2. Find the Laplace transform of

$f(t) = \begin{cases} t; 0 \leq t \leq a \\ 2a - t\,; a \leq t \leq 2a \end{cases}$ given that $f(t + 2a) = f(t)$. **Ans:** $\frac{1}{s^2}\tanh\left(\frac{s\pi}{2}\right)$

3. Find the Laplace transform of

$f(t) = \begin{cases} \frac{t}{a}; 0 \leq t \leq a \\ \frac{2a-t}{a}\,; a \leq t \leq 2a \end{cases}$ given that $f(t + 2a) = f(t)$. **Ans:** $\frac{1}{as^2}\tanh\left(\frac{sa}{2}\right)$

4. Find the Laplace transform of

$f(t) = \begin{cases} sint; 0 < t < \pi \\ 0\,; \pi < t < 2\pi \end{cases}$ $f(t + 2\pi) = f(t)$ **Ans:** $\frac{1}{(1-e^{-\pi s})(s^2+1)}$

5.6 INVERSE LAPLACE TRANSFORM

Definition

If the Laplace transform of a function $f(t)$ is $F(s)ie., L[f(t)] = F(s)$,then$f(t)$ is called an inverse Laplace transform of $F(s)$ and we write symbolically$f(t) = L^{-1}[F(s)]$, where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform of elementary functions

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}\left[\frac{1}{s^2}\right] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ *if n is an integer*	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
$L[sinat] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{sinat}{a}$
$L[cosat] = \frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = cosat$

$L[sinhat] = \frac{a}{s^2 - a^2}$	$L^{-1}\left[\frac{1}{s^2 - a^2}\right] = \frac{sinhat}{a}$
$L[cosat] = \frac{s}{s^2 - a^2}$	$L^{-1}\left[\frac{s}{s^2 - a^2}\right] = coshat$

Result on inverse Laplace transform

Result: 1 Linear property

$L[f(t)] = F(s)$ and $L[g(t)] = G(s)$,then $L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Where a and b are constants.

Proof:

We know that $L[aF(s) \pm bG(s)] = aL[F(s)] \pm bL[G(s)]$

$$= a\,F(s) \pm b\,G(s)$$

$(i.e.)a\,F(s) \pm b\,G(s) = L[af(t) \pm bg(t)]$

Operating L^{-1} on both sides, we get

$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$

$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

$\because f(t) = L^{-1}[F(s)]$
$\because g(t) = L^{-1}[G(s)]$

Result: 2 First shifting property

(i) $L^{-1}\left[F(s+a) = e^{-at}L^{-1}[F(s)]\right]$

(ii) $L^{-1}\left[F(s-a) = e^{at}L^{-1}[F(s)]\right]$

Proof:

Let $L[e^{-at}f(t)] = F[s+a]$

Operating L^{-1} on both sides, we get

$e^{-at}f(t) = L^{-1}\left[F[s+a]\right]$

$L^{-1}\left[F[s+a]\right] = e^{-at}L^{-1}[F(s)]$

Result: 3 Multiplication by s.

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$, then $L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) = sF(s)$

Operating L^{-1} on both sides, we get

$f'(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}f(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}L^{-1}[F(s)] = L^{-1}[sF(s)]$

$\therefore L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Result: 4 Division by s.

$$\boldsymbol{L^{-1}}\left[\frac{F(s)}{s}\right] = \int_{\boldsymbol{0}}^{\boldsymbol{t}} L^{-1}[F(s)]dt$$

Proof:

We know that $L\left[\int_0^t f\,(t)dt\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}F(s)$

Operating L^{-1} on both sides ,we get

$$\int_0^t f\,(t)dt = L^{-1}\left[\frac{1}{s}F(s)\right]$$

$$\int_0^t L^{-1}[F(s)]\,dt = L^{-1}\left[\frac{1}{s}F(s)\right]$$

$$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

Result: 5 Inverse Laplace transform of derivative

$$\boldsymbol{L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds}L[f(t)] = \frac{-d}{ds}F(s)$

Operating L^{-1} on both sides ,we get

$$tf(t) = -L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$f(t) = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Result: 6 Inverse Laplace transform of integral

$$\boldsymbol{L^{-1}[F(s)] = t}L^{-1}\left[\int_s^\infty F(s)ds\right]$$

Proof:

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))\,ds$

$$= \int_s^\infty F(s)\,ds$$

Operating L^{-1} on both sides, we get

$$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$f(t) = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

Problems under inverse Laplace transform of elementary functions

Example: 5.39 Find the inverse Laplace for the following

(i) $\boldsymbol{\frac{1}{2s+3}}$ **(ii)** $\boldsymbol{\frac{1}{4s^2+9}}$ **(iii)** $\boldsymbol{\frac{s^3-3s^2+7}{s^4}}$ **(iv)** $\boldsymbol{\frac{3s+5}{s^2+36}}$

Solution:

(i) $L^{-1}\left[\frac{1}{2s+3}\right] = L^{-1}\left[\frac{1}{2\left[s+\frac{3}{2}\right]}\right]$

$$= \frac{1}{2}e^{-\frac{-3t}{2}}$$

(ii) $L^{-1}\left[\frac{1}{4s^2+9}\right] = L^{-1}\left[\frac{1}{4\left[s^2+\frac{9}{4}\right]}\right]$

$$= \frac{1}{4}L^{-1}\left[\frac{1}{\left[s^2+\frac{9}{4}\right]}\right]$$

$$= \frac{1}{4}\frac{1}{3/2}\sin\frac{3}{2}t$$

$$= \frac{1}{6}\sin\frac{3}{2}t$$

(iii) $L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = L^{-1}\left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4}\right]$

$$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 7L^{-1}\left[\frac{1}{s^4}\right]$$

$$L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = 1 - 3t + \frac{7t^3}{3!}$$

(iv) $L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3L^{-1}\left[\frac{s}{s^2+36}\right] + 5L^{-1}\left[\frac{1}{s^2+36}\right]$

$$L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3\cos 6t + \frac{5\sin 6t}{6}$$

Inverse Laplace transform using First shifting theorem

$$\boldsymbol{L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]}$$

Example: 5.40 Find the inverse Laplace transform for the following:

(i)	$\frac{1}{(s+2)^2}$	**(ii)**	$\frac{1}{(s-3)^4}$	**(iii)**	$\frac{1}{(s+3)^2+9}$	**(iv)**	$\frac{1}{s^2-2s+2}$
(v)	$\frac{1}{s^2-4s+13}$	**(vi)**	$\frac{s+2}{(s+2)^2+25}$	**(vii)**	$\frac{s+2}{s^2+4s+20}$	**(viii)**	$\frac{s}{(s+3)^2}$
(ix)	$\frac{s}{(s-4)^3}$	**(x)**	$\frac{s}{s^2-2s+2}$	**(xi)**	$\frac{2s+3}{s^2+6s+25}$	**(xii)**	$\frac{s}{s^2+6s-7}$

Solution:

(i) $L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}t$

(ii) $L^{-1}\left[\frac{1}{(s-3)^4}\right] = e^{3t}L^{-1}\left[\frac{1}{s^4}\right] = e^{-2t}\frac{t^3}{3!}$

(iii) $L^{-1}\left[\frac{1}{(s+3)^2+9}\right] = e^{-3t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{-3t}\frac{\sin 3t}{3}$

(iv) $L^{-1}\left[\frac{1}{s^2-2s+2}\right] = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] = e^{t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{t}\sin t$

(v) $L^{-1}\left[\frac{1}{s^2-4s+13}\right] = L^{-1}\left[\frac{1}{(s-2)^2+9}\right] = e^{2t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{2t}\frac{\sin 3t}{3}$

(vi) $L^{-1}\left[\frac{s+2}{(s+2)^2+25}\right] = e^{-2t}L^{-1}\left[\frac{s}{s^2+25}\right] = e^{-2t}\cos 5t$

(vii) $L^{-1}\left[\frac{s+2}{s^2+4s+20}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+16}\right]$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+16}\right] = e^{-2t}cos4t$$

(viii) $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2}\right]$

$$= L^{-1}\left[\frac{s+3}{(s+3)^2}\right] - L^{-1}\left[\frac{3}{(s+3)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s+3}\right] - 3L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= e^{-3t} - 3e^{-3t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-3t} - 3e^{-3t}t$$

(ix) $L^{-1}\left[\frac{s}{(s-4)^3}\right] = L^{-1}\left[\frac{s-4+4}{(s-4)^3}\right]$

$$= L^{-1}\left[\frac{s-4}{(s-4)^3}\right] + L^{-1}\left[\frac{4}{(s-4)^3}\right]$$

$$= L^{-1}\left[\frac{1}{(s-4)^2}\right] + 4L^{-1}\left[\frac{1}{(s-4)^3}\right]$$

$$= e^{4t}L^{-1}\left[\frac{1}{s^2}\right] + 4e^{4t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= e^{4t}t + 4e^{4t}\frac{t^2}{2!}$$

$$= e^{4t}t + 2e^{4t}t^2$$

(x) $L^{-1}\left[\frac{s}{s^2-2s+2}\right] = L^{-1}\left[\frac{s}{(s-1)^2+1}\right] = L^{-1}\left[\frac{s-1+1}{(s-1)^2+1}\right]$

$$= L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] + L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= e^{t}L^{-1}\left[\frac{s}{s^2+1}\right] + e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{s}{s^2-2s+2}\right] = e^{t}cost + e^{t}sint$$

(xi) $L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = L^{-1}\left[\frac{2s+3}{(s+3)^2+16}\right] = L^{-1}\left[\frac{2(s+3-3)+3}{(s+3)^2+16}\right]$

$$= L^{-1}\left[\frac{2(s+3)-6+3}{(s+3)^2+16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{2s-3}{s^2+16}\right]$$

$$= e^{-3t}\left[2L^{-1}\left[\frac{s}{s^2+16}\right] - 3L^{-1}\left[\frac{1}{s^2+16}\right]\right]$$

$$L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = e^{-3t}\left(2cos4t - \frac{3sin4t}{4}\right)$$

(xii) $L^{-1}\left[\frac{s}{s^2+6s-7}\right] = L^{-1}\left[\frac{s}{(s+3)^2-16}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2-16}\right]$

$$= e^{-3t}L^{-1}\left[\frac{s-3}{s^2-16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{s}{s^2-16}\right] - 3e^{-3t}L^{-1}\left[\frac{1}{s^2-16}\right]$$

$$L^{-1}\left[\frac{s}{s^2+6s-7}\right] = e^{-3t}\left[cosh4t - \frac{3sinh4t}{4}\right]$$

Exercise: 5.7

Find the inverse Laplace transform for the following:

1. $\frac{2s-3}{s^2+5^2}$ **Ans:** $2cos5t - \frac{3sin5t}{5}$
2. $\frac{3s+5}{s^2+16}$ **Ans:** $3cos4t + \frac{5sin4t}{4}$
3. $\frac{1}{4s^2+9}$ **Ans:** $\frac{1}{6}sin\frac{3}{2}t$
4. $\frac{1}{(s+4)^5}$ **Ans:** $e^{-4t}\frac{t^4}{4!}$
5. $\frac{1}{s^2-4s+13}$ **Ans:** $\frac{e^{2t}}{3}sin3t$

Inverse using the formula

$$\boldsymbol{L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

Note: This formula is used when $F(s)$ is $\cot^{-1}\emptyset(s)$ or $\tan^{-1}\emptyset(s)$ or $log\emptyset(s)$

Example: 5.41 Find the inverse Laplace transform for the following

(i) $\boldsymbol{cot^{-1}\left(\frac{s}{a}\right)}$ **(ii)** $\boldsymbol{tan^{-1}\left(\frac{a}{s}\right)}$ **(iii)** $\boldsymbol{cot^{-1}as}$

(iv) $\boldsymbol{tan^{-1}(s+a)}$ **(v)** $\boldsymbol{log\left(\frac{s+a}{s+b}\right)}$ **(vi)** $\boldsymbol{cot^{-1}\left(\frac{2}{s+1}\right)}$ **(vii)** $\boldsymbol{tan^{-1}\left(\frac{2}{s^2}\right)}$

Solution:

(i) $L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{s}{a}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+\frac{s^2}{a^2}}\left(\frac{1}{a}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{-1}{\frac{a^2+s^2}{a^2}}\left(\frac{1}{a}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{t}sinat$$

(ii) $L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{a}{s}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{a}{s}\right)^2}\left(\frac{-a}{s^2}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{1}{\frac{s^2+a^2}{s^2}}\left(\frac{-a}{s^2}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{1}{t}sinat$$

(iii) $L^{-1}[cot^{-1}as] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(cot^{-1}(as))\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+a^2s^2}(a)\right] = \frac{1}{t}L^{-1}\left[\frac{a}{a^2\left(s^2+\frac{1}{a^2}\right)}\right]$$

$$= \frac{1}{at}L^{-1}\left[\frac{1}{s^2+\frac{1}{a^2}}\right] = \frac{1}{at}\left[\frac{sin\frac{1}{a}t}{\frac{1}{a}}\right]$$

$$L^{-1}[cot^{-1}as] = \frac{1}{t}sin\frac{t}{a}$$

(iv) $L^{-1}[tan^{-1}(s+a)] = e^{-at}L^{-1}[tan^{-1}s]$

$$= e^{-at}\left[\frac{-1}{t}L^{-1}\left[\frac{d}{ds}(tan^{-1}s)\right]\right]$$

$$= e^{-at}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$= \frac{-1}{t}e^{-at}L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-e^{-at}}{t}sint$$

(v) $L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(log\left(\frac{s+a}{s+b}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(\log(s+a) - \log(s+b))\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= \frac{-1}{t}[e^{-at} - e^{-bt}]$$

$$L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right] = \frac{-1}{t}[e^{-at} - e^{-bt}]$$

(vi) $L^{-1}\left[cot^{-1}\left(\frac{2}{s+1}\right)\right] = e^{-t}L^{-1}\left[cot^{-1}\left(\frac{2}{s}\right)\right]$

$$= e^{-t}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{2}{s}\right)\right)\right]$$

$$= e^{-t}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{-1}{1+\frac{4}{s^2}}\left(\frac{-2}{s^2}\right)\right] = -\frac{e^{-t}}{t}L^{-1}\left[\frac{1}{\frac{s^2+4}{s^2}}\left(\frac{2}{s^2}\right)\right]$$

$$= -\frac{e^{-t}}{t}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{2}{s+1}\right)\right] = -\frac{e^{-t}}{t}sin2t$$

(vii) $L^{-1}\left[tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{2}{s^2}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(\frac{-4}{s^3}\right)\right] = \frac{4}{t}L^{-1}\left[\frac{1}{\frac{s^4+4}{s^4}}\left(\frac{1}{s^3}\right)\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{s^4+4}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2)^2+2^2}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right] \qquad \boxed{\because a^2+b^2 = (a+b)^2 - 2ab}$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{-4s}\left(\frac{1}{s^2+2+2s} - \frac{1}{s^2+2-2s}\right)\right] \qquad \because \left\{\begin{array}{c}\frac{1}{(s^2+ax+b)(s^2+ax+c)} \\ = \frac{1}{c-b}\left[\frac{1}{s^2+ax+b} - \frac{1}{s^2+ax+c}\right]\end{array}\right\}$$

$$= \frac{-1}{t}L^{-1}\left[\left(\frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}\right]$$

$$= \frac{-1}{t}\left(e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] - e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$= \frac{-1}{t}(e^{-t}sint - e^{t}sint)$$

$$= \frac{sint}{t}(e^{-t} - e^{t})$$

$$= \frac{sint}{t}2sinht$$

$$L^{-1}\left[tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{2sintsinht}{t}$$

Inverse using the formula

$$\boldsymbol{L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]}$$

Example: 5.42 Find $\boldsymbol{L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]}$

Solution:

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]\cdots(1)$$

$$L^{-1}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = L^{-1}\frac{d}{ds}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(\log(s^2+a^2) - \log(s^2+b^2))\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s^2+a^2}2s - \frac{1}{s^2+b^2}2s\right]$$

$$= \frac{-2}{t}L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$= \frac{-2}{t}[cosat - cosbt]$$

$$= \frac{2}{t}[cosbt - cosat]$$

Substituting in (1), we get

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}\left[\frac{2}{t}[cosbt - cosat]\right]$$

$$= 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

Inverse using the formula

$$\boldsymbol{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt}$$

This formula is used when $F(s) = \frac{one\ term}{s(another\ term)}$

Example: 5.43 Find $\boldsymbol{L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]}$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2+a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{-cosat}{a}\right]_0^t$$

$$= \frac{-1}{a^2}[cosat]_0^t$$

$$= \frac{-1}{a^2}(cosat - cos0) = \frac{-1}{a^2}(cosat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-cosat}{a^2}$$

Example: 5.44 Find $L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2-a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinhat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{coshat}{a}\right]_0^t$$

$$= \frac{1}{a^2}[coshat]_0^t$$

$$= \frac{1}{a^2}(coshat - cosh0) = \frac{1}{a^2}(coshat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = \frac{coshat-1}{a^2}$$

Example: 5.45 Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+a)}\right]dt$$

$$= \int_0^t e^{-at}dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{-1}{a}(e^{-at} - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1-e^{-at}}{a}$$

Inverse using Partial Fraction

Example: 5.46 Find $L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right]$

Solution:

$$\frac{s-2}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$= \frac{A(s+2)(s-1)+Bs(s-1)+Cs(s+2)}{s(s+2)(s-1)}$$

$$A(s+2)(s-1) + Bs(s-1) + Cs(s+2) = s - 2 \cdots (1)$$

Put $s = 0$ in (1)	Put $s = -2$ in (1)	Put $s = 1$ in (1)
$A(2)(-1) = -2$	$B(-2)(-3) = -4$	$3C = -1$

$\Rightarrow A = 1$ $\Rightarrow B = \frac{-4}{6} = \frac{-2}{3}$ $\Rightarrow C = \frac{-1}{3}$

$$\therefore \frac{s-2}{s(s+2)(s-1)} = \frac{1}{s} - \frac{2}{s+2} - \frac{1}{3(s-1)}$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = L^{-1}\left[\frac{1}{s}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right]$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = 1 - \frac{2}{3}e^{-2t} - \frac{1}{3}e^{t}$$

Example: 5.47 Find $L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right]$

Solution:

$$\frac{2s-3}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 2s - 3 \cdots (1)$$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	Equating the coefficient of s^2
		$A + B = 0$
$A = -1$	$C = 1$	$B = -A \Rightarrow B = 1$

$$\therefore \frac{2s-3}{(s-1)(s-2)^2} = \frac{-1}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right] + \left[\frac{1}{(s-2)^2}\right]$$

$$= -e^{t} + e^{2t} + e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$\therefore L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -e^{t} + e^{2t} + e^{2t}t$$

Example: 5.48 Find the inverse Laplace transform of $\boldsymbol{\frac{5s^2-15s-11}{(s+1)(s-2)^3}}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{C}{(s-2)^3}$$

$$= \frac{A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)}{(s-1)(s-2)^3}$$

$$A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1) = 5s^2 - 15s - 11 \cdots (1)$$

Put $s = -1$ in (1)	Put $s = 2$ in (1)	Equating the coefficient of s^3
$A(-27) = 9$	$D(3) = -21$	$A + B = 0$
$A = \frac{9}{-27} \Rightarrow A = \frac{-1}{3}$	$D = \frac{-21}{3} = -7$	$B = -A \Rightarrow B = \frac{1}{3}$

Put $s = 0$ in (1), we get

$$-8A + 4B - 2C + D = -11$$

$$\frac{8}{3}+\frac{4}{3}-2C-7=-11$$

$$4-2C=7-11$$

$$-2C=-8\Rightarrow C=4$$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1}{3(s+1)}+\frac{1}{3(s-2)}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}\frac{t^2}{2}$$

Example: 5.49 Find the inverse Laplace transform of $\frac{4s+5}{(s+1)(s^2+4)}$

Solution:

$$\frac{4s+5}{(s+1)(s^2+4)}=\frac{A}{s+1}+\frac{Bs+c}{s^2+4}$$

$$=\frac{A(s^2+4)+(Bs+c)(s+1)}{(s+1)(s^2+4)}$$

$$A(s^2+4)+(Bs+c)(s+1)=4s+5\cdots\cdots(1)$$

Put $s=-1$in (1)	Equating coefficients of s^2term in (1)	Put $s=0$in (1)
$A(1+4)+0=4(-1)+5$	$A+B=0$	$A(4)+C=5$
$A(5)=1\Rightarrow A=\frac{1}{5}$	$B=-A\Rightarrow B=\frac{-1}{5}$	$C=5-4A=5-\frac{4}{5}$
		$=\frac{25-4}{5}=\frac{21}{5}$

$$\therefore \frac{4s+5}{(s+1)(s^2+4)}=\frac{\frac{1}{5}}{s+1}+\frac{\frac{-1}{5}s+\frac{21}{5}}{s^2+4}$$

$$=\frac{1}{5(s+1)}-\frac{s}{5(s^2+4)}+\frac{21}{5}\frac{1}{(s^2+4)}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right]=\frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right]+\frac{21}{5}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}cos2t+\frac{21}{5}\frac{sin2t}{2}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right]=\frac{1}{5}e^{-t}-\frac{1}{5}cos2t+\frac{21}{10}sin2t$$

Exercise: 5.8

Find the Inverse Laplace transforms using partial fraction for the following

1. $\frac{1}{(s+1)(s+3)}$ **Ans:** $\frac{1}{2}(e^{-t}-e^{-3t})$
2. $\frac{1}{s(s+1)(s+2)}$ **Ans:** $\frac{1}{2}(e^{-2t}-2e^{-t}+1)$
3. $\frac{54-3s-5}{(s+1)(s^2-3s+2)}$ **Ans:** $2e^{-t}+2\,e^{\frac{3t}{2}}cosh\frac{t}{2}2+8e^{\frac{3t}{2}}sinh\frac{t}{2}$

5. 7 INITIAL AND FINAL VALUE THEOREMS

Initial value theorem

Statement: If $L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$\therefore sF(s) = L[f'(t)] + f(0)$

$$= \int_0^\infty e^{-st} f'(t)dt + f(0)$$

Taking limit as $s \to \infty$ on both sides, we have

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt + f(0)\right]$$

$$= \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty}[e^{-st} f'(t)]dt + f(0)$$

$$= 0 + f(0) \qquad \because e^{-\infty} = 0$$

$$= f(0)$$

$$= \lim_{t\to 0} f(t)$$

$\therefore \lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$

Final value theorem

Statement: If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$,then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$\therefore sF(s) = L[f'(t)] + f(0)$

$$= \int_0^\infty e^{-st} f'(t)dt + f(0)$$

Taking limit as $s \to 0$ on both sides, we have

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\int_0^\infty e^{-st} f'(t)dt + f(0)\right]$$

$$= \lim_{s\to 0}\left[\int_0^\infty e^{-st} f'(t)dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s\to 0}[e^{-st} f'(t)]dt + f(0)$$

$$= \int_0^\infty f'(t)dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0)$$

$$= f(\infty)$$

$$= \lim_{t\to\infty} f(t)$$

$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Example: 5.50 Verify the initial value theorem for the function $f(t) = ae^{-bt}$

Solution:

Given $f(t) = ae^{-bt}$

$F(s) = L[f(t)]$

$= L[ae^{-bt}]$

$= a\ \frac{1}{s+b}$

$sF(s) = \frac{as}{s+b}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt}$$

$$= a \cdots\cdots\cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \left[\frac{as}{s+b}\right]$$

$$= \lim_{s\to\infty} \left[\frac{as}{s\left(1+\frac{b}{s}\right)}\right] == \lim_{s\to\infty} \left[\frac{a}{\left(1+\frac{b}{s}\right)}\right]$$

$$= a \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Example: 5.51Verify the initial value theorem and Final value theorem for the function $\boldsymbol{f(t) = 1 + e^{-t}[sint + cost]}$.

Solution:

Given $f(t) = 1 + e^{-t}[sint + cost]$

$F(s) = L[f(t)]$

$= L\big[1 + e^{-t}[sint + cost]\big]$

$= L[1] + L\big[e^{-t}[sint + cost]\big]$

$= L[1] + L[sint + cost]_{s\to s+1}$

$= \frac{1}{s} + \left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1}$

$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$

$F(s) = \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$

$sF(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\lim_{t\to 0} f(t) = \lim_{t\to 0}\big[1 + e^{-t}[sint + cost]\big]$

$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (1)$

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$

$$= 1 + \lim_{s\to\infty}\left[\frac{1}{s\left(1+\frac{2}{s}+\frac{2}{s^2}\right)} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(1 + e^{-t}[sint + cost])$$

$$= 1 + 0 = 1 \cdots\cdots\cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$$

$$= 1 + 0 + 0 = 1 \cdots\cdots\cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: 5.52 Verify the initial value theorem and Final value theorem for the function

$\boldsymbol{f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right]}$

Solution:

Given $f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right] \cdots (1)$

$$= \int_0^t L^{-1}\left[\frac{1}{(s+2)^2}\right] dt = \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^2}\right] dt$$

$$= \int_0^t e^{-2t}t\, dt$$

$$= \int_0^t te^{-2t}\, dt$$

$$= \left[t\left(\frac{e^{-2t}}{-2}\right) - \frac{(1)e^{-2t}}{(-2)^2}\right]_0^t$$

$$= -t\frac{e^{-2t}}{2} - \frac{e^{-2t}}{4} - 0 + \frac{1}{4}$$

$$\therefore f(t) = \frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}$$

From (1), $F(s) = \frac{1}{s(s+2)^2}$

$$sF(s) = \frac{1}{(s+2)^2}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$= \frac{1}{4} - 0 - \frac{1}{4} = 0$$

$$\therefore \lim_{t\to 0} f(t) = 0 \cdots (2)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{1}{(s+2)^2} = 0$$

$$\therefore \lim_{s\to\infty} sF(s) = 0 \cdots (3)$$

From (2) and (3), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\therefore$ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$= \frac{1}{4} - 0 - 0 = \frac{1}{4} \cdots (4)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{4} \cdots (5)$$

From (4) and (5), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$\therefore$ Final value theorem is verified

Example: 5.53 Verify the initial value theorem and Final value theorem for the function $\boldsymbol{f(t) = e^{-t}(t+2)^2}$

Solution:

Given $f(t) = e^{-t}(t+2)^2$

$$= e^{-t}(t^2 + 4t + 4)$$

$$F(s) = L[f(t)]$$

$$= L[e^{-t}(t^2 + 4t + 4)]$$

$$= L[t^2 + 4t + 4]_{s\to s+1}$$

$$= [L(t^2) + 4L(t) + 4L(1)]_{s\to s+1}$$

$$= \left[\frac{2!}{s^3} + 4\frac{1}{s^2} + 4\frac{1}{s}\right]_{s\to s+1}$$

$$= \frac{2}{(s+1)^3} + 4\frac{1}{(s+1)^2} + 4\frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[e^{-t}(t^2 + 4t + 4)]$$

$$= 4 \cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2s}{s^3\left(1+\frac{1}{s}\right)^3} + \frac{4s}{s^2\left(1+\frac{1}{s}\right)^2} + \frac{4s}{s\left(1+\frac{1}{s}\right)}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2}{s^2\left(1+\frac{1}{s}\right)^3} + \frac{4}{s\left(1+\frac{1}{s}\right)^2} + \frac{4}{\left(1+\frac{1}{s}\right)}\right]$$

$$= 0 + 0 + 4$$

$$= 4 \cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [e^{-t}(t^2 + 4t + 4)]$$

$$= 0 \cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= 0 \cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: 5.54 If $L[f(t)] = \frac{1}{s(s+1)}$, find the $\lim_{t\to 0} f(t)$ and $\lim_{t\to\infty} f(t)$ using initial and final value theorems.

Solution:

Given $L[f(t)] = \frac{1}{s(s+1)} \cdots (1)$

$ie., F(s) = \frac{1}{s(s+1)} => sF(s) = \frac{1}{(s+1)}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$= \lim_{s\to\infty} \frac{1}{(s+1)} = 0$$

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$= \lim_{s\to 0} \frac{1}{(s+1)} = 1$$

Exercise: 5.9

1. Verify the initial value theorem for the function $f(t) = e^{-t} sint$
2. Verify the initial value theorem for the function $f(t) = \sin^2 t$
3. Verify the initial value theorem for the function $f(t) = 1 + e^{-t} + t^2$
4. Verify the Final value theorem for the function $f(t) = 1 - e^{-at}$
5. Verify the Final value theorem for the function $f(t) = t^2 e^{-3t}$

5.8 CONVOLUTION THEOREM

Definition: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and defined by $f(t) * g(t) = \int_0^t f(u)g(t-u)du.$

State and prove Convolution theorem

Statement: If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then $L[f(t)] * L[g(t)] = F(s)G(s)$

Proof:

We have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$$L[f(t) * g(t)] = \int_0^\infty [f(t) * g(t)]\, e^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)du e^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st}dudt \cdots (1)$$

Now we have no change the order of integration.

$u = 0, u = t; t = 0, t = \infty$

Change of order is . Draw horizontal strip PQ

At P, $t = u$, At A $u = \infty$

$$L[f(t) * g(t)] = \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st}dtdu$$

$$= \int_0^\infty f(u)\left[\int_u^\infty g(t-u)e^{-st}dt\right]du \cdots (2)$$

Put $t - u = x \cdots (3)$

$t = u + x \Rightarrow dt = dx$

When $t = u; (3) \Rightarrow x = 0$

When $t = \infty; (3) \Rightarrow x = \infty$

$$(2) \Rightarrow L[f(t) * g(t)] = \int_0^\infty f(u)\left[\int_0^\infty g(x)e^{-s(u+x)}dx\right]du$$

$$= \int_0^\infty f(u)\left[\int_0^\infty g(x)e^{-su}e^{-sx}dx\right]du$$

$$= \int_0^\infty f(u)e^{-su}du \int_0^\infty g(x)e^{-sx}dx$$

$$= L[f(u)]L[g(x)]$$

$\therefore L[f(t) * g(t)] = F(s)G(s)$

Note: Convolution theorem is very useful to compute inverse Laplace transform of product of two terms

Convolution theorem is $L[f(t) * g(t)] = F(s)G(s)$

$$L^{-1}[F(s)G(s)] = f(t) * g(t)$$

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems under Convolution theorem

Example: 5.55 Find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)}\right] * L^{-1}\left[\frac{1}{(s+b)}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}e^{-b(t-u)}\, du$$

$$= e^{-bt}\int_0^t e^{-au}\, e^{bu}du$$

$$= e^{-bt}\int_0^t e^{(b-a)u}\, du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right]$$

$$= \frac{e^{-bt}}{b-a}\left[e^{bt-at} - 1\right]$$

$$= \frac{1}{b-a}\left[e^{-bt+bt-at} - e^{-bt}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

Example: 5.56 Find the inverse Laplace transform $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+b^2)}\right]$$

$$= cosat * cosbt$$

$$= \int_0^t cosau\ cosb(t-u)du$$

$$= \int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2}du$$

$$= \frac{1}{2}\int_0^t (\cos(au+bt-bu) + \cos(au-bt+bu))\,du$$

$$= \frac{1}{2}\int_0^t [\cos(a-b)\,u + bt + \cos(a+b)\,u - bt]du$$

$$= \frac{1}{2}\left[\frac{sin[(a-b)u+bt]}{a-b} + \frac{sin[(a+b)u+bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)sinat+(a-b)sinat-(a+b)sinbt+(a-b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2asinat-2bsinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(asinat-bsinbt)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinat-bsinbt}{a^2-b^2}$$

Example: 5.57 Find the inverse Laplace transform $\frac{1}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+b^2)}\right]$$

$$= \frac{1}{a}sinat * \frac{1}{b}sinbt$$

$$= \frac{1}{ab}\int_0^t sinau\ sinb(t-u)du$$

$$= \frac{1}{ab}\int_0^t \frac{\cos(au-bt+bu)-\cos(au+bt-bu)}{2}du$$

$$= \frac{1}{2ab}\int_0^t (\cos(au-bt+bu)-\cos(au+bt-bu))\,du$$

$$= \frac{1}{2}\int_0^t [\cos[(\mathrm{a}+\mathrm{b})\mathrm{u}-\mathrm{bt}]-\cos[(\mathrm{a}-\mathrm{b})\mathrm{u}+\mathrm{bt}]]du$$

$$= \frac{1}{2ab}\left[\frac{sin[(a+b)u-bt]}{a+b}-\frac{sin[(a-b)u+bt]}{a-b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin(at+bt-bt)}{a+b}-\frac{\sin(at-bt+bt)}{a-b}+\frac{sinbt}{a+b}+\frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{\sin at}{a+b}-\frac{\sin at}{a-b}-\frac{sinbt}{a+b}+\frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{(a-b)sinat-(a+b)sinat+(a-b)sinbt+(a+b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{-2bsinat+2asinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{2(asinbt-bsinat)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinbt-bsinat}{ab(a^2-b^2)}$$

Example: 5.58 Find the inverse Laplace transform $\frac{s}{(s^2+4)(s^2+9)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{(s^2+4)}\ \frac{s}{(s^2+9)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+4)}\right] * L^{-1}\left[\frac{s}{(s^2+9)}\right]$$

$$= \frac{1}{2}sin2t * cos3t$$

$$= \frac{1}{2}\int_0^t sin2u\ cos3(t-u)du$$

$$= \frac{1}{2}\int_0^t \frac{\sin(2u+3t-3u)+\sin(2u-3t+3u)}{2}du$$

$$= \frac{1}{4}\int_0^t [\sin(3\mathrm{t}-u)+\sin(5\mathrm{u}-3\mathrm{t})]du$$

$$= \frac{1}{4}\left[\frac{-\cos(3t-u)}{-1}-\frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{\cos(3\mathrm{t}-\mathrm{t})}{1}-\frac{\cos(5\mathrm{t}-3t)}{5}-\frac{cos3t}{1}+\frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[cos2t-\frac{\cos 2t}{5}-cos3t+\frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[\frac{5cos2t-cos2t-5cos3t+cos3t}{5}\right]$$

$$= \frac{1}{20}[4cos2t-4cos3t]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{cos2t-cos3t}{5}$$

Example: 5.59 Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * cosat$$

$$= \frac{1}{a}\int_0^t sinau\ cosa(t-u)du$$

$$= \frac{1}{a}\int_0^t \frac{\sin(au+at-au)+\sin(au-at+au)}{2} du$$

$$= \frac{1}{2a}\int_0^t [sinat + \sin(2au - at)]du$$

$$= \frac{1}{2a}\left[\int_0^t sinatdu + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat\int_0^t du + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat(u)_0^t - \left(\frac{\cos(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos(2at-at)}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos at}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\ tsinat$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{tsinat}{2a}$$

Example: 5.60 Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * \frac{1}{a} sinat$$

$$= \frac{1}{a^2}\int_0^t sinau\ sina(t-u)du$$

$$= \frac{1}{a^2}\int_0^t \frac{\cos(au-at+au)-\cos(au+at-au)}{2} du$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at) - cosat]du$$

$$= \frac{1}{2a^2}\left[\int_0^t \cos(2au-at)du - \int_0^t cosat\,du\right]$$

$$= \frac{1}{2a^2}\left[\int_0^t \cos(2au-at)du - cosat\int_0^t du\right]$$

$$= \frac{1}{2a^2}\left[\left(\frac{\sin(2au-at)}{2a}\right)_0^t - cosat(u)_0^t\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2at-at)}{2a} - \frac{\sin(-at)}{2a} - tcosat\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} + \frac{\sin at}{2a} - t\cos at\right]$$

$$= \frac{1}{2a^2}\left[\frac{2\sin at}{2a} - t\cos at\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$$

Example: 5.61 Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au\, \cos a(t-u) du$$

$$= \int_0^t \frac{\cos(au+at-au)+\cos(au-at+au)}{2} du$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au - at)] du$$

$$= \frac{1}{2}\left[\int_0^t \cos at du + \int_0^t \cos(2au - at)\, du\right]$$

$$= \frac{1}{2}\left[\cos at \int_0^t du + \int_0^t \cos(2au - at)\, du\right]$$

$$= \frac{1}{2}\left[\cos at (u)_0^t + \left(\frac{\sin(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin(2at-at)}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{2\sin at}{2a}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$$

Example: 5.62 Find $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+2^2)}\ \frac{s}{(s^2+2^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+2^2)}\right] * L^{-1}\left[\frac{s}{(s^2+2^2)}\right]$$

$$= \cos 2t * \cos 2t$$

$$= \int_0^t \cos 2u\, \cos 2(t-u) du$$

$$= \int_0^t \frac{\cos(2u+2t-2u)+\cos(2u-2t+2u)}{2} du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos(4u - 2t)] du$$

$$= \frac{1}{2}\left[\int_0^t \cos 2t du + \int_0^t \cos(4u - 2t)\, du\right]$$

$$= \frac{1}{2}\left[cos2t \int_0^t du + \int_0^t \cos(4u - 2t)\, du\right]$$

$$= \frac{1}{2}\left[cos2t(u)_0^t + \left(\frac{\sin(4u-2t)}{4}\right)_0^t\right]$$

$$= \frac{1}{2}\left[tcos2t + \frac{\sin(4t-2t)}{4} - \frac{\sin(-2t)}{4}\right]$$

$$= \frac{1}{2}\left[tcos2t + \frac{\sin 2t}{4} + \frac{sin2t}{4}\right]$$

$$= \frac{1}{2}\left[tcos2t + \frac{2\sin 2t}{4}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left[tcos2t + \frac{\sin 2t}{2}\right]$$

Example: 5.63 Find $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = L^{-1}\left[\frac{1}{s}\ \frac{1}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= 1 * \frac{\sin 2t}{2}$$

$$= \frac{\sin 2t}{2} * 1$$

$$= \int_0^t \frac{\sin 2u\ (1)}{2} du$$

$$= \left[\frac{-cos2u}{4}\right]_0^t = \frac{1}{4}(-cos2t + 1)$$

$$= \frac{1}{4}(1 - cos2t)$$

Example: 5.64 Find the inverse Laplace transform $\frac{s+2}{(s^2+4s+13)^2}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{s+2}{s^2+4s+13}\ \frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{s^2+4s+13}\right] * L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] * L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right] * e^{-2t}L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-2t}cos3t * \frac{e^{-2t}sin3t}{3}$$

$$= \int_0^t e^{-2u}cos3u\, e^{-2(t-u)}\, \frac{sin3(t-u)}{3} du$$

$$= \int_0^t e^{-2u}cos3u\, e^{-2t+2u}\, \frac{sin(3t-3u)}{3} du$$

$$= \frac{1}{3}\int_0^t e^{-2u-2t+2u}cos3u \sin(3t-3u)\, du$$

$$= \frac{e^{-2t}}{3}\int_0^t \frac{\sin(3u+3t-3u) - \sin(3u-3t+3u)}{2} du$$

$$= \frac{e^{-2t}}{6}\int_0^t [sin3t - \sin(6u - 3t)]du$$

$$= \frac{e^{-2t}}{6}\left[\int_0^t sin3tdu - \int_0^t \sin(6u - 3t)\, du\right]$$

$$= \frac{e^{-2t}}{6}\left[sin3t\int_0^t du - \int_0^t \sin(6u - 3t)\, du\right]$$

$$= \frac{e^{-2t}}{6}\left[sin3t(u)_0^t + \left(\frac{\cos(6u-3t)}{6}\right)_0^t\right]$$

$$= \frac{e^{-2t}}{6}\left[tsin3t + \frac{\cos(6t-3t)}{6} - \frac{\cos(-3t)}{6}\right]$$

$$= \frac{e^{-2t}}{6}\left[tsin3t + \frac{\cos 3t}{6} - \frac{cos3t}{6}\right]$$

$$= \frac{e^{-2t}}{6}\ tsin3t$$

$$\therefore L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{6}\ tsin3t$$

Example: 5.65 Find the inverse Laplace transform $\frac{1}{(s+1)(s^2+4)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+4)(s+1)}\right] = L^{-1}\left[\frac{1}{s+1}\ \frac{1}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= e^{-t} * cos2t$$

$$= \int_0^t e^{-(t-u)} cos2u\, du$$

$$= e^{-t}\int_0^t e^u cos2u\, du$$

$$\boxed{\because \int e^{at} cosbtdt = \frac{e^{at}}{a^2 + b^2}(acosbt + bsinbt)}$$

$$= e^{-t}\left[\frac{e^u}{1^2+2^2}(cos2u + 2sin2u)\right]_0^t$$

$$= \frac{e^{-t}}{5}[e^t(cos2t + 2sin2t) - e^0(cos0 - 0)]$$

$$= \frac{e^{-t}}{5}[e^t(cos2t + 2sin2t) - 1]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+4)(s+1)}\right] = \frac{e^{-t}}{5}[e^t(cos2t + 2sin2t) - 1]$$

Exercise: 5.10

Find the inverse Laplace transforms using convolution theorem for the following

1. $\frac{1}{s(s^2+1)}$ **Ans:** $1 - cost$
2. $\frac{s}{(s^2+4)^2}$ **Ans:** $\frac{1}{8}\left[\frac{\sin 2t}{2} - tcos2t\right]$
3. $\frac{s^2}{(s^2+4)^2}$ **Ans:** $\frac{1}{2}\left[tcos2t + \frac{\sin 2t}{2}\right]$
4. $\frac{1}{(s+1)(s^2+1)}$ **Ans:** $\frac{1}{2}[e^{-t} + sint - cost]$
5. $\frac{1}{(s+1)(s^2+4)}$ **Ans:** $-\frac{1}{5}e^{-t} + \frac{1}{5}cos2t - \frac{1}{10}sin2t$

5.9 SOLUTION OF DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM TECHNIQUE

There are so many methods to solve a linear differential equation. If the initial conditions are known, then Laplace transform technique is easier to solve the differential equation. The Laplace transform transforms the differential equation into an algebraic equation.

$$L[y'(t)] = sL[y(t)] - y(0)$$
$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Problems using Partial Fraction

Example: 5.66 Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

Given $x'' - 3x' + 2x = 2; x(0) = 0; x'(0) = 5$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$
$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0; x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$
$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$
$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$
$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$
$$(s-1)(s-2)\bar{x} = \frac{2}{s} + 5$$
$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

Consider $\frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)+Bs(s-2)+Cs(s-1)}{s(s-1)(s-2)}$$
$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

Put $s = 0$ in (1)	Put $s = 1$ in (1)	Put $s = 2$ in (1)
$A(-1)(-2) = 2$	$B(1)(-1) = 7$	$C(2)(1) = 2 + 10$
$A = 1$	$B = -7$	$C = 6$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$\therefore \bar{x} = \frac{1}{s} - 7\frac{1}{s-1} + 6\frac{1}{s-2}$

$x(t) = L^{-1}\left[\frac{1}{s}\right] - 7L^{-1}\left[\frac{1}{s-1}\right] + 6L^{-1}\left[\frac{1}{s-2}\right]$

$x(t) = 1 - 7e^{t} + 6e^{2t}$

Example: 5.67 Using Laplace transform solve the differential equation $\boldsymbol{y'' - 3y' - 4y = 2e^{-t}}$**, with** $\boldsymbol{y(0) = 1 = y'(0)}$.

Solution:

Given $y'' - 3y' - 4y = 2e^{-t}$; with $y(0) = 1 = y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2\frac{1}{s+1}$

Substituting $y(0) = 1 = y'(0)$.

$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2+s(s+1)-2(s+1)}{s+1}$

$= \frac{2+s^2+s-2s-2}{s+1}$

$(s+1)(s-4)\bar{y} = \frac{s^2-s}{s+1}$

$\bar{y} = \frac{s^2-s}{(s+1)(s+1)(s-4)}$

$\bar{y} = \frac{s^2-s}{(s+1)^2(s-4)}$

Consider $\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4)+B(s-4)+C(s+1)^2}{(s+1)^2(s-4)}$

$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \cdots (1)$

Put $s = -1$ in (1)	Put $s = 4$ in (1)	equating the coefficients of s^2, we get
$-5B = 1 + 1$	$25C = 16 - 4$	$A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$
$B = \frac{-2}{5}$	$C = \frac{12}{25}$	$A = \frac{13}{25}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$y(t) = \frac{13}{25} L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25} L^{-1}\left[\frac{1}{s-4}\right]$

$y(t) = \frac{13}{25} e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} e^{4t}$

Example: 5.68 Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$, with $y(0) = 1$ and $y'(0) = 0$ using Laplace transform.

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ and $y'(0) = 1$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$

$[s^2 L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$

Substituting $y(0) = 1$ and $y'(0) = 0$.

$[s^2 L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1 + s(s+1) - 3(s+1)}{s+1}$

$= \frac{1 + s^2 + s - 3s - 3}{s+1}$

$(s-1)(s-2)\bar{y} = \frac{s^2 - 2s - 2}{s+1}$

$\bar{y} = \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}$

Consider $\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$

$\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$

$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \cdots (1)$

Put $s = -1$ in (1)	put $s = 1$ in (1)	put $s = 2$ in (1)
$6A = 1 + 2 - 2$	$-2B = 1 - 4$	$3C = 4 - 4 - 2$
$A = \frac{1}{6}$	$B = \frac{3}{2}$	$C = \frac{-2}{3}$

$\therefore \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$

$y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}$

Example: 5.69 Using Laplace transform solve the differential equation $y'' + 2y' - 3y = sint$, with $y(0) = y'(0) = 0$.

Solution:

Given $y'' + 2y' - 3y = sint$ with $y(0) = 0 = y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L(sint)$

$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] - 3L[y(t)] = \frac{1}{s^2+1}$

Substituting $y(0) = 0 = y'(0)$.

$[s^2L[y(t)] - 0 - 0] + 2[sL[y(t)] - 0] - 3L[y(t)] = \frac{1}{s^2+1}$

$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$

$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$

$[s^2 + 2s - 3]\bar{y} = \frac{1}{s^2+1}$

$(s-1)(s+3)\bar{y} = \frac{1}{s^2+1}$

$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$

Consider $\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s^2+1)(s+3)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$

$A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1 \cdots (1)$

Put $s = 1$ in (1)	Put $s = -3$ in (1)	equating the coefficients of s^2, we get
$8A = 0 + 1$	$B(-4)(10) = 1$	$A + B + C = 0 \Rightarrow C = -A - B = \frac{-1}{8} + \frac{1}{40}$
$A = \frac{1}{8}$	$B = \frac{-1}{40}$	$C = \frac{-1}{10}$

Put $s = 0$ in (1), we get

$3A - B - 3D = 1 \Rightarrow \frac{3}{8} + \frac{1}{40} - 3D = 1$

$3D = \frac{3}{8} + \frac{1}{40} - 1$

$3D = \frac{15+1-40}{40} \Rightarrow D = \frac{-24}{40\times3} \Rightarrow D = \frac{-1}{5}$

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} + \frac{\left(\frac{-1}{10}\right)s - \frac{1}{5}}{s^2+1}$

$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$

$y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$

$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(cost - 2sint)$

Example: 5.70 Using Laplace transform solve the differential equation$y'' - 3y' + 2y = 4e^{2t}$, with$y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ $and y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$

Substituting$y(0) = -3$ and $y'(0) = 5$.

$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$

Put$L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$

$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$

$[s^2 - 3s + 2]\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$(s-1)(s-2)\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$\bar{y} = \frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2}$

Consider $\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2+B(s-1)(s-2)+C(s-1)}{(s-1)(s-2)^2}$

$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \cdots (1)$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	equating the coefficients ofs^2, we get
$A = 4 - 11$	$C = 4 + 0$	$A + B = -3 \Rightarrow -7 + B = -3$
$A = -7$	$C = 4$	$B = 4$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$y(t) = -7L^{-1}\left[\frac{1}{(s-1)}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$

$$= -7e^{t} + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = -7e^{t} + 4e^{2t} + 4e^{2t}t$$

Example: 5.71 Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with$y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2\ andy'(0) = -2$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$$

Substituting $y(0) = 2\ andy'(0) = -2$.

$$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1 + (2s-10)(s-2)}{s-2}$$

$$\bar{y} = \frac{1 + (2s-10)(s-2)}{(s-2)(s^2-4s+8)}$$

$$= \frac{1 + (2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$$

Consider $\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$

$$= \frac{A[(s-2)^2+4] + B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$$

$$A[(s-2)^2 + 4] + B[(s-2) + C](s-2) = 1 + (2s-10)(s-2) \cdots (1)$$

Put $s = 2$ in (1)	Put $s = 0\ in\ (1)$	equating the coefficients ofs^2, we get
$4A = 1 + 0$	$8A + 4B - 2C = 21$	$A + B = 2 \Rightarrow \frac{1}{4} + B = 2$
$A = \frac{1}{4}$	$C = -6$	$B = \frac{7}{4}$

$$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2) - 6}{(s-2)^2+4}$$

$$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4}\frac{(s-2)}{(s-2)^2+4} - 6\frac{1}{(s-2)^2+4}$$

$$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{(s-2)}\right] + \frac{7}{4}L^{-1}\left[\frac{(s-2)}{(s-2)^2+4}\right] - 6L^{-1}\left[\frac{1}{(s-2)^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}\left[\frac{s}{s^2+4}\right] - 6e^{2t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 6e^{2t}\frac{sin2t}{2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 3e^{2t}sin2t$$

Problems without using Partial Fraction

Example: 5.72 Solve using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$

Solution:

Given $x'' - 2x' + x = e^t; x(0) = 2; x'(0) = -1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$$

Substituting $x(0) = 2; x'(0) = -1$

$$[s^2L[x(t)] - 2s + 1] - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 2s\bar{x} + \bar{x} = \frac{1}{s-1} + 2s - 5$$

$$[s^2 - 2s + 1]\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$\bar{x} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^tL^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2} + 2e^t + 2e^tt - 5e^tt$$

$$\therefore x = \frac{t^2e^t}{2} + 2e^t - 3e^tt$$

Example: 5.73 Solve the following differential equation using Laplace transform

$(D^2 - 2D + 1)y = t^2e^t$ Given $y(0) = 2$ and $Dy(0) = 3$

Solution:

Given $(D^2 - 2D + 1)y = t^2e^t$ with $y(0) = 2$ and $Dy(0) = 3$

$ie., D^2y - 2Dy + y = t^2e^t$

$y'' - 2y' + y = t^2e^t$ With $y(0) = 2$ and $y'(0) = 3$

Apply Laplace transform on both sides, we get

$$L[y''(t)] - 2L[y'(t)] + L[y(t)] = L(t^2e^t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 2[sL[y(t)] - y(0)] + L[y(t)] = L[t^2]_{s\to s-1}$$

Substituting $y(0) = 2\ and y'(0) = 3$.

$$[s^2L[y(t)] - 2s - 3] - 2[sL[y(t)] - 2] + L[y(t)] = \left[\frac{2!}{s^3}\right]_{s\to s-1}$$

$$s^2L[y(t)] - 2s - 3 - 2sL[y(t)] + 4 + L[y(t)] = \frac{2}{(s-1)^3}$$

$$s^2L[y(t)] - 2sL[y(t)] + L[y(t)] = \frac{2}{(s-1)^3} + 2s - 1$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} - 2s\bar{y} + \bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$[s^2 - 2s + 1]\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$(s-1)^2\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$\bar{y} = \frac{2}{(s-1)^5} + \frac{2s}{(s-1)^2} - \frac{1}{(s-1)^2}$$

$$y(t) = L^{-1}\left[\frac{2}{(s-1)^5}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= 2e^tL^{-1}\left[\frac{1}{s^5}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= 2e^t\frac{t^4}{4!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - e^tt$$

$$= 2e^t\frac{t^4}{24} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - e^tt$$

$$= e^t\frac{t^4}{12} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - e^tt$$

$$= e^t\frac{t^4}{12} + 2e^t + 2e^tt - e^tt$$

$$\therefore x = \frac{t^4e^t}{12} + 2e^t + e^tt$$

Example: 5.74 Solve using Laplace transform $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$, given that $y(0) = 0$ and $y'(0) = 0$

Solution:

Given $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$ with $y(0) = 0$ and $y'(0) = 0$

$y'' + 6y' + 9y = 6t^2e^{-3t}$ With $y(0) = 0$ and $y'(0) = 0$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + 6L[y'(t)] + 9L[y(t)] = 6L(t^2e^{-3t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 6[sL[y(t)] - y(0)] + 9L[y(t)] = 6L[t^2]_{s\to s+3}$$

Substituting $y(0) = 0\ and y'(0) = 0$.

$$[s^2L[y(t)] - 0 - 0] + 6[sL[y(t)] - 0] + 9L[y(t)] = 6\left[\frac{2!}{s^3}\right]_{s\to s+3}$$

$$s^2L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

$$s^2L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} + 6s\bar{y} + 9\bar{y} = \frac{12}{(s+3)^3}$$

$$[s^2 + 6s + 9]\bar{y} = \frac{12}{(s+3)^3}$$

$$(s+3)^2\bar{y} = \frac{12}{(s+3)^3}$$

$$\bar{y} = \frac{12}{(s+3)^5}$$

$$y(t) = L^{-1}\left[\frac{12}{(s+3)^5}\right] = 12e^{-3t}L^{-1}\left[\frac{1}{s^5}\right]$$

$$= 12e^{-3t}\frac{t^4}{4!}$$

$$\therefore y = \frac{t^4e^{-3t}}{2}$$

Example: 5.75 Solve $\boldsymbol{\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}sint;\ x(0) = 0}$ **and** $\boldsymbol{x'(0) = 1}$

Solution:

Given $x'' + 2x' + 5x = e^{-t}sint; x(0) = 0; x'(0) = 1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] + 2L[x'(t)] + 5L[x(t)] = L(e^{-t}sint)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] + 2[sL[x(t)] - x(0)] + 5L[x(t)] = L[sint]_{s\to s+1}$$

Substituting $x(0) = 0; x'(0) = 1$

$$[s^2L[x(t)] - 0 - 1] + 2[sL[x(t)] - 0] + 5L[x(t)] = \left[\frac{1}{s^2+1}\right]_{s\to s+1}$$

$$s^2L[x(t)] + 2sL[x(t)] + 5L[x(t)] - 1 = \frac{1}{(s+1)^2+1}$$

$$s^2L[x(t)] + 2sL[x(t)] + 5L[x(t)] = \frac{1}{(s+1)^2+1} + 1$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} + 2s\bar{x} + 5\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{s^2+2s+2} + 1$$

$$\bar{x} = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{5-2}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right] + \frac{1}{s^2+2s+5}$$

$$\boxed{\frac{1}{(s^2+ax+b)(s^2+ax+c)} = \frac{1}{c-b}\left[\frac{1}{s^2+ax+b} - \frac{1}{s^2+ax+c}\right]}$$

$$= \frac{1}{3}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right] + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{3(s^2+2s+2)} - \frac{1}{3(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$\bar{x} = \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)}$$

$$x(t) = \frac{1}{3}L^{-1}\left[\frac{1}{(s^2+2s+2)}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s^2+2s+5)}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{2}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{3}e^{-t}sint + \frac{2}{3}e^{-t}\frac{sin2t}{2}$$

$$\therefore x = \frac{1}{3}e^{-t}[sint + sin2t]$$

Example: 5.76 Solve using Laplace transform $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that $= 4$,$y' = -2$ when $t = 0$

Solution:

Given $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

$y'' + y' = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + L[y'(t)] = L(t^2) + L(2t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + [sL[y(t)] - y(0)] = \frac{2}{s^3} + 2\frac{1}{s^2}$$

Substituting $y(0) = 4\, and y'(0) = -2.$

$$[s^2L[y(t)] - 4s + 2] + [sL[y(t)] - 4] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2L[y(t)] + sL[y(t)] - 4s + 2 - 4 = \frac{2+2s}{s^3}$$

$$s^2L[y(t)] + sL[y(t)] = \frac{2(1+s)}{s^3} + 4s + 2$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} + s\bar{y} = \frac{2(1+s)}{s^3} + 2(2s+1)$$

$$s(s^2 + s)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s+1)$$

$$s(s+1)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s+1)$$

$$\bar{y} = \frac{2(s+1)}{s^4(s+1)} + \frac{2(2s+1)}{s(s+1)}$$

$$= \frac{2}{s^4} + 2\left[\frac{s+(s+1)}{s(s+1)}\right]$$

$$= \frac{2}{s^4} + 2\left[\frac{s}{s(s+1)} + \frac{s+1}{s(s+1)}\right]$$

$$= \frac{2}{s^4} + 2\left[\frac{1}{s+1} + \frac{1}{s}\right]$$

$$\bar{y} = \frac{2}{s^4} + \frac{2}{s+1} + \frac{2}{s}$$

$$y(t) = 2L^{-1}\left[\frac{2}{s^4}\right] + 2L^{-1}\left[\frac{1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s}\right]$$

$$= 2\frac{t^3}{3!} + 2e^{-t} + 2(1)$$

$$\therefore y = \frac{t^3}{3} + 2e^{-t} + 2$$

Example: 5.77 Solve using Laplace transform $\boldsymbol{\frac{d^2x}{dt^2} + 9x = cos2t}$**, if** $(\mathbf{0}) = \mathbf{1}$ **;** $\boldsymbol{x\left(\frac{\pi}{2}\right) = -1}$

Solution:

Given $x'' + 9x = cos2t; x(0) = 1; x\left(\frac{\pi}{2}\right) = -1$

Since $x'(0)$ is not given assume $x'(0) = k$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] + L[x(t)] = L(cos2t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] + 9L[x(t)] = L(cos2t)$$

Substituting $x(0) = 1; x\left(\frac{\pi}{2}\right) = -1$

$$[s^2L[x(t)] - s - k] + 9L[x(t)] = \frac{s}{s^2+4}$$

$$s^2L[x(t)] + 9L[x(t)] = \frac{s}{s^2+4} + s + k$$

$$[s^2 + 9]L[x(t)] = \frac{s}{s^2+4} + s + k$$

Put $L[x(t)] = \bar{x}$

$$[s^2 + 9]\bar{x} = \frac{s}{s^2+4} + s + k$$

$$\bar{x} = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{9-4}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{5}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$\bar{x} = \frac{s}{5(s^2+4)} + \frac{(5s-s)}{5(s^2+9)} + \frac{k}{s^2+9}$$

$$= \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$x(t) = \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{4}{5}L^{-1}\left[\frac{s}{s^2+9}\right] + kL^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{1}{5}cos2t + \frac{4}{5}cos3t + k\frac{sin3t}{3} \cdots (1)$$

Given $x\left(\frac{\pi}{2}\right) = -1$

Put $t = \frac{\pi}{2}\ in\ (1)$

$$(1) \Rightarrow x\left(\frac{\pi}{2}\right) = \frac{1}{5}cos\frac{2\pi}{2} + \frac{4}{5}cos\frac{3\pi}{2} + k\frac{sin\frac{3\pi}{2}}{3}$$

$$-1 = \frac{1}{5}(-1) + 0 + \frac{k}{3}(-1)$$

$-\frac{k}{3} = \frac{1}{5} - 1 \Rightarrow -\frac{k}{3} = \frac{-4}{5} \Rightarrow k = \frac{12}{5}$

$= \frac{1}{5}cos2t + \frac{4}{5}cos3t + \frac{12}{5}\frac{sin3t}{3}$

$\therefore x(t) = \frac{1}{5}[cos2t + 4cos3t + 4sin3t]$

Exercise: 5.11

1. Solve using Laplace transform $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$, given that $= 0$,$\frac{dy}{dt} = 2$ when $t = 0$

Ans:$-1 - \frac{1}{6}e^{-5t} + \frac{5}{6}e^{t}$

2. Using Laplace transform solve the differential equation $y'' + 5y' + 6y = 2$, with $y(0) = 0 = y'(0)$. Where $y' = \frac{dy}{dt}$ **Ans:**$y(t) = \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}$

3. Using Laplace transform solve the differential equation $y'' + 4y' + 3y = e^{-t}$, with $y(0) = 1;\ y'(0) = 0.$ **Ans:**$y(t) = \frac{-1}{4}e^{-3t} - \frac{5}{4}e^{-t} + \frac{1}{2}te^{-t}$

4. Solve using Laplace transform $\frac{d^2y}{dt^2} + y = sint$ given $= 1$,$\frac{dy}{dt} = 0$ when $t = 0$

Ans:$y(t) = sint - tcost$

5. Solve using Laplace transform$\frac{d^2y}{dt^2} + 9y = cos2t$, if $y(0) = 1$;$y\left(\frac{\pi}{2}\right) = -1$

Ans:$y(t) = \frac{1}{5}[cos2t + 4cos3t + 4sin3t]$