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Class 9, Chapter 10, Exercise 5.3

Q. $\angle PQR = 100°$, where $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ are points on a circle with centre $\mathbf{O}$. Find $\angle OPR$
Solution: Let, we have a unit circle with center at origin, i.e. $\mathbf{O}$, and radius $r = 1$. Then let following points be on the circle

O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
R	$\begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Q	$\begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) \end{pmatrix}$	$\begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$
P	$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$	$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

TABLE I: points

Using the theorem from appendix of matrix analysis book, Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{1}$$

be points on a unit circle. Then

$$\cos\angle PRQ = \frac{(\mathbf{R}-\mathbf{P})^\top(\mathbf{R}-\mathbf{Q})}{\|\mathbf{R}-\mathbf{P}\|\|\mathbf{R}-\mathbf{Q}\|} \tag{2}$$

$$= \cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{3}$$

For our question we have 3 points

$$\mathbf{P} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix}, \tag{4}$$

on a unit circle. Then using this theorem, we get

$$\cos\angle PQR = \frac{(\mathbf{Q}-\mathbf{P})^\top(\mathbf{Q}-\mathbf{R})}{\|\mathbf{Q}-\mathbf{P}\|\|\mathbf{Q}-\mathbf{R}\|} \tag{5}$$

$$= \cos\left(\frac{\theta-0}{2}\right) \tag{6}$$

As per given condition, we have

$$\angle PQR = 100° \tag{7}$$

$$\implies \cos 100° = \cos\left(\frac{\theta-0}{2}\right) \tag{8}$$

$$\theta = 200° \tag{9}$$

$$\mathbf{P} = \begin{pmatrix} \cos 200° \\ \sin 200° \end{pmatrix} \tag{10}$$

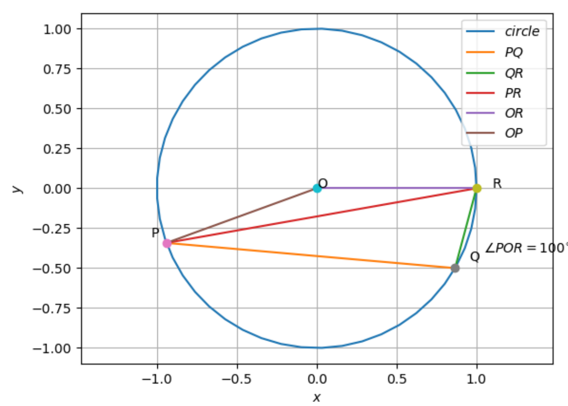

Fig. 1: circle

Now, let's check the $\angle OPR$

$$\cos\angle OPR = \frac{(\mathbf{P}-\mathbf{O})^\top(\mathbf{P}-\mathbf{R})}{\|\mathbf{P}-\mathbf{O}\|\|\mathbf{P}-\mathbf{R}\|} \tag{11}$$

$$\tag{12}$$

$$\|\mathbf{P}-\mathbf{O}\| = \sqrt{\left(\cos^2 200^\circ + \sin^2 200^\circ\right)} = 1 \quad (13)$$

$$\|\mathbf{P}-\mathbf{R}\| = \sqrt{(\cos 200^\circ - 1)^2 + \left(\sin^2 200^\circ\right)} \quad (14)$$

$$= 2\sin 100^\circ \quad (15)$$

So,

$$\cos\angle OPR = \frac{\begin{pmatrix}\cos 200^\circ \\ \sin 200^\circ\end{pmatrix}\begin{pmatrix}\cos 200^\circ - 1 & \sin 200^\circ\end{pmatrix}}{2\sin 100^\circ} \quad (16)$$

$$= \frac{1-\cos 200^\circ}{2\sin 100^\circ} \quad (17)$$

$$= \sin 100^\circ \quad (18)$$

$$\implies \angle OPR = \cos^{-1}\sin 100^\circ = 10^\circ \quad (19)$$