Towards High-Order Differentiation using Semi-Implicit Euler Discretization

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Introduction

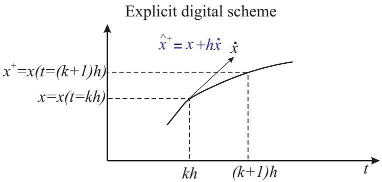
General overview

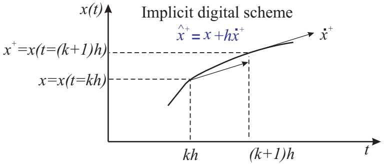
- Introduced by the work of Brogliato *et al.*[1], the *implicit discretization method* is well adapted to sliding-mode controllers, high-order differentiation and more generally to differential inclusion $\dot{x} \in F(x)$
- It aims to replace the sign function by an *implicit projector* with very promising results[2] including
  - reduction of the chattering effect
  - robustness of the control under lower sampling frequencies
  - preservation of the global stability

1. V. Acary, B. Brogliato "Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems" Systems and Control Letters, Elsevier, 2010, 59 (5), pp.284-295.
2. M.R. Mojallizadeh et al. "A survey on the discrete-time differentiators in closed-loop control systems : experiments on an electro-pneumatic system" Control Engineering Practice, 2023, 136 (105546), pp.1-23.

Introduction

Explicit vs implicit Euler scheme : toward a projector-based solution





✓ To estimate x^+, we need to know $\dot{x}^+$

- Linear case : example $\dot{x} = -a\,x$, $a > 0$
  1. Explicit : $x^+ = (1 - h\,a)x \implies$ stable if $h < \frac{2}{a}$ (small h)
  2. Implicit : $x^+ = \dfrac{x}{1 + h\,a} \implies$ stable $\forall$ h if $O(h^2) \simeq 0$

- Sliding mode case :
  1. Explicit : Chattering
  2. Implicit : No Chattering (Projector-based solution)

Toward implicit & projector-based differentiation

Extension of the sign properties from the sliding mode control : the implicit scheme

✓ The exact discretized system, under a sampling-time h, is controlled by the *implicit projector* $\mathcal{N}_{\lambda,h}$ that gives

$$\begin{cases} x_{k+1} = x_k + h\,u_{k+1} \\ u_{k+1} = -\lambda\,\mathrm{sgn}(x_{k+1}) \end{cases} \tag{1}$$

where $\mathrm{sgn}(x_{k+1})$ is evaluated thanks to the operator $\mathcal{N}_{\lambda,\,h}$ with $\lambda > 0$ that is defined as

$$\begin{cases} |x_k| < \lambda h \rightarrow \mathcal{N}_{\lambda,\,h} = \dfrac{x_k}{\lambda h} \qquad (i.e. \quad x_{k+1} = 0) \\ |x_k| \geq \lambda h \rightarrow \mathcal{N}_{\lambda,\,h} = \mathrm{sgn}(x_k) \qquad (i.e. \quad x_{k+1} \neq 0) \end{cases} \tag{2}$$

Towards implicit & projector-based differentiation

✓ Given the state variable x_k, the backward Euler semi-implicit scheme reads

$$\begin{cases} x_{k+1} = x_k + h\, u_{k+1} \\ u_{k+1} = -|x_k|^{\alpha} \underbrace{\mathcal{N}_{\lambda,\,h}}_{\mathrm{sgn}(x_{k+1})} \end{cases}$$

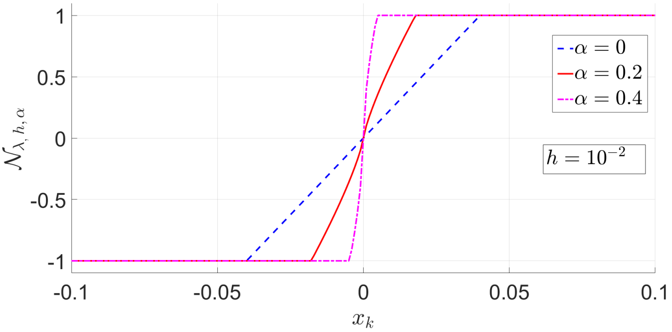
- Depending on x_k, u_{k+1} belongs either to the saturation mode[3] defined by $u_{k+1} = -\lambda\,\mathrm{sgn}(x_k)$, or u_{k+1} belongs to the linear mode and corresponds to a $1/\lambda$-contraction of $\frac{x_k}{\lambda h}$
- The term $|x_k|^{\alpha}$ is the explicit part that overcomes the chattering phenomena w.r.t. the homogeneous exponent α

The sign function is generalized by $\mathcal{N}_{\lambda,\,h}$ whose purpose is to anticipate the next x_{k+1} step

3. The $\mathrm{sgn}(x)$ function verifies : if $x > 0$, then $+1$; if $x < 0$ then -1 ; if $x = 0$ then $]-1, 1[$.

Towards implicit & projector-based differentiation

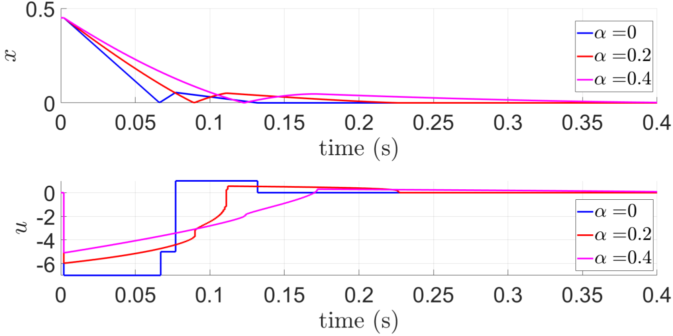
Extension of the sign properties from the sliding mode control : basic semi-implicit control / representation of the projector



Examples of representation of $\mathcal{N}_{\lambda,\boldsymbol{h},\alpha}$ versus α

Towards implicit & projector-based differentiation

Extension of the sign properties from the sliding mode control : basic semi-implicit control / results[4]



State variable x (top) and control input u (bottom) versus time (s), for different values of α

4. L. Michel et al. "Semi-Implicit Euler Discretization for Homogeneous Observer-based Control : one dimensional case" IFAC-PapersOnLine, Volume 53, Issue 2, 2020, Pages 5135-5140.

First & Sec. order semi-implicit differentiation

✓ Considering the error $\epsilon_1 = x - z_1$ and $\alpha \in [0.5\ , 1[$, the first order differentiation (SIHD1) reads

$$\begin{cases} z_1^+ = z_1 + h\left(z_2^+ + \lambda_1 \mu |\epsilon_1|^{\alpha} \mathcal{N}_1(\epsilon_1)\right) \\ \\ z_2^+ = z_2 + E_1^+ h \lambda_2 \mu^2 |\epsilon_1|^{2\alpha - 1} \mathcal{N}_2(\epsilon_1) \end{cases} \tag{3}$$

where the projector $\mathcal{N}_q$, $q = 1...2$ and the activation of the flag E_1^+ are such as

$$\mathcal{N}_q(\epsilon_1) := \begin{cases} \epsilon_1 \in SD \ \ \rightarrow \mathcal{N}_q = \dfrac{\lceil \epsilon_1 \rfloor^{q(1-\alpha)}}{\lambda_q (\mu h)^q}, \ \ E_q^+ = 1 \\ \\ \epsilon_1 \notin SD \ \ \rightarrow \mathcal{N}_q = \operatorname{sign}(\epsilon_1), \ \ E_q^+ = 0 \end{cases} \tag{4}$$

with the domain of attraction $SD = \{\epsilon_1 \,/\, |\epsilon_1| \leq (\lambda_1 \mu h)^{\frac{1}{q(1-\alpha)}}\}$

First & Sec. order semi-implicit differentiation / Application to robotics

SIHD2-based (real-time) estimation in open-loop

Cable Parallel Robot prototype located at LS2N, Nantes, France

https://www.youtube.com/watch?v=I-IOcAGha3o

Third order semi-implicit differentiation

✓ Extension to third order estimation (position, velocity, acc. and jerk) incl. correction terms that refine the Taylor expansion [5]

$$
\begin{cases}
z_4^+ = z_4 + E_1^+ E_2^+ E_3^+ h\left(\lambda_4 \mu^4 \left|e_1\right|^{4\,\alpha-3} \mathcal{N}_4\right) \\
z_3^+ = z_3 + E_1^+ E_2^+ h\left(z_4^+ + \lambda_3 \mu^3 \left|e_1\right|^{3\,\alpha-2} \mathcal{N}_3\right) \\
z_2^+ = z_2 + E_1^+ h\left(z_3^+ \boldsymbol{- E_3^+ h \frac{1}{2} z_4^+} + \lambda_2 \mu^2 \left|e_1\right|^{2\,\alpha-1} \mathcal{N}_2\right) \\
z_1^+ = z_1 + h\left(z_2^+ \boldsymbol{- E_2^+ \frac{h}{2} z_3^+ + E_3^+ E_2^+ \frac{h^2}{3!} z_4^+} + \lambda_1 \mu |e_1|^{\alpha} \mathcal{N}_1\right)
\end{cases}
\tag{5}
$$

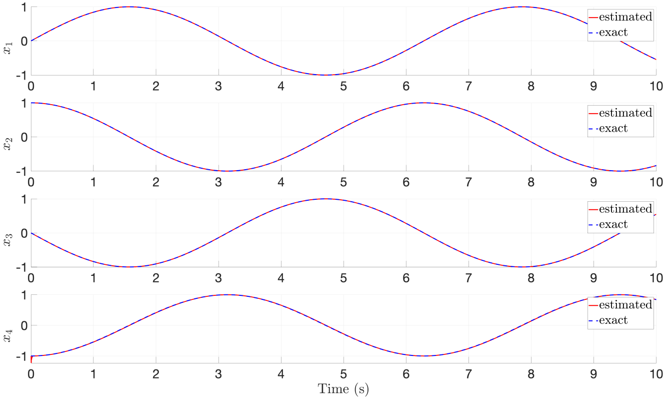
https://github.com/LoicMichelControl/SemiImplicitDifferentiation

~~Analysis of convergence of~~ the sec. order differentiator [6]

5. L. Michel and J.P. Barbot "A note about high-order semi-implicit differentiation : application to a numerical integration scheme with Taylor-based compensated error" arXiv :2408.00497, Aug. 2024.

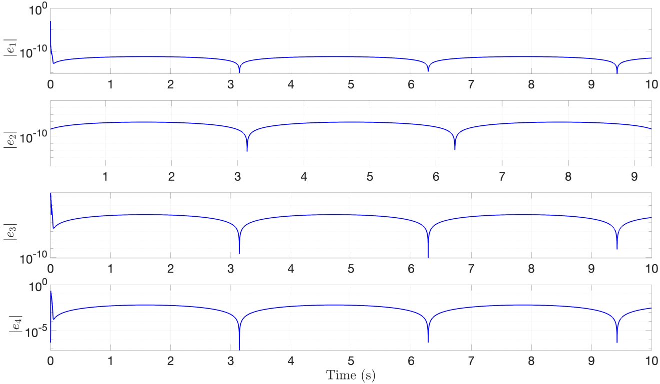
6. L. Michel et al. "Semi-Implicit Homogeneous Euler Differentiator for a Second-Order System : Validation on Real Data" 60th IEEE Conference on Decision and Control (CDC), Austin, TX, USA, 2021, pp. 5911-5917.

Third order semi-implicit differentiation



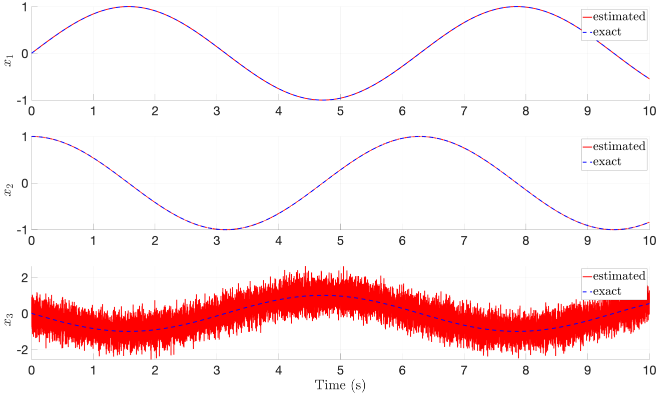
Example of SIHD3-based differentiation of a sine function without extra noise

Third order semi-implicit differentiation



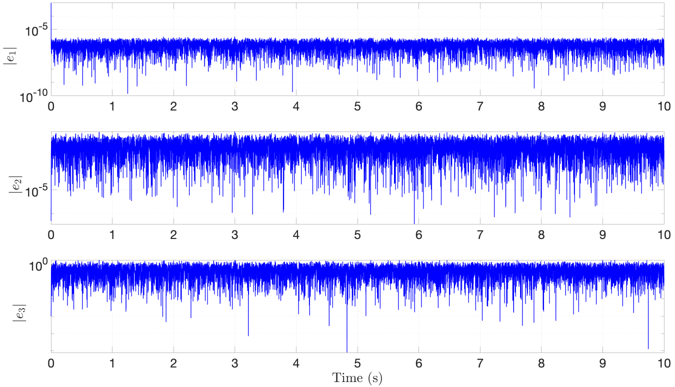
Example of SIHD3-based differentiation of a sine function without extra noise - estimation errors

Third order semi-implicit differentiation



Example of SIHD3-based differentiation of a sine function including extra noise (of order 10^{-6})

Third order semi-implicit differentiation

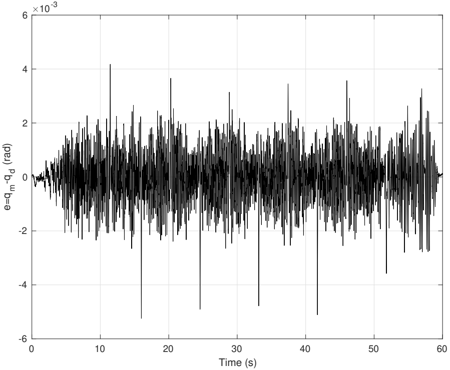


Example of SIHD3-based differentiation of a sine function including extra noise (of order 10^{-6}) - estimation errors

Application to closed-loop robotics

PID-based Γ-torque control (tracking of *desired* variables)

$$\Gamma = \Im\left(\dot{x}_2^d - K_v(z_2 - x_2^d) - K_p(x_1 - x_1^d) - K_i \int_0^t (x_1 - x_1^d)\mathrm{d}\tau\right)$$



Example of evolution of the tracking error for an orbital movement