12.10.3.13

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Class 12, Chapter 10, Exercise 3.13

13) If are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$, find the value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$.
Solution: The inner product of given unit vectors with sum of all unit vectors (Which is given to be 0) will be zero.

$$\mathbf{a}^\top(\mathbf{a}+\mathbf{b}+\mathbf{c}) = 0 \tag{1}$$
$$\Rightarrow (\mathbf{a}^\top\mathbf{b}+\mathbf{a}^\top\mathbf{c}) = -1 \tag{2}$$
$$\mathbf{b}^\top(\mathbf{a}+\mathbf{b}+\mathbf{c}) = 0 \tag{3}$$
$$\Rightarrow (\mathbf{b}^\top\mathbf{c}+\mathbf{b}^\top\mathbf{a}) = -1 \tag{4}$$
$$\mathbf{c}^\top(\mathbf{a}+\mathbf{b}+\mathbf{c}) = 0 \tag{5}$$
$$\Rightarrow (\mathbf{c}^\top\mathbf{a}+\mathbf{c}^\top\mathbf{b}) = -1 \tag{6}$$

Adding equations (2),(4) and(6) we get

$$\Rightarrow 2\times(\mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a}) = -3 \tag{7}$$
$$\Rightarrow (\mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a}) = \frac{-3}{2} \tag{8}$$
$$\Rightarrow \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a} = \frac{-3}{2} \tag{9}$$

Let's verify with a numerical example.

$$\mathbf{a} = \begin{pmatrix}\cos(0)\\ \sin(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{10}$$
$$\mathbf{b} = \begin{pmatrix}\cos(2\pi/3)\\ \sin(2\pi/3)\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix} \tag{11}$$
$$\mathbf{c} = \begin{pmatrix}\cos(4\pi/3)\\ \sin(4\pi/3)\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{pmatrix} \tag{12}$$
$$\implies \mathbf{a}+\mathbf{b}+\mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{13}$$
$$\mathbf{a}^\top\mathbf{b} = \begin{pmatrix}1\\0\end{pmatrix}^\top \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix} = -\frac{1}{2} \tag{14}$$
$$\mathbf{b}^\top\mathbf{c} = \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}^\top \begin{pmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{pmatrix} = -\frac{1}{2} \tag{15}$$
$$\mathbf{c}^\top\mathbf{a} = \begin{pmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{pmatrix}^\top \begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2} \tag{16}$$
$$\implies \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a} = -\frac{3}{2} \tag{17}$$

Verified.