MODULE V

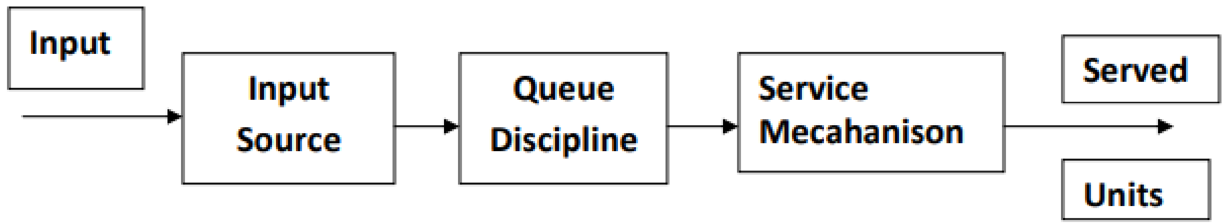
QUEUING AND SIMULATION

Queuing Model

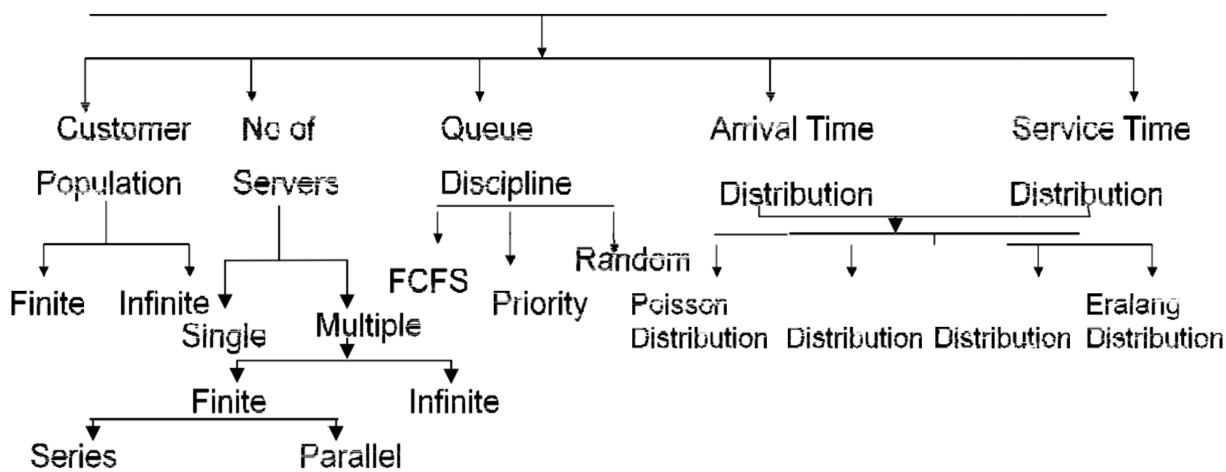
Introduction

Queues are a part of everyday life. We all wait in queues to buy a movie ticket, to make bank deposit, pay for groceries, mail a package, obtain a food in a cafeteria, to have ride in an amusement park and have become adjustment to wait but still get annoyed by unusually long waits.

The Queuing models are very helpful for determining how to operate a queuing system in the most effective way if too much service capacity to operate the system involves excessive costs. The models enable finding an appropriate balance between the cost of service and the amount of waiting.



Information required to solve the queuing problem:



Characteristics of the queuing system

- Input source
- Queue discipline
- Service mechanism

(a) Input source

One characteristic of the input source is the size. The size is the total number of units that might require service from time to time. It may be assumed to be finite or infinite. The customer assumption is that they generate according to 'Poisson Distribution' at a certain average rate. Therefore, the equivalent assumption is that they generate according to exponential distribution between consecutive arrivals. To solve the problems use & assume customer population as ∞.

(b) Queue Discipline

A queue is characterized by maximum permissible number of units that it contains. Queues are called finite or infinite, according to whether number is finite or infinite. The service discipline refers to the order in which number of queues is selected for service.

Ex: It may be FIFO, random or priority; FIFO is usually assumed unless stated otherwise.

(c) Service mechanism

The time elapsed from the commencement of the service to its completion for a unit at the service facility is known as service time usually; service time follows as exponential distribution. Classification of queuing models using Kendal & Lee notations. Generally, any queuing models may be completely specified in the following symbolic form.

$$a / b / c : d / e$$

a $\rightarrow$ Type of distribution of inter – arrival time

b $\rightarrow$ Type of distribution of inter – service time

c $\rightarrow$ Number of servers

d $\rightarrow$ Capacity of the system

e $\rightarrow$ Queue discipline

M $\rightarrow$ Arrival time follows Poisson distribution and service time follows an exponential distribution

Model I : M / M / 1 : / FCFS

Where M Arrival time follows a Poisson distribution

M $\rightarrow$ Service time follows a exponential distribution

1 $\rightarrow$ Single service model

∞ $\rightarrow$ Capacity of the system is infinite

FCFS $\rightarrow$ Queue discipline is first come first served

Model II : M / M / 1 : N / FCFS

Where N $\rightarrow$ Capacity of the system is finite

Model III : M / M / 1 : / SIRO

Where SIRO $\rightarrow$ Service in random order

Model IV : M / O / 1 : / FCFS

Where D $\rightarrow$ Service time follows a constant distribution or is deterministic

Model V : M / G / 1 : / FCFS

Where G $\rightarrow$ Service time follows a general distribution or arbitrary distribution

Model VI : M / E_k / 1 : / FCFS

Where E_k $\rightarrow$ Service time follows Erlang distribution with K phases.

Model VII : M / M / K : / FCFS

Where K $\rightarrow$ Multiple Server model

Model VIII : M / M / K : N / FCFS

Formulas: 1. Utilization factor traffic intensity /

Utilization parameter / Busy period

$$\rho = \frac{\lambda}{\mu}$$

Where λ = Mean arrival rate ; μ= mean service rate

Note : $\mu > \lambda$ in single server model only

2. Probability that exactly zero units are in the system

$$P_o = 1 - \frac{\lambda}{\mu}$$

3. Probability that exactly 'n' units in the system

$$P_n = P_o\left(\frac{\lambda}{\mu}\right)^n$$

4. Probability that n or more units in the system

$$P_{n\ or\ more} = \left(\frac{\lambda}{\mu}\right)^n$$

more then 'n' means n should be n+1

5. Expected number of units in the queue / queue length

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

6. Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

7. Expected number of units in the system

$$L = L_q + \frac{\lambda}{\mu}$$

8. Expected waiting time in the system

$$W = W_q + \frac{1}{\mu}$$

9. Expected number of units in queue that from time to time – (OR) non - empty queue size

$$D = \frac{\mu}{\mu - \lambda}$$

10. Probability that an arrival will have to wait in the queue for service

Probability $= 1 - P_o$

11. Probability that an arrival will have to wait in the queue more than w (where w > o), the waiting time in the queue

$$\text{Probability} = \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)w}$$

12. Probability that an arrival will have to wait more than v(v > o) waiting time in the system is

$$= e^{(\lambda-\mu)^v}$$

13. Probability that an arrival will not have to wait in the queue for service = P_o

Problems

1. Arrivals at a telephone both are considered to be Poisson at an average time of 8 min between our arrival and the next. The length of the phone call is distributed exponentially, with a mean of 4 min.

Determine

a)Expected fraction of the day that the phone will be in use.
b)Expected number of units in the queue
c)Expected waiting time in the queue.
d)Expected number of units in the system.
e)Expected waiting time in the system
f)Expected number of units in queue that from time to time.
g)What is the probability that an arrival will have to wait in queue for service?
h)What is the probability that exactly 3 units are in system
i) What is the probability that an arrival will not have to wait in queue for service?
j) What is the probability that there are 3 or more units in the system?
k)What is the probability that an arrival will have to wait more than 6 min in queue for service?
l) What is the probability that more than 5 units in system
m) What is the probability that an arrival will have to wait more than 8 min in system?
n)Telephone ccompany will install a second booth when convinced that an arrival would have to wait for attest 6 min in queue for phone. By how much the flow of arrival is increased in order to justify a second booth.

Solution:

The mean arrival rate = λ = 1/8 x 60 = 7.5 / hour.
The mean service = μ = x 60 = 15 / hour.

a) Fraction of the day that the phone will be in use

$$\rho = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

(b) The expected number pf units in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{7.5^2}{15(15-7.5)}$$
$$L_q = 0.5\,(units)\,person$$

(c) Expected waiting time in the queue

(c) Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{0.5}{7.5} = 0.066\ hrs$$

(d) Expected number of units in the system:-

$L = L_q + \lambda / \mu$

$= 0.5 + 0.5$

$L = 1$ person

(e) Expected waiting time in the system

$$W = Wq + \frac{1}{\mu}$$

$$= 0.066 + \frac{1}{15} = 0.133$$

(f) Expected number of units in the queue that form from time to time:-

$$D = \frac{\mu}{\mu - \lambda}$$

$$= \frac{15}{15 - 7.5} = 2\ \ persons$$

(g) Probability that an arrival will have to wait in the system:-

$$P_{ro} = 1 - P_o$$

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right)$$

$$P_{ro} = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

(h) The Probability that exactly zero waits in the system:-

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - 0.5 \ = \ 0.5$$

(i) The probability that exactly 3 units in the system:-

$$P_n = P_o - \left(\frac{\lambda}{\mu}\right)^n \qquad\qquad n = 3$$

$$P_3 = 0.5(0.5)^3 = 0.0625$$

(j) Probability that an arrival will not have to wait for service:-

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= \ 0.5$$

(k) Probability that 3 or more units in the system:-

$$P_{n\ or\ more} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 3$$

$$P_{n\ or\ more} = 0.5^3 = 0.125$$

(l) Probability that an arrival will have to wait more than 6mins in queue for service

$$P_{ro} = \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)\omega}$$

$$\omega = 6\min = \frac{6}{60} hrs$$

$$P_{ro} = 0.5\, e^{(7.5-15)\frac{6}{60}}$$

$$P_{ro} = 0.236$$

(m) Probability that more than 5 units in the system

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 6$$

$$P_{ro} = 0.5^6 = 0.015$$

(n) Probability that an arrival will directly enter for service

P_o= 0.5

(O) Probability that arrival will have to wait more than 8mins in the system.

V = 8 / 60 hrs

$$P_{ro} = e^{(\lambda-\mu)v}$$

$$= e^{(7.5-15)\frac{8}{60}}$$

$$= 0.367$$

(p)

$$W_q = \frac{6}{60}\, hrs \quad = 0.1\, hr$$

$$\therefore \quad W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\mu(\mu-\lambda)\lambda}$$

$$0.1 = \frac{\lambda}{15(15-\lambda)}$$

$$\therefore \quad \lambda = 9\, per\ hour.$$

To justify a second booth should be increased from 7.5 to 9 per hour

Simulation

Simulation:

Simulation is an experiment conducted on a model of some system to collect necessary information on the behavior of that system.

The representation of reality in some physical form or in some form of Mathematical equations is called Simulations. Simulations are imitation of reality.

For example:

1, Children cycling park with various signals and crossing is a simulation of a read model traffic system

2. Planetarium

3. Testing an air craft model in a wind tunnel.

Need for simulation:

- Consider an example of the queuing system, namely the reservation system of a transport corporation.
- The elements of the system are booking counters (servers) and waiting customers (queue).
- Generally the arrival rate of customers follow a Poisson distribution and the service time follows exponential distribution.
- Then the queuing model (M/M/1) : (GD/∞ /∞) can be used to find the standard results.

But in reality, the following combinations of distributions may exist.

1. Arrived rate does not follow Poisson distribution, but the service rate follows an exponential distribution.

2. Arrival rate follows a Poisson distribution and the service rate does not follow exponential distribution.

3. Arrival rate does not follows Poisson distribution and the service time also does not follow exponential distribution.

In each of the above cases, the standard model (M/M/1): (G/D/∞ /∞) cannot be used. The last resort to find the solution for such a queuing problem is to use simulation.

Advantage of simulation:

1. Simulation is Mathematically less complicated
2. Simulation is flexible
3. It can be modified to suit the changing environments.
4. It can be used for training purpose
5. It may be less expensive and less time consuming in a quite a few real world situations.

Limitations of Simulation:

1. Quantification or Enlarging of the variables maybe difficult.
2. Large number of variables make simulations unwieldy and more difficult.
3. Simulation may not. Yield optimum or accurate results.
4. Simulation are most expensive and time consuming model.

Steps in simulation:

1. Identify the measure of effectiveness.
2. Decide the variables which influence the measure of effectiveness and choose those variables, which affects the measure of effectiveness significantly.
3. Determine the probability distribution for each variable in step 2 and construct the cumulative probability distribution.
4. Choose an appropriate set of random numbers.
5. Consider each random number as decimal value of the cumulative probability distribution.
6. Use the simulated values so generated into the formula derived from the measure of effectiveness.
7. Repeat steps 5 and 6 until the sample is large enough to arrive at a satisfactory and reliable decision.

Uses of Simulation:

Simulation is used for solving

1. Inventory Problem
2. Queuing Problem
3. Training Programs etc

General Purpose Languages used for Simulation:

FORTRAN: Probably more models than any other language

PASCAL; Not an universal as FORTRAN

MODULA: Many improvements over PASCAL

ADA: Department of Defense attempt at standardization

C, C++: Object-Oriented Programming Language

PSPICE: Simulation Software

MAT LAB: MATrix LABoratory : High Level Languages (Mathematical and Graphical Subroutines)

SIMULINK: Used to Model, Analyze and Simulate Dynamic Systems using block diagrams

Problem :

Customers arrive at a milk booth for the required service. Assume that inter – arrival and service time are constants and given by 1.5 and 4 minutes respectively. Simulate the system by hand computations for 14 minutes.

(i) What is the waiting time per customer?
(ii) What is the percentage idle time for the facility?
(Assume that the system starts at t = 0)

Solution :

First customer arrives at the service center at t = 0
His departure time after getting service = 0 + 4 = 4 minutes.
Second customer arrives at time t = 1.5 minutes
he has to wait = 4 – 1.5 = 2.5 minutes.
Third customer arrives at time t = 3 minutes
he has to wait for = 8-3 = 5 minutes
Fourth customer arrives at time t = 4.5 minutes and he has to wait for 12 – 4.5 = 7.5 minutes.
During this 4.5 minutes, the first customer leaves in 4 minutes after getting service and the second customer is getting service.
Fifth customer arrives at t = 6 minutes
he has to wait 14 – 6 = 8 minutes
Sixth customer arrives at t = 7.5 minutes
he has to wait 14 – 7.5 = 6.5 minutes
Seventh customer arrives at t = 9 minutes
he has to wait 14 – 9 = 5 minutes
During this 9 minutes the second customer leaves the service in 8th minute and third person is to get service in 9th minute.
Eighth customer arrives at t = 10.5 minutes
he has to wait 14 – 10.5 = 3.5 minutes
Ninth customer arrives at t = 12 minutes
he has to wait 14 – 12 = 2 minutes
But by 12th minute the third customer leaves the Service
10th Customer arrives at t = 13.5 minutes
he has to wait 14-13.5 = 0.5 minute
From this simulation table it is clear that

(i) Average waiting time for 10 customers = $\dfrac{2.5+5+7.5+8+6.5+5.0+3.5+2+0.5}{10}$

$= \dfrac{40.5}{10} = 4.05$

(ii) Average waiting time for 9 customers who are in waiting for service $\dfrac{40.5}{9} = 4.5$ minutes.

But the average service time is 4 minutes which is less that the average waiting time, the percentage of idle time for service = 0%