FOURIER SERIES

- A function *f* (*x*) is called a **periodic function** if *f* (*x*) is defined for all real *x*, except possibly at some points, and if there is some positive number *p*, called a **period** of *f* (*x*) such that

$$f(x+p) = f(x) \qquad\qquad for\ all\ x$$

- Familiar periodic functions are the *cosine, sine, tangent*, and *cotangent*. Examples of functions that are not periodic are $x, x^2, x^3, e^x, \cos hx\ etc.$ to mention just a few.

If *f(x)* has a period of *p* then it has also a period of *2p*

$$f(x+2p) = f\{(x+p)+p\} = f(x+p) = f(x)$$

Or in general we can write

$$f(x+np) = f(x)$$

- A Fourier series is defined as an expansion of a real function or representation of a real function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0, a_n, and\ b_n$ are constants, called the **Fourier coefficients** of the series. We see that each term has the period of 2π Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π.

- The **Fourier coefficients** of *f(x)*, given by the **Euler formulas**

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad\qquad n = 1, 2, 3, \dots.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad\qquad n = 1, 2, 3, \dots$$

The above Fourier series is given for period 2π. The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $= 2L$. Then we can introduce a new variable *v* such that ,$f(x)$ as a function of *v*, has period 2π.

- If we set

$$x = \frac{p}{2\pi} v \ \Rightarrow v = \frac{2\pi}{p} x \ \Rightarrow v = \frac{\pi}{L} x$$

This means $v = \pm\pi$ corresponds to $x = \pm L$. This represent f, as function of v has a period of 2π. Hence the Fourier series is

$$f(v) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nv + \sum_{n=1}^{\infty} b_n \sin nv$$

- Now using $v = \frac{\pi}{L} x$ Fourier series for the period of *(-L, L)* is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \frac{\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin n \frac{\pi}{L} x$$

This is Fourier series we obtain for a function of *f(x)* period $2L$ the Fourier series.

The coefficient is given by

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t) dt,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx,$$

- $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx,$

SOME IMPORTANT RESULTS

- $\int e^{\,ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx$
- $\int e^{\,ax} \cos bx\, dx = \frac{e^{\,ax}}{a^2+b^2}(a \cos bx + b\sin bx$
- $\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$
- $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$
- $\int_{-\infty}^{\infty} \frac{\sin mx}{(x-b)^2+a^2}\, dx = \frac{\pi}{a}\, e^{-am} \sin bm\,, \quad [m > 0]$

FOURIER INTEGRAL

- Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only.
- Since, of course, many problems involve functions that are ***nonperiodic*** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to “Fourier integrals.”

FOURIER INTEGRAL THEOREM

Fourier integral theorem states that $f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos u(t-x)dt\,du$

Proof. We know that Fourier series of a function f (x) in (-c, c) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^\infty b_n \sin\frac{n\pi x}{c}$$

Where a_0 , a_n *and* b_n are given by

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)dt,$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt,$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt,$$

Substituting the values of a_0 , a_n *and* b_n in above equation, we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^\infty \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt\cos\frac{n\pi x}{c} + \sum_{n=1}^\infty \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt, \sin\frac{n\pi x}{c}$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^\infty \frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c} + \sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^\infty \frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\left(\frac{n\pi t}{c} - \frac{n\pi x}{c}\right)\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^\infty \frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi}{c}(t-x)\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\left\{1 + 2\sum_{n=1}^\infty \cos\frac{n\pi}{c}(t-x)\right\}dt$$

Since cosine functions are even functions i.e., $\cos(-\theta) = \cos\theta$ the expression

$$\left\{1 + 2\sum_{n=1}^\infty \cos\frac{n\pi}{c}(t-x)\right\} = \sum_{n=-\infty}^\infty \cos\frac{n\pi}{c}(t-x)$$

We now let the parameter *c* approach infinity, transforming the finite interval [-*c, c*] into the infinite interval (-∞ to +∞). We set

$$\frac{n\pi}{c} = \omega, and\ \frac{\pi}{c} = d\omega \qquad with\ c \to \infty$$

Then we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\left\{\int_{-\infty}^\infty d\omega\cos\omega(t-x)\right\}dt$$

On simplifying

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(t-x)\,d\omega\,dt \qquad\qquad Proved$$

FOURIER SINE AND COSINE INTEGRALS

$f(x) = \frac{2}{\pi} \int_0^\infty \sin \omega x \, du \int_0^\infty f(t) \sin \omega t \, dt$ (Fourier Sine Integrals)

$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \, du \int_0^\infty f(t) \cos \omega t \, dt$ (Fourier Cosine Integrals)

Proof: We can write

$$\cos \omega(t - x) = \cos(\omega t - \omega x) = \cos \omega t \cos \omega x + \sin \omega t \sin \omega x$$

Using this expansion in Fourier integral theorem, we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \cos \omega(t - x) \, d\omega \, dt$$

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t)(\cos \omega t \cos \omega x + \sin \omega t \sin \omega x) d\omega \, dt$$

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t)(\cos \omega t \cos \omega x \, d\omega \, dt + \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \sin \omega t \sin \omega x \, d\omega \, dt$$

Now to solve the above equation, we have two different cases, using the following conditions

$$\int_{-a}^{a} f(x) dx = 0 \quad \text{for odd function}$$

And

$$\int_{-a}^{a} f(x) dx = 2 \int_0^a f(x) \, dx \quad \text{for even function}$$

Case I: when *f(t)* is even function: this means

$$\Rightarrow f(t) \sin \omega t \quad \text{is odd function and}$$
$$f(t) \cos \omega t \quad \text{is even function}$$

Hence

$$\frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \sin \omega t \sin \omega x \, d\omega \, dt = 0$$

And

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t)(\cos \omega t \cos \omega x \, d\omega \, dt = \frac{2}{\pi} \int_0^\infty \cos \omega x \, d\omega \int_{-\infty}^\infty f(t) \cos \omega t \, dt$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \, du \int_0^\infty f(t) \cos \omega t \, dt$$

Case II: If *f(t)* is odd function: this means

$$\Rightarrow f(t) \sin \omega t \quad \text{is even function and}$$
$$f(t) \cos \omega t \quad \text{is odd function}$$

Hence

$$\frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \omega t \cos \omega x \, d\omega \, dt = 0$$

And

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \sin \omega t \sin \omega x \, d\omega \, dt = \frac{2}{\pi} \int_{0}^\infty \sin \omega x \, d\omega \int_{-\infty}^\infty f(t) \sin \omega t \, dt$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \omega x \, du \int_0^\infty f(t) \sin \omega t \, dt$$

This is known as Fourier sine integral.

FOURIER'S COMPLEX INTEGRALS

We know from Fourier integral theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos \omega(t-x)\, d\omega\, dt$$

Now adding

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)dt \int_{-\infty}^{\infty} \sin \omega(t-x)\, d\omega = 0$$

Since

$$\int_{-\infty}^{\infty} \sin\, \omega(t-x) d\omega = 0 \qquad \textit{because of odd function}$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos \omega(t-x)\, d\omega\, dt + \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)dt \int_{-\infty}^{\infty} \sin \omega(t-x)\, d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, dt \left[\int_{-\infty}^{\infty} \cos \omega(t-x) + i \sin \omega(t-x)\right] \, d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, dt [\int_{-\infty}^{\infty} e^{i\omega(t-x)}\,]\ d\omega$$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\, d\omega \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt}$$

This relation is known as Fourier's complex Integral.

Example 1. Express the following function

$$f(x) = \begin{cases} 1 & when\ \ x \le 1 \\ 0 & when\ \ x > 1 \end{cases}$$

as a Fourier integral. Hence evaluate

$$\int_0^{\infty} \frac{\sin u \cos ux}{u} du$$

Solution: we know the Fourier Integral theorem, the Fourier Integral of a function $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)cos\ \omega(t-x)\, d\omega\, dt$$

Using $\omega = u$ we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)cos\ u(t-x)\, du\, dt$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1} cos\ u(t-x)\, dt\, du \qquad since\ f(t) = 1$$

Now integrating w.r.t. *t* we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin u(t-x)}{u}\right]_{-1}^{1} du$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin u(1-x) + \sin u(1+x)}{u}\right] du$$

Now using $\sin C + sinD = 2\, sin\frac{C+D}{2} cos\frac{C-D}{2}$ and solving it we will get

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin u \cos ux}{u} du$$

We can rewrite this

$$\int_0^{\infty}\frac{\sin u \cos ux}{u} du = \frac{\pi}{2} f(x)$$

$$\int_0^{\infty}\frac{\sin u \cos ux}{u} du = \begin{cases} \frac{\pi}{2} \times 1 = \frac{\pi}{2}, & for\ x < 1 \\ \frac{\pi}{2} \times 0 = 0, & for\ x > 1 \end{cases}$$

For *x=1*, which is a point of discontinuity of *f(x)*, value of integral $= \frac{\frac{\pi}{2}+0}{2} = \frac{\pi}{4}$

FOURIER TRANSFORMS

From the Fourier complex integral we know that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\, d\omega \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

We can rewrite the above expression as follows using $\omega = s$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, ds\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]$$

Now using $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = F(s)$ in above equation, we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, F(s) ds$$

Where $F(s)$ is called the Fourier Transform of $f(x)$.

And $f(x)$ is called the Inverse Fourier transform of $F(s)$.

Thus , we obtain the definition of Fourier transform is

$$\boldsymbol{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{ist} dt}$$

$$\boldsymbol{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}. F(s) ds}$$

FOURIER SINE TRANSFORMS

We know that from Fourier sine integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx \, ds \int_0^\infty f(t) \sin st \, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \, ds \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t) \sin st \, dt\right]$$

Now putting $F(s) = \sqrt{\frac{2}{\pi}\int_0^\infty f(t) \sin st \, dt}$

We have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx \, ds \, F(s)$$

In above equation $F(s)$ is called Fourier Sine transform of $f(x)$

$$\boldsymbol{F(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t) \sin st \, dt}$$

And $f(x)$ given below is known as inverse Fourier Sine transform of $F(s)$

$$\boldsymbol{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(s) \sin sx \, ds}$$

FOURIER COSINE TRANSFORM

From Fourier cosine integral we know that

$$f(x) = \frac{2}{\pi}\int_0^\infty cos\,\omega x\,du \int_0^\infty f(t)\,cos\,\omega t\,dt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,ds\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt\right]$$

Now putting $F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,ds\,F(s)$$

In above equation $F(s)$ is called Fourier cosine transform of $f(x)$

$$\boldsymbol{F(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt}$$

And $f(x)$ given below is known as inverse Fourier cosine transform of $F(s)$

$$\boldsymbol{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,F(s)ds}$$

Example 2: Find the Fourier transform of e^{-ax^2} , where a>0.

Solution : The Fourier transform of f(x):

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

Hence

$$F\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2+isx} dx$$

$$\Rightarrow F\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2-\frac{s^2}{4a}+isx+\frac{s^2}{4a}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{is}{2\sqrt{a}}\right)^2-\frac{s^2}{4a}}\, dx$$

$$\Rightarrow F\left\{e^{-ax^2}\right\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{is}{2\sqrt{a}}\right)^2}\, dx$$

$Putting\ x\sqrt{a} - \frac{is}{2\sqrt{a}} = u \Rightarrow dx = \frac{du}{\sqrt{a}}$ in above expression we get,

$$\Rightarrow F\left\{e^{-ax^2}\right\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-u^2}\, du \qquad \left[since \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\right]$$

$$\Rightarrow F\left\{e^{-ax^2}\right\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi a}} \sqrt{\pi} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2a}} \qquad Ans.$$

Example 3: Find the Fourier transform of

$$f(x) = \begin{cases} 2 & for\ \ |x| < a \\ 0 & for\ \ |x| > a \end{cases}$$

Solution: We know that the Fourier transform of a function is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Using the given value of *f(x)* we get,

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 2e^{isx} dx = \frac{2}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx =$$

$$F\{f(x)\} = \frac{2}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{2}{\sqrt{2\pi}\, is} \left[e^{ias} - e^{-ias}\right] = \frac{4}{\sqrt{2\pi}\, s} \frac{\left[e^{ias} - e^{-ias}\right]}{2i}$$

$$F\{f(x)\} = \frac{4}{\sqrt{2\pi}\, s} \sin as = 2\sqrt{\frac{2}{\pi}} \frac{\sin sa}{s} \qquad Ans.$$

Example 4: Find Fourier Sine transform of $\frac{1}{x}$.

Solution: We have to find the Fourier sine transform of $f(x) = \frac{1}{x}$

We know that from Fourier sine transform

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx$$

Now using the value of $f(x) = \frac{1}{x}$, we get,

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{1}{x}\sin sx\, dx$$

$$now\ using\ sx = t \Rightarrow dx = \frac{dt}{s}$$

We get $= \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\sin t}{t}\, dt = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) \quad \Rightarrow since\ \int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$

Hence $F_s\,[f(x)] = \sqrt{\frac{\pi}{2}} \quad Ans.$

Example 5: Find the Fourier Sine Transform of e^{-ax}.

Solution: Here, $f(x) = e^{-ax}$.

The Fourier sine transform of $f(x)$:

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

On putting the value of $f(x)$ in (1), we get

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

On Integrating by parts, we get

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}[-a\, sinsx - s\, cos\, sx]\right]_0^\infty$$

$$using \left[\int_0^\infty e^{\,ax} \sin bx\, dx = \frac{e^{\,ax}}{a^2+b^2}(a \sin bx - bcos\, bx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{a^2+s^2}\,(-s)\right] = \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right) \qquad Ans.$$

Example 6: Find the Fourier Cosine Transform of $f(x) = 5e^{-2x} + 2e^{-5x}$

Solution: The Fourier Cosine Transform of $f(x)$ is given by

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx$$

Putting the value of $f(x)$, we get

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty (5e^{-2x} + 2e^{-5x})\cos sx\, dx$$

$$= 5\int_0^\infty e^{-2x}\cos sx\, dx + 2\int_0^\infty e^{-5x}\cos sx\, dx$$

$$using\ \left[\int_0^\infty e^{\,ax}\, cos\, bx\, dx = \frac{e^{\,ax}}{a^2+b^2}(a\, cos\, bx + b\sin bx\right]$$

$$= 5\left[\frac{e^{-2x}}{(-2)^2+s^2}\ (-2\cos sx + s\sin sx\right]_0^\infty + 2\left[\frac{e^{-5x}}{(-5)^2+s^2}\ (-5\cos sx + s\sin sx\right]_0^\infty$$

$$= 5\left[0 - \frac{1}{4+s^2}(-2)\right] + 2\left[0 - \frac{1}{25+s^2}(-5)\right] = 5\left(\frac{2}{s^2+4}\right) + 2\left(\frac{5}{s^2+25}\right)$$

$$= 10\left(\frac{1}{s^2+4} + \frac{1}{s^2+25}\right) \qquad Ans.$$

PROPERTIES OF FOURIER TRANSFORMS

9.12.1 LINEAR PROPERTY: If $F_1(s)$ and $F_2(s)$ are Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively then

$$F[af_1(x) + b\, f_2(x)] = a\, F_1(s) + b\, F_2(s) \qquad \textit{where a and b are constants.}$$

Proof: we know from the definition of Fourier transform

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

We can write

$$F_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{isx}\, dx$$

And

$$F_2(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x) e^{isx}\, dx$$

Now

$$F[af_1(x) + b\, f_2(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af_1(x) + b\, f_2(x)] e^{isx}\, dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{isx}\, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x)] e^{isx}\, dx$$

$$\Rightarrow F[af_1(x) + b\, f_2(x)] = a\, F_1(s) + b\, F_2(s) \qquad \textit{Proved}$$

CHANGE OF SCALE PROPERTY

We know that Fourier transform equation is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Proof: we know

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\Rightarrow F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx \qquad \left[now\ put\ ax = t \Rightarrow dx = \frac{dt}{a}\right]$$

We have

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\frac{s}{a}t} \frac{dt}{a} = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} dt$$

$$\Rightarrow F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad Proved$$

SHIFTING PROPERTY

The Fourier transform equation is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Then

$$F\{f(x-a)\} = e^{isa} \; F(s)$$

Proof: Given

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

then

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

$$Put \; (x-a) = u \;\Rightarrow x = u + a \;\; and \; dx = du$$

We have

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{is(u+a)} du = e^{isa} \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{isu} du$$

$$\Rightarrow F\{f(x-a)\} = e^{isa} \; F(s) \qquad\qquad Proved$$