Filter Design

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1 Introduction:

Designing the corresponding FIR and IIR filter realizations for a given filter number is our task. Below are the parameters for this band pass filter.

2 Specifications of the filter:

2.1 The digital filter:

1. **Pass band:**
   Range of frequency pass band is:

$$(4 + 0.6(j))kHz \text{ to } (4 + 0.6(j + 2))kHz$$

$$j = (r - 11000) \mod \sigma \tag{1}$$

   where σ is the sum of digits of roll number and r is roll number.

$$r = 11004 \tag{2}$$

$$\sigma = 6 \tag{3}$$

$$j = 4 \tag{4}$$

   substituting $j = 4$ gives the pass band range for our band pass filter as 6.4 kHz - 7.6 kHz. Hence, the un-normalized discrete time filter pass band frequencies are F_{p1} = 6.4 kHz and F_{p2} = 7.6 kHz.
   The corresponding normalized digital filter pass band frequencies are

$$\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.317\pi \tag{5}$$

$$\omega_{p2} = 2\pi\frac{F_{p2}}{F_s} = 0.292\pi \tag{6}$$

2. **Tolerances:** The pass band (δ_1) and stop band (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

3. **Stop band:**The *transition band* for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband. Hence, the un-normalized *stopband* frequencies are $F_{s1} = 7.6+0.3 = 7.9$ kHz and $F_{s2} = 6.4-0.3 = 6.1$ kHz. The corresponding normalized frequencies are $\omega_{s1} = 0.329\pi$ and $\omega_{s2} = 0.254\pi$.

2.2 The filter design:

In the bi-linear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency(ω) :

$$\Omega = \tan\frac{\omega}{2} \tag{7}$$

Using this relation, we obtain the analog pass band and stop band frequencies as: $\Omega_{p1} = 0.5913$, $\Omega_{p2} = 0.702$ and $\Omega_{s1} = 0.5662$, $\Omega_{s2} = 0.7361$ respectively.

3 The IIR Filter Design:

We are supposed to design filters whose stop band is monotonic and pass band equiripple. Hence, we use the Chebyschev approximation to design our band pass IIR filter.

3.1 The analog filter:

(i) Low Pass Filter Specifications: Let $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{8}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.49167$ and $B = \Omega_{p2} - \Omega_{p1} == 0.0977$.

Substituting Ω_{s1} and Ω_{s2} in (**??**) we obtain the stop band edges of low pass filter

$$\Omega_{Ls1} = \frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}} = 1.4689 \tag{9}$$

$$\Omega_{Ls2} = \frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}} = -1.5459 \tag{10}$$

And we choose the minimum of these two stop band edges

$$\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.4689. \tag{11}$$

(ii) The Low Pass Chebyschev Filter Parameters: The magnitude of frequency response of the low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{12}$$

Here c_N denote the chebyshev polynomials for a particular order N of the filter.

$$c_N(x) = \cosh(N \cosh^{-1} x), x = \Omega_L \tag{13}$$

$$c_0(x) = 1 \tag{14}$$

$$c_1(x) = x \tag{15}$$

There exists a relation from which all the polynomials can be found out is given as:

$$c_{N+2} = 2xc_{N+1} - c_N \tag{16}$$

Imposing the band restrictions on (**??**)

$$|H_{a,LP}(j\Omega_L)|^2 < \delta_2 \text{ for } \Omega_L = \Omega_{Ls} \tag{17}$$

$$1 - \delta_1 < |H_{a,LP}(j\Omega_L)|^2 < 1 \text{ for } \Omega_L = \Omega_{Lp} \tag{18}$$

we obtain :

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \leq \epsilon \leq \sqrt{D_1},$$
$$N \geq \left\lceil \frac{\cosh^{-1} \sqrt{D_2/D_1}}{\cosh^{-1} \Omega_{Ls}} \right\rceil, \tag{19}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$ and $\lceil . \rceil$ is known as the ceiling operator .

Parameter	Value
D_1	0.384
D_2	43.44
N	4
$c_4(x)$	$8x^4 + 8x^2 + 1$

Table 1: Parameter Table

we get $N \geq 4$ and $0.278 \leq \epsilon \leq 0.61$
The below c-code generates the text file and python code plots (**??**) for different values of ϵ .
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_design/codes/plot1.c$
python code:
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_design/codes/plot1.py$

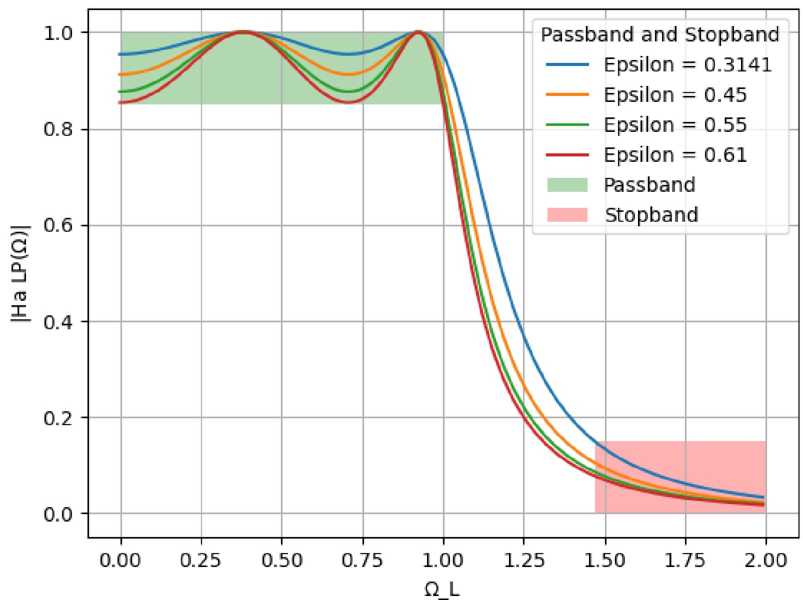

Figure 1: The Analog Low-Pass Frequency Response for $0.3141 \leq \epsilon \leq 0.61$

In Fig. **??** we can observe the equiripple behaviour in passband and monotonic behaviour in stopband. As the value of ϵ increases the value of $|H_{a,LP}(j\Omega_L)|$ decreases.

(iii) The Low Pass Chebyschev Filter: The next step in design is to find an expression for magnitude response in s domain.
Using $s = j\Omega$ or in this case $s_L = j\Omega_L$ we obtain:

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\frac{s_L}{j})} \tag{20}$$

To find poles equate the denominator to zero:

$$1 + \epsilon^2 c_N^2 \left(\frac{s_L}{j}\right) = 0 \quad \text{where} \qquad c_N(x) = cos\left(Ncos^{-1}(x)\right) \tag{21}$$

On solving (**??**) we obtain poles :

$$s_k = -\Omega_{Lp} \sin(A_k) \sinh(B_k) - j\Omega_{Lp} \cos(A_k) \cosh(B_k) \tag{22}$$

where k is the index of the pole and

$$A_k = (2k+1)\frac{\pi}{2N} \tag{23}$$

$$B_k = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{24}$$

The below code computes the values of s_k and stores it in a text file.
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_design/codes/plot2.c$
python code for plotting

https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_design/codes/plot2.py$
The poles obtained are formulated in the table below:

$Pole$	$Value$
$s[1]$	$0.1621 - j1.0033$
$s[2]$	$0.3913 - j0.4156$
$s[3]$	$0.3913 + j0.4156$
$s[4]$	$0.1621 + j1.0033$
$s[5]$	$-0.1621 - j1.0033$
$s[6]$	$-0.3913 - j0.4156$
$s[7]$	$-0.3913 + j0.4156$
$s[8]$	$-0.1621 + j1.0033$

Table 2: Values of s_k

The below code plots the pole-zero plot.

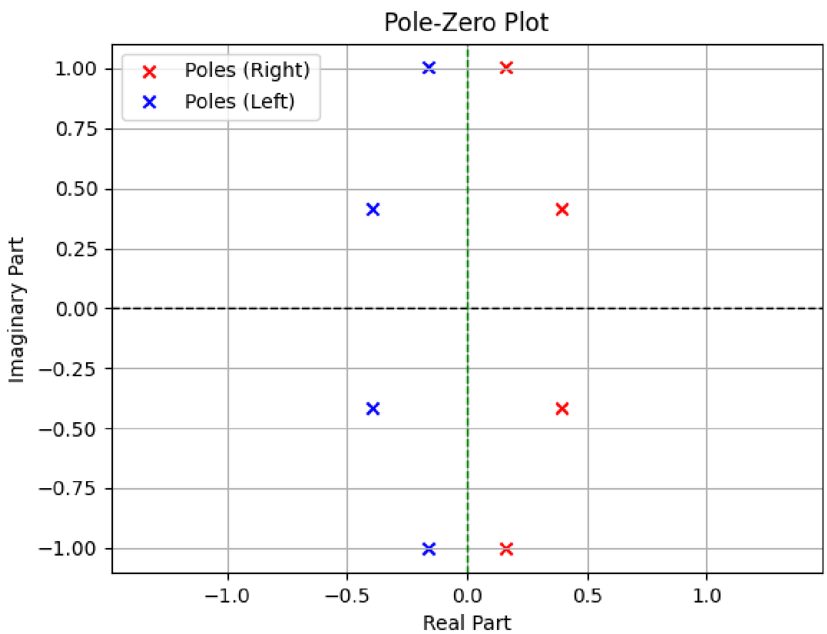

Figure 2: The Pole zero plot and all the poles lie on an ellipse. The left and right poles have been identified as shown.

The poles in the left half of the plane are considered in the design as we intend to design a stable system.
Therefore the magnitude response is written as :-

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{25}$$

where G_{LP} is the gain of the Low pass filter. Refer to Table **??** for s_k values.
We know that from (**??**):-

$$\left|H_{a,LP}(s_L)\right| = \frac{1}{\sqrt{1+\epsilon^2}} \text{at } \Omega_L = 1 \implies s_L = j \tag{26}$$

Substituting respective values in (**??**) we get $G_{LP} = 0.3125$

$$H_{a,LP}(s_L) = \frac{0.3125}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{27}$$

$$= \frac{0.3125}{s^4 + 1.1068s^3 + 1.61245s^2 + 0.91397s + 0.33656} \tag{28}$$

code for plotting for Band-pass filter
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_{d}esign/codes/plot3.py$

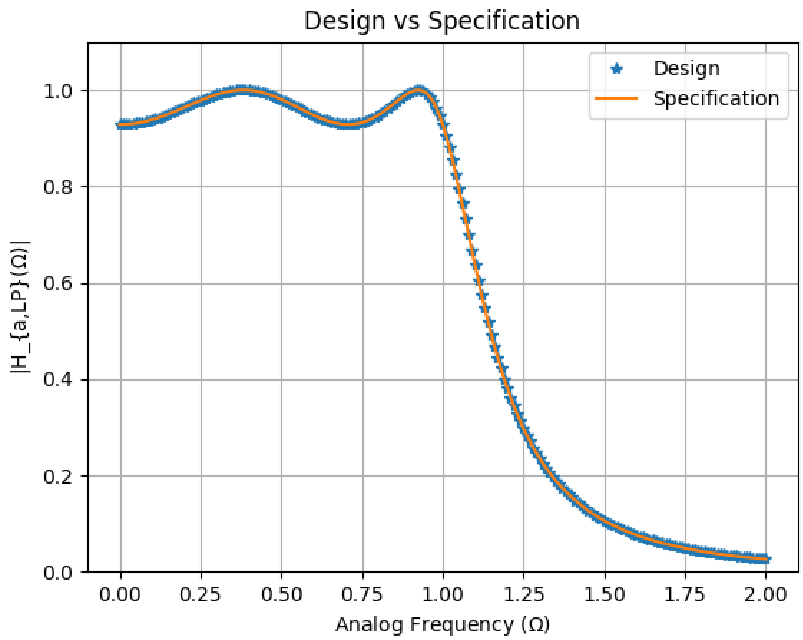

Figure 3: Design vs Specification corresponding to (**??**) and (**??**)

(iv) The Band Pass Chebyschev Filter: After verifying design with the required specifications the next step in design is to jump to required type of filter using frequency transformation.

$$s_L = \frac{s^2 + \Omega_0^2}{Bs} \tag{29}$$

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{30}$$

As there is one to one correspondence between the filters so $\Omega = \Omega_{p1}$ should correspond to Ω_{Lp}

$$s = j\Omega_{p1} \tag{31}$$

$$s_L = \frac{(j\Omega_{p1})^2 + \Omega_0^2}{B(j\Omega_{p1})} \tag{32}$$

$$\left|H_{a,BP}(j\Omega_{p1})\right| = 1 \tag{33}$$

$$G_{BP}\left|H_{a,LP}(s_L)\right| = 1 \tag{34}$$

Substituting (**??**) in (**??**) we obtain Gain of required bass pass filter:

$$G_{BP} = 1.0771 \tag{35}$$

Thus the response in s domain

$$H_{a,BP}(s) = \frac{9.8138 \times 10^{-5} s^4}{s^8 + 0.108s^7 + 0.982s^6 + 0.079s^5 + 0.358s^4 + 0.0192s^3 + 0.0574s^2 + 0.0015s + 0.0034} \tag{36}$$

The expressions in the s-domain and gain factors are computed by writing a Python code.
In Figure 3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positive as well as negative frequencies. We find that the pass band and stop band frequencies in the figure match well with those obtained analytically through the bi-linear transformation. Code:
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_{d}esign/codes/plot4.py$

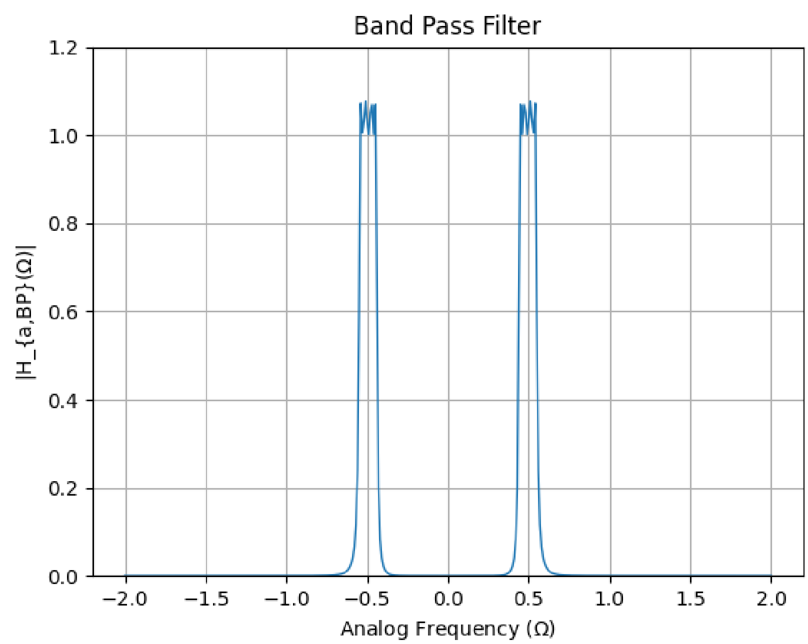

Figure 4: The Analog Band pass Magnitude Response from (**??**).The filter design specifications are satisfied

3.2 The Digital Filter:

From the bi-linear transformation, we obtain the digital band pass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{37}$$

Substituting $s = \frac{1-z^{-1}}{1+z^{-1}}$ in (**??**) and calculating expression using a python code we get :

$$H_{d,BP}(z) = \frac{G\left(z^{-8} - 4z^{-6} + 6z^{-4} - 4z^{-2} + 1.0\right)}{2.61 - 12.43z^{-1} + 32.16z^{-2} - 53.24z^{-3} + 61.99z^{-4} - 50.98z^{-5} + 29.48z^{-6} - 10.91z^{-7} + 2.19z^{-8}} \tag{38}$$

where $G = 9.8138 \times 10^{-5}$ Code for the plotting Digital Band pass filter
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_design/codes/plot5.py$

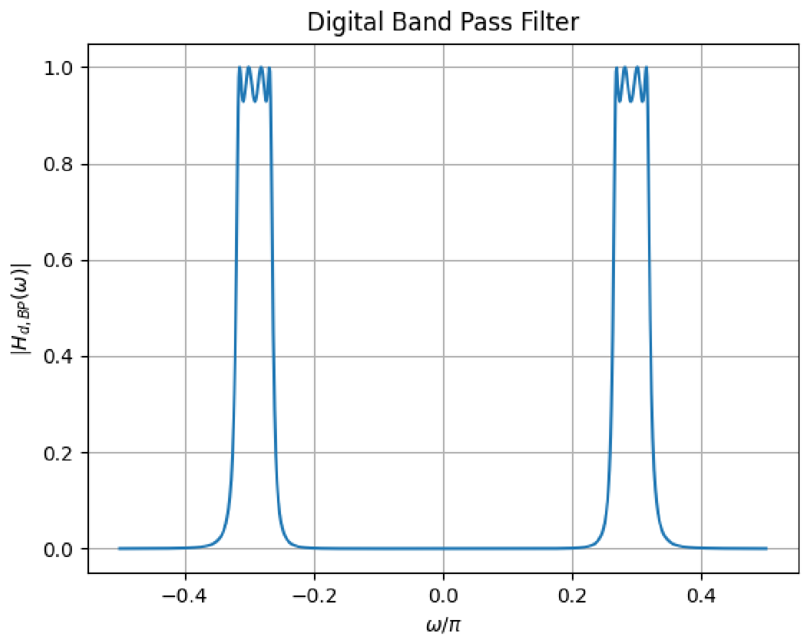

Figure 5: Digital Specifications are met. Pass band and stop band frequencies are same

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) low pass equivalent using the Kaiser window and then converting it to a causal band pass filter.

4.1 The Equivalent Low pass Filter

The low pass filter has a pass band frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stop band tolerance is $\delta = 0.15$.The cutoff-frequency is given by :

$$\omega_l = \frac{\omega_{p1} - \omega_{p2}}{2} \tag{39}$$

$$= 0.025\pi \tag{40}$$

Code:

https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_design/codes/plot6.py$

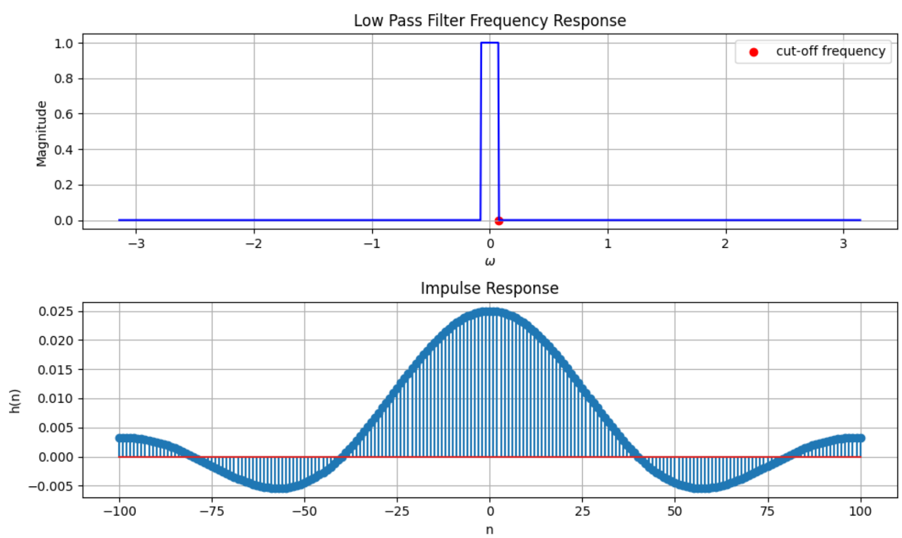

Figure 6: Frequency response and impulse response of an ideal Low Pass Filter

The impulse response of ideal Low Pass Filter is given by :

$$h(n) = \begin{cases} \frac{w_l}{\pi}, & \text{if } n = 0 \\ \frac{\sin(w_l n)}{n\pi}, & \text{if } n \neq 0 \end{cases} \tag{41}$$

From (**??**) we conclude that $h(n)$ for an ideal Low Pass Filter is not causal and can neither be made causal by introducing a finite delay. And $h(n)$ do not converge and hence the system is unstable.

4.2 Kaiser Window

Therefore we move on windowing the impulse response.A window function is chosen and multiplied. The Kaiser window is defined as

$$w(n) = \begin{cases} \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, & -N \le n \le N, \quad \beta > 0 \\ 0 & \text{otherwise,} \end{cases}$$

(i) N is chosen according to

$$N \ge \frac{A-8}{4.57\Delta\omega}, \tag{42}$$

where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \ge 48$.

(ii) β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases} \tag{43}$$

The window function is defined as :

$$w(n) = \begin{cases} 1, & \text{for } -48 \le n \le 48 \\ 0, & \text{otherwise} \end{cases} \tag{44}$$

Therefore the desired impulse response is :

$$h_{lp} = h_n w_n \tag{45}$$

$$h(n) = \begin{cases} \frac{\sin(w_l n)}{n\pi}, & \text{for } -48 \le n \le 48 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

Code:
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_design/codes/plot7.py$

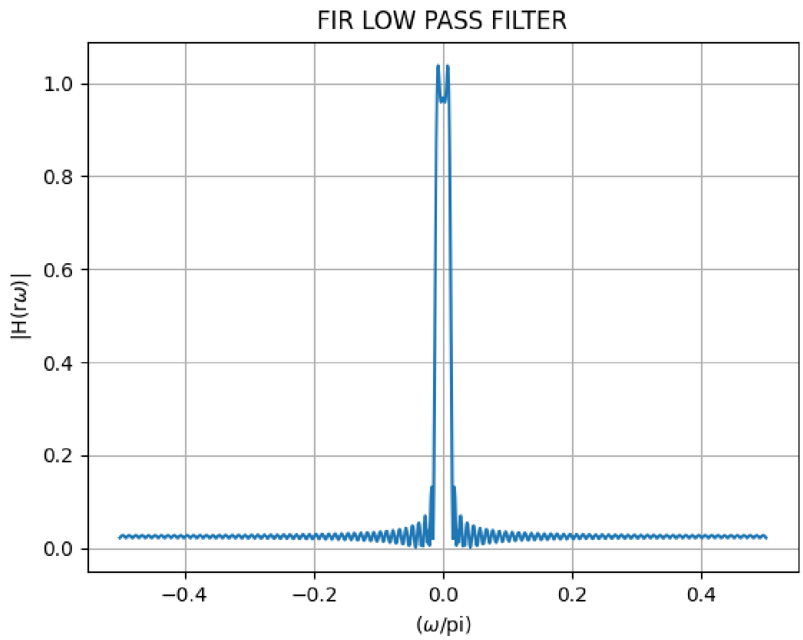

Figure 7: Magnitude Response of Low Pass Filter after using Kaiser Window

4.3 The Equivalent Band Pass Filter

A Band-Pass Filter (BPF) can be obtained by subtracting the magnitude response of a Low-Pass Filter (LPF) with cutoff frequency ω_{p1} from another LPF magnitude response with cutoff frequency ω_{p2}.

$$h_{BP}(n) = \begin{cases} \frac{\sin(w_{p2}n)}{n\pi} - \frac{\sin(\omega_{p1}n)}{n\pi}, & \text{for } n \neq 0 \\ \frac{\omega_{p2}-\omega_{p1}}{\pi} & \text{for } n = 0 \end{cases} \tag{47}$$

$$\frac{\sin(\omega_{p2}n)}{n\pi} - \frac{\sin\left(\omega_{p1}n\right)}{n\pi} = 2\cos\left(\frac{\omega_{p2}n + \omega_{p1}n}{2}\right)\sin\left(\frac{\omega_{p2}n - \omega_{p1}n}{2}\right) \tag{48}$$

$$= \frac{2\cos(0.292n\pi)\sin(0.025n\pi)}{n\pi} \tag{49}$$

Multipying by window function we get :

$$h_{BP}(n) = \begin{cases} \frac{2\cos(0.292n\pi)\sin(0.025n\pi)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

Code:
https://github.com/Lohith12321/signals-and-systems/blob/main/filter$_{d}esign/codes/plot8.py$

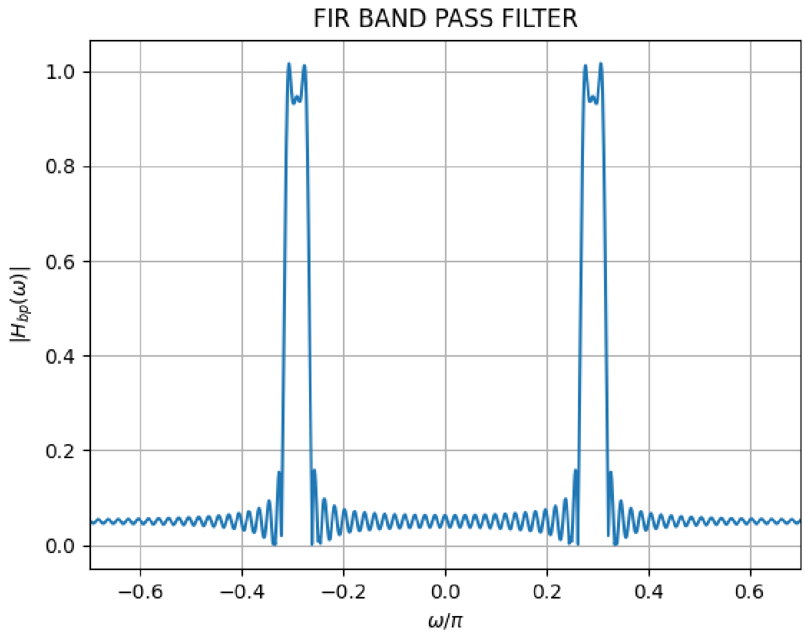

Figure 8: Magnitude Response of Band Pass Filter after using Kaiser Window