• Forma generală a soluției unei ecuații cvasiliniare

$$\sum_{k=1}^{n} a_k(x,u)\,\partial_k u = g(x,u)$$ → ecuație cvasiliniară cu derivate parțiale de ordinul întâi, unde $a_k, g: D_1 \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $k=\overline{1,n}$

este: $f(\varphi_1(x,u), \dots, \varphi_n(x,u)) = 0$

unde $\varphi_1, \varphi_2, \dots, \varphi_n$ sunt integrale prime independente ale sistemului caracteristic:

$$\frac{dx_1}{a_1(x,u)} = \frac{dx_2}{a_2(x,u)} = \dots = \frac{dx_n}{a_n(x,u)} = \frac{du}{g(x,u)}$$

1) Se cere forma generală a soluției pentru ecuațiile următoare:

a) $x_1^2\partial_1 u + x_2^2 \partial_2 u = 2(x_1+x_2)$.

$n=2$

$a_1(x,u) = x_1^2$

$a_2(x,u) = x_2^2$

$g(x,u) = 2(x_1+x_2)$

Sistemul caracteristic: $\dfrac{dx_1}{x_1^2} = \dfrac{dx_2}{x_2^2} = \dfrac{du}{2(x_1+x_2)}$

integralele prime se determină din 2 rapoarte din sistemul caracteristic sau din rapoarte care se obțin din sistemul caracteristic prin operații permise.

$$\Rightarrow \frac{dx_1}{x_1^2} = \frac{dx_2}{x_2^2} \Rightarrow \int \frac{dx_1}{x_1^2} = \int \frac{dx_2}{x_2^2} \Rightarrow \int \frac{1}{x_1^2}\,dx_1 = \int \frac{1}{x_2^2}\,dx_2 \Rightarrow$$

$$\Rightarrow -\frac{1}{x_1} = \frac{-1}{x_2} + c_1 \Rightarrow \frac{1}{x_2} - \frac{1}{x_1} = \underset{\text{const.}}{c_1} \Rightarrow \boxed{\varphi_1(x,u) = \frac{1}{x_2} - \frac{1}{x_1}}$$

$$\Rightarrow \frac{x_2\,dx_1}{x_1^2 x_2} = \frac{x_1\,dx_2}{x_2^2 x_1} = \frac{du}{2(x_1+x_2)} \Rightarrow \frac{x_2\,dx_1 + x_1\,dx_2}{x_1^2x_2 + x_2^2x_1} = \frac{du}{2(x_1+x_2)} \Rightarrow$$

$$\Rightarrow \frac{d(x_1x_2)}{x_1x_2\cdot\cancel{(x_1+x_2)}} = \frac{du}{2\cancel{(x_1+x_2)}} \Rightarrow du = 2\cdot\frac{d(x_1x_2)}{x_1x_2} \Rightarrow du = 2\frac{dy}{y} \Rightarrow$$

notăm $y = x_1x_2$

$$\Rightarrow \int du = \int 2\frac{dy}{y} \Rightarrow u = 2\cdot \ln|y| + c_2 \Rightarrow u - \ln y^2 = \underset{\text{const.}}{c_2}$$

$$\boxed{\varphi_2(x,u) = u - \ln(x_1x_2)^2}$$

Soluția generală în formă implicită este $f\left(\frac{1}{x_2} - \frac{1}{x_1},\ u - \ln(x_1x_2)^2\right) = 0$.

Exemple de soluții: fie $f(z_1,z_2) = z_1 + z_2 \Rightarrow \frac{1}{x_2} - \frac{1}{x_1} + u - \ln(x_1x_2)^2 = 0$.

$\Rightarrow u(x_1,x_2) = \ln(x_1x_2)^2 - \frac{1}{x_2} + \frac{1}{x_1}$

b) $x_1 \partial_1 u + x_2 \partial_2 u = (x_1 + x_2) u$

$n = 2$

$a_1(x,u) = x_1$

$a_2(x,u) = x_2$

$g(x,u) = (x_1 + x_2) \cdot u$

Sistemul caracteristic: $\dfrac{dx_1}{x_1} = \dfrac{dx_2}{x_2} = \dfrac{du}{(x_1+x_2)u}$.

$\to \dfrac{dx_1}{x_1} = \dfrac{dx_2}{x_2} \Rightarrow \int \dfrac{dx_1}{x_1} = \int \dfrac{dx_2}{x_2} \Rightarrow \ln|x_1| = \ln|x_2| + c_1 \Rightarrow e^{\ln|x_1|} = e^{\ln|x_2| + c_1} \Rightarrow$

$\Rightarrow |x_1| = |x_2| \cdot e^{c_1} \Rightarrow x_1 = \pm x_2 \cdot e^{c_1} \Rightarrow \dfrac{x_1}{x_2} = \underbrace{\pm e^{c_1}}_{\text{const.}}$

$$\boxed{\varphi_1(x,u) = \frac{x_1}{x_2}}$$

$\to \dfrac{dx_1}{x_1} = \dfrac{dx_2}{x_2} = \dfrac{du}{(x_1+x_2)\cdot u} \Rightarrow \dfrac{dx_1 + dx_2}{x_1 + x_2} = \dfrac{du}{(x_1+x_2)\cdot u} \Rightarrow \dfrac{d(x_1+x_2)}{\cancel{x_1+x_2}} = \dfrac{du}{\cancel{(x_1+x_2)}\cdot u}$

$\Rightarrow$ not. $x_1 + x_2 = y \Rightarrow \dfrac{du}{u} = dy \Rightarrow \int \dfrac{du}{u} = \int dy \Rightarrow \ln|u| = y + C_2 \Rightarrow$

$\Rightarrow \ln|u| - (x_1 + x_2) = \underbrace{C_2}_{\text{const.}} \Rightarrow$

$\Rightarrow \ln|u| - \ln e^{x_1+x_2} = C_2 \Rightarrow \ln \dfrac{|u|}{e^{x_1+x_2}} = C_2 \Rightarrow \dfrac{|u|}{e^{x_1+x_2}} = e^{C_2}$

$\Rightarrow \dfrac{u}{e^{x_1+x_2}} = \underbrace{\pm e^{C_2}}_{\text{const.}} \Rightarrow \boxed{\varphi_2(x,u) = \dfrac{u}{e^{x_1+x_2}}}$

Forma generală a soluției (forma implicită!) este: $f\left(\dfrac{x_1}{x_2}, \dfrac{u}{e^{x_1+x_2}}\right) = 0$.

• Problema Cauchy pentru ecuații cvasiliniare cu derivate parțiale de ord. I

$$\begin{cases} \sum_{k=1}^{n} a_k(x,u)\, \partial_k u = g(x,u) \\ u(x) = u_0(x) \text{ pe } S = \{x \in \mathbb{R}^n \mid h(x) = 0\}. \end{cases}$$

2) Să se determine soluțiile următoarelor probleme Cauchy

$$\begin{cases} (x_1 + 3x_2)\,\partial_1 u + (x_1 - x_2)\,\partial_2 u = u + x_1 + x_2 \\ u(x_1,x_2) = 3x_1 \text{ pe } S = \{x \in \mathbb{R}^2 \mid 2x_1 - x_2 = 0\}. \end{cases}$$

$n = 2$.

$a_1(x,u) = x_1 + 3x_2$

$a_2(x,u) = x_1 - x_2$

$g(x,u) = u + x_1 + x_2$

$u_0(x_1,x_2) = 3x_1$

$h(x) = 2x_1 - x_2 \Rightarrow x_2 = 2x_1$

Parametrizare pentru S:

$$S: \begin{cases} x_1 = \alpha_1(s) = s \\ x_2 = \alpha_2(s) = 2s \end{cases} \quad (\text{din } h).$$

$$\varphi(s) = u_0(\alpha_1(s), \alpha_2(s)) = u_0(s, 2s) = 3s$$

Condiția 1: $\operatorname{rang}\begin{pmatrix} \frac{\partial \alpha_1}{\partial s} \\ \frac{\partial \alpha_2}{\partial s} \end{pmatrix} = 1$

$$\left.\begin{array}{l} \frac{\partial \alpha_1}{\partial s} = 1 \\ \frac{\partial \alpha_2}{\partial s} = 2 \end{array}\right\} \Rightarrow \operatorname{rang}\begin{pmatrix} \frac{\partial \alpha_1}{\partial s} \\ \frac{\partial \alpha_2}{\partial s} \end{pmatrix} = \operatorname{rang}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \ \checkmark$$

Condiția 2: $\begin{vmatrix} a_1((\alpha_1(s), \alpha_2(s)), \varphi(s)) & \frac{\partial \alpha_1}{\partial s}(s) \\ a_2((\alpha_1(s), \alpha_2(s)), \varphi(s)) & \frac{\partial \alpha_2}{\partial s}(s) \end{vmatrix} \neq 0$

$$\begin{vmatrix} a_1((s, 2s), 3s) & 1 \\ a_2((s, 2s), 3s) & 2 \end{vmatrix} = \begin{vmatrix} s + 3\cdot 2s & 1 \\ s - 2s & 2 \end{vmatrix} = \begin{vmatrix} 7s & 1 \\ -s & 2 \end{vmatrix} = 14s + s = 15s \neq 0$$

dacă $\underline{\underline{s \neq 0}}$.

Sistemul caracteristic:

$$\begin{cases} \frac{dx_1}{dt} = x_1 + 3x_2 & (a_1(x, u)) \\ \frac{dx_2}{dt} = x_1 - x_2 & (a_2(x, u)) \\ \frac{du}{dt} = u + x_1 + x_2 & (g(x, u)) \\ x_1(0) = s & (\alpha_1(s)) \\ x_2(0) = 2s & (\alpha_2(s)) \\ u(0) = 3s & (\varphi(s)) \end{cases}$$

Rezolv: $\begin{cases} x_1' = x_1 + 3x_2 \\ x_2' = x_1 - x_2 \end{cases}$ cu condițiile inițiale $\begin{cases} x_1(0) = s \\ x_2(0) = 2s \end{cases}$

Forma matriceală: $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}}_{A \to \text{ constantă}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$x' = Ax.$$

Obs. $x' = Ax$ poate fi rezolvat cu metoda:

Dacă avem sistemul liniar $\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{cases}$, $a_{12} \neq 0$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

atunci x_1 este soluția ecuației diferențiale cu coeficienți constanți:

$$x_1'' = trA \cdot x_1' - \det A \cdot x_1$$

iar x_2 se determină din prima ecuație din sistem, $x_2 = \frac{x_1' - a_{11}x_1}{a_{12}}$, $a_{12} \neq 0$.

$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

$\det A = -1 - 3 = -4$

$tr A = 1 + (-1) = 0.$

$\Rightarrow x_1'' = 0 \cdot x_1' - (-4) \cdot x_1 \Rightarrow x_1'' = 4x_1 \Rightarrow x_1'' - 4x_1 = 0.$

Ecuația caracteristică atașată: $r^2 - 4 = 0 \Rightarrow r = \pm 2$

$r_1 = 2,\ m_1 = 1,\ \varphi_1(t) = e^{2t}$

$r_2 = -2,\ m_2 = 1,\ \varphi_2(t) = e^{-2t}$

$x_1(t) = c_1 \cdot e^{2t} + c_2 e^{-2t}$, $c_1, c_2 \in \mathbb{R}$.

$x_2 = \frac{x_1' - 1 \cdot x_1}{3} = \frac{1}{3} \cdot \left(c_1 \cdot e^{2t} \cdot 2 + c_2 \cdot e^{-2t} \cdot (-2) - (c_1 e^{2t} + c_2 e^{-2t}) \right)$

$\Rightarrow x_2(t) = \frac{1}{3} \cdot \left(c_1 \cdot e^{2t} - 3 \cdot c_2 e^{-2t} \right)$

$\begin{cases} x_1(0) = s \\ x_2(0) = 2s \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = s \\ \frac{1}{3}(c_1 - 3c_2) = 2s \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = s \\ c_1 - 3c_2 = 6s \end{cases}$ $-$

$4c_2 = -5s \Rightarrow c_2 = \frac{-5}{4}s$

$c_1 = s - c_2 = s + \frac{5}{4}s = \frac{9}{4}s.$

$\begin{cases} \tilde{x}_1(t,s) = \frac{9s}{4} \cdot e^{2t} - \frac{5s}{4} e^{-2t} = \frac{s}{4}\left(9e^{2t} - 5e^{-2t}\right) \\ \tilde{x}_2(t,s) = \frac{1}{3} \cdot \left(\frac{9s}{4} e^{2t} - 3 \cdot \frac{-5s}{4} e^{-2t} \right) = \frac{s}{12}\left(9e^{2t} + 15e^{-2t}\right) = \frac{s}{4}\left(3e^{2t} + 5e^{-2t}\right) \end{cases}$

$\frac{du}{dt} = u + x_1 + x_2 \Rightarrow \frac{du}{dt} = u + \frac{s}{4}\left(9e^{2t} - 5e^{-2t} + 3e^{2t} + 5e^{-2t}\right)$

$\Rightarrow \frac{du}{dt} = u + \frac{s}{4}\left(12e^{2t} + 0\right) \Rightarrow \frac{du}{dt} = u + \frac{s}{4} \cdot 12e^{2t} \Rightarrow \underbrace{\frac{du}{dt} = u + 3se^{2t}}_{\text{ec. afină}}$,

$\frac{du}{dt} = \underbrace{a(t)}_{1} \cdot u + \underbrace{b(t)}_{3se^{2t}}$

Căutăm o soluție particulară $u_0(t) = me^{2t} \Rightarrow (me^{2t})' = me^{2t} + 3se^{2t}$

$\Rightarrow me^{2t} \cdot 2 = me^{2t} + 3se^{2t} \Rightarrow me^{2t} = 3se^{2t} \Rightarrow m = 3s \Rightarrow$

$\Rightarrow u_0(t) = 3s\,e^{2t}$

$u(t) = \bar{u}(t) + \underbrace{3se^{2t}}_{u_0(t)}$, $\bar{u} \to$ sol. $\frac{d\bar{u}}{dt} = \bar{u} \Rightarrow$ $\bar{u}(t) = c\cdot e^{\int 1\,dt} \Rightarrow$

$\Rightarrow \bar{u}(t) = ce^{t}$

$u(t) = ce^{t} + 3se^{2t}$ $\qquad u(0) = c + 3s \Rightarrow c + 3s = 3s \Rightarrow c = 0.$

$u(0) = 3s$

$\Rightarrow \tilde{u}(t, s) = 3se^{2t}.$

Soluția parametrică : $\begin{cases} x_1 = \frac{s}{4}(9e^{2t} - 5e^{-2t}) \\ x_2 = \frac{s}{4}(3e^{2t} + 5e^{-2t}) \\ u = 3se^{2t} \end{cases}$

$x_1 + x_2 = \frac{s}{4}\cdot(9e^{2t} \cancel{-5e^{-2t}} + 3e^{2t} + \cancel{5e^{-2t}})$

$\Rightarrow x_1 + x_2 = \frac{s}{4}\cdot 12e^{2t} = 3se^{2t}$

$u = x_1 + x_2$

$\underline{u(x_1, x_2) = x_1 + x_2}.$

3) Să se determine $u: \mathbb{R}^2 \to \mathbb{R}$ a.î.

$$\begin{cases} (\partial_1 u)^2 - (\partial_2 u)^2 - 2u = 0 \\ u(x,y) = (x+y)^2 \quad \text{pe} \quad S = \{(x,y) \in \mathbb{R}^2 \mid x-1=0\} \end{cases}$$

$$F(x,y,u,\partial_1 u, \partial_2 u) = (\partial_1 u)^2 - (\partial_2 u)^2 - 2u$$

$$u_0(x,y) = (x+y)^2$$

$$F(x,y,u,p,q) = p^2 - q^2 - 2u$$

$$g(x,y) = x-1.$$

Pas 1

- $x = \alpha_1(s)$
  $y = \alpha_2(s)$
  $\varphi(s) = u_0(\alpha_1(s), \alpha_2(s))$

deci $S: g(x,y) = x-1 = 0 \Rightarrow \begin{cases} x = 1 = \alpha_1(s) \\ y = s = \alpha_2(s) \end{cases}$

$$\begin{cases} \alpha_1(s) = 1 \\ \alpha_2(s) = s \end{cases}, \ s \in \mathbb{R}$$

$$\varphi(s) = u_0(1,s) = (1+s)^2$$

Pas 2

- $\begin{cases} F(\alpha_1(s), \alpha_2(s), \varphi(s), \gamma_1, \gamma_2) = 0 \\ \gamma_1 \cdot \alpha_1'(s) + \gamma_2 \cdot \alpha_2'(s) = \varphi'(s) \end{cases} \Rightarrow \begin{cases} \gamma_1^2 - \gamma_2^2 - 2\cdot(1+s)^2 = 0 \\ \gamma_1 \cdot 0 + \gamma_2 \cdot 1 = 2(s+1) \Rightarrow \gamma_2 = 2(s+1) \end{cases}$

$$\Rightarrow \gamma_1^2 - 4(s+1)^2 - 2(1+s)^2 = 0$$

$$\gamma_1^2 = 6(1+s)^2 \Rightarrow \gamma_1 = \sqrt{6}(1+s) \quad \text{sau} \quad \gamma_1 = -\sqrt{6}(1+s)$$

Pas 3

- $F(x,y,u,p,q) = p^2 - q^2 - 2u = 0.$

$$\frac{\partial F}{\partial p} = 2p, \quad \frac{\partial F}{\partial q} = -2q, \quad \frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial u} = -2$$

Sistemul caracteristic:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial F}{\partial p} \\ \dfrac{dy}{dt} = \dfrac{\partial F}{\partial q} \\ \dfrac{dp}{dt} = -\dfrac{\partial F}{\partial x} - p\dfrac{\partial F}{\partial u} \\ \dfrac{dq}{dt} = -\dfrac{\partial F}{\partial y} - q\dfrac{\partial F}{\partial u} \\ \dfrac{du}{dt} = p\dfrac{\partial F}{\partial p} + q\dfrac{\partial F}{\partial q} \end{cases}$$

$$\frac{dx}{dt} = 2p \qquad x(0) = \alpha_1(s) = 1$$

$$\frac{dy}{dt} = -2q \qquad y(0) = \alpha_2(s) = s$$

$$\frac{dp}{dt} = -0 - p\cdot(-2) \qquad p(0) = \chi_1(s) = 2(1+s)$$

$$\frac{dq}{dt} = -0 - q\cdot(-2) \qquad q(0) = \chi_2(s) = \sqrt{6}(1+s)$$

$$\frac{du}{dt} = p\cdot 2p + q\cdot(-2q) \qquad u(0) = \varphi(s) = (1+s)^2$$

$$\underset{\text{ec. lin. omogenă.}}{\frac{dp}{dt} = 2p} \Rightarrow p(t) = c_1\cdot e^{2t},\ p(0) = c_1\cdot e^0 = 2(s+1) \Rightarrow c_1 = 2(s+1)$$

$$\underline{p(t,s) = 2(1+s)\cdot e^{2t}}$$

$$\frac{dx}{dt} = 2\cdot 2(1+s)\cdot e^{2t} \Rightarrow x = \int 4(s+1)\cdot e^{2t}\,dt = 4(s+1)\cdot\frac{e^{2t}}{2} + c_2$$

$$x(0) = 1 \Rightarrow 1 = 2(s+1)\,e^0 + c_2 \Rightarrow c_2 = 1 - 2(s+1)$$

$$x(t,s) = 2(s+1)\cdot e^{2t} + 1 - 2(s+1) = \underline{2(s+1)(e^{2t}-1) + 1}$$

$$\frac{dq}{dt} = 2q \Rightarrow q(t) = c_3\, e^{2t}$$

$$q(0) = \sqrt{6}(1+s) \Rightarrow c_3\cdot e^0 = \sqrt{6}(s+1) \Rightarrow c_3 = \sqrt{6}(s+1)$$

$$\underline{q(t,s) = \sqrt{6}\cdot(s+1)\,e^{2t}}$$

$$\frac{dy}{dt} = -2\cdot\sqrt{6}(s+1)\,e^{2t} \Rightarrow y = \int(-2\sqrt{6})(s+1)\,e^{2t}\,dt = -2\sqrt{6}(s+1)\int e^{2t}\,dt =$$

$$= -2\sqrt{6}(s+1)\cdot\frac{e^{2t}}{2} + c_4 = -\sqrt{6}(s+1)e^{2t} + c_4$$

$$y(0) = s \Rightarrow -\sqrt{6}(s+1) + c_4 = s \Rightarrow c_4 = s + \sqrt{6}(s+1)$$

$$\underline{y(t,s) = \sqrt{6}(s+1)(1-e^{2t}) + s}$$

$$\frac{du}{dt} = 2p^2 - 2q^2 \Rightarrow \frac{du}{dt} = 2\cdot 4(1+s)^2\cdot e^{4t} - 2\cdot 6(s+1)^2 e^{4t}$$

$$\frac{du}{dt} = -4(1+s)^2 e^{4t} \Rightarrow u = \int -4(s+1)^2 e^{4t}\,dt$$

$$= -4(s+1)^2\int e^{4t}\,dt = -4(s+1)^2\,\frac{e^{4t}}{4} + c_5 = -(s+1)^2 e^{4t} + c_5.$$

$$u(0) = -(s+1)^2 + c_5 = (s+1)^2 \Rightarrow c_5 = 2(s+1)^2$$

$u(t,s) = (s+1)^2(2-e^{4t})$

Soluția parametrică $\begin{cases} x(t,s) \overset{\circledast}{=} 2(s+1)(e^{2t}-1)+1 \;\Big|\cdot\sqrt{6} \\ y(t,s) = \sqrt{6}(s+1)(1-e^{2t})+s \;\Big|\cdot 2 \\ u(t,s) = (s+1)^2(2-e^{4t}) \end{cases} \overset{(+)}{\Rightarrow}$

$\Rightarrow x\sqrt{6}+2y = 2\sqrt{6}(s+1)(\cancel{e^{2t}}\cancel{-1}\cancel{+1}\cancel{-e^{2t}})+\sqrt{6}+2s$

$\Rightarrow x\sqrt{6}+2y = \sqrt{6}+2s \Rightarrow s = \dfrac{x\sqrt{6}+2y-\sqrt{6}}{2}$ → înlocuim în $\circledast$ →

$\Rightarrow x = 2\cdot\left(\dfrac{x\sqrt{6}+2y-\sqrt{6}}{2}+1\right)\cdot(e^{2t}-1)+1$

$\Rightarrow x-1 = (x\sqrt{6}+2y-\sqrt{6}+2)(e^{2t}-1) \Rightarrow$

$\Rightarrow e^{2t} = \dfrac{x(1+\sqrt{6})+2y-\sqrt{6}+1}{x\sqrt{6}+2y-\sqrt{6}+2}$ → înlocuim în u.

$u(x,y) = \left(\dfrac{x\sqrt{6}+2y-\sqrt{6}}{2}+1\right)^2\cdot\left(2-\left(\dfrac{x(1+\sqrt{6})+2y-\sqrt{6}+1}{x\sqrt{6}+2y-\sqrt{6}+2}\right)^2\right)$

$u(x,y) = \dfrac{(x\sqrt{6}+2y-\sqrt{6}+2)^2}{4}\cdot\left\{\dfrac{2(x\sqrt{6}+2y-\sqrt{6}+2)^2-(x(1+\sqrt{6})+2y-\sqrt{6}+1)^2}{(x\sqrt{6}+2y-\sqrt{6}+2)^2}\right\}$

$u(x,y) = \dfrac{(x\sqrt{12}+2\sqrt{2}y-\sqrt{12}+2\sqrt{2})^2-(x+x\sqrt{6}+2y-\sqrt{6}+1)^2}{4}$

$u(x,y) = \frac{1}{4}(12x^2+8y^2+12+8+8\sqrt{6}xy-24x+8\sqrt{6}x-8\sqrt{6}y+16y-8\sqrt{6}-$
$-x^2-6x^2-4y^2-6-1-2\sqrt{6}x^2-4xy+2x\sqrt{6}-2x-4\sqrt{6}xy+12x-2\sqrt{6}x+4\sqrt{6}y-$
$-4y+2\sqrt{6})$.

$u(x,y) = \frac{1}{4}(5x^2-2\sqrt{6}x^2+4y^2+13-6\sqrt{6}+4\sqrt{6}xy-4xy+8\sqrt{6}x-14x-4\sqrt{6}y+12y)$

Verificăm $u(x,y) = (x+y)^2$ pt. $x=1$.

$u(1,y) = \frac{1}{4}(5-\cancel{2\sqrt{6}}+4y^2+13-\cancel{6\sqrt{6}}+\cancel{4\sqrt{6}y}-4y+\cancel{8\sqrt{6}}-14-\cancel{4\sqrt{6}y}+12y)$
$= \frac{1}{4}(4y^2+8y+4) = y^2+2y+1 = (y+1)^2$ ✓