Homework 6

PHYS209

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(1) Problem 1

(1.1)

We have a differential equation:

$$x^2 f'''(x) + \frac{(4+x)x}{2} f''(x) + \frac{4-x}{2} f'(x) - \frac{1}{2} f(x) = 0 \tag{1.1}$$

And we are already told that one of our solutions is $1/x^k$ for some k. Lets use the method of reduction of order to reduce the order of given differential equation. For this we will substitue in the following:

$$f(x) = \frac{g(x)}{x^k} \tag{1.2}$$

Finding derivatives of $f(x)$:

$$f'(x) = \frac{g'(x)x^k - kg(x)x^{k-1}}{x^{2k}} \tag{1.3}$$

$$f''(x) = \frac{g''(x)x^k - 2kg'(x)x^{k-1} + k(k+1)g(x)x^{k-2}}{x^{3k}} \tag{1.4}$$

$$f'''(x) = \frac{g'''(x)x^k - 3kg''(x)x^{k-1} + 3k(k+1)g'(x)x^{k-2} - k(k+1)(k+2)g(x)x^{k-3}}{x^{4k}} \tag{1.5}$$

For us to be able to reduce the order of differential equation, we need $g(x)$ to not be present ion our equation, so coefficient of $g(x)$ must be zero:

$$x^2 \left[-k(k+1)(k+2)x^{-k-3}\right] + \frac{(4+x)x}{2} \left[k(k+1)x^{-k-2}\right] - \frac{4-x}{2} \left[-kx^{-k-1}\right] - \frac{1}{2} x^{-k} = 0 \tag{1.6}$$

Multiplying both sides by $2x^{k+1}$:

$$2k^3 + 2k^2 - 4k + xk^2 + x = 0 \tag{1.7}$$

After refactoring we get:

$$(k-1)\left(2k^2 + 4k - xk - x\right) = 0 \tag{1.8}$$

Giving us three solutions:

$$k_1 = 1 \quad k_{2,3} = \frac{1}{2}\left(-2 \pm \sqrt{4+4x}\right) \tag{1.9}$$

Since only k_1 is x-independent, we will use it to reduce the order of our differential equation.

(1.2)

After substituting $k = 1$ into our equation, we get:

$$xg'''(x) + \left[-3 + \frac{4+x}{2}\right] g''(x) + \left[6\frac{1}{x} - \frac{4-x}{x} + \frac{-4+x}{2x}\right] g'(x) = 0 \tag{1.10}$$

Multiplying both sides by 2 we get our final form of differential equation for $g(x)$:

$$2xg'''(x) + (x-2)g''(x) - g'(x) = 0 \tag{1.11}$$

(1.3)

Now substituting $g'(x) = h(x)$ we get:

$$2xh'(x) + (x-2)h(x) - h'(x) = 0 \tag{1.12}$$

To solve this equation we need to rewrite it in form:

$$(\alpha(x)h'(x) + h(x)) + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)h'(x) + h(x)\right] = 0 \tag{1.13}$$

Let's open the brackets and simplify:

$$\begin{aligned}
&2xh''(x) - xh'(x) + 2h'(x) - h(x) = 0 \quad \text{multiply by -1} \\
&-2xh'(x) + xh'(x) - 2h'(x) + h(x) = 0 \\
&(-2h'(x) + h(x)) + (-2xh''(x) + xh'(x)) = 0 \\
&(-2h'(x) + h(x)) + x\left(-2h''(x) + h'(x)\right) = 0 \\
&(-2h'(x) + h(x)) + x\frac{\mathrm{d}}{\mathrm{d}x}\left[-2h'(x) + h(x)\right] = 0
\end{aligned} \tag{1.14}$$

Which is in form 1.13 with $\alpha(x) = -2$ and $\beta(x) = x$.

(1.4)