Probleme Cauchy ptr. ec. dif. de ordin 1

① Fie problema Cauchy: $\begin{cases} \frac{dx}{dt} = t\sin x, & (t,x) \in [-1,1] \times [0, \frac{\pi}{2}] \\ x(0) = \frac{\pi}{4} \end{cases}$ (1)

a) Verificare ipoteze teorema de existență și unicitate

b) Calculați $\varphi_0, \varphi_1, \varphi_2$ din șirul aproximărilor succesive.

c) Determinarea soluției problemei Cauchy (1).

d) Pentru $t \in [0, \frac{\pi}{2}]$, construiți o schemă numerică de ordin 2 cu $N+1$ puncte echidistante.

---

a) 1) $\Delta = [-1,1] \times [0, \frac{\pi}{2}]$

$f: \Delta \to \mathbb{R}, \quad f(t,x) = t\sin x$

$t_0 = 0, \quad x_0 = \frac{\pi}{4}$

$\exists a, b > 0$ a.î. $\Delta_{a,b} = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] \subset \Delta$

$\Delta_{a,b} = [-a, a] \times [\frac{\pi}{4} - b, \frac{\pi}{4} + b] \subset [-1,1] \times [0, \frac{\pi}{2}]$

Fie $a = \frac{1}{2}, \ b = \frac{\pi}{4} \Rightarrow [-\frac{1}{2}, \frac{1}{2}] \times [0, \frac{\pi}{2}] \subset [-1,1] \times [0, \frac{\pi}{2}]$ (A)

$\forall a \in (0,1), \forall b \in (0, \frac{\pi}{4})$ verifică.

2) $f \to$ continuă în ambele variabile (A), operăm cu funcții elementare (continue)

3) $\frac{\partial f}{\partial x} f(t,x) = t(\sin x)'_x = t \cdot \cos x \to$ continuă

$M = \sup\limits_{(t,x) \in \Delta_{a,b}} |f(t,x)| = \sup\limits_{\substack{t \in [-a,a] \\ x \in [\frac{\pi}{4}-b, \frac{\pi}{4}+b] \subset [0, \frac{\pi}{2}]}} |t| \cdot |\sin x| = a \cdot \sin\left(\frac{\pi}{4} + b\right)$ (sin e s. cresc. pe $[0, \frac{\pi}{2}]$)

b) Șirul aproximărilor succesive: $\begin{cases} \varphi_0(t) = x_0 = \frac{\pi}{4} \\ \varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\, ds, & n \in \mathbb{N}^* \end{cases}$

$\Rightarrow \varphi_{n+1}(t) = \frac{\pi}{4} + \int_{t_0}^{t} s \cdot \sin(\varphi_n(s))\, ds$

$\varphi_1(t) = \frac{\pi}{4} + \int_{t_0}^{t} s \cdot \sin(\varphi_0(s))\, ds = \frac{\pi}{4} + \int_{t_0}^{t} s \cdot \sin\frac{\pi}{4}\, ds =$

$= \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \int_0^t s\, ds = \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \frac{s^2}{2}\Big|_0^t = \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \frac{t^2}{2} - \frac{\sqrt{2}}{2} \cdot 0$

$= \frac{\pi}{4} + \frac{\sqrt{2}}{4} t^2$

$\varphi_1(t) = \frac{\pi}{4} + \frac{\sqrt{2}}{4} t^2$

$\varphi_2(t) = \frac{\pi}{4} + \int_0^t s \cdot \sin(\varphi_1(s))\, ds = \frac{\pi}{4} + \int_0^t s \cdot \sin\left(\frac{\pi}{4} + \frac{\sqrt{2}}{4} s^2\right) ds$

$\frac{\pi}{4} + \frac{\sqrt{2}}{4} s^2 = z \Rightarrow \frac{\sqrt{2}}{4} \cdot 2s\, ds = dz \Rightarrow \frac{\sqrt{2}}{2} s\, ds = dz$

$s = 0, z = \frac{\pi}{4}$; $s = t, z = \frac{\pi}{4} + \frac{\sqrt{2}}{4} t^2$

$\varphi_2(t) = \frac{\pi}{4} + \int_{\frac{\pi}{4}}^{\frac{\pi}{4} + \frac{\sqrt{2}}{4} t^2} \frac{2}{\sqrt{2}} \sin z\, dz = \frac{\pi}{4} + \sqrt{2} \cdot (-\cos z) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{4} + \frac{\sqrt{2}}{4} t^2} \Rightarrow$

$\Rightarrow \varphi_2(t) = \frac{\pi}{4} - \sqrt{2} \cos\left(\frac{\pi}{4} + \frac{\sqrt{2}}{4} t^2\right) + \sqrt{2} \cdot \underbrace{\cos\frac{\pi}{4}}_{=\frac{\sqrt{2}}{2}}$

$\varphi_2(t) = \frac{\pi}{4} - \sqrt{2} \cdot \cos\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2} t^2\right) + 1.$

c) $\frac{dx}{dt} = t \cdot \sin x$ → ecuație cu variabile separabile

sol. staționare: $\sin x = 0 \Rightarrow x \equiv 0$.

$\frac{dx}{\sin x} = t \cdot dt$

$\int \frac{1}{\sin x} dx = \int t\, dt$

$\operatorname{tg} \frac{x}{2} = s \Rightarrow x = 2 \cdot \operatorname{arctg} s \Rightarrow dx = \frac{2}{1+s^2} ds.$

$\sin x = \frac{2 \cdot \operatorname{tg}\frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{2s}{1+s^2}$

$\int \frac{1+s^2}{2s} \cdot \frac{2}{1+s^2} ds = \int \frac{1}{s} ds = \ln|s| = \ln\left|\operatorname{tg}\frac{x}{2}\right|$

$\Rightarrow \ln\left|\operatorname{tg}\frac{x}{2}\right| = \frac{t^2}{2} + c, \ c \in \mathbb{R}$

$\left|\operatorname{tg}\frac{x}{2}\right| = e^{\frac{t^2}{2} + c} \Rightarrow \operatorname{tg}\frac{x}{2} = \pm e^{\frac{t^2}{2} + c}$

$\Rightarrow \begin{cases} x_1(t) = 2 \cdot \operatorname{arctg}\left(e^{\frac{t^2}{2} + c}\right) \\ x_2(t) = 2 \operatorname{arctg}\left(-e^{\frac{t^2}{2} + c}\right), \quad c \in \mathbb{R} \\ x_0(t) \equiv 0 \quad x(0) = \frac{\pi}{4} \text{ fals.} \end{cases}$

$\boxed{\begin{array}{l} 1 = \operatorname{tg}\frac{\pi}{4} = \frac{2 \operatorname{tg}\frac{\pi}{8}}{1 - \operatorname{tg}^2\frac{\pi}{8}} \Rightarrow \operatorname{tg}^2\frac{\pi}{8} + 2\operatorname{tg}\frac{\pi}{8} - 1 = 0 \\ \Rightarrow \operatorname{tg}\frac{\pi}{8} = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2} \to -1 + \sqrt{2} \\ \text{ptr. că } \operatorname{tg}\frac{\pi}{8} > 0! \end{array}}$

x_1: $x_1(0) = \frac{\pi}{4} \Rightarrow 2 \operatorname{arctg} e^c = \frac{\pi}{4} \Rightarrow \operatorname{arctg} e^c = \frac{\pi}{8} \Rightarrow e^c = \operatorname{tg}\frac{\pi}{8} \Rightarrow c = \ln\left(\operatorname{tg}\frac{\pi}{8}\right)$

x_2: $x_2(0) = \frac{\pi}{4} \Rightarrow 2 \operatorname{arctg}(-e^c) = \frac{\pi}{4} \Rightarrow \underbrace{-e^c}_{<0} = \underbrace{\operatorname{tg}\frac{\pi}{8}}_{>0}$ Fals!

Soluția: $x(t) = 2 \cdot \operatorname{arctg}\left(e^{\frac{t^2}{2} + \ln(\sqrt{2} - 1)}\right)$

d) Schema de ordin 2 construită prin metoda Taylor:

$$\begin{cases} x_0 \\ x_{j+1} = x_j + h \cdot \phi_2(h, t_j, x_j), \quad j=\overline{0,N-1} \end{cases}$$

unde $\phi_2(h,t_j,x_j) = f(t_j,x_j) + \frac{h}{2}\left[\frac{\partial f}{\partial t}(t_j,x_j) + \frac{\partial f}{\partial x}(t_j,x_j) \cdot f(t_j,x_j)\right]$

$f(t,x) = t \cdot \sin x$

$\frac{\partial f}{\partial t}(t,x) = \sin x$

$\frac{\partial f}{\partial x}(t,x) = t \cdot \cos x$

$\Rightarrow \phi_2(h,t,x) = t \sin x + \frac{h}{2}[\sin x + t\cos x \cdot t \sin x]$

$$\Rightarrow \begin{cases} x_0 \\ x_{j+1} = x_j + h \cdot \left[t_j \sin x_j + \frac{h}{2} \cdot \sin x_j (1 + t_j^2 \cdot \cos x_j)\right], \quad j=\overline{0,N-1} \end{cases}$$

$t_j = t_0 + j \cdot h, \quad h = \frac{T}{N}$

$t_0 = 0$

$[0, \frac{\pi}{2}] = [t_0, t_0 + T] \Rightarrow T = \frac{\pi}{2}$

$h = \frac{\pi}{2N}$

$t_j = 0 + j \cdot \frac{\pi}{2N} = \frac{\pi}{2N} \cdot j, \quad j = \overline{0,N}$

2. Fie problema Cauchy: $\begin{cases} \frac{dx}{dt} = 2x + t, \quad (t,x) \in \mathbb{R}^2 \\ x(0) = 1 \end{cases}$

a) Se cere șirul de aproximări succesive

b) Soluția problemei.

c) Pentru $N=2$, $t \in [0,1]$, calculați aproximarea soluției în $t=1$ folosind metoda Euler cu ~~puncte~~ echidistante.

a) Verificare ipoteze teorema de existență + unicitate:

1) $f: \Delta \to \mathbb{R}, \ \Delta = \mathbb{R}^2, \ f(t,x) = 2x + t$

$t_0 = 0, \ x_0 = 1.$

$\exists\, a, b > 0$ a.î. $D_{a,b} = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] \subset D$

$D_{a,b} = [-a, a] \times [1-b, 1+b] \subset \mathbb{R}^2$

Fie $a = 1,\ b = 5 \rightarrow [-1, 1] \times [-4, 6] \subset \mathbb{R}^2$ (A)

$\forall a \in \mathbb{R},\ \forall b \in \mathbb{R} \rightarrow$ (A)

2) $f \rightarrow$ continuă în ambele variabile (A), operăm cu funcții continue

3) $\frac{\partial f}{\partial x} f(t, x) = 2 \rightarrow$ continuă

$t_0 = 0,\ x_0 = 1$

$$\begin{cases} \varphi_0(t) = x_0 = 1 \\ \varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\,ds, \quad n \in \mathbb{N}^* \end{cases}$$

$\Rightarrow \varphi_{n+1}(t) = 1 + \int_0^t (s + 2\cdot\varphi_n(s))\,ds,\ \forall n \in \mathbb{N}^*$

$\varphi_1(t) = 1 + \int_0^t (s + 2\cdot\varphi_0(s))\,ds = 1 + \int_0^t (s+2)\,ds = 1 + \frac{s^2}{2}\Big|_0^t + 2s\Big|_0^t$

$\Rightarrow \varphi_1(t) = 1 + \frac{t^2}{2} + 2t$

$\varphi_2(t) = 1 + \int_0^t (s + 2\varphi_1(s))\,ds = 1 + \int_0^t \left(s + 2\cdot\left(1 + \frac{s^2}{2} + 2s\right)\right) ds =$

$= 1 + \frac{s^2}{2}\Big|_0^t + 2s\Big|_0^t + \frac{2\cdot s^3}{2\cdot 3}\Big|_0^t + 2\cdot 2\cdot\frac{s^2}{2}\Big|_0^t =$

$= 1 + \frac{t^2}{2} + 2t + 2\cdot\frac{t^3}{3!} + 2^2\cdot\frac{t^2}{2}$

$\varphi_3(t) = 1 + \int_0^t (s + 2\varphi_2(s))\,ds = 1 + \int_0^t \left(s + 2\cdot\left(1 + \frac{s^2}{2} + 2s + 2\cdot\frac{s^3}{3!} + 2^2\cdot\frac{s^2}{2}\right)\right) ds$

$= 1 + \frac{s^2}{2}\Big|_0^t + 2s\Big|_0^t + 2\cdot\frac{s^3}{2\cdot 3}\Big|_0^t + 2\cdot 2\cdot\frac{s^2}{2}\Big|_0^t + 2^2\cdot\frac{s^4}{3!\cdot 4}\Big|_0^t + 2^3\cdot\frac{s^3}{2!\cdot 3}\Big|_0^t$

$= 1 + \underline{\frac{t^2}{2}} + 2t + \underline{2\cdot\frac{t^3}{3!}} + 2^2\cdot\frac{t^2}{2!} + \underline{2^2\cdot\frac{t^4}{4!}} + 2^3\cdot\frac{t^3}{3!}$

$\varphi_n(t) = \left(1 + \frac{2t}{1!} + \frac{2^2 t^2}{2!} + \frac{2^3\cdot t^3}{3!} + \dots + \frac{2^n t^n}{n!}\right) + \left(\frac{t^2}{2!} + 2\cdot\frac{t^3}{3!} + \frac{2^2 t^4}{4!} + \dots + 2^{n-1}\cdot\frac{t^{n+1}}{(n+1)!}\right)$

Scanned with CamScanner

Demonstrăm prin inducție.

Pp. $p(n)$ – (A).

dem. $p(n+1)$ – (A), adică

$$\varphi_{n+1}(t)=\left(1+\frac{2t}{1!}+\frac{2^2t^2}{2!}+\ldots+\frac{2^{n+1}t^{n+1}}{(n+1)!}\right)+\left(\frac{t^2}{2!}+2\cdot\frac{t^3}{3!}+\ldots+2^{n}\frac{t^{n+2}}{(n+2)!}\right)$$

Calculăm φ_{n+1} din rel. de recurență:

$$\varphi_{n+1}(t)=1+\int_0^t\left[s+2\cdot\left(1+\frac{2s}{1!}+\ldots+\frac{2^n s^n}{n!}\right)+2\cdot\left(\frac{s^2}{2!}+2\cdot\frac{s^3}{3!}+\ldots+2^{n-1}\cdot\frac{s^{n+1}}{(n+1)!}\right)\right]ds$$

$$=1+\frac{s^2}{2}\Big|_0^t+\left(2s+\frac{2^2\cdot s^2}{1!\cdot 2}+\ldots+\frac{2^{n+1}}{n!}\cdot\frac{s^{n+1}}{n+1}\right)\Big|_0^t+\left(\frac{2s^3}{2!\cdot 3}+\frac{2^2\cdot s^4}{3!\cdot 4}+\ldots+2^n\cdot\frac{s^{n+2}}{(n+1)!(n+2)}\right)\Big|_0^t$$

$$=\left(1+\frac{2t}{1!}+\frac{2^2\cdot t^2}{2!}+\ldots+\frac{2^{n+1}t^{n+1}}{(n+1)!}\right)+\left(\frac{t^2}{2!}+\frac{2t^3}{3!}+2^2\frac{t^4}{4!}+\ldots+2^n\cdot\frac{t^{n+2}}{(n+2)!}\right)\Rightarrow$$

$\Rightarrow p(n+1)$ – (A) $\Rightarrow p(n)$ – (A), $n\in\mathbb{N}^*$.

e) $N=2$, $t\in[0,1]$

$f:\mathbb{R}^2\to\mathbb{R}$, $f(t,x)=2x+t$.

$t_0=0$, $x_0=1$.

$[t_0,t_0+T]=[0,T]$, $T>0$.

$t\in[0,1]\Rightarrow T=1$, $N=2$, $h=\frac{T}{N}=\frac{1}{2}$.

$t_j=t_0+jh=0+j\cdot\frac{1}{2}$, $j=\overline{0,2}$

$t_1=\frac{1}{2}$

$t_2=1$

$x_{j+1}=x_j+h\cdot f(t_j,x_j)$, $j=\overline{0,N-1}$

$x_1=x_0+h\cdot f(t_0,x_0)=1+\frac{1}{2}\cdot f(0,1)=1+\frac{1}{2}\cdot(2\cdot 1+0)=1+1=2.$

$x_2=x_1+h\cdot f(t_1,x_1)=2+\frac{1}{2}\cdot f\left(\frac{1}{2},2\right)=2+\frac{1}{2}\cdot\left(2\cdot 2+\frac{1}{2}\right)=2+2+\frac{1}{4}=\frac{17}{4}.$

f) $\frac{dx}{dt}=2x+t$ → ec. afină

$\bar{x}'=2\bar{x}$ ⇒ ec. lin. asociată cu soluția $\bar{x}(t)=c\cdot e^{\int 2\,dt}=c\cdot e^{2t}$, $c\in\mathbb{R}$

$x(t)=c(t)\cdot e^{2t}\Rightarrow\left(c(t)\cdot e^{2t}\right)'=2\cdot c(t)\cdot e^{2t}+t\Rightarrow$

$\Rightarrow c'(t)\cdot e^{2t}+\cancel{c(t)\cdot e^{2t}\cdot 2}=\cancel{2c(t)\cdot e^{2t}}+t\Rightarrow c'(t)=\frac{t}{e^{2t}}$

$$c(t) = \int \frac{t}{e^{2t}} dt = \int t \cdot e^{-2t} dt$$

$$\left.\begin{array}{l} f(t) = t \\ g'(t) = e^{-2t} \end{array}\right\} \Rightarrow \begin{array}{l} f'(t) = 1 \\ g(t) = \int e^{-2t} dt = -\frac{1}{2} e^{-2t} \end{array}$$

$$c(t) = -\frac{1}{2} \cdot te^{-2t} + \int \frac{1}{2} e^{-2t} dt = -\frac{1}{2} te^{-2t} + \frac{1}{2} \cdot \int e^{-2t} dt =$$

$$= -\frac{1}{2} t \cdot e^{-2t} + \frac{1}{2} \cdot \frac{-1}{2} \cdot e^{-2t} + k = -\frac{1}{2} te^{-2t} - \frac{1}{4} e^{-2t} + k,$$

$k \in \mathbb{R}$.

$$\Rightarrow x(t) = \left(-\frac{1}{2} te^{-2t} - \frac{1}{4} e^{-2t} + k\right) \cdot e^{2t}$$

$$x(t) = -\frac{1}{2} t - \frac{1}{4} + k \cdot e^{2t}, \; k \in \mathbb{R}.$$

$$x(0) = 1 \Rightarrow -\frac{1}{2} \cdot 0 - \frac{1}{4} + k \cdot e^{0} = 1 \Rightarrow$$

$$\Rightarrow -\frac{1}{4} + k = 1 \Rightarrow k = 1 + \frac{1}{4} = \frac{5}{4}$$

$\Rightarrow$ Soluția este $x(t) = -\frac{1}{2} t - \frac{1}{4} + \frac{5}{4} e^{2t}$.

③ Se cere șirul aproximărilor succesive pentru

$$\begin{cases} \frac{dx}{dt} = t x \\ x(0) = 1 \end{cases}$$

$t_0 = 0,\ x_0 = 1$

$$\begin{cases} \varphi_0(t) = x_0 = 1 \\ \varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\, ds,\ n \in \mathbb{N}^* \end{cases}$$

$$\varphi_{n+1}(t) = 1 + \int_0^t s \cdot \varphi_n(s)\, ds$$

$$\varphi_1(t) = 1 + \int_0^t s \cdot 1\, ds = 1 + \frac{s^2}{2}\Big|_0^t = 1 + \frac{t^2}{2}$$

$$\varphi_2(t) = 1 + \int_0^t s \cdot \left(1 + \frac{s^2}{2}\right) ds = 1 + \int_0^t \left(s + \frac{s^3}{2}\right) ds = 1 + \frac{s^2}{2}\Big|_0^t + \frac{s^4}{2 \cdot 4}\Big|_0^t$$

$$= 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4} = 1 + \frac{t^2}{2^1} + \frac{t^4}{2^2 \cdot 2!}$$

$$\varphi_3(t) = 1 + \int_0^t s \cdot \left(1 + \frac{s^2}{2} + \frac{s^4}{2 \cdot 4}\right) ds = 1 + \frac{s^2}{2}\Big|_0^t + \frac{s^4}{2 \cdot 4}\Big|_0^t + \frac{s^6}{2 \cdot 4 \cdot 6}\Big|_0^t =$$

$$= 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4} + \frac{t^6}{2 \cdot 4 \cdot 6} = 1 + \frac{t^2}{2} + \frac{t^4}{2^2 \cdot 2!} + \frac{t^6}{2^3 \cdot 3!}$$

$$\varphi_n(t) = 1 + \frac{t^2}{2} + \frac{t^4}{2^2 \cdot 2!} + \frac{t^6}{2^3 \cdot 3!} + \dots + \frac{t^{2n}}{2^n \cdot n!}$$

Demonstrăm prin inducție, pp. $p(n)$ - (A) și dem. $p(n+1)$.

$$\varphi_{n+1}(t) = 1 + \int_0^t s \cdot \left(1 + \frac{s^2}{2} + \frac{s^4}{2^2 \cdot 2!} + \frac{s^6}{2^3 \cdot 3!} + \dots + \frac{s^{2n}}{2^n \cdot n!}\right) ds =$$

$$= 1 + \frac{s^2}{2}\Big|_0^t + \frac{s^4}{2 \cdot 4}\Big|_0^t + \frac{s^6}{2^2 \cdot 2! \cdot \underbrace{6}_{2 \cdot 3}}\Big|_0^t + \frac{s^8}{2^3 \cdot 3! \cdot \underbrace{8}_{2 \cdot 4}}\Big|_0^t + \dots + \frac{s^{2n+2}}{2^n \cdot n! \cdot \underbrace{(2n+2)}_{2(n+1)}}\Big|_0^t =$$

$$= 1 + \frac{t^2}{2} + \frac{t^4}{2^2 \cdot 2!} + \frac{t^6}{2^3 \cdot 3!} + \frac{t^8}{2^4 \cdot 4!} + \dots + \frac{t^{2n+2}}{2^{n+1} \cdot (n+1)!}$$

$$\Rightarrow p(n+1) - (A) \Rightarrow p(n) - (A),\ n \in \mathbb{N}^*.$$