— Ecuații de tipul $\frac{dx}{dt} = g\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$ unde

a_1, a_2 nu sunt zero simultan și
b_1, b_2 nu sunt zero simultan

Notăm $d = a_1 b_2 - a_2 b_1$

Cazul $d = 0 \to$ sch. var. $a_1 t + b_1 x = y$, dacă $b_1 \neq 0$ sau $a_2 t + b_2 x = y$, dacă $b_2 \neq 0$

Cazul $d \neq 0 \to$ se rezolvă sistemul $\begin{cases} a_1 t + b_1 x + c_1 = 0 \\ a_2 t + b_2 x + c_2 = 0 \end{cases}$ cu sol. $t_0 = \dots, x_0 = \dots$

Se face sch. var. $\begin{cases} s = t - t_0 \\ y = x - x_0 \end{cases} \Rightarrow \begin{cases} t = s + t_0 \\ x = y + x_0 \end{cases}$

Exemplul 1. $\frac{dx}{dt} = \frac{2t - x + 1}{4t - 2x + 3}$, unde $(t, x) \in D_1 \subset \{(t,x) \mid 4t - 2x + 3 > 0\}$

$a_1 = 2,\ b_1 = -1,\ c_1 = 1$
$a_2 = 4,\ b_2 = -2,\ c_2 = 3$

$d = 2\cdot(-2) - 4\cdot(-1) = -4 + 4 = 0.$

$b_1 \neq 0 \Rightarrow$ sch. var. $y = a_1 t + b_1 x$

$y = 2t - x \Rightarrow x = 2t - y$

$(2t - y)' = \frac{2t - (2t - y) + 1}{4t - 2(2t - y) + 3}$

$2 - y' = \frac{y + 1}{2y + 3} \Rightarrow y' = 2 - \frac{y+1}{2y+3}$

$y' = \frac{4y + 6 - y - 1}{2y + 3} \Rightarrow \frac{dy}{dt} = \frac{3y + 5}{2y + 3} \to$ ec. cu var. sep.

sol. staționare: $\frac{3y+5}{2y+3} = 0 \Rightarrow y = -\frac{5}{3}$

$\int \frac{2y+3}{3y+5}\, dy = \int dt \Rightarrow \frac{1}{3}\int \frac{6y+9}{3y+5}\, dy = \frac{1}{3}\int \frac{2(3y+5) - 1}{3y+5}\, dy =$

$= \frac{1}{3}\left(\int 2dy - \int \frac{1}{3y+5}\, dy\right) = \frac{1}{3}\left(2y - \frac{1}{3}\ln|3y+5|\right) + c \Rightarrow$

$\Rightarrow \frac{1}{3}\left(2y - \frac{1}{3}\ln|3y+5|\right) = t+c,\ c\in\mathbb{R}.$

$$\begin{cases} \frac{2}{3}(2t-x) - \frac{1}{9}\ln|6t-3x+5| = t+c,\ c\in\mathbb{R} & \rightarrow \text{mulțimea sol. implicite pentru ecuația în } (t,x). \\ x(t) = 2t+\frac{5}{3} & \rightarrow \text{sol. particulară} \end{cases}$$

Exemplul 2. $\frac{dx}{dt} = \frac{3t+x-5}{2t-x}$, $(t,x)\in D_1 \subset \{(t,x) \mid 2t-x>0\}$.

$a_1=3,\ b_1=1,\ c_1=-5$

$a_2=2,\ b_2=-1,\ c_2=0.$

$d = 3\cdot(-1)-2\cdot 1 = -5 \neq 0.$

Rezolvăm sistemul $\begin{cases} 3t+x-5=0 \\ 2t-x=0 \end{cases}$ +

$5t-5=0 \Rightarrow t_0=1,\ x_0=2.$

Facem sch. var. $\begin{cases} s=t-t_0 \\ y=x-x_0 \end{cases} \Rightarrow \begin{cases} t=s+1 \\ x=y+2 \end{cases}$

$x(t)=y(s(t))$, $\frac{dx}{dt}(t) = \frac{dy}{ds}(s(t))\cdot \underbrace{s'(t)}_{=1} = \frac{dy}{ds}(s(t))$

$$\frac{dy}{ds} = \frac{3(s+1)+y+2-5}{2(s+1)-(y+2)} \Rightarrow \frac{dy}{ds} = \frac{3s+y}{2s-y} \Rightarrow \frac{dy}{ds} = \frac{3+\frac{y}{s}}{2-\frac{y}{s}}$$

ec. omogenă

Sch. var. $z=\frac{y}{s}$, $y=s\cdot z$

$(s\cdot z)' = \frac{3+z}{2-z}$

$$z+sz' = \frac{3+z}{2-z} \Rightarrow sz' = \frac{3+z}{2-z} - z \Rightarrow sz' = \frac{3+z-2z+z^2}{2-z}$$

$\Rightarrow z' = \frac{1}{s}\cdot\frac{z^2-z+3}{2-z}$ ec. cu var. sep. cu sol. st.: $z^2-z+3=0$

$\Delta = 1-12<0,$

nu avem sol. staționare

$$\frac{2-z}{z^2-z+3}\,dz = \frac{1}{s}\,ds.$$

$$\int \frac{2-z}{3-z+z^2}\,dz = \int \frac{2-z}{(z-\frac{1}{2})^2-\frac{1}{4}+3}\,dz = \int \frac{2-z}{(z-\frac{1}{2})^2+\frac{11}{4}}\,dz$$

sch. var. $z-\frac{1}{2}=v \Rightarrow dz=dv$

$$z = v+\frac{1}{2}$$

$$\int \frac{2-v-\frac{1}{2}}{v^2+\frac{11}{4}}\,dv = \frac{3}{2}\int \frac{1}{v^2+\left(\frac{\sqrt{11}}{2}\right)^2}\,dv - \int \frac{v}{v^2+\frac{11}{4}}\,dv =$$

$$= \frac{3}{2}\cdot\frac{1}{\frac{\sqrt{11}}{2}}\,\mathrm{arctg}\left(\frac{2v}{\sqrt{11}}\right) - \frac{1}{2}\cdot\ln\left(v^2+\frac{11}{4}\right)+c$$

$$\frac{3}{\sqrt{11}}\,\mathrm{arctg}\left(\frac{2z-1}{\sqrt{11}}\right) - \frac{1}{2}\ln\left(z^2-z+3\right) = \ln|s|+c,\ c\in\mathbb{R}$$

$$\frac{3\sqrt{11}}{11}\,\mathrm{arctg}\left(\frac{\left(2\frac{y}{s}-1\right)\sqrt{11}}{11}\right) - \frac{1}{2}\ln\left(\frac{y^2}{s^2}-\frac{y}{s}+3\right) = \ln|s|+c,\ c\in\mathbb{R}$$

$$\frac{3\sqrt{11}}{11}\,\mathrm{arctg}\left(\frac{\left(2\frac{x-2}{t-1}-1\right)\sqrt{11}}{11}\right) - \frac{1}{2}\ln\left(\frac{(x-2)^2}{(t-1)^2}-\frac{(x-2)}{(t-1)}+3\right) = \ln|t-1|+c,$$

$c\in\mathbb{R}$.

↳ mulțimea soluțiilor implicite.

Ecuații de ordin superior care admit reducerea ordinului

1) $F(t, x^{(k)}, x^{(k+1)}, \dots, x^{(n)}) = 0$, $k \geq 1$.

Schimbare de variabilă $y = x^{(k)} \Rightarrow F(t, y, y', \dots, y^{(n-k)}) = 0$.

omogenă

2) $F(t, \frac{x'}{x}, \frac{x''}{x}, \dots, \frac{x^{(n)}}{x}) = 0$.

Schimbare de variabilă $y = \frac{x'}{x} \Rightarrow G(t, y, y', \dots, y^{(n-1)}) = 0$

autonomă

3) $F(x, x', \dots, x^{(n)}) = 0$

Schimbare de variabilă $x' = y(x) \Rightarrow G(x, y, y', \dots, y^{(n-1)}) = 0$

4) Ecuații Euler, $F(x, tx', t^2x'', \dots, t^n x^{(n)}) = 0$

Schimbare de variabilă $|t| = e^s \Rightarrow G(y, y', \dots, y^{(n)}) = 0$.

• Ecuații liniare de ordinul al II-lea cu coeficienți constanți

$x'' + ax' + bx = 0$, $a, b \in \mathbb{R}$.

Asociem ecuația caracteristică $\lambda^2 + a\lambda + b = 0$ $\begin{cases} \lambda_1 \\ \lambda_2 \end{cases}$

Sol. generală:

- $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2 \Rightarrow x(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t}$, $c_1, c_2 \in \mathbb{R}$
- $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \Rightarrow x(t) = c_1 \cdot e^{\lambda t} + c_2 \cdot t e^{\lambda t}$, $c_1, c_2 \in \mathbb{R}$
- $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R} \Rightarrow x(t) = c_1 \cdot e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \cdot \sin(\beta t)$, $c_1, c_2 \in \mathbb{R}$.

1) Să se determine soluția generală:

a) $t x'' + x' + t = 0$. → tipul 1)

Sch. var. $y = x'$

$t y' + y + t = 0 \Rightarrow \frac{dy}{dt} = -\frac{1}{t} y + (-1)$, $t \neq 0$, pp $t > 0$.

→ ec. afină

Ec. lin. asociată: $\overline{y}' = -\frac{\overline{y}}{t}$ cu soluția $\overline{y}(t) = c \cdot e^{\int -\frac{1}{t} dt} =$

$= c \cdot e^{-\ln t} = \frac{c}{t}$

Căutăm sol. de forma $y(t) = \frac{c(t)}{t}$

$\left(\frac{c(t)}{t}\right)' = -\frac{c(t)}{t^2} - 1 \Rightarrow \frac{c'(t) \cdot t - c(t)}{t^2} = \frac{-c(t)}{t^2} - 1 \Rightarrow$

$\Rightarrow \frac{c'(t) \cdot t}{t^2} = -1 \Rightarrow c'(t) = -t$

$c(t) = \int -t \, dt = -\frac{t^2}{2} + k, k \in \mathbb{R}$

$y(t) = \frac{1}{t}\left(-\frac{t^2}{2} + k\right), k \in \mathbb{R}$

$y(t) = -\frac{t}{2} + \frac{k}{t}, k \in \mathbb{R}$

$\parallel$

$x'(t) \Rightarrow x(t) = \int -\frac{t}{2} + \frac{k}{t} \, dt = -\frac{t^2}{4} + k \ln t + c, \; k, c \in \mathbb{R}.$

b) $t^2 x'' - t x' - 3x = 0$. $\rightarrow$ ec. Euler.

Sch. var. $|t| = e^s$ $\begin{cases} t = e^s, t > 0 \\ t = -e^s, t < 0 \end{cases}$

Pp. $t > 0, t = e^s$

$x(e^s) = y(s)$

Derivăm, $x'(e^s) \cdot e^s = y'(s) \Rightarrow x'(e^s) = e^{-s} \cdot y'(s)$

Derivăm încă o dată

$x''(e^s) \cdot e^s = -e^{-s} y'(s) + e^{-s} y''(s) \Rightarrow$

$\Rightarrow x''(e^s) = -e^{-2s} y'(s) + e^{-2s} y''(s)$

Înlocuim

$\Rightarrow e^{2s}\left(-e^{-2s} y'(s) + e^{-2s} y''(s)\right) - e^s \cdot e^{-s} \cdot y'(s) - 3y(s) = 0$

$\Rightarrow -y'(s) + y''(s) - y'(s) - 3y(s) = 0$

$\Rightarrow y''(s) - 2y'(s) - 3y(s) = 0 \rightarrow$ ec. lin. de ord II cu coef constanți

$\lambda^2 - 2\lambda - 3 = 0, \; \lambda_1 = 3$

$\lambda_2 = -1$

Soluția generală: $y(s) = c_1 e^{3s} + c_2 e^{-s}, \; c_1, c_2 \in \mathbb{R}.$

$\cdot\ y(s) = x(e^s) \Rightarrow y(\ln t) = x(t)$

$e^s = t$

$\Updownarrow$

$s = \ln t$

$x(t) = c_1 \cdot e^{3\ln t} + c_2 e^{-\ln t} = c_1 t^3 + c_2 t^{-1},\ c_1, c_2 \in \mathbb{R}.$