Digital Signal Processing
EE3900: Linear Systems and Signal Processing
Indian Institute of Technology Hyderabad

Lokesh Badisa
AI21BTECH11005

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CONTENTS

1. SOFTWARE INSTALLATION

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2. DIGITAL FILTER

2.1 Download the sound file from

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/Sound_Noise
    .wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
wget https://raw.githubusercontent.com/
    LokeshBadisa/EE3900−Linear−Systems−
    and−Signal−Processing/main/codes/2.3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3. DIFFERENCE EQUATION

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.1

```
wget https://github.com/LokeshBadisa/
    EE3900/tree/main/codes/3.1.py
```

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

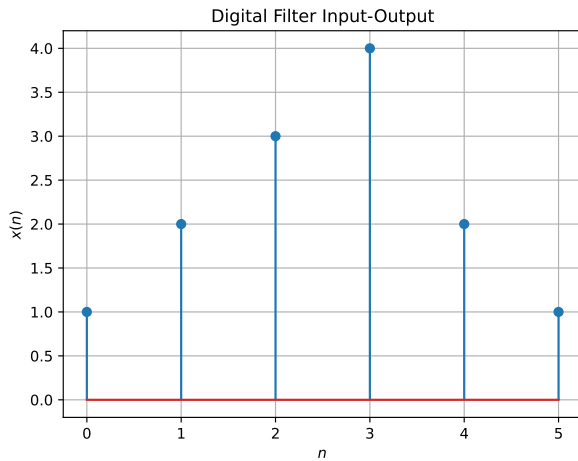Sketch $y(n)$.
Solution: The following code yields Fig. 5.3.

Fig. 3.1.

wget https://github.com/LokeshBadisa/
EE3900−Linear−Systems−and−Signal−
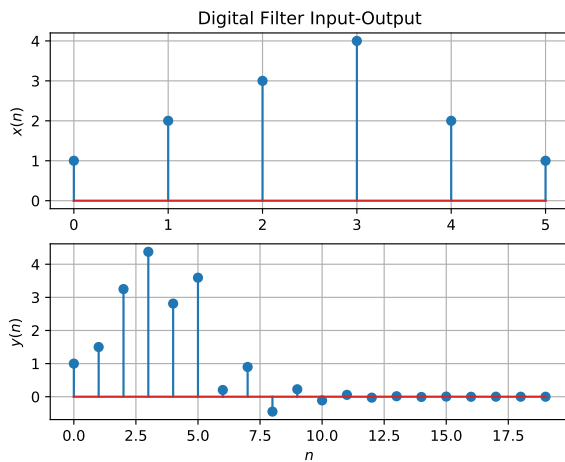Processing/blob/main/codes/3.2.py

Fig. 3.2.

4. Z-TRANSFORM

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.6)$$

4.2 Obtain $X(z)$ for x(n) define in problem 3.1.
Solution: From (4.1),

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.7)$$

$$= \sum_{n=0}^{\infty} x(n)z^{-n} \quad (4.8)$$

$$= 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + 2 \cdot z^{-5} + 1 \cdot z^{-6} \quad (4.9)$$

$$= \frac{z^5 + 2z^4 + 3z^3 + +4z^2 + 2z + 1}{z^6} \quad (4.10)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.11)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.12)$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.13)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.14)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.15)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.16)$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \quad (4.17)$$

and from (4.15),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.18}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.19}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.20}$$

Solution: From (4.1),

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} u(n) \left(\frac{a}{z}\right)^{-n} \tag{4.21}$$

From (4.15),

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{-n} \tag{4.22}$$

$$= \frac{1}{1 - \frac{a}{z}} \tag{4.23}$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.24}$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \tag{4.25}$$

Plot $|H(e^{j\omega})|$. Is it periodic? If so, find the period. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution:

$$H(e^{j\omega}) = H(z = e^{j\omega}) \tag{4.26}$$

$$= \frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \tag{4.27}$$

$$= \frac{2\left(1 + e^{-2j\omega}\right)}{2 + e^{-j\omega}} \tag{4.28}$$

$$= \frac{2\left(1 + \cos 2\omega - j\sin 2\omega\right)}{2 + \cos\omega - j\sin\omega} \tag{4.29}$$

$$= \frac{2\left|\left(2\cos^2\omega - j\sin 2\omega\right)\right|}{\left|2 + \cos\omega - j\sin\omega\right|} \tag{4.30}$$

$$= \frac{2\sqrt{4\cos^4\omega + 4\sin^2\omega\cos^2\omega}}{\sqrt{(2 + \cos\omega)^2 + \sin^2\omega}} \tag{4.31}$$

$$= \frac{4\cos\omega}{\sqrt{5 + 4\cos\omega}} \tag{4.32}$$

$$|H(e^{j\omega})| = \frac{|4\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.33}$$

Since $|H(e^{j\omega})|$ is function of cosine we can say it is periodic. And from the plot 4.6 we can say that it is symmetric about $\omega = 0$ (even function) and it is periodic with period 2π. You can find the same from the theoritical expression $|H(e^{j\omega})|$,

$$H(e^{j\omega}) = H(e^{j(-\omega)}) \text{ (cos is an even function)} \tag{4.34}$$

And to find period, the period of $|\cos(\omega)|$ is π and the period of $\sqrt{5 + 4\cos(\omega)}$ is 2π. So the period of division of both will be 2π. This gives us the period of $|H(e^{j\omega})|$ as 2π The following code plots Fig. 4.6.

wget https://github.com/LokeshBadisa/
EE3900−Linear−Systems−and−Signal−
Processing/blob/main/codes/4.6.py

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$.
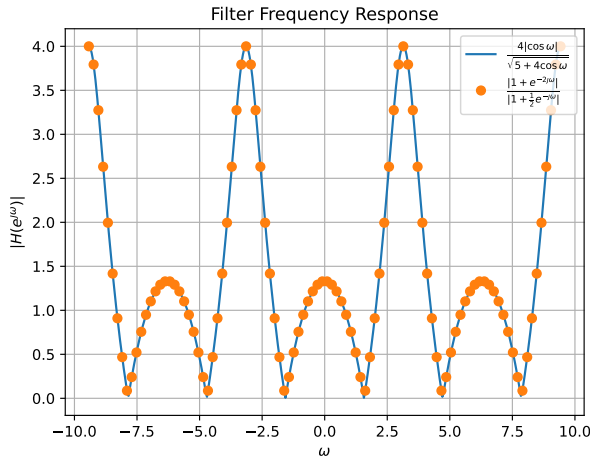
Fig. 4.6. $|H(e^{j\omega})|$

Solution:

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} \, dw \tag{4.35}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{j\omega k} e^{-j\omega n} \, dw \tag{4.36}$$

$$Integral = \begin{cases} 0 & n \neq k \\ 2\pi & n = k \end{cases} \tag{4.37}$$

$$= h(n)\frac{1}{2\pi}2\pi \tag{4.38}$$

$$= h(n) \tag{4.39}$$

Hence Proved.

5. IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.13). **Solution:**

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$
\begin{array}{r}
2x - 4 \\
\frac{1}{2}x + 1 \overline{)\; x^2 \qquad\quad + 1} \\
\underline{-\,x^2 - 2x} \\
-2x + 1 \\
\underline{2x + 4} \\
5
\end{array}
$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)\left(-4 + 2z^{-1}\right) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.4}$$

On applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.5}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.6}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.7}$$

$$\frac{5}{1 + \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.8}$$

$$\tag{5.9}$$

Therefore,

$$h(n) = -4\delta(n) + 2\delta(n-1) + 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.10}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.11}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.13),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.12}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.13}$$

using (4.20) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution:
The following code plots Fig. 5.2.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/5.3.py
```

$$\lim_{n \to \infty} h(n) = \lim_{n \to \infty} \left(-\frac{1}{2}\right)^n u(n) + \lim_{n \to \infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.14}$$
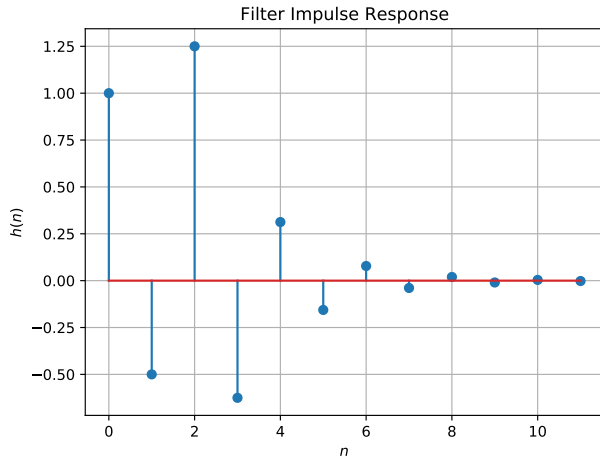
Fig. 5.2. $h(n)$ as the inverse of $H(z)$
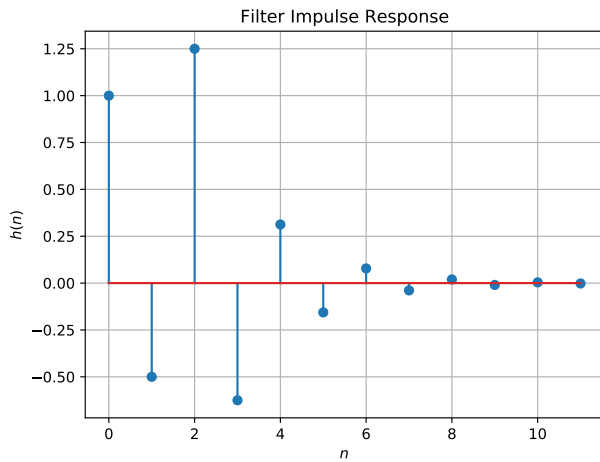
Fig. 5.3.

$$h(n) = \begin{cases} 5\left(\frac{-1}{2}\right)^n & n \geq 2 \\ \left(\frac{-1}{2}\right)^n & 0 \leq n < 2 \\ 0 & n < 0 \end{cases} \tag{5.15}$$

Maximum value and minimum value are always bounded in this case. $\therefore$ $h(n)$ is bounded

5.4 Is it convergent? Justify using the ratio test.
Solution:

$$h(n + 1) = \left(-\frac{1}{2}\right)^{n+1} u(n + 1) + \left(-\frac{1}{2}\right)^{n-1} u(n - 1) \tag{5.16}$$

According to ratio test, L is given by

$\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right|$, if $L < 1$ then h(n) is convergent.

$$L = \lim_{n\to\infty} \left|\frac{h(n + 1)}{h(n)}\right| \tag{5.17}$$

$$= \lim_{n\to\infty} \left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n + 1) + \left(-\frac{1}{2}\right)^{n-1} u(n - 1)}{\left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n - 2)}\right| \tag{5.18}$$

$$= \left|\frac{-\frac{1}{2} + -\frac{1}{2}^{-1}}{1 + -\frac{1}{2}^{-2}}\right| \tag{5.19}$$

$$= \left|\frac{-\frac{1}{2} - 2}{1 + 4}\right| \tag{5.20}$$

$$= \left|\frac{\frac{-5}{2}}{5}\right| \tag{5.21}$$

$$= \frac{1}{2} \tag{5.22}$$

As $L < 1$, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the impulse response in (5.11)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} h(n) \tag{5.24}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n - 2) \tag{5.25}$$

$$= 2\left(\frac{1}{1 + \frac{1}{2}}\right) \tag{5.26}$$

$$= \frac{4}{3} \tag{5.27}$$

As $\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3}$ is less than ∞, the system defined by (3.2) is stable for the impulse response in (5.1).

5.6 Verify the above result using a python code.
Solution: The following code determines if it is convergent or not:

wget https://github.com/LokeshBadisa/
EE3900−Linear−Systems−and−Signal−
Processing/blob/main/codes/5.5.py

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.28)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7.
Note that this is the same as Fig. 5.2.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/5.7.py
```

Computing,
$h(0) = 1$
$h(1) = -\frac{1}{2}h(0)$
$h(2) = -\frac{1}{2}h(1) + 1$
Parallely, $h(n) = -\frac{1}{2}h(n-1)$
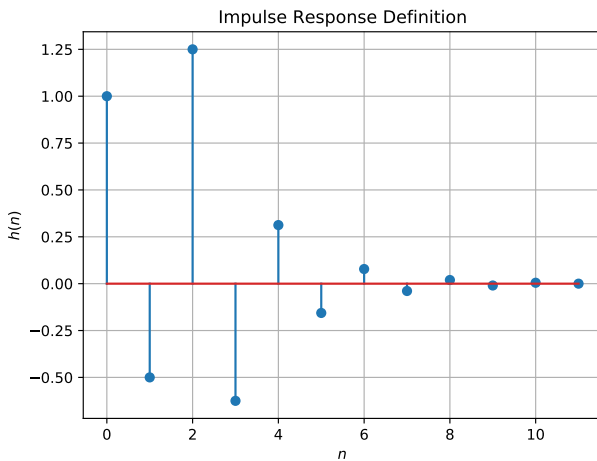
Fig. 5.7. $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.29)$$

Comment. The operation in (5.29) is known as
convolution.
Solution: The following code plots Fig. 5.8.
Note that this is the same as $y(n)$ in Fig. 5.3.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/5.8.py
```
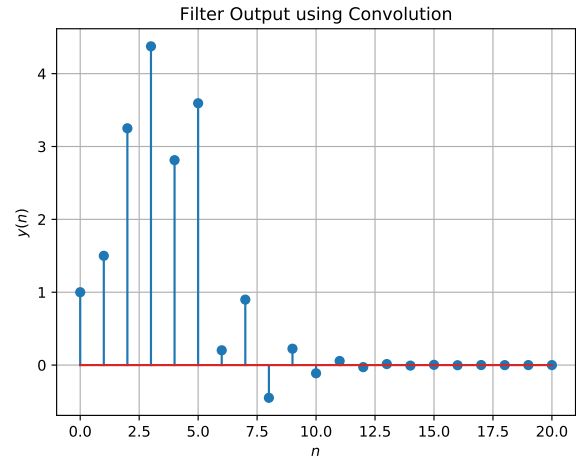

Fig. 5.8. $y(n)$ from the definition of convolution

5.9 Express the above convolution using a toeplitz matrix. **Solution:**

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \quad (5.30)$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ h_{m-1} & . & . & . & h_2 & h_1 \\ h_m & h_{m-1} & . & . & . & h_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

$$(5.31)$$

$$\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 & 0 \\ \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 \\ \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 \\ \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 \\ \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} \\ 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} \\ 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} \\ 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{64} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1. \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.8125 \\ 3.59375 \\ 0.203125 \\ 0.9375 \\ -0.390625 \\ 0.3125 \\ 0. \\ 0.078125 \end{pmatrix}$$

$$(5.32)$$

And this is what we got in (5.29)

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.33)$$

Solution: Substitute $k \rightarrow n - k$ then

$$y(n) = x(n) * h(n) \tag{5.34}$$

$$= \sum_{n=-\infty}^{\infty} x(k)h(n-k) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.35}$$

6. DFT AND FFT

6.1 Compute

$$X(k) \overset{\triangle}{=} \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$. **Solution:** The following code plots Fig. 6.1.

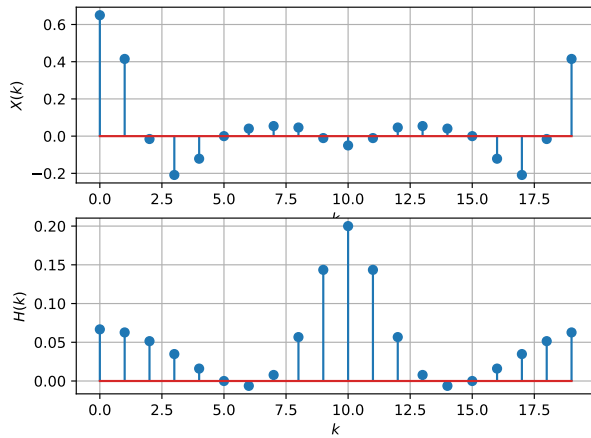Fig. 6.1. $X(k), H(k)$ from the DFT

wget https://github.com/LokeshBadisa/
   EE3900−Linear−Systems−and−Signal−
   Processing/blob/main/codes/xkhkdft.py

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig. 6.2.

wget https://github.com/LokeshBadisa/
   EE3900−Linear−Systems−and−Signal−
   Processing/blob/main/codes/ykdft.py

6.3 Compute

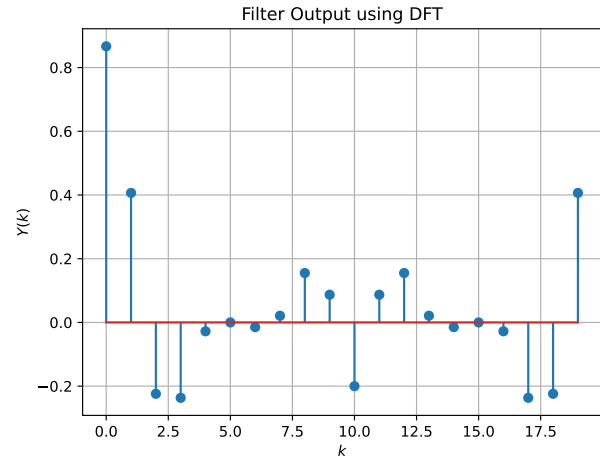$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Fig. 6.2. $Y(k)$ from the DFT

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 5.3.

wget https://github.com/LokeshBadisa/
   EE3900−Linear−Systems−and−Signal−
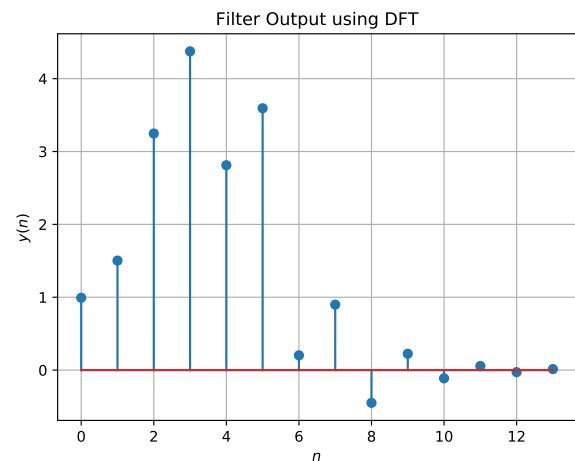   Processing/blob/main/codes/yndft.py

Fig. 6.3. $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT. **Solution:** The following code calculates $X(k), H(k)$ and $y(n)$.

wget https://github.com/LokeshBadisa/
   EE3900−Linear−Systems−and−Signal−
   Processing/blob/main/codes/6.4.py

6.5 Wherever possible, express all the above equations as matrix equations.

6.6 Verify the above equations by generating the DFT matrix in python.

7. EXERCISES

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m) y(n-m) = \sum_{k=0}^{N} b(k) x(n-k) \quad (7.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.