

Digital Signal Processing

EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

Circuits and Transforms

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7 Oct 2022

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2. Draw the circuit using latex-tikz.

Solution: The following code yields Fig.2.2

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
Tikz_Circuits/2.2.tex
```

1. DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2. LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2 \mu C$.

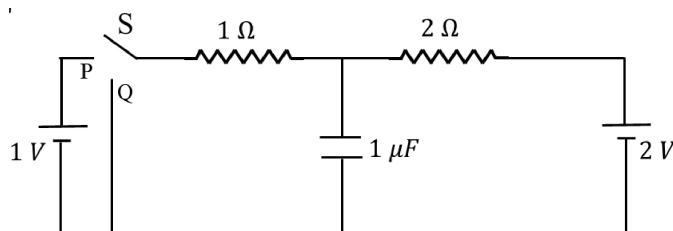


Fig. 2.1.

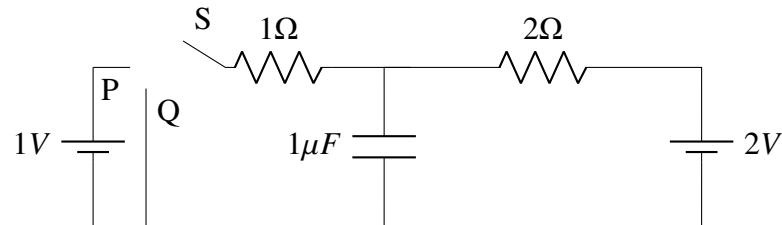


Fig. 2.2. Given Circuit

3. Find q_1 .

Solution: Before switching S to Q: Calculating current,

$$1 - i - 2i - 2 = 0 \quad (2.1)$$

$$3i = -1 \Rightarrow i = \frac{-1}{3} \quad (2.2)$$

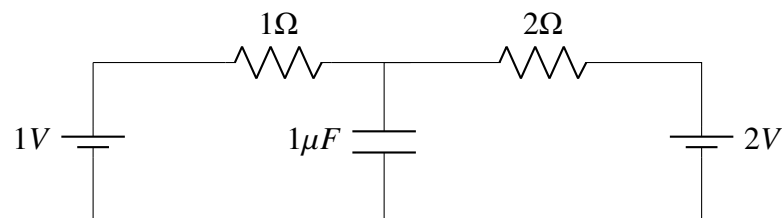


Fig. 2.3. Before switching S to Q

Potential Difference between capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3} \quad (2.3)$$

$$q_1 = \frac{4}{3} \cdot 1 \quad (2.4)$$

$$= \frac{4}{3} \mu C \quad (2.5)$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We know that from definition of Laplace Transform,

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt U(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.6)$$

Using (1.1),

$$U(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.7)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.8)$$

$$= -\left(0 - \frac{1}{s}\right) \quad (2.9)$$

$$= \frac{1}{s} \quad (2.10)$$

ROC is $Re(s) > 0$ since $e^{-st} < \infty$ for $t \rightarrow \infty$ The following command plots the ROC of above Laplace Transform.

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
codes/2.4.py
python3 2.4.py
```

5. Show that

$$e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \quad (2.11)$$

and find the ROC. **Solution:** From (2.6),

$$F(s) = \int_0^{\infty} u(t)e^{-at}e^{-st} dt \quad (2.12)$$

$$= \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.13)$$

$$= \int_0^{\infty} e^{-(s+a)t} dt \quad (2.14)$$

$$= -\left(0 - \frac{1}{s+a}\right) \quad (2.15)$$

$$= \frac{1}{s+a} \quad (2.16)$$

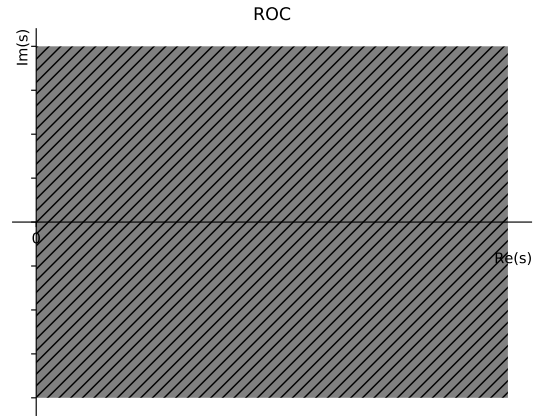


Fig. 2.4.

ROC is

$$s + a > 0 \Rightarrow s > -a \quad (2.17)$$

The following command plots the ROC of above Laplace Transform.

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
codes/2.5.py
python3 2.5.py
```

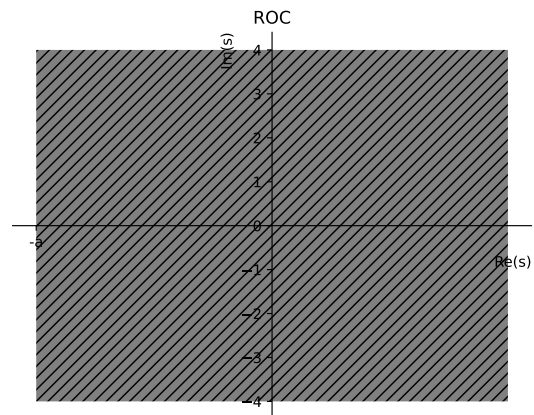


Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. 3.3 where

$$u(t) \xrightarrow{\mathcal{L}} V_1(s) \quad (2.18)$$

$$2u(t) \xrightarrow{\mathcal{L}} V_2(s) \quad (2.19)$$



Fig. 2.6.

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

$$R_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} \Omega \quad (2.20)$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} V \quad (2.21)$$

$$V_{C_0}(s) = V_S(s) \frac{C_0}{C_0 + R_{eff}} \quad (2.22)$$

$$= \left(\frac{4}{3s} \right) \left(\frac{\frac{1}{s}}{\frac{1}{s} + \frac{2}{3}} \right) \quad (2.23)$$

$$= \frac{6}{s(4s + 9)} \quad (2.24)$$

7. Find $v_{C_0}(t)$. Plot using python.

Solution: Using (2.24),

$$\frac{6}{s(4s + 9)} = \frac{4}{3s} - \frac{2}{9 + 4s} \quad (2.25)$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-\infty}} V_{C_0}(t) \quad (2.26)$$

$$\mathcal{L}^{-1}[V_{C_0}(s)] = \mathcal{L}^{-1} \left[\frac{4}{3s} - \frac{2}{9 + 4s} \right] \quad (2.27)$$

$$= \mathcal{L}^{-1} \left[\frac{4}{3s} \right] - \mathcal{L}^{-1} \left[\frac{2}{9 + 4s} \right] \quad (2.28)$$

Since,

$$\mathcal{L}^{-1} \left[\frac{1}{s} \right] = u(t) \quad (2.29)$$

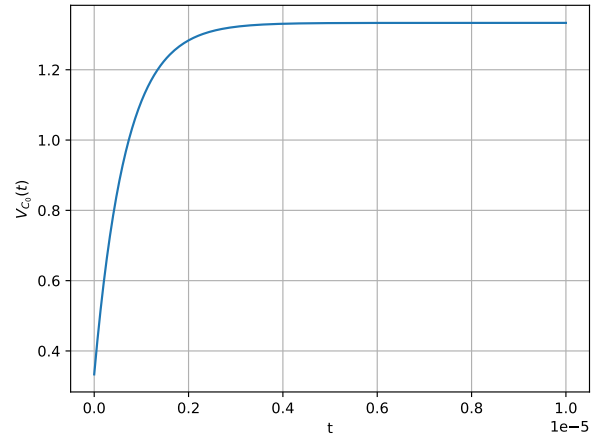
$$\mathcal{L}^{-1} \left[\frac{1}{s - a} \right] = e^{at} u(t) \quad (2.30)$$

Using the above equations,

$$V_{C_0}(t) = \frac{4}{3} \left(1 - e^{-\frac{3}{4}t} \right) u(t) \quad (2.31)$$

The following command plots the above equation.

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
codes/2.7.py
python3 2.7.py
```

Fig. 2.7. Plot of $V_{C_0}(t)$

8. Verify your result using ngspice. **Solution:** The following command plots the ROC of above Laplace Transform.

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
codes/2.8.cir
ngspice 2.8.cir
```

3. INITIAL CONDITIONS

1. Find q_2 in Fig. 3.3.

Solution: At steady state, $V_{C_0} = V_{1\Omega}$

$$V_{C_0} = \frac{q_2}{C} = V_{1\Omega} = \frac{2}{1 + 2} = \frac{2}{3}$$

$$q_2 = \frac{2}{3} \mu C$$

2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz. **Solution:** The following command plots the ROC of above Laplace Transform.

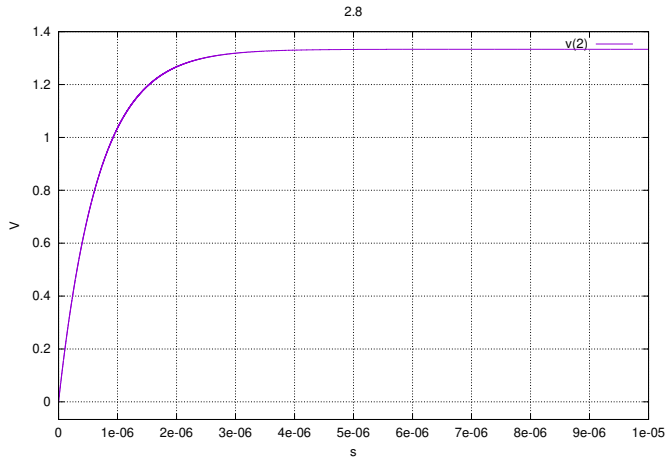


Fig. 2.8.

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
Tikz_Circuits/3.2.tex
```

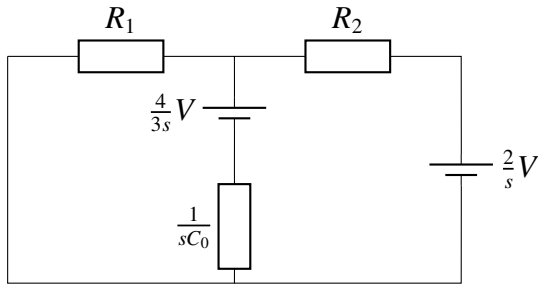


Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$

Solution: Using KCL at node in Fig. 3.1

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0 \left(V - \frac{4}{3s} \right) = 0 \quad (3.1)$$

$$\Rightarrow V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (3.2)$$

4. $v_{C_0}(t) = ?$ Plot using python.

Solution: From (3.2),

$$V_{C_0}(s) = \frac{4}{3} \left(\frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.3)$$

Taking an inverse Laplace Transform,

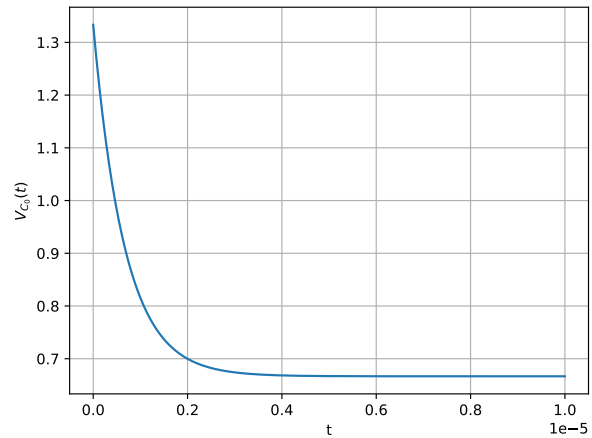
$$v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t) \quad (3.4)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.5)$$

The following command plots the above equation.

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
codes/3.4.py
python3 3.4.py
```

Fig. 3.2. Plot of $V_{C_0}(t)$

5. Verify your result using ngspice. **Solution:** The following command plots Fig.3.3

```
wget https://raw.githubusercontent.com/
LokeshBadisa/EE3900-Linear-Systems-
and-Signal-Processing/main/Circuits/
codes/3.5.cir
ngspice 3.5.cir
```

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3} V \quad (3.6)$$

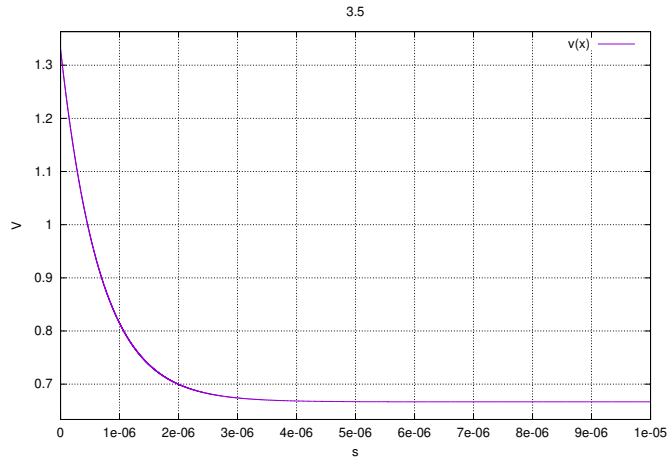


Fig. 3.3.

Using (3.5),

$$v_{C_0}(0+) = \lim_{t \rightarrow 0+} v_{C_0}(t) = \frac{4}{3}V \quad (3.7)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3}V \quad (3.8)$$