NCERT Discrete-11.9.4-5

EE22BTECH11004 - Allu Lohith

Find the sum of n terms of this sequence:

$$5^2 + 6^2 + 7^2 ... + 20^2$$

Solution: The standard z transforms,

Parameter	Description	Formulae/Value
n	Iteration number starting from zero till 15	-
$x(n)$	General term of the sequence from $n = 0$ to $n = 15$	$(n + 5)^2\ u(n)$
$x(0)$	First term of the sequence	5

TABLE I
PARAMETERS

$$u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, |z| > 1 \qquad (1)$$

$$nu(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, |z| > 1 \qquad (2)$$

$$n^2u(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}, |z| > 1 \qquad (3)$$

As

$$x(n) = (n^2 + 10n + 25)u(n) \qquad (4)$$

The z transform of general term can be written as ,

$$X(z) = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} + 10\frac{z^{-1}}{(1 - z^{-1})^2} + \frac{25}{1 - z^{-1}} \qquad (5)$$

$$X(z) = \frac{16z^{-2} - 39z^{-1} + 25}{(1 - z^{-1})^3}; |z| > 1 \qquad (6)$$

On convolution for finding the sum

$$y(n) = x(n) * u(n) \qquad (7)$$

On z transform,

$$Y(z) = X(z) \cdot U(z) \qquad (8)$$

$$= \left(\frac{16z^{-2} - 39z^{-1} + 25}{(1 - z^{-1}))^3}\right) \cdot \frac{1}{1 - z^{-1}} \qquad (9)$$

$$\implies Y(z) = \frac{16z^{-2} - 39z^{-1} + 25}{(1 - z^{-1})^4}; \quad |z| > 1 \qquad (10)$$

Using the contour integration to find the inverse z transform,

$$y(n) = \oint_c Y(z) \cdot z^{n-1} dz \qquad (11)$$

$$y(21) = \oint_c \left(\frac{16z^{-2} - 39z^{-1} + 25}{(1 - z^{-1})^4}\right) z^{14} dz \qquad (12)$$

As there are four poles from observation, so $m = 4$

$$y(21) = \frac{1}{(m - 1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} ((z - a)^m f(z)) \qquad (13)$$

$$= \frac{1}{3!} \lim_{z \to 1} \frac{d^3}{dz^3} \left((z - 1)^4 \frac{(16z^{-2} - 39z^{-1} + 25)}{(1 - z^{-1})^4} z^{14}\right) \qquad (14)$$

$$= \frac{1}{6} \lim_{z \to 1} \frac{d^3}{dz^3} \left((16z^{-2} - 39z^{-1} + 25) z^{18}\right) \qquad (15)$$

$$= \frac{1}{6} \lim_{z \to 1} \frac{d^3}{dz^3} \left(16z^{16} - 39z^{17} + 25z^{18}\right) \qquad (16)$$

$$= \frac{1}{6} (16 \times 18 \times 17 \times 16 + 14 \times 17 \times 16 \times 15) \qquad (17)$$

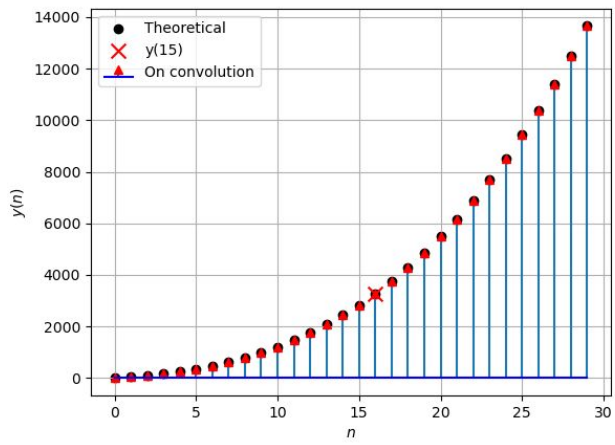$$\implies y(21) = 2840 \qquad (18)$$

Hence the sum of the terms of the sequence is 2840.

Fig. 1. Simulation v/s theoretical