11.11.2.5

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Class 11, Chapter 11, Exercise 2.5

Q. Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum $y^2 = 10x$

Solution: The given equation of the parabola can be rearranged as

$$y^2 - 10x = 0 \tag{1}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2}$$

Comparing coefficients of (1) and (2),

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = -\begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

From (3), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 0 \tag{6}$$

$$\lambda_2 = 1 \tag{7}$$

and the eigenvector matrix

$$\mathbf{P} = \mathbf{I}. \tag{8}$$

$$\therefore \mathbf{n} = \sqrt{\lambda_2}\mathbf{p_1} \tag{9}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{10}$$

Since

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}}, \tag{11}$$

Substituting values of $\mathbf{u}, \mathbf{n}, \lambda_2$ and f in (11), we get

$$c = \frac{5^2 - 1(0)}{-2\begin{pmatrix} 5 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = -\frac{5}{2} \tag{12}$$

$$\tag{13}$$

The focus $\mathbf{F}$ of parabola is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{14}$$

$$= \frac{-\frac{5}{2}(1)^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}}{1} \tag{15}$$

$$= \begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix} \tag{16}$$

The directrix is given by

$$\mathbf{n}^\top\mathbf{x} = c \tag{17}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = -\frac{5}{2} \tag{18}$$

$$\tag{19}$$

The equation for the axis of parabola passing through $\mathbf{F}$ and orthogonal to the directrix is given as

$$\mathbf{m}^\top(\mathbf{x} - \mathbf{F}) = 0 \tag{20}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of the directrix.

$$\because \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{21}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix}\right) = 0 \tag{22}$$

$$\text{or, } \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{23}$$

The latus rectum of a parabola is given by

$$
\begin{aligned}
l = \frac{\eta}{\lambda_2} = -\frac{2\mathbf{u}^\top \mathbf{p_1}}{\lambda_2} \qquad (24)\\
= -\frac{2\begin{pmatrix} -5 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{1} \qquad (25)\\
= 10 \text{ units} \qquad (26)
\end{aligned}
$$