Homework 11

PHYS209

Hikmat Gulaliyev

22.12.2023

(1) Problem 1

(1.1)

By eulers identity:

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ \cos(\theta) &= \frac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin(\theta) &= \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{aligned} \tag{1.1}$$

Using these we can find fourier transform of sine and cosine functions. We can denote F.T as $\mathcal{F}$.

$$\begin{gathered} \mathcal{F}\cos = \int_{-\infty}^{\infty} \cos(x)e^{-ikx}dx = \int_{-\infty}^{\infty} \frac{e^{ix}+e^{-ix}}{2}e^{-ikx}dx = \\ \frac{1}{2}\int_{-\infty}^{\infty} e^{i(1-k)x}dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{i(-1-k)x}dx \end{gathered} \tag{1.2}$$

Using $\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk}dx$, and since Dirawc-delta is even:

$$\mathcal{F}\cos = \pi\left[\delta(k-1)+\delta(k+1)\right] \tag{1.3}$$

Similarly:

$$\begin{gathered} \mathcal{F}\sin = \int_{-\infty}^{\infty} \sin(x)e^{-ikx}dx = \int_{-\infty}^{\infty} \frac{e^{ix}-e^{-ix}}{2i}e^{-ikx}dx = \\ \frac{1}{2i}\int_{-\infty}^{\infty} e^{i(1-k)x}dx - \frac{1}{2i}\int_{-\infty}^{\infty} e^{i(-1-k)x}dx \end{gathered} \tag{1.4}$$

$$\mathcal{F}\sin = \pi i\left[\delta(k+1)-\delta(k-1)\right] \tag{1.5}$$

(1.2)

Fourier transform can be written as:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(y)e^{ikx}dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(z)e^{-ikz}dz\right]e^{ikx}dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)e^{ik(x-z)}dzdy \end{aligned} \tag{1.6}$$

Changing order of integration:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)e^{ik(x-z)}dydz \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\int_{-\infty}^{\infty} e^{ik(x-z)}dydz \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)2\pi\delta(x-z)dz \\ &= \int_{-\infty}^{\infty} f(z)\delta(x-z)dz \\ &= f(x) \end{aligned} \tag{1.7}$$

(1.3)

To find the F.T. of $m(t) = \cos(ft)s(t)$:

$$\begin{aligned} M(k) = \int_{-\infty}^{\infty} m(t)e^{-ikt}dt = \int_{-\infty}^{\infty}\cos(ft)s(t)e^{-ikt}dt = \\ \int_{-\infty}^{\infty}\frac{e^{ift}+e^{-ift}}{2}s(t)e^{-ikt}dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{i(f-k)t}s(t)dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(f+k)t}s(t)dt \\ = \frac{1}{2}\left[S(f-k)+S(f+k)\right] \end{aligned} \tag{1.8}$$

(2) Problem 2

(2.1)

Since fundamental period of $\cos(x)$ is 2π, and fundamental period of $\sin(2x)$ is π, fundamental period of $f(x) = \cos(x)+\sin(2x)$ is 2π the least common denominator of 2π and π. Therefore $f(x)$ is periodic with period 2π, meaning $f(x+2\pi) = f(x)$. Meaning smallest value for T is 2π.

(2.2)

Let $f(x) = \cos(x) + \sin(2x)$, and T $= 4\pi$. Using $f_p(x) = f(x - T\left\lfloor \frac{x}{T} \right\rfloor)$, we can find $f_p(x)$ for x $= \{10, 11, 21, 22\}$:

$$\begin{aligned} f_p(10) &= f\left(10 - 4\pi\left\lfloor\frac{10}{4\pi}\right\rfloor\right) = f(10) \\ f_p(11) &= f\left(11 - 4\pi\left\lfloor\frac{11}{4\pi}\right\rfloor\right) = f(11) \\ f_p(21) &= f\left(21 - 4\pi\left\lfloor\frac{21}{4\pi}\right\rfloor\right) = f(21-4\pi) \\ f_p(22) &= f\left(22 - 4\pi\left\lfloor\frac{22}{4\pi}\right\rfloor\right) = f(22-4\pi) \end{aligned} \tag{2.9}$$

(2.3)

Since $g(x) = x + x^2 + x^3$, its fundamental period is 2π, L $= \pi$ and therefore F.T of $g(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_m\cos(mx) + b_m\sin(mx)\right] \tag{2.10}$$

From Euler-Fourier formulas, we can evaluate coefficients. Starting with a_0 is:

$$\begin{gathered}\frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\cos(0)dx = \\ \frac{1}{\pi}\int_{-\infty}^{\infty} x + x^2 + x^3 dx = \\ \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right]_{-\pi}^{\pi} = \\ \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} + \frac{\pi^4}{4} - \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^4}{4}\right] = \frac{2\pi^2}{3}\end{gathered} \tag{2.11}$$

For a_1:

$$\begin{aligned}&\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(x)dx = \\ &\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(x) + x^2\cos(x) + x^3\cos(x)dx = \\ &\frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos(x)dx + \int_{-\infty}^{\infty} x^2\cos(x)dx + \int_{-\infty}^{\infty} x^3\cos(x)dx\right] =\end{aligned} \tag{2.12}$$

Calculating each integral separately:

$$\begin{gathered}\int_{-\pi}^{\pi} x\cos(x)dx = x\sin(x) + \cos(x)\Big|_{-\pi}^{\pi} = 0 \\ \int_{-\pi}^{\pi} x^2\cos(x)dx = -2[-x\cos(x) + \sin(x)]\Big|_{-\pi}^{\pi} = -4\pi \\ \int_{-\pi}^{\pi} x^3\cos(x)dx = 3x^2\cos(x) - 6x\sin(x) - 6\cos(x)\Big|_{-\pi}^{\pi} = 0\end{gathered} \tag{2.13}$$

Therefore:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(x)dx = \frac{1}{\pi}\left[0 - 4\pi + 0\right] = -4 \tag{2.14}$$

Now finding b_1:

$$\begin{gathered}\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(x)dx = \\ \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x) + x^2\sin(x) + x^3\sin(x)dx = \\ \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin(x)dx + \int_{-\pi}^{\pi} x^2\sin(x)dx + \int_{-\pi}^{\pi} x^3\sin(x)dx\right] =\end{gathered} \tag{2.15}$$

Calculating each integral separately:

$$\begin{gathered}\int_{-\pi}^{\pi} x\sin(x)dx = \sin(x) - x\cos(x)\Big|_{-\pi}^{\pi} = 2\pi \\ \int_{-\pi}^{\pi} x^2\sin(x)dx = 0 \\ \int_{-\pi}^{\pi} x^3\sin(x)dx = (3x^2-2)\sin(x) - x(x^2-6)\cos(x)\Big|_{-\pi}^{\pi} = 2\pi^3 - 12\pi\end{gathered} \tag{2.16}$$

Therefore:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(x)dx = \frac{1}{\pi}\left[2\pi + 0 + 2\pi^3 - 12\pi\right] = 2\pi^2 - 10 \tag{2.17}$$

Giving us our final result:

$$f(x) = \frac{2\pi^2}{3} - 4\cos(x) + (2\pi^2 - 10)\sin(x) + ... \tag{2.18}$$