1) Se dă sistemul: $\begin{cases} x_1' = -x_1 + x_2 - 2x_3 \\ x_2' = 4x_1 + x_2 + e^{-t} \\ x_3' = 2x_1 + x_2 - x_3 \end{cases}$

a) Scrieți sistemul în formă matriceală, $x' = Ax + b(t)$.

b) Determinați mulțimea soluțiilor sistemului.

c) Găsiți soluția care verifică $x_1(0) = 1$, $x_2(0) = -1$ și $x_3(0) = 2$.

a)
$$\underbrace{\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}}_{x'} = \underbrace{\begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 \\ e^{-t} \\ 0 \end{pmatrix}}_{b(t)}$$

b) Sistemul liniar omogen atașat: $\bar{x}' = A\bar{x}$

Valorile proprii ale matricei A: $\det(A - \lambda I_3) = 0$.

$$\begin{vmatrix} -1-\lambda & 1 & -2 \\ 4 & 1-\lambda & 0 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)^2 \cdot (1-\lambda) - 8 + 0 - (-2)\cdot 2(1-\lambda) - 0 - 4(-1-\lambda) = 0$$

$$\Rightarrow (1 + 2\lambda + \lambda^2)(1-\lambda) \cancel{-8} \cancel{+4} \cancel{-4\lambda} \cancel{+4} \cancel{+4\lambda} = 0$$

$$\Rightarrow (1+\lambda)^2 \cdot (1-\lambda) = 0 \Rightarrow \begin{cases} 1-\lambda = 0 \Rightarrow \lambda_1 = 1,\ m_1 = 1 \\ (1+\lambda)^2 = 0 \Rightarrow \lambda_2 = -1,\ m_2 = 2. \end{cases}$$

- $\lambda_1 = 1, m_1 = 1$. determinăm vectorul propriu $u \in \mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$ asociat valorii proprii λ_1.

$$Au = \lambda_1 u \Rightarrow \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} -u_1 + u_2 - 2u_3 = u_1 \\ 4u_1 + u_2 = u_2 \\ 2u_1 + u_2 - u_3 = u_3 \end{cases} \Rightarrow u_1 = 0. \Rightarrow \begin{cases} u_2 = 2u_3 \\ u_2 = 2u_3 \end{cases} \Rightarrow u = \begin{pmatrix} 0 \\ 2u_3 \\ u_3 \end{pmatrix} = u_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \varphi_1(t) = e^{\lambda_1 t} \cdot u = e^t \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2e^t \\ e^t \end{pmatrix}$$

- $\lambda_2 = -1, m_2 = 2 \rightarrow$ determinăm $p_0, p_1 \in \mathbb{R}^3$ nu amândoi nuli a.î. $\varphi(t) = (p_0 + p_1 t)e^{\lambda_2 t}$ să fie soluția a sist. $\bar{x}' = A\bar{x}$

$$\Rightarrow ((p_0+p_1t)e^{-t})' = A\cdot(p_0+p_1t)e^{-t} \Rightarrow$$

$$\Rightarrow (p_0+p_1t)'\cdot e^{-t} + (p_0+p_1t)\cdot(e^{-t})' = A\cdot(p_0+p_1t)e^{-t}$$

$$\Rightarrow p_1\cdot e^{-t} + (p_0+p_1t)\cdot(-e^{-t}) = A(p_0+p_1t)e^{-t}$$

$$\Rightarrow (p_1-p_0-p_1t)e^{-t} = A(p_0+p_1t)e^{-t} \Big/ :e^{-t} \Rightarrow$$

$$\Rightarrow p_1-p_0-p_1t = Ap_0 + Ap_1t,\ \forall t\in\mathbb{R} \Rightarrow \begin{cases} -p_1 = Ap_1 \\ p_1-p_0 = Ap_0 \end{cases} \Rightarrow \begin{cases} 0_{\mathbb{R}^3} = Ap_1+p_1 \\ p_1 = Ap_0+p_0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 0_{\mathbb{R}^3} = (A+I_3)p_1 \\ p_1 = (A+I_3)p_0 \Big/ \cdot(A+I_3) \text{ la stg.} \end{cases} \quad (A+I_3 \text{ nu e inv.}) \Rightarrow \underbrace{(A+I_3)p_1}_{0_{\mathbb{R}^3}} = (A+I_3)^2 p_0 \Rightarrow$$

$$\Rightarrow (A+I_3)^2 p_0 = 0_{\mathbb{R}^3} \Rightarrow p_0 \in \ker((A+I_3)^2)$$

$$(A+I_3)^2 = \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & -8 \\ 4 & 4 & -4 \end{pmatrix}$$

Fie $p_0 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$; $\begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & -8 \\ 4 & 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{cases} 0=0 \\ 8v_1+8v_2-8v_3=0 \\ 4v_1+4v_2-4v_3=0 \Big/ \cdot 2 \end{cases} \Rightarrow v_1+v_2-v_3=0 \Rightarrow v_3 = v_1+v_2$$

$$\Rightarrow p_0 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_1+v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ v_2 \end{pmatrix} = v_1\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + v_2\cdot\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\ker((A+I_3)^2) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$ - bază în $\ker((A+I_3)^2)$

$$* \ p_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow p_1 = (A+I_3)p_0 = \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$$

$$\Rightarrow \varphi_2(t) = (p_0+p_1t)e^{-t} = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2t \\ 4t \\ 2t \end{pmatrix}\right)e^{-t} = \begin{pmatrix} (1-2t)e^{-t} \\ 4te^{-t} \\ (1+2t)e^{-t} \end{pmatrix}$$

$\rightarrow p_0 = \begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow p_1 = (A+I_3)p_0 = \begin{pmatrix}0&1&-2\\4&2&0\\2&1&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\1\end{pmatrix}$

$$\Rightarrow \varphi_3(t) = (p_0 + p_1 t)e^{-t} = \left(\begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}-1\\2\\1\end{pmatrix}t\right)e^{-t} = \begin{pmatrix}-te^{-t}\\(1+2t)e^{-t}\\(1+t)e^{-t}\end{pmatrix}$$

Sistemul fundamental de soluții: $\{\varphi_1, \varphi_2, \varphi_3\}$

Matricea de soluții: $\Phi(t) = \begin{pmatrix}|&|&|\\ \varphi_1&\varphi_2&\varphi_3\\ |&|&|\end{pmatrix} = \begin{pmatrix}0&(1-2t)e^{-t}&-te^{-t}\\2e^{t}&4te^{-t}&(1+2t)e^{-t}\\e^{t}&(1+2t)e^{-t}&(1+t)e^{-t}\end{pmatrix}$

Mulțimea soluțiilor pentru $\bar{x}' = A\cdot\bar{x}$ este $S_1 = \{\bar{x}(t) = \Phi(t)\cdot c \mid c \in \mathbb{R}^3\}$.

Aplicăm metoda variației constantelor: determinăm $c: \mathbb{R}\to\mathbb{R}^3$, $c = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ a.î. $x(t) = \Phi(t)\cdot c(t)$ să fie soluție a sistemului afin $x' = Ax + b(t)$

$$\Rightarrow (\Phi(t)\cdot c(t))' = A\cdot\Phi(t)\cdot c(t) + b(t) \Rightarrow$$

$$\Rightarrow \cancel{\Phi'(t)\cdot c(t)} + \Phi(t)\cdot c'(t) = \cancel{A\cdot\Phi(t)\cdot c(t)} + b(t)$$

Știm $\Phi'(t) = A\Phi(t) \Rightarrow \Phi(t)\cdot c'(t) = b(t)$

$$\begin{pmatrix}0&(1-2t)e^{-t}&-te^{-t}\\2e^{t}&4te^{-t}&(1+2t)e^{-t}\\e^{t}&(1+2t)e^{-t}&(1+t)e^{-t}\end{pmatrix}\cdot\begin{pmatrix}c_1'(t)\\c_2'(t)\\c_3'(t)\end{pmatrix} = \begin{pmatrix}0\\e^{-t}\\0\end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases}(1-2t)\cancel{e^{-t}}c_2'(t) - t\cancel{e^{-t}}c_3'(t) = 0 \quad \Rightarrow (1-2t)c_2'(t) = t\cdot c_3'(t) \quad (*)\\ 2e^{t}c_1'(t) + 4te^{-t}c_2'(t) + (1+2t)e^{-t}c_3'(t) = e^{-t}\\ e^{t}\cdot c_1'(t) + (1+2t)e^{-t}c_2'(t) + (1+t)e^{-t}c_3'(t) = 0 \;\Big|\cdot(-2)\end{cases} \Bigg\}\overset{(+)}{\Rightarrow}$$

$$\Rightarrow (4t - 2 - 4t)e^{-t}c_2'(t) + (1+2t-2-2t)e^{-t}c_3'(t) = e^{-t}$$

$$\Rightarrow -2\cancel{e^{-t}}c_2'(t) - \cancel{e^{-t}}c_3'(t) = \cancel{e^{-t}}$$

$$\Rightarrow \begin{cases}-2c_2'(t) - c_3'(t) = 1 \;\Big|\cdot t\\ (*)\ (1-2t)c_2'(t) - tc_3'(t) = 0\end{cases} \Rightarrow \begin{cases}-2tc_2'(t) - tc_3'(t) = t\\ (1-2t)c_2'(t) - tc_3'(t) = 0\end{cases} \; -$$

$$-c_2'(t) = t \Rightarrow c_2'(t) = -t \Rightarrow$$

$\Rightarrow c_2(t) = -\int t\,dt = -\frac{t^2}{2} + k_2,\ k_2 \in \mathbb{R}$

$c_3'(t) = -2c_2'(t) - 1 = -2\cdot(-t) - 1 = 2t-1$

$c_3(t) = \int 2t-1\,dt = t^2 - t + k_3,\ k_3 \in \mathbb{R}$

ec.(1) $e^t c_1'(t) + (1+2t)e^{-t}\cdot(-t) + (1+t)e^{-t}\cdot(2t-1) = 0$

$e^t \cdot c_1'(t) + e^{-t}(\cancel{-t} - \cancel{2t^2} + \cancel{2t} - 1 + \cancel{2t^2} - \cancel{t}) = 0$

$\Rightarrow e^t c_1'(t) = +e^{-t} \Rightarrow c_1'(t) = +\frac{e^{-t}}{e^t} = +e^{-2t}$

$c_1(t) = \int e^{-2t}\,dt = \frac{e^{-2t}}{-2} + k_1,\ k_1 \in \mathbb{R}$

Soluția este $x(t) = \begin{pmatrix} 0 & (1-2t)e^{-t} & -te^{-t} \\ 2e^t & 4te^{-t} & (1+2t)e^{-t} \\ e^t & (1+2t)e^{-t} & (1+t)e^{-t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}e^{-2t} + k_1 \\ -\frac{t^2}{2} + k_2 \\ t^2 - t + k_3 \end{pmatrix},\ k_1, k_2, k_3 \in \mathbb{R}$

c) $X(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

dar $X(0) = \begin{pmatrix} 0 & 1\cdot e^{-0} & 0 \\ 2e^0 & 4\cdot 0 & 1\cdot e^0 \\ e^0 & 1\cdot e^0 & 1\cdot e^0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2}e^0 + k_1 \\ -0 + k_2 \\ 0 - 0 + k_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} + k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow \begin{cases} k_2 = 1 \\ 2\cdot(-\frac{1}{2} + k_1) + k_3 = -1 \\ -\frac{1}{2} + k_1 + k_2 + k_3 = 2 \end{cases} \Rightarrow \begin{cases} k_2 = 1 \\ -1 + 2k_1 + k_3 = -1 \\ -\frac{1}{2} + k_1 + k_2 + k_3 = 2 \end{cases} \Rightarrow \begin{cases} k_2 = 1 \\ 2k_1 + k_3 = 0 \\ k_1 + 1 + k_3 = 2 + \frac{1}{2} \end{cases}$

$\Rightarrow \begin{cases} 2k_1 + k_3 = 0 \\ k_1 + k_3 = \frac{3}{2} \end{cases} \overset{(-)}{\Rightarrow} k_1 = -\frac{3}{2},\ k_3 = 3,\ k_2 = 1.$

Sol. este: $x(t) = \begin{pmatrix} 0 & (1-2t)e^{-t} & -te^{-t} \\ 2e^t & 4te^{-t} & (1+2t)e^{-t} \\ e^t & (1+2t)e^{-t} & (1+t)e^{-t} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2}e^{-2t} - \frac{3}{2} \\ -\frac{t^2}{2} + 1 \\ t^2 - t + 3 \end{pmatrix}$

2) Fie sistemul $\begin{cases} x_1' = 3t^2 x_2 \\ x_2' = 3t^2 x_1 \end{cases}$.

a) Arătați că $\varphi_1(t) = \begin{pmatrix} e^{-t^3} \\ -e^{-t^3} \end{pmatrix}$ este soluție a sistemului.

b) Folosind reducerea dimensiunii sistemului, determinați mulțimea soluțiilor sistemului și o soluție $\varphi(t)$ a.î. $\{\varphi_1, \varphi_2\}$ să fie sistem fundamental de soluții.

a) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 \\ 3t^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x' = A(t) \cdot x$

φ_1 - sol. $\Leftrightarrow \varphi_1'(t) = A(t)\varphi_1(t) \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} e^{-t^3} \cdot (-t^3)' \\ -e^{-t^3} \cdot (-t^3)' \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 \\ 3t^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-t^3} \\ -e^{-t^3} \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} e^{-t^3} \cdot (-3t^2) \\ -e^{-t^3} \cdot (-3t^2) \end{pmatrix} = \begin{pmatrix} -3t^2 e^{-t^3} \\ 3t^2 e^{-t^3} \end{pmatrix}$ (A), $\forall t \in \mathbb{R}$

b) $n = 2$.

$m = 1 \to$ numărul de soluții independente.
$< n!$

$\det(\varphi_{11}(t)) = \det(e^{-t^3}) = e^{-t^3} \neq 0, \forall t \in \mathbb{R}$

Facem schimbarea de variabilă $\boxed{x = Z(t) y}$, unde $Z(t) = \begin{pmatrix} e^{-t^3} & 0 \\ -e^{-t^3} & 1 \end{pmatrix}$

φ_1 ; se completează cu vectori din baza canonică din $\mathbb{R}^2$

$\det(Z(t)) = e^{-t^3} \neq 0 \Rightarrow \exists (Z(t))^{-1}$

Sistemul devine $(Z(t) \cdot y)' = A(t) \cdot Z(t) \cdot y \Rightarrow$

$\Rightarrow Z(t)' \cdot y + Z(t) \cdot y' = A(t) \cdot Z(t) \cdot y \Rightarrow$

$\Rightarrow Z(t) \cdot y' = A(t) \cdot Z(t) \cdot y - Z'(t) \cdot y \Rightarrow Z(t) \cdot y' = (A(t) Z(t) - Z'(t)) \cdot y$

$$\Rightarrow y' = Z(t)^{-1} \cdot \left(A(t) \cdot Z(t) - Z'(t)\right) y$$

$$Z(t) = \begin{pmatrix} e^{-t^3} & 0 \\ -e^{-t^3} & 1 \end{pmatrix}$$

$$Z'(t) = \begin{pmatrix} -3t^2 e^{-t^3} & 0 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix} ; \quad Z(t)^T = \begin{pmatrix} e^{-t^3} & -e^{-t^3} \\ 0 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow Z(t)^* = \begin{pmatrix} 1 & 0 \\ e^{-t^3} & e^{-t^3} \end{pmatrix} \Rightarrow Z^{-1}(t) = \frac{1}{\det Z(t)} \cdot Z^*(t) = \frac{1}{e^{-t^3}} \cdot Z^*(t) =$$

$$= e^{t^3} \cdot \begin{pmatrix} 1 & 0 \\ e^{-t^3} & e^{-t^3} \end{pmatrix} = \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow y' = \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & 3t^2 \\ 3t^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-t^3} & 0 \\ -e^{-t^3} & 1 \end{pmatrix} - \begin{pmatrix} -3t^2 e^{-t^3} & 0 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix} \right) y$$

$$\Rightarrow y' = \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} -3t^2 e^{-t^3} & 3t^2 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix} - \begin{pmatrix} -3t^2 e^{-t^3} & 0 \\ 3t^2 e^{-t^3} & 0 \end{pmatrix} \right) y \Rightarrow$$

$$\Rightarrow y' = \begin{pmatrix} e^{t^3} & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3t^2 \\ 0 & 0 \end{pmatrix} y$$

$$\Rightarrow y' = \begin{pmatrix} 0 & 3t^2 e^{t^3} \\ 0 & 3t^2 \end{pmatrix} \cdot y$$

fie $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 e^{t^3} \\ 0 & 3t^2 \end{pmatrix} \cdot \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} y_1'(t) = 3t^2 e^{t^3} y_2(t) \\ y_2'(t) = 3t^2 y_2(t) \end{cases} \Rightarrow$ ec. liniară cu soluția $y_2(t) = c \cdot e^{\int 3t^2 dt} = c \cdot e^{t^3}$

$$\Rightarrow y_1'(t) = 3t^2 e^{t^3} \cdot c e^{t^3} = 3ct^2 e^{2t^3} \Rightarrow$$

$$y_1(t) = \int 3ct^2 e^{2t^3} dt = 3c \int t^2 e^{2t^3} dt =$$

sch. var. $s = 2t^3 \Rightarrow ds = 6t^2 dt \Rightarrow t^2 dt = \frac{1}{6} ds$

$$= 3c \int \frac{1}{6} e^s ds = \frac{3c}{6} e^s + k = \frac{3c}{6} e^{2t^3} + k = \frac{c}{2} e^{2t^3} + k$$

$$x(t) = \begin{pmatrix} e^{-t^3} & 0 \\ -e^{-t^3} & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{c}{2} e^{2t^3} + k \\ c e^{t^3} \end{pmatrix} = \begin{pmatrix} \frac{c}{2} e^{t^3} + k e^{-t^3} \\ -\frac{c}{2} e^{t^3} - k e^{-t^3} + c e^{t^3} \end{pmatrix} =$$

$$= c \cdot \underbrace{\begin{pmatrix} \frac{1}{2} e^{t^3} \\ \frac{1}{2} e^{t^3} \end{pmatrix}}_{\varphi_2(t)} + k \cdot \underbrace{\begin{pmatrix} e^{-t^3} \\ -e^{-t^3} \end{pmatrix}}_{\varphi_1(t)}, \quad c, k \in \mathbb{R}$$

$\Rightarrow$ Sistemul fundamental de soluții: $\left\{ \begin{pmatrix} \frac{1}{2} e^{t^3} \\ \frac{1}{2} e^{t^3} \end{pmatrix}, \begin{pmatrix} e^{-t^3} \\ -e^{-t^3} \end{pmatrix} \right\}$.