Digital Signal Processing
EE3900: Linear Systems and Signal Processing
Indian Institute of Technology Hyderabad

Pingala

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Abstract—This document contains solution to Pingala Series Assignment given regarding EE3900 course]

1. JEE 2019

Let α and β ($\alpha > \beta$) be the roots of the equation $z^2 - z - 1 = 0$. Define,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \qquad (1.1)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \qquad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \qquad (1.3)$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \qquad (1.4)$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \qquad (1.5)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \qquad (1.6)$$

Download the Python code using

```
$ wget https://raw.githubusercontent.com/goats−9/
    ee3900−assignmnets/main/pingala/codes/1.py
```

and run it using

```
$ python3 1.py
```

2. Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \qquad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

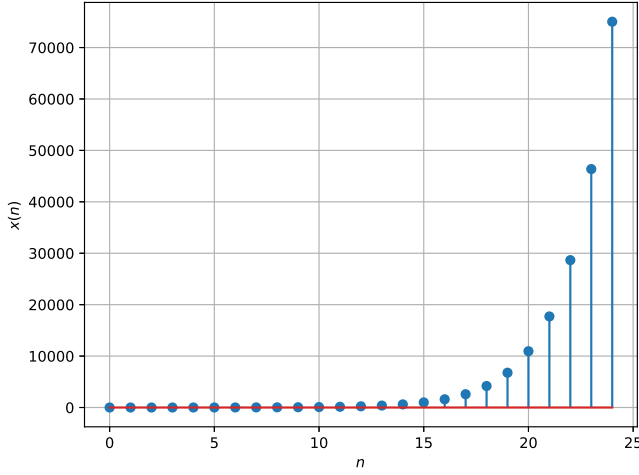$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \qquad (2.2)$$

Generate a stem plot for $x(n)$.
Solution: Download the Python code using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ee3900−assignmnets/main/
    pingala/codes/2_1.py
```

and run it using

```
$ python3 2_1.py
```

Fig. 2.2. Plot of $x(n)$

2.3 Find $X^+(z)$.

Solution: Taking the one-sided Z-transform on both sides of (2.2),

$$\mathcal{Z}^+[x(n+2)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n)]$$

(2.3)

$$z^2 X^+(z) - z^2 x(0) - z x(1) = z X^+(z) - z x(0) + z X^+(z)$$

(2.4)

$$\left(z^2 - z - 1\right) X^+(z) = z^2$$

(2.5)

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

(2.6)

$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \quad |z| > \alpha$$

(2.7)

2.4 Find $x(n)$.

Solution: Expanding $X^+(z)$ in (2.7) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta) z^{-1}} \left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right]$$

(2.8)

$$= \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n+1}$$

(2.9)

$$= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1}$$

(2.10)

$$= \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} z^{-k}$$

(2.11)

where $k := n + 1$. Thus,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n)$$
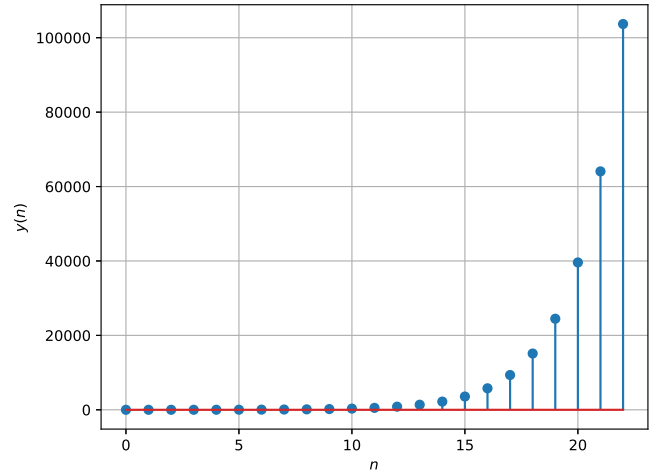
(2.12)

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \quad (2.13)$$

Solution: Download the Python code using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ee3900−assignmnets/main/
    pingala/codes/2_2.py
```

and run it using

```
$ python3 2_2.py
```


Fig. 2.5. Plot of $y(n)$

2.6 Find $Y^+(z)$.

Solution: Taking the one-sided Z-transform on both sides of (2.13),

$$\mathcal{Z}^+[y(n)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n-1)]$$

(2.14)

$$Y^+(z) = z X^+(z) - z x(0) + z^{-1} X^+(z) + z x(-1)$$

(2.15)

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z$$

(2.16)

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha$$

(2.17)

since $x(n) = 0 \ \forall \ n < 0$.

2.7 Find $y(n)$.

Solution: Using (2.7),

$$Y^+(z) = (1 + 2z^{-1}) \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.18}$$

$$= \sum_{n=0}^{\infty} x(n)z^{-n} + \sum_{n=1}^{\infty} 2x(n-1)z^{-n} \tag{2.19}$$

$$= x(0) + \sum_{n=1}^{\infty} (x(n) + 2x(n-1))z^{-n} \tag{2.20}$$

Thus, $y(0) = x(0) = 1$ and for $n \geq 1$, using the fact that α and β are the roots of the equation $z^2 - z - 1 = 0$,

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta} \tag{2.21}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta} \tag{2.22}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta(\alpha^n + \beta^n)}{\alpha - \beta} \tag{2.23}$$

$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \tag{2.24}$$

$$= \alpha^{n+1} + \beta^{n+1} \tag{2.25}$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.22) with the definition of b_n, we see that $y(n) = b_{n+1}$. Hence, $b_n = \alpha^n + \beta^n$.

3. POWER OF THE Z TRANSFORM

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution: From (2.12), and noting that $x(n) = 0 \, \forall \, n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.4}$$

$$= x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$[x(n+1) - 1]u(n) \tag{3.7}$$

Solution: From (2.12),

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \tag{3.8}$$

and so, using the definition of $u(n)$,

$$a_{n+2} - 1 = [x(n+1) - 1]u(n) \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10}X^+(10) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.11}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.12}$$

$$= \frac{1}{10}X^+(z) \tag{3.13}$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.15}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right)u(n) \tag{3.16}$$

and find $W(z)$.
Solution: Putting $n = k+1$ in (3.15) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right)u(k) \tag{3.17}$$

Hence, (3.15) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right)u(n) = y(n) \tag{3.18}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.19}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10}Y^+(10) \tag{3.20}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.21}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.22}$$

$$= \frac{1}{10} Y^+(z) \tag{3.23}$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.24}$$

3.6 Solve the JEE 2019 problem.
   Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.25}$$

But

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z) \tag{3.26}$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{3.27}$$

$$= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \tag{3.28}$$

$$\overset{\mathcal{Z}}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \tag{3.29}$$

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \tag{3.30}$$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \tag{3.31}$$

$$\tag{3.32}$$

From (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.33}$$

We have already established the remaining options in order in the problems (3.3), (2.7), (3.5). Therefore, options 1, 2, and 3 are correct and option 4 is incorrect.