— continuare —

1) $x^2+(x')^2-2xx''=0$ → ecuație autonomă

Facem schimbarea de variabilă $x'=y(x)$, derivăm →

$\Rightarrow x''=y'(x)\cdot x'=y'(x)\cdot y(x)$

$\Rightarrow x^2+y^2(x)-2x\cdot y'(x)\cdot y(x)=0$

$2x\cdot y'(x)\cdot y(x)=x^2+y^2(x) \Rightarrow$

$\Rightarrow y'(x)=\dfrac{x^2}{2x\cdot y(x)}+\dfrac{y^2(x)}{2x\cdot y(x)}=\dfrac{x}{2y(x)}+\dfrac{y(x)}{2x}$

deci $\dfrac{dy}{dx}=\dfrac{1}{2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)$ → ecuație Bernoulli,

$y'(x)=\underbrace{\dfrac{1}{2x}}_{a(x)}\cdot y+\underbrace{\dfrac{x}{2}}_{b(x)}\cdot y^{-1}$, $\alpha=-1$

Scriem ec. liniară asociată: $\bar{y}'=\dfrac{1}{2x}\bar{y}$

cu soluția generală, $\bar{y}(x)=c\cdot e^{\int\frac{1}{2x}dx}=c\cdot e^{\frac{1}{2}\ln x}=c\cdot x^{\frac{1}{2}}=c\sqrt{x}$

Căutăm sol. de forma $y(x)=c(x)\cdot\sqrt{x}$ (metoda var. constantelor)

$(c(x)\sqrt{x})'=\dfrac{1}{2x}\cdot c(x)\cdot\sqrt{x}+\dfrac{x}{2}\cdot\dfrac{1}{c(x)\cdot\sqrt{x}}$

$\Rightarrow c'(x)\sqrt{x}+\cancel{\dfrac{1}{2\sqrt{x}}\cdot c(x)}=\cancel{\dfrac{1}{2x}\cdot c(x)\sqrt{x}}+\dfrac{x}{2}\cdot\dfrac{1}{c(x)\cdot\sqrt{x}}$

$\Rightarrow c'(x)\cdot\sqrt{x}=\dfrac{\sqrt{x}}{2\cdot c(x)} \Rightarrow c'(x)=\dfrac{1}{2c(x)} \Rightarrow \dfrac{dc}{dx}=\dfrac{1}{2c}$ → ec. cu variabile separabile

cu soluțiile staționare: $\dfrac{1}{2c}=0$ (F)

$2c\,dc=dx \Rightarrow \int 2c\,dc=\int dx \Rightarrow c^2=x+k$, $k\in\mathbb{R}$

$\Rightarrow c(x)=\pm\sqrt{x+k}$, $k\in\mathbb{R}$.

deci $\underset{x'(t)}{\underset{\parallel}{y(x)}}=\pm\sqrt{x+k}\cdot\sqrt{x}$, $k\in\mathbb{R}$

$x'(t)=\pm\sqrt{x(t)+k}\cdot\sqrt{x(t)}$, $k\in\mathbb{R}$

$\dfrac{dx}{dt}=\sqrt{x+k}\cdot\sqrt{x} \Rightarrow \dfrac{dx}{dt}=\sqrt{x^2+kx}$ → ec. cu var. sep

cu sol. staționare: $\sqrt{x^2+kx}=0 \Rightarrow x(x+k)=0 \Rightarrow \begin{cases} x=0 \\ x=-k, k\in\mathbb{R}\end{cases}$

$$\int \frac{dx}{\sqrt{x^2+kx}} = \int dt$$

$$\int \frac{1}{\sqrt{(x+\frac{k}{2})^2-\frac{k^2}{4}}}\,dx = \int \frac{1}{\sqrt{w^2-\frac{k^2}{4}}}\,dw = \ln\left|w+\sqrt{w^2-\frac{k^2}{4}}\right| + c,\ c\in\mathbb{R}$$

$$x+\frac{k}{2} = w,\ dx = dw$$

$$= \ln\left|x+\frac{k}{2}+\sqrt{x^2+kx}\right| + c,\ c\in\mathbb{R}.$$

$$\Rightarrow \begin{cases} \ln\left|x+\frac{k}{2}+\sqrt{x^2+kx}\right| = t+c,\ c\in\mathbb{R} \\ x_1(t)\equiv 0 \\ x_2(t) = k,\ k\in\mathbb{R} \end{cases}$$

În celălalt caz, $x' = -\sqrt{x^2+kx}$, $-\ln\left|x+\frac{k}{2}+\sqrt{x^2+kx}\right| = t+c$, $c\in\mathbb{R}$

2) $t\,x\,x'' + t(x')^2 - x x' = 0$, $t>0$

Observăm că $x(t)\equiv 0$ verifică ecuația.

Pp. $x\neq 0$ în continuare, $t\frac{x''}{x} + t\frac{(x')^2}{x^2} - \frac{x'}{x} = 0$ (am împărțit prin x^2).

$$\Rightarrow t\frac{x''}{x} + t\cdot\left(\frac{x'}{x}\right)^2 - \frac{x'}{x} = 0 \rightarrow \text{ec. omogenă.}$$

Schimbare de variabilă $y = \frac{x'}{x}$

$$x'(t) = x(t)\cdot y(t)$$

derivăm, $x''(t) = x'(t)\cdot y(t) + x(t)\cdot y'(t)$

$$\Rightarrow \frac{x''(t)}{x(t)} = \underbrace{\frac{x'(t)}{x(t)}}_{y(t)}\cdot y(t) + y'(t)$$

$$\Rightarrow \frac{x''(t)}{x(t)} = y^2(t) + y'(t)$$

$$\Rightarrow t\cdot(y^2(t)+y'(t)) + t\cdot y^2(t) - y(t) = 0$$

$$t\cdot y'(t) + 2t\,y^2(t) - y(t) = 0$$

$$y'(t) = \frac{y(t)}{t} - 2y^2(t)$$

$$y' = \frac{1}{t}\cdot y - 2y^2 \rightarrow \text{Bernoulli},\ \alpha = 2.$$

Ec. liniară asociată: $\bar{y}' = \frac{1}{t} \cdot \bar{y}$

cu soluția $\bar{y}(t) = c \cdot e^{\int \frac{1}{t} dt} = c \cdot e^{\ln t} = c \cdot t,\ c \in \mathbb{R}$.

Căutăm soluții de forma $y(t) = c(t) \cdot t$ (metoda variației constantelor)

$(c(t) \cdot t)' = \frac{1}{t} \cdot c(t) \cdot t - 2 \cdot c^2(t) \cdot t^2$

$c'(t) \cdot t + \cancel{c(t)} = \cancel{c(t)} - 2c^2(t) \cdot t^2$

$c'(t) = -2c^2(t) \cdot t$

$c' = -2c^2 \cdot t \rightarrow$ ec. cu variabile separabile

cu sol. staționare: $-2c^2 = 0 \Rightarrow c_0(t) \equiv 0$.

$\frac{dc}{dt} = -2c^2 \cdot t \Rightarrow \frac{dc}{-2c^2} = t\,dt \Rightarrow \int \frac{dc}{-2c^2} = \int t\,dt \Rightarrow$

$\Rightarrow -\frac{1}{2} \int \frac{1}{c^2} dc = \int t\,dt \Rightarrow -\frac{1}{2} \cdot \left(-\frac{1}{c}\right) = \frac{t^2}{2} + k,\ k \in \mathbb{R}$

$\frac{1}{2c} = \frac{t^2 + k}{2},\ k \in \mathbb{R} \Rightarrow 2c = \frac{2}{t^2 + k},\ k \in \mathbb{R}$, de unde $c = \frac{1}{t^2 + k},\ k \in \mathbb{R}$

$$\begin{cases} y(t) = \frac{t}{t^2 + k},\ k \in \mathbb{R} \\ y_0(t) \equiv 0. \end{cases}$$

$\frac{x'}{x} = 0 \Rightarrow x' = 0 \Rightarrow x_c(t) \equiv c,\ c \in \mathbb{R}$

și $\frac{x'}{x} = \frac{t}{t^2 + k} \Rightarrow x' = \frac{t}{t^2 + k} \cdot x \rightarrow$ ec. liniară cu soluția:

$x(t) = g \cdot e^{\int \frac{t}{t^2 + k} dt} = g \cdot e^{\frac{1}{2} \int \frac{1}{s} ds} =$

$t^2 + k = s \Rightarrow 2t\,dt = ds \Rightarrow t\,dt = \frac{1}{2} ds$

$= g \cdot e^{\frac{1}{2} \ln s} = g \cdot e^{\frac{1}{2} \ln(t^2 + k)} = g \cdot (t^2 + k)^{\frac{1}{2}} = g\sqrt{t^2 + k},\ g \in \mathbb{R},\ k \in \mathbb{R}$.

$$\begin{cases} x_c(t) \equiv c,\ c \in \mathbb{R} \\ x_{g,k}(t) = g\sqrt{t^2 + k},\ k, g \in \mathbb{R} \end{cases}$$

3) Să se determine mulțimea soluțiilor ecuației:

$x = t\cdot(x')^2 + (x')^3$.

ec. Lagrange: $x = t\cdot\varphi(x') + \psi(x')$

$\varphi(x') = (x')^2$, $\psi(x') = (x')^3$, $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ derivabile

Fie $\boxed{p = x'}$ $\Rightarrow$ $x = tp^2 + p^3$

Derivăm ecuația: $x' = t'p^2 + t\cdot(p^2)' + (p^3)' \Rightarrow$

$\Rightarrow \underbrace{x'}_{p} = p^2 + 2tpp' + 3p^2\cdot p' \Rightarrow$

$\Rightarrow p - p^2 = p'(2tp + 3p^2) \Rightarrow \dfrac{dp}{dt} = \dfrac{p-p^2}{2tp+3p^2}$

$(t,p) \xrightarrow[\text{ecuația răsturnată}]{} (p,t)$

$\dfrac{dt}{dp} = \dfrac{2tp+3p^2}{p-p^2} \Rightarrow \dfrac{dt}{dp} = \dfrac{\cancel{p}(2t+3p)}{\cancel{p}(1-p)} \Rightarrow \dfrac{dt}{dp} = \dfrac{2t+3p}{1-p}$

$\dfrac{dt}{dp} = \dfrac{2}{1-p}\cdot t + \dfrac{3p}{1-p}$.

Verific dacă $p=0$ este soluție. $p = 0 \Rightarrow x' = 0 \Rightarrow x = c$, $c\in\mathbb{R}$

$c = t\cdot(c')^2 + (c')^3 \Rightarrow c = 0 \Rightarrow$

$x(t) \equiv 0$ e sol.

$p \neq 0 \Rightarrow \dfrac{dt}{dp} = \dfrac{2}{1-p}\cdot t + \dfrac{3p}{1-p}$ ec. afină

Scriem ec. liniară asociată: $\dfrac{d\bar{t}}{dp} = \dfrac{2}{1-p}\cdot\bar{t} \Rightarrow \bar{t}(p) = c\cdot e^{\int\frac{2}{1-p}dp} =$

$= c\cdot e^{2\cdot(-1)\cdot\ln|1-p|} = c\cdot|1-p|^{-2} = c\cdot\dfrac{1}{(1-p)^2}$, $c\in\mathbb{R}$

Aplicăm metoda variației constantelor,

căutăm soluții de forma: $t(p) = c(p)\cdot\dfrac{1}{(1-p)^2}$

$\left(c(p)\cdot\dfrac{1}{(1-p)^2}\right)' = \dfrac{2}{1-p}\cdot c(p)\dfrac{1}{(1-p)^2} + \dfrac{3p}{1-p}$

$\Rightarrow c'(p)\cdot\dfrac{1}{(1-p)^2} + c(p)\cdot\dfrac{-2\cancel{(1-p)}\cdot(-1)}{(1-p)^{\cancel{4}\,3}} = \dfrac{2\cdot c(p)}{(1-p)^3} + \dfrac{3p}{1-p}$

se reduc

$\Rightarrow c'(p) = \dfrac{3p}{1-p}\cdot(1-p)^2 \Rightarrow c'(p) = 3p(1-p) \Rightarrow$

$c'(p) = 3p - 3p^2$

$c(p) = \int 3p - 3p^2 dp = \frac{3p^2}{2} - p^3 + k, k \in \mathbb{R}$

$t(p) = \left(\frac{3p^2}{2} - p^3 + k\right) \cdot \frac{1}{(1-p)^2}, k \in \mathbb{R}$

Mulțimea soluțiilor parametrice : $\begin{cases} x = tp^2 + p^3 \\ t = \left(\frac{3p^2}{2} - p^3 + k\right) \cdot \frac{1}{(1-p)^2}, k \in \mathbb{R} \end{cases}$

5) $x = 2tx' - (x')^2$

ec. Lagrange : $x = t \cdot \varphi(x') + \psi(x')$

$\varphi(x') = 2x'$

$\psi(x') = -(x')^2$, $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ derivabile

Fie $p = x' \Rightarrow \quad x = 2tp - p^2$

derivăm $\Rightarrow \quad x' = 2 \cdot t' \cdot p + 2tp' - 2p \cdot p'$

$\underset{p}{x'} = 2p + 2tp' - 2pp'$

$\Rightarrow -p = p'(2t - 2p) \Rightarrow p' = \frac{-p}{2t - 2p} \Rightarrow$

$\Rightarrow \frac{dp}{dt} = \frac{-p}{2t-2p}$, scriem ec. răsturnată, $\frac{dt}{dp} = \frac{2t-2p}{-p}$

$\Rightarrow \frac{dt}{dp} = \frac{2p - 2t}{p} \Rightarrow \frac{dt}{dp} = 2 + t \cdot \left(-\frac{2}{p}\right)$

Verific dacă $p = 0$ este soluție : $p = 0 \Rightarrow x' = 0 \Rightarrow x = c, c \in \mathbb{R}$

$c = 2t \cdot 0 - 0 \Rightarrow c = 0 \Rightarrow x(t) \equiv 0$ e sol.

$p \neq 0 \Rightarrow \frac{dt}{dp} = 2 + t \cdot \left(-\frac{2}{p}\right)$ ec. afină

ec. lin. asociată : $\frac{d\bar{t}}{dp} = \bar{t} \cdot \left(-\frac{2}{p}\right) \Rightarrow \bar{t}(p) = c \cdot e^{\int -\frac{2}{p} dp} =$

$= c \cdot e^{-2 \ln p} = c \cdot p^{-2} = \frac{c}{p^2}, c \in \mathbb{R}$

Căutăm sol. de forma : $t(p) = \frac{c(p)}{p^2}$

$\left(\frac{c(p)}{p^2}\right)' = 2 + \frac{c(p)}{p^2} \cdot \left(-\frac{2}{p}\right)$

$\Rightarrow \frac{c'(p) \cdot p^2 - c(p) \cdot 2p}{p^4} = 2 - \frac{2c(p)}{p^3} \Rightarrow \frac{c'(p)}{p^2} = 2 \Rightarrow$

$\Rightarrow c'(p) = 2p^2 \Rightarrow c(p) = 2 \cdot \frac{p^3}{3} + k,\ k \in \mathbb{R}$

$$t(p) = \frac{2p^3}{3} \cdot \frac{1}{p^2} + \frac{k}{p^2} = \frac{2p}{3} + \frac{k}{p^2}\ k \in \mathbb{R}$$

Mulțimea soluțiilor parametrice: $\begin{cases} x = 2tp - p^2 \\ t = \frac{2p}{3} + \frac{k}{p^2},\ k \in \mathbb{R} \end{cases}$