1) Fie sistemul $\begin{cases} x_1' = \frac{1}{t}x_1 + \frac{2}{t}x_2 \\ x_2' = -\frac{2}{t}x_1 - \frac{3}{t}x_2 + \ln t \end{cases}, t>0$.

a) Scrieți forma matriceală a sistemului.

b) Arătați că prin schimbarea de variabilă $t=e^s$ se obține un sistem cu coeficienți constanți pentru partea liniară.

c) Determinați soluția generală pentru sistemul de la b), apoi, pentru sistemul inițial determinați soluția generală și soluția care verifică: $\begin{cases} x_1(1)=2 \\ x_2(1)=1 \end{cases}$

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a) $x' = A(t)\cdot x + b(t)$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & \frac{2}{t} \\ -\frac{2}{t} & \frac{-3}{t} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \ln t \end{pmatrix}$$

(Obs2-curs 8) Dacă sistemul liniar omogen are $A(t)$ a.î. printr-o schimbare de variabilă să devină sistem cu coeficienți constanți, atunci se aplică metoda cu valori proprii, revenind apoi asupra schimbării de variabilă.

$$A(t) = \frac{1}{t}\cdot\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} = \frac{1}{t}\cdot B, \quad b(t) = \begin{pmatrix} 0 \\ \ln t \end{pmatrix}$$

$A : (0,\infty) \longrightarrow \mathcal{M}_2(\mathbb{R})$

$b : (0,\infty) \longrightarrow \mathbb{R}^2$

$$B = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$

b) $x' = \frac{1}{t}Bx + b(t)$

$(t,x) \xrightarrow{t=e^s} (s,y)$ $\qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$t = e^s \Rightarrow s = \ln t$

$x(t) = y(s(t))$

$s(t) = \ln t \Rightarrow s'(t) = \frac{1}{t}$

$x'(t) = (y(s(t)))' = y'(s(t))\cdot s'(t) = y'(s)\cdot\frac{1}{t} = \frac{1}{e^s}\cdot y'(s)$

Sistemul devine: $\frac{1}{e^s} y' = \frac{1}{e^s} By + b(e^s) \Big| \cdot e^s \Rightarrow y' = By + \underbrace{e^s \cdot b(e^s)}_{\tilde{b}(s)}$

$$\Rightarrow y' = By + \tilde{b}(s), \text{ cu } \tilde{b}(s) = e^s \cdot b(e^s) = \begin{pmatrix} e^s \cdot 0 \\ e^s \ln e^s \end{pmatrix} = \begin{pmatrix} 0 \\ s e^s \end{pmatrix}$$

c) Rezolvăm sistemul $y' = By + \tilde{b}(s)$, $B = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$

Rezolvăm sistemul liniar omogen atașat, $\bar{y}' = B\bar{y}$.

Valorile proprii ptr. matricea B: $\det(B - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -2 & -3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)(-3-\lambda) - 2\cdot(-2) = 0 \Rightarrow -3 + 3\lambda - \lambda + \lambda^2 + 4 = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda_1 = -1,\ m_1 = 2 \Rightarrow$$

determinăm $p_0, p_1 \in \mathbb{R}^2$, nu amândoi nuli a.î. $\varphi(s) = (p_0 + p_1 s) e^{-s}$ să verifice sistemul liniar omogen $\bar{y}' = B\bar{y}$.

$$\Rightarrow ((p_0 + p_1 s) e^{-s})' = B \cdot (p_0 + p_1 s) e^{-s}$$

$$\Rightarrow (p_0 + p_1 s)' \cdot e^{-s} + (p_0 + p_1 s) \cdot (e^{-s})' = B \cdot (p_0 + p_1 s) e^{-s}$$

$$\Rightarrow p_1 \cdot e^{-s} + (p_0 + p_1 s) \cdot (-e^{-s}) = B \cdot (p_0 + p_1 s) e^{-s} \Big| : e^{-s}$$

$$\Rightarrow p_1 + (p_0 + p_1 s) \cdot (-1) = B \cdot (p_0 + p_1 s)$$

$$\Rightarrow p_1 - p_0 - p_1 s = Bp_0 + Bp_1 s.$$

identificăm coeficienții puterilor lui s $\Rightarrow \begin{cases} p_1 - p_0 = Bp_0 \\ -p_1 = Bp_1 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} p_1 = p_0 + Bp_0 \\ 0_{\mathbb{R}^2} = Bp_1 + p_1 \end{cases} \Rightarrow \begin{cases} p_1 = (I_2 + B) p_0 \\ 0_{\mathbb{R}^2} = (B + I_2) \cdot p_1 \end{cases} \Big| \cdot (B + I_2) \text{ la stânga} \Rightarrow$$

$$\Rightarrow \underbrace{(B + I_2) p_1}_{0_{\mathbb{R}^2}} = (B + I_2)^2 p_0 \Rightarrow (B + I_2)^2 \cdot p_0 = 0_{\mathbb{R}^2} \Rightarrow$$

$p_0 \in \ker((B+I_2)^2) = \{v \in \mathbb{R}^2 \mid (B+I_2)^2 v = 0_{\mathbb{R}^2}\}.$

$$(B+I_2)^2 = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2$$

fie $p_0 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (A), $\forall\, v_1, v_2 \in \mathbb{R}$

$$\Rightarrow p_0 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\ker((B+I_2)^2) = \mathbb{R}^2 = \left\langle \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right\rangle$$

- $p_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow p_1 = (B+I_2) \cdot p_0 = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$$\Rightarrow \varphi_1(s) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot s \right) e^{-s} = \begin{pmatrix} 1+2s \\ -2s \end{pmatrix} \cdot e^{-s} = \begin{pmatrix} (1+2s)e^{-s} \\ -2s e^{-s} \end{pmatrix}$$

- $p_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow p_1 = (B+I_2) \cdot p_0 = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \Rightarrow$

$$\Rightarrow \varphi_2(s) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot s \right) e^{-s} = \begin{pmatrix} 2s \\ 1-2s \end{pmatrix} e^{-s} = \begin{pmatrix} 2s e^{-s} \\ (1-2s)e^{-s} \end{pmatrix}$$

Matricea fundamentală de soluții este: $\phi(s) = \begin{pmatrix} (1+2s)e^{-s} & 2s e^{-s} \\ -2s e^{-s} & (1-2s)e^{-s} \end{pmatrix}$

$$\Rightarrow \bar{y} = \phi(s) \cdot C, \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2$$

Aplicăm metoda variației constantelor, $y(s) = \phi(s) \cdot C(s)$ - soluție a sistemului afin $y' = By + \tilde{b}(s)$, unde $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} : (-\infty, \infty) \to \mathbb{R}^2$

$s = \ln t,\ t \in (0, +\infty)$

$$\Rightarrow (\phi(s) \cdot C(s))' = B \cdot \phi(s)\, C(s) + \tilde{b}(s)$$

$$\Rightarrow \cancel{\phi'(s) \cdot C(s)} + \phi(s) \cdot C'(s) = \cancel{B\phi(s)\, C(s)} + \tilde{b}(s), \quad \phi'(s) = B\phi(s)$$

$$\Rightarrow \phi(s) \cdot C'(s) = \tilde{b}(s) \Rightarrow$$

$$\Rightarrow \begin{pmatrix} (1+2s)e^{-s} & 2s e^{-s} \\ -2s e^{-s} & (1-2s)e^{-s} \end{pmatrix} \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ se^{s} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} (1+2s)e^{-s} \cdot c_1' + 2s e^{-s} \cdot c_2' = 0 \quad \Big| : e^{-s} \Rightarrow (1+2s)c_1' + 2s\, c_2' = 0 \\ -2s e^{-s} c_1' + (1-2s) e^{-s} c_2' = s e^{s} \end{cases} \quad (+)$$

$$\Downarrow \quad e^{-s} \cdot c_1' + e^{-s} c_2' = s e^{s} \Big/ \cdot e^{s} \Rightarrow c_1' + c_2' = s e^{2s} \Rightarrow c_2' = s e^{2s} - c_1' \Rightarrow$$

$$\Rightarrow (1+2s)c_1' + 2s(se^{2s} - c_1') = 0 \Rightarrow c_1' \cdot (1+2s-2s) + 2s^2 e^{2s} = 0$$

$$\Rightarrow c_1' = -2s^2 e^{2s} \Rightarrow c_1(s) = \int -2s^2 e^{2s}\, ds = \text{ integrăm prin părți,}$$

$$f(s) = -2s^2 \qquad f'(s) = -4s$$

$$g'(s) = e^{2s} \Rightarrow g(s) = \int e^{2s} ds = \frac{e^{2s}}{2}$$

$$c_1(s) = -2s^2 \cdot \frac{e^{2s}}{2} - \int (-4s) \cdot \frac{e^{2s}}{2}\, ds = -s^2 e^{2s} + 2\int s e^{2s}\, ds$$

integrăm prin părți, $f(s) = s \qquad f'(s) = 1$

$$g'(s) = e^{2s} \Rightarrow g(s) = \int e^{2s} = \frac{e^{2s}}{2}$$

$$c_1(s) = -s^2 e^{2s} + 2 \cdot \left(s \cdot \frac{e^{2s}}{2} - \int 1 \cdot \frac{e^{2s}}{2} ds \right) = -s^2 e^{2s} + s \cdot e^{2s} - 2 \cdot \frac{1}{2} \int e^{2s} ds =$$

$$= -s^2 e^{2s} + s e^{2s} - \int e^{2s} ds = -s^2 e^{2s} + s e^{2s} - \frac{e^{2s}}{2} + k_1,\ k_1 \in \mathbb{R}.$$

$$c_2' = s e^{2s} - c_1' = s e^{2s} + 2s^2 e^{2s}$$

$$c_2' = (s + 2s^2) e^{2s}$$

$$c_2(s) = \int (s+2s^2) e^{2s}\, ds = \text{ integrăm prin părți}$$

$$f(s) = s + 2s^2 \qquad f'(s) = 1 + 4s$$

$$g'(s) = e^{2s} \Rightarrow g(s) = \int e^{2s} ds = \frac{e^{2s}}{2}$$

$$c_2(s) = (s + 2s^2) \cdot \frac{e^{2s}}{2} - \int (1+4s) \cdot \frac{e^{2s}}{2}\, ds$$

→ integrăm prin părți

$f(s) = 1+4s \qquad f'(s) = 4$

$g'(s) = \dfrac{e^{2s}}{2} \quad \Rightarrow \quad g(s) = \displaystyle\int \frac{e^{2s}}{2}\,ds = \frac{e^{2s}}{4}.$

$$c_2(s) = (s+2s^2)\cdot\frac{e^{2s}}{2} - \left[(1+4s)\cdot\frac{e^{2s}}{4} - \int 4\cdot\frac{e^{2s}}{4}\,ds\right] \Rightarrow$$

$$c_2(s) = (s+2s^2)\cdot\frac{e^{2s}}{2} - (1+4s)\cdot\frac{e^{2s}}{4} + \int e^{2s}\,ds$$

$$c_2(s) = (s+2s^2)\cdot\frac{e^{2s}}{2} - (1+4s)\cdot\frac{e^{2s}}{4} + \frac{e^{2s}}{2} + k_2,\ k_2\in\mathbb{R}$$

$$c_2(s) = \frac{e^{2s}}{4}\cdot\left(2s+4s^2-1-4s+2\right)+k_2,\ k_2\in\mathbb{R}$$

$$c_2(s) = \frac{e^{2s}}{4}\cdot(4s^2-2s+1)+k_2,\ k_2\in\mathbb{R}$$

Soluția sistemului afin $y' = By+\tilde{b}(s)$ este:

$$y(s) = \phi(s)\cdot C(s) = \begin{pmatrix} (1+2s)e^{-s} & 2se^{-s} \\ -2se^{-s} & (1-2s)e^{-s} \end{pmatrix}\begin{pmatrix} \frac{e^{2s}}{2}\cdot(-2s^2+2s-1)+k_1 \\ \frac{e^{2s}}{4}\cdot(4s^2-2s+1)+k_2 \end{pmatrix}$$

$$y(s) = e^{-s}\cdot\begin{pmatrix} 1+2s & 2s \\ -2s & 1-2s \end{pmatrix}\cdot\begin{pmatrix} \frac{e^{2s}}{4}(-4s^2+4s-2)+k_1 \\ \frac{e^{2s}}{4}(4s^2-2s+1)+k_2 \end{pmatrix}$$

$$y(s) = e^{-s}\cdot\begin{pmatrix} 1+2s & 2s \\ -2s & 1-2s \end{pmatrix}\cdot\frac{e^{2s}}{4}\cdot\begin{pmatrix} -4s^2+4s-2 \\ 4s^2-2s+1 \end{pmatrix} + \underbrace{e^{-s}\cdot\begin{pmatrix} 1+2s & 2s \\ -2s & 1-2s \end{pmatrix}}_{\phi(s)}\cdot\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$y(s) = \frac{e^{s}}{4}\cdot\begin{pmatrix} \cancel{-4s^2}+\cancel{4s}-2\cancel{-8s^3}+\cancel{8s^2}\cancel{-4s}+\cancel{8s^3}\cancel{-4s^2}+2s \\ \cancel{8s^3}\cancel{-8s^2}+\cancel{4s}+\cancel{4s^2}\cancel{-2s}+1\cancel{-8s^3}+\cancel{4s^2}\cancel{-2s} \end{pmatrix} + \phi(s)\cdot\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$y(s) = \frac{e^{s}}{4}\cdot\begin{pmatrix} 2s-2 \\ 1 \end{pmatrix} + \phi(s)\cdot\begin{pmatrix} k_1 \\ k_2 \end{pmatrix},\ k_1,k_2\in\mathbb{R}$$

Revenim la schimbarea de variabilă,

$x(t) = y(\ln t) \Rightarrow$

$$\Rightarrow x(t) = \frac{e^{\ln t}}{4}\begin{pmatrix} 2\cdot\ln t-2 \\ 1 \end{pmatrix} + \phi(\ln t)\cdot\begin{pmatrix} k_1 \\ k_2 \end{pmatrix},\ k_1,k_2\in\mathbb{R}$$

$$x(t) = \frac{t}{4}\cdot\begin{pmatrix} 2\ln t-2 \\ 1 \end{pmatrix} + \begin{pmatrix} (1+2\ln t)\cdot e^{-\ln t} & 2\ln t\, e^{-\ln t} \\ -2\ln t\, e^{-\ln t} & (1-2\ln t)e^{-\ln t} \end{pmatrix}\cdot\begin{pmatrix} k_1 \\ k_2 \end{pmatrix},\ k_1,k_2\in\mathbb{R}$$

$$e^{-\ln t} = (e^{\ln t})^{-1} = t^{-1} = \frac{1}{t}$$

$$\Rightarrow x(t) = \frac{t}{4}\begin{pmatrix} 2\ln t - 2 \\ 1 \end{pmatrix} + \begin{pmatrix} (1+2\ln t)\cdot\frac{1}{t} & \frac{2\ln t}{t} \\ -\frac{\ln t}{t} & \frac{1-2\ln t}{t}\end{pmatrix}\cdot\begin{pmatrix}k_1\\k_2\end{pmatrix}, \quad k_1, k_2 \in \mathbb{R}$$

$$\begin{cases} x_1(1) = 2 \\ x_2(1) = 1 \end{cases} \Rightarrow x(1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x(1) = \frac{1}{4}\cdot\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix} -\frac{2}{4} \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + k_1 \\ \frac{1}{4} + k_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -\frac{1}{2} + k_1 = 2 \\ \frac{1}{4} + k_2 = 1 \end{cases} \Rightarrow \begin{array}{l} k_1 = \frac{1}{2} + 2 = \frac{5}{2} \\ k_2 = 1 - \frac{1}{4} = \frac{3}{4} \end{array}$$

$$x(t) = \frac{t}{4}\begin{pmatrix} 2\ln t - 2 \\ 1 \end{pmatrix} + \begin{pmatrix} (1+2\ln t)\cdot\frac{1}{t} & \frac{2\ln t}{t} \\ -\frac{\ln t}{t} & \frac{1-2\ln t}{t}\end{pmatrix}\cdot\begin{pmatrix}\frac{5}{2}\\ \frac{3}{4}\end{pmatrix}$$