Semi-Implicit Euler Discretization of Homogeneous Differentiators

L. Michel[1], M. Ghanes[1], F. Plestan[1], Y. Aoustin[2] and JP Barbot[13]

[1] Ecole Centrale de Nantes-LS2N, UMR 6004 CNRS
[2] Université de Nantes-LS2N, UMR 6004 CNRS
[3] ENSEA Quartz Laboratory EA 7393

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Outline

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     - Implicit Euler Scheme
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$$
{}^{1}\begin{cases} \dot{z}_1 = z_2 + \lambda_1 \mu \lceil e_1 \rfloor^{\alpha} \\ \dot{z}_2 = \lambda_2 \mu^2 \lceil e_1 \rfloor^{2\alpha - 1} \end{cases} \tag{1}
$$

- $y = x_1$ is the measure to be differentiated, $\dot{y} = x_2$ and so on...
- $e_1 = x_1 - z_1$, $\lceil \bullet \rfloor^{\alpha} = |\bullet|^{\alpha}\mathrm{sign}(\bullet)$ with $\alpha = [0.5,\ 1]$
- $e_2 = x_2 - z_2$, $e = (e_1, e_2)^T$
- $\theta^{-m}\Lambda_r^{-1} f(\Lambda_r e) = f(e) = \dot{e} \implies m = \frac{\alpha - 1}{\alpha}$ (homogeneity[2]), with $\Lambda_r = diag(\theta^{r_1}, \theta^{r_2})$
- $\lambda_i > 0$, $i = 1, 2 \rightarrow$ eigenvalues sufficiently stable
- $\mu \rightarrow$ sufficiently large to cancel the effect of the perturbation

Levant, Perruquetti, Rosier, Hermes

[1] xx

[2] xx

... could be solved using e.g. an analogue computer

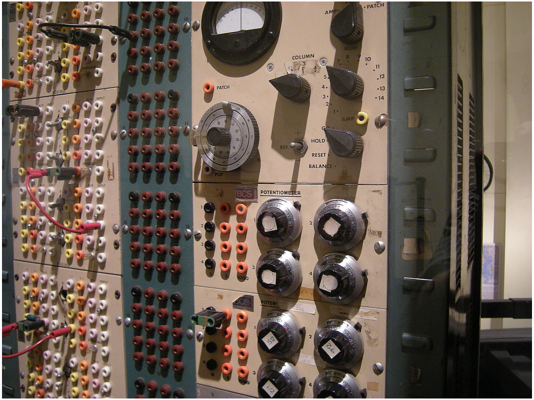
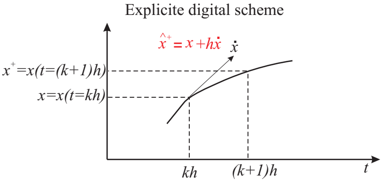
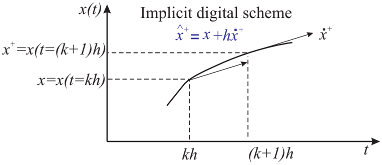


Figure: Analogue computer (Wikimedia Commons)

✓ Chattering: analogue rise/fall times and complex implementation!

HOW TO DISCRETIZE ? (Explicit vs Implicit Euler Schemes)





✓ To estimate x^+, we need to know $\dot{x}^+$

- Linear case : example $\dot{x} = -a\,x$, $a > 0$
  1. Explicit : $x^+ = (1 - h\,a)x \implies$ stable if $h < \frac{2}{a}$ (small h)
  2. Implicit : $x^+ = \dfrac{x}{1 + h\,a} \implies$ stable $\forall\ h$ if $O(h^2) \simeq 0$
- Sliding mode case :
  1. Explicit : Chattering
  2. Implicit : No Chattering (Projector-solution[3])

[3] V. Acary, B. Brogliato. Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems. Systems and Control Letters, Elsevier, 2010, 59 (5), pp.284-295.

- Explicit Euler discretization of homogeneous differentiator

$$\begin{cases} z_1^+ = z_1 + h\left(z_2 + \lambda_1 \mu \lceil e_1 \rfloor^{\alpha}\right) \\ z_2^+ = z_2 + h\left(\lambda_2 \mu^2 \lceil e_1 \rfloor^{2\alpha - 1}\right) \end{cases} \tag{2}$$

where $e_1 := x_1 - z_1$

✓ Runs well for h small but chattering and accuracy problems

- Explicit discrete homogeneous differentiator: correction terms[4]

$$\begin{cases} z_1^+ = z_1 + hz_2 + \frac{h^2}{2!} z_3 + h\lambda_1 \mu \lceil e_1 \rfloor^{2/3} \\ z_2^+ = z_2 + hz_3 + h\lambda_2 \mu^2 \lceil e_1 \rfloor^{1/3} \\ z_3^+ = z_3 + h\lambda_3 \mu^3 \operatorname{sign}(e_1) \end{cases} \tag{3}$$

where $e_1 := x_1 - z_1$

✓ Better accuracy !

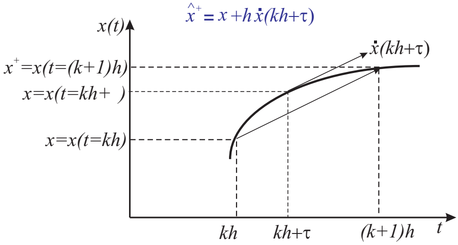
[4] A. Levant, M. Livne, X. Yu, Sliding-Mode-Based Differentiation and Its Application, IFAC-PapersOnLine, Volume 50, Issue 1, 2017

- Implicit Euler discretization of homogeneous differentiator

$$\begin{cases} z_1^+ = z_1 + h\left(z_2^+ + \lambda_1 \mu \lceil e_1^+ \rfloor^{\alpha}\right) \\ z_2^+ = z_2 + h\left(\lambda_2 \mu^2 \lceil e_1^+ \rfloor^{2\alpha - 1}\right) \end{cases} \tag{4}$$

where $e_1 := x_1 - z_1$

✓ No solutions for $e_1^+ = 0$ if $e_1 \neq 0$



✓ Solutions : Semi-Implicit Euler discretization

- Semi-Implicit scheme based continuity requirement (i.e. $\mathrm{sign}(e_1^+) = \mathrm{sign}(e_1)$, w.r.t (4))[5]

- Semi-Implicit scheme based quasi-linearization (i.e. $\lceil e_1 \rfloor^{\alpha} = |e_1|^{\alpha-1} e_1^+$, w.r.t (2))[6]

- Semi-Implicit scheme based projectors [7]

✓ Comparisons can be found between some methods[8]. Hereafter, we present the projector-based methods

[5] A. Polyakov, D. Efimov, and W. Perruquetti, "Homogeneous differentiator design using implicit Lyapunov function method," in Proc. IEEE Eur. Control Conf., 2014, pp. 293-288.

[6] M.Wetzlinger, M. Reichhartinger, M. Horn, L. Fridman, and J. A. Moreno, "Semi-implicit discretization of the uniform robust exact differentiator" in Proc. IEEE 58th Conf. Decis. Control, 2019, pp. 5995-6000.

[7] L. Michel, M. Ghanes, F. Plestan, Y. Aoustin and J. -P. Barbot, "Semi-Implicit Homogeneous Euler Differentiator for a Second-Order System: Validation on Real Data," 2021 60th IEEE Conference on Decision and Control (CDC), 2021, pp. 5911-591.

[8] M. R. Mojallizadeh, B. Brogliato, A. Polyakov, S. Selvarajan, L. Michel, F. Plestan, M. Ghanes, J-P. Barbot, Y. Aoustin. Discrete-time differentiators in closed-loop control systems: experiments on electropneumatic system and rotary inverted pendulum. [Research Report] INRIA Grenoble. 2022. hal- 03125960v2

From (4), $\lceil e_1^+ \rfloor$ is replaced by $|e_1|\mathrm{sign}(e_1^+)$

$$\begin{cases} z_1^+ = z_1 + h\left(z_2^+ + \lambda_1\mu|e_1|^\alpha \mathrm{sign}(e_1^+)\right) \\ z_2^+ = z_2 + E_1^+ h\left(\lambda_2\mu^2|e_1|^{2\,\alpha-1}\mathrm{sign}(e_1^+)\right) \end{cases} \tag{5}$$

$$\begin{cases} e_1^+ = e_1 + h\left(e_2^+ - \lambda_1\mu|e_1|^\alpha \mathcal{N}_1\right) \\ e_2^+ = e_2 + h\,\ddot{y} - E_1^+ h\left(\lambda_2\mu^2|e_1|^{2\,\alpha-1}\mathcal{N}_1\right) \end{cases} \tag{6}$$

where $\mathrm{sign}(e_1^+) = \mathcal{N}_1$ is defined as

$$\mathcal{N}_1, E_1^+ := \begin{cases} e_1 \in SD \quad \to \mathcal{N}_1 = \frac{\lceil e_1 \rfloor^{1-\alpha}}{\lambda_1\mu h}, \quad E_1^+ = 1 \\ e_1 \notin SD \quad \to \mathcal{N}_1 = \mathrm{sign}(e_1), \quad E_1^+ = 0 \end{cases} \tag{7}$$

$$SD = \{e_1 \,/\, |e_1| \leq (\lambda_1\mu h)^{\frac{1}{1-\alpha}}\}$$

Assumption 1: $\exists\ \ddot{y}_M > 0$ such that $|\ddot{y}(t)| < \ddot{y}_M\ \forall\ t > 0$.

Assumption 2 :

1. $\exists\ \dot{y}_M > 0$ such that $\forall\ t > 0$, $|\dot{y}(t)| < \dot{y}_M$;
2. the acceleration $\ddot{y}(t)|$ is slowly varying, that implies that for sufficient small $h > 0$, $\ddot{y}^+ \simeq \ddot{y}$.

Remark: For $e_1 \in SD$, then $e_1^+ = h\, e_2^+$. The second raw of (6), after one sampling, becomes:

$$e_2^+ = e_2 + h\,\ddot{y} - \frac{\lambda_1}{\lambda_2}\mu\, h^\alpha \lceil e_2 \rfloor^\alpha \tag{8}$$

To stay on SD, the bound of e_1 has to satisfy this condition:

$$\frac{1}{2}(\lambda_1 \mu)^{\frac{1}{1-\alpha}} h^{\frac{\alpha}{1-\alpha}} > \dot{y}_M$$

.

- Chattering analysis of (8)

Let be the change of coordinate $e_2 = e_c + e_{2eq}$, then we can characterize the limit cycle:

$$e_c^+ = e_c + h(a - b\lceil e_c + e_{2eq} \rfloor^{\alpha}) \tag{9}$$

with $a = \ddot{y}$ and $b = \frac{\lambda_1}{\lambda_2}\mu\, h^{\alpha-1}$, and where e_{2eq} verifes the following $e_{2eq}^+ = e_{2eq} + h(a - b\lceil e_{2eq} \rfloor^{\alpha}) \implies e_{2eq} = \lceil \frac{a}{b} \rfloor^{\frac{1}{\alpha}}$

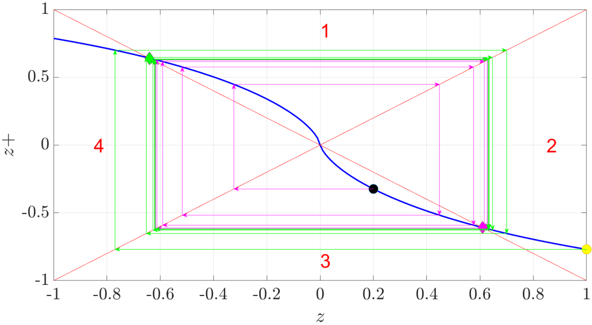


Figure: e_c^+ function of e_c in blue.

Case $\alpha < 1$, $ha = 10^{-3}$, $hb = 1.7783$, $\alpha = 0.75$

The intersection point is given by:

$$-e_{cint} = e_{cint} + h(a - b\lceil e_{cint} + e_{2eq}\rfloor^{\alpha})$$

which gives

$$2e_{cint} = hb\lceil e_{cint} + e_{eq}\rfloor^{\alpha} - ha \tag{10}$$

Replacing e_{2eq} by $\lceil \frac{a}{b}\rfloor^{\frac{1}{\alpha}}$, (10) becomes:

$$2e_{cint} = hb\lceil e_{cint} + (\frac{a}{b})^{\frac{1}{\alpha}}\rfloor^{\alpha} - ha$$

or again

$$2e_{cint} = h(\lceil b^{\frac{1}{\alpha}} e_{cint} + a^{\frac{1}{\alpha}}\rfloor^{\alpha} - a) \tag{11}$$

and setting $e_{cint} = w^{\frac{1}{\alpha}}$, (11) gives the e_2 limit of the attraction domain in case of $\alpha < 1$

$$2e_{cint} = h(\lceil (bw)^{\frac{1}{\alpha}} + a^{\frac{1}{\alpha}}\rfloor^{\alpha} - a) \tag{12}$$

The case $\alpha > 1$ (fixed-time convergence)

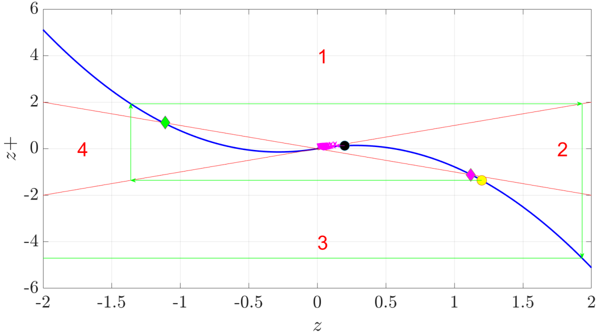


Figure: e_c^+ function of e_c in blue. Case $\alpha > 1$, $ha = 10^{-3}$, $hb = 1.7783$, $\alpha = 2$

Theorem

Under assumptions 1 and 2 hold and $e_{1M} = h\, e_{2M} \in SD$. *Then, for* $h > 0$, $\exists\ \lambda_1 > 0$, $\lambda_2 > 0$ *and* μ *such that the differentiation error dynamics* (6) *converge in finite time to*

$$SD_{1,2} \;=\; \{e_1, e_2 \mid e_1 \in SD_1 \;\; and \;\; e_2 \in SD_2\}$$

with

$$\begin{aligned} SD1 &= \{e_1 \mid |e_1| \le max\{h\, e_{2O1}, h\, e_{2O2}\}\} \\ SD2 &= \{e_2 \mid |e_2| \le max\{e_{2O1}, e_{2O2}\}\}\,. \end{aligned}$$

where

$$e_{2O1} = \left|\frac{\ddot{y}_M \lambda_1}{\lambda_2 \mu h^{\alpha-1}} - \frac{\lambda_2 \mu h^{\alpha}}{\lambda_1}\right|, \quad e_{2O2} = \frac{\ddot{y}_M \lambda_1}{\lambda_2 \mu h^{\alpha-1}} + \frac{\lambda_2 \mu h^{\alpha}}{2\lambda_1}.$$

To damp the oscillations on e_2, a 2-Proj. version **SIHD-2** is proposed as follows:

$$\left\{\begin{array}{l} z_1^+ = z_1 + h\left(z_2^+ + \lambda_1 \mu |e_1|^\alpha \mathcal{N}_1\right) \\ \\ z_2^+ = z_2 + E_1^+ h\left(\lambda_2 \mu^2 |e_1|^{2\,\alpha-1} \mathcal{N}_2\right) \end{array}\right. \tag{13}$$

$$\left\{\begin{array}{l} e_1^+ = e_1 + h\left(e_2^+ - \lambda_1 \mu |e_1|^\alpha \mathcal{N}_1\right) \\ \\ e_2^+ = e_2 + h\,\ddot{y} - E_1^+ h\left(\lambda_2 \mu^2 |e_1|^{2\,\alpha-1} \mathcal{N}_2\right) \end{array}\right. \tag{14}$$

where $\mathcal{N}_1$ is defined in (7) and as on SD we have $e_1 = h\,e_2$, $\mathcal{N}_2$ reads:

$$\mathcal{N}_2 := \left\{\begin{array}{l} e_1 \in SD' < \lambda_2 \mu^2 h^2 \rightarrow \mathcal{N}_2 = \frac{\lceil e_1 \rfloor^{2-2\alpha}}{\lambda_2 h^2 \mu^2} \\ \\ e_1 \in SD' \notin \rightarrow \mathcal{N}_2 = \mathrm{sign}(\mathrm{e}_1) \end{array}\right. \tag{15}$$

$$SD' = \{e_1 \in SD / \, |e_1| \leq (\lambda_1 \mu^2 h^2)^{\frac{1}{2(1-\alpha)}} \equiv |e_2| \leq (\lambda_1 \mu^2)^{\frac{1}{2(1-\alpha)}} h^{\frac{\alpha}{1-\alpha}} \}$$

Theorem

Suppose that assumptions 1-2 hold. Then for $h > 0$, there exist $\lambda_1 > 0$, $\lambda_2 > 0$ and μ such that the differentiation error dynamics (14) converge in finite time to

$$SD'_{1,2} = \{e_1, e_2 \mid e_1 \in SD'_1 \text{ and } e_2 \in SD'_2\}$$

with

$$\begin{aligned} SD'1 &= \left\{e_1 \mid |e_1| \leq h^2\, \ddot{y}_M\right\}, \\ SD'2 &= \left\{e_2 \mid |e_2| \leq h\, \ddot{y}_M\right\}. \end{aligned}$$

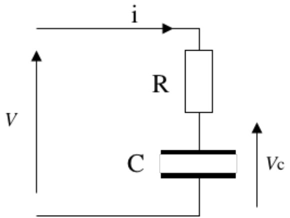


Figure: RC Circuit

- Differentiate v_C of a RC series circuit ($R = 100\,\Omega$, $C = 100\,\mu$F)
- v_C a sine function of frequency $\omega = 64.71$ rad/s ($f = 10.3$ Hz)
- Due to measurement noise, the projectors $\mathcal{N}_1$ and $\mathcal{N}_2$ are modified.

$$\mathcal{N}_1 := \begin{cases} (1-\theta)|e_1|^{1-\alpha} < \lambda_1 \mu h & \rightarrow \mathcal{N}_1 = \frac{(1-\theta)\lceil e_1 \rfloor^{1-\alpha}}{\lambda_1 h \mu} \\ (1-\theta)|e_1|^{1-\alpha} \geq \lambda_1 \mu h & \rightarrow \mathcal{N}_1 = \mathrm{sign(e_1)} \end{cases}$$

and

$$\mathcal{N}_2 := \begin{cases} (1-\theta)\left|e_1\right|^{2-2\alpha} < \lambda_2 \mu^2 h^2 \rightarrow \mathcal{N}_2 = \frac{(1-\theta)\lceil e_1 \rfloor^{2-2\alpha}}{\lambda_2 h^2 \mu^2} \\ (1-\theta)\left|e_1\right|^{2-2\alpha} \geq \lambda_2 \mu^2 h^2 \rightarrow \mathcal{N}_2 = \mathrm{sign(e_1)} \end{cases}$$

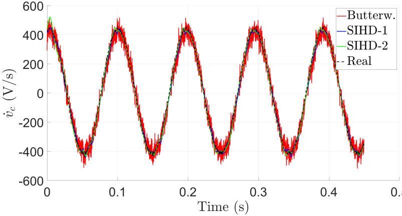


Figure: Differentiated signals for $h_1 = 2\,10^{-4}$ s.

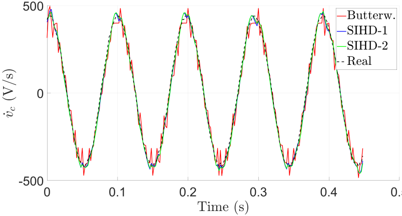


Figure: Differentiated signals for $h_2 = 2\,10^{-3}$ s.

RC CIRCUIT : RESULTS

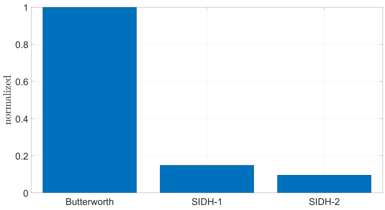


Figure: Evaluation of SSE (Some of Square Error) index for $h_1 = 2\,10^{-4}$ s.

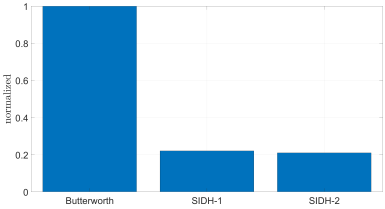


Figure: Evaluation of normalized SSE index $h_2 = 2\,10^{-3}$ s.

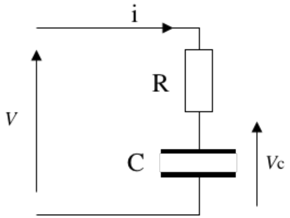
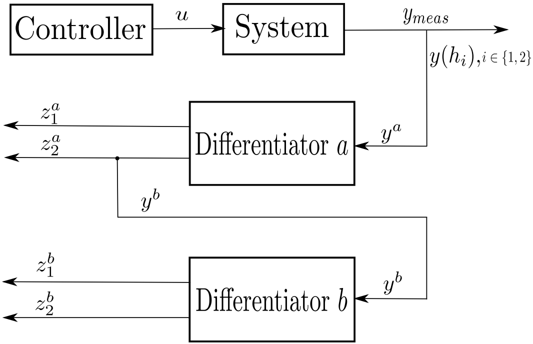


Figure: RLC Circuit

- Differentiate twice v_C of a RLC series circuit ($R = 100\,\Omega$, $C = 100\,\mu$F, $L =$)
- v_C a sine function of frequency $\omega = 64.71$ rad/s ($f = 10.3$ Hz)
- Cascaded iterative of homogeneous differentiators "a" and "b" structure allows a certain flexibility using different choices of homogeneous exponents α_a and α_b for the differentiators using the same structure


Controller
u
System
y_{meas}
$y(h_i), i \in \{1,2\}$
z_1^a
z_2^a
Differentiator a
y^a
y^b
z_1^b
z_2^b
Differentiator b
y^b