Random Numbers

AI1110: Probability and Random Variables

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1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat

Solution: Download the C source code by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/1.1.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 1.1.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: Download the following python code that plots Fig. 1.2

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/1.2.py
```

Run the code by executing

```
python3 1.2.py
```

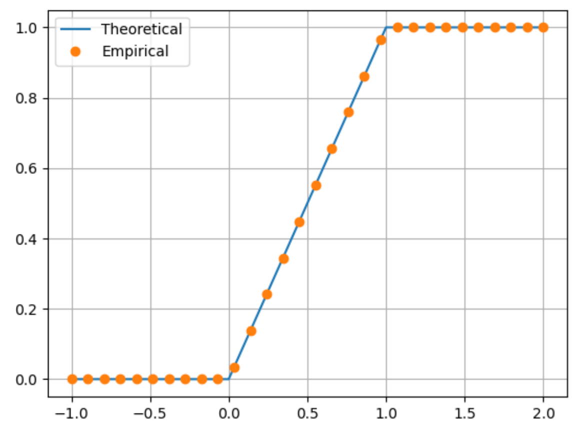

Fig. 1.2. The CDF of U

1.3 Find a theoretical expression for $F_U(x)$

Solution: The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.3}$$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{x} 0\, dx = 0 \tag{1.4}$$

If c,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} 1\, dx \tag{1.5}$$

$$= 0 + x \tag{1.6}$$

$$= x \tag{1.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x = \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{1} 1\ \mathrm{d}x + \int_{1}^{x} 0\ \mathrm{d}x \quad (1.8)$$

$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x = 0 + 1 + 0 \quad (1.9)$$

$$= 1 \quad (1.10)$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad (1.11)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \quad (1.12)$$

and its variance as

$$\mathrm{Var}[U] = E[U - E[U]]^2 \quad (1.13)$$

Write a C program to find the mean and variance of U

Solution: Download the C source code by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/1.4.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 1.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.500007 \quad (1.14)$$

$$\mu_{\text{the}} = 0.500000 \quad (1.15)$$

$$\sigma^2_{\text{emp}} = 0.083301 \quad (1.16)$$

$$\sigma^2_{\text{the}} = 0.083333 \quad (1.17)$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k \mathrm{d}F_U(x) \quad (1.18)$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x\ \mathrm{d}F_U(x) \quad (1.19)$$

On differentiating the CDF of U, we get

$$\mathrm{d}F_U(x) = \begin{cases} 0 & x < 0 \\ \mathrm{d}x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \quad (1.20)$$

$$\therefore E[U] = \int_{0}^{1} x\ \mathrm{d}x = \frac{1}{2} = 0.5 \quad (1.21)$$

Similarly,

$$\therefore E\left[U^2\right] = \int_{0}^{1} x^2\ \mathrm{d}x = \frac{1}{3} \quad (1.22)$$

Now, the variance of U is given by

$$\mathrm{Var}[U] \quad (1.23)$$

$$= E[U - E[U]]^2 \quad (1.24)$$

$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \quad (1.25)$$

By linearity of expectation, we have

$$E\left[U^2\right] + E[-2UE[U]] + E\left[(E[U])^2\right] \quad (1.26)$$

$$= E\left[U^2\right] - 2E[U]E[U] + (E[U])^2 \quad (1.27)$$

$$= E\left[U^2\right] - (E[U])^2 \quad (1.28)$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.29)$$

$$= \frac{1}{12} \approx 0.083333 \quad (1.30)$$

2. Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the C source code by executing the following commands

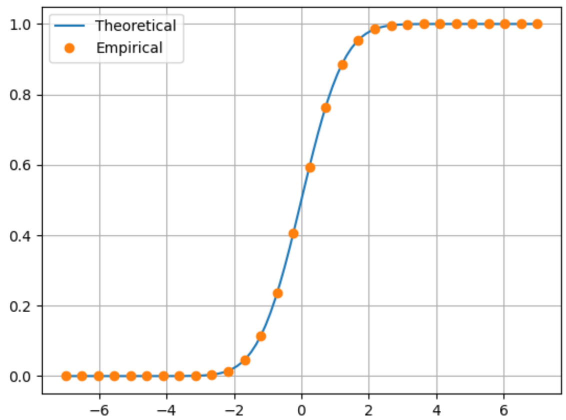

Fig. 2.2. The CDF of X

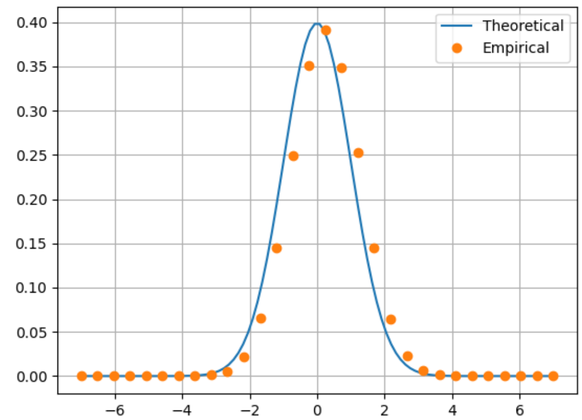

Fig. 2.3. The PDF of X

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/2.1.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 2.1.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the following python code that plots Fig. 2.2

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/2.2.py
```

Run the code by executing

```
python3 2.2.py
```

Every CDF is monotone increasing and right-continuous. Furthermore,

$$\lim_{x\to-\infty} F_X(x) = 0 \qquad \lim_{x\to\infty} F_X(x) = 1 \tag{2.2}$$

Thus, every CDF is bounded between 0 and 1 and hence, convergent.

In this case, the CDF is also left-continuous. Therefore, X is a continuous random variable.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.3}$$

What properties does the PDF have?
Solution: Download the following python code that plots Fig. 2.2

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/2.3.py
```

Run the code by executing

```
python3 2.3.py
```

Every PDF is bounded between 0 and 1 and

$$\int_{-\infty}^{\infty} p_X(x)\, dx = 1 \tag{2.4}$$

In this case, the PDF is symmetric about $x = 0$

2.4 Find the mean and variance of X by writing a C program
Solution: Download the C source code by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/2.4.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc -lm 2.4.c
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.000294 \tag{2.5}$$
$$\mu_{\text{the}} = 0.000000 \tag{2.6}$$
$$\sigma^2_{\text{emp}} = 0.999560 \tag{2.7}$$
$$\sigma^2_{\text{the}} = 1.000000 \tag{2.8}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.9}$$

repeat the above exercise theoretically

Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) \mathrm{d}x \tag{2.10}$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.11}$$

Now, let

$$g(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.12}$$
$$\implies g(-x) = \frac{-x}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \tag{2.13}$$
$$= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.14}$$
$$= -g(x) \tag{2.15}$$

Thus, $g(x)$ is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) \mathrm{d}x = 0 \tag{2.16}$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x) \mathrm{d}x \tag{2.17}$$
$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.18}$$
$$= 2\int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.19}$$

since $\frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function

Using integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.20}$$
$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x\right)\Bigg|_0^\infty$$
$$- \sqrt{\frac{2}{\pi}} \int_0^\infty 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.21}$$

Substitute $t = \frac{x^2}{2} \implies \mathrm{d}t = x\mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \int \exp(-t) \mathrm{d}t \tag{2.22}$$
$$= -\exp(-t) \tag{2.23}$$
$$= -\exp\left(-\frac{x^2}{2}\right) \tag{2.24}$$

Now,

$$-x \exp\left(-\frac{x^2}{2}\right)\Bigg|_0^\infty = 0 - 0 = 0 \tag{2.25}$$
$$\because \lim_{x\to\infty} x \exp\left(-\frac{x^2}{2}\right) = \lim_{x\to\infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \tag{2.26}$$

as exponential function grows much faster than a polynomial function

Also,

$$\int_0^\infty -\exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.27}$$
$$\xleftrightarrow{x=t\sqrt{2}} \int_0^\infty -\exp(-t^2) \mathrm{d}t \sqrt{2} \tag{2.28}$$
$$= -\sqrt{2} \int_0^\infty \exp(-t^2) \mathrm{d}t \tag{2.29}$$
$$= -\sqrt{2} \frac{\sqrt{\pi}}{2} \tag{2.30}$$
$$= -\sqrt{\frac{\pi}{2}} \tag{2.31}$$

Therefore,

$$E\left[X^2\right] = 0 - \sqrt{\frac{2}{\pi}} \left(-\sqrt{\frac{\pi}{2}}\right) \tag{2.32}$$
$$= 1 \tag{2.33}$$
$$\therefore \mathrm{Var}[X] = E\left[X^2\right] - (E[X])^2 \tag{2.34}$$
$$= 1 - 0 \tag{2.35}$$
$$= 1 \tag{2.36}$$

3. From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF

Solution: Download the C source code by executing the following commands

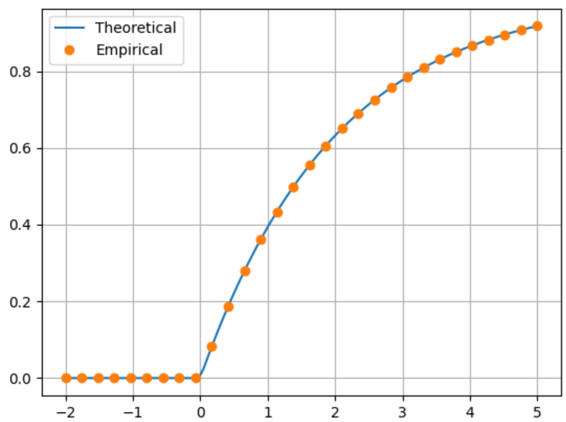

Fig. 3.1. The CDF of V

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/3.1.c
```

Compile and run the C program by executing the following

```
cc −lm 3.1.c
./a.out
```

Download the following python code that plots Fig. 3.1

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/3.1.py
```

Run the code by executing

```
python3 3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$

Solution: We have

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(\ln(1-U) \ge -\frac{x}{2}\right) \tag{3.4}$$
$$= \Pr\left(1-U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= \Pr\left(U \le 1-\exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$
$$= F_U\left(1-\exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

Now,

$$0 \le 1-\exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \ge 0 \tag{3.8}$$
$$1-\exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \tag{3.9}$$

Therefore,

$$F_V(x) = \begin{cases} 1-\exp\left(-\frac{x}{2}\right) & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{3.10}$$

4. Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/4.1.c
```

Compile and run the C program by executing the following

```
cc −lm 4.1.c
./a.out
```

4.2 Find the CDF of T

Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$

Since $U_1, U_2 \in [0,1] \implies U_1 + U_2 \in [0,2]$
Therefore, if $t \ge 2$, then $U_1 + U_2 \le t$ is always true and if $t < 0$, then $U_1 + U_2 \le t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \le t \implies U_2 \le t - x \tag{4.3}$$

If $0 \le t \le 1$, then x can take all values in $[0,t]$

$$F_T(t) = \int_0^t \Pr(U_2 \le t-x)\, p_{U_1}(x)\mathrm{d}x \tag{4.4}$$
$$= \int_0^t F_{U_2}(t-x) p_{U_1}(x)\mathrm{d}x \tag{4.5}$$

$$0 \le x \le t \implies 0 \le t-x \le t \le 1 \tag{4.6}$$
$$\implies F_{U_2}(t-x) = t-x \tag{4.7}$$

$$F_T(t) = \int_0^t (t-x)\cdot 1 \cdot \mathrm{d}x \tag{4.8}$$
$$= tx - \frac{x^2}{2}\bigg|_0^t \tag{4.9}$$
$$= \frac{t^2}{2} \tag{4.10}$$

If $1 < t < 2$, x can only take values in $[0,1]$ as $U_1 \le 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x)\cdot 1 \cdot \mathrm{d}x \tag{4.11}$$

$$0 \le x \le t-1 \implies 1 \le t-x \le t \tag{4.12}$$
$$t-1 \le x \le 1 \implies 0 < t-1 \le t-x \le 1 \tag{4.13}$$

$$F_T(t) = \int_0^{t-1} 1\mathrm{d}x + \int_{t-1}^1 (t-x)\mathrm{d}x \tag{4.14}$$
$$= t-1+t(1-(t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \tag{4.15}$$
$$= t-1+2t-t^2-\frac{1}{2}+\frac{t^2}{2}+\frac{1}{2}-t \tag{4.16}$$
$$= -\frac{t^2}{2}+2t-1 \tag{4.17}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \ge 2 \end{cases} \tag{4.18}$$

4.3 Find the PDF of T
Solution: The PDF of T is given by

$$p_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}F_T(t) \tag{4.19}$$
$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2-t & 1 < t < 2 \\ 0 & t \ge 2 \end{cases} \tag{4.20}$$

4.4 Find the theoretical expressions for the PDF and CDF of T
Solution: The theoretical expressions for the CDF and PDF have been found in problems 4.2 and 4.3 respectively

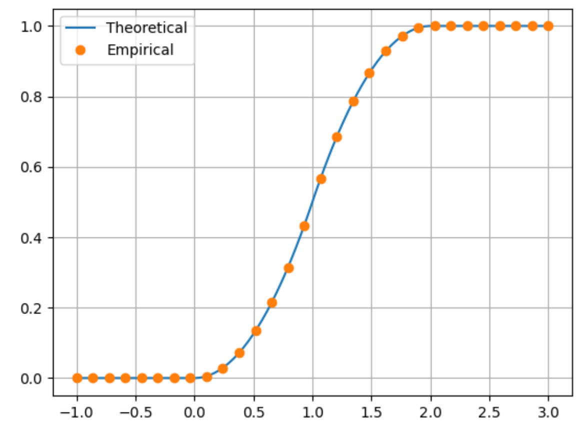

Fig. 4.2. The CDF of T

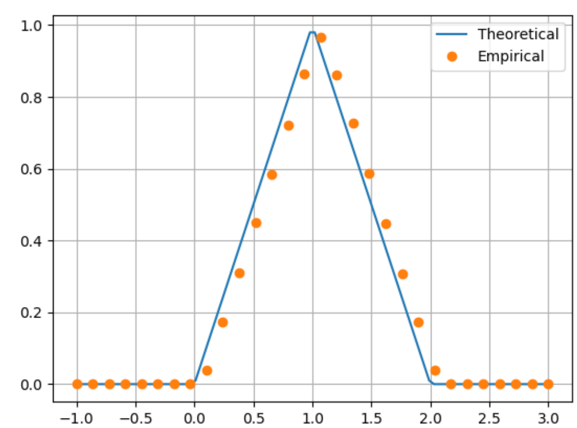

Fig. 4.3. The PDF of T

4.5 Verify your results through a plot
Solution: Download the following python codes that plot Fig. 4.2 and Fig. 4.3

```
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/4.2.py
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/4.3.py
```

Run the codes by executing

```
python3 4.2.py
python3 4.3.py
```

5. Maximum Likelihood

5.1 Generate equiprobable $X \in \{-1, 1\}$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/5.1.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 5.1.c
./a.out
```

5.2 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB, $X \in \{-1, 1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$

Solution: Download the C source code by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/5.2.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 5.2.c
./a.out
```

5.3 Plot Y

Solution: Download the following python code that plots Fig. 5.3

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/5.3.py
```

Run the code by executing

```
python3 5.3.py
```

5.4 Guess how to estimate X from Y

Solution:

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{5.2}$$

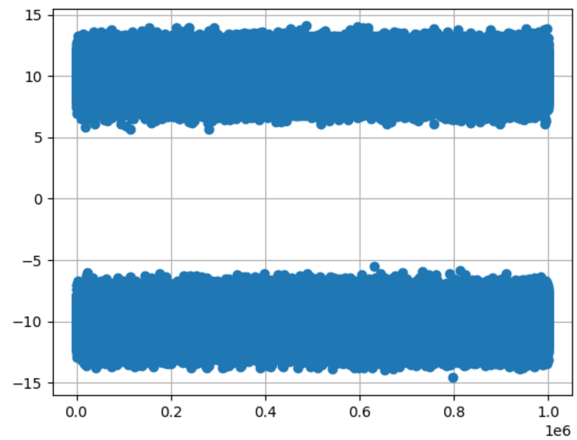

Fig. 5.3. Plot of Y

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution:

$$\Pr\left(\hat{X} = -1 | X = 1\right) = \Pr\left(Y < 0 | X = 1\right) \tag{5.5}$$
$$= \Pr\left(A + N < 0\right) \tag{5.6}$$
$$= \Pr\left(N < -A\right) \tag{5.7}$$
$$= 1 - \Pr\left(N > -A\right) \tag{5.8}$$
$$= 1 - Q(-A) \tag{5.9}$$
$$= Q(A) \tag{5.10}$$

where $Q(x) = \Pr\left(N > x\right)$ is the Q-function

$$Q(x) = 1 - Q(-x) \qquad \forall x \in \mathbb{R} \tag{5.11}$$

$$\Pr\left(\hat{X} = 1 | X = -1\right) = \Pr\left(Y > 0 | X = -1\right) \tag{5.12}$$
$$= \Pr\left(-A + N > 0\right) \tag{5.13}$$
$$= \Pr\left(N > A\right) \tag{5.14}$$
$$= Q(A) \tag{5.15}$$

5.6 Find P_e assuming that X has equiprobable symbols

Solution:

$$P_e = \Pr\left(X = -1\right) P_{e|0} + \Pr\left(X = 1\right) P_{e|1} \tag{5.16}$$

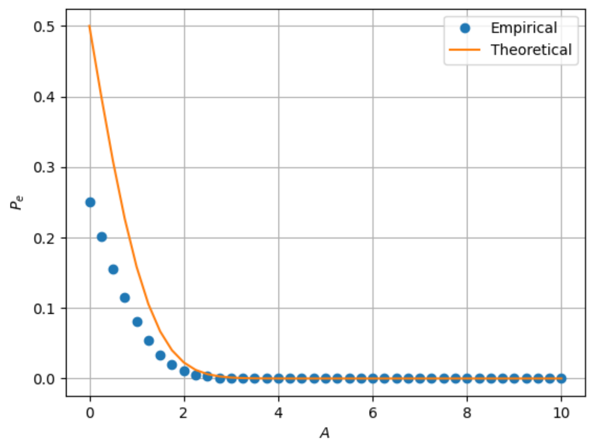

Fig. 5.7. Plot of P_e

Since X has equiprobable symbols, $\Pr(X=-1) = \Pr(X=1) = \frac{1}{2}$

$$P_e = \frac{1}{2}Q(A) + \frac{1}{2}Q(A) \tag{5.17}$$
$$= Q(A) \tag{5.18}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB

Solution: Download the following python code that plots Fig. 5.7

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/5.7.py
```

Run the code by executing

```
python3 5.7.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e

Solution:

$$X = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \tag{5.19}$$

$$\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) && (5.20)\\
&= \Pr\left(Y < \delta | X = 1\right) && (5.21)\\
&= \Pr\left(A + N < \delta\right) && (5.22)\\
&= \Pr\left(N < \delta - A\right) && (5.23)\\
&= Q(A - \delta) && (5.24)\\
P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) && (5.25)\\
&= \Pr\left(Y > \delta | X = -1\right) && (5.26)\\
&= \Pr\left(-A + N > \delta\right) && (5.27)\\
&= \Pr\left(N > \delta + A\right) && (5.28)\\
&= Q(A + \delta) && (5.29)\\
& && (5.30)
\end{aligned}$$

Now, P_e is given by

$$\begin{aligned}
P_e &= \Pr\left(X = -1\right) P_{e|0} + \Pr\left(X = 1\right) P_{e|1} && (5.31)\\
&= \frac{1}{2}Q(A-\delta) + \frac{1}{2}Q(A+\delta) && (5.32)\\
&= \frac{Q(A-\delta) + Q(A+\delta)}{2} && (5.33)\\
&= g(\delta) && (5.34)
\end{aligned}$$

On differentiating g with respect to δ, we get

$$g'(\delta) = \frac{Q'(A+\delta) - Q'(A-\delta)}{2} \tag{5.35}$$

Recall the definition of $Q(x)$

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \tag{5.36}$$

$$\implies Q'(x) = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{5.37}$$

Thus,

$$g'(\delta) = \frac{\exp\left(-\frac{(A-\delta)^2}{2}\right) - \exp\left(-\frac{(A+\delta)^2}{2}\right)}{2\sqrt{2\pi}} \tag{5.38}$$

$$\begin{aligned}
g'(\delta) = 0 &\implies (A-\delta)^2 = (A+\delta)^2 && (5.39)\\
&\implies |A-\delta| = |A+\delta| && (5.40)\\
&\implies \delta = 0 && (5.41)
\end{aligned}$$

$$\begin{aligned}
g''(\delta) = &\frac{(A-\delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(A-\delta)^2}{2}\right)\\
&+ \frac{(A+\delta)}{2\sqrt{2\pi}} \exp\left(-\frac{(A+\delta)^2}{2}\right) && (5.42)
\end{aligned}$$

$$g''(0) = \frac{A}{\sqrt{2\pi}} \exp\left(-\frac{A^2}{2}\right) > 0 \quad (\because A > 0) \tag{5.43}$$

Therefore, $\hat{\delta} = 0$ is a minima and it is what minimizes P_e

5.9 Repeat the above exercise when

$$p_X(-1) = p \tag{5.44}$$

Solution:

$$P_e = p_X(1)P_{e|0} + p_X(-1)P_{e|1} \tag{5.45}$$
$$= (1-p)Q(A-\delta) + pQ(A+\delta) \tag{5.46}$$
$$= g(\delta) \tag{5.47}$$

On differentiating g with respect to δ, we get

$$g'(\delta) = \frac{(1-p)\exp\left(-\frac{(A-\delta)^2}{2}\right) - p\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\sqrt{2\pi}} \tag{5.48}$$

$g'(\delta) = 0$ when

$$(1-p)\exp\left(-\frac{(A-\delta)^2}{2}\right) = p\exp\left(-\frac{(A+\delta)^2}{2}\right) \tag{5.49}$$

$$\implies \exp\left(\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) = \frac{p}{1-p} \tag{5.50}$$

$$\implies \exp(2A\delta) = \frac{p}{1-p} \tag{5.51}$$

$$\therefore \hat{\delta} = \frac{1}{2A}\ln\frac{p}{1-p} \tag{5.52}$$

5.10 Repeat the above exercise using the MAP criterion

Solution: The PDF of $X|Y$ is given by

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} \tag{5.53}$$

Assuming X has equiprobable symbols, $p_X(x) = \frac{1}{2}, \quad x = -1, 1$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1) \tag{5.54}$$

$$= \frac{1}{2}\Pr(-A+N=y) + \frac{1}{2}\Pr(A+N=y) \tag{5.55}$$

$$= \frac{p_N(y+A) + p_N(y-A)}{2} \tag{5.56}$$

$$= \frac{\exp\left(-\frac{(y+A)^2}{2}\right) + \exp\left(-\frac{(y-A)^2}{2}\right)}{2\sqrt{2\pi}} \tag{5.57}$$

$$\tag{5.58}$$

Now,

$$p_{X|Y}(1|y) = \frac{\Pr(A+N=y)\,p_X(1)}{p_Y(y)} \tag{5.59}$$

$$= \frac{p_N(y-A)p_X(1)}{p_Y(y)} \tag{5.60}$$

$$= \frac{\exp\left(-\frac{(y-A)^2}{2}\right)}{2\left(\exp\left(-\frac{(y+A)^2}{2}\right) + \exp\left(-\frac{(y-A)^2}{2}\right)\right)} \tag{5.61}$$

$$= \frac{1}{2\left(1 + \exp\left(\frac{(y-A)^2-(y+A)^2}{2}\right)\right)} \tag{5.62}$$

$$= \frac{1}{2(1+\exp(-2Ay))} \tag{5.63}$$

Similarly,

$$p_{X|Y}(-1|y) = \frac{1}{2(1+\exp(2Ay))} \tag{5.64}$$

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \tag{5.65}$$

$$\iff \frac{1}{2(1+\exp(-2Ay))} > \frac{1}{2(1+\exp(2Ay))} \tag{5.66}$$

$$\iff \exp(-2Ay) < \exp(2Ay) \tag{5.67}$$

$$\iff y > 0 \tag{5.68}$$

And $p_{X|Y}(1|y) < p_{X|Y}(-1|y) \iff y < 0$
Therefore, $X = 1$ is more probable than $X = -1$ when $Y > 0$ and vice versa

Consider now a general Bernoulli random variable X with $p_X(-1) = p, p_X(1) = 1 - p$

$$p_Y(y) = p_X(-1)p_{Y|X}(y|-1) + p_X(1)p_{Y|X}(y|1) \tag{5.69}$$

$$= pp_N(y + A) + (1 - p)p_N(y - A) \tag{5.70}$$

$$= \frac{p\exp\left(-\frac{(y+A)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-A)^2}{2}\right)}{\sqrt{2\pi}} \tag{5.71}$$

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1)p_X(1)}{p_Y(y)} \tag{5.72}$$

$$= \frac{(1-p)\exp\left(-\frac{(y-A)^2}{2}\right)}{p\exp\left(-\frac{(y+A)^2}{2}\right) + (1-p)\exp\left(-\frac{(y-A)^2}{2}\right)} \tag{5.73}$$

$$= \frac{1-p}{1-p+p\exp(-2Ay)} \tag{5.74}$$

Similarly,

$$p_{X|Y}(-1|y) = \frac{p}{p + (1-p)\exp(2Ay)} \tag{5.75}$$

Now,

$$p_{X|Y}(1|y) > p_{X|Y}(-1|y) \tag{5.76}$$

$$\iff \frac{1-p}{1-p+p\exp(-2Ay)} > \frac{p}{p+(1-p)\exp(2Ay)} \tag{5.77}$$

$$\iff 1 + \frac{p}{1-p}\exp(-2Ay) < 1 + \frac{1-p}{p}\exp(2Ay) \tag{5.78}$$

$$\iff \exp(4Ay) > \left(\frac{p}{1-p}\right)^2 \tag{5.79}$$

$$\iff y > \frac{1}{2A}\ln\frac{p}{1-p} = \hat{\delta} \tag{5.80}$$

and

$$p_{X|Y}(1|y) < p_{X|Y}(-1|y) \tag{5.81}$$

$$\iff \frac{1-p}{1-p+p\exp(-2Ay)} < \frac{p}{p+(1-p)\exp(2Ay)} \tag{5.82}$$

$$\iff 1 + \frac{p}{1-p}\exp(-2Ay) > 1 + \frac{1-p}{p}\exp(2Ay) \tag{5.83}$$

$$\iff \exp(4Ay) < \left(\frac{p}{1-p}\right)^2 \tag{5.84}$$

$$\iff y < \frac{1}{2A}\ln\frac{p}{1-p} = \hat{\delta} \tag{5.85}$$

Therefore, $X = 1$ is more probable than $X = -1$ when $Y > \hat{\delta}$ and vice versa

6. Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/6.1.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 6.1.c
./a.out
```

Download the following python code that plots Fig. 6.1

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/6.1.py
```

Run the code by executing

```
python3 6.1.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α

Solution: Let $R \geq 0, \Theta \in [0, 2\pi]$

$$X_1 = R\cos\Theta \tag{6.3}$$

$$X_2 = R\sin\Theta \tag{6.4}$$

such that $V = X_1^2 + X_2^2 = R^2$

The Jacobian matrix transforming R, Θ to X_1, X_2 is defined as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.5}$$

$$= \begin{pmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{pmatrix} \tag{6.6}$$

$$\implies \left|\mathbf{J}\right| = R\cos^2\Theta + R\sin^2\Theta = R \tag{6.7}$$

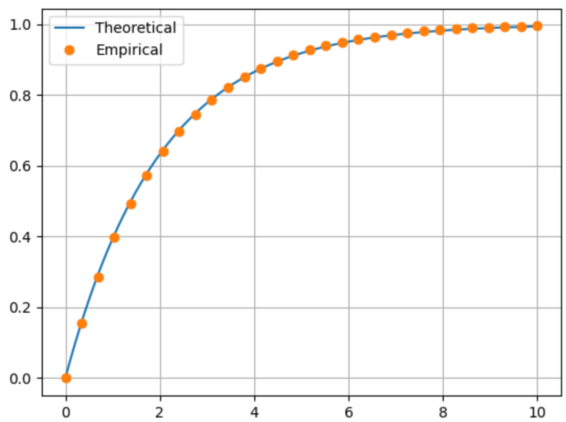

Fig. 6.1. CDF of V

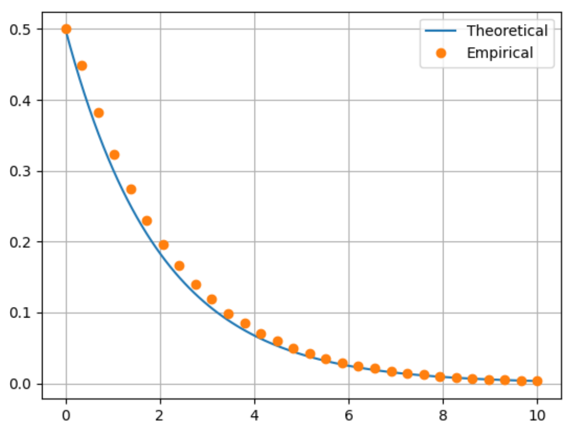

Fig. 6.1. PDF of V

Then,

$$p_{R,\Theta}(r,\theta) = p_{X_1,X_2}(x_1,x_2)\left|\mathbf{J}\right| \tag{6.8}$$
$$= p_{X_1,X_2}(x_1,x_2)\, r \tag{6.9}$$
$$= r p_{X_1}(x_1) p_{X_2}(x_2) \tag{6.10}$$

since X_1 and X_2 are independent

$$p_{R,\Theta}(r,\theta) = r\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_1^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_2^2}{2}\right) \tag{6.11}$$
$$= \frac{r}{2\pi}\exp\left(-\frac{(x_1+x_2)^2}{2}\right) \tag{6.12}$$
$$= \frac{r}{2\pi}\exp\left(-\frac{r^2}{2}\right) \tag{6.13}$$

The marginal distribution of R is given by

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta)\,\mathrm{d}\theta \tag{6.14}$$
$$= \frac{r}{2\pi}\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi}\mathrm{d}\theta \tag{6.15}$$
$$= r\exp\left(-\frac{r^2}{2}\right) \quad \text{for } r \geq 0 \tag{6.16}$$

The CDF of R is thus given by

$$F_R(r) = \Pr(R \leq r) \tag{6.17}$$
$$= \int_0^r p_R(r)\mathrm{d}r \tag{6.18}$$
$$= \int_0^r r\exp\left(-\frac{r^2}{2}\right)\mathrm{d}r \tag{6.19}$$
$$= -\int_0^r \exp\left(-\frac{r^2}{2}\right)(-r\mathrm{d}r) \tag{6.20}$$
$$= -\int_0^r \exp\left(-\frac{r^2}{2}\right)\mathrm{d}\left(-\frac{r^2}{2}\right) \tag{6.21}$$
$$= -\exp\left(-\frac{r^2}{2}\right)\Bigg|_0^r \tag{6.22}$$
$$= 1-\exp\left(-\frac{r^2}{2}\right) \quad \text{for } r \geq 0 \tag{6.23}$$

Now, $V = R^2$, hence the CDF of V is given by

$$F_V(x) = \Pr(V \leq x) \tag{6.24}$$
$$= \Pr\left(R^2 \leq x\right) \tag{6.25}$$
$$= \Pr\left(|R| \leq \sqrt{x}\right) \tag{6.26}$$
$$= \Pr\left(R \leq \sqrt{x}\right) \quad (\because R \geq 0) \tag{6.27}$$
$$= F_R(\sqrt{x}) \tag{6.28}$$
$$= 1-\exp\left(-\frac{x}{2}\right) \quad \text{for } x \geq 0 \tag{6.29}$$

And the PDF of V is given by

$$p_V(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_V(x) \tag{6.30}$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(1-\exp\left(-\frac{x}{2}\right)\right) \tag{6.31}$$
$$= \frac{1}{2}\exp\left(-\frac{x}{2}\right) \tag{6.32}$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.33}$$

$$p_V(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.34}$$

$$\therefore \alpha = \frac{1}{2} \tag{6.35}$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.36}$$

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/6.3.c
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc -lm 6.3.c
./a.out
```

Download the following python code that plots Fig. 6.3

```
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/6.3.py
```

Run the code by executing

```
python3 6.3.py
```

The CDF of A for $x \geq 0$ is given by

$$F_A(x) = \Pr(A \leq x) \tag{6.37}$$
$$= \Pr\left(\sqrt{V} \leq x\right) \tag{6.38}$$
$$= \Pr\left(V \leq x^2\right) \tag{6.39}$$
$$= F_V(x^2) \tag{6.40}$$
$$= 1 - \exp\left(-\frac{x^2}{2}\right) \tag{6.41}$$

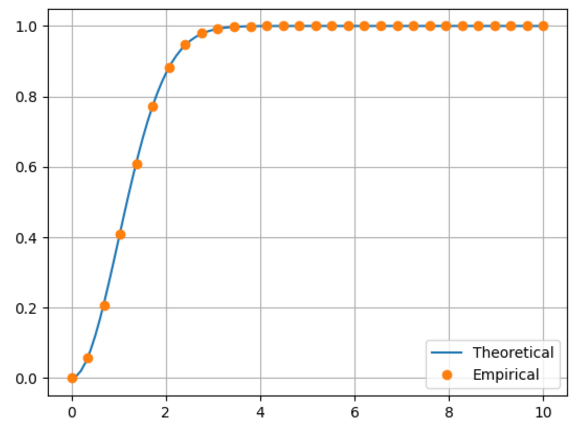

Fig. 6.3. CDF of A

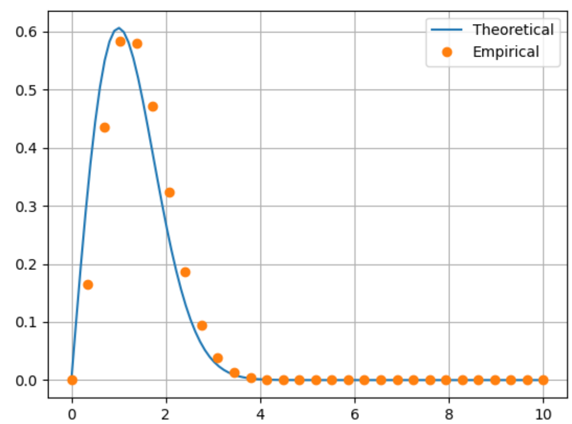

Fig. 6.3. PDF of A

The PDF of A is given by

$$p_A(x) = \frac{d}{dx}F_A(x) \tag{6.42}$$
$$= \frac{d}{dx}\left(1 - \exp\left(-\frac{x^2}{2}\right)\right) \tag{6.43}$$
$$= x\exp\left(-\frac{x^2}{2}\right) \tag{6.44}$$

Therefore,

$$F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.45}$$

$$p_A(x) = \begin{cases} x\exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.46}$$

7. Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N \tag{7.2}$$

where A is Rayleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in \{-1,1\}$ for $0 \le \gamma \le 10$ dB

Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/7.1.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 7.1.c
./a.out
```

Download the following python code that plots Fig. 7.4

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/7.1.py
```

Run the code by executing

```
python3 7.1.py
```

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

Solution:

$$P_e(N) = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.3}$$
$$= \Pr\left(Y < 0 | X = 1\right) \tag{7.4}$$
$$= \Pr\left(A + N < 0\right) \tag{7.5}$$
$$= \Pr\left(A < -N\right) \tag{7.6}$$
$$= F_A(-N) \tag{7.7}$$

For Gaussian random variables with $\sigma = 1$, we found the CDF of A to be

$$F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2}\right) & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.8}$$

For Gaussian random variable Y with a general σ,

$$Y = \sigma X \tag{7.9}$$
$$\implies X = \frac{Y}{\sigma} \tag{7.10}$$

$$F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.11}$$

$$\implies p_A(x) = \begin{cases} \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right) & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.12}$$

$$E\left[A^2\right] = \int_0^\infty x^2 \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x \tag{7.13}$$

Substitute $-\frac{x^2}{2\sigma^2} = u \implies \mathrm{d}u = -\frac{x}{\sigma^2}\mathrm{d}x$

$$E\left[A^2\right] = \int_0^{-\infty} 2\sigma^2 u \exp(u) \mathrm{d}u \tag{7.14}$$
$$= 2\sigma^2 \left(u\exp(u) - \exp(u)\right)\Big|_0^{-\infty} \tag{7.15}$$
$$= 2\sigma^2 \tag{7.16}$$
$$= \gamma \tag{7.17}$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \le 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.18}$$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x) \mathrm{d}x \tag{7.19}$$

Find $P_e = E\left[P_e(N)\right]$

Solution:

$$P_e = \int_{-\infty}^{0} P_e(x) p_N(x) \mathrm{d}x \tag{7.20}$$
$$= \int_{-\infty}^{0} \left(1 - \exp\left(-\frac{x^2}{\gamma}\right)\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{7.21}$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-x^2\left(\frac{1}{2} + \frac{1}{\gamma}\right)\right) \mathrm{d}x \tag{7.22}$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{(2+\gamma)x^2}{2\gamma}\right) \mathrm{d}x \quad (7.23)$$

Now,

$$\int_{-\infty}^{0} \exp\left(-\frac{x^2}{2a^2}\right) \mathrm{d}x = \int_{0}^{\infty} \exp\left(-\frac{x^2}{2a^2}\right) \mathrm{d}x \quad (7.24)$$

$$= a\sqrt{\frac{\pi}{2}} \quad (7.25)$$

Therefore,

$$P_e = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} - \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\gamma}{\gamma+2}} \quad (7.26)$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+2}} \quad (7.27)$$

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.
Solution: Download the following python code that plots Fig. 7.4

```
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/7.1.py
```

Run the code by executing

```
python3 7.1.py
```

The graphs in both cases coincide and thus

$$\Pr\left(\hat{X} = -1 | X = 1\right) = E\left[P_e(N)\right] \quad (7.28)$$

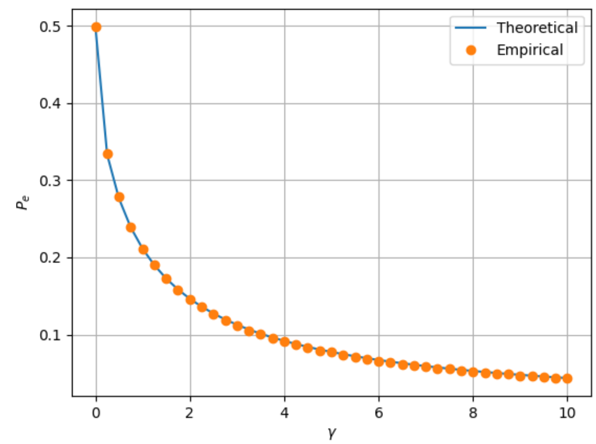

Fig. 7.4. Plot of P_e

8. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1) \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot
Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/8.1.c
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc -lm 8.1.c
./a.out
```

Download the following python code that plots Fig. 8.1

```
wget https://github.com/LokeshBadisa/AI1110
    -Probability-and-Random-Variables/
    blob/main/Random-Numbers/codes/8.1.py
```

Run the code by executing

```
python3 8.1.py
```

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$
Solution: Let

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (8.5)$$

Then, the line $y_2 - y_1 = 0$ seems to separate the two data sets. Thus,

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{s}_0 & y_2 - y_1 < 0 \\ \mathbf{s}_1 & y_2 - y_1 > 0 \end{cases} \quad (8.6)$$

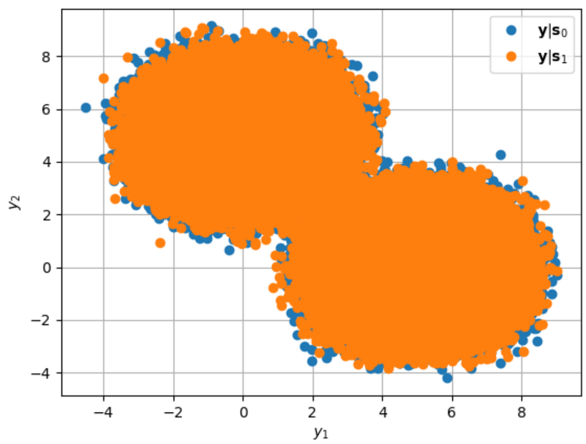

Fig. 8.1. Plot of $\mathbf{y}|\mathbf{s}_0$ and $\mathbf{y}|\mathbf{s}_1$ for $A = 5$ dB

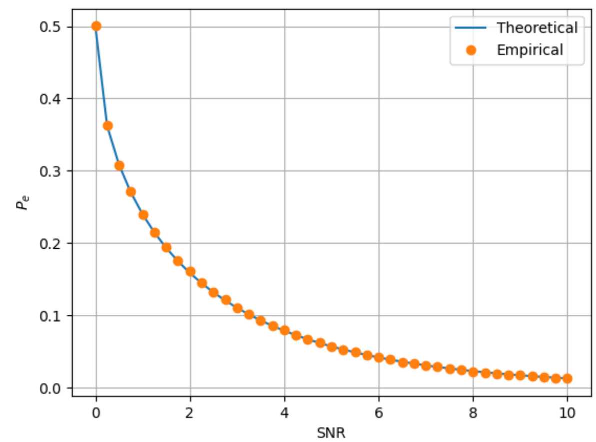

Fig. 8.3. Plot of P_e

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.7}$$

with respect to the SNR from 0 to 10 dB
Solution: Download the C source code to generate the data by executing the following commands

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/8.3.c
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/
    header.h
```

Compile and run the C program by executing the following

```
cc −lm 8.3.c
./a.out
```

Download the following python code that plots Fig. 8.3

```
wget https://github.com/LokeshBadisa/AI1110
    −Probability−and−Random−Variables/
    blob/main/Random−Numbers/codes/8.3.py
```

Run the code by executing

```
python3 8.3.py
```

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph

Solution:

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.8}$$
$$= \Pr\left(y_2 - y_1 > 0 | x_1 = 1, x_2 = 0\right) \tag{8.9}$$
$$= \Pr\left((A(0) + n_2) - (A(1) - n_1) > 0\right) \tag{8.10}$$
$$= \Pr\left(n_2 - n_1 > A\right) \tag{8.11}$$

Since $n_1, n_2 \sim \mathcal{N}(0,1)$, $n_2 - n_1$ also has a Gaussian distribution because a linear combination of Gaussian random variables is also a Gaussian random variable.

$$E\left[n_2 - n_1\right] = E\left[n_2\right] - E\left[n_1\right] \tag{8.12}$$
$$= 0 - 0 \tag{8.13}$$
$$= 0 \tag{8.14}$$
$$\mathrm{Var}\left[n_2 - n_1\right] = \mathrm{Var}\left[n_2\right] + \mathrm{Var}\left[n_1\right] \tag{8.15}$$
$$= 1 + 1 \tag{8.16}$$
$$= 2 \tag{8.17}$$
$$\therefore n_2 - n_1 \sim \mathcal{N}(0,2) \tag{8.18}$$

Thus,

$$P_e = \Pr\left(n_2 - n_1 > A\right) \tag{8.19}$$
$$= \int_A^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x \tag{8.20}$$

Substitute $\frac{x}{\sigma} = u \implies \mathrm{d}x = \sigma \mathrm{d}u$

$$P_e = \int_{\frac{A}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{u^2}{2}\right) \sigma \mathrm{d}u \tag{8.21}$$
$$= \int_{\frac{A}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u \tag{8.22}$$
$$= Q\left(\frac{A}{\sigma}\right) \tag{8.23}$$

Therefore,

$$P_e = Q\left(\frac{A}{\sqrt{2}}\right) = Q\left(\sqrt{\frac{\text{SNR}}{2}}\right) \tag{8.24}$$

as

$$\text{SNR} = A^2 \tag{8.25}$$

Now, consider a threshold δ while estimating $\hat{\mathbf{x}}$.

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{s}_0 & y_2 - y_1 < \delta \\ \mathbf{s}_1 & y_2 - y_1 > \delta \end{cases} \tag{8.26}$$

We have to find the δ that minimizes P_e

$$P_{e|\mathbf{s}_1} = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.27}$$
$$= \Pr\left(y_2 - y_1 > \delta | x_1 = 1, x_2 = 0\right) \tag{8.28}$$
$$= \Pr\left((A(0) + n_2) - (A(1) + n_1) > \delta\right) \tag{8.29}$$
$$= \Pr\left(n_2 - n_1 > A + \delta\right) \tag{8.30}$$
$$= Q\left(\frac{A+\delta}{\sqrt{2}}\right) \tag{8.31}$$

Similarly,

$$P_{e|\mathbf{s}_0} = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_0 | \mathbf{x} = \mathbf{s}_1\right) \tag{8.32}$$
$$= \Pr\left(y_2 - y_1 < \delta | x_1 = 0, x_2 = 1\right) \tag{8.33}$$
$$= \Pr\left(A + n_2 - n_1 < \delta\right) \tag{8.34}$$
$$= \Pr\left(n_2 - n_1 < \delta - A\right) \tag{8.35}$$
$$= \Pr\left(n_2 - n_1 > A - \delta\right) \tag{8.36}$$
$$= Q\left(\frac{A-\delta}{\sqrt{2}}\right) \tag{8.37}$$

Assuming $\mathbf{x}$ has equiprobable symbols,

$$\Pr\left(\mathbf{x} = \mathbf{s}_0\right) = \Pr\left(\mathbf{x} = \mathbf{s}_1\right) = \frac{1}{2} \tag{8.38}$$

$$P_e = \Pr\left(\mathbf{x} = \mathbf{s}_0\right) P_{e|\mathbf{s}_1} + \Pr\left(\mathbf{x} = \mathbf{s}_1\right) P_{e|\mathbf{s}_0} \tag{8.39}$$
$$= \frac{1}{2} P_{e|\mathbf{s}_1} + \frac{1}{2} P_{e|\mathbf{s}_0} \tag{8.40}$$
$$= \frac{1}{2}\left(Q\left(\frac{A+\delta}{\sqrt{2}}\right) + Q\left(\frac{A+\delta}{\sqrt{2}}\right)\right) \tag{8.41}$$

On differentiating with respect to δ, we get

$$\frac{\mathrm{d}P_e}{\mathrm{d}\delta} = \frac{1}{2\sqrt{2}} Q'\left(\frac{A+\delta}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}} Q'\left(\frac{A-\delta}{\sqrt{2}}\right) \tag{8.42}$$
$$= \frac{1}{4\pi}\left(\exp\left(-\frac{(A-\delta)^2}{4}\right) - \exp\left(-\frac{(A+\delta)^2}{4}\right)\right) \tag{8.43}$$

Thus,

$$\frac{\mathrm{d}P_e}{\mathrm{d}\delta} = 0 \implies (A-\delta)^2 = (A+\delta)^2 \tag{8.44}$$
$$\implies |A-\delta| = |A+\delta| \tag{8.45}$$
$$\implies \delta = 0 \tag{8.46}$$

$$\frac{\mathrm{d}^2 P_e}{\mathrm{d}\delta^2} = \frac{A-\delta}{8\pi}\exp\left(-\frac{(A-\delta)^2}{4}\right) + \frac{A+\delta}{8\pi}\exp\left(-\frac{(A+\delta)^2}{4}\right) \tag{8.47}$$

$$\left.\frac{\mathrm{d}^2 P_e}{\mathrm{d}\delta^2}\right|_{\delta=0} = \frac{A}{4\pi}\exp\left(-\frac{A^2}{4}\right) \tag{8.48}$$
$$> 0 \quad (\because A > 0) \tag{8.49}$$

Therefore, $\hat{\delta} = 0$ is a minima that minimizes P_e and our original guess was justified.