Ecuații diferențiale liniare de ordin n

1) Să se determine soluția generală a ecuației:

$$x'' - 2x' + x = 2te^t$$

Ec. liniară neomogenă cu coef. constanți, $x^{(2)} = a_0 x + a_1 x^{(1)} + g(t)$

$n = 2$, $a_0 = -1$, $a_1 = 2$, $g(t) = 2te^t$

Ec. liniară omogenă atașată: $\bar{x}'' - 2\bar{x}' + \bar{x} = 0$ (cu coef. ct.)

Scriem ec. caracteristică: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = 1, m_1 = 2$ 2 sol.

$\varphi_1(t) = e^{r_1 t} = e^t$

$\varphi_2(t) = te^{r_1 t} = te^t$

Sistemul fundamental de soluții este $\{e^t, te^t\}$

Soluția ec. lin. omogene este: $\bar{x}(t) = c_1 e^t + c_2 te^t$, $c_1, c_2 \in \mathbb{R}$

Aplicăm metoda variației constantelor: $x(t) = c_1(t) e^t + c_2(t) te^t$ - sol. ec. liniare neomogene (afine) și determinăm $c_1, c_2 : \mathbb{R} \to \mathbb{R}$.

Sistemul afin asociat ec. afine: $\begin{cases} c_1' \cdot \varphi_1(t) + c_2' \varphi_2(t) = 0 \\ c_1' \varphi_1'(t) + c_2' \varphi_2'(t) = \underbrace{2te^t}_{g(t)} \end{cases}$

$\Rightarrow \begin{cases} c_1' \cdot e^t + c_2' \cdot te^t = 0 \\ c_1' (e^t)' + c_2' \cdot (te^t)' = 2te^t \end{cases} \Rightarrow \begin{cases} c_1' e^t + c_2' te^t = 0 \\ c_1' e^t + c_2' (e^t + te^t) = 2te^t \end{cases}$ $(-)$

$c_2' \cdot (\cancel{te^t} - e^t - \cancel{te^t}) = -2te^t$

$\Rightarrow c_2' \cdot (-e^t) = -2te^t \Rightarrow c_2' = +2t \Rightarrow c_2(t) = \int 2t\, dt = t^2 + k_2$, $k_2 \in \mathbb{R}$

$c_1' \cancel{e^t} + 2t \cdot t\cancel{e^t} = 0 \Rightarrow c_1' = -2t^2 \Rightarrow c_1(t) = \int -2t^2\, dt = -2\frac{t^3}{3} + k_1$, $k_1 \in \mathbb{R}$

Soluția generală a ecuației afine este:

$$x(t) = c_1(t) \cdot \varphi_1(t) + c_2(t) \cdot \varphi_2(t)$$

$$\Rightarrow x(t) = \left(-\frac{2t^3}{3} + k_1\right) \cdot e^t + (t^2 + k_2) te^t =$$

$$= \underbrace{k_1 e^t + k_2 te^t}_{\bar{x}(t)} + \underbrace{\frac{-2t^3}{3} e^t + t^3 e^t}_{\varphi_0(t)}$$

(Soluția ec. liniare omogene) (soluție particulară)

Dacă ni se dă $\varphi_0$, atunci $x(t) = \bar{x}(t) + \varphi_0(t)$ este soluția ec. afine.

4) $x^{(3)} = -3x^{(2)} - 3x^{(1)} - x$ → ecuație liniară omogenă cu coeficienți constanți.

Scriem ecuația caracteristică, $r^3 = -3r^2 - 3r - 1$

$\Rightarrow r^3 + 3r^2 + 3r + 1 = 0 \Rightarrow (r+1)^3 = 0 \Rightarrow r_1 = -1, m_1 = 3 \ (=n)$

$\varphi_1(t) = e^{-t} \quad (e^{r_1 t})$

$\varphi_2(t) = te^{-t} \quad (te^{r_1 t})$

$\varphi_3(t) = t^2 e^{-t} \quad (t^2 e^{r_1 t})$

$\Rightarrow x(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}, \ c_1, c_2, c_3 \in \mathbb{R}$.

3) $x^{(5)} + 8x^{(3)} + 16x^{(1)} = 32$.

$n = 5$, $g(t) = 32 = \text{const}$; $x^{(0)}$ nu apare

Căutăm o sol. $\varphi_0(t) = \alpha t, \alpha \in \mathbb{R}$.

$\varphi_0'(t) = \varphi_0^{(1)}(t) = (\alpha t)' = \alpha$

$\varphi_0''(t) = \varphi_0^{(2)}(t) = (\alpha)' = 0, \quad \varphi_0^{(3)}(t) = \varphi_0^{(4)}(t) = \dots = 0$.

$\varphi_0$ - sol. $\Rightarrow \varphi_0^{(5)}(t) + 8\varphi_0^{(3)}(t) + 16\varphi_0'(t) = 32$

$0 + 8 \cdot 0 + 16 \cdot \alpha = 32 \Rightarrow \alpha = 2 \Rightarrow \varphi_0(t) = 2t$

Deci $x(t) = \bar{x}(t) + 2t = \bar{x}(t) + 2t$, cu $\bar{x}$ sol. a ec. liniare omogene $x^{(5)} + 8x^{(3)} + 16x^{(1)} = 0$

Ec. caracteristică: $r^5 + 8r^3 + 16r = 0$.

$\Rightarrow r(r^4 + 8r^2 + 16) = 0 \Rightarrow \quad r_1 = 0, m_1 = 1 \Rightarrow \varphi_1(t) = e^{0t} = 1$.

sau $r^4 + 8r^2 + 16 = 0 \Rightarrow$

$(r^2+4)^2 = 0 \quad \Rightarrow r^2 = -4 \Rightarrow \quad r_2 = 2i, m_2 = 2$

$r_3 = -2i = \bar{r}_2$, $m_3 = 2$.

Verific $n = \sum m_i$

$5 = 1+2+2$ ✓

• $r_2 = 2i, m_2 = 2, r_3 = \bar{r}_2 \Rightarrow \varphi_2(t) = \text{Re}(e^{2it}) = \text{Re}(\cos 2t + i\sin 2t) = \cos 2t$

$\varphi_3(t) = \text{Im}(e^{2it}) = \text{Im}(\cos 2t + i\sin 2t) = \sin 2t$

$\varphi_4(t) = \text{Re}(t \cdot e^{2it}) = t\cos 2t$

$\varphi_5(t) = \text{Im}(t \cdot e^{2it}) = t \cdot \sin 2t$

Sistem fundamental de soluții ptr. ec. liniară omogenă:

$\{1, \cos 2t, \sin 2t, t\cdot\cos 2t, t\cdot\sin 2t\}$.

Sol. ec. lin. omogene este $\bar{x}(t) = c_1 \cdot 1 + c_2 \cdot \cos 2t + c_3 \cdot \sin 2t + c_4 t \cdot \cos 2t + c_5 t \sin 2t$, $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$

$$x(t) = \bar{x}(t) + \varphi_0(t) = \underbrace{c_1 + c_2 \cos 2t + c_3 \cdot \sin 2t + c_4 t \cdot \cos 2t + c_5 t \sin 2t}_{\bar{x}} + \underbrace{2t}_{\varphi_0}$$

Soluția ec. afine.

4) $(2t+3)^3 x^{(3)} + 4(2t+3)^2 x^{(2)} + 4(2t+3)x^{(1)} - 8x = 8(2t+3)^3$

este ecuație Euler generală, $(\alpha t+\beta)^n x^{(n)} = \sum_{k=0}^{n-1} \alpha_k (\alpha t+\beta)^k \cdot x^{(k)} + g(t)$

$n = 3$, $\alpha = 2$, $\beta = 3$, $\alpha_0 = 8$, $\alpha_1 = -4$, $\alpha_2 = -4$, $g(t) = 8 \cdot (2t+3)^3$.

$g : (-\frac{3}{2}, +\infty) \to \mathbb{R}$

Schimbarea de variabilă $|\alpha t + \beta| = e^s \Rightarrow |2t+3| = e^s$

$(t,x) \longrightarrow (s,y)$

$2t+3 = e^s$, $t > -\frac{3}{2}$

$s = \ln(2t+3)$

$x(t) = y(s(t))$

$s'(t) = (\ln(2t+3))' = \frac{1}{2t+3} \cdot (2t+3)' = \frac{2}{2t+3}$.

$x'(t) = (y(s(t)))' = y'(s(t)) \cdot s'(t) \Rightarrow x' = y'(s) \cdot \frac{2}{2t+3} \Rightarrow (2t+3)x' = 2y'$

$x''(t) = \left(y'(s) \cdot \frac{2}{2t+3}\right)' = y''(s) \cdot s'(t) \cdot \frac{2}{2t+3} + y'(s) \cdot \left(\frac{2}{2t+3}\right)' =$

$= y''(s) \cdot \frac{4}{(2t+3)^2} + y'(s) \cdot \frac{-4}{(2t+3)^2} = \frac{4}{(2t+3)^2}(y''(s) + y'(s)) \Rightarrow$

$(2t+3)^2 \cdot x'' = 4(y'' - y')$

$x'''(t) = \left(\frac{4}{(2t+3)^2} \underbrace{(y''(s) - y'(s))}\right)' =$

$= \frac{4' \cdot (2t+3)^2 - 4 \cdot ((2t+3)^2)'}{(2t+3)^4} \cdot (y''(s) - y'(s)) + \frac{4}{(2t+3)^2} \cdot (y''(s) - y'(s))' =$

$= \frac{-4 \cdot 2(2t+3) \cdot 2}{(2t+3)^4} \cdot (y''(s) - y'(s)) + \frac{4}{(2t+3)^2} \cdot (y'''(s) \cdot s'(t) - y''(s) \cdot s'(t)) =$

$= \frac{-16}{(2t+3)^3} \cdot (y''(s) - y'(s)) + \frac{4}{(2t+3)^2} \cdot \frac{2}{2t+3} \cdot (y'''(s) - y''(s)) =$

$= \frac{-16}{(2t+3)^3} \cdot (y''(s) - y'(s)) + \frac{8}{(2t+3)^3} \cdot (y'''(s) - y''(s))$

$\Rightarrow (2t+3)^3 x''' = 8(y''' - y'' - 2y'' + 2y')$

$\Rightarrow (2t+3)^3 x''' = 8\cdot(y''' - 3y'' + 2y')$

Ecuația devine:

$8(y''' - 3y'' + 2y') + 4\cdot 4\cdot(y'' - y') + 4\cdot 2y' - 8y = 8(e^s)^3 \,/ :8$

$y''' - 3y'' + 2y' + 2y'' - 2y' + y' - y = e^{3s}$

$y''' - y'' + y' - y = e^{3s}$ (*)

căutăm sol. particulară, $\varphi_0(s) = \alpha\cdot e^{3s}, \alpha \in \mathbb{R}$,

$\varphi_0'(s) = \alpha\cdot e^{3s}\cdot 3 = 3\alpha e^{3s}$

$\varphi_0''(s) = (3\alpha e^{3s})' = 3\alpha(e^{3s})' = 3\alpha\cdot e^{3s}(3s)' = 9\alpha e^{3s}$

$\varphi_0'''(s) = (9\alpha e^{3s})' = 9\alpha(e^{3s})' = 9\alpha\cdot e^{3s}(3s)' = 9\alpha\cdot e^{3s}\cdot 3 = 27\alpha e^{3s}$

$\varphi_0$ - sol. $\Rightarrow 27\alpha e^{3s} - 9\alpha e^{3s} + 3\alpha e^{3s} - \alpha e^{3s} = e^{3s} \,/ : e^{3s} \Rightarrow$

$\Rightarrow 27\alpha - 9\alpha + 3\alpha - \alpha = 1$

$\Rightarrow 20\alpha = 1 \Rightarrow \alpha = \frac{1}{20}$, deci $\varphi_0(s) = \frac{1}{20}e^{3s}$.

Soluția ecuației (*) este $y(s) = \bar{y}(s) + \varphi_0(s)$, unde $\bar{y}$ e sol. ec. liniare omogene:

$\bar{y}''' - \bar{y}'' + \bar{y}' - \bar{y} = 0.$

Scriem ec. caracteristică: $r^3 - r^2 + r - 1 = 0 \Rightarrow r^2(r-1) + (r-1) = 0$

$\Rightarrow (r^2+1)(r-1) = 0 \Rightarrow r_1 = 1, m_1 = 1$

$r_2 = i, m_2 = 1$

$r_3 = -i, m_3 = 1$

$r_1 = 1, m_1 = 1, \quad \varphi_1(s) = e^{r_1 s} = e^s$

$r_2 = i, m_2 = 1, r_3 = \bar{r}_2 \Rightarrow \varphi_2(s) = \mathrm{Re}(e^{is}) = \mathrm{Re}(\cos s + i\sin s) = \cos s$

$\varphi_3(s) = \mathrm{Im}(e^{is}) = \mathrm{Im}(\cos s + i\sin s) = \sin s$

$\Rightarrow \bar{y}(s) = c_1 e^s + c_2\cdot\cos s + c_3\cdot\sin s, \quad c_1, c_2, c_3 \in \mathbb{R}$

$y(s) = \frac{1}{20}e^{3s} + c_1 e^s + c_2\cdot\cos s + c_3\cdot\sin s, \quad c_1, c_2, c_3 \in \mathbb{R}$

$x(t) = y(s(t))$

$\Rightarrow x(t) = y(\ln(2t+3)) \Rightarrow x(t) = \frac{1}{20}\cdot(2t+3)^3 + c_1\cdot(2t+3) +$

$+ c_2\cdot\cos(\ln(2t+3)) + c_3\cdot\sin(\ln(2t+3)), \quad c_1, c_2, c_3 \in \mathbb{R}$

b) Aflați soluția care verifică $\begin{cases} x(-1)=1 \\ x'(-1)=2 \\ x''(-1)=0 \end{cases}$

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$x(-1)=1$

$x(-1) = \frac{1}{20}\cdot(-2+3)^3 + c_1\cdot(-2+3) + c_2\cdot\cos(\ln(-2+3)) + c_3\cdot\sin(\ln(-2+3))$

$x(-1) = \frac{1}{20}\cdot 1 + c_1\cdot 1 + c_2\cdot\cos(\ln 1) + c_3\cdot\sin(\ln 1)$

$x(-1) = \frac{1}{20} + c_1 + c_2\cdot\underbrace{\cos 0}_{1} + c_3\underbrace{\sin 0}_{0}$

$x(-1) = \frac{1}{20} + c_1 + c_2 = 1. \quad (1)$

$x'(t) = \frac{1}{20}\cdot((2t+3)^3)' + c_1\cdot(2t+3)' + c_2\cdot\cos(\ln(2t+3))' + c_3\cdot\sin(\ln(2t+3))'$

$x'(t) = \frac{1}{20}\cdot 3\cdot(2t+3)^2\cdot(2t+3)' + c_1\cdot 2 + c_2\cdot(-\sin(\ln(2t+3)))\cdot(\ln(2t+3))'$
$+ c_3\cdot\cos(\ln(2t+3))\cdot(\ln(2t+3))'$

$x'(t) = \frac{3}{\cancel{20}_{10}}\cdot(2t+3)^2\cdot\cancel{2} + 2c_1 - c_2\cdot\sin(\ln(2t+3))\cdot\frac{1}{2t+3}\cdot(2t+3)' +$
$+ c_3\cdot\cos(\ln(2t+3))\cdot\frac{1}{2t+3}\cdot(2t+3)'$

$x'(t) = \frac{3}{10}\cdot(2t+3)^2 + 2c_1 - \frac{2\cdot c_2}{2t+3}\cdot\sin(\ln(2t+3)) + \frac{2c_3}{2t+3}\cdot\cos(\ln(2t+3))$

$x'(-1) = \frac{3}{10}\cdot 1 + 2c_1 - \frac{2c_2}{1}\cdot\sin(\ln 1) + \frac{2c_3}{1}\cdot\cos(\ln 1)$

$x'(-1) = \frac{3}{10} + 2c_1 - 2c_2\underbrace{\sin 0}_{0} + 2c_3\cdot\underbrace{\cos 0}_{1}$

$x'(-1) = \frac{3}{10} + 2c_1 + 2c_3 = -1 \quad (2)$

$x''(t) = \frac{3}{10}\cdot 2\cdot(2t+3)\cdot(2t+3)' + 0 - \frac{0 - 2c_2\cdot(2t+3)'}{(2t+3)^2}\cdot\sin(\ln(2t+3)) -$

$-\frac{2c_2}{2t+3}\cdot\cos(\ln(2t+3))\cdot(\ln(2t+3))' + \frac{0 - 2c_3(2t+3)'}{(2t+3)^2}\cdot\cos(\ln(2t+3)) +$

$+\frac{2c_3}{2t+3}\cdot(-\sin(\ln(2t+3)))\cdot(\ln(2t+3))' =$

$= \frac{12}{10}\cdot(2t+3) + \frac{4c_2}{(2t+3)^2}\cdot\sin(\ln(2t+3)) - \frac{2c_2}{2t+3}\cdot\cos(\ln(2t+3))\cdot\frac{2}{2t+3}$

$-\frac{4c_3}{(2t+3)^2}\cdot\cos(\ln(2t+3)) - \frac{4c_3}{(2t+3)^2}\cdot(-\sin(\ln(2t+3)))$

$$x''(-1) = \frac{12}{10} + 4c_2 \cdot \sin(\ln 1) - 2c_2 \cdot \cos(\ln 1) \cdot 2 - 4c_3 \cdot \cos(\ln 1)$$
$$+ 4c_3 \sin(\ln 1) =$$

$$= \frac{12}{10} + 4c_2 \cdot \sin 0 - 2c_2 \cdot \cos 0 \cdot 2 - 4c_3 \cdot \cos 0 + 4c_3 \cdot \sin 0 =$$
$$= \frac{12}{10} + 0 - 4c_2 - 4c_3 = \frac{12}{10} - 4c_2 - 4c_3 = 0 \quad (3)$$

$$\begin{cases} c_1 + c_2 = \frac{19}{20} \\ 2c_1 + 2c_3 = -\frac{13}{10} \\ 4c_2 + 4c_3 = \frac{12}{10} \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = \frac{19}{20} \\ c_1 + c_3 = -\frac{13}{20} \\ c_2 + c_3 = \frac{3}{10} \end{cases}$$

$$2(c_1 + c_2 + c_3) = \frac{12}{20} \Rightarrow c_1 + c_2 + c_3 = \frac{6}{20}.$$

$$(1) \Rightarrow c_3 = \frac{-13}{20}$$
$$(2) \Rightarrow c_2 = \frac{19}{20}$$
$$(3) \Rightarrow c_1 = 0$$

solutia:
$$x(t) = \frac{1}{20} \cdot (2t+3)^3 + \frac{19}{20} \cdot \cos(\ln(2t+3)) - \frac{13}{20} \cdot \sin(\ln(2t+3)).$$