Capital Asset Pricing Model. Although the estimation of betas that has been discussed can directly be translated to the betas of multi-factor models as we discussed at the end of Chapter 5 and as we will discuss further in Chapter 12, the CAPM has been our main point of departure. Apart from identifying the market return as the single factor that drives expected returns (next to the risk free interest rate), a key characteristic of the CAPM is also that it is - by construction - a one-period model. At the same time, in virtually every valuation exercise, the analyst needs to discount a stream of future expected cash flows, beyond one period.

In this part of the book we will discuss the discounting of multi-period cash flows. To this end, Chapter 11 will first analyze what assumptions need to be made in order to apply the one-period CAPM in a multi-period setting. In that chapter we will also go into the question what is meant with one period, and how the discounting period is related to the estimation of the necessary parameters in the discount rate. Chapter 12 will discuss intertemporal models that are designed for multi-period problems. In that chapter we will derive the Intertemporal CAPM, which is a multi-factor alternative to the CAPM. Chapter 13 will deal with the empirical implementation of the Intertemporal CAPM, in particular using the Fama-French three-factor model.

CHAPTER 11

Multi-Period and Longer Horizon Discounting

In this chapter we will start by analyzing assumptions for the cash flow process that allow us to apply the CAPM in a multi-period setting. At one level, this is a theoretical exercise, but it is important that we understand the type of assumptions we need to make (beyond the ones we already made when deriving the CAPM in Chapter 5) in order to be able to apply the CAPM. Understanding these assumptions will also help us to better motivate the choices we make in the valuation process in modeling and estimating the necessary parameters - and ultimately make better choices.

A second topic we will discuss is the meaning of "one period". In particular, the CAPM is a one-period model but it does not specify the length of a period. Likewise, we model expected cash flows for multiple periods, but we also need to define the length of the discounting period. For instance, we often use a specific data frequency to estimate for instance betas, which implicitly use a month or a quarter as one period, whereas cash flows are often modeled annually. Thus, the frequency of our estimation of betas does not coincide with the frequency of our cash flow projections. We therefore need a proper understanding of how discount rate parameters such as betas and the market risk premium translate from one frequency (e.g. monthly) to another (e.g. annual).

11.1. Discounting Multi-Period Cash Flows

11.1.1. Revisiting the One Period Case. [1]In Chapter 9 we introduced the valuation of a single cash flow using both risk adjusted discounting and certainty equivalent valuation in a one period setting. In this section we will extend this to a multi-period setting. Whereas in Chapter 9 the focus was on deriving the value at time zero, V_0, of a cash flow that was expected at time one, CF_1, in this chapter we want to know the time zero value V_0 of a cash flow that is expected at some future date t, CF_t, where t may be (much) later than time one, i.e. $t \geq 1$. We will revert to uncertainty in the cash flow itself (instead of revenues), i.e., ε_t, which is defined from

$$CF_t = E_{t-1}\left[CF_t\right]\left(1+\varepsilon_t\right). \tag{11.1.1}$$

Thus, ε_t, is the (percentage) unexpected part of the realized cash flow CF_t relative to what we expect it to be one period earlier, E_{t-1}. Rational expectations imply that $E_{t-1}\left[\varepsilon_t\right] = 0$. Below we will see what additional assumptions for ε_t are required.

We revert to the uncertainty in the (free) cash flow ε_t rather than the uncertainty in revenues u_t which was the main focus of the previous chapter. Recall that the main motivation to study the innovation in revenues is the availability of good quality data: sales data are usually available at a relatively high frequency (quarterly) and of good quality, which is not necessarily the case for cash flow data. However, the focus in valuations is on cash flow, not revenues, and the ultimate goal is to understand the risk profile of cash flows. We also discussed in Chapter 9 that the risk profile of revenues and cash flows are related through operating leverage however, which in a bit simplified analysis comes down to:

$$\varepsilon_t = OL \times u_t = \frac{Gross\ Margin}{EBIT} \times u_t. \tag{11.1.2}$$

Thus, knowing the risk profile of revenues together with operating leverage allows us to determine the risk profile of cash flows. The risk profile of revenues is relatively easy to estimate, as discussed in Chapter 10.

From the risk adjusted discount rate and certainty equivalent value in Chapter 9, we know we can state the time $t-1$ value of a cash flow to be received at time t with a risk adjusted discount rate as:

$$V_{t-1} = \frac{E_{t-1}\left[CF_t\right]}{1+E_{t-1}\left[R_t\right]} = \frac{E_{t-1}\left[CF_t\right]}{1+k_{t-1}}. \tag{11.1.3}$$

[1]This section builds on Fama (1977).

Note that R_t refers to the cash flow here, implying we are working at the (unlevered) firm level. As we are working in a multi-period framework, we now need to be precise in our time subscripts. To avoid confusion: A time subscript t to a variable means that the variable is *known* as of time t. Thus, the cash flow is eventually known when it is realized at time t. The discount rate or expected return to be applied for the period $t-1 \to t$ is known at time $t-1$, etc. The notation k_{t-1} and $E_{t-1}[R_t]$ imply that we explicitly allow for time-variation in discount rates. Applying the CAPM we can write

$$E_{t-1}[R_t] = Rf_{t-1} + \beta_A MRP_{t-1}, \tag{11.1.4}$$

$$\beta_A = \frac{Cov[R_t, Rm_t]}{Var[Rm_t]} = \frac{Cov\left[\frac{CF_t}{V_{t-1}}, Rm_t\right]}{Var[Rm_t]},$$

$$MRP_{t-1} = E_{t-1}[Rm_t - Rf_{t-1}].$$

We do not put a time subscript to β, which means that we assume that (co)variances are not changing over time. This assumption can be relaxed as well, but it is empirically less relevant for valuation purposes. Following the steps in Chapter 9, Equation (11.1.3), together with the CAPM, implies a certainty equivalent value

$$V_{t-1} = \frac{E_{t-1}[CF_t](1 - A_{m,t-1}Cov[\varepsilon_t, Rm_t])}{1 + Rf_{t-1}}. \tag{11.1.5}$$

11.1.2. The Multi-Period Case. Equations (11.1.3) (together with (11.1.4)) and (11.1.5) are two equivalent ways of expressing the one-period discounted value V_{t-1} of a cash flow to be received at t. Equations (11.1.3) and (11.1.4) state it with a risk adjusted discount rate, whereas Equation (11.1.5) states it as a certain equivalent cash flow discounted at the risk free interest rate. These expressions are based on the CAPM. Here we make the important assumption that not only are variances and covariances constant over time, also the market parameters Rf_t and $A_{m,t}$ are non-stochastic[2], for every period t. Since in the CAPM $A_{m,t} = E_t[Rm_{t+1} - Rf_t] / Var[Rm_{t+1}]$, this effectively means that we assume the market risk premium MRP_t to be non-stochastic as well. For now, we make the even stronger assumption that A_m is constant. Fama (1977) and Fama & MacBeth(1974) make clear that such assumptions are necessary in order for the CAPM to hold every period. In the next chapter, when discussing the Intertemporal CAPM, we will verify that such assumptions indeed imply that the CAPM holds every period.

[2]Note that non-stochastic does not mean that these parameters are constant, only that they change in predictable ways.

In technical language, the assumptions on constant (co)variances, expected returns, and risk free interest rates is known as a *constant investment opportunity set,* as opposed to a *stochastic* investment opportunity set. All securities that we can invest in are characterized by their expected returns and (co)variances, offering us a set of investment opportunities - as we studied in Chapter 5. The fact that we assume these expected returns and (co)variances to be constant is to say that we assume the investment opportunities to be the same every every period, or constant. Just as the one-period CAPM can be motivated by the assumption that security returns can be characterized by a normal distribution, so the multi-period version of the CAPM can be motivated by the (additional) assumption of a constant investment opportunity set.

Note that such non-stochastic behavior of Rf_t and MRP_t is at odds with the empirical evidence on these parameters as discussed in Chapter 6. For now, however, making the assumption that the CAPM holds every period and the implied consequences for these market parameters, allows us to focus on the risk profile of the cash flows, and the assumptions we need to make for the cash flow process, to justify a constant risk adjusted discount rate that adheres to the CAPM.

In order to analyze the multi-period case, we start by making assumptions about the cash flow process. Specifically, we need to model how our expectations about the cash flow change over time. Following Fama (1977), we assume the expected cash flow evolves as

$$E_1 [CF_t] = E_0 [CF_t] (1 + \varepsilon_1), \tag{11.1.6}$$

$$E_2 [CF_t] = E_1 [CF_t] (1 + \varepsilon_2), \tag{11.1.7}$$

$$\vdots$$

$$E_s [CF_t] = E_{s-1} [CF_t] (1 + \varepsilon_s) \tag{11.1.8}$$

$$\vdots$$

$$CF_t = E_{t-1} [CF_t] (1 + \varepsilon_t). \tag{11.1.9}$$

The time-subscripts for the expectations operator E_s highlight that each time we take the expectation of the cash flow CF_t at time s, taking all available information until time s into account.[3] For the realized cash flow, from the

[3] Thus, $E_s [CF_t]$is the expectation of CF_t *conditional* on the information that is known at time s, $E_s [CF_t] = E [CF_t \mid I_s]$. I_s denotes the information available at time s.

perspective of time zero, this means

$$CF_t = E_0\left[CF_t\right] \times (1+\varepsilon_1) \times (1+\varepsilon_2) \ldots \times (1+\varepsilon_t).$$

Again, ε_s ($s = 1, 2, \ldots, t$) is the per period unexpected change in the expected cash flow. As the actual cash flow is only realized in the last period t, it is natural to think of the ε_s's as the news or updates about the expected cash flow every period, which is clearly seen from Equations (11.1.6)-(11.1.9).

We additionally assume that the $\varepsilon_s' s$ are uncorrelated with each other and have a constant correlation with the market return only in the same period - thus, ε_s is only correlated with the market return Rm_s and not with market returns in any other period. We also assume that ε_s has expectation zero (which is the rational expectations assumption we already made) and the same standard deviation σ_ε in every period. Not all of these assumptions are strictly necessary (and Fama (1977) indeed makes weaker assumptions), but for our purpose they provide the most convenient setting for the analysis.

Armed with this cash flow process, we can now analyze the multi-period valuation of a future cash flow. We start from the value of the cash flow CF_t with a value at time $t-1$, V_{t-1} as above, and work backwards in time through $t-2, t-3, \ldots$ until we arrive at time zero. The value at time $t-2$ is

$$V_{t-2} = \frac{E_{t-2}\left[V_{t-1}\right]}{1+E_{t-2}\left[R_{t-1}\right]} = \frac{E_{t-2}\left[V_{t-1}\right] - A_m Cov\left[V_{t-1}, Rm_{t-1}\right]}{1+Rf_{t-2}}, \tag{11.1.10}$$

where the second step follows again by assuming that the CAPM also holds from $t-2 \to t-1$ and following similar steps as above (and in Chapter 9). We can combine this certainty equivalent value with the one of V_{t-1} in (11.1.5). With the additional assumption that $A_{m,t-1} = A_m$ (constant) and that Rf_{t-1} is non-stochastic, all uncertainty in V_{t-1} is reflected in $E_{t-1}\left[CF_t\right]$. This is an important result: going from $t-1$ to the final time t, the uncertainty is about the *realization* of CF_t. At time time $t-1$ the analyst has a rational expectation about the cash flow, $E_{t-1}\left[CF_t\right]$, and the uncertainty to be resolved at time t is in the difference between the actual cash flow CF_t and this expectation, which is driven by ε_t: $CF_t = E_{t-1}\left[CF_t\right](1+\varepsilon_t)$. Starting at time $t-2$, the uncertainty in going to time $t-1$ is in the adjustment of the *expectation* of this cash flow: how does the expectation of CF_t change from $t-2$ to $t-1$?

With all these additional assumptions and writing $Cov\left[\varepsilon_s, Rm_S\right] = \sigma_{\varepsilon m}, \forall s$, we can now substitute V_{t-1} from (11.1.5) into V_{t-2} in (11.1.10) and use from the cash flow process that $E_{t-1}[CF_t] = E_{t-2}[CF_t](1+\varepsilon_{t-1})$ to obtain:

$$V_{t-2} = \frac{E_{t-2}[V_{t-1}] - A_m Cov[V_{t-1}, Rm_{t-1}]}{1+Rf_{t-2}}$$
$$= \frac{E_{t-2}[E_{t-1}[CF_t](1-A_m\sigma_{\varepsilon m})](1-A_m\sigma_{\varepsilon m})}{(1+Rf_{t-2})(1+Rf_{t-1})}$$

(11.1.11)
$$\Rightarrow V_{t-2} = \frac{E_{t-2}[CF_t](1-A_m\sigma_{\varepsilon m})^2}{(1+Rf_{t-2})(1+Rf_{t-1})} = \frac{E_{t-2}[CF_t](1-\beta_{\varepsilon m}MRP)^2}{(1+Rf_{t-2})(1+Rf_{t-1})}.$$

The last step uses $A_m = MRP/Var[Rm]$.

Before proceeding to time zero, note that the denominator of the last equation (11.1.11) combines two one-period interest rates in different time periods. Although the assumption we made is that the interest rates are non-stochastic, at time $t-2$, the combination of these two one-period interest rates implies that we can combine them into a single two-period interest rate if we put the expectations hypothesis of interest rates to use. Using the expectations hypothesis and the notation introduced in Chapter 5, from a time $t-2$ perspective we have:

$$E_{t-2}[(1+Rf_{t-1})(1+Rf_{t-2})] =$$
$$E_{t-2}\left[\left(1+Rf(1)_{t-1}\right)\left(1+Rf(1)_{t-2}\right)\right] = \left(1+Rf(2)_{t-2}\right)^2.$$

Recall that $Rf(m)_t$ is the (per period) interest rate at time t for a maturity of m periods. Note that this is again an assumption, not a definition: the last step only holds if the expectations hypothesis of interest rates holds. We are deliberately loose here in the use of non-stochastic interest rates, expectations, and the expectations hypothesis. Nonetheless, framing the valuation in the way we do here, it makes intuitive sense to discount the certainty equivalent cash flow in (11.1.11) with a two-period interest rate:

$$V_{t-2} = \frac{E_{t-2}[CF_t](1-\beta_{\varepsilon m}MRP)^2}{\left(1+Rf(2)_{t-2}\right)^2}.$$

It is now a small step to repeat all this going backwards to time zero. We leave the actual calculations as an exercise to the reader, but it is straightforward to show that the value of the cash flow at time zero equals:

(11.1.12)
$$V_0 = \frac{E_0[CF_t](1-\beta_{CF}MRP)^t}{\left(1+Rf(t)_0\right)^t}.$$

Here we denote $\beta_{\varepsilon m} = \beta_{CF}$ as in Chapter 9. This can also be expressed using a risk adjusted discount rate:

$$V_0 = \frac{E_0\left[CF_t\right]}{\left(1+k_0\right)^t}, \tag{11.1.13}$$

$$k_0 = Rf\left(t\right)_0 + \beta_A MRP. \tag{11.1.14}$$

The two valuation approaches are equivalent. It is important to notice however that - as in Chapter 9 - the meaning of the betas is different. β_{CF} is the regression slope coefficient of a regression of cash flow news ε_t on the market returns Rm_t. It relates the cash flow CF_t to its expectation $E_{t-1}\left[CF_t\right]$, or - equivalently - it relates the value V_t to its expectation $E_{t-1}\left[V_t\right]$. β_A on the other hand, is the regression slope coefficient of the change in value V_t/V_{t-1} on the market returns Rm_t. Thus, the denominator is now V_{t-1} not $E_{t-1}\left[V_t\right]$. The relation between the two betas was already given in Equations (9.1.10) and (9.1.11) in Chapter 9, and these still hold, as can readily be checked:

$$\beta_A = \beta_{CF}\left(\frac{1+Rf\left(t\right)_0}{1-\beta_{CF}MRP}\right), \tag{11.1.15}$$

$$\beta_{CF} = \frac{\beta_A}{1+Rf\left(t\right)_0+\beta_A MRP}. \tag{11.1.16}$$

Note that in all these latter results we explicitly write the t-period risk free interest $Rf\left(t\right)_0$. This is the risk free interest rate at time zero with a maturity t that matches the maturity of the cash flow CF_t. Thus, as already discussed in Chapter 6, the multi-period framework developed here, together with the (loosely interpreted) expectations hypothesis for interest rates, implies that in discounting different cash flows, each cash flow (for different times t) has its own risk free interest rate with maturity t, i.e., $Rf\left(t\right)$.

11.2. Estimation Issues

From the multi-period analysis in the previous section, two estimation issues arise that we will discuss below. First, as the multi-period discounting implies that cash flows that occur at different times require different risk free interest rates (with different maturities), the question arises how to choose the interest rate and the implied market risk premium. Second, as the cash flows - and the revenues from which they are derived - occur at later periods, we need to make additional assumptions on how we can estimate asset betas from revenue betas.

11.2.1. The Risk Free Interest Rate and the Market Risk Premium. From Equations (11.1.12) and (11.1.14) we see that a cash flow at time t, valued at time zero, should be discounted using the t-period risk free interest rate. In the certainty equivalent valuation a risk premium is deduced from the expected cash flow, whereas in the risk-adjusted discount rate a risk premium is included in the discount rate. The implication is that each cash flow, depending on its timing t has its own discount rate as a different risk free interest rate $Rf\left(t\right)$ applies to each period.

As already discussed in Chapter 6 however, in valuations the analyst will often want to use a single discount rate for all future expected cash flows instead of different discount rates for each cash flow. With a constant risk premium this means we want to use one risk free interest rate, for only one maturity, instead of the full term structure of interest rates where the maturity of the interest rates are matched with the maturity of the cash flows. In case of bond valuations this single interest rate would be the yield to maturity. In Chapter 6 it was argued that the best choice for this single interest rate is the risk free interest rate whose maturity equals the yield duration or double duration in Equation (6.1.8).

The analysis of the yield to maturity and duration in Chapter 6 is for the case where the discount rate is the interest rate, i.e., there is no risk premium or risk adjustment in the discount rate. The advantage of the certainty equivalent valuation is that the risk adjustment is in the cash flows, which are then discounted at the risk free interest rate. This puts us in the same setting as in Section 6.1. Thus, defining the certainty equivalent cash flow as

$$CEC_t = E\left[CF_t\right]\left(1-\beta_{CF}MRP\right)^t,$$

the yield duration of a series of cash flows is as in (6.1.8):

$$DurY = \frac{\sum_t t^2 \times PV\left(CEC_t\right)}{\sum_t t \times PV\left(CEC_t\right)},$$

where the present values PV are calculated with the risk free interest rates $Rf(t)$. The interest rate with a maturity $DurY$ is then (approximately) the yield to maturity, $y \approx Rf(DurY)$, that can be used as a single interest in (11.1.12) and (11.1.14).

From Exhibit 6.4 we know that for growing perpetual cash flows, which are often the basis of company valuations, the duration is in most cases at least 20 years, suggesting that we should use a very long term interest rate. A caveat is in order however. The analysis in Section 6.1 used a constantly growing perpetual stream of cash flows with a positive growth rate g: $E_0[CF_t] = CF_0(1+g)^t$. Using the certainty equivalent cash flow, we obtain:

$$CEC_t = CF_0(1+g)^t(1-\beta_{CF}MRP)^t = CF_0(1+ceg)^t, \tag{11.2.1}$$

where ceg is the *certainty equivalent growth rate*, a combination of the growth rate and the risk adjustment:

$$ceg = (1+g)(1-\beta_{CF}MRP) - 1.$$

This certainty equivalent growth rate can easily be negative when the risk premium $\beta_{CF}MRP$ exceeds the growth rate g. Exhibit 11.1 therefore repeats Panel B of Exhibit 6.4, showing the yield duration $DurY$ in Equation (6.1.9) allowing for negative ceg's.

$yield \setminus ceg$	-10%	-8%	-6%	-4%	-2%	0%	2%	4%	6%	8%	10%
1%	17	21	28	39	66	201					
2%	16	19	25	33	50	101					
3%	15	18	22	28	40	68	205				
4%	14	16	20	25	34	51	103				
5%	13	15	18	22	29	41	69	209			
6%	12	14	17	20	26	34	52	105			
7%	12	13	15	18	23	30	42	70	213		
8%	11	13	14	17	21	26	35	53	107		
9%	10	12	14	16	19	23	30	43	72	217	
10%	10	11	13	15	17	21	27	36	54	109	

Exhibit 11.1: Yield duration for different combinations of yield to maturity and certainty equivalent growth (ceg).

Exhibit 11.1 shows that the result from Chapter 6 that yield durations mostly exceed 20 years no longer holds. For high risk premiums (negative certainty equivalent growth) combined with high interest rates (yield to maturity), durations are often well below 20 years. However, for the combinations in Exhibit 11.1 the duration never falls short of 10 years, implying that when using a single interest rate in valuations, maturities of at least 10 years should be chosen.

The Market Risk Premium. A caveat is in order regarding the use of the market risk premium. Given the discussion on the interest rates, the analyst may wonder which interest should be used in determining the market risk premium MRP. Here it is natural to take the interest rate that coincides with the discounting period. Thus, if we model annual cash flows and discount then with annual discount rates, as is the most common valuation practice, the market risk premium should reflect an annual premium:

$$MRP\,(1\text{ year}) = E\left[Rm\,(1\text{ year}) - Rf\,(1\text{ year})\right].$$

Even if the cash flow to be discounted occurs in five years - or if the duration of all cash flows is five years - and we would use the five-year interest rate $Rf\,(5)$ in (11.1.12) and (11.1.14), the annual discounting implies we are considering annual investment returns. The MRP in the discount rate should reflect a premium that is earned each year, not a five year holding period. $MRP\,(1\text{ year})$ is therefore the premium of a *one year* stock market investment over a *one year* interest rate, not a longer term interest rate.

This one year premium is also implicit in the market risk aversion A_m that we use in our CAPM-based analysis and that we assume to be constant. From Chapter 5 we know that

$$A_m = \frac{E\left[Rm - Rf\right]}{Var\left[Rm\right]} = \frac{MRP}{Var\left[Rm\right]}.$$

The market risk aversion and the variance of the market portfolio are related to the one period investment horizon. When discounting annual cash flows the investment horizon is one year and these parameters reflect annual risk aversion and annual variance. Unless we have good reason to believe that either the market risk aversion will be different in, say, five years, or the variance of the market portfolio will change in the future, there is no reason to adjust the market risk premium. A constant A_m and a constant $Var\left[Rm\right]$ imply a constant MRP within the CAPM. And even if either A_m or $Var\left[Rm\right]$ are expected to change over time, there is no reason why they should change according to the term structure of interest rates. Adjusting MRP with the interest rate that matches the maturity of the cash flow would imply that the risk aversion A_m and/or the variance $Var\left[Rm\right]$ should also be adjusted according to the term structure of interest rates. This looks artificial at best.

Within the CAPM it is most natural to think of the market risk premium MRP that is required by investors independent of the level of the interest rate. Thus, if $MRP = 5.0\%$ for instance, this premium is added to the risk free interest rate, irrespective of the level of the risk free interest rate. If the one year risk free interest rate is 2.0%, the required one year total market return is $2.0 + 50 = 7.0\%$, whereas if the one year risk free interest rate is 4.0%, the required one year market return is $4.0 + 5.0 = 9.0\%$.[4] This is not to say that the market risk premium (and variance and risk aversion) cannot change, but there is no mechanical relation between changes in these parameters and the term structure of interest rates.

11.2.2. Revenue Betas and Cash Flow Betas. In Chapter 10 we outlined the estimation of cash flow betas, β_{CF}, with betas derived from revenues or sales, β_{Sales}. Whereas the beta of interest, β_{CF}, aims to find the relation between unexpected cash flows versus the market return (ε_t versus Rm_t), the estimation in Chapter 10 started from the relation between unexpected sales versus the market return (u_t versus Rm_t). Knowing this relation with sales we can easily derive the relation with cash flow using Operating Leverage, as in Equation (11.1.2):

$$\varepsilon_t = OL \times u_t.$$

Thus, we can base the cash flow beta on an estimated sales beta and then use operting leverage to adjust it. Equation (11.1.2) immediately implies:

$$\beta_{CF} = \frac{Cov\left[\varepsilon_t, Rm_t\right]}{Var\left[Rm_t\right]} = \frac{OL \times Cov\left[u_t, Rm_t\right]}{Var\left[Rm_t\right]} = OL \times \beta_{Sales}.$$

This is Equation (9.2.2). In order to derive the sales beta, we assume a similar process for the evolution of expected sales as we assumed for the cash flow itself in Equations (11.1.7)-(11.1.10):

$$E_s\left[Sales_t\right] = E_{s-1}\left[Sales_t\right]\left(1 + u_s\right), s = 1....t.$$

In Chapter 10 we developed a way to construct expectations for revenues, giving us $E_{t-1}\left[Sales_t\right]$, which then allowed us to use the realized $Sales_t$ versus this expectation to measure u_t and determine its beta with respect to the market.

[4] This is consistent with the analysis in, for instance, Campbell & Viceira (2002), Chapters 3 and 4.

Note though, that in running this regression to estimate β_{Sales}, we make an additional assumption that

$$Cov\left[u_t, Rm_t\right] = \sigma_{um},$$
$$u_t = \frac{Sales_t - E_{t-1}\left[Sales_t\right]}{E_{t-1}\left[Sales_t\right]}, \forall t.$$

This constant covariance assumption is different from the assumptions above. The assumptions in relation to (11.1.7)-(11.1.10) say that for given revenues (cash flow) in period t, the adjustment in expectations regarding these revenues (cash flow) evolves in such a way that $Cov\left[u_s, Rm_s\right] = \sigma_{um}, 1 \leq s \leq t$. That is, the expectations for a specific period t sales evolve gradually over time such that the change in expectations for this specific sales (cash flow) has constant covariance with the market. Each period sales may have its own variance and its own covariance with the market though. To be precise, in the analysis in this chapter so far there is no need to assume that $Cov\left[u_t, Rm_t\right]$ is the same for every period t. However, when we want to estimate β_{Sales} using the regression

$$u_t = \alpha + \beta_{Sales} Rm_t + e_t,$$

there is an implicit assumption that $Cov\left[u_t, Rm_t\right]$ is constant. Thus, in order to estimate β_{Sales}, we need to make the additional assumption that the relation between unexpected (realized) sales and the market is constant. All time variation is captured in the process of the expectation $E_{t-1}\left[Sales_t\right]$ (using seasonalities and time trends e.g.), but not in the (co)variances.[5]

Having said that, we may still add more granularity in estimating β_{CF} for a specific cash flow. First, a company may have different lines of business that each have their own risk profile. For instance, a car manufacturer often consists of a pure manufacturing part, where the β_{Sales} depend on the sales of cars (or even a particular type of car), and a leasing company part. The leasing company is a financial service company that is likely to have a different risk profile than the pure manufacturing part. Indeed, the evidence in Exhibit 10.5 suggests that the pure manufacturing part has a (revenue based) asset beta of about 0.98, whereas the leasing part would have a much lower asset beta of about 0.43.

Second, even assuming that β_{Sales} is constant for a specific line of business, still allows the analyst to derive cash flow betas, β_{CF}, that are different for different cash flows evolving over time as the cost structure of the company is expected to change. Thus, if the company is expected, or modeled, to have a

[5]Of course, using standard time series regression techniques, we can make β_{Sales} time varying by assuming additional processes like $\beta\left(t\right) = \beta_0 + \beta_1 x_t$, where x_t is a predetermined variable like the temperature one month ago.

change in operating leverage, this also implies a change in β_{CF}. We can make this explicit by adding a time index to operating leverage: $OL(t)$. This $OL(t)$ is the (expected) operating leverage that applies to a specific timed cash flow CF_t. The beta that applies to this specific period t cash flow is then given by:

$$\beta(t)_{CF} = OL(t) \times \beta_{Sales}. \tag{11.2.2}$$

Although perhaps subtle, we write $OL(t)$, not OL_t, to make sure that in the (cash flow) modeling this operating leverage is not stochastic, but planned for - and thus pre-determined or expected. This notation is similar to our notation of $Rf(m)_t$ where the interest rate for a maturity m is known at time t. I.e., once we are at time t, $Rf(m)_t$ is no longer stochastic.

The analysis above highlights that when applying CAPM-based discounting (or for any other model for that matter) in a multi-period setting, many - often implicit and subtle - assumptions are made. This section is not meant to be exhaustive, and the exposition is deliberately loose at some points. The important message is that multi-period discounting using a one period model like the CAPM, as well as the estimation of the necessary parameters, require the analyst to make additional assumptions she needs to be aware of and needs to validate.

11.3. Longer Horizon Discounting

In the previous section we discussed a number of issues that arise when applying a one period model like the CAPM in multi-period discounting exercises. Even when only considering one period, and thus working within the CAPM framework, there are horizon issues to consider however. In particular, although the CAPM is a useful and important one period model, it is silent about what is one period. Thus the horizon that refers to one period is not known a priori. Acknowledging this, the problem faced by the analyst is that she usually estimates the necessary market parameters such as the market risk premium and beta using a given frequency or return horizon (usually monthly or quarterly as we have also done in previous chapters), whereas cash flow projections - and therefore the discounting - use a different frequency or return horizon (typically annually). The question therefore naturally arises how to translate parameters estimated with monthly or quarterly data to ones that can be used when discounting cash flows that are modeled and discounted annually.

In this section we assume the CAPM holds every period. Based on the CAPM the required return or cost of capital for security, company, or asset (or even single cash flow) i is

$$E\left[R_i\right] = k_i = Rf + \beta_i MRP.$$

Thus, there are three parameters that need to be estimated and that need to be made consistent with the discounting horizon: the risk free rate, the market risk premium, and the beta.

11.3.1. The Risk Free Interest Rate. For the risk free rate, the “short term” logically coincides with the the discounting period in the valuation analysis. Thus, if cash flows are modeled and discounted annually, $Rf(1)$ is the one-year interest rate. Having said that, the discussion in the previous section and in Chapter 6 implies that we also incorporate the term structure of interest rates and adjust the risk free interest for the various cash flows according to their maturity. Thus, when a cash flow is expected five years from now, we would start from the five year interest rate $Rf\left(5\right)$. If the first cash flow is closer to the valuation date, as is often the case, a similar adjustment is made. For instance, if cash flows are modeled as occurring at the end of the year, 31 December, but the valuation date is 30 September, the first cash flow occurs after one quarter and the corresponding interest rate is $Rf\left(1/4\right)$.

11.3.2. The Market risk premium. The second parameter is the market risk premium $MRP = E\left[Rm - Rf\right]$. As discussed in Section 6.2 there are different ways to estimate this, even unconditionally. As discussed in the Section

11.1, and as will be spelled out in more detail in Chapter 12, in order for the CAPM to hold in a multi-period setting, we need to assume a constant investment opportunity set. In statistical terms, this means we assume that returns are independently and identically distributed ($i.i.d.$) over time. In more common language, we assume that the mean (expected) returns, their standard deviations and correlations are constant over time for all securities (stocks). In this section we will uphold that assumption when discussing the effect of the horizon on the market risk premium and betas.

We define a K-period (compounded) return as

$$1 + R_{t\to t+K} = \prod_{s=1}^{K} (1 + R_{t+s}) \, ,$$

where $R_{t+s}, s = 1, 2, ..., K$ are the one period returns. For example, for monthly one period returns R_{t+s} the annual return is $R_{t\to t+12}$, and $K = 12$. When returns are independent and identically distributed ($i.i.d.$), the expected return over K periods is

$$1 + E\left[R_{t\to t+K}\right] = \left(1 + E\left[R_t\right]\right)^K . \tag{11.3.1}$$

For example, in "Workbook Chapter 11.xls", the sheet "Period Returns", the average stock return of Asbury Automotive Group (ABG) in the period May 2002 - March 2022 was 2.09% per month. If we take this as the expected monthly return, the expected annual return on the stock of Asbury is

$$E\left[R_{Annual}\right] = \left(1 + E\left[R_{Monht}\right]\right)^{12} - 1 = 1.021^{12} - 1 = 28.2\%.$$

Simply multiplying the monthly average by 12, a common approximation to calculate annual returns, would yield 25.1%, a difference of more than three percentage points. For a perpetual stream of expected cash flows, such a difference would translate in a valuation difference of about 11%.

In the same Workbook, we also show monthly returns on the CRSP value-weighted stock index, along with the one month interest rate and the one year interest rate for the period January 1962 - November 2022. Using these returns, we find that the average monthly market return is 0.90% and the average one month risk free interest rate is 0.36% (both expressed per month). Thus, the monthly market risk premium can be estimated as

$$MRP_{Month} = \overline{Rm}_{Month} - \overline{Rf}_{Month} = 0.90 - 0.36 = 0.54\%.$$

In order to convert this to an annual estimate of the market risk premium, we annualize the average market return using (9.2.1) and then confront it with the average one year interest rate. The expected annual market return equals

$$E\left[Rm_{Annual}\right] = \left(1 + E\left[Rm_{Monht}\right]\right)^{12} - 1 = 1.090^{12} - 1 = 11.38\%.$$

The average one year interest rate during the sample is 4.91%, implying that based on the data in Workbook Chapter 11.xls, the estimate of the annual market risk premium equals

$$MRP_{Annual} = 11.38 - 4.91 = 6.5\%$$

For the current calculations we will use this estimate as an illustration, but recall that we extensively discussed the estimation of the market risk premium in Chapter 6. The estimates in Chapter 6 were already based on annual data, so no adjustment for the change in return horizon is necessary. Using Equation (11.3.1) gives a higher market risk premium than simply multiplying a monthly number by 12 because of the compounding. At the same time, the use of the one year interest rate instead of a monthly interest rate, lowers the estimated premium as part of the term premium (the difference between long and short term interest rates) is in the one year interest rate versus the one month rate. This makes perfect sense though. When evaluating annual cash flows, assuming they are received once per year, there is a horizon effect that reflects interest rate risk rather than stock market risk.

11.3.3. *Advanced: Betas. The third parameter to be discussed is the beta in the CAPM equation. When discounting annual cash flows we are (implicitly) using one year holding period returns. When estimating betas with monthly (or quarterly) returns, the resulting beta reflects the relation between monthly (quarterly) returns. This may not necessarily coincide with the covariation that is present in annual returns. We denote betas estimated at a return horizon K as

$$\beta\left(K\right)_i = \frac{Cov\left[R_{i,t\to t+K}, Rm_{t\to t+K}\right]}{Var\left[Rm_{t\to t+K}\right]}.$$

We want to relate betas estimated at one horizon, e.g., a monthly $\beta\left(1\right)_i$, to betas at longer (or shorter) horizons, e.g., annual $\beta\left(12\right)_i$. Exact formulas that relate betas derived at different data frequencies in case returns are $i.i.d.$ have been derived previously[6], but are a bit cumbersome to implement. An approximate relation that can easily be extended to returns that are not $i.i.d.$

[6]E.g. Levhari & Levy (1977), Cohen et al. (1983), Handa et al. (1989), and Bessembinder et al. (2022).

and that is very close to the exact formulas is[7]:

$$\beta(K)_i = \beta(1)_i \frac{(1+E[R_{i,t}])^{K-1}}{(1+E[Rm_t])^{K-1}}. \tag{11.3.2}$$

Given estimates of betas and expected returns on short horizon returns, we can use the CAPM itself to get and estimate of $E[R_{i,t}]$ and then calculate the longer horizon beta. To illustrate, let's use the example of the stock of Asbury again, for which we found (using the data in Workbook Chapter 11.xls) with monthly data that $\beta(1) = 1.99$. The one month risk free rate in November 2022 was 0.29%, and we estimate the risk premium in the sample in Workbook Chapter 11.xls as 0.54% per month. If we take these numbers as given, we can estimate the monthly expected return for Asbury and the annual beta:

$$E[R_{ABG,t}] = Rf + \beta(1)_{ABG} MRP_{Month} = 0.29 + 1.99 \times 0.54 = 1.37\%,$$

$$\beta(12)_{ABG} = 1.99 \times \frac{1.0137^{11}}{(1+0.0029+0.0054)^{11}} = 2.11.$$

Note that the expected return estimate of Asbury in the first line is 1.37% per month, not per year. The resulting estimate of the annual beta of Asbury is thus clearly higher than the monthly estimate.

Using similar calculations, Exhibit 11.2, Panel A, shows betas for monthly, annual, and five-year horizon returns. The column $\beta(1)_p$ shows the beta associated with the percentile p in the first column, assuming that betas are from a normal distribution with mean 1.0 and standard deviation 0.5[8] (recall the discussion on this Chapters 5 and 7). Using Equation (11.3.2) together with the CAPM and $Rf = 0.29\%$ (per month) and $MRP = 0.54\%$ (per month), columns three and four show the implied annual and five year betas $\beta(12)_p$ and $\beta(60)_p$ respectively.

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[7] See Karehnke & De Roon (2023)

[8] For instance, for the 75th percentile, $\beta(1)_{75}$ can be found in Excel as $\beta(1)_{75} = norminv(0.75, 1.0, 0.5) = 1.34$.

	Panel A: CAPM, $i.i.d.$			Panel B: Empirical		
p	$\beta(1)_p$	$\beta(12)_p$	$\beta(60)_p$	$\beta(1)_p$	$\beta(12)_p$	$\beta(60)_p$
1%	-0.16	-0.15	-0.11	-0.18	-0.03	0.21
5%	0.18	0.17	0.14	0.17	0.19	0.17
10%	0.36	0.35	0.29	0.38	0.36	0.28
25%	0.66	0.65	0.60	0.69	0.78	0.81
50%	1.00	1.00	1.00	1.03	1.07	1.21
75%	1.34	1.36	1.49	1.36	1.56	2.06
90%	1.64	1.70	2.01	1.71	1.84	3.21
95%	1.82	1.91	2.37	2.02	2.41	5.13
99%	2.16	2.32	3.13	2.59	3.27	6.64

Exhibit 11.2: Distribution of betas for different horizons ($K = 1, 12, 60$). Panel A is for the case where returns are $i.i.d.$ and the CAPM holds. For the monthly horizon ($K = 1$) we assume $Rf = 0.29\%$ and $MRP = 0.54\%$ per month. Panel B provides empirical estimates of the beta distributions.

Looking at the annual betas $\beta(12)$ relative to the monthly ones, the differences are minor for betas below one. For all percentiles below 50%, the difference never exceeds 0.01, as we also found for Asbury For betas above one however, the differences become bigger, especially for the higher percentiles. For instance, for the 90% percentile the exhibit shows $\beta(1)_{90} = 1.64$ whereas $\beta(12)_{90} = 1.70$. Such differences become meaningful and they increase as we go further into the right tail of the distribution of betas. For the five-year betas, $\beta(60)_p$, the differences become even more significant, both for betas below and above one. For instance, $\beta(1)_{25} = 0.66$, whereas $\beta(60)_{25} - 0.60$. The differences are much more pronounced for betas above one. For instance, for the mirror percentile we see $\beta(1)_{75} = 1.34$ (0.34 above the mean instead of below the mean of one), whereas $\beta(60)_{75} = 1.49$. This difference is about three times as big as for the 25 percentile beta.

Although calculations like the ones in Exhibit 11.2 Panel A do give us insight and can easily be applied in valuation settings, they depend on the assumptions that returns are $i.i.d.$ and that the CAPM holds. To relax these assumptions, Panel B of Exhibit 11.2 shows empirical estimates of the beta distributions for different horizons.[9] Based on individual US stock returns for the period January 1970 - March 2022, betas are estimated for different return horizons.[10] The

[9]This is based on based on Karehnke & De Roon (2023).

[10]A difficulty with long horizon returns is that, even over a fifty year sample period like this, there only a few independent long horizons for each stock. Exhibit 11.2 therefore uses the bootstrap

table should be read conditional on the monthly betas. For instance, at the 25th percentile in the empirical beta distribution, the one month beta is 0.69. For companies that have a monthly beta of 0.69, the last two columns show that their annual beta (on average) is 0.78 and their five-year beta 0.81. Thus, in this way Panel B of Exhibit 11.2 can be used to translate monthly betas into annual or five year ones. For intermediate betas we can use linear interpolation to estimate their longer horizon beta. As an example, for Asbury we found the monthly beta to be equal to 1.99, which in Panel B of the exhibit is in between the monthly beta of 1.71 and 2.01. The corresponding five year betas are 3.21 and 5.13 respectively. A linear interpolation would then give as an estimate for the annual beta of Asbury:

$$\beta\,(60)_{ABG} = 3.21 + \frac{1.99 - 1.71}{2.01 - 1.71} \times (5.13 - 3.21) = 3.43.$$

From Panel B of Exhibit 11.2 it is clear that the difference between annual and five year betas versus their monthly estimate is much bigger empirically than what follows from the $i.i.d.$assumption combined with the CAPM in Panel B. For instance, for the 75th percentile the monthly betas are close (1.36 versus 1.34), but the annual ones are already 0.14 apart (1.84 versus 1.70) and the five year beta is about 1.4 times as big (2.06 versus 1.49). We therefore conclude that we should be careful in applying betas estimated at one horizon to discounting at a different horizon, as the difference in the implied cost of capital may be substantial.

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methodology to create longer samples and more observations per stocks. Bootstrapping is like repeat sampling from an existing dataset.

Further Reading

- Campbell, J.Y., & L.M. Viceira, 2002, *Strategic Asset Allocation*, Oxford University Press.