Homework 9

PHYS209

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(1)

(1.1)

Let us have differential equation:

$$f''(x) - (x+1)f(x) = 0 \tag{1.1}$$

We can solve given differentialequation by using $f(x) = \sum_{k=0}^{\infty} a_k (x+1)^k$. From term by term differentiation we get:

$$\begin{aligned} f'(x) &= \sum_{k=0}^{\infty} a_k k (x+1)^{k-1} \\ f''(x) &= \sum_{k=0}^{\infty} a_k k(k-1)(x+1)^{k-2} \end{aligned} \tag{1.2}$$

Using (1.2) in (1.1) we get:

$$\sum_{k=0}^{\infty} a_k k(k-1)(x+1)^{k-2} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.3}$$

To simplfy the equation we can shift the index of first summation by 3:

$$\sum_{k=-3}^{\infty} a_{k+3}(k+3)(k+2)(x+1)^{k+1} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.4}$$

Since value of the first sum for $k = -3$ and $k = -2$ is zero, we can start the first sum from $k = -1$:

$$\sum_{k=-1}^{\infty} a_{k+3}(k+3)(k+2)(x+1)^{k+1} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.5}$$

Let's evaluate the first sum for $k = -1$, and get it out of the sum:

$$2a_2(x+1)^0 + \sum_{k=0}^{\infty} a_{k+3}(k+3)(k+2)(x+1)^{k+1} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.6}$$

Since 1.6 must hold for all ordinary points, therefore all $x \in \mathbb{C}$, it must also hold for $x = -1$. Therefore:

$$2a_2 + 0 - 0 = 0 \Rightarrow a_2 = 0 \tag{1.7}$$

Now we can simplify (1.6):

$$\sum_{k=0}^{\infty} a_{k+3}(k+3)(k+2)(x+1)^{k+1} - \sum_{k=0}^{\infty} a_k(x+1)^{k+1} = 0 \tag{1.8}$$

Bringing to sums together:

$$\sum_{k=0}^{\infty} \left[a_{k+3}(k+3)(k+2) - a_k\right](x+1)^{k+1} = 0 \tag{1.9}$$

(1.2)

From orthogonality condition we know that:

$$a_{k+3}(k+3)(k+2) - a_k = 0 \tag{1.10}$$

However this condition doesn't give us direct relationship between a_k and a_{k+1}. Instead, a relationship between a_k and a_{k+3}:

$$a_{k+3} = \frac{a_k}{(k+3)(k+2)} \tag{1.11}$$

So than, we can write all a_k for $k > 2$ in terms of a_0, a_1 and a_2:

$$\begin{gathered} a_0 = a_0 \\ a_1 = a_1 \\ a_2 = 0 \\ a_3 = \frac{a_0}{3 \cdot 2} \\ a_4 = \frac{a_1}{4 \cdot 3} \\ a_5 = 0 \end{gathered} \tag{1.12}$$

From previous we can write general form of a_k as follows:

$$a_k = \begin{cases} \dfrac{a_0}{(2\cdot 3)(5\cdot 6)...(3n-1)\cdot 3n} & k = 3n \\ \dfrac{a_1}{(3\cdot 4)(6\cdot 7)...3n\cdot(3n+1)} & k = 3n+1 \\ 0 & k = 3n+2 \end{cases} \tag{1.13}$$

(1.3)

Inserting a_k into $f(x)$ we get:

$$f(x) = c_1 f_1(x) + c_2 f_2(x)$$
$$f(x) = a_0 \sum_0^\infty \left(\frac{(x+1)^{3n}}{(2\cdot 3)(5\cdot 6)...(3n-1)\cdot 3n}\right) + a_1 \sum_0^\infty \left(\frac{(x+1)^{3n+1}}{(3\cdot 4)(6\cdot 7)...3n\cdot(3n+1)}\right) \tag{1.14}$$

These functions are actually pretty common in physics, and they are called Airy functions, and by convention called, $Ai(x)$ and $Bi(x)$

(1.4)

We now need to prove linear independence of $Ai(x)$ and $Bi(x)$. To do so we compute Wronskian:

$$W(x) = \begin{vmatrix} Ai(x) & Bi(x) \\ Ai'(x) & Bi'(x) \end{vmatrix} \tag{1.15}$$

Where $Ai'(x)$ and $Bi'(x)$ are derivatives of $Ai(x)$ and $Bi(x)$ respectively, and are given by:

$$\begin{aligned} Ai'(x) &= \sum_0^\infty \left(\frac{3n(x+1)^{3n-1}}{(2\cdot 3)(5\cdot 6)...(3n-1)\cdot 3n}\right) \\ Bi'(x) &= \sum_0^\infty \left(\frac{(3n+1)(x+1)^{3n}}{(3\cdot 4)(6\cdot 7)...3n\cdot(3n+1)}\right) \end{aligned} \tag{1.16}$$

It can be proven that $W(x)$ is constant.

$$
\begin{gathered}
W(x) = Ai(x)Bi'(x) - Ai'(x)Bi(x) \\
W'(x) = Ai'(x)Bi'(x) + Ai(x)Bi''(x) - Ai''(x)Bi(x) - Ai'(x)Bi'(x) \\
Ai''(x) = (x+1)Ai(x) \\
Bi''(x) = (x+1)Bi(x) \\
W'(x) = (x+1)Ai(x)Bi'(x) + (x+1)Ai(x)Bi'(x) - \\
(x+1)Ai(x)Bi'(x) - (x+1)Ai(x)Bi'(x) \\
W'(x) = 0
\end{gathered} \tag{1.17}
$$

Therefore $W(x)$ is constant. Since $W(x)$ is constantç we can evaluate it at any point. Let's evaluate it at $x = -1$: $Ai(-1) = 1$, $Bi(-1) = 0$, $Ai'(-1) = 0$, $Bi'(-1) = 1$, therefore, Wronskian at $x = -1$ is 0