

11.11.5.3

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CLASS 11, CHAPTER 11, EXERCISE 5.3

Q. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

Solution: Uniformly loaded suspension bridge cable hangs in the form of a parabola facing upwards.

O	Lowest point of cable	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
d	Length of the cable	100 m
d_1	Length of longest wire	30 m
d_2	Length of shortest wire	6 m
A	End point of cable	$\begin{pmatrix} \frac{d}{2} \\ d_1 - d_2 \end{pmatrix}$
B	End point of cable	$\begin{pmatrix} -\frac{d}{2} \\ d_1 - d_2 \end{pmatrix}$

TABLE I: points

This will give us a setup similar to figure 1,

Here A and B are the points on the parabola where the cable is attached to the roadway, i.e. longest wire is attached at this points. And vertex of parabola O is point where shortest wire is attached, which is 6m from the ground.

With the assumption of point O being $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we'll get

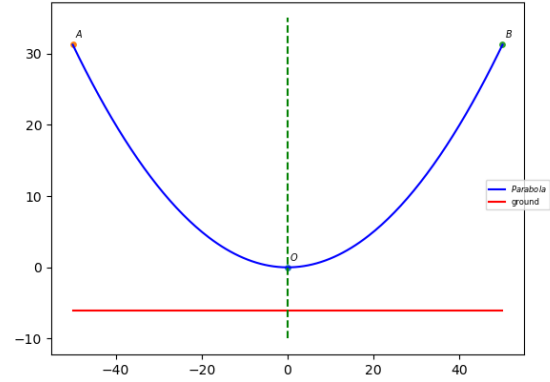


Fig. 1: Representation of parabola with vertex at origin.

$$\mathbf{A} = \begin{pmatrix} \frac{d}{2} \\ d_1 - d_2 \end{pmatrix} = \begin{pmatrix} 50 \\ 24 \end{pmatrix} \quad (1)$$

$$\mathbf{B} = \begin{pmatrix} -\frac{d}{2} \\ d_1 - d_2 \end{pmatrix} = \begin{pmatrix} -50 \\ 24 \end{pmatrix} \quad (2)$$

The generic equation of conic is

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3)$$

As conic is upward facing parabola,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4)$$

Points **O**, **A**, and **B** are on conic, so we have

$$\mathbf{O}^T \mathbf{V} \mathbf{O} + 2\mathbf{u}^T \mathbf{O} + f = 0 \quad (5)$$

$$\mathbf{A}^T \mathbf{V} \mathbf{A} + 2\mathbf{u}^T \mathbf{A} + f = 0 \quad (6)$$

$$\mathbf{B}^T \mathbf{V} \mathbf{B} + 2\mathbf{u}^T \mathbf{B} + f = 0 \quad (7)$$

Rewrite the equations as

$$2\mathbf{O}^\top \mathbf{u} + f = -\mathbf{O}^\top \mathbf{V}\mathbf{O} \quad (8)$$

$$2\mathbf{A}^\top \mathbf{u} + f = -\mathbf{A}^\top \mathbf{V}\mathbf{A} \quad (9)$$

$$2\mathbf{B}^\top \mathbf{u} + f = -\mathbf{B}^\top \mathbf{V}\mathbf{B} \quad (10)$$

This can be formulated as matrix, as follows:

$$\begin{pmatrix} 2\mathbf{O}^\top & 1 \\ 2\mathbf{A}^\top & 1 \\ 2\mathbf{B}^\top & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = - \begin{pmatrix} \mathbf{O}^\top \mathbf{V}\mathbf{O} \\ \mathbf{A}^\top \mathbf{V}\mathbf{A} \\ \mathbf{B}^\top \mathbf{V}\mathbf{B} \end{pmatrix} \quad (11)$$

Substituting the values of \mathbf{O} , \mathbf{A} , and \mathbf{B} in the above equation, we get

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \\ \begin{pmatrix} 100 & 48 \end{pmatrix} & 1 \\ \begin{pmatrix} -100 & 48 \end{pmatrix} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = - \begin{pmatrix} 0 \\ -2500 \\ -2500 \end{pmatrix} \quad (12)$$

$$\Rightarrow f = 0 \text{ and } \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{625}{12} \end{pmatrix} \quad (13)$$

So, equation of parabola is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & -\frac{625}{12} \end{pmatrix} \mathbf{x} = 0 \quad (14)$$

At a point λ_1 m from middle, i.e.

$$\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \quad (15)$$

Substitute this in parabola equation, we get

$$\begin{aligned} & (\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2)^\top \mathbf{V} (\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) \\ & + 2\mathbf{u}^\top (\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) + f = 0 \end{aligned} \quad (16)$$

$$\begin{aligned} & \lambda_1^2 \mathbf{e}_1^\top \mathbf{V} \mathbf{e}_1 + \lambda_2^2 \mathbf{e}_2^\top \mathbf{V} \mathbf{e}_2 + \lambda_1 \lambda_2 \mathbf{e}_1^\top \mathbf{V} \mathbf{e}_2 + \\ & \lambda_1 \lambda_2 \mathbf{e}_2^\top \mathbf{V} \mathbf{e}_1 + 2\lambda_1 \mathbf{u}^\top \mathbf{e}_1 + 2\lambda_2 \mathbf{u}^\top \mathbf{e}_2 + f = 0 \end{aligned} \quad (17)$$

Substituting the values of \mathbf{V} , \mathbf{u} , \mathbf{e}_1 , \mathbf{e}_2 and f

$$\lambda_1^2 - \frac{6}{625} \lambda_2 = 0 \quad (18)$$

$$(19)$$

It is given that $\lambda_1 = 18$

$$\Rightarrow \lambda_2 = \frac{6}{625} (18)^2 = \frac{1944}{625} \quad (20)$$

\Rightarrow Length of a supporting wire attached to the roadway 18m from the middle is

$$= \lambda_2 + d_2 = \frac{1944}{625} + 6 = \frac{5694}{625} \text{ m} \quad (21)$$

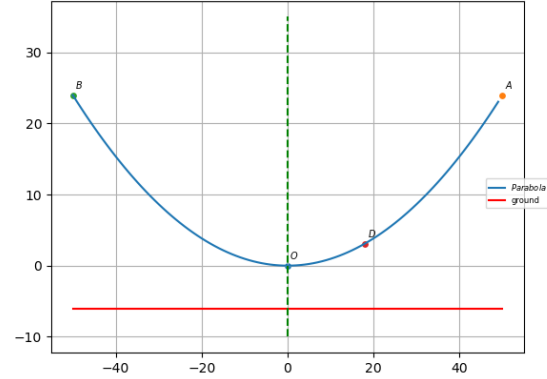


Fig. 2: Parabola