Digital Signal Processing

EE3900: Linear Systems and Signal Processing

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CONTENTS

1. Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2. Digital Filter

2.1 Download the sound file from

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/Sound_Noise
    .wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer.
Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
wget https://raw.githubusercontent.com/
    LokeshBadisa/EE3900−Linear−Systems−
    and−Signal−Processing/main/codes/2.3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3. Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.1

```
wget https://github.com/LokeshBadisa/
    EE3900/tree/main/codes/3.1.py
```

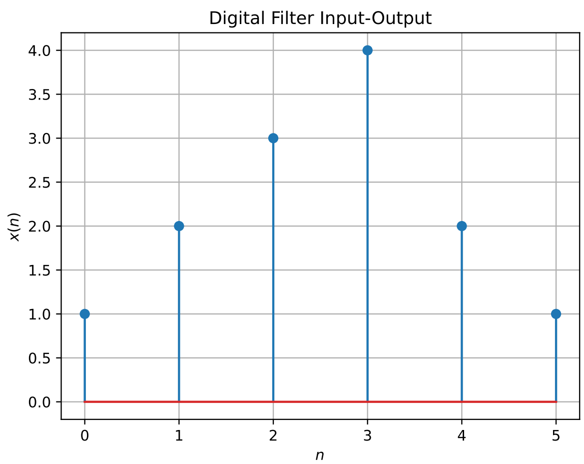

Fig. 3.1.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch $y(n)$.
Solution: The following code yields Fig. 5.3.

```
wget https://github.com/LokeshBadisa/
    EE3900-Linear-Systems-and-Signal-
    Processing/blob/main/codes/3.2.py
```

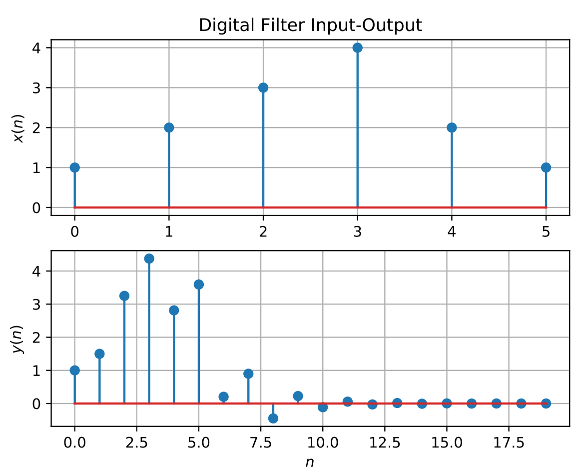

Fig. 3.2.

4. Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.6)$$

4.2 Obtain $X(z)$ for x(n) define in problem 3.1. **Solution:** From (4.1),

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.7)$$
$$= \sum_{n=0}^{\infty} x(n)z^{-n} \quad (4.8)$$
$$= 1\cdot z^{-1} + 2\cdot z^{-2} + 3\cdot z^{-3} + 4\cdot z^{-4} + 2\cdot z^{-5} + 1\cdot z^{-6} \quad (4.9)$$
$$= \frac{z^5 + 2z^4 + 3z^3 + +4z^2 + 2z + 1}{z^6} \quad (4.10)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.11)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.12)$$
$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \quad (4.13)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.14)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.15)$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.16}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.17}$$

and from (4.15),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.18}$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.19}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.20}$$

Solution: From (4.1),

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} u(n) \left(\frac{a}{z}\right)^{-n} \tag{4.21}$$

From (4.15),

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{-n} \tag{4.22}$$

$$= \frac{1}{1-\frac{a}{z}} \tag{4.23}$$

$$= \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.24}$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.25}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Is it periodic? If so,find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution:

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right) \tag{4.26}$$

$$= \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}} \tag{4.27}$$

$$= \frac{2\left(1+e^{-2\jmath\omega}\right)}{2+e^{-\jmath\omega}} \tag{4.28}$$

$$= \frac{2\left(1+\cos 2\omega - \jmath n 2\omega\right)}{2+\cos\omega - \jmath n\omega} \tag{4.29}$$

$$= \frac{2\left|\left(2\cos^2\omega - \jmath n 2\omega\right)\right|}{\left|2+\cos\omega - \jmath n\omega\right|} \tag{4.30}$$

$$= \frac{2\sqrt{4\cos^4\omega + 4n^2\omega\cos^2\omega}}{\sqrt{\left(2+\cos\omega\right)^2 + n^2\omega}} \tag{4.31}$$

$$= \frac{4\cos\omega}{\sqrt{5+4\cos\omega}} \tag{4.32}$$

$$\left|H\left(e^{\jmath\omega}\right)\right| = \frac{|4\cos\omega|}{\sqrt{5+4\cos\omega}} \tag{4.33}$$

Since $\left|H(e^{j\omega})\right|$ is function of cosine we can say it is periodic.And from the plot 4.6 we can say that it is symmetric about $\omega = 0$ (even function) and it is periodic with period 2π.You can find the same from the theoritical expression $\left|H\left(e^{j\omega}\right)\right|$,

$$H(e^{\jmath\omega}) = H(e^{j(-\omega)}) \text{ (cos is an even function)} \tag{4.34}$$

And to find period, the period of $|\cos(\omega)|$ is π and the period of $\sqrt{5+4\cos(\omega)}$ is 2π. So the period of division of both will be 2π. This gives us the period of $\left|H(e^{j\omega})\right|$ as 2π The following code plots Fig. 4.6.

```
wget https://github.com/LokeshBadisa/
    EE3900-Linear-Systems-and-Signal-
    Processing/blob/main/codes/4.6.py
```

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.

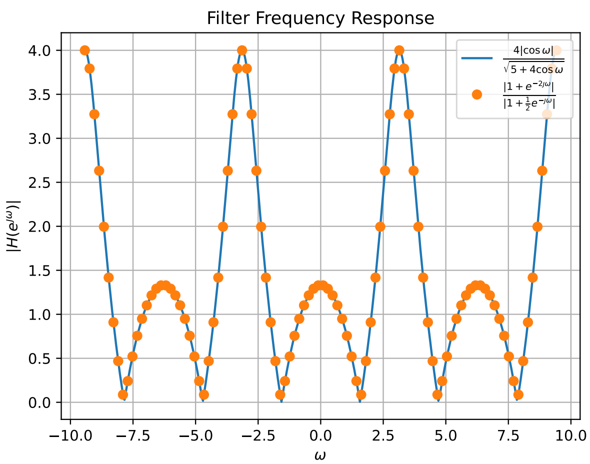

Fig. 4.6. $|H(e^{\jmath\omega})|$

Solution:

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{\jmath\omega k}\, dw \tag{4.35}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{\jmath\omega k} e^{-\jmath\omega n}\, dw \tag{4.36}$$

$$Integral = \begin{cases} 0 & n \neq k \\ 2\pi & n = k \end{cases} \tag{4.37}$$

$$= h(n)\frac{1}{2\pi}2\pi \tag{4.38}$$

$$= h(n) \tag{4.39}$$

Hence Proved.

5. Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.13). **Solution:**

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{r|l} & \quad 2x - 4 \\ \hline \frac{1}{2}x + 1) & x^2 \quad\quad + 1 \\ & -x^2 - 2x \\ \hline & -2x + 1 \\ & 2x + 4 \\ \hline & 5 \end{array}$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)\left(-4 + 2z^{-1}\right) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.4}$$

On applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.5}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.6}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.7}$$

$$\frac{5}{1 + \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.8}$$

$$\tag{5.9}$$

Therefore,

$$h(n) = -4\delta(n) + 2\delta(n-1) + 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.10}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.11}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.13),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.12}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.13}$$

using (4.20) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution:
The following code plots Fig. **??**.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/5.3.py
```

$$\lim_{n\to\infty} h(n) = \lim_{n\to\infty}\left(-\frac{1}{2}\right)^n u(n) + \lim_{n\to\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.14}$$

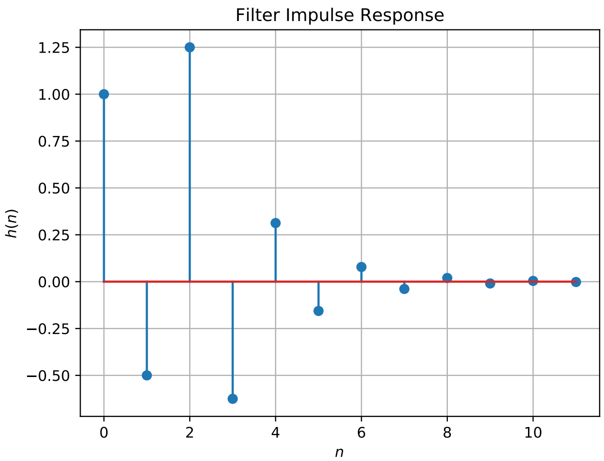

Fig. 5.3.

$$h(n) = \begin{cases} 5\left(\frac{-1}{2}\right)^n & n \geq 2 \\ \left(\frac{-1}{2}\right)^n & 0 \leq n < 2 \\ 0 & n < 0 \end{cases} \tag{5.15}$$

Maximum value and minimum value are always bounded in this case. $\therefore$ $h(n)$ is bounded

5.4 Is it convergent? Justify using the ratio test.
Solution:

$$h(n+1) = \left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1) \tag{5.16}$$

According to ratio test, L is given by $\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right|$, if $L < 1$ then h(n) is convergent.

$$L = \lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| \tag{5.17}$$

$$= \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)}\right| \tag{5.18}$$

$$= \left|\frac{-\frac{1}{2} + -\frac{1}{2}^{-1}}{1 + -\frac{1}{2}^{-}2}\right| \tag{5.19}$$

$$= \left|\frac{-\frac{1}{2} - 2}{1+4}\right| \tag{5.20}$$

$$= \left|\frac{\frac{-5}{2}}{5}\right| \tag{5.21}$$

$$= \frac{1}{2} \tag{5.22}$$

As $L < 1$, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the impulse response in (5.11)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} h(n) \tag{5.24}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.25}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.26}$$

$$= \frac{4}{3} \tag{5.27}$$

As $\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3}$ is less than ∞, the system defined by (3.2) is stable for the impulse response in (5.1).

5.6 Verify the above result using a python code.
Solution: The following code determines if it is convergent or not:

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/5.5.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.28}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. **??**.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/5.7.py
```

Computing,
$h(0) = 1$
$h(1) = -\frac{1}{2}h(0)$
$h(2) = -\frac{1}{2}h(1) + 1$
Parallely, $h(n) = -\frac{1}{2}h(n-1)$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.29}$$

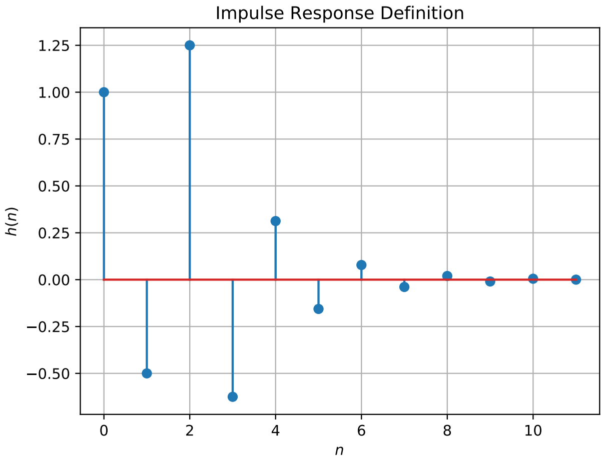

Fig. 5.7. $h(n)$ from the definition

Comment. The operation in (5.29) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 5.3.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/5.8.py
```

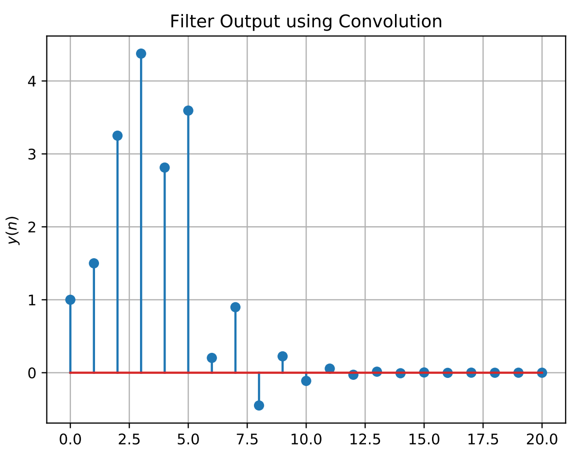

Fig. 5.8. $y(n)$ from the definition of convolution

5.9 Express the above convolution using a toeplitz matrix. **Solution:**

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{5.30}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ h_{m-1} & . & . & . & h_2 & h_1 \\ h_m & h_{m-1} & . & . & . & h_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix} \tag{5.31}$$

$$\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 & 0 \\ \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 \\ \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 \\ \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 \\ \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} \\ 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} \\ 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} \\ 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{64} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1. \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.8125 \\ 3.59375 \\ 0.203125 \\ 0.9375 \\ -0.390625 \\ 0.3125 \\ 0. \\ 0.078125 \end{pmatrix} \tag{5.32}$$

And this is what we got in (5.29)

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.33}$$

Solution: Substitute $k \to n - k$ then

$$y(n) = x(n) * h(n) \tag{5.34}$$

$$= \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.35}$$

6. DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$. **Solution:** The following code plots Fig. 6.1.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/xkhkdft.py
```

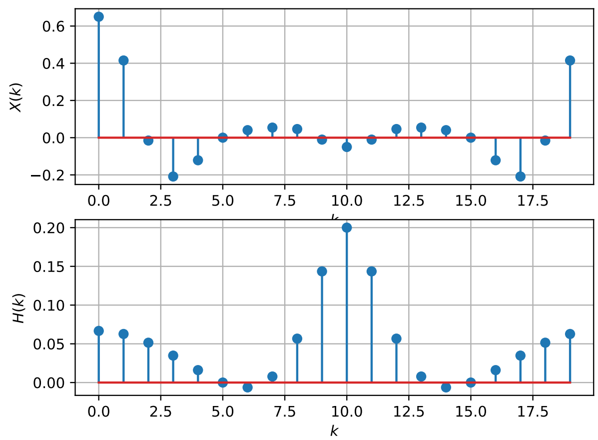

Fig. 6.1. $X(k), H(k)$ from the DFT

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig. 6.2.

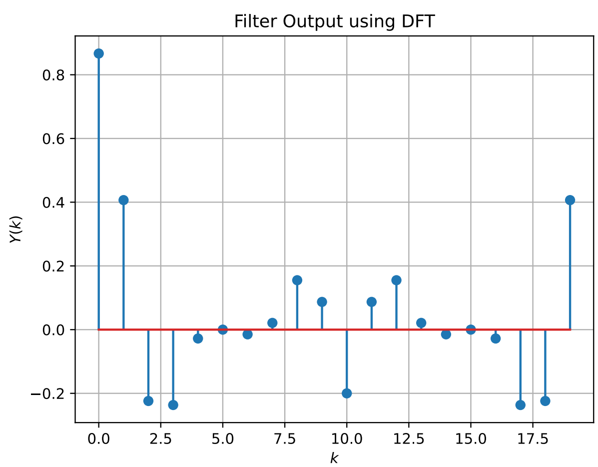

Fig. 6.2. $Y(k)$ from the DFT

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/ykdft.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 5.3.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/yndft.py
```

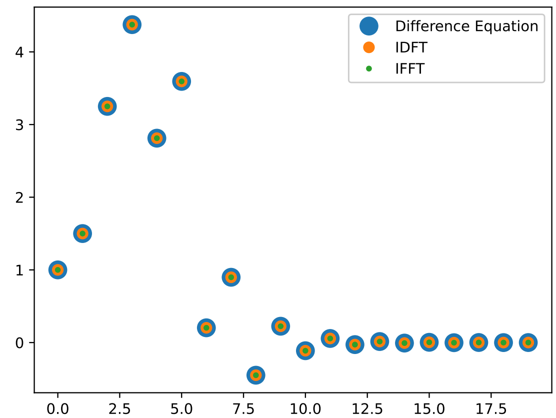

Fig. 6.3. $y(n)$ from the DFT

If you observe both the graphs, they are close to each other but not same.

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT. **Solution:** The following code calculates $X(k), H(k)$ and$y(n)$.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/6.4.py
```

7. FFT

A. Definitions

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}] \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

B. Problems

1. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution: Given

$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$W_N^2 = e^{2(-j2\pi/N)} \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$

Which is same as saying

$$W_N^{k(2n)} = W_{N/2}^{kn} \tag{7.12}$$

2. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.13}$$

Solution: As $\mathbf{I}_2$ is a 2×2 identity matrix which means

$$\mathbf{I}_2\mathbf{A} = \mathbf{A}\mathbf{I}_2 = \mathbf{A} \tag{7.14}$$

Consider

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.15}$$

where

$$\mathbf{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.16}$$

$$\mathbf{D}_2 = diag(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \tag{7.17}$$

Also,

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.18}$$

$$\mathbf{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.19}$$

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \tag{7.20}$$

$$\therefore \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \tag{7.21}$$

From (7.13), using (7.21) and (7.18)

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.22}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.23}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.24}$$

which is same as $\mathbf{F}_4$.

$$\therefore \mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.25}$$

3. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.26}$$

Solution:
Consider the following properties:

$$W_N^{k(2n+1)} = W_N^k W_{N/2}^{kn} \tag{7.27}$$

$$W_N^{k+N/2} = e^{-j2\pi[k+N/2]/N} = e^{-j2\pi k/N}e^{-j\pi} \tag{7.28}$$
$$= -W_N^k \tag{7.29}$$

Consider $X[k]$,

$$X[k] = \sum_{n=0}^{N-1} W_N^{kn} \tag{7.30}$$

$$= \sum_{n=0}^{N/2-1} \left[x[2n] W_N^{k(2n)} + x[2n+1] W_N^{k(2n+1)} \right] \tag{7.31}$$

$k = 0, \cdots, N-1$
which is gathering odd and even terms seperately which allows us to write

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{kn} \tag{7.32}$$

$$= Y[k] + W_N^k Z[k] \; k = 0, \cdots, N-1 \tag{7.33}$$

where $Y[k]$ and $Z[k]$ are DFTs of length N/2 of even numbered sequence $\{x[2n]\}$ and of the odd numbered sequence $\{x[2n]\}$, respectively. Here, we cannot proceed for $k \geq N/2$. So, for $k \geq N/2$:

$$X[k+N/2] = Y[k+N/2] + W_N^{k+N/2} Z[k+N/2] \tag{7.34}$$

Using the periodicity of $Y[k]$ and $Z[k]$ and (7.28),

$$X[k+N/2] = Y[k+N/2] - W_N^k Z[k] \tag{7.35}$$

Using (7.33) and (7.35), and linear transformation upon them

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{Y_{N/2}} \\ \mathbf{Z_{N/2}} \end{bmatrix} \tag{7.36}$$

where $\mathbf{I_{N/2}}$ is a unit matrix and $\mathbf{D_{N/2}}$ is a diagonal matrix with elements as $\left\{W_N^k, k = 0, \cdots, N/2-1\right\}$, both of dimension $N/2 \times N/2$ .Whereas
$\mathbf{Y_N/2}$ = DFT of even terms of $\mathbf{X_N} = \mathbf{F_{N/2}x_{even}}$
$\mathbf{Z_N/2}$ = DFT of odd terms of $\mathbf{X_N} = \mathbf{F_{N/2}x_{odd}}$
which gets us to

$$\begin{bmatrix} \mathbf{Y_{N/2}} \\ \mathbf{Z_{N/2}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{x} \tag{7.37}$$

which is permuting into combinations of even and odd terms of a sequence.As we have permuted it into odd and even parts, we have reverse to the process. So, a permutation matrix is multiplied.

$$\begin{bmatrix} \mathbf{Y_{N/2}} \\ \mathbf{Z_{N/2}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P_N x} \tag{7.38}$$

Replacing in (7.36),

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P_N x} \tag{7.39}$$

As $\mathbf{X_N} = \mathbf{F_N x}$,

$$\mathbf{F_N x} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P_N x} \tag{7.40}$$

Applying $\mathbf{x^{-1}}$,

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.41}$$

4. Find

$$\mathbf{P}_4 \mathbf{x} \tag{7.42}$$

Solution: From (7.18),

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.43}$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.44}$$

After proper zero padding of $\mathbf{P}_4$,

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.45}$$

$$\mathbf{P}_4 \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.46}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{7.47}$$

5. Show that

$$\mathbf{X} = \mathbf{F}_N \mathbf{x} \tag{7.48}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Given $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.49}$$

$$\mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \tag{7.50}$$

$$\mathbf{F}_N = \begin{bmatrix} 1 & 1 & ! & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.51}$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \tag{7.52}$$

Upon linear transformation over k,

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & ! & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.53}$$

$$\therefore \mathbf{X} = \mathbf{F}_N \mathbf{x} \tag{7.54}$$

6. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.55}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.56}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.57}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.58}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.60}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.61}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.62}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.63}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.64}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.65}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.66}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.67}$$

7. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.68}$$

compte the DFT using (7.48) **Solution:**

$$\mathbf{F}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \tag{7.69}$$

Using (7.68),

$$\mathbf{X} = \mathbf{F}_6\mathbf{x} \tag{7.70}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.71}$$

$$= \begin{bmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{bmatrix} \tag{7.72}$$

8. Repeat the above exercise using the FFT after zero padding **x**.
   Solution: **x** after padding is

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.73}$$

Using (7.41),

$$\mathbf{F}_8 = \begin{bmatrix} \mathbf{I}_4 & \mathbf{D}_4 \\ \mathbf{I}_4 & -\mathbf{D}_4 \end{bmatrix} \begin{bmatrix} \mathbf{F}_4 & 0 \\ 0 & \mathbf{F}_4 \end{bmatrix} \mathbf{P}_8 \tag{7.74}$$

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.75}$$

$$\mathbf{F}_2 = \begin{bmatrix} \mathbf{I}_1 & \mathbf{D}_1 \\ \mathbf{I}_1 & -\mathbf{D}_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 & 0 \\ 0 & \mathbf{F}_1 \end{bmatrix} \mathbf{P}_2 \tag{7.76}$$

$$\mathbf{F_1} = [1] \tag{7.77}$$

Calculating $\mathbf{F_2}$,

$$\mathbf{F_2} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{D_1F_1} \\ \mathbf{F}_1 & -\mathbf{D_1F_1} \end{bmatrix} \mathbf{P_2} \tag{7.78}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.79}$$

Calculating $\mathbf{F_4}$,

$$\mathbf{D}_2 = diag(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \tag{7.80}$$

$$\mathbf{D_2F_2} = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.81}$$

$$= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \tag{7.82}$$

$$\mathbf{F_4} = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D_2F_2} \\ \mathbf{F}_2 & -\mathbf{D_2F_2} \end{bmatrix} \mathbf{P_4} \tag{7.83}$$

$$\mathbf{F_4} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -j & j \\ 1 & 0 & -1 & -1 \\ 0 & 1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.84}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \tag{7.85}$$

Calculating $\mathbf{F_8}$,

$$\mathbf{D_4} = diag\left(1, W_8, W_8^2, W_8^3\right) \tag{7.86}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \tag{7.87}$$

$$\mathbf{D_4F_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \tag{7.88}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ -1 & 1 & 0 & -j \\ 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \end{bmatrix} \tag{7.89}$$

$F_8 = \mathbf{ABP_8}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 1 & 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1+j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & j \\ 0 & 0 & 0 & 1 & 0 & \frac{-1+j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} \end{bmatrix} \tag{7.90}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -j & 1 & j & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & j & 1 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & j & -1 & j \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & -j & -1 & j \end{bmatrix} \tag{7.91}$$

$$\mathbf{F_8} = \tag{7.92}$$

And $\mathbf{P_8}$ is a permutation matrix.

$$\mathbf{X} = \begin{bmatrix} 13 \\ -4-8j \\ j \\ 2-2j \\ -1 \\ 2+2j \\ -j \\ -4+8j \end{bmatrix} \tag{7.93}$$

9. Write a C program to compute the 8-point FFT.

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/7.9.c
gcc 7.9.c −lm
```

8. Exercises

Answer the following questions by looking at the python code in Problem 2.3

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \tag{8.2}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.3}$$

For obtaining the discrete Fourier transform, put $z = \jmath\frac{2\pi i}{I}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$
Download the following Python code that does the above

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/7.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \tag{8.4}$$

The difference equation is then given by

$$\mathbf{a}^{\top}\mathbf{y} = \mathbf{b}^{\top}\mathbf{x} \tag{8.5}$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

Download the following Python code

```
wget https://github.com/LokeshBadisa/
    EE3900−Linear−Systems−and−Signal−
    Processing/blob/main/codes/7.2.py
```

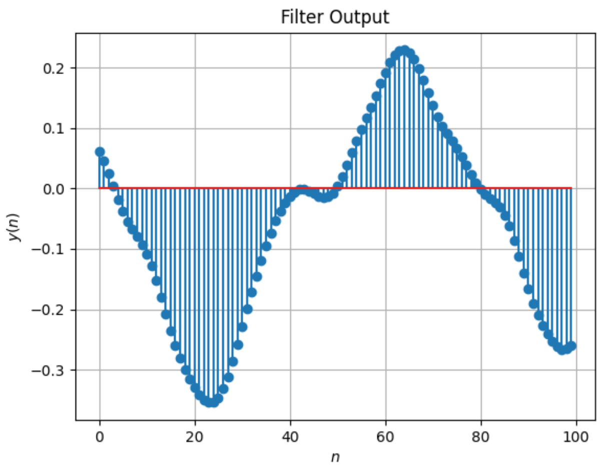

Fig. 8.2. Plot of $y(n)$

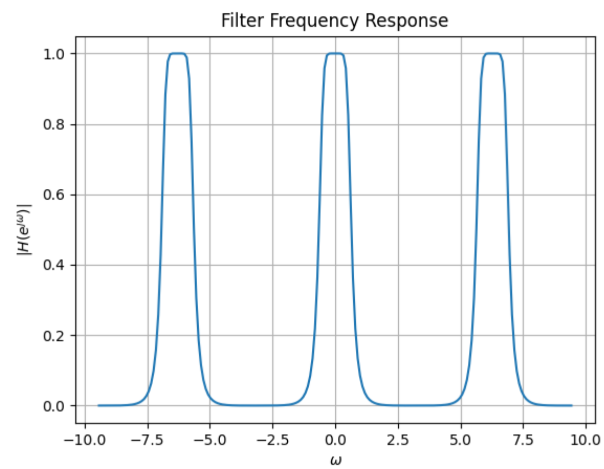

Fig. 8.2. Plot of $|H(e^{j\omega})|$

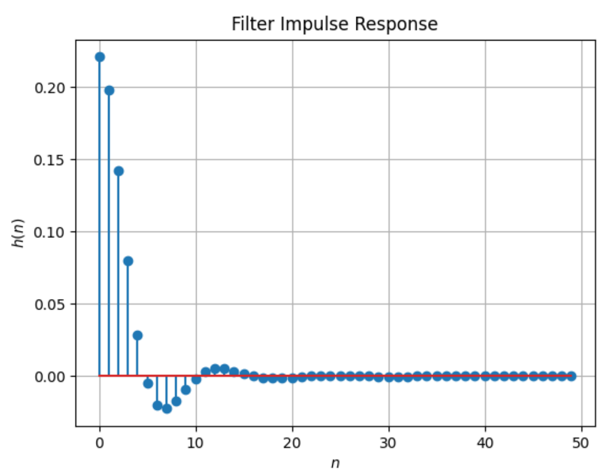

Fig. 8.2. Plot of $h(n)$

8.3 What is the sampling frequency of the input signal?
Solution: The sampling frequency of the input signal is 44 100 Hz = 44.1 kHz

8.4 What is the type, order and cutoff frequency of the above Butterworth filter?
Solution:
Type: low-pass
Order: 4
Cutoff frequency: 4000 Hz = 4 kHz

8.5 Modify the code with different input parameters to get the best possible output.
Solution:
Order: 10
Cutoff frequency: 3000 Hz = 3 kHz