Homework 4

PHYS209

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(1) Problem 1

(1.1)

To solve $\text{🦇}_a \cdot \text{👻}_\triangle = \text{🎃}$ we can necessary substitutions and get:

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x} - a\right) x^\triangle = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^\triangle - a x^\triangle = 0 \tag{1.1}$$

$$\triangle x x^{\triangle - 1} = a x^\triangle$$

Giving us $\triangle = a$

(1.2)

Considering definitions of 💀_a, 💀_b and commutator operator, following can be achieved:

$$\left[\underset{a}{\text{💀}}, \underset{b}{\text{💀}}\right] \cdot f = \underset{a}{\text{💀}} \cdot \left(\underset{b}{\text{💀}} \cdot f\right) - \underset{b}{\text{💀}} \cdot \left(\underset{a}{\text{💀}} \cdot f\right) = \underset{a}{\text{💀}} \cdot (f' - bf) - \underset{b}{\text{💀}} \cdot (f' - af) =$$
$$= (f' - bf)' - a(f' - bf) - (f' - af)' + b(f' - af) =$$
$$= f'' - bf' - af + abf - f'' + af' + bf' - abf = x \to 0 \tag{1.2}$$

Which proves and $\left[\text{💀}_a, \text{💀}_b\right] \cdot f = \text{🎃}$ therefore 💀_a and 💀_b commute.

(1.3)

To prove commutativity 🦇_a and 🦇_b, using their definitions:

$$[\text{🦇}_a, \text{🦇}_b] \cdot f = \text{🦇}_a(\text{🦇}_b \cdot f) - \text{🦇}_b(\text{🦇}_a \cdot f) = \text{🦇}_a \cdot \left(x\frac{\mathrm{d}}{\mathrm{d}x} - b\right) f - \text{🦇}_b \cdot \left(x\frac{\mathrm{d}}{\mathrm{d}x} - a\right) f =$$
$$= x(xf' - bf)' - a(xf' - bf) - x(xf' - af)' + b(xf' - af) =$$
$$xf' + x^2 f'' - xbf' - axf' + abf - xf' - x^2 f'' + af'x + bxf' - abf' = x \to 0 \tag{1.3}$$

Proving that $[\mathbat_a, \mathbat_b] \cdot f = \pumpkin$ therefore $\mathbat_a$ and $\mathbat_b$ commute.

(1.4)

To check whether $\mathbat_b$ and $\skull_a$ commute, we can use their definitions:

$$\left[\bigskull_a, \mathbat_b\right] \cdot f = \bigskull_a \cdot (\mathbat_b \cdot f) - \mathbat_b \cdot \left(\bigskull_a \cdot f\right) = \bigskull_a \cdot (xf' - b)\, f - \mathbat_b \cdot (f' - af) =$$
$$= (xf' - bf')' - a(xf' - bf) - (f' - af)' + b(f' - af) =$$
$$f' + xf'' - bf' - axf' + abf - xf'' + axf' + bf' - abf = x \to f' \quad (1.4)$$

Since commutator isn't equal to 0, these operators don't commute.

(1.5)

Since commutability of differential operator $\mathbat$ is proved in previous sections, we can use it to solve given differential equation:

$$\mathbat_{a_1} \cdot \mathbat_{a_2} \cdots \mathbat_{a_n} \cdot f = \pumpkin \quad (1.5)$$

We can find one solution as:

$$\begin{aligned} \left(x\frac{\mathrm{d}}{\mathrm{d}x} - a_1\right) f &= 0 \\ xf' &= a_1 f \\ \frac{f'}{f} &= \frac{a_1}{x} \\ \ln(f) &= a_1 \ln(x) + C \\ f &= c_1 x^{a_1} \end{aligned} \quad (1.6)$$

Becouse of commutability, since we have n operators, we can find n solutions:

$$f = c_1 x^{a_1} + c_2 x^{a_2} + \cdots + c_n x^{a_n} \quad (1.7)$$

(2) Problem 2

(2.1)

Given our differential equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \pi\tan(\pi x)\frac{\mathrm{d}}{\mathrm{dx}} + \pi^4\cos(\pi x)^2\right) f(x) = 0 \tag{2.8}$$

New substitution parameter $u(x)$ can be defined as:

$$u(x) = \int \sqrt{q(x)}\mathrm{d}x = \int \sqrt{\pi^4\cos(\pi x)^2}\mathrm{d}x = \int \pi^2\cos(\pi x)\mathrm{d}x = \pi\left|\sin(\pi x)\right| + C \tag{2.9}$$

However it doesn't matter if $u(x) = \pi\sin(\pi x)$ or $u(x) = -\pi\sin(\pi x)$ since $\frac{q(x)}{u'(x)}$ can be choosen to be equal to -1 or 1.
For our differential equation to be rewritten as linear differential equation with constant coefficients, following condition must be satisfied:

$$\frac{q'(x) + 2p(x)q(x)}{2q(x)^{3/2}} = const \tag{2.10}$$

Substituting $q(x)$ and $p(x)$ with their values we get:

$$\frac{-2\pi^5\cos(\pi x)\sin(\pi x) + 2\pi^5\tan(\pi x)\cos(\pi x)^2}{2\left(\pi^2\cos(\pi x)\right)^3} = 0 \tag{2.11}$$

Since, condition 2.10 is satisfied.

(2.2)

To rewrite our differential equation as linear differential equation with constant coefficients, we need to find substitutions for $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}f}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = u''\frac{\mathrm{d}f}{\mathrm{d}u} + u'\frac{\mathrm{d}^2 f}{\mathrm{d}u^2} \tag{2.12}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = u'\frac{\mathrm{d}f}{\mathrm{d}u} \tag{2.13}$$

(2.3)

Now using substitutions from previous section, we can rewrite our differential equation in terms of u:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{u'' + u'p(x)}{(u')^2}\frac{\mathrm{d}}{\mathrm{d}u} + \frac{q(x)}{(u')^2}\right) f(x) = 0 \tag{2.14}$$

Where u'' and u' are:

$$u' = \pi^2 \cos(\pi x) \tag{2.15}$$

$$u'' = -\pi^3 \sin(\pi x) \tag{2.16}$$

Making substitutions:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{-\pi^3 \sin(\pi x) + \pi^2 \cos(\pi x)\pi \tan(\pi x)}{\pi^4 \cos(\pi x)^2}\frac{\mathrm{d}}{\mathrm{d}u} + \frac{\pi^4 \cos(\pi x)^2}{\pi^4 \cos(\pi x)^2}\right) f(x) = 0 \tag{2.17}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right) f(x) = 0 \tag{2.18}$$

(2.4)

To solve this ordinary linear differential equation with constant coefficients, we need to find and solve its characteristic equation:

$$\begin{aligned} r^2 + 1 &= 0 \\ r_{1,2} &= \pm i \end{aligned} \tag{2.19}$$

Using roots of characteristic equation, we can find general solution of our differential equation:

$$f(u(x)) = c_1 e^{iu} + c_2 e^{-iu} = d_1 \cos(u) + d_2 \sin(u) \tag{2.20}$$

(2.5)

Writing our answer again in terms of x:

$$f(x) = d_1 \cos(\pi \sin(\pi x)) + d_2 \sin(\pi \sin(\pi x)) \tag{2.21}$$

(2.6)

Assuming given initial conditions $f(x=0) = 12000212\sqrt{2}$ and $f'(x=0) = 17101711\sqrt{2}\pi^2$ we can determine values of d_1 and d_2:

$$\begin{aligned} f(x=0) &= d_1 = 12000212\sqrt{2} \\ f'(x=0) &= d_2\pi^2 = 17101711\sqrt{2}\pi^2 \end{aligned} \tag{2.22}$$

Giving us our final solution:

$$f(x) = 12000212\sqrt{2}\cos(\pi\sin(\pi x)) + 17101711\sqrt{2}\sin(\pi\sin(\pi x)) \tag{2.23}$$

Plugging in $x = \frac{\arcsin(1/4)}{\pi}$ we get:

$$f\left(\frac{\arcsin(1/4)}{\pi}\right) = 29101923 \tag{2.24}$$