Advanced Machine Learning

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Assignment 1

Deadline: Friday, 22^{nd} of April 2022

Upload your solutions at: https://tinyurl.com/AML-2022-ASSIGNMENT1

1. **(0.5 points)** Give an example of a finite hypothesis class $\mathcal{H}$ with $\text{VCdim}(\mathcal{H}) = 2022$. Justify your choice.

2. **(0.5 points)** What is the maximum value of the natural even number n, $n = 2m$, such that there exists a hypothesis class $\mathcal{H}$ with n elements that shatters a set C of $m = \frac{n}{2}$ points? Give an example of such an $\mathcal{H}$ and C. Justify your answer.

3. **(0.75 points)** Let $\mathcal{X} = \mathbb{R}^2$ and consider $\mathcal{H}$ the set of axis aligned rectangles with the center in origin O(0, 0). Compute the $VCdim(\mathcal{H})$.

4. **(1 point)** Let $\mathcal{X} = \mathbb{R}^2$ and consider $\mathcal{H}_\alpha$ the set of concepts defined by the area inside a right triangle ABC with two catheti AB and AC parallel to the axes (Ox and Oy), and with the ratio AB/AC $= \alpha$ (fixed constant > 0). Consider the realizability assumption. Show that the class $\mathcal{H}_\alpha$ is (ϵ, δ)-PAC learnable by giving an algorithm A and determining an upper bound on the sample complexity $m_H(\epsilon, \delta)$ such that the definition of PAC-learnability is satisfied.

5. **(1.25 points)** Consider $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, where:

$$\mathcal{H}_1 = \{h_{\theta_1} : \mathbb{R} \to \{0,1\} \mid h_{\theta_1}(x) = \mathbf{1}_{[x \geq \theta_1]}(x) = \mathbf{1}_{[\theta_1, +\infty)}(x), \theta_1 \in \mathbb{R}\},$$

$$\mathcal{H}_2 = \{h_{\theta_2} : \mathbb{R} \to \{0,1\} \mid h_{\theta_2}(x) = \mathbf{1}_{[x < \theta_2]}(x) = \mathbf{1}_{(-\infty, \theta_2)}(x), \theta_2 \in \mathbb{R}\},$$

$$\mathcal{H}_3 = \{h_{\theta_1,\theta_2} : \mathbb{R} \to \{0,1\} \mid h_{\theta_1,\theta_2}(x) = \mathbf{1}_{[\theta_1 \leq x \leq \theta_2]}(x) = \mathbf{1}_{[\theta_1, \theta_2]}(x), \theta_1, \theta_2 \in \mathbb{R}\}.$$

Consider the realizability assumption.

a) Compute VCdim($\mathcal{H}$).

b) Show that $\mathcal{H}$ is PAC-learnable.

c) Give an algorithm A and determine an upper bound on the sample complexity $m_{\mathcal{H}}(\epsilon, \delta)$ such that the definition of PAC-learnability is satisfied.

6. **(1 point)** A decision list may be thought of as an ordered sequence of if-then-else statements. The sequence of conditions in the decision list is tested in order, and the answer associated with the first satisfied condition is output.

More formally, a *k-decision list* over the boolean variables $x_1, x_2, \ldots, x_n$ is an ordered sequence $L = \{(c_1, b_1), (c_2, b_2), \ldots, (c_l, b_l)\}$ and a bit b, in which each c_i is a conjunction of at most k literals over $x_1, x_2, \ldots, x_n$ and each $b_i \in \{0, 1\}$. For any input $a \in \{0, 1\}^n$, the value $L(a)$ is defined to be b_j where j is the smallest index satisfying $c_j(a) = 1$; if no such index exists, then $L(a) = b$. Thus, b is the "default" value in case a falls off the end of the list. We call b_i the bit associated with the condition c_i.

The next figure shows an example of a *2-decision list* along with its evaluation on a particular input.

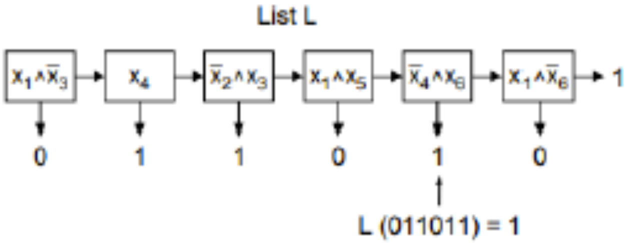


Figure 1: *A 2-decision list and the path followed by an input. Evaluation starts at the leftmost item and continues to the right until the first condition is satisfied, at which point the binary value below becomes the final result of the evaluation.*

Show that the VC dimension of 1-decision lists over $\{0, 1\}^n$ is lower and upper bounded by linear functions, by showing that there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that:

$$\alpha \cdot n + \beta \leq VCdim(\mathcal{H}_{1-decision\ list}) \leq \gamma \cdot n + \delta$$

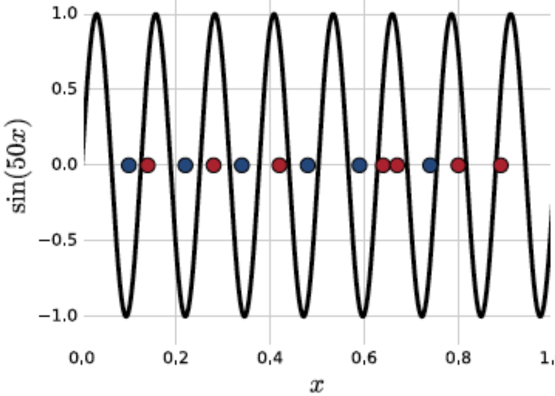
Hint: Show that 1-decision lists over $\{0, 1\}^n$ compute linearly separable functions (halfspaces).

Recap - VCdim($\mathcal{H}_{\text{sin}}$)

VCdim($\mathcal{H}_{\text{thresholds}}$) = 1, VCdim($\mathcal{H}_{\text{intervals}}$) = 2, VCdim($\mathcal{H}_{\text{lines}}$) = 3, VCdim($\mathcal{H}_{\text{rec}^2}$) = 4

Consider $\mathcal{H} = \mathcal{H}_{\text{sin}}$ be the set of sin functions:

$\mathcal{H}_{\text{sin}} = \{h_\theta: \mathbf{R} \to \{0,1\} \mid h_\theta(x) = \lceil \sin(\theta x) \rceil, \theta \in \mathrm{R}\}, \lceil -1 \rceil = 0$



Show that VCdim($\mathcal{H}_{\text{sin}}$) = ∞ based on the following lemma:

Let x $\in$ (0, 1) and let $0.x_1x_2x_3\ldots$ be the binary representation of x. Then, for any natural number m, provided that there exist $k \geq m$ such that $x_k = 1$, we have:

$$\lceil \sin(2^m \pi x) \rceil = 1 - x_m$$

Recap - VCdim($\mathcal{HS}_0^{\text{n}}$)

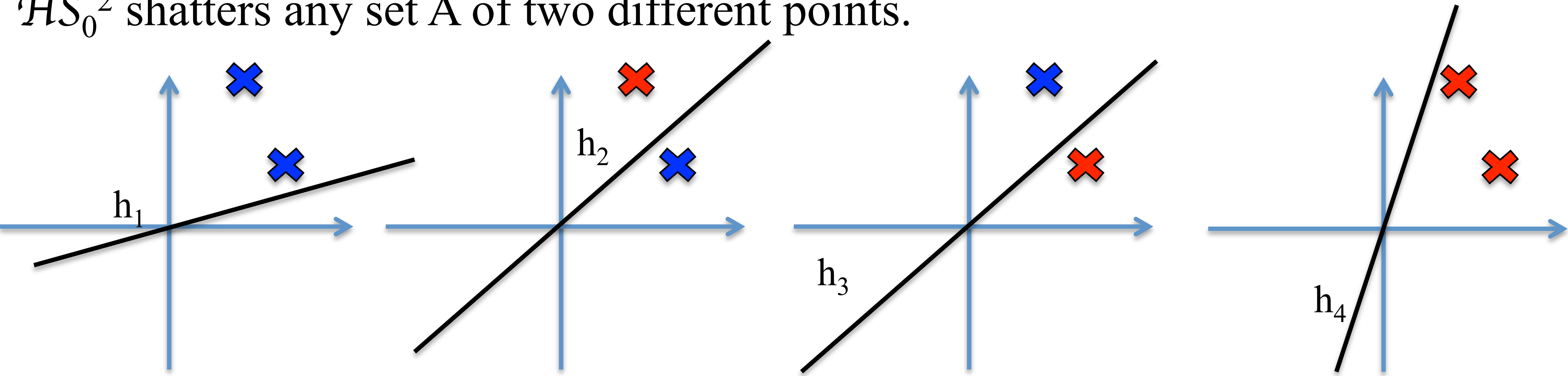
$\mathcal{HS}_0^{\text{n}} = \{h_{\text{w},0}\text{: } \mathbf{R}^{\text{n}} \rightarrow \{-1, 1\},\ h_{w,0}(x) = sign\left(\sum_{i=1}^{n} w_i x_i\right) \mid \text{w} \in \mathbf{R}^{\text{n}}\}$

For n = 2 we have:

$\mathcal{HS}_0^2 = \{\text{h}_{\text{w1,w2}}\text{: } \mathbf{R}^2 \rightarrow \{-1, 1\},\ \text{h}_{\text{w1,w2}}(\text{x}) = \text{sign}(\text{w}_1\text{x}_1 + \text{w}_2\text{x}_2) \mid (\text{w}_1,\text{w}_2) \in \mathbf{R}^2\}$

What is the VCdim(HS_0^2) ?

$\mathcal{HS}_0^2$ shatters any set A of two different points.



label (-1,-1)

label (1,-1)

label (-1,1)
Change the signs for the coefficient in h$_2$

label (1,1)

Does HS_0^2 shatter a set A of three points?

Difficult to reason geometrically… choose the algebraic proof.

Recap - VCdim($\mathcal{HS}_0^n$)

Proof:

1st part – show that VCdim($\mathcal{HS}_0^n$) ≥ n

A = {e_1, e_2, …, e_n}, the orthonormal basis of $\mathbf{R}^n$ is shattered by $\mathcal{HS}_0^n$.

2nd part – show that VCdim($\mathcal{HS}_0^n$) < n + 1

Any set A = {x_1, x_2, …, x_{n+1}} of *n* + 1 points in $\mathbf{R}^n$ cannot be shattered by $\mathcal{HS}_0^n$. Provide an algebraic proof, based on the fact that {x_1, x_2, …, x_{n+1}} are linearly dependent in $\mathbf{R}^n$.

So, VCdim($\mathcal{HS}_0^n$) = n

Similarly, it can be shown that VCdim($\mathcal{HS}^n$) = n + 1

The fundamental theorem of statistical learning

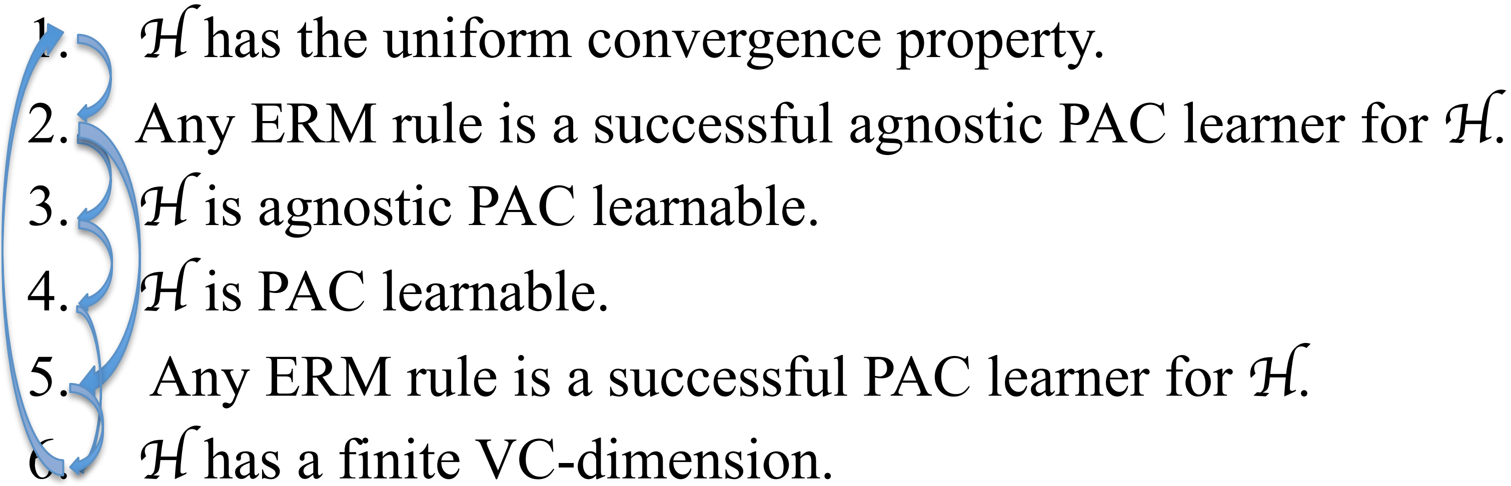
The fundamental theorem of statistical learning

Theorem (The Fundamental Theorem of Statistical Learning).
Let $\mathcal{H}$ be a hypothesis class of functions from a domain $\mathcal{X}$ to $\{0,1\}$ and let the loss function be the $0-1$ loss. Then, the following statements are equivalent:

1. $\mathcal{H}$ has the uniform convergence property.
2. Any ERM rule is a successful agnostic PAC learner for $\mathcal{H}$.
3. $\mathcal{H}$ is agnostic PAC learnable.
4. $\mathcal{H}$ is PAC learnable.
5. Any ERM rule is a successful PAC learner for $\mathcal{H}$.
6. $\mathcal{H}$ has a finite VC-dimension.

A finite VC- dimension guarantees learnability*. *Hence, the VC-dimension characterizes PAC learnability.

Proof



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4. $\mathcal{H}$ is PAC learnable.
5. Any ERM rule is a successful PAC learner for $\mathcal{H}$.
6. $\mathcal{H}$ has a finite VC-dimension.

Proof:

1 → 2 follows from lecture 4: uniform convergence property → every sample S is ε-representative → ERM is a successful agnostic PAC learner

2 → 3, 3 → 4 (lecture 5), 2 → 5 follow immediately from the definition

4 → 6 (lecture 5), 5 → 6 – follow from the No-Free Lunch theorem

Need to prove 6 → 1 (the hardest part)

Remember – lecture 4: uniform convergence property

Definition (*uniform convergence*)

A hypothesis class $\mathcal{H}$ has the *uniform convergence property* wrt a domain $\mathcal{Z}=\mathcal{X} \times \mathcal{Y}$, loss fct. ℓ if:

- there exists a function $m_H^{UC} : (0,1)^2 \rightarrow \mathrm{N}$
- such that for all (ε, δ) ∈(0,1)²
- and for any probability distribution $\mathcal{D}$ over $\mathcal{Z}$

if S is a sample of m ≥ $m_H^{UC}(\varepsilon,\delta)$ examples drawn i.i.d. according to $\mathcal{D}$, then, with probability of at least 1 – δ, S is ε-representative.

Definition (ε – representative sample)

A sample S is called ε – representative wrt domain $\mathcal{Z}$, hypothesis class $\mathcal{H}$, loss function ℓ and distribution $\mathcal{D}$ if:

$$\forall h \in \mathcal{H}, \;\; |L_S(h) - L_{\mathcal{D}}(h)| \le \epsilon.$$

Lemma

Let S be a sample that is ε/2 – representative wrt domain $\mathcal{Z}$, hypothesis class $\mathcal{H}$, loss function ℓ and distribution $\mathcal{D}$. Then any output of $\mathrm{ERM}_{\mathcal{H}}(S)$ i.e any $h_S \in \mathrm{argmin}_h L_S(h)$ satisfies:

$$L_{\mathcal{D}}(A(S)) \le \min_{h\in\mathcal{H}} L_{\mathcal{D}}(h) + \epsilon$$

Proof for 6 → 1

We want to prove that finite VC-dimension → *uniform convergence property*

Two steps:

1. (Sauer's lemma) If $\mathrm{VCdim}(\mathcal{H}) \leq d < \infty$, then even though $\mathcal{H}$ might be infinite, when restricting it to a finite set $C \subseteq \mathcal{X}$, its "effective" size, $|\mathcal{H}_C|$, is only $O(|C|^d)$. That is, the size of $\mathcal{H}_C$ grows polynomially rather than exponentially with $|C|$.

2. we have shown in lecture 4 that finite hypothesis classes enjoy the uniform convergence property. We generalize this result and show that uniform convergence holds whenever the hypothesis class has a "small effective size." By "small effective size" we mean classes for which $|\mathcal{H}_C|$ grows polynomially with $|C|$.

The Growth function

Definition

Let $\mathcal{H}$ be a hypothesis class. Then the growth function of $\mathcal{H}$, denoted by τ_H, where $\tau_{\mathcal{H}}$: **N**→**N**, is defined as:

$$\tau_H(m) = \max_{C \subseteq X : |C| = m} |H_C|$$

In other words, $\tau_H(m)$ is the maximum number of different functions from a set C of size m to {0,1} that can be obtained by restricting $\mathcal{H}$ to C.

Observation: if VCdim($\mathcal{H}$) = d then for any $m \leq d$ we have $\tau_{\mathcal{H}}(m) = 2^m$. In such cases, $\mathcal{H}$ induces all possible functions from C to {0,1}.

What happens when m becomes larger than the VC-dimension?

Answer given by the Sauer's lemma: the growth function $\tau_{\mathcal{H}}$ increases polynomially rather than exponentially with m.

The Growth function

Definition

Let $\mathcal{H}$ be a hypothesis class. *$\tau_H(m)$ is the maximum number of different functions from a set C of size m to {0,1} that can be obtained by restricting $\mathcal{H}$ to C.*

$$\tau_H(m) = \max_{C \subseteq X : |C| = m} |H_C|$$

Example: consider $\mathcal{H} = \mathcal{H}_{\text{thresholds}}$ be the set of threshold functions over the real line $\mathcal{H}_{\text{thresholds}} = \{h_a : R \rightarrow \{0, 1\}, h_a(x) = \mathbf{1}_{[x<a]}, a \in \mathbf{R}\}$, $|\mathcal{H}_{\text{thresholds}}| = \infty$. We know that VCdim($\mathcal{H}$) = 1.

Consider C = $\{c_1, c_2, \ldots, c_m\}$ a set of m points, with $c_i < c_j$. What is *$\tau_H(m)$?*

The Growth function

Definition

Let $\mathcal{H}$ be a hypothesis class. *$\tau_H(m)$ is the maximum number of different functions from a set C of size m to {0,1} that can be obtained by restricting $\mathcal{H}$ to C.*

$$\tau_H(m) = \max_{C \subseteq X : |C| = m} |H_C|$$

Example:

Consider C = {c_1, c_2, …, c_m} a set of m points, with $c_i < c_j$. What is $\tau_H(m)$?

$\mathcal{H}_C$ can have at most m+1 different functions: take $a_1 < c_1 < a_2 < c_2 < \ldots < a_m < c_m < a_{m+1}$, than we will have $|\mathcal{H}_C|$ = {h_{a_1}, h_{a_2}, …, h_{a_m+1}} = m+1 as

h_{a_1} labels points c_1, c_2, …, c_m with labels (0, 0, 0, …., 0 , 0)

h_{a_2} labels points c_1, c_2, …, c_m with labels (1, 0, 0, …., 0 , 0)

h_{a_3} label points c_1, c_2, …, c_m with labels (1, 1, 0, …., 0 , 0)

……..

h_{a_m} label points c_1, c_2, …, c_m with labels (1, 1, 1, …., 1 , 0)

h_{a_m+1} label points c_1, c_2, …, c_m with labels (1, 1, 1, …., 1 , 1)

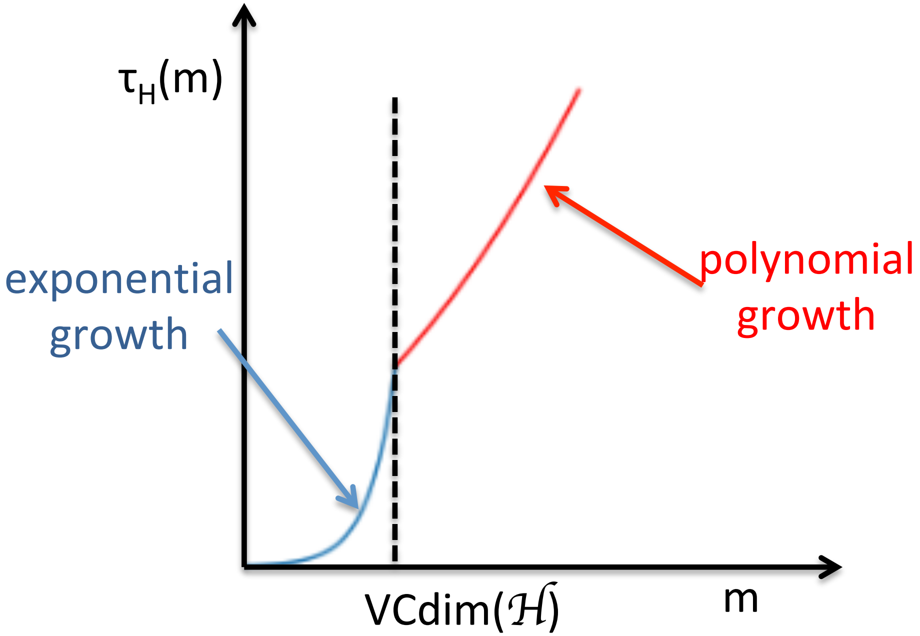
The Sauer's lemma

Lemma (Sauer – Shelah – Perles)

Let $\mathcal{H}$ be a hypothesis class with VCdim($\mathcal{H}$) $\leq d < \infty$. Then, for all m , we have that:

$$\tau_H(m) \leq \sum_{i=0}^{d} C_m^i$$

In particular, if $m > d + 1$ then $\tau_{\mathcal{H}}(m) \leq (em/d)^d = O(m^d)$



The Sauer's lemma - proof

Lemma (Sauer – Shelah – Perles)

Let $\mathcal{H}$ be a hypothesis class with VCdim($\mathcal{H}$) $\leq$ d $< \infty$. Then, for all m , we have that:

$$\tau_H(m) \leq \sum_{i=0}^{d} C_m^i$$

In particular, if m > d +1 then $\tau_{\mathcal{H}}$(m) $\leq$ (em/d)^d = O(m^d)

Proof

To prove the lemma it suffices to prove the following stronger claim:

For any C = {c$_1$, c$_2$, . . . , c$_m$} we have:

$|\mathcal{H}_C| \leq |\{B \subseteq C: \mathcal{H}$ shatters B$\}|$, for all $\mathcal{H}$ a hypothesis class

The reason why this claim is sufficient to prove the lemma is that if VCdim($\mathcal{H}$) $\leq$ d then no set B whose size is larger than d is shattered by $\mathcal{H}$ and therefore:

$$\tau_H(m) = \max_{C \subseteq X: |C|=m} |H_C| \leq \max_{C \subseteq X: |C|=m} \left|\{B \subseteq C : |B| \leq d\}\right| \leq \sum_{i=0}^{d} C_m^i$$

The Sauer's lemma - proof

We will employ induction over the size of C

First step: Fix $\mathcal{H}$ and consider $|C| = 1$.

If $|\mathcal{H}_C| = 1 \leq |\{B \subseteq C: \mathcal{H} \text{ shatters } B\}| = 1$ ($\mathcal{H}$ shatters the empty set).

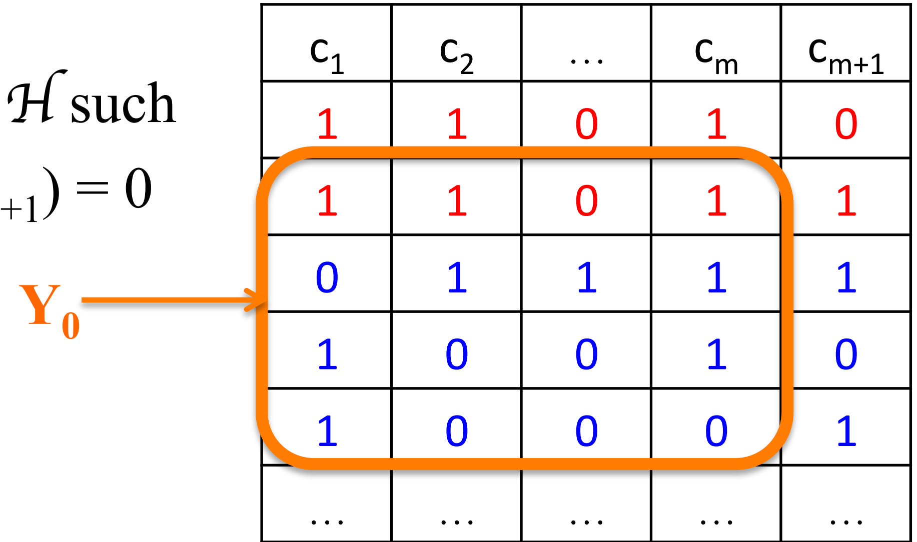
If $|\mathcal{H}_C| = 2 \leq |\{B \subseteq C: \mathcal{H} \text{ shatters } B\}| = 2$ ($\mathcal{H}$ shatters the empty set and C)

Induction step:

Assume the claim holds for $|C| \leq m$ and prove it for $|C| = m+1$.

Fix $\mathcal{H}$ and consider $C = \{c_1, c_2, . . . , c_m, c_{m+1}\}$ and $C' = \{c_1, c_2, . . . , c_m\}$, so $C = C' \cup \{c_{m+1}\}$

Take $Y_0 = \{g: C' \rightarrow \{0, 1\} |$ exists $h \in \mathcal{H}$ such that $h(c) = g(c)$ for all $c \in C'$ and $h(c_{m+1}) = 0$ OR $h(c_{m+1}) = 1\}$



c_1	c_2	…	c_m	c_{m+1}
1	1	0	1	0
1	1	0	1	1
0	1	1	1	1
1	0	0	1	0
1	0	0	0	1
…	…	…	…	…

So, $Y_0 = \mathcal{H}_{C'}$

The Sauer's lemma - proof

We will employ induction over the size of C

First step: Fix $\mathcal{H}$ and consider $|C| = 1$.

If $|\mathcal{H}_C| = 1 \leq |\{B \subseteq C: \mathcal{H}$ shatters $B\}| = 1$ ($\mathcal{H}$ shatters the empty set).

If $|\mathcal{H}_C| = 2 \leq |\{B \subseteq C: \mathcal{H}$ shatters $B\}| = 2$ ($\mathcal{H}$ shatters the empty set and C)

Induction step:

Assume the claim holds for $|C| \leq m$ and prove it for $|C| = m+1$.

Fix $\mathcal{H}$ and consider $C = \{c_1, c_2, . . . , c_m, c_{m+1}\}$ and $C' = \{c_1, c_2, . . . , c_m\}$.

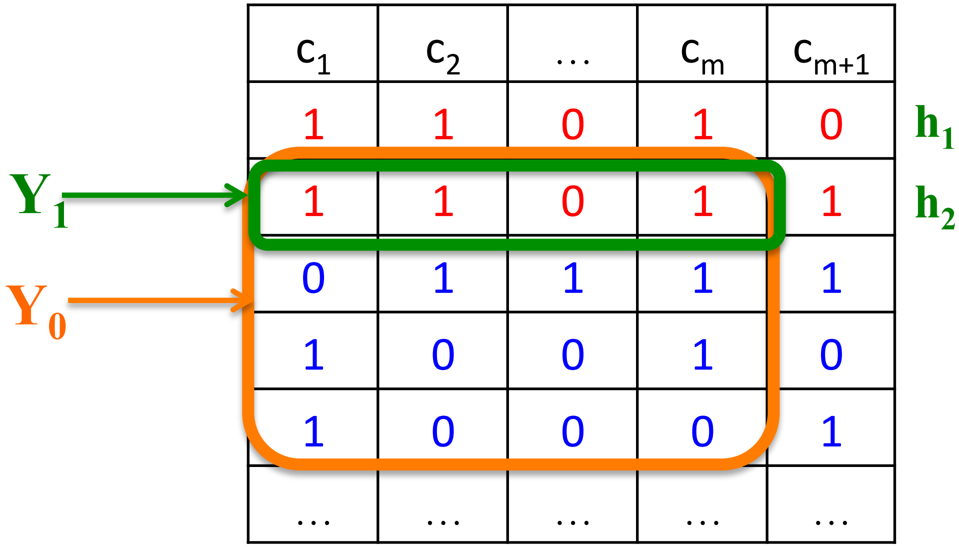
Take $Y_0 = \{g: C' \rightarrow \{0, 1\}|$ exists $h \in \mathcal{H}$ such that $h(c) = g(c)$ for all $c \in C'$ and $h(c_{m+1}) = 0$ OR $h(c_{m+1}) = 1\} = \mathcal{H}_{C'}$

If there exists two different function h_1 and h_2 in $\mathcal{H}$ that agree with g on C' then they will disagree on c_{m+1}: $h_1(c_{m+1}) \neq h_2(c_{m+1})$. They are two different functions in $\mathcal{H}$ but they will be counted only once in Y_0.

The Sauer's lemma - proof

Take $Y_0 = \{g: C' \rightarrow \{0, 1\} |$ exists $h \in \mathcal{H}$ such that $h(c) = g(c)$ for all $c \in C'$ and $h(c_{m+1}) = 0$ OR $h(c_{m+1}) = 1\} = \mathcal{H}_{C'}$

Take $Y_1 = \{g: C' \rightarrow \{0, 1\} |$ exists $h_1, h_2 \in \mathcal{H}$ such that $h_1(c) = g(c)$ for all $c \in C'$ and $h_1(c_{m+1}) = 0$ AND $h_2(c) = g(c)$ for all $c \in C'$ and $h_2(c_{m+1}) = 1\}$



c_1	c_2	…	c_m	c_{m+1}	
1	1	0	1	0	h_1
1	1	0	1	1	h_2
0	1	1	1	1	
1	0	0	1	0	
1	0	0	0	1	
…	…	…	…	…	

The Sauer’s lemma - proof

Take $Y_0 = \{g\text{: C'} \rightarrow \{0, 1\} |$ exists $h \in \mathcal{H}$ such that $h(\text{c}) = g(\text{c})$ for all $\text{c} \in \text{C'}$ and $h(\text{c}_{\text{m+1}}) = 0$ OR $h(\text{c}_{\text{m+1}}) = 1\} = \mathcal{H}_{\text{C'}}$

Take $Y_1 = \{g\text{: C'} \rightarrow \{0, 1\} |$ exists $h_1, h_2 \in \mathcal{H}$ such that $h_1(\text{c}) = g(\text{c})$ for all $\text{c} \in \text{C'}$ and $h_1(\text{c}_{\text{m+1}}) = 0$ AND $h_2(\text{c}) = g(\text{c})$ for all $\text{c} \in \text{C'}$ and $h_2(\text{c}_{\text{m+1}}) = 1\}$

We have that $Y_1 \subseteq Y_0$

Y_1 contains only those restriction $h_{C'}$ that come from two different functions h_1 and h_2 from $\mathcal{H}$

Y_0 might contain restrictions $h_{C'}$ that come from a single h from H.

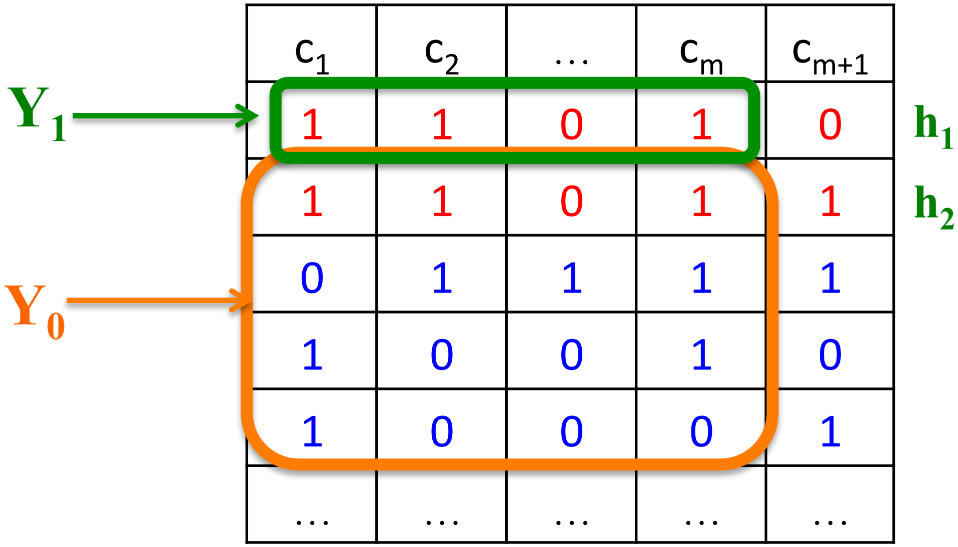
For simplicity let’s assume that $\text{C} = \mathcal{X}$, $\mathcal{X}$ is the domain of $\mathcal{H}$.

We have that $|\text{H}| = |Y_0| + |Y_1|$

The Sauer's lemma - proof

Take $Y_0 = \{g: C' \rightarrow \{0, 1\} |$ exists $h \in \mathcal{H}$ such that $h(c) = g(c)$ for all $c \in C'$ and $h(c_{m+1}) = 0$ OR $h(c_{m+1}) = 1\} = \mathcal{H}_{C'}$

Take $Y_1 = \{g: C' \rightarrow \{0, 1\} |$ exists $h_1, h_2 \in \mathcal{H}$ such that $h_1(c) = g(c)$ for all $c \in C'$ and $h_1(c_{m+1}) = 0$ AND $h_2(c) = g(c)$ for all $c \in C'$ and $h_2(c_{m+1}) = 1\}$



c_1	c_2	…	c_m	c_{m+1}	
1	1	0	1	0	h_1
1	1	0	1	1	h_2
0	1	1	1	1	
1	0	0	1	0	
1	0	0	0	1	
…	…	…	…	…	

The Sauer’s lemma - proof

Now, we will apply our induction hypothesis on Y_0

$|Y_0| = |\mathcal{H}_{C'}| \leq |\{B \subseteq C' : \mathcal{H} \text{ shatters } B\}| = |\{B \subseteq C : \mathcal{H} \text{ shatters } B \text{ and } c_{m+1} \notin B\}|$

Take $\mathcal{H}' = \{h_1 \in \mathcal{H}$ such that there exists $h_2 \in \mathcal{H}$ s. t. for all $c \in C'$ we have $h_1(c) = h_2(c)$ but $h_1(c_{m+1}) \neq h_2(c_{m+1})\}$

Then $Y_1 = \mathcal{H}'_{C'}$ = set of function on C’ with two extensions on c_{m+1}

Use the induction hypothesis here, on Y_1 :

$|Y_1| = |\mathcal{H}'_{C'}| \leq |\{B \subseteq C' : \mathcal{H}' \text{ shatters } B\}| = |\{B \subseteq C : \mathcal{H} \text{ shatters } B \text{ and } c_{m+1} \in B\}|$

So, we have that $|\mathcal{H}| = |\mathcal{H}_C| \leq |\{B \subseteq C : \mathcal{H} \text{ shatters } B\}|$

$\tau_{\mathcal{H}}$ grows polynomially

Corollary

Let H be a hypothesis class with VCdim(H) = d. Then for all m ≥ d:

$$\tau_H(m) \leq \left(\frac{em}{d}\right)^d = O(m^d)$$

Proof:

From the Sauer lemma we have:

$$\tau_H(m) \leq \sum_{i=0}^{d} C_m^i \underset{m \geq d}{\leq} \sum_{i=0}^{d}\left(C_m^i \times \left(\frac{m}{d}\right)^{d-i}\right) \underset{m \geq d}{\leq} \sum_{i=0}^{m}\left(C_m^i \times \left(\frac{m}{d}\right)^{d-i}\right) = \left(\frac{m}{d}\right)^d \sum_{i=0}^{m}\left(C_m^i \times \left(\frac{d}{m}\right)^{i}\right)$$

$$\tau_H(m) \leq \left(\frac{m}{d}\right)^d \sum_{i=0}^{m}\left(C_m^i \times \left(\frac{d}{m}\right)^{i}\right) \underset{\text{Newton's binomial formula}}{=} \left(\frac{m}{d}\right)^d \left(1+\frac{d}{m}\right)^m \underset{1-x \leq e^{-x}}{\leq} \left(\frac{m}{d}\right)^d \left(e^{\frac{d}{m}}\right)^m = \left(\frac{em}{d}\right)^d$$

Proof for 6 → 1

We want to prove that finite VC-dimension → *uniform convergence property*

Two steps:

1. (Sauer's lemma) If VCdim($\mathcal{H}$) = d < ∞, then even though $\mathcal{H}$ might be infinite, when restricting it to a finite set C ⊆ $\mathcal{X}$, its "effective" size, $|\mathcal{H}_C|$, is only O($|C|^d$). That is, the size of $\mathcal{H}_C$ grows polynomially rather than exponentially with |C|.

2. we have shown in lecture 4 that finite hypothesis classes enjoy the uniform convergence property. We generalize this result and show that uniform convergence holds whenever the hypothesis class has a "small effective size." By "small effective size" we mean classes for which $|\mathcal{H}_C|$ grows polynomially with |C|.

Uniform converge holds for $\mathcal{H}$ with small effective size

Theorem

Let $\mathcal{H}$ be a class and let $\tau_{\mathcal{H}}$ be its growth function. Then, for every $\mathcal{D}$ and every $\delta \in (0,1)$, with probability of at least $1 - \delta$ over the choice of $S \sim \mathcal{D}^m$ we have:

$$\left|L_D(h) - L_S(h)\right| \leq \frac{4 + \sqrt{\log(\tau_H(2m))}}{\delta\sqrt{2m}}$$

Proof:

- in the book, is beyond the scope of this lecture

Proof for 6 → 1

We want to prove that finite VC-dimension → *uniform convergence property.*

Combine the last result with Sauer lemma: $\tau_{\mathcal{H}}(\mathrm{m}) \leq (\mathrm{em/d})^{\mathrm{d}} = O(\mathrm{m}^{\mathrm{d}})$ to obtain: for every $\mathcal{D}$ and every $\delta \in (0,1)$, with probability of at least $1 - \delta$ over the choice of $S \sim \mathcal{D}^m$ we have:

$$\left|L_D(h) - L_S(h)\right| \leq \frac{4+\sqrt{\log(\tau_H(2m))}}{\delta\sqrt{2m}} \underset{\uparrow}{\leq} \frac{4+\sqrt{d\log(2em/d)}}{\delta\sqrt{2m}} \underset{\uparrow}{\leq} \frac{2\sqrt{d\log(2em/d)}}{\delta\sqrt{2m}}$$

Sauer lemma

$\tau_{\mathcal{H}}(2\mathrm{m}) \leq (2\mathrm{em/d})^{\mathrm{d}}$

consider m such that

$4^2 \leq d\log(2em/d)$

$$\left|L_D(h) - L_S(h)\right| \leq \frac{1}{\delta}\frac{\sqrt{2d\log(2em/d)}}{\sqrt{m}} < \varepsilon$$

This leads (see the calculation in the book) to:

$$m \geq 4\frac{2d}{(\delta\varepsilon^2)}\log\left(\frac{2d}{\delta\varepsilon^2}\right) + \frac{4d\log(2\varepsilon/d)}{(\delta\varepsilon^2)}$$

Proof for 6 → 1

We want to prove that finite VC-dimension → *uniform convergence property.*

for every $\mathcal{D}$ and every δ ∈ (0,1), with probability of at least 1 − δ over the choice of S ~ $\mathcal{D}^m$ we have that if:

$$m \geq 4\frac{2d}{(\delta\varepsilon^2)}\log(\frac{2d}{\delta\varepsilon^2}) + \frac{4d\log(2\varepsilon/d)}{(\delta\varepsilon^2)}$$

then the sample S is ε-representative

$$\left|L_D(h) - L_S(h)\right| \leq \frac{1}{\delta}\frac{\sqrt{2d\log(2em/d)}}{\sqrt{m}} < \varepsilon$$

So, we have that: $m_H^{UC}(\varepsilon,\delta) \leq 4\frac{2d}{(\delta\varepsilon^2)}\log(\frac{2d}{\delta\varepsilon^2}) + \frac{4d\log(2\varepsilon/d)}{(\delta\varepsilon^2)}$

The derived bound is not the tightest possible, there exist another bound much tighter (see next).

The fundamental theorem of statistical learning – quantitative version

Theorem

Let $\mathcal{H}$ be a hypothesis class of functions from a domain $\mathcal{X}$ to $\{0,1\}$ and let the loss function be the 0−1 loss. Assume that VCdim($\mathcal{H}$) = d <∞. Then, there are absolute constants C_1, C_2 such that:

1. $\mathcal{H}$ has the uniform convergence property with sample complexity:

$$C_1 \frac{d + \log(1/\delta)}{\epsilon^2} \leq m_{\mathcal{H}}^{UC}(\epsilon, \delta) \leq C_2 \frac{d + \log(1/\delta)}{\epsilon^2}$$

2. $\mathcal{H}$ is agnostic PAC learnable with sample complexity:

$$C_1 \frac{d + \log(1/\delta)}{\epsilon^2} \leq m_{\mathcal{H}}(\epsilon, \delta) \leq C_2 \frac{d + \log(1/\delta)}{\epsilon^2}$$

3. $\mathcal{H}$ is PAC learnable with sample complexity:

$$C_1 \frac{d + \log(1/\delta)}{\epsilon} \leq m_{\mathcal{H}}(\epsilon, \delta) \leq C_2 \frac{d \log(1/\epsilon) + \log(1/\delta)}{\epsilon}$$

The VC dimension determines (along with ε, δ) the samples complexities of learning a class. It gives us a lower and an upper bound.

Intuition for deriving the lower bounds

The PAC case (realizable case)

$$C_1\frac{d+\log(1/\delta)}{\epsilon}\le m_{\mathcal{H}}(\epsilon,\delta)\le C_2\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon}$$

Pick a set A = $\{x_1, x_2, \ldots, x_d\}$ of size d (=VCdim($\mathcal{H}$)) that is shattered by $\mathcal{H}$. Choose the following (adversarial) probability distribution $\mathcal{D}$ over $\mathcal{X}$:

$\mathcal{D}(x_1) = 1\text{-}4\varepsilon$, $\mathcal{D}(x_i) = 4\varepsilon/(d\text{-}1)$, i = 2,3,…,d, $\mathcal{D}(x) = 0$, for all x in $\mathcal{X} \setminus A$

By the No Free Lunch theorem as long as a sample S hits B = $\{x_2,\ldots x_d\}$ at most (d-1)/2 times, the probability of making an error over B is ≥ 1/4. This happens because we see less then half of the domain B points. So, our expected error with respect to $\mathcal{D}$ is 4ε/4 = ε.

If the sample S has size *m*, then roughly $4m\varepsilon$ points will hit B = $\{x_2, \ldots, x_d\}$. So, to make less than ε errors we need to have 4mε > (d-1)/2, m > (d-1)/8ε