Digital Signal Processing
EE3900: Linear Systems and Signal Processing
Indian Institute of Technology Hyderabad

Pingala

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CONTENTS

1. JEE 2019

Let α and β $(\alpha > \beta)$ be the roots of the equation $z^2 - z - 1 = 0$. Define,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution:

https://raw.githubusercontent.com/
   LokeshBadisa/EE3900−Linear−Systems−
   and−Signal−Processing/main/pingala/
   codes/1.1.py

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution:

https://raw.githubusercontent.com/
   LokeshBadisa/EE3900−Linear−Systems−
   and−Signal−Processing/main/pingala/
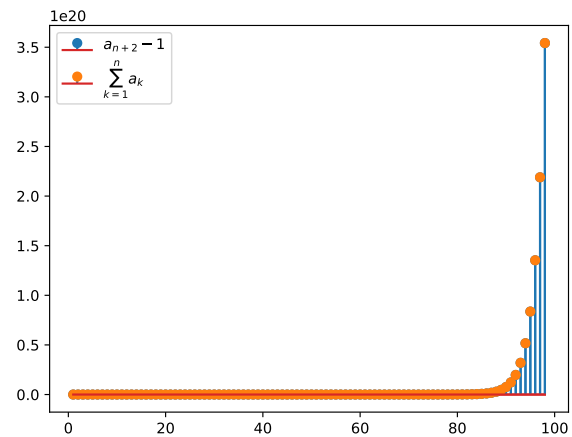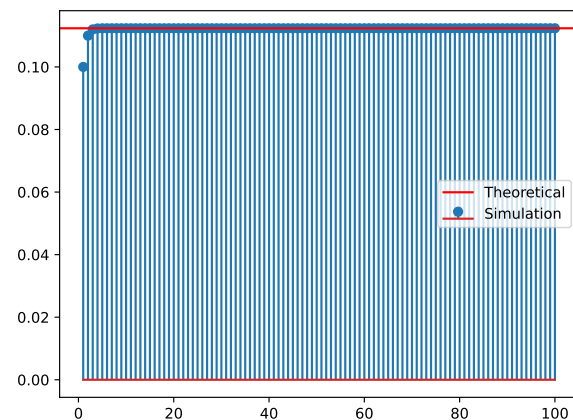   codes/1.2.py

Fig. 1.1.

Fig. 1.2.

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \qquad (1.5)$$

Solution:

https://raw.githubusercontent.com/
LokeshBadisa/EE3900−Linear−Systems−
and−Signal−Processing/main/pingala/
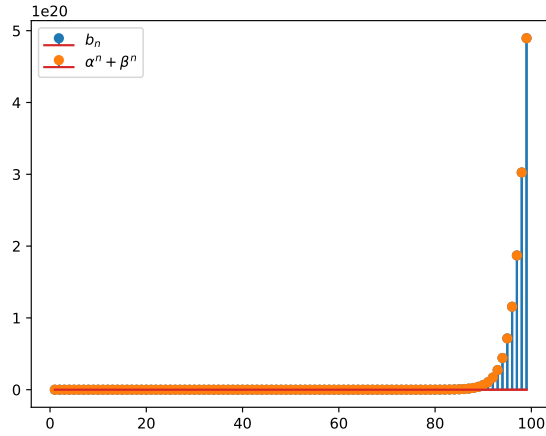codes/1.3.py

Fig. 1.4.

Fig. 1.3.

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \qquad (1.6)$$

Solution:

https://raw.githubusercontent.com/
LokeshBadisa/EE3900−Linear−Systems−
and−Signal−Processing/main/pingala/
codes/1.4.py

https://raw.githubusercontent.com/
LokeshBadisa/EE3900−Linear−Systems−
and−Signal−Processing/main/pingala/
codes/2.1.py

2. PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \qquad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n + 2) = x(n + 1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \qquad (2.2)$$
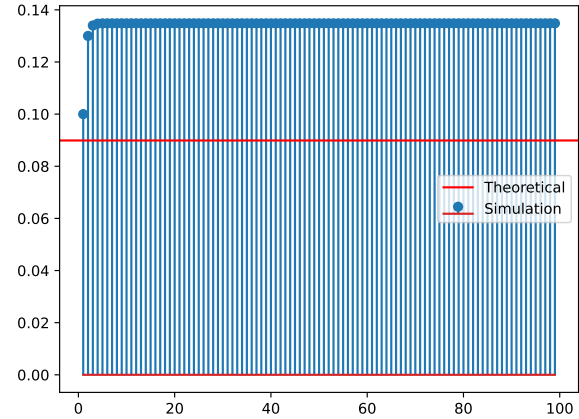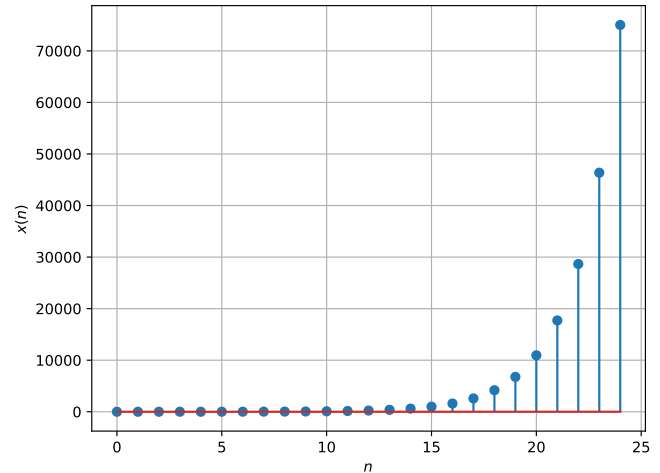
Generate a stem plot for $x(n)$.
Solution:

Fig. 2.2. Plot of $x(n)$

2.3 Find $X^+(z)$.
Solution: Taking the one-sided Z-transform on

both sides of (2.2),

$$\mathcal{Z}^+[x(n+2)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n)]$$
(2.3)

$$z^2 X^+(z) - z^2 x(0) - zx(1) = zX^+(z) - zx(0) + zX^+(z)$$
(2.4)

$$\left(z^2 - z - 1\right) X^+(z) = z^2$$
(2.5)

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$
(2.6)

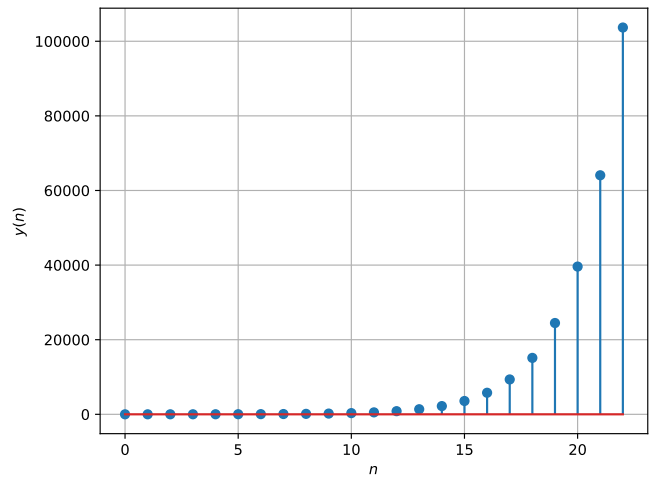$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \quad |z| > \alpha$$
(2.7)

2.4 Find $x(n)$.

Solution: Expanding $X^+(z)$ in (2.7) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta) z^{-1}} \left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right]$$
(2.8)

$$= \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n+1}$$
(2.9)

$$= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1}$$
(2.10)

$$= \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} z^{-k}$$
(2.11)

where $k := n + 1$. Thus,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n)$$
(2.12)

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0$$
(2.13)

Solution:

2.6 Find $Y^+(z)$.

Fig. 2.5. Plot of $y(n)$

Solution: Taking the one-sided Z-transform on both sides of (2.13),

$$\mathcal{Z}^+[y(n)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n-1)]$$
(2.14)

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1)$$
(2.15)

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z$$
(2.16)

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha$$
(2.17)

since $x(n) = 0 \; \forall \; n < 0$.

2.7 Find $y(n)$.

Solution: Using (2.7),

$$Y^+(z) = (1 + 2z^{-1}) \sum_{n=0}^{\infty} x(n) z^{-n}$$
(2.18)

$$= \sum_{n=0}^{\infty} x(n) z^{-n} + \sum_{n=1}^{\infty} 2x(n-1) z^{-n}$$
(2.19)

$$= x(0) + \sum_{n=1}^{\infty} (x(n) + 2x(n-1)) z^{-n}$$
(2.20)

Thus, $y(0) = x(0) = 1$ and for $n \geq 1$, using the fact that α and β are the roots of the equation

$$z^2 - z - 1 = 0,$$

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta} \quad (2.21)$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta} \quad (2.22)$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta\left(\alpha^n + \beta^n\right)}{\alpha - \beta} \quad (2.23)$$

$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \quad (2.24)$$

$$= \alpha^{n+1} + \beta^{n+1} \quad (2.25)$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.22) with the definition of b_n, we see that $y(n) = b_{n+1}$. Hence, $b_n = \alpha^n + \beta^n$.

3. Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \quad (3.1)$$

Solution: From (2.12), and noting that $x(n) = 0 \ \forall \ n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \quad (3.2)$$

$$= \sum_{k=-\infty}^{n-1} x(k) \quad (3.3)$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \quad (3.4)$$

$$= x(n) * u(n-1) \quad (3.5)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.6)$$

can be expressed as

$$[x(n+1) - 1]u(n) \quad (3.7)$$

Solution: From (2.12),

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \quad (3.8)$$

and so, using the definition of $u(n)$,

$$a_{n+2} - 1 = [x(n+1) - 1]u(n) \quad (3.9)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.10)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \quad (3.11)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad (3.12)$$

$$= \frac{1}{10} X^+(z) \quad (3.13)$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \quad (3.14)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.15)$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right)u(n) \quad (3.16)$$

and find $W(z)$.

Solution: Putting $n = k+1$ in (3.15) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right)u(k) \quad (3.17)$$

Hence, (3.15) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right)u(n) = y(n) \quad (3.18)$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.19)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.20)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \quad (3.21)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad (3.22)$$

$$= \frac{1}{10} Y^+(z) \quad (3.23)$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \quad (3.24)$$

3.6 Solve the JEE 2019 problem.

Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \qquad (3.25)$$

But

$$x(n) * u(n-1) \overset{Z}{\rightleftharpoons} X(z)z^{-1}U(z) \qquad (3.26)$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \qquad (3.27)$$

$$= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \qquad (3.28)$$

$$\overset{Z}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \qquad (3.29)$$

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \qquad (3.30)$$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \qquad (3.31)$$

$$\qquad (3.32)$$

From (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \qquad (3.33)$$

We have already established the remaining options in order in the problems (3.3), (2.7), (3.5). Therefore, options 1, 2, and 3 are correct and option 4 is incorrect.