

11.10.3.10

Lokesh Surana

CLASS 11, CHAPTER 10, EXERCISE 3.10

Q. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h .

Solution: Let the point \mathbf{P} be the foot of the perpendicular on the line $7x - 9y - 19 = 0$ from point $\begin{pmatrix} h \\ 3 \end{pmatrix}$ (Let's say point \mathbf{O}). The optimization problem can be expressed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \quad (1)$$

$$\text{s.t. } \mathbf{n}^\top \mathbf{x} = c \quad (2)$$

where

$$\mathbf{n} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}, c = 19 \quad (3)$$

The line equation can be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4)$$

where

$$\mathbf{m} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} \quad (5)$$

Using the parametric form, Substituting (4) in (1), the optimization problem becomes

$$\min_{\lambda} \|\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\|^2 \quad (6)$$

$$\begin{aligned} \Rightarrow \min_{\lambda} f(\lambda) &= \lambda^2 \|\mathbf{m}\|^2 + \\ &2\lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \|\mathbf{A} - \mathbf{O}\|^2 \end{aligned} \quad (7)$$

\therefore the coefficient of $\lambda^2 > 0$, (7) is a convex function. Thus,

$$f''(\lambda) = 2\|\mathbf{m}\|^2 \quad (8)$$

$$\therefore f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \quad (9)$$

yielding

$$f'(\lambda_{min}) = 2\lambda_{min}\|\mathbf{m}\|^2 + 2(\mathbf{A} - \mathbf{O})^\top \mathbf{m} = 0 \quad (10)$$

$$\lambda_{min} = -\frac{(\mathbf{A} - \mathbf{O})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \quad (11)$$

It is given that the line through the points $\begin{pmatrix} h \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ intersects the line $7x - 9y - 19 = 0$ at right angle.

And the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is on the line $7x - 9y - 19 = 0$.

From equation (4)

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} + \lambda_{min} \begin{pmatrix} 9 \\ 7 \end{pmatrix} \quad (12)$$

$$\Rightarrow \lambda_{min} = \frac{1}{7} \quad (13)$$

Substituting the values of \mathbf{A} , \mathbf{O} , λ_{min} and \mathbf{m} in equation (11)

$$\frac{1}{7} = -\frac{\left(\begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} - \begin{pmatrix} h \\ 3 \end{pmatrix}\right)^\top \begin{pmatrix} 9 \\ 7 \end{pmatrix}}{\left\|\begin{pmatrix} 9 \\ 7 \end{pmatrix}\right\|^2} \quad (14)$$

$$\Rightarrow \frac{130}{7} = -\frac{171}{7} + 9h + 21 \quad (15)$$

$$\Rightarrow h = \frac{22}{9} \quad (16)$$

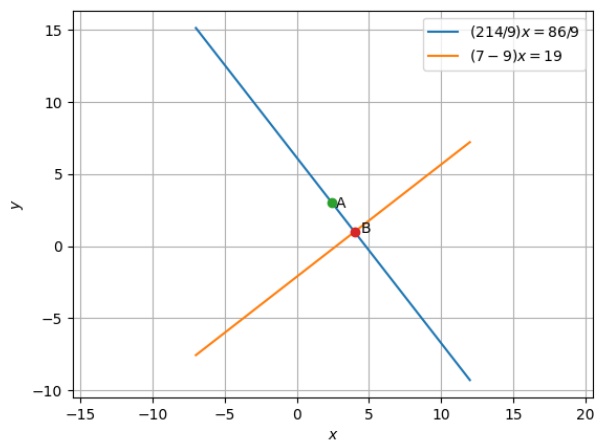


Fig. 1