11.10.3.10

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CLASS 11, CHAPTER 10, EXERCISE 3.10

Q. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.

Solution: Let the point $\mathbf{P}$ be the foot of the perpendicular on the line $7x - 9y - 19 = 0$ from point $\begin{pmatrix} h \\ 3 \end{pmatrix}$ (Let's say point $\mathbf{O}$). The optimization problem can be expressed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \tag{1}$$

$$\text{s.t.} \quad \mathbf{n}^T \mathbf{x} = c \tag{2}$$

where

$$\mathbf{n} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}, \; c = 19 \tag{3}$$

The line equation can be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

where

$$\mathbf{m} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}, \; \mathbf{A} = \begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} \tag{5}$$

Using the parameric form, Substituting (4) in (1), the optimization problem becomes

$$\min_{\lambda} \|\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\|^2 \tag{6}$$

$$\begin{aligned} \implies \min_{\lambda} f(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + \\ 2\lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \|\mathbf{A} - \mathbf{O}\|^2 \end{aligned} \tag{7}$$

$\because$ the coefficient of $\lambda^2 > 0$, (7) is a convex function. Thus,

$$f''(\lambda) = 2\|\mathbf{m}\|^2 \tag{8}$$

$$\because f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \tag{9}$$

yielding

$$f'(\lambda_{min}) = 2\lambda_{min} \|\mathbf{m}\|^2 + 2(\mathbf{A} - \mathbf{O})^\top \mathbf{m} = 0 \tag{10}$$

$$\lambda_{min} = -\frac{(\mathbf{A} - \mathbf{O})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \tag{11}$$

It is given that the line through the points $\begin{pmatrix} h \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ intersects the line $7x - 9y - 19 = 0$ at right angle. And the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is on the line $7x - 9y - 19 = 0$.

From equation (4)

$$\implies \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} + \lambda_{min} \begin{pmatrix} 9 \\ 7 \end{pmatrix} \tag{12}$$

$$\implies \lambda_{min} = \frac{1}{7} \tag{13}$$

Substituting the values of $\mathbf{A}$, $\mathbf{O}$, λ_{min} and $\mathbf{m}$ in equation (11)

$$\frac{1}{7} = -\frac{\left(\begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} - \begin{pmatrix} h \\ 3 \end{pmatrix} \right)^\top \begin{pmatrix} 9 \\ 7 \end{pmatrix}}{\left\| \begin{pmatrix} 9 \\ 7 \end{pmatrix}^2 \right\|} \tag{14}$$

$$\implies \frac{130}{7} = -\frac{171}{7} + 9h + 21 \tag{15}$$

$$\implies h = \frac{22}{9} \tag{16}$$

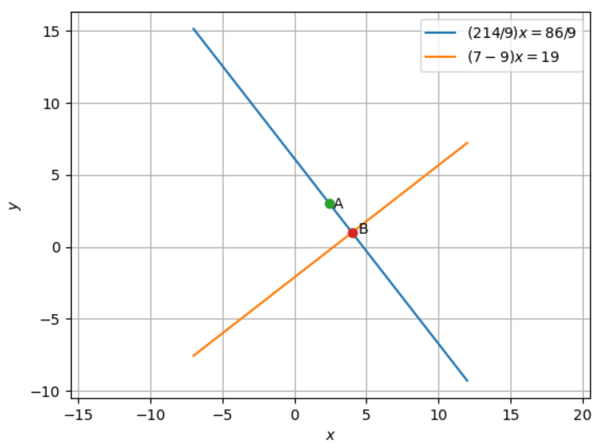

Fig. 1