1) Să se rezolve ecuația diferențială:

$$\frac{dx}{dt} = \frac{2t(x^2+5x+6)}{t^2+4}, \quad x \in \mathbb{R}, t \in \mathbb{R}.$$

$a:\mathbb{R}\to\mathbb{R},\ a(t)=\frac{2t}{t^2+4},\quad b:\mathbb{R}\to\mathbb{R},\ b(x)=x^2+5x+6$

Este o ec. dif. cu variabile separabile cu soluțiile staționare date de:

$x^2+5x+6=0 \Rightarrow x_1=-2 \qquad x_1(t)=-2,\ t\in\mathbb{R},$

$x_2=-3. \qquad x_2(t)=-3,\ t\in\mathbb{R}$

Separăm variabilele: $\frac{dx}{x^2+5x+6} = \frac{2t}{t^2+4}dt.$

integrăm: $\int\frac{dx}{x^2+5x+6} = \int\frac{2t}{t^2+4}dt$

$$\int\frac{1}{x^2+5x+6}dx = \int\frac{1}{(x+3)(x+2)}dx = \int\frac{(x+3)-(x+2)}{(x+3)(x+2)}dx = \int\frac{1}{x+2}dx - \int\frac{1}{x+3}dx$$

$$= \ln|x+2| - \ln|x+3| + c_1,\ c_1\in\mathbb{R}$$

$$\int\frac{2t}{t^2+4}dt = \ln(t^2+4) + c_2,\ c_2\in\mathbb{R}$$

deci $\ln|x+2|-\ln|x+3|+c_1 = \ln(t^2+4)+c_2$, not. $c_2-c_1=k\in\mathbb{R}$

$$\ln\left|\frac{x+2}{x+3}\right| = \ln(t^2+4)+\underbrace{k}_{\ln c,\ c\in\mathbb{R}_+^*},\ k\in\mathbb{R}$$

$$\ln\left|\frac{x+2}{x+3}\right| = \ln(t^2+4)+\ln c,\ c\in\mathbb{R}_+^* \Rightarrow \ln\left|\frac{x+2}{x+3}\right| = \ln\left[(t^2+4)\cdot c\right],\ c\in\mathbb{R}_+^* \Rightarrow$$

$$\Rightarrow \left|\frac{x+2}{x+3}\right| = c(t^2+4),\ c\in\mathbb{R}_+^*$$

$$\Rightarrow \frac{x+2}{x+3} = \underbrace{\pm c}(t^2+4),\ c\in\mathbb{R}_+^* \Rightarrow \frac{x+2}{x+3} = c(t^2+4),\ c\in\mathbb{R}^*.$$

$x+2=(x+3)\cdot c(t^2+4),\ c\in\mathbb{R}^*$

$x(1-c(t^2+4)) = 3c(t^2+4)-2,\ c\in\mathbb{R}^*$

$$x=\frac{3c(t^2+4)-2}{1-c(t^2+4)},\ c\in\mathbb{R}^*$$

$$\begin{cases} x_1(t) = -2 \\ x_2(t) = -3 \\ x_k(t) = \dfrac{3k(t^2+4)-2}{1-k(t^2+4)}, \quad k \in \mathbb{R}^* \end{cases}$$

2) $\dfrac{dx}{dt} = \dfrac{2t\,x\ln x}{(t^2+1)\ln(\ln x)}, \quad x \in (3,+\infty), t \in \mathbb{R}$

$a : \mathbb{R} \to \mathbb{R},\ a(t) = \dfrac{2t}{t^2+1}, \quad b : (3,+\infty) \to \mathbb{R},\ b(x) = \dfrac{x\ln x}{\ln(\ln x)}$

Este o ecuație cu variabile separabile.

Soluțiile staționare: $\dfrac{x\ln x}{\ln(\ln x)} = 0 \Rightarrow x\ln x = 0 \Rightarrow x = 0 < 3$ (F)

$\ln x > \ln 3 > \ln 1 = 0$

Separăm variabilele: $\dfrac{\ln(\ln x)}{x\ln x}dx = \dfrac{2t}{t^2+1}dt$.

integrăm: $\displaystyle\int \frac{\ln(\ln x)}{x\ln x}dx = \int \frac{2t}{t^2+1}dt$.

$\displaystyle\int \frac{\ln(\ln x)}{x\ln x}dx = \int \frac{\ln s}{s}ds = \int p\,dp = \frac{p^2}{2} + c = \frac{\ln^2 s}{2} + c = \frac{\ln^2(\ln x)}{2} + c,$ $c \in \mathbb{R}$

sch. var. $s = \ln x \Rightarrow ds = \frac{1}{x}dx$

sch var $p = \ln s \Rightarrow dp = \frac{1}{s}ds$

$\displaystyle\int \frac{2t}{t^2+1}dt = \int \frac{1}{s}ds = \ln s = \ln(t^2+1) + c, \quad c \in \mathbb{R}$

$s = t^2+1 \Rightarrow ds = 2t\,dt$

$\Rightarrow \dfrac{\ln^2(\ln x)}{2} = \ln(t^2+1) + c, \ c \in \mathbb{R}$

$\Rightarrow \ln^2(\ln x) = 2\ln(t^2+1) + c$

$\Rightarrow \ln(\ln x) = \sqrt{2\ln(t^2+1)+c} \quad \left(\begin{array}{l} \ln x > \ln 3 > \ln e = 1 \\ \ln(\ln x) > \ln 1 = 0 \end{array}\right)$

$\ln x = e^{\sqrt{2\ln(t^2+1)+c}}$

$x = e^{e^{\sqrt{2\ln(t^2+1)+c}}}$

$x_c(t) = e^{e^{\sqrt{2\ln(t^2+1)+c}}}, \quad c \in \mathbb{R}.$

3) $\frac{dx}{dt} = x \cdot tg t + \cos t$, $t \in (0, \frac{\pi}{2})$, $x \in \mathbb{R}$.

Este o ecuație afină, $a(t) = tg t$, $b(t) = \cos t$, $a, b : (0, \frac{\pi}{2}) \to \mathbb{R}$ continue

Folosim metoda variației constantelor.

Scriem ecuația liniară asociată: $\frac{d\bar{x}}{dt} = \bar{x} \cdot tg t$, cu soluția:

$\bar{x}(t) = c \cdot e^{\int a(t) dt} = c \cdot e^{\int tg t \, dt} = c \cdot e^{-\ln|\cos t|} = c \cdot \left(e^{\ln|\cos t|}\right)^{-1} =$

$= c \cdot |\cos t|^{-1} = c \cdot \frac{1}{|\cos t|}$, $c \in \mathbb{R}$

$t \in (0, \frac{\pi}{2}) \Rightarrow \bar{x}(t) = c \cdot \frac{1}{\cos t}$, $c \in \mathbb{R}$.

Căutăm soluții de forma $x(t) = c(t) \cdot \frac{1}{\cos t}$.

$\left(c(t) \cdot \frac{1}{\cos t}\right)' = c(t) \cdot \frac{1}{\cos t} \cdot tg t + \cos t$

$\Rightarrow c'(t) \cdot \frac{1}{\cos t} + \cancel{c(t) \cdot \frac{-1}{\cos^2 t} (-\sin t)} = \cancel{c(t) \cdot \frac{1}{\cos t} \cdot \frac{\sin t}{\cos t}} + \cos t$

$\Rightarrow c'(t) \cdot \frac{1}{\cos t} = \cos t \Rightarrow c'(t) = \cos^2 t$

$c(t) = \int \cos^2 t \, dt = \int \frac{1}{2}(1 + \cos 2t) \, dt = \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) + k$, $k \in \mathbb{R}$

$\cos 2t = 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{1 + \cos 2t}{2}$

$x(t) = \left(\frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) + k\right) \cdot \frac{1}{\cos t}$, $k \in \mathbb{R}$

4) $x' + x^2 - 2x \sin t + \sin^2 t - \cos t = 0$, $\varphi_0(t) = \sin t$.

Este o ec. Riccati, $x' = -x^2 + 2x \sin t + \cos t - \sin^2 t$

$a(t) = -1$, $b(t) = 2\sin t$, $c(t) = \cos t - \sin^2 t$, $a, b, c : \mathbb{R} \to \mathbb{R}$ cont.

Facem schimbarea de variabilă, $y(t) = x(t) - \varphi_0(t)$

$y(t) = x(t) - \sin t \Rightarrow x(t) = y(t) + \sin t$.

$(y(t)+\sin t)' + (y(t)+\sin t)^2 - 2(y(t)+\sin t)\cdot \sin t + \sin^2 t - \cos t = 0$

$\Rightarrow y'(t) + \cancel{\cos t} + y^2(t) + \cancel{2\cdot y(t)\cdot \sin t} + \cancel{\sin^2 t} - \cancel{2y(t)\cdot \sin t} - \cancel{2\sin^2 t} + \cancel{\sin^2 t} - \cancel{\cos t} = 0$

$\Rightarrow y'(t) + y^2(t) = 0 \Rightarrow y'(t) = -y^2(t) \rightarrow$ este o ec. cu variabile separabile

cu sol. staționare date de : $-y^2(t) = 0 \Rightarrow y_1(t) \equiv 0$.

$$\frac{dy}{dt} = -y^2$$

Separăm variabilele : $\frac{dy}{-y^2} = dt$

Integrăm : $\int \frac{dy}{-y^2} = \int dt \Rightarrow \frac{1}{y} = t+k, k \in \mathbb{R} \Rightarrow$

$$y(t) = \frac{1}{t+k}, k \in \mathbb{R}$$

$$\begin{cases} x_1(t) = \sin t \\ x_k(t) = \sin t + \frac{1}{t+k}, k \in \mathbb{R} \end{cases}$$