

# Digital Signal Processing

## EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

### Circuits and Transforms

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2. Draw the circuit using latex-tikz.

**Solution:** The following code yields Fig.2.2

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
Tikz_Circuits/2.2.tex
```

#### 1. DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of  $g(t)$  is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

#### 2. LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes  $q_1 \mu C$ . Then S is switched to position Q. After a long time, the charge on the capacitor is  $q_2 \mu C$ .

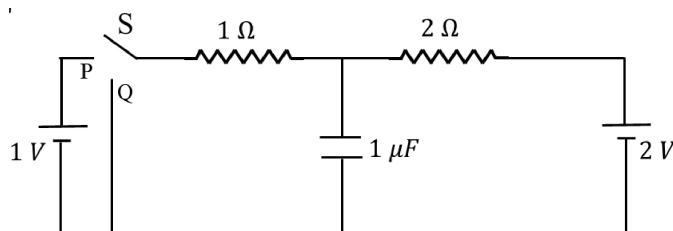


Fig. 2.1.

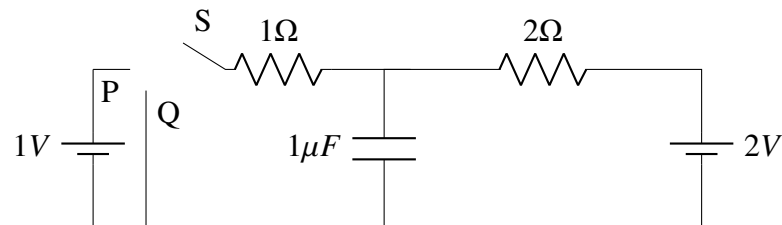


Fig. 2.2. Given Circuit

3. Find  $q_1$ .

**Solution:** Before switching S to Q: Calculating current,

$$1 - i - 2i - 2 = 0 \quad (2.1)$$

$$3i = -1 \Rightarrow i = \frac{-1}{3} \quad (2.2)$$

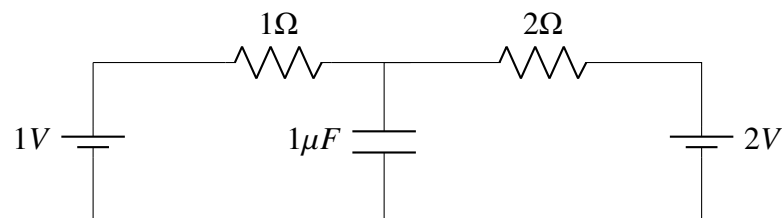


Fig. 2.3. Before switching S to Q

Potential Difference between capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3} \quad (2.3)$$

$$q_1 = \frac{4}{3} \cdot 1 \quad (2.4)$$

$$= \frac{4}{3} \mu C \quad (2.5)$$

4. Show that the Laplace transform of  $u(t)$  is  $\frac{1}{s}$  and find the ROC.

**Solution:** We know that from definition of Laplace Transform,

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt U(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.6)$$

Using (1.1),

$$U(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.7)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.8)$$

$$= -\left(0 - \frac{1}{s}\right) \quad (2.9)$$

$$= \frac{1}{s} \quad (2.10)$$

ROC is  $Re(s) > 0$  since  $e^{-st} < \infty$  for  $t \rightarrow \infty$

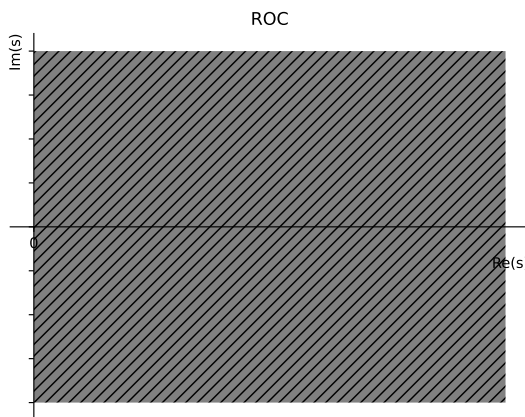


Fig. 2.4.

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \quad (2.11)$$

and find the ROC. **Solution:** From (2.6),

$$F(s) = \int_0^{\infty} u(t)e^{-at}e^{-st} dt \quad (2.12)$$

$$= \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.13)$$

$$= \int_0^{\infty} e^{-(s+a)t} dt \quad (2.14)$$

$$= -\left(0 - \frac{1}{s+a}\right) \quad (2.15)$$

$$= \frac{1}{s+a} \quad (2.16)$$

ROC is

$$s+a > 0 \Rightarrow s > -a \quad (2.17)$$

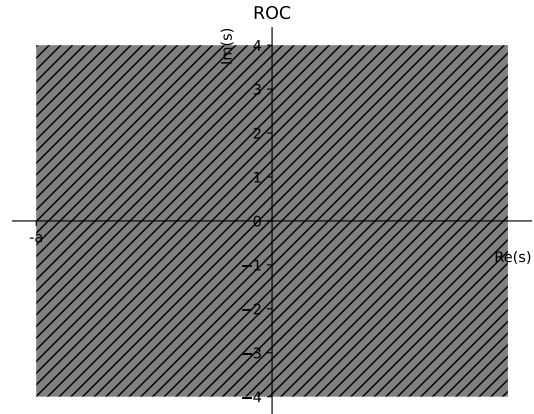


Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. 3.3 where

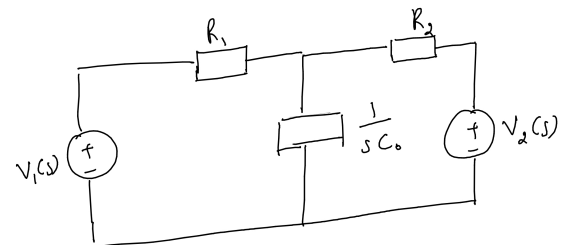


Fig. 2.6.

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \quad (2.18)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \quad (2.19)$$

Find the voltage across the capacitor  $V_{C_0}(s)$ .

**Solution:**

$$R_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} \Omega \quad (2.20)$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} V \quad (2.21)$$

$$V_{C_0}(s) = V_S(s) \frac{C_0}{C_0 + R_{eff}} \quad (2.22)$$

$$= \left( \frac{4}{3s} \right) \left( \frac{\frac{1}{s}}{\frac{1}{s} + \frac{2}{3}} \right) \quad (2.23)$$

$$= \frac{6}{s(4s + 9)} \quad (2.24)$$

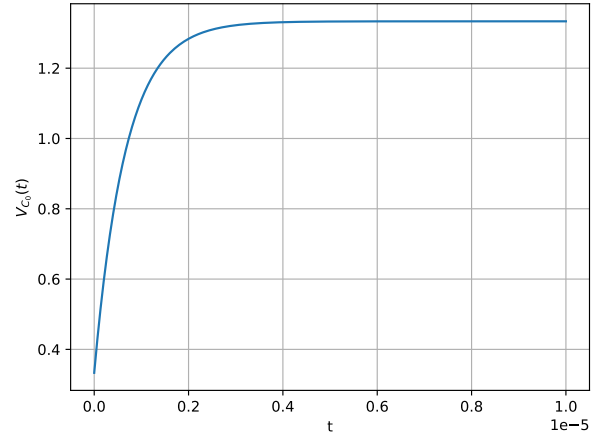


Fig. 2.7. Plot of  $V_{C_0}(t)$

7. Find  $v_{C_0}(t)$ . Plot using python.

**Solution:** Using (2.24),

$$\frac{6}{s(4s + 9)} = \frac{4}{3s} - \frac{2}{9 + 4s} \quad (2.25)$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-\infty}} V_{C_0}(t) \quad (2.26)$$

$$\mathcal{L}^{-1}[V_{C_0}(s)] = \mathcal{L}^{-1}\left[\frac{4}{3s} - \frac{2}{9 + 4s}\right] \quad (2.27)$$

$$= \mathcal{L}^{-1}\left[\frac{4}{3s}\right] - \mathcal{L}^{-1}\left[\frac{2}{9 + 4s}\right] \quad (2.28)$$

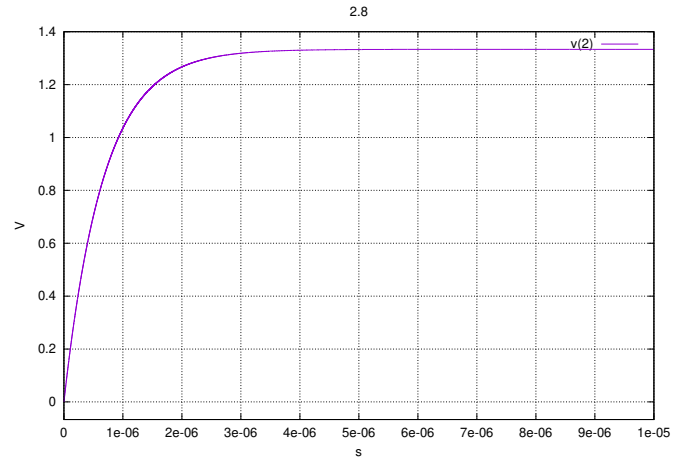


Fig. 2.8.

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \quad (2.29)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s - a}\right] = e^{at}u(t) \quad (2.30)$$

Using the above equations,

$$V_{C_0}(t) = \frac{4}{3} \left(1 - e^{-\frac{3}{2}t}\right) u(t) \quad (2.31)$$

8. Verify your result using ngspice. **Solution:**

### 3. INITIAL CONDITIONS

1. Find  $q_2$  in Fig. 3.3.

**Solution:** At steady state,  $V_{C_0} = V_{1\Omega}$

$$V_{C_0} = \frac{q_2}{C} = V_{1\Omega} = \frac{2}{1 + 2} = \frac{2}{3}$$

$$q_2 = \frac{2}{3} \mu C$$

2. Draw the equivalent  $s$ -domain resistive circuit when S is switched to position Q. Use variables  $R_1, R_2, C_0$  for the passive elements. Use latex-tikz. **Solution:**

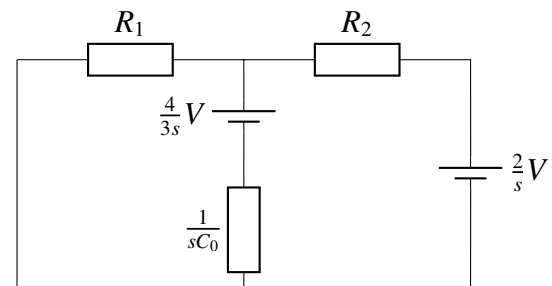


Fig. 3.1. After switching S to Q

3.  $V_{C_0}(s) = ?$

**Solution:** Using KCL at node in Fig. 3.1

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \quad (3.1)$$

$$\Rightarrow V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.2)$$

4.  $v_{C_0}(t) = ?$  Plot using python.

**Solution:** From (3.2),

$$V_{C_0}(s) = \frac{4}{3} \left( \frac{1}{\frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( \frac{1}{s} - \frac{1}{\frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.3)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) + \frac{2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( 1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t) \quad (3.4)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3} \left( 1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.5)$$

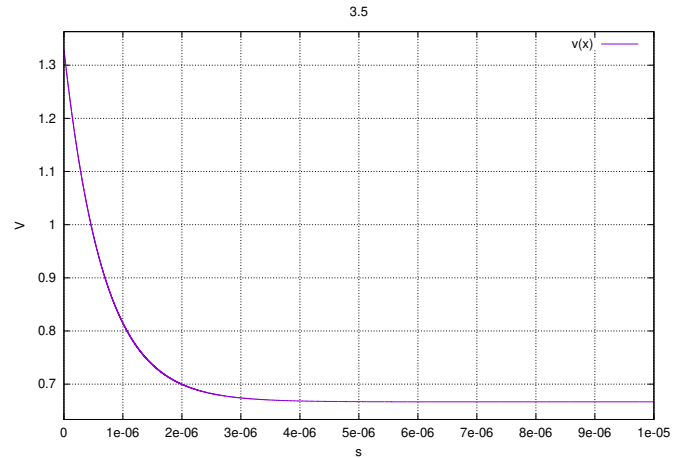


Fig. 3.3.

Using (3.5),

$$v_{C_0}(0+) = \lim_{t \rightarrow 0+} v_{C_0}(t) = \frac{4}{3} V \quad (3.7)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3} V \quad (3.8)$$

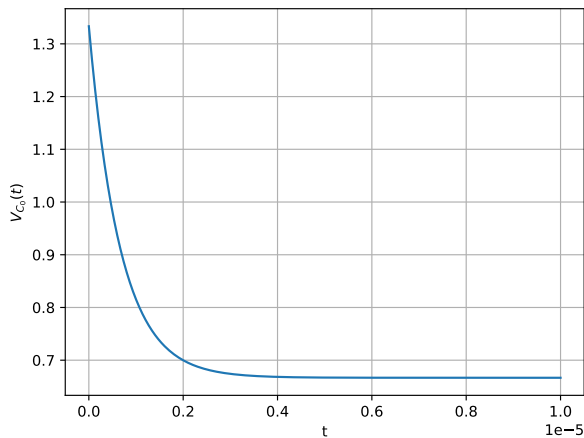


Fig. 3.2. Plot of  $V_{C_0}(t)$

5. Verify your result using ngspice. **Solution:**

6. Find  $v_{C_0}(0-)$ ,  $v_{C_0}(0+)$  and  $v_{C_0}(\infty)$ .

**Solution:** From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3} V \quad (3.6)$$