Să se rezolve ecuația:

1) $x'' + 4x' + 5x = 0$

Este o ecuație liniară de ordin II cu coef. constanți.

Ecuația caracteristică este: $\lambda^2 + 4\lambda + 5 = 0$

$$\Delta = 4^2 - 4\cdot 5 = 16 - 20 = -4 < 0.$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2i}{2} \begin{cases} -2+i \\ -2-i \end{cases}$$

Soluția este $x(t) = c_1 \cdot e^{-2t} \cos t + c_2 \cdot e^{-2t} \sin t.$

2) $x'' = \sqrt{1+(x')^2}$. Să se determine mulțimea soluțiilor.

ec. tip. $F(t, x^{(k)}, x^{(k+1)}, \dots, x^{(n)}) = 0,\ k \geq 1.$

Se face schimbarea de variabilă $y = x'$

$$y(t) = x'(t) \Rightarrow y'(t) = x''(t)$$

$y' = \sqrt{1+y^2}$

$\frac{dy}{dt} = \sqrt{1+y}$ → ec. cu variabile separabile

cu sol. staționare: $\sqrt{1+y^2} = 0 \Rightarrow 1+y^2 = 0$ (F)

$\frac{dy}{\sqrt{1+y^2}} = dt \Rightarrow \int \frac{dy}{\sqrt{1+y^2}} = \int dt$

$\ln|y + \sqrt{y^2+1}| = t + c,\ c \in \mathbb{R} \Rightarrow |\underbrace{y + \sqrt{y^2+1}}_{>0}| = e^{t+c},\ c \in \mathbb{R}$

$\Rightarrow y + \sqrt{y^2+1} = e^{t+c} \Rightarrow \sqrt{y^2+1} = e^{t+c} - y \Rightarrow$

$\Rightarrow y^2 + 1 = e^{2(t+c)} + y^2 - 2e^{t+c} \cdot y \Rightarrow y = \frac{e^{2(t+c)} - 1}{2e^{(t+c)}} = \frac{e^{2(t+c)}}{2e^{(t+c)}} - \frac{1}{2e^{(t+c)}}$

$= \frac{e^{(t+c)}}{2} - \frac{1}{2e^{(t+c)}}$, not. $e^c = k > 0$

$$\underset{x'}{y} = \frac{k}{2} \cdot e^t - \frac{1}{2k} e^{-t},\ k > 0.$$

$x'(t) = \frac{k}{2} e^t - \frac{1}{2k} e^{-t},\ k > 0$

$x(t) = \int \frac{k}{2} e^t - \frac{1}{2k} e^{-t}\, dt = \frac{k}{2} e^t + \frac{1}{2k} e^{-t} + c,\ k, c \in \mathbb{R},\ k > 0$

3) $t^2x'' - 2t\,xx' + tx' = 0$. Să se det. mulțimea soluțiilor.

$$t^2x'' - 2x\cdot tx' + tx' = 0 \rightarrow \text{ec. Euler}$$

Se face schimbarea de variabilă $|t| = e^s$ $\begin{cases} t = e^s, t>0 \\ t = -e^s, t<0. \end{cases}$

pp. $t>0$, $t = e^s$.

$$x(t) = x(e^s) = y(s)$$

derivăm, $x'(e^s)\cdot e^s = y'(s) \Rightarrow x'(e^s) = e^{-s}\cdot y'(s)$

derivăm din nou, $x''(e^s)\cdot e^s = -e^{-s}\cdot y'(s) + e^{-s}\cdot y''(s)$

$$\Rightarrow x''(e^s) = -e^{-2s}y'(s) + e^{-2s}y''(s)$$

$$\Rightarrow e^{2s}\cdot(-e^{-2s}y' + e^{-2s}y'') - 2\cdot y\cdot e^s\cdot e^{-s}\cdot y' + e^s\cdot e^{-s}\cdot y' = 0$$

$$\Rightarrow \cancel{-y'} + y'' - 2yy' + \cancel{y'} = 0.$$

$\rightarrow y'' - 2yy' = 0$. $\rightarrow$ ec. autonomă

Sch. var. $z(y) = y' \overset{\text{derivăm}}{\Rightarrow} y'' = z'(y)\cdot y' = z'(y)\cdot z(y)$

$z'(y)\cdot z(y) - 2\cdot y\cdot z(y) = 0 \rightarrow$ $z = 0 \Rightarrow y' = 0 \Rightarrow y = c, c\in\mathbb{R}$

$\Rightarrow x = c, c\in\mathbb{R}$

dacă $z \neq 0 \rightarrow z'(y) = \dfrac{2y\,z(y)}{z(y)}$

$\Rightarrow z'(y) = 2y \Rightarrow z(y) = \int 2y\,dy = y^2 + k, k\in\mathbb{R}$

$\Rightarrow y'(s) = y^2 + k,\ k \in \mathbb{R}$

$\frac{dy}{ds} = y^2 + k,\ k \in \mathbb{R} \to$ ec. cu var. separabile

cu sol. staționare: $y^2 + k = 0 \Rightarrow y^2 = \pm\sqrt{k},\ k \in \mathbb{R}$ deci $y = ct = x$

(avem deja sol.)

$$\frac{dy}{y^2+k} = ds$$

$$\int \frac{1}{y^2+k}\,dy = \int ds,\quad k \in \mathbb{R}$$

Pp. $k > 0 \Rightarrow \frac{1}{\sqrt{k}} \cdot \operatorname{arctg} \frac{y}{\sqrt{k}} = s + k_1,\ k, k_1 \in \mathbb{R}$

$$\operatorname{arctg} \frac{y}{\sqrt{k}} = s\sqrt{k} + k_1\sqrt{k},\quad k, k_1 \in \mathbb{R},\ k > 0.$$

$$\frac{y}{\sqrt{k}} = \operatorname{tg}(s\sqrt{k} + k_1\sqrt{k}) \Rightarrow y = \sqrt{k} \cdot \operatorname{tg}(s\sqrt{k} + k_1\sqrt{k}),\quad k, k_1 \in \mathbb{R},\ k > 0.$$

$t = e^s \Rightarrow \ln t = s$

$y(s) = x(e^s) \Rightarrow y(\ln t) = x(t)$

$$\begin{cases} x(t) = \sqrt{k} \cdot \operatorname{tg}(\ln t \cdot \sqrt{k} + k_1\sqrt{k}), & k, k_1 \in \mathbb{R},\ k > 0 \\ x \equiv 2,\ 2 \in \mathbb{R} \end{cases}$$

pentru $k < 0$: $\int \frac{1}{y^2 - (-k)}\,dy = \frac{1}{2\sqrt{-k}} \cdot \ln\left|\frac{y - \sqrt{-k}}{y + \sqrt{-k}}\right| = s + k_1,\ k, k_1 \in \mathbb{R},\ k < 0.$

$$\Rightarrow \ln\left|\frac{y - \sqrt{-k}}{y + \sqrt{-k}}\right| = 2\sqrt{-k}\,(s + k_1),\quad k, k_1 \in \mathbb{R},\ k < 0.$$

$$\frac{y - \sqrt{-k}}{y + \sqrt{-k}} = \pm e^{2\sqrt{-k}(s+k_1)}$$

$y(1 - e^{2\sqrt{-k}(s+k_1)}) = \sqrt{-k} \cdot e^{2\sqrt{-k}(s+k_1)} + \sqrt{-k}$ | $y(1 + e^{2\sqrt{-k}(s+k_1)}) = -\sqrt{-k}\,e^{2\sqrt{-k}(s+k_1)} + \sqrt{-k}$

$y = \frac{\sqrt{-k}\,e^{2\sqrt{-k}(s+k_1)} + \sqrt{-k}}{1 - e^{2\sqrt{-k}(s+k_1)}}$ | $y = \frac{-\sqrt{-k}\,e^{2\sqrt{-k}(s+k_1)} + \sqrt{-k}}{1 + e^{2\sqrt{-k}(s+k_1)}}$

$$x(t) = y(\ln t) = \frac{\sqrt{-k}\,e^{2\sqrt{-k}(\ln t + k_1)} + \sqrt{-k}}{1 - e^{2\sqrt{-k}(\ln t + k_1)}},\quad k, k_1 \in \mathbb{R},\ k < 0.$$

sau $x(t) = \frac{-\sqrt{-k}\,e^{2\sqrt{-k}(\ln t + k_1)} + \sqrt{-k}}{1 + e^{2\sqrt{-k}(\ln t + k_1)}}$

4) $x'' \cdot \cos x + \underline{(x')^2 \sin x} - x' = 0,\ x \in (0, \frac{\pi}{2})$

t - nu apare $\Rightarrow$ ec. autonomă

Sch. var. : $x' = y(x)$

$$x'' = y'(x) \cdot x' = y'(x) \cdot y(x)$$

$$y y' \cdot \cos x + y^2 \sin x - y = 0.$$

dacă $y = 0 \Rightarrow x' = 0 \Rightarrow x = c,\ c \in \mathbb{R}$

dacă $y \neq 0 \Rightarrow y' = \frac{y - y^2 \sin x}{y \cos x} = \frac{1 - y \sin x}{\cos x} = \frac{1}{\cos x} + y \cdot \left(-\frac{\sin x}{\cos x}\right)$

$$\Rightarrow y' = \underbrace{(-\operatorname{tg} x)}_{a(x)} \cdot y + \underbrace{\frac{1}{\cos x}}_{b(x)} \rightarrow \text{ec. afină}$$

Scriem ec. lin. atașată : $\bar{y}' = -\operatorname{tg} x \cdot \bar{y}$ cu soluția $\bar{y}(x) = c \cdot e^{\int -\operatorname{tg} x\, dx} =$

$$= c \cdot e^{\ln|\cos x|} = c \cdot |\cos x| = c \cdot \cos x$$

Aplicăm met. variației constantelor, căutăm soluții de forma:

$y(x) = c(x) \cdot \cos x$

$$(c(x) \cdot \cos x)' = -\operatorname{tg} x \cdot c(x) \cdot \cos x + \frac{1}{\cos x}$$

$$\Rightarrow c'(x) \cdot \cos x - c(x) \cdot \sin x = -\frac{\sin x}{\cancel{\cos x}} \cdot c(x) \cdot \cancel{\cos x} + \frac{1}{\cos x}$$

se reduc

$$\Rightarrow c'(x) \cdot \cos x = \frac{1}{\cos x} \Rightarrow c'(x) = \frac{1}{\cos^2 x}$$

$$c(x) = \int \frac{1}{\cos^2 x}\, dx = \operatorname{tg} x + k,\ k \in \mathbb{R}$$

$y(x) = (\operatorname{tg} x + k) \cdot \cos x,\ k \in \mathbb{R}$

$y(x) = \frac{\sin x}{\cos x} \cdot \cos x + k \cdot \cos x$

$\left.\begin{array}{l} y(x) = \sin x + k \cos x,\ k \in \mathbb{R} \\ y(x) = x' \end{array}\right\} \Rightarrow x' = \sin x + k \cos x,\ k \in \mathbb{R}$

$$\frac{dx}{dt} = \sin x + k \cos x \rightarrow \text{ec. cu var. sep.}$$

$\sin x + k\cos x = 0 \Rightarrow \sin x = -k\cos x \to tg\, x = -k \Rightarrow \underset{\substack{\downarrow \\ \text{sol. staționară}}}{x} = arctg(-k) \in (0, \frac{\pi}{2})$ ptr. $k<0$

$$\int \frac{dx}{\sin x + k\cos x} = \int dt$$

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}, \quad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}, \quad tg\frac{x}{2} = s \Rightarrow x = 2arctg\, s \Rightarrow$$

$$dx = \frac{2}{1+s^2}ds$$

$$\sin x = \frac{2s}{1+s^2}, \quad \cos x = \frac{1-s^2}{1+s^2}$$

$$\int \frac{\frac{2}{1+s^2}}{\frac{2s}{1+s^2} + k\cdot\frac{1-s^2}{1+s^2}}ds = \int \frac{2}{2s+k(1-s^2)}ds = 2\cdot\int\frac{1}{-ks^2+2s+k}ds =$$

$$= -\frac{2}{k}\int\frac{1}{s^2-\frac{2}{k}\cdot s-1}ds = -\frac{2}{k}\cdot\int\frac{1}{\left(s^2-2\cdot s\cdot\frac{1}{k}+\frac{1}{k^2}\right)-\frac{1}{k^2}-1}ds =$$

$$= -\frac{2}{k}\int\frac{1}{(s-\frac{1}{k})^2-\frac{1+k^2}{k^2}}ds = -\frac{2}{k}\cdot\int\frac{1}{u^2-\left(\frac{\sqrt{1+k^2}}{k}\right)^2}du = -\frac{2}{k}\cdot\frac{1}{2\frac{\sqrt{1+k^2}}{k}}\ln\left|\frac{u-\frac{\sqrt{1+k^2}}{k}}{u+\frac{\sqrt{1+k^2}}{k}}\right| + c, \; c\in\mathbb{R}$$

$$u = s-\frac{1}{k} \Rightarrow du = ds$$

$$= \frac{-1}{\sqrt{1+k^2}}\ln\left|\frac{uk-\sqrt{1+k^2}}{uk+\sqrt{1+k^2}}\right| + c, \; k, c\in\mathbb{R}$$

$$= -\frac{1}{\sqrt{1+k^2}}\ln\left|\frac{(s-\frac{1}{k})\cdot k-\sqrt{1+k^2}}{(s-\frac{1}{k})\cdot k+\sqrt{1+k^2}}\right| + c =$$

$$= -\frac{1}{\sqrt{1+k^2}}\cdot\ln\left|\frac{sk-1-\sqrt{1+k^2}}{sk-1+\sqrt{1+k^2}}\right| + c = \underbrace{\frac{-1}{\sqrt{1+k^2}}\ln\left|\frac{k\,tg\frac{x}{2}-1-\sqrt{1+k^2}}{k\,tg\frac{x}{2}-1+\sqrt{1+k^2}}\right|}_{E(x)} + c$$

Soluția: $E(x) \equiv t+c$, $k, c\in\mathbb{R}$