

# Digital Signal Processing

## EE3900: Linear Systems and Signal Processing

### Indian Institute of Technology Hyderabad

## Fourier Series

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<i>Abstract—This manual provides a simple introduction to Fourier Series</i>		

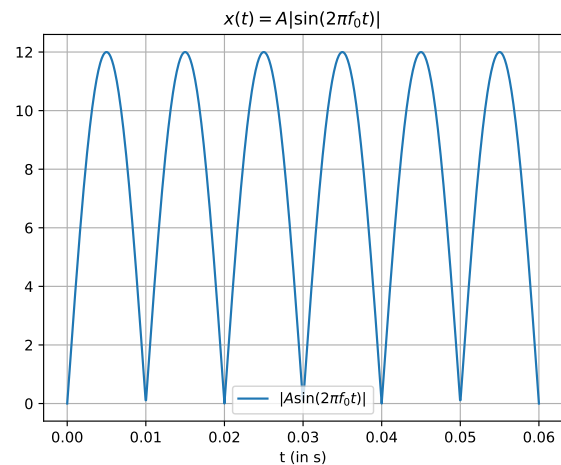


Fig. 1.1.

### 1. PERIODIC FUNCTION

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \quad (1.1)$$

#### 1.1 Plot $x(t)$ .

**Solution:**

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/charger/
codes/1.1.py
python3 1.1.py
```

#### 1.2 Show that $x(t)$ is periodic and find its period.

**Solution:** If a signal  $x(t)$  is periodic then

$$x(t + T) = x(t) \quad (1.2)$$

where  $T$  is known as fundamental period. Since  $|\sin\theta|$  function is periodic,  $x(t)$  is also periodic.

$$\text{Fundamental Period} = T = \frac{1}{2} \left( \frac{2\pi}{2\pi f_0} \right) \quad (1.3)$$

$$= \frac{1}{2f_0} \quad (1.4)$$

### 2. FOURIER SERIES

Consider  $A_0 = 12$  and  $f_0 = 50$  for all numerical calculations.

#### 2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.1)$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \quad (2.2)$$

**Solution:** From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.3)$$

Multitply  $e^{-j2\pi l f_0 t}$  on both sides

$$x(t)e^{-j2\pi l f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} e^{-j2\pi l f_0 t} \quad (2.4)$$

Integrate on both sides with respect to 't' between  $-T$  to  $T$  where  $T$  is fundamental time period of  $x(t)$ .

Using (1.4),

$$T = \frac{1}{2f_0} \quad (2.5)$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t} dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-l)f_0 t} dt \quad (2.6)$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t} dt \quad (2.7)$$

The above integral:

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t} dt = \begin{cases} 0 & k \neq l \\ \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1 dt & k = l \end{cases} \quad (2.8)$$

$$\therefore \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t} dt = \left(\frac{1}{f_0}\right) c_k \quad (2.9)$$

$$\therefore c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t} dt \quad (2.10)$$

2.2 Find  $c_k$  for (1.1)

**Solution:**  $c_k$  can be calculated even simpler by using

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t} dt \quad (2.11)$$

$x(t) = A_0 \sin(2\pi f_0 t)$  in 0 to  $\frac{1}{2f_0}$  region.  
Also,

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (2.12)$$

Using (2.12),

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \left( \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} \right) e^{-j2\pi k f_0 t} dt \quad (2.13)$$

$$= A_0 f_0 \int_0^{\frac{1}{2f_0}} \left( \frac{e^{j2\pi(1-k)f_0 t} - e^{j2\pi(-1-k)f_0 t}}{j} \right) dt \quad (2.14)$$

$$= A_0 f_0 \left[ \frac{e^{j2\pi(1-k)f_0 t}}{-2\pi(1-k)f_0} \Big|_0^{\frac{1}{2f_0}} - \frac{e^{j2\pi(-1-k)f_0 t}}{-2\pi(-1-k)f_0} \Big|_0^{\frac{1}{2f_0}} \right] \quad (2.15)$$

$$= A_0 \left[ \frac{e^{j\pi(1-k)} - 1}{2\pi(k-1)} - \frac{e^{-j\pi(1+k)} - 1}{2\pi(k+1)} \right] \quad (2.16)$$

$$= \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k = \text{even} \\ 0 & k = \text{odd} \end{cases} \quad (2.17)$$

2.3 Verify (1.1) using python.

**Solution:**

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/charger/
codes/2.3.py
python3 2.3.py
```

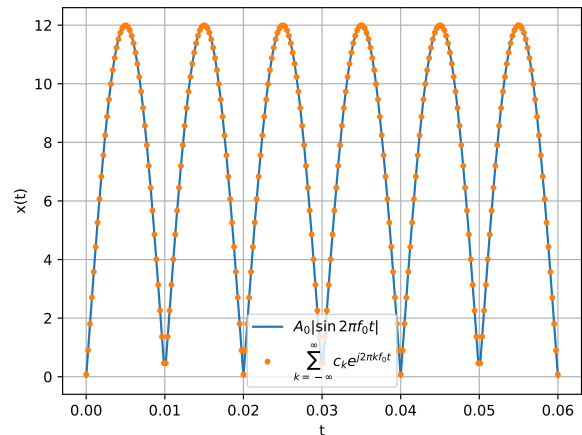


Fig. 2.3.

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \quad (2.18)$$

and obtain the formulae for  $a_k$  and  $b_k$ .

**Solution:** Using (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.19)$$

As,

$$e^{j2\pi k f_0 t} = \cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t) \quad (2.20)$$

Substituting leads to

$$x(t) = \sum_{k=-\infty}^{\infty} c_k [\cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t)] \quad (2.21)$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t) \quad (2.22)$$

$$= \sum_{k=-\infty}^{-1} [c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)] + c_0 + \sum_{k=1}^{\infty} [c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)] \quad (2.23)$$

$$= \sum_{k=1}^{\infty} [c_{-k} \cos(2\pi k f_0 t) - j c_{-k} \sin(2\pi k f_0 t)] + c_0 + \sum_{k=1}^{\infty} [c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)] \quad (2.24)$$

$$= c_0 + \sum_{k=1}^{\infty} [(c_k + c_{-k}) \cos(2\pi k f_0 t) + j(c_k - c_{-k}) \sin(2\pi k f_0 t)] \quad (2.25)$$

Replacing  $(c_k + c_{-k}) \rightarrow a_k$  and  $j(c_k - c_{-k}) \rightarrow b_k$ ,

$$= c_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \quad (2.26)$$

$$= \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \quad (2.27)$$

$$\therefore a_k = \begin{cases} c_k + c_{-k} & k \neq 0 \\ c_0 & k = 0 \end{cases} \quad (2.28)$$

$$b_k = j(c_k - c_{-k}) \quad (2.29)$$

Using (2.2),

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \quad (2.30)$$

$$c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t} dt \quad (2.31)$$

$$a_k = c_k + c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) [e^{-j2\pi k f_0 t} + e^{j2\pi k f_0 t}] dt \quad (2.32)$$

$$= 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos(2\pi k f_0 t) dt \quad (2.33)$$

Parallely,

$$b_k = -2jf_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin(2\pi k f_0 t) dt \quad (2.34)$$

2.5 Find  $a_k$  and  $b_k$  for (1.1)

**Solution:** Using (2.28) and (2.29) with (2.17),

$$a_k = c_k + c_{-k} = \begin{cases} \frac{4A_0}{\pi} \frac{(1-k^2)}{2} & k = \text{even} \\ \frac{4A_0}{\pi} \sin(2\pi k f_0 t) & k = 0 \\ 0 & k = \text{odd} \end{cases} \quad (2.35)$$

2.6 Verify (2.18) using python.

**Solution:**

wget <https://raw.githubusercontent.com/LokeshBadisa/EE3900-Linear-Systems-and-Signal-Processing/main/charger/codes/2.6.py>  
python3 2.3.py

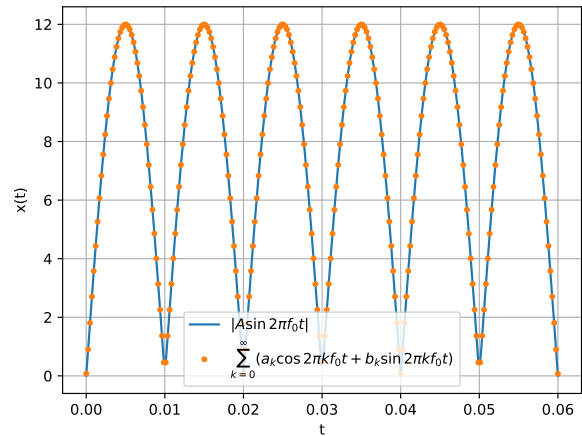


Fig. 2.6.

### 3. FOURIER TRANSFORM

3.1

$$\delta(t) = 0, \quad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3.2)$$

3.2 The Fourier Transform of  $g(t)$  is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \quad (3.3)$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \quad (3.4)$$

$$(3.5)$$

**Solution:**

$$\mathcal{F}\{g(t - t_0)\} = \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi ft} dt \quad (3.6)$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f(t - t_0)} dt \quad (3.7)$$

$$= G(f) e^{-j2\pi f t_0} \quad (3.8)$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.9)$$

**Solution:** From the definition of Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \quad (3.10)$$

Replace  $t \rightarrow f$ ,

$$g(f) = \int_{-\infty}^{\infty} G(t) e^{j2\pi ft} dt \quad (3.11)$$

Replace  $f \rightarrow -f$ ,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi ft} dt \quad (3.12)$$

$$= \mathcal{F}\{G(t)\} \quad (3.13)$$

$$\therefore G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.14)$$

3.5  $\delta(t) \xleftrightarrow{\mathcal{F}} ?$

**Solution:**

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt \quad (3.15)$$

$$= \int_{-\infty}^{\infty} \delta(t) dt \quad (3.16)$$

$$(3.17)$$

Since  $e^{-j2\pi ft} = 1$  for  $t=0$  and remaining integrand is zero for  $t \neq 0$ .

$$= \int_{-\infty}^{\infty} \delta(t) dt \quad (3.18)$$

$$= 1 \quad (3.19)$$

3.6  $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$

**Solution:**

$$\mathcal{F}\{e^{-j2\pi f_0 t}\} = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi(f+f_0)t} dt \quad (3.20)$$

$$= \int_{-\infty}^{\infty} \mathcal{F}\{\delta(t)\} e^{-j2\pi(f+f_0)t} dt \quad (3.21)$$

Using (3.9),

$$= \delta(-(f+f_0)) = \delta(f+f_0) \quad (3.22)$$

3.7  $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$

**Solution:**

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \mathcal{F}\left\{\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right\} \quad (3.23)$$

Using (3.22),

$$= \frac{\delta(f+f_0) + \delta(f-f_0)}{2} \quad (3.24)$$

3.8 Find the Fourier Transform of  $x(t)$  and plot it. Verify using python. Using (2.1),

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}\right\} \quad (3.25)$$

$$X(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_0) \quad (3.26)$$

The following verifies the above equations:

```
wget https://github.com/LokeshBadisa/
EE3900-Linear-Systems-and-Signal-
Processing/blob/main/charger/codes/3.8.py
python3 3.8.py
```

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(t) \quad (3.27)$$

Verify using python.

**Solution:**

$$\mathcal{F}\{\text{rect}(t)\} = \int_{-\infty}^{\infty} \text{rect}(t) e^{-j2\pi ft} dt \quad (3.28)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j2\pi ft} dt \quad (3.29)$$

$$= \frac{e^{-j2\pi ft}}{-j2\pi f} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \quad (3.30)$$

$$= \text{sinc}(t) \quad (3.31)$$

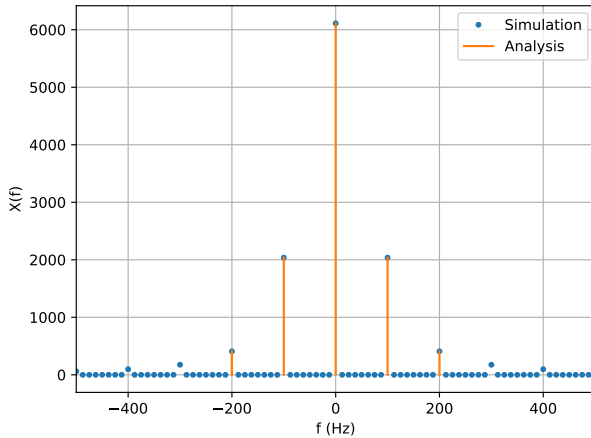


Fig. 3.8.

```
wget https://github.com/LokeshBadisa/
EE3900-Linear-Systems-and-Signal-
Processing/blob/main/charger/codes/3.9.py
python3 3.9.py
```

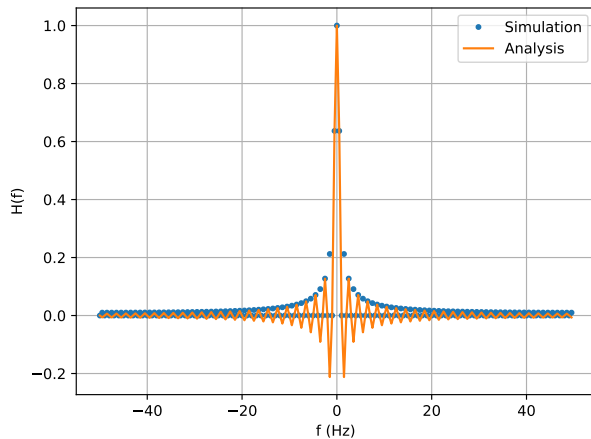


Fig. 3.9.

3.10  $\text{sinc}(t) \xleftrightarrow{\mathcal{F}} ?$ . Verify using python.

**Solution:** Using (3.31), (3.9) and even property of rect function,

$$\text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}(f) \quad (3.32)$$

```
wget https://github.com/LokeshBadisa/
EE3900-Linear-Systems-and-Signal-
Processing/blob/main/charger/codes/3.10.
py
```

```
python3 3.10.py
```

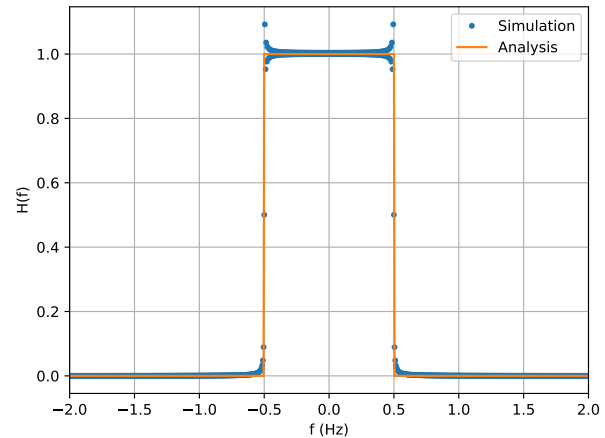


Fig. 3.10.

#### 4. FILTER

4.1 Find  $H(f)$  which transforms  $x(t)$  to DC 5V.

**Solution:**

$$x[t] \longrightarrow \boxed{h[t]} \longrightarrow x[t] * h[t]$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow X(f)H(f)$$

$$X(f)H(f) = V_0\delta(f) \quad (4.1)$$

Above equation indicates that  $H(f)$  will pass  $X(f)$  for  $f=0$ .

$\therefore H(f)$  should be a low pass filter.

$$|H(f)| = \frac{V_0}{\left(\frac{2A_0}{\pi}\right)} = \frac{V_0\pi}{2A_0} \quad (4.2)$$

$$H(f) = \frac{V_0\pi}{2A_0} \quad \text{in} \quad -2f_0 \leq f \leq 2f_0 \quad (4.3)$$

$$H(f) = \frac{V_0\pi}{2A_0} \text{rect}\left(\frac{f}{4f_0}\right) \quad (4.4)$$

4.2 Find  $h(t)$ .

**Solution:** Using (4.4) and (3.32),

$$h(t) = \frac{2V_0\pi f_0}{A_0} \text{sinc}(4f_0 t) \quad (4.5)$$

4.3 Verify your result using through convolution.

**Solution:**

```
wget https://github.com/LokeshBadisa/
EE3900-Linear-Systems-and-Signal-
Processing/blob/main/charger/codes/4.3.py
python3 4.3.py
```

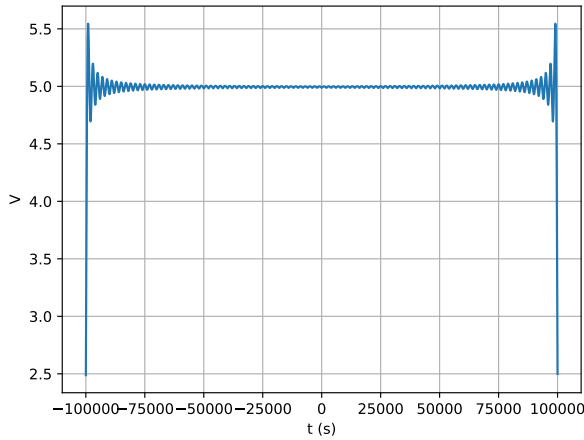


Fig. 4.3.

## 5. FILTER DESIGN

### 5.1 Design a Butterworth filter for $H(f)$ .

**Solution:**

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2n}}} \quad (5.1)$$

$n$  = Order of the filter

$$A = 10 \log_{10} |H(f)|^2 \quad (5.2)$$

$$A_1 = -10 \log_{10} \left[ 1 + \left(\frac{f_1}{f_c}\right)^{2n} \right] \quad (5.3)$$

$$A_2 = -10 \log_{10} \left[ 1 + \left(\frac{f_2}{f_c}\right)^{2n} \right] \quad (5.4)$$

Solving for  $n$ ,

$$n = \frac{\log \left( \frac{10^{-A_1/10} - 1}{10^{-A_2/10} - 1} \right)}{2 \log \left( \frac{f_1}{f_2} \right)} \quad (5.5)$$

Assuming values,

$$A_1 = -1 \text{ dB} \quad (5.6)$$

$$A_2 = -10 \text{ dB} \quad (5.7)$$

$$f_1 = 50 \text{ Hz} \quad (5.8)$$

$$f_2 = 100 \text{ Hz} \quad (5.9)$$

We get,

$$n = 2.5596 \approx 3 \quad (5.10)$$

Putting  $n = 3$  in above equations, We get

$$f'_1 = 62.628 \text{ Hz} \quad (5.11)$$

$$f'_2 = 69.336 \text{ Hz} \quad (5.12)$$

$$f_c = \sqrt{f'_1 f'_2} = 65.897 \text{ Hz} \quad (5.13)$$

### 5.2 Design a Chebyshev filter for $H(f)$ .

**Solution:**

$$|H_n(f)| = \frac{1}{\sqrt{1 + \epsilon^2 c_n^2 \left(\frac{f}{f_0}\right)}} \quad (5.14)$$

where

$c_n$  =  $n$ th order Chebyshev polynomial,

$\epsilon$  = ripple factor which is related to passband ripple in  $\delta$  as  $\sqrt{10^{\delta/10} - 1}$

$$A_1 = -10 \log_{10} \left[ 1 + \epsilon^2 c_n^2 \left(\frac{f}{f_0}\right) \right] \quad (5.15)$$

$$\Rightarrow c_n \left(\frac{f}{f_0}\right) = \frac{\sqrt{10^{-\frac{A_1}{10}} - 1}}{\epsilon} \quad (5.16)$$

$$\Rightarrow n = \frac{\cosh^{-1} \left( \frac{\sqrt{10^{-\frac{A_1}{10}} - 1}}{\epsilon} \right)}{\cosh^{-1} \left( \frac{f}{f_0} \right)} \quad (5.17)$$

Considering

$$f_0 = 65 \text{ Hz} \quad (5.18)$$

$$f = 120 \text{ Hz} \quad (5.19)$$

$$A_1 = 10 \text{ dB} \quad (5.20)$$

$$\delta = 0.2 \text{ dB} \quad (5.21)$$

$$\epsilon = 0.217 \quad (5.22)$$

On solving we get  $n = 2.71302$ ,  $\Rightarrow n = 3$ .

### 5.3 Design a circuit for your Butterworth filter.

**Solution:** Using Cauer Topology,

$$C_k = 2 \sin \left[ \frac{(2k-1)}{2n} \pi \right] \quad (5.23)$$

$$L_k = 2 \sin \left[ \frac{(2k-1)}{2n} \pi \right] \quad (5.24)$$

### 5.4 Design a circuit for your Chebyshev filter.

**Solution:** the normalised values obtained are  $C_1 = 1.2276F$ ,  $L_2 = 1.1525H$ ,  $C_3 = 1.2276F$ . De-normalizing the values we get:

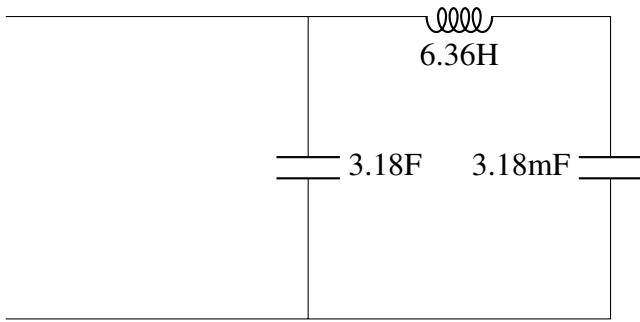


Fig. 5.3.

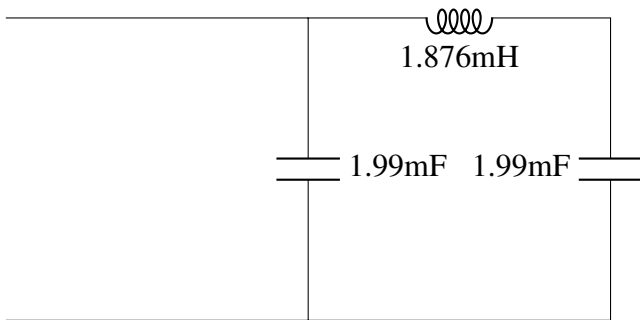


Fig. 5.4.