Homework 5

PHYS209

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(1) Problem 1

(1.1)

Given our differential equations in form $f''(x) + 2f'(x) + f(x) = 0$, we can find the roots of the characteristic equation $r^2 + 2r + 1 = 0$ to be $r = -1$. This gives us the general solution:

$$f(x) = c_1 e^{-x} + c_2 x e^{-x} \tag{1.1}$$

(1.2)

To prove that our differential equation $x^6 f''(x) + 3x^5 f'(x) - f(x) = 0$ can be written in form $\left(x^{a_1}\frac{\mathrm{d}}{\mathrm{dx}} + b_1\right) \cdot \left(x^{a_2}\frac{\mathrm{d}}{\mathrm{dx}} + b_2\right) \cdot f(x) = 0$:

$$\begin{aligned}
&\left(x^{a_1}\frac{\mathrm{d}}{\mathrm{dx}} + b_1\right) \cdot \left(x^{a_2}\frac{\mathrm{d}}{\mathrm{dx}} + b_2\right) f(x) = \left(x^{a_1}\frac{\mathrm{d}}{\mathrm{dx}} + b_1\right) \cdot \left(x^{a_2} f' + b_2 f\right) \\
&= x^{a_1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{a_2} f' + b_2 f\right) + b_1\left(x^{a_2} f' + b_2 f\right) \\
&= x^{a_1}\left(x^{a_2} f'' + a_2 x^{a_2 - 1} f' + b_2 f'\right) + b_1\left(x^{a_2} f' + b_2 f\right) \\
&= x^{a_1 + a_2} f'' + a_2 x^{a_1 + a_2 - 1} f' + b_2 x^{a_1} f' + b_1 x^{a_2} f' + b_1 b_2 f \\
&= x^{a_1 + a_2} f'' + \left(a_2 x^{a_1 + a_2 - 1} + b_2 x^{a_1} + b_1 x^{a_2}\right) f' + b_1 b_2 f
\end{aligned} \tag{1.2}$$

From 1.2, it can be seen that $a_1 + a_2 = 6$, $b_1 b_2 = -1$, and $a_1 = a_2 = 3$. Giving us:

$$\left(x^3\frac{\mathrm{d}}{\mathrm{dx}} + 1\right) \cdot \left(x^3\frac{\mathrm{d}}{\mathrm{dx}} - 1\right) \cdot f(x) = 0 \tag{1.3}$$

(1.3)

Since our operators commute, we can write our differential equation as we can solve our differential equation for each operator separately:

$$\begin{aligned}
&\left(x^3\frac{\mathrm{d}}{\mathrm{d}x}+1\right)\cdot f(x)=0\\
&x^3f'+f=0\\
&\frac{f'}{f}=-\frac{1}{x^3}\\
&\ln(f)=\frac{1}{2x^2}+c_1\\
&f(x)=c_2e^{1/2x^2}
\end{aligned} \tag{1.4}$$

Solving similarly for second operator:

$$f(x)=c_2e^{-1/2x^2} \tag{1.5}$$

Giving us our final solution:

$$f(x)=c_1e^{1/2x^2}+c_2e^{-1/2x^2} \tag{1.6}$$

(1.4)

Given our differential equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}}{\mathrm{d}x}+e^{-2x}\right)f(x)=e^{-2x} \tag{1.7}$$

We can get our substitution parameter $u(x)$ by solving:

$$u(x)=\int\sqrt{e^{-2x}}\mathrm{d}x=\int e^{-x}\mathrm{d}x=-e^{-x}+c_1 \tag{1.8}$$

Finding substitutions for $\frac{\mathrm{d}^2 f}{\mathrm{d}^2 x}$ and $\frac{\mathrm{d}f}{\mathrm{d}x}$:

$$\begin{aligned}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} &= u''\frac{\mathrm{d}f}{\mathrm{d}u} + u'\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}\\ \frac{\mathrm{d}f}{\mathrm{d}x} &= u'\frac{\mathrm{d}f}{\mathrm{d}u}\end{aligned} \tag{1.9}$$

After substituting into our differential equation, we get:

$$\left(\left(u'(x)^2\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \left(u'' + u'(x)\right)\frac{\mathrm{d}}{\mathrm{d}u} + e^{-2x}\right) f(x) = e^{-2x} \tag{1.10}$$

Since $u'(x) = e^{-x}$, $u''(x) = -e^{-x}$, and dividing both sides by $u'(x)^2$:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right) f(u) = 1 \tag{1.11}$$

Solving for $f(u)$, first we find the homogeneous solution:

$$\begin{aligned}&\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right) f(u) = 0\\ &x^2 + 1 = 0\\ &x = \pm i\\ &f(u) = c_1 e^{iu} + c_2 e^{-iu}\\ &f(u) = d_1 \cos(u) + d_2 \sin(u)\end{aligned} \tag{1.12}$$

Now we can find the particular solution by finding impulse response and using convolution:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right) \mathbb{i}(u) = \delta(u) \tag{1.13}$$

Laplace transforming both sides:

$$\left(s^2 + 1\right) \mathbb{I}(s) = 1 \tag{1.14}$$

$$\mathbb{I}(s) = \frac{1}{s^2 + 1} \tag{1.15}$$

Inverse Laplace transforming:

$$\mathbb{i}(u) = \sin(u) \tag{1.16}$$

Now we can use convolution to find the particular solution:

$$f_p(u) = \int_0^{\infty} \mathbb{i}(\tau)\mathrm{d}\tau \tag{1.17}$$

However, since impulse response is zero for $u < 0$, we can simplify our integral:

$$f_p(u) = \int_0^{u} \mathbb{i}(\tau)\mathrm{d}\tau = \int_0^{u} \sin(\tau)\mathrm{d}\tau = -\cos(u) + 1 \tag{1.18}$$

Giving us our final solution:

$$f(u) = a_1\cos(u) + a_2\sin(u) + 1 \tag{1.19}$$

Going back to our original variable x:

$$f(x) = a_1\cos(-e^{-x}) + a_2\sin(-e^{-x}) + 1 \tag{1.20}$$

(1.5)

To solve given differential equation, we can use $h(x) = g^{(2)(x)}$:

$$g^{(4)}(x) + 2g^{(3)}(x) + g^{(2)}(x) = 0 \tag{1.21}$$

Giving us:

$$h''(x) + 2h'(x) + h(x) = 0 \tag{1.22}$$

Solving for $h(x)$ using characteristic equation:

$$h(x) = c_1we^{-x} + c_2xe^{-x} \tag{1.23}$$

And now to find $g(x)$ we can integrate $h(x)$ twice:

$$g'(x) = \int h(x)\mathrm{dx} = \int c_1e^{-x} + c_2xe^{-x}\mathrm{dx} = c_1e^{-x} + c_2xe^{-x} + c_3 \tag{1.24}$$

$$g(x) = \int g'(x)\mathrm{dx} = \int c_1e^{-x} + c_2xe^{-x} + c_3\mathrm{dx} = d_1e^{-x} + d_2xe^{-x} + d_3x + d_4 \tag{1.25}$$

Giving us our final solution:

$$g(x) = d_1 e^{-x} + d_2 x e^{-x} + d_3 x + d_4 \tag{1.26}$$

(1.6)

If general third order differential equation is given as:

$$\left(p(x)\frac{\mathrm{d}^3}{\mathrm{d}x^3} + q(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r(x)\frac{\mathrm{d}}{\mathrm{d}x} + s(x) \right) f(x) = 0 \tag{1.27}$$

For our differential equation to be exact or in other words:

$$\frac{\mathrm{d}}{\mathrm{dx}} \left(\left[a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x} + c(x) \right] f(x) \right) = 0 \tag{1.28}$$

We need to have:

$$\begin{aligned} a &= p \\ b &= q - \frac{\mathrm{d}p}{\mathrm{d}x} \\ c &= \int s \mathrm{d}x \end{aligned} \tag{1.29}$$

And therefore condition becomes:

$$r = q' - p'' + \int s \mathrm{d}x \tag{1.30}$$

For our given differential equation:

$$\begin{aligned} p(x) &= x \\ q(x) &= 1 \\ r(x) &= 1/x \\ s(x) &= -1/x^2 \end{aligned} \tag{1.31}$$

Substituting into 1.30:

$$\begin{aligned} r &= q' - p'' + \int s\mathrm{d}x \\ \frac{1}{x} &= 0 - 0 + \int -\frac{1}{x^2}\mathrm{d}x \\ \frac{1}{x} &= \frac{1}{x} \end{aligned} \tag{1.32}$$

Which proves that our differential equation is exact.

(1.7)

Given our differential equation $(x-1)f''(x) - xf'(x) + f(x) = 0$ and of its solutions e^x, we can write it as first order differential equation, by using $f(x) = e^x g(x)$:

$$\begin{aligned} f'(x) &= e^x g'(x) + e^x g(x) \\ f''(x) &= e^x g''(x) + 2e^x g'(x) + e^x g(x) \end{aligned} \tag{1.33}$$

Substituting into our differential equation:

$$\begin{aligned} &(x-1)\left[g''(x)e^x + 2g'(x)e^x + g(x)e^x\right] - x\left[g'(x)e^x + g(x)e^x\right] + g(x)e^x = 0 \\ &(x-1)g''(x) + (2x-3)g'(x) = 0 \end{aligned} \tag{1.34}$$

Replacing $g'(x)$ with $h(x)$:

$$(x-1)h'(x) + (2x-3)h(x) = 0 \tag{1.35}$$

Which proves we can rewrite it as first order differential equation.