Homework 7

PHYS209

Hikmat Gulaliyev

24.11.2023

(1) Problem 2

Let us have following matrices A and B:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{1.1}$$

Let us also have following matrix C, which is a product of A and B:

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \tag{1.2}$$

We know from matrix multiplication formula $c_{ij} = \sum_k^n a_{ik} b_{kj}$:

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{aligned} \tag{1.3}$$

From formula of determinant of a 2x2 matrix:

$$\begin{aligned} \det(A) &= a_{11}a_{22} - a_{12}a_{21} \\ \det(B) &= b_{11}b_{22} - b_{12}b_{21} \\ \det(C) &= c_{11}c_{22} - c_{12}c_{21} \end{aligned} \tag{1.4}$$

So for us to prove that $\det(C) = \det(A)\det(B)$, we need to prove that:

$$c_{11}c_{22} - c_{12}c_{21} = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) \tag{1.5}$$

Substituting 1.3 into the left side of the equation and simplifying:

$$\begin{aligned}(a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22})-(a_{11}b_{12}+a_{12}b_{22})(a_{21}b_{11}+a_{22}b_{21}) &= \\ =\cancel{a_{11}a_{21}b_{11}b_{12}}+a_{11}a_{22}b_{11}b_{22}+a_{12}a_{21}b_{12}b_{21}+\cancel{a_{12}a_{22}b_{21}b_{22}} & \\ -\cancel{a_{11}a_{21}b_{11}b_{12}}-a_{11}a_{22}b_{12}b_{21}-a_{12}a_{21}b_{11}b_{12}-\cancel{a_{12}a_{22}b_{21}b_{22}} &= \\ =a_{11}a_{22}b_{11}b_{22}+a_{12}a_{21}b_{12}b_{21}-a_{11}a_{22}b_{12}b_{21}-a_{12}a_{21}b_{11}b_{12} &=\end{aligned} \tag{1.6}$$

Opening the brackets and simplifying right side of the equation:

$$\begin{aligned}&(a_{11}a_{22}-a_{12}a_{21})(b_{11}b_{22}-b_{12}b_{21})= \\ &=a_{11}a_{22}b_{11}b_{22}+a_{12}a_{21}b_{12}b_{21}-a_{11}a_{22}b_{12}b_{21}-a_{12}a_{21}b_{11}b_{12}\end{aligned} \tag{1.7}$$

From this we can see that left side of the equation is equal to the right side of the equation, which proves that $\det(C)=\det(A)\det(B)$.

(2) Problem 3

Let us find Wronskian of the following functions:

$$\begin{aligned}f_1(x)&=\frac{1}{x} \\ f_2(x)&=\frac{e^{-x^2/2}}{x} \\ f_3(x)&=x-4\end{aligned} \tag{2.8}$$

By definition of Wronskian:

$$W=\begin{vmatrix}f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x)\end{vmatrix}=\begin{vmatrix}\frac{1}{x} & \frac{e^{-x^2/2}}{x} & x-4 \\ -\frac{1}{x^2} & \frac{e^{-x/2}(x+2)}{2x^2} & 1 \\ \frac{2}{x^3} & \frac{e^{-x/2}(x^2+4x+8)}{4x^3} & 0\end{vmatrix} \tag{2.9}$$

Putting x = 1 into the Wronskian, we get:

$$W(1) = \begin{vmatrix} 1 & \frac{e^{-1/2}}{1} & -3 \\ -1 & \frac{-3e^{-1/2}}{2} & 1 \\ 2 & \frac{-13e^{-1/2}}{4} & 0 \end{vmatrix} = -\frac{1}{2\sqrt{e}} \tag{2.10}$$