11.11.5.3

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Class 11, Chapter 11, Exercise 5.3

Q. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

Solution: Uniformly loaded suspension bridge cable hangs in the form of a parabola facing upwards.

$\mathbf{O}$	Lowest point of cable	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
d	Length of the cable	100 m
d_1	Length of longest wire	30 m
d_2	Length of shortest wire	6 m
$\mathbf{A}$	End point of cable	$\begin{pmatrix} \frac{d}{2} \\ d_1 - d_2 \end{pmatrix}$
$\mathbf{B}$	End point of cable	$\begin{pmatrix} -\frac{d}{2} \\ d_1 - d_2 \end{pmatrix}$

TABLE I: points

This will give us a setup similar to figure 1,

Here A and B are the points on the parabola where the cable is attached to the roadway, i.e. longest wire is attached at this points. And vertex of parabola O is point where shortest wire is attached, which is 6m from the ground. With the assumption of point O being $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we'll get

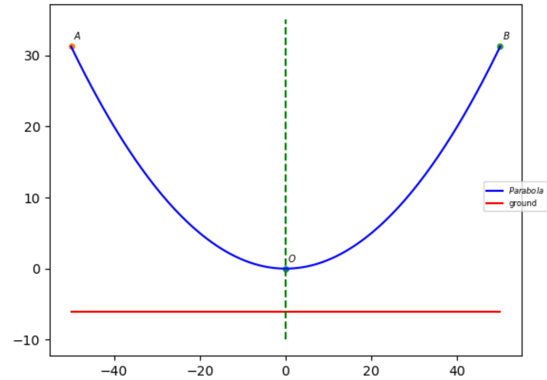

Fig. 1: Representation of parabola with vertex at origin.

$$\mathbf{A} = \begin{pmatrix} \frac{d}{2} \\ d_1 - d_2 \end{pmatrix} = \begin{pmatrix} 50 \\ 24 \end{pmatrix} \tag{1}$$

$$\mathbf{B} = \begin{pmatrix} -\frac{d}{2} \\ d_1 - d_2 \end{pmatrix} = \begin{pmatrix} -50 \\ 24 \end{pmatrix} \tag{2}$$

The generic equation of conic is

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3}$$

As conic is upward facing parabola,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

Points $\mathbf{O}$, $\mathbf{A}$, and $\mathbf{B}$ are on conic, so we have

$$\mathbf{O}^\top \mathbf{V} \mathbf{O} + 2\mathbf{u}^\top \mathbf{O} + f = 0 \tag{5}$$

$$\mathbf{A}^\top \mathbf{V} \mathbf{A} + 2\mathbf{u}^\top \mathbf{A} + f = 0 \tag{6}$$

$$\mathbf{B}^\top \mathbf{V} \mathbf{B} + 2\mathbf{u}^\top \mathbf{B} + f = 0 \tag{7}$$

Rewrite the equations as

$$2\mathbf{O}^\top \mathbf{u} + f = -\mathbf{O}^\top \mathbf{V}\mathbf{O} \tag{8}$$
$$2\mathbf{A}^\top \mathbf{u} + f = -\mathbf{A}^\top \mathbf{V}\mathbf{A} \tag{9}$$
$$2\mathbf{B}^\top \mathbf{u} + f = -\mathbf{B}^\top \mathbf{V}\mathbf{B} \tag{10}$$

This can be formulated as matrix, as follows:

$$\begin{pmatrix} 2\mathbf{O}^\top & 1 \\ 2\mathbf{A}^\top & 1 \\ 2\mathbf{B}^\top & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = -\begin{pmatrix} \mathbf{O}^\top \mathbf{V}\mathbf{O} \\ \mathbf{A}^\top \mathbf{V}\mathbf{A} \\ \mathbf{B}^\top \mathbf{V}\mathbf{B} \end{pmatrix} \tag{11}$$

Substituting the values of **O**, **A**, and **B** in the above equation, we get

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \\ \begin{pmatrix} 100 & 48 \end{pmatrix} & 1 \\ \begin{pmatrix} -100 & 48 \end{pmatrix} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = -\begin{pmatrix} 0 \\ -2500 \\ -2500 \end{pmatrix} \tag{12}$$

$$\implies f = 0 \text{ and } \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{625}{12} \end{pmatrix} \tag{13}$$

So, equation of parabola is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -\frac{625}{12} \end{pmatrix} \mathbf{x} = 0 \tag{14}$$

At a point λ_1 m from middle, i.e.

$$\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \tag{15}$$

Substitute this in parabola equation, we get

$$(\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2)^\top \mathbf{V} (\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) + 2\mathbf{u}^\top (\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) + f = 0 \tag{16}$$

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \mathbf{V} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + 2\mathbf{u}^\top \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + f = 0 \tag{17}$$

$$2\mathbf{u}^\top \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -\left(f + \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \mathbf{V} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) \tag{18}$$

Substituting $\mathbf{u}, \lambda_1, f, \mathbf{V}$, we get

$$2\begin{pmatrix} 0 & -\frac{625}{12} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{19}$$

$$\implies \lambda_2 = \frac{6}{625} (\lambda_1)^2 \tag{20}$$

It is given that $\lambda_1 = 18$

$$\implies \lambda_2 = \frac{6}{625} (18)^2 = \frac{1944}{625} \tag{21}$$

$\implies$ Length of a supporting wire attached to the roadway $18m$ from the middle is

$$= \lambda_2 + d_2 = \frac{1944}{625} + 6 = \frac{5694}{625} m \tag{22}$$

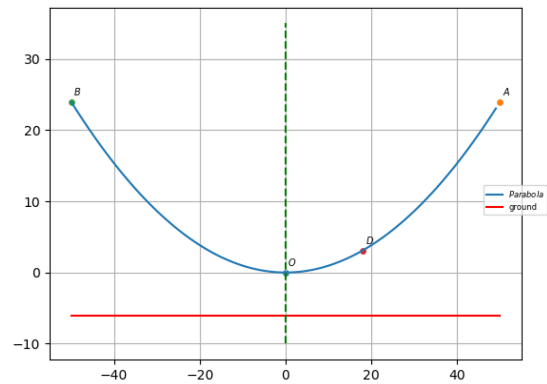

Fig. 2: Parabola