Ecuaţii Riccati

$x' = a(t)\cdot x^2 + b(t)\cdot x + c(t)$, $\quad a, b, c : I \subseteq \mathbb{R} \to \mathbb{R}$ continue

$\exists\ \varphi_0(\cdot)$ soluţie particulară a.î. $y = x - \varphi_0(t)$ → schimbare de variabilă.

1) $t x' - (2t+1)\cdot x + x^2 + t^2 = 0$, $\quad \varphi_0(t)$ - polinom gr I

$x' = -\frac{1}{t} x^2 + \frac{2t+1}{t} \cdot x - t$, $\quad t \neq 0$, pp. $t > 0$.

Fie $\varphi_0(t) = at + b$, $a, b \in \mathbb{R}$ - soluţia particulară

$(at+b)' = -\frac{1}{t}\cdot(at+b)^2 + \frac{2t+1}{t}\cdot(at+b) - t$

$\Rightarrow a = -\frac{1}{t}(a^2t^2 + 2abt + b^2) + \frac{2t+1}{t}(at+b) - t$

$\Rightarrow at = \underline{-a^2t^2} - 2abt - b^2 + \underline{2at^2} + at + 2bt + b \underline{-t^2}$

$\Rightarrow t^2(-a^2 + 2a - 1) + t\cdot(-2ab + a + 2b - a) - b^2 + b = 0$, $\forall t \in D_{\varphi_0}$ dom. de def. al lui φ_0.

$$\Rightarrow \begin{cases} -a^2 + 2a - 1 = 0 \\ -2ab + \cancel{a} + 2b - \cancel{a} = 0 \\ -b^2 + b = 0 \end{cases} \Rightarrow \begin{array}{l} a = 1 \\ 0 = 0 \\ b(b-1) = 0 \Rightarrow b = 1 \text{ sau } b = 0 \end{array}$$

$\Rightarrow \varphi_0(t) = t$ sau $\varphi_0(t) = t + 1$.

$y(t) = x(t) - t$

$\Rightarrow y(t) + t = x(t)$

$\Rightarrow t\cdot(y(t)+t)' - (2t+1)\cdot(y(t)+t) + (y(t)+t)^2 + t^2 = 0$

$\Rightarrow t\cdot y'(t) + \cancel{t} - \cancel{2t y(t)} - \cancel{2t^2} - y(t) \cancel{-t} + y^2(t) + \cancel{2\cdot y(t)\cdot t} + \cancel{t^2} + \cancel{t^2} = 0$

$\Rightarrow t\cdot y'(t) - y(t) + y^2(t) = 0 \Rightarrow$

$\Rightarrow y'(t) = -\frac{1}{t}\cdot y^2(t) + \frac{1}{t}\cdot y(t) \qquad y' = -\frac{1}{t} y^2 + \frac{1}{t}\cdot y$

$y'(t) = \frac{1}{t}\cdot y(t) - \frac{1}{t} y^2(t)$ → ecuaţie Bernoulli, $\alpha = 2$

sau, <u>mai simplu</u>, ec. cu var. separabile,

$y' = \frac{y - y^2}{t}$ ⇒ cu sol. staţionare date de

$y - y^2 = 0 \Rightarrow y_1(t) \equiv 0$, $y_2(t) \equiv 1$

$$\frac{y'}{y-y^2} = \frac{1}{t} \Rightarrow \int \frac{1}{y-y^2}\,dy = \int \frac{1}{t}\,dt$$

$$\Rightarrow -\int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \int \frac{1}{t}\,dt$$

$$\Rightarrow -\ln|y-1| + \ln|y| = \ln t + k, \ k\in\mathbb{R}$$

$$\Rightarrow \ln\left|\frac{y}{y-1}\right| = \ln t + \underline{k}, \ k\in\mathbb{R}$$

$$\ln\left|\frac{y}{y-1}\right| = \ln t + \underline{\ln c}, \ c\in\mathbb{R}_+^* \Rightarrow \left|\frac{y}{y-1}\right| = t\cdot c, \ c\in\mathbb{R}_+^* \Rightarrow$$

$$\Rightarrow \frac{y}{y-1} = c\cdot t, \ c\in\mathbb{R}^*$$

$$y = cty - ct \Rightarrow y = \frac{-ct}{1-ct}, \ c\in\mathbb{R}^*$$

$$\text{deci } y = \frac{ct}{1+ct}, \ c\in\mathbb{R}^*$$

$$\begin{cases} x_c(t) = y_c(t) + t = \frac{ct}{1+ct} + t, \ c\in\mathbb{R}^* \\ x_1(t) = y_1(t) + t = t \\ x_2(t) = y_2(t) + t = 1+t \end{cases}$$

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2) $x' = \underline{x^2 - 2xe^t + e^{2t}} + e^t, \quad \varphi_0(t) = e^t$

$$y(t) = x(t) - \varphi_0(t)$$

$$y(t) = x(t) - e^t \Rightarrow x(t) = y(t) + e^t$$

$$(y(t) + e^t)' = (y(t)+e^t)^2 - 2\cdot(y(t)+e^t)\cdot e^t + e^{2t} + e^t$$

$$y'(t) + \cancel{e^t} = y^2(t) + \cancel{2\cdot y(t)\cdot e^t} + \cancel{e^{2t}} - \cancel{2y(t)\cdot e^t} - \cancel{2e^{2t}} + \cancel{e^{2t}} + \cancel{e^t}$$

$\Rightarrow y'(t) = y^2(t) \Rightarrow \frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \to$ ec. cu var. separab.

cu sol staționare $y^2(t) = 0 \Rightarrow y_1(t) \equiv 0$

$$\int \frac{dy}{y^2} = \int dt \Rightarrow -\frac{1}{y} = t+k, \ k\in\mathbb{R} \Rightarrow y_k(t) = \frac{-1}{t+k}, \ k\in\mathbb{R}$$

$$\begin{cases} x_1(t) = e^t \\ x_k(t) = \frac{-1}{t+k} + e^t, \ k\in\mathbb{R} \end{cases}$$

3) Ec. omogenă $x' = f\left(\frac{x}{t}\right)$

Sch. variabilă $y = \frac{x}{t}$.

$$2t^2 x' = t^2 + x^2$$

$$x' = \frac{1}{2} + \frac{1}{2}\cdot\left(\frac{x}{t}\right)^2, \; t \neq 0, \text{ pp. } t > 0.$$

$$y = \frac{x}{t}$$

$$y(t) = \frac{x(t)}{t} \Rightarrow x(t) = t \cdot y(t)$$

$$(t \cdot y(t))' = \frac{1}{2} + \frac{1}{2}\cdot\left(\frac{t\cdot y(t)}{t}\right)^2 \Rightarrow y(t) + t\cdot y'(t) = \frac{1}{2} + \frac{1}{2}y^2(t)$$

$$\Rightarrow y'(t) = \frac{1}{2t} - \frac{y(t)}{t} + \frac{1}{2t}\cdot y^2(t)$$

$$y'(t) = \frac{1 - 2y(t) + y^2(t)}{2t}$$

$$y' = \frac{(y-1)^2}{2t}$$ → ecuații cu variabile separabile

cu sol. staționare: $(y-1)^2 = 0 \Rightarrow y = 1 \Rightarrow y_1(t) \equiv 1$.

$$\frac{dy}{dt} = \frac{(y-1)^2}{2t} \Rightarrow \frac{dy}{(y-1)^2} = \frac{dt}{2t} \Rightarrow$$

$$\Rightarrow \int \frac{dy}{(y-1)^2} = \int \frac{dt}{2t} \Rightarrow \frac{-1}{y-1} = \frac{1}{2}\cdot \ln t + k, \; k \in \mathbb{R}$$

$$\Rightarrow \frac{1}{y-1} = -\frac{1}{2}\ln t - k, \; k \in \mathbb{R} \Rightarrow$$

$$y - 1 = \frac{1}{-\frac{1}{2}\ln t - k}$$

$$y = 1 - \frac{1}{\frac{1}{2}\ln t + k}, \quad k \in \mathbb{R}$$

$$y_k(t) = 1 - \frac{1}{\frac{1}{2}\ln t + k}, \; k \in \mathbb{R}$$

$$\begin{cases} x_1(t) = t \\ x_k(t) = t - \dfrac{t}{\frac{1}{2}\ln t + k}, \; k \in \mathbb{R} \end{cases}$$

4) $x' = \dfrac{x + \sqrt{tx}}{t}$.

$x' = \dfrac{x}{t} + \dfrac{\sqrt{tx}}{t}$

$x' = \dfrac{x}{t} + \sqrt{\dfrac{tx}{t^2}}$

$x' = \dfrac{x}{t} + \sqrt{\dfrac{x}{t}}$ → ecuație omogenă, $t \neq 0$, pp. $t > 0$.

$y = \dfrac{x}{t} \Rightarrow x = y \cdot t$

$x(t) = t \cdot y(t)$

$(t \cdot y(t))' = \dfrac{t \cdot y(t)}{t} + \sqrt{\dfrac{t\,y(t)}{t}}$

$\Rightarrow \cancel{y(t)} + t y'(t) = \cancel{y(t)} + \sqrt{y(t)} \Rightarrow t \cdot y'(t) = \sqrt{y(t)} \Rightarrow y'(t) = \sqrt{y(t)} \cdot \dfrac{1}{t}$

$y' = \sqrt{y} \cdot \dfrac{1}{t}$ → ecuație cu variabile separabile

cu sol. staționare $\sqrt{y} = 0 \Rightarrow y = 0 \Rightarrow y_1(t) \equiv 0$.

$\dfrac{dy}{dt} = \sqrt{y} \cdot \dfrac{1}{t} \Rightarrow \dfrac{dy}{\sqrt{y}} = \dfrac{dt}{t} \Rightarrow \int \dfrac{dy}{\sqrt{y}} = \int \dfrac{dt}{t} \Rightarrow$

$\Rightarrow \dfrac{\sqrt{y}}{\frac{1}{2}} = \ln t + k$, $k \in \mathbb{R}$

$2\sqrt{y} = \ln t + k$, $k \in \mathbb{R}$

$\sqrt{y} = \frac{1}{2} \ln t + k$, $k \in \mathbb{R} \Rightarrow y_k(t) = \left(\frac{1}{2} \ln t + k\right)^2$, $k \in \mathbb{R}$

$x_k(t) = t \cdot \left(\frac{1}{2} \ln t + k\right)^2$, $k \in \mathbb{R}$

$x_1(t) = 0$.

5) $t x' = x + t \cdot \cos^2 \frac{x}{t}$

$x' = \frac{x}{t} + \cos^2 \frac{x}{t}$ → ecuație omogenă, $t \neq 0$, pp. $t > 0$

Facem schimbarea de variabilă $y = \frac{x}{t}$

$x(t) = t \cdot y(t)$

$\Rightarrow (t \cdot y(t))' = \frac{t \cdot y(t)}{t} + \cos^2 \frac{t \cdot y(t)}{t}$

$\Rightarrow \cancel{y(t)} + t \cdot y'(t) = \cancel{y(t)} + \cos^2 y(t)$

$\Rightarrow t \cdot y'(t) = \cos^2 y(t)$

$\Rightarrow y' = (\cos^2 y) \cdot \frac{1}{t}$ → ec. cu variabile separabile

cu sol. staționară dată de $\cos^2 y = 0 \Rightarrow y = \frac{\pi}{2} \Rightarrow y_1(t) = \frac{\pi}{2}$

$\frac{dy}{dt} = (\cos^2 y) \cdot \frac{1}{t} \Rightarrow \frac{dy}{\cos^2 y} = \frac{1}{t} dt$

$\int \frac{dy}{\cos^2 y} = \int \frac{1}{t} dt$

$\operatorname{tg} y = \ln t + k, k \in \mathbb{R}$

$\Rightarrow y(t) = \operatorname{arctg}(\ln t + k), k \in \mathbb{R}$

$x_1(t) = t \cdot \frac{\pi}{2}$

$x_k(t) = t \cdot \operatorname{arctg}(\ln t + k), k \in \mathbb{R}$.