Slo.1: The Runge-Kutta (RK4) method to solve the differential equation $y' = 1 - t^2 + y$, over the interval $0 \leq t \leq 2$, with the initial condition $y(0) = 0.5$, and with $n = 10$ steps is given by:

$$
\begin{aligned}
h &= \frac{2-0}{10} \\
t_{i+1} &= t_i + h \\
k_1 &= h(1 - t_i^2 + y_i) \\
k_2 &= h\left(1 - \left(t_i + \frac{h}{2}\right)^2 + \left(y_i + \frac{k_1}{2}\right)\right) \\
k_3 &= h\left(1 - \left(t_i + \frac{h}{2}\right)^2 + \left(y_i + \frac{k_2}{2}\right)\right) \\
k_4 &= h\left(1 - (t_i + h)^2 + (y_i + k_3)\right) \\
y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

Where h is the step size, t_i and y_i are the values of t and y at the i-th step respectively, and k_1, k_2, k_3, k_4 are intermediate slopes.

Execution

Standard Output

```
For Iteration: 0 Value of t = 0, y(t) = 0.8292933333333333
For Iteration: 1 Value of t = 0.2, y(t) = 1.2140762106666667
For Iteration: 2 Value of t = 0.4, y(t) = 1.6489220170416
For Iteration: 3 Value of t = 0.6000000000000001, y(t) = 2.1272026849479433
For Iteration: 4 Value of t = 0.8, y(t) = 2.6408226927287513
For Iteration: 5 Value of t = 1, y(t) = 3.17989417023223
For Iteration: 6 Value of t = 1.2000000000000002, y(t) = 3.732340072854979
For Iteration: 7 Value of t = 1.4000000000000001, y(t) = 4.283409498318405
For Iteration: 8 Value of t = 1.6, y(t) = 4.815085694579433
For Iteration: 9 Value of t = 1.8, y(t) = 5.305363000692653
For Iteration: 10 Value of t = 2, y(t) = 5.305363000692653
```

Graph of RNN and LSTM cell

Sol.2: To solve Berger's equation using the fourth-order Runge-Kutta (RK4) method with central differences for approximating derivatives, we need to discretize the space and time domains. Let's denote x_i as the spatial grid points and t_n as the temporal grid points. Then, we can use the following discretization:

$$x_i = i \cdot \Delta x, \quad i = 0, 1, 2, \ldots, N_x$$

$$t_n = n \cdot \Delta t, \quad n = 0, 1, 2, \ldots, N_t$$

where Δx and Δt are the spatial and temporal step sizes, respectively. Now, let's apply the RK4 method to solve the equation.

1. Define initial and boundary conditions. 2. Update the solution using RK4 in each time step.

Let's denote u_i^n as the numerical solution at spatial point x_i and temporal point t_n.

The RK4 update formula for the given equation becomes:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}
k_1 &= -\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} u_i^n + \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \\
k_2 &= -\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \left(u_i^n + \frac{\Delta t}{2} k_1 \right) + \nu \frac{u_{i+1}^n - 2\left(u_i^n + \frac{\Delta t}{2} k_1\right) + u_{i-1}^n}{\Delta x^2} \\
k_3 &= -\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \left(u_i^n + \frac{\Delta t}{2} k_2 \right) + \nu \frac{u_{i+1}^n - 2\left(u_i^n + \frac{\Delta t}{2} k_2\right) + u_{i-1}^n}{\Delta x^2} \\
k_4 &= -\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \left(u_i^n + \Delta t k_3 \right) + \nu \frac{u_{i+1}^n - 2\left(u_i^n + \Delta t k_3\right) + u_{i-1}^n}{\Delta x^2}
\end{aligned}$$

```
Time 0.000: [0.00000000e+00 3.42020143e-01 6.42787610e-01 8.66025404e-01
 9.84807753e-01 9.84807753e-01 8.66025404e-01 6.42787610e-01
 3.42020143e-01 1.22464680e-16]
Time 0.200: [0.         0.30847314 0.58903779 0.81392744 0.95627889 0.99228346
 0.90557929 0.69411822 0.37756032 0.        ]
Time 0.400: [0.         0.28081652 0.54221483 0.76344817 0.92121436 0.98901047
 0.93869431 0.74812848 0.41930192 0.        ]
Time 0.600: [0.         0.25766457 0.50145609 0.71595392 0.88231772 0.97580393
 0.9629968  0.80294378 0.46826187 0.        ]
Time 0.800: [0.         0.23801979 0.46588559 0.67207732 0.84188148 0.95437354
 0.97683122 0.85590237 0.52540993 0.        ]
Time 1.000: [0.         0.22115152 0.43470792 0.63197352 0.80161502 0.92689821
 0.97961681 0.90370336 0.59152206 0.        ]
Time 1.200: [0.         0.20651513 0.40723817 0.5955217  0.76265005 0.89559175
 0.97196566 0.94276442 0.66698589 0.        ]
Time 1.400: [0.         0.19369777 0.38290182 0.56246241 0.72564345 0.86237919
 0.95552183 0.96975756 0.75158037 0.        ]
Time 1.600: [0.         0.18238146 0.36122277 0.53248181 0.69090896 0.82873212
 0.93258375 0.98221328 0.84427641 0.        ]
Time 1.800: [0.         0.17231771 0.34180816 0.50525881 0.65853681 0.79564643
 0.9056412  0.9790297  0.94312576 0.        ]
```