1) Se cere mulțimea soluțiilor următorului sistem de ecuații diferențiale:

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 3x_2 - 2x_1 \end{cases}, \quad x \in \mathbb{R}^2, \ x = (x_1, x_2)$$

Forma matriceală a sistemului:

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = -2x_1 + 3x_2 \end{cases} \quad \underbrace{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}}_{x'} = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}, \quad x' = Ax$$

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

Determinăm valorile proprii ale matricei A:

$$\det(A - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (1-\lambda)\cdot(3-\lambda) - 1\cdot(-2) = 0 \Rightarrow 3 - \lambda - 3\lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

$$\Delta = (-4)^2 - 4\cdot 1\cdot 5 = 16 - 20 = -4 < 0$$

$$\lambda_{1,2} = \frac{-(-4) \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$\lambda_1 = 2+i, \ m_1 = 1$

$\lambda_2 = 2-i, \ m_2 = 1.$

$n = 2 \ (x \in \mathbb{R}^2), \ m_1 + m_2 = 1+1 = 2$

$\lambda_1 = 2+i, \ m_1 = 1,$ determinăm $u \in \mathbb{C}^2 \setminus \{0\}$ a.î. $Au = \lambda_1 u$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (2+i) \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} u_1 + u_2 = (2+i)u_1 \\ -2u_1 + 3u_2 = (2+i)u_2 \end{cases} \Rightarrow \begin{cases} u_1(-1-i) + u_2 = 0 \\ -2u_1 + (1-i)u_2 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} u_2 = (1+i)\cdot u_1 \\ u_2 = \dfrac{2}{1-i} u_1 = \dfrac{2(1+i)}{\underbrace{1-(-1)}_{i^2}} u_1 = \dfrac{\cancel{2}(1+i)}{\cancel{2}} u_1 = (1+i) u_1 \end{cases}$$

$$\Rightarrow u_2 = (1+i)\cdot u_1$$

$$u = \begin{pmatrix} u_1 \\ (1+i)\,u_1 \end{pmatrix} = u_1 \cdot \begin{pmatrix} 1 \\ 1+i \end{pmatrix}, \; u_1 \in \mathbb{C}$$

Avem 2 soluții în sistemul fundamental, corespunzător pentru λ_1 și $\bar{\lambda}_1 = \lambda_2$.

$$\varphi_1(t) = \mathrm{Re}(u \cdot e^{\lambda_j t}) \qquad \text{deci} \quad \varphi_1(t) = \mathrm{Re}\left(e^{\lambda_1 t} \cdot \begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right)$$

$$\varphi_2(t) = \mathrm{Im}(u \cdot e^{\lambda_j \cdot t}) \qquad \varphi_2(t) = \mathrm{Im}\left(e^{\lambda_1 t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right)$$

$$e^{\lambda_1 t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{(2+i)t} \cdot \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = e^{2t} \cdot e^{it} \cdot \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] =$$

$$= e^{2t} \cdot (\cos t + i \cdot \sin t) \cdot \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] =$$

$$\left(e^{it} = e^{0 + i \cdot t} = \underbrace{e^{0}}_{1}(\cos t + i \cdot \sin t) = \cos t + i \cdot \sin t \right); \quad \left(e^{a+ib} = e^{a}(\cos b + i \sin b)\right) \; a, b \in \mathbb{R}$$

$$= \underbrace{e^{2t} \cdot \left[\cos t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]}_{\mathrm{Re}\left(e^{\lambda_1 t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right)} + i \cdot \underbrace{e^{2t} \cdot \left[\sin t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cos t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]}_{\mathrm{Im}\left(e^{\lambda_1 t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right)}$$

$$\varphi_1(t) = e^{2t} \cdot \left[\cos t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = e^{2t} \cdot \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}$$

și $$\varphi_2(t) = e^{2t} \cdot \left[\sin t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cos t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = e^{2t} \cdot \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}$$

$$S_A = \{ c_1 \varphi_1 + c_2 \varphi_2 \mid c_1, c_2 \in \mathbb{R} \}$$

sau $\Rightarrow$ $$X(t) = \underbrace{\phi(t)}_{\text{matricea fundamentală de soluții}} = \begin{pmatrix} \varphi_1 & \varphi_2 \end{pmatrix} = e^{2t}\begin{pmatrix} \cos t & \sin t \\ \cos t - \sin t & \sin t + \cos t \end{pmatrix}$$

Verific $\det X(t) \neq 0$.

$$\det X(t) = e^{2t} \cdot \cos t \cdot e^{2t}(\sin t + \cos t) - e^{2t} \cdot \sin t \cdot e^{2t} \cdot (\cos t - \sin t) =$$

$$= e^{4t}(\cancel{\cos t \cdot \sin t} + \cos^2 t - \cancel{\sin t \cdot \cos t} + \sin^2 t) = e^{4t} \cdot 1 = e^{4t} \neq 0 \quad \forall t \in \mathbb{R}$$

$$S_A = \left\{ x(t) = \phi(t) \cdot C \;\middle|\; C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2 \right\}$$

2) Det. mulțimea soluțiilor următorului sistem de ecuații diferențiale:

$$\begin{cases} x_1' = 2x_1 - x_2 - x_3 \\ x_2' = 3x_1 - 2x_2 - 3x_3 \\ x_3' = -x_1 + x_2 + 2x_3 \end{cases}, \quad n=3.$$

Scriem forma matriceală:

$$\underbrace{\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}}_{X'} = \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X}$$

Determinăm valorile proprii:

$$\det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 \cdot (-2-\lambda) - 3 - 3 - (-1)^2 \cdot (-2-\lambda)$$

$$-(-3)(2-\lambda) - (-1)\cdot 3 \cdot (2-\lambda) = (4 - 4\lambda + \lambda^2)\cdot(-2-\lambda) - \cancel{3} - \cancel{3} + 2 + \lambda + \cancel{6} - 3\lambda +$$

$$+ 6 - 3\lambda = -\cancel{8} + 8\lambda - 2\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 + \cancel{2} - 2\lambda + \cancel{6} - 3\lambda =$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = \lambda(-\lambda^2 + 2\lambda - 1) = -\lambda \cdot (\lambda^2 - 2\lambda + 1) =$$

$$= -\lambda \cdot (\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = 0, \ m_1 = 1$$

$$\lambda_2 = 1, \ m_2 = 2.$$

- $\lambda_1 = 0$, $m_1 = 1$. Determinăm vectorul propriu $u \in \mathbb{R}^3 \setminus \{0\}$ coresp. valorii proprii λ_1:

$$Au = \lambda_1 u \Rightarrow \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2u_1 - u_2 - u_3 \overset{(1)}{=} 0 \\ 3u_1 - 2u_2 - 3u_3 \overset{(2)}{=} 0 \\ -u_1 + u_2 + 2u_3 \overset{(3)}{=} 0 \end{cases}$$

$$\begin{vmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{vmatrix} = -8 - 3 - 3 + 2 + 6 + 6 = -14 + 14 = 0.$$

$$\Delta_p = \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} = -4 + 3 = -1 \neq 0 \Rightarrow$$ u_1, u_2 – var. princ., u_3 – var. secundară
(1), (2) – ec. principale, (3) – ec. secundară

det. caracteristic

$$\Delta_c = \begin{vmatrix} 2 & -1 & 0 \\ 3 & -2 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 0 \Rightarrow$$ sistem compatibil nedeterminat

$$\Rightarrow \begin{cases} 2u_1 - u_2 = u_3 \\ 3u_1 - 2u_2 = 3u_3 \end{cases} \Rightarrow u_2 = 2u_1 - u_3 \Rightarrow$$

$$3u_1 - 2(2u_1 - u_3) = 3u_3 \Rightarrow 3u_1 - 4u_1 + 2u_3 = 3u_3 \Rightarrow -u_1 = u_3$$

$$\Rightarrow u_1 = -u_3$$

$$u_2 = 2u_1 - u_3 = 2\cdot(-u_3) - u_3 = -3u_3$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -u_3 \\ -3u_3 \\ u_3 \end{pmatrix} = u_3 \cdot \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

Sol. în sist. fundam. este $\varphi_1(t) = e^{\lambda_1 t} \cdot \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

$\lambda_1 = 0.$

- $\lambda_2 = 1,\ m_2 = 2.$

Det $p_0, p_1 \in \mathbb{R}^3$ vectori nu amândoi nuli a.î. $\varphi(t) = (p_0 + p_1 t)e^{\lambda_2 t}$ să fie

$\underset{m_{\lambda}-1 = m_2 - 1 = 2-1 = 1}{\overset{}{p_1}}$

soluție a sistemului inițial $(x' = Ax)$

$$\Rightarrow \left((p_0 + p_1 t)e^t\right)' = A\cdot(p_0 + p_1 t)e^t$$

$$\Rightarrow (p_0 + p_1 t)'\cdot e^t + (p_0 + p_1 t)\cdot(e^t)' = (Ap_0 + Ap_1 t)\cdot e^t$$

$$\Rightarrow (p_1 + p_0 + p_1 t)\cancel{e^t} = (Ap_0 + t\cdot Ap_1)\cdot\cancel{e^t}$$

$$\Rightarrow p_1 + p_0 + p_1 t = Ap_0 + t\cdot Ap_1$$

Identific. coeficienții puterilor lui t: $\begin{cases} p_1 + p_0 = Ap_0 \\ p_1 = Ap_1 \end{cases} \Rightarrow \begin{cases} p_1 = (A - I_3)p_0 \\ 0_{\mathbb{R}^3} = (A - I_3)p_1 \end{cases}$

Înmulțim prima ecuație cu $(A - I_3)$ la stânga $\Rightarrow$

$$\Rightarrow \underbrace{(A - I_3)p_1}_{0_{\mathbb{R}^3}} = (A - I_3)\cdot(A - I_3)p_0 \Rightarrow 0_{\mathbb{R}^3} = (A - I_3)^2 p_0 \Rightarrow$$

$$\Rightarrow p_0 \in \operatorname{Ker}\left((A - I_3)^2\right)$$

$$\operatorname{Ker}\left((A - I_3)\right)^2 = \{ v \in \mathbb{R}^3 \mid (A - I_3)^2 v = 0_{\mathbb{R}^3} \}.$$

$$(A - I_3)^2 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix}$$

$$(A-I_3)^2 v = 0_{\mathbb{R}^3} \Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} -v_1+v_2+v_3=0 \\ -3v_1+3v_2+3v_3=0 \\ v_1-v_2-v_3=0 \end{cases} \Rightarrow -v_1+v_2+v_3=0 \Rightarrow v_1=v_2+v_3$$

$$v = \begin{pmatrix} v_2+v_3 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + v_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; v_2, v_3 \in \mathbb{R}.$$

$$\ker((A-I_3)^2) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$ - bază pentru $\ker((A-I_3)^2)$

$$\rightarrow p_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow p_1 = (A-I_3)p_0 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\varphi_2(t) = (p_0 + p_1 t) \cdot e^t = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot e^t$$

$$\rightarrow p_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow p_1 = (A-I_3)p_0 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \varphi_3(t) = (p_0 + p_1 t) \cdot e^t = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^t$$

Matricea fundamentală de soluții este $X(t) = \phi(t) = \begin{pmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{pmatrix}$

$$\Rightarrow S_A = \left\{ x(t) = \phi(t) \cdot C \;\middle|\; C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$$

Verific $\det X(t) \neq 0$.

$$-e^{2t} + 0 + 0 - e^{2t} - 0 + 3e^{2t} = e^{2t} \neq 0, \; \forall t \in \mathbb{R}.$$

3) $\begin{cases} x_1' = 2x_1 - x_2 - x_3 + e^t \\ x_2' = 3x_1 - 2x_2 - 3x_3 + 1 \\ x_3' = -x_1 + x_2 + 2x_3 - e^{2t} \end{cases}$ → Determinați mulțimea soluțiilor.

Forma matriceală va fi: $x' = A x + b(t)$ (sistem afin de ecuații diferențiale)

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}, \quad b(t) = \begin{pmatrix} e^t \\ 1 \\ -e^{2t} \end{pmatrix}$$

Se determină soluția sistemului liniar omogen atașat: $\overline{x}' = A\overline{x}$.
(sistemul de la exe. 2) $\Rightarrow \overline{x}(t) = \phi(t) \cdot C, \ C \in \mathbb{R}^3$ și

$$\phi(t) = \begin{pmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{pmatrix}, t \in \mathbb{R}$$

Aplicăm metoda variației constantelor, det. $c: \mathbb{R} \to \mathbb{R}^3$ a.î.
$x(t) = \phi(t) \cdot c(t)$ să fie soluția sistemului afin $x' = Ax + b(t)$

$\Rightarrow (\phi(t) \cdot c(t))' = A \cdot \phi(t) \cdot c(t) + b(t)$

$\Rightarrow \phi'(t) \cdot c(t) + \phi(t) \cdot c'(t) = A \cdot \phi(t) \cdot c(t) + b(t)$

Știm $\phi'(t) = A\phi(t) \Rightarrow$ se reduc $\Rightarrow \phi(t) \cdot c'(t) = b(t)$

$$\Rightarrow \begin{pmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{pmatrix} \cdot \begin{pmatrix} c_1'(t) \\ c_2'(t) \\ c_3'(t) \end{pmatrix} = \begin{pmatrix} e^t \\ 1 \\ -e^{2t} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} -c_1'(t) + e^t \cdot c_2'(t) + e^t \cdot c_3'(t) = e^t \\ -3c_1'(t) + e^t \cdot c_2'(t) = 1 \\ c_1'(t) + e^t \cdot c_3'(t) = -e^{2t} \end{cases}$$

$\Rightarrow c_2'(t) = \dfrac{1 + 3c_1'(t)}{e^t} = e^{-t} + 3e^{-t} c_1'(t)$

$\Rightarrow c_3'(t) = \dfrac{-e^{2t} - c_1'(t)}{e^t} = -e^t - c_1'(t) e^{-t}$

$\Rightarrow -c_1'(t) + e^t(e^{-t} + 3e^{-t} c_1'(t)) + e^t \cdot (-e^t - c_1'(t) \cdot e^{-t}) = e^t$

$\Rightarrow -c_1'(t) + 1 + 3c_1'(t) - e^{2t} - c_1'(t) = e^t$

$c_1'(t) = e^{2t} + e^t - 1$

$c_1(t) = \int e^{2t} + e^t - 1 \, dt = \dfrac{e^{2t}}{2} + e^t - t + k_1, \ k_1 \in \mathbb{R}$

$$c_2'(t) = e^{-t} + 3e^{-t}(e^{2t} + e^{t} - 1) =$$
$$= e^{-t} + 3e^{t} + 3 - 3e^{-t} = 3e^{t} + 3 - 2e^{-t}$$
$$c_2(t) = \int 3e^{t} + 3 - 2e^{-t}\,dt = 3e^{t} + 3t + 2e^{-t} + k_2,\ k_2 \in \mathbb{R}$$
$$c_3'(t) = -e^{t} - e^{-t}\cdot(e^{2t} + e^{t} - 1) = -e^{t} - e^{t} - 1 + e^{-t} = -2e^{t} - 1 + e^{-t}$$
$$c_3(t) = \int -2e^{t} - 1 + e^{-t}\,dt = -2e^{t} - t - e^{-t} + k_3,\ k_3 \in \mathbb{R}$$
$$S_{A,b} = \left\{ x(t) = \phi(t)\cdot C(t) = \begin{pmatrix} -1 & e^{t} & e^{t} \\ -3 & e^{t} & 0 \\ 1 & 0 & e^{t} \end{pmatrix} \begin{pmatrix} \frac{e^{2t}}{2} + e^{t} - t + k_1 \\ 3e^{t} + 3t + 2e^{-t} + k_2 \\ -2e^{t} - t - e^{-t} + k_3 \end{pmatrix} \middle|\ k_1, k_2, k_3 \in \mathbb{R} \right\}$$