Unit - II

NUMERICAL DIFFERENTIATION AND INTEGRATION

Introduction:- The Process of Computing the value of derivative $\frac{dy}{dx}$ for some particular value of x from given data when the actual form of data is not known is known as "Numerical differentiation".

If the values of argument (x values) are equally spaced and $\frac{dy}{dx}$ is required near at the begining of the table, we use Newton's forward interpolation formula. If we require ~~required~~ the data at the end of the table, we use Newton's backward interpolation formula.

Derivatives using Newton's forward difference formula:-

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right) \longrightarrow (1)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right) \quad (2)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right) \rightarrow (3)$$

Derivatives using Newton's backward interpolation formula :-

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \quad (1)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right] \quad (2)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \rightarrow (3)$$

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Problems

1) Find the first and second derivatives of the function tabulated below at the point $x = 1.5$.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.0	13.625	24.0	38.875	59.0

Soln:- ~~The difference~~ since $x = 1.5$ appears in the begining of the table. So we use Newton's forward interpolation formula.

$\therefore$ The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375 (y_0)					
		3.625 (Δy_0)				
			3 $(\Delta^2 y_0)$			
2.0	7.0			0.75 $(\Delta^3 y_0)$		
		6.625			0 $(\Delta^4 y_0)$	
2.5	13.625		3.75			
				0.75		0 $(\Delta^5 y_0)$
		10.375				
			4.5		0	
3.0	24.0					
		14.875		0.75		
			5.25			
3.5	38.875					
		20.125				
4.0	59.0					

Here $x_0 = 1.5$, $y_0 = 3.375$ and $h = 0.5$.

By Newton's forward difference formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right)$$

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$$\therefore \left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left(3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75) - \frac{1}{4}(0) + \frac{1}{5}(0)\right)$$

$$= \frac{1}{0.5}(3.625 - 1.5 + 0.25) = \frac{2.375}{0.5} = 4.75.$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.5} = \frac{1}{(0.5)^2}\left(3 - 0.75 + \frac{11}{12}(0) - \frac{5}{6}(0)\right]$$

$$= \frac{2.25}{0.25} = 9.$$

P) For the table below find $f'(1.76)$ and $f(1.72)$

x	1.72	1.73	1.74	1.75	1.76
$f(x)$	0.17907	0.17728	0.17552	0.17377	0.17204

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Soln:- Since we require $f'(x)$ at $x = 1.72$, we use Newton's Forward formula and to get $f'(x)$ at $x = 1.76$ we use Newton's backward formula.

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.72	0.17907 (y_0)				
		-0.00179 (Δy_0)			
			0.00003 $(\Delta^2 y_0)$		
1.73	0.17728			-0.00002 $(\Delta^3 y_0)$	
		-0.00126			0.00003 $(\Delta^4 y_0)$ $(\nabla^4 y_n)$
			0.00001		
1.74	0.17552			0.00001 $(\nabla^3 y_n)$	
		-0.00175			
			0.00002 $(\nabla^2 y_n)$		
1.75	0.17377				
		-0.00173 (∇y_n)			
1.76	0.17204 (y_n)				

Here $x_0 = 1.72$, $y_0 = 0.17907$ and $h = 0.01$

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By Newton's forward formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right)$$

$$\left(\frac{dy}{dx}\right)_{x=1.72} = \frac{1}{0.01}\left(-0.00179 - \frac{1}{2}(0.00003) + \frac{1}{3}(-0.00002) - \frac{1}{4}(0.00003)\right)$$

$$= -0.1819$$

By Newton's backward difference formula.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.76} = \frac{1}{0.01}\left[-0.00173 + \frac{1}{2}(0.00002) + \frac{1}{3}(0.00001) + \frac{1}{4}(0.00003)\right]$$

$$= -0.1709.$$

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P) The population of a certain town (as obtained from census) data) is shown in the following data

Year	1951	1961	1971	1981	1991
Population (in thousands)	19.96	39.65	58.81	77.21	94.61

Estimate the rate of growth of the population in the year 1981

sol:- Since $x = 1981$ is towards the end, we shall ~~but~~ use Newton's backward difference formula.

We know that rate of growth of population is given by the first derivative of y.

The difference table is

Year (x)	Population (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1951	19.96				
		19.69			
1961	39.65		-0.53		
		19.16		-0.23 $(\nabla^3 y_n)$	
1971	58.81		-0.76 $(\nabla^2 y_n)$		-0.01
		18.4 (∇y_n)		-0.24	
1981	77.21 (y_n)		-1		
		17.4			
1991	94.61				

By using Newton's backward interpolation formula.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right)$$

$$\left(\frac{dy}{dx}\right)_{x=1981} = \frac{1}{10}\left(18.4 + \frac{1}{2}(-0.76) + \frac{1}{3}(-0.23)\right)$$

$$= \frac{1}{10}(18.4 - 0.3 - 0.076)$$

$$= \frac{17.943}{10} = 1.7943 \simeq 1.8 \text{ thousands/year.}$$

NUMERICAL INTEGRATION:-

If the function $y = f(x)$ is not known explicitly, but we are given a set of values of the function $y = f(x)$ corresponding to the same values of x, then we apply numerical integration method to find

$$\int_{x_0}^{x_0+nh} f(x)\,dx$$

(i) Simpson's $\frac{1}{3}$ rule:-

The formula for Simpson's $1/3$ rule is given by

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\Big]$$

(ii) Simpson's $3/8$ rule:-

The formula for Simpson's $3/8$ rule is given by

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+y_{10}+y_{11}+\cdots) + 2(y_3+y_6+y_9+y_{12}+\cdots)\Big]$$

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Note:- (i) when the number of subintervals (n) is even, then we use simpson's 1/3 rule for solving the problem.

(ii) when the number of sub intervals (n) is multiple of 3, then we use Simpson's 3/8 rule for solving the problem.

P) Evaluate $\int_0^2 e^{-x^2} dx$ using simpson's rule by taking $h = 0.25$

Sol^n:- The values of $y = f(x) = e^{-x^2}$ are given below.

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x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
x^2	0	0.0625	0.25	0.5625	1	1.5625	2.25	3.0625	4
$y=e^{-x^2}$	$e^{-0}=1$ (y_0)	$e^{-0.0625}=$ 0.9394 (y_1)	$e^{-0.25}=$ 0.7788 (y_2)	$e^{-0.5625}=$ 0.5697 (y_3)	$e^{-1}=$ 0.3678 (y_4)	$e^{-1.5625}=$ 0.2096 (y_5)	$e^{-2.25}=$ 0.1054 (y_6)	$e^{-3.0625}=$ 0.04677 (y_7)	$e^{-4}=$ 0.0183 (y_8)

Here the number of intervals $(n) = 8$ (even number)

So we use Simpson's $1/3$ rule.

$\therefore$ By using Simpson's $1/3$ rule.

$$\int_0^2 e^{-x^2}\,dx = \frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$= \frac{0.25}{3}\left[(1+0.0183)+4(0.9394+0.5697+0.2096+0.04677)+2(0.7788+0.3678+0.1054)\right]$$

$$= \underline{0.8816}$$

P) By using Simpson's $1/3$ rule with 10 subdivisions, evaluate $\int_0^4 e^{x}\,dx$.

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$\underline{\underline{\text{Sol}^n}}$:- Here $a = 0$, $b = 4$.

$\cdot$ n = number of subdivisions = 10.

$$\therefore h = \frac{b-a}{n} = \frac{4-0}{10} = \frac{4}{10} = 0.4.$$

Hence the table is

x	0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
$y = e^x$	1 (y_0)	1.4918 (y_1)	2.2255 (y_2)	3.3201 (y_3)	4.9530 (y_4)	7.3890 (y_5)	11.0232 (y_6)	16.4446 (y_7)	24.5391 (y_8)	36.5982 (y_9)	54.5981 (y_{10})

$\therefore$ By Simpson's $1/3$ rule.

$$\int_0^4 e^x\,dx = \frac{h}{3}\Big[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\Big]$$

$$= \frac{0.4}{3}\Big[(1 + 54.5981) + 4(1.4918 + 3.3201 + 7.3890 + 16.4446 + 36.59782) + 2(2.255 + 4.9530 + 11.0232 + 24.5385)\Big]$$

$$= \frac{0.4}{3}\Big[55.5981 + 4(65.2437) + 2(42.7342)\Big]$$

$$= \frac{0.4}{3}(402.013) = \underline{\underline{53.6055}}$$

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P) Using simpson's 3/8 rule, evaluate $\int_0^6 \frac{dx}{1+x^2}$ by dividing the range into 6 equal parts.

Solⁿ:- Here $a=0$, $b=6$ and n = number of subintervals $=6$.

$$\therefore h = \frac{b-a}{n} = \frac{6-0}{6} = 1.$$

∴ The values of x and y are tabulated below:

x	0	1	2	3	4	5	6
$y=\frac{1}{1+x^2}$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.058824 y_4	0.03846 y_5	0.027027 y_6

∴ By simpson's 3/8 rule

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3(1)}{8}\left[(1+0.027027)+3(0.5+0.2+0.058824+0.03846)+2(0.1)\right]$$

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$$= \frac{3}{8}\left[1.027027 + 2.391852 + 0.2\right]$$

$$= \frac{3}{8}\left[3.618879\right]$$

$$= 1.35708$$

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P) Evaluate $\int_0^{\pi/2} e^{\sin x}\,dx$ correct to four decimal places by simpson's 3/8 rule.

Soln: Here $b-a = \pi/2 - 0 = \pi/2$.

Simpson's 3/8 rule is applicable only when n is a multiple of 3.

So we divide $[0, \frac{\pi}{2}]$ into six equal parts.

$$\therefore h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}.$$

The value of $y = e^{\sin x}$ are tabulated as follows.

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}=\frac{\pi}{6}$	$\frac{3\pi}{12}=\frac{\pi}{4}$	$\frac{4\pi}{12}=\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12}=\frac{\pi}{2}$
$\sin x$	0	0.2588	0.5	0.7071	0.8660	0.9659	1
$y=e^{\sin x}$	1 y_0	1.2954 y_1	1.6487 y_2	2.0281 y_3	2.3774 y_4	2.6252 y_5	2.7183 y_6

By simpson's 3/8 rule

$$\int_0^{\pi/2} e^{\sin x}\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3(\pi/12)}{8}\left[(1+2.7183)+3(1.2954+1.6487+2.3774+2.6272)+2(2.0281)\right]$$

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$$= \frac{3\pi}{96}(3.7183 + 2\,3.8461 + 4.0562)$$

$$= \frac{3\pi}{96}(31.6206)$$

$$\simeq 3.1043$$

p) The table shows the temparature $f(t)$ as a function of time.

t	1	2	3	4	5	6	7
$f(t)$	81	75	80	83	78	70	60

Use simpson's $1/3$ method to estimate $\int_1^7 f(t)\,dt$.

Solⁿ:- Here $h=1$ and $y_0=81$, $y_1=75$, $y_2=80$, $y_3=83$, $y_4=78$, $y_5=70$, $y_6=60$.

By simpson's $1/3$ rule.

$$\int_1^7 f(t)\,dt = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(81+60)+4(75+83+70)+2(80+78)\right]$$

$$= \frac{1}{3}\left[141+912+316\right] = \frac{1369}{3} = 456.3333$$