Inner Product spaces

Contents: Inner product, Norm, Distance, Inner product spaces, Orthogonal and Orthonormal basis, Gram-Schmidt orthogonalization, Single value decomposition for square matrices.

Inner Product Spaces: In the discussion of vector spaces, F is considered as an arbitrary field. In this module we will consider vector space V(F) with F as the field of real numbers or as the field of complex numbers.

We are defining inner product of vector space V(F). First of all, we state some important properties of complex numbers below.

The set of complex numbers,

$\mathbb{C} = \{x + iy \mid x , y \in \mathbb{R} \; and \; i = \sqrt{-1}\}$

If z is a complex number, then $z = x + iy$ where $x, y \in R$ and $i^2 = -1.$ Here 'x' is called real part of z and 'y' is called imaginary part of z.

We write $x = Re\, z \; and \; y = Im\, z.$

The modulus of $z = x + iy$ denoted by $|z|$ is the non-negative real number $\sqrt{x^2 + y^2}$.

The conjugate of $z = x + iy$, denoted by $\bar{z}$ is the complex number $x - iy.$

- $z = 0 \Rightarrow x + iy = 0 \Rightarrow x = 0 , y = 0.$
- $z_1 = x_1 + iy_1 and\; z_2 = x_2 + iy_2 \; then\; z_1 = z_2 \Leftrightarrow x_1 = x_2 and\; y_1 = y_2.$
- If $z_1 = x_1 + iy_1 and\; z_2 = x_2 + iy_2$ then $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2).$
- If $z_1 = x_1 + iy_1 and\; z_2 = x_2 + iy_2$ then $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.
- If $z \in \mathbb{C}$ then $\bar{\bar{z}} = z$ and $|z| = |\bar{z}|.$
- $z + \bar{z} = 2\, Re\, z$ and $z - \bar{z} = 2\, Im\, z.$
- If $z = x + iy$ then $z\,\bar{z} = x^2 + y^2 = |z|^2.$

- If $z_1, z_2 \in \mathbb{C}$ then $|z_1 + z_2| \leq |z_1| + |z_2|$.
- If $z_1, z_2 \in \mathbb{C}$ then $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1} \pm \overline{z_2}$

Inner Product Space: An inner product on a vector space V is an operation that assigns to every pair of vectors u and v in V to a real number $< u, v >$ such that the following properties hold for al vectors u, v and w in V and the scalar c:

1) $< u, v > = < v, u >$ (Symmetry)
2) $< u, v + w > =< u, v > + < u, w >$ (Distributivity)
3) $< cu, v > = c < u, v >$
4) $< u, u > \geq 0$ and $< u, u \geq 0$ iff $u = 0$ (Positive Definite Property)

A Vector Space with an inner product is called an inner product space.

Example: $\mathbb{R}^n$ with usual dot product. Here V= $\mathbb{R}^n$ with the usual dot product $< u, v > = u.v = u_1v_1 + u_2v_2 + \cdots u_nv_n$ is an inner product space.

Example 1: Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$, Show that

$< u, v \geq 2u_1v_1 + 3u_2v_2$ defines an inner product.

Solution:

Here, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

I) $< u, v \geq 2u_1 v_1 + 3u_2v_2 = 2v_1u_1 + 2v_2u_2 =< v, u >$

II) Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Then $< u, v + w >= 2u_1(v_1 + w_1) + 3u_2(v_2 + w_2)$

$= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2$

$= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 2u_2w_2)$

Hence, we get $< u, v + w > = < u, v > + < u, w >$. Which proves the 2nd property.

III) If c is a scalar, then,

$$< cu, v > = 2(cu_1)v_1 + 3(cu_2)v_2$$

$$= c(2u_1v_1) + c(3u_2v_2)$$
$$= c(2u_1v_1 + 3u_2v_2) = c < u, v >$$

Which proves the 3rd property.

IV)$< u, u >= 2u_1u_1 + 3u_2u_2 = 2u_1^2 + 3u_2^2 \geq 0$ and it is clear that $< u, u > = 2u_1^2 + 3u_2^2 = 0 \Rightarrow u_1 = u_2 = 0$. This verifies the property 4th.

Hence $< u, v >= 2u_1v_1 + 3u_2v_2$ defines an inner product.

Example 2: let A be a symmetric, positive definite $n \times n$ matrix and let u, v be vectors in $\mathbb{R}^n$, Show that $< u, v >= u^T Av$ defines an inner product.

Solution:

I)$< u, v >= u^T Av = uAv = Avu$

$$= A^T vu$$

$$= (v^T A)^T u$$

$$= v^T Au$$

So $< u, v >=< v, u >$.

II) Again $< u, v + w >= u^T A(v + w)$

$$= u^T Av + u^T Aw$$

$$=< u, v > +< u, w >$$

III) If c is a scalar, then $< cu, v > = (cu)^T Av$

$$= c(u^T Av)$$

So, $< cu, v > = c < u, v >$

IV) $< u, u >= u^T A\, u > 0, \forall u \neq 0$

So $< u, u >= u^T Au = 0 \Leftrightarrow u = 0$

Hence $< u, v > = u^T A\, v$ defines an inner product.

Example 3: In P_2, let $p(x) = a_0 + a_1x + a_2x^2, q(x) = b_0 + b_1x + b_2x^2$ .Show that $< p(x), q(x) >= a_0b_0 + a_1b_1 + a_2b_2$ defines an inner product on P_2.(Try this)

Example 4: Let f and g be in $c[a, b]$,the vector space of all continuous functions on the closed interval $[a, b]$. Show that $< f, g >= \int_a^b f(x)g(x)dx$ defines an inner product on $c[a, b]$. (Try this)

Example 5: Suppose we consider $P[0,1]$, the vector space of all polynomials on the interval $[0,1]$. Then using the inner product of example 4, we have $f(x) = x^2 and\ g(x) = 1 + x$.

$$< x^2, 1 + x >= \int_0^1 x^2(1 + x)dx$$

$$= \int_0^1 (x^2 + x^3)dx$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Consider the inner product space $C[0,1]$. Compute the following inner products:

a) $< 1, x >$ b) $< x, x^2 >$ c) $< 1 + x, 2 + x^2 >$

Solution:

a) $< 1, x >= \int_0^1 1.xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

b)$< x, x^2 >= \int_0^1 x^3dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

c)$< 1 + x, 2 + x^2 > = \int_0^1 (1 + x)(2 + x^2)dx$

$$= \int_0^1 (2 + x^2 + 2x + x^3)dx$$

$$= \left[2x + \frac{x^3}{3} + x^2 + \frac{x^4}{4}\right]_0^1$$

$$= 2 + \frac{1}{3} + 1 + \frac{1}{4} = \frac{43}{12}$$

Norm: Let u be a vector in an inner product space. Then the norm of u is defined as $||u|| = \sqrt{< u, u >}$.

Problems:

1) Calculate $||\alpha||$ if I) $\alpha = (1, -2,5)$ II) $\alpha = (4,1,8)$

Solution:

I) Here $\alpha = (1, -2,5)$, So $||\alpha|| = \sqrt{1^2 + (-2)^2 + (5)^2} = \sqrt{30}$

II) Here $\alpha = (4,1,8)$, So $||\alpha|| = \sqrt{(4^2) + (1)^2 + (8)^2} = 9$

2) Calculate the norm of $f(x) = x^2$ in $C[-1,1]$.

Solution:

In the vector space $C[-1,1]$, the inner product is defined as

$$< f, g > = \int_{-1}^{1} f(x)g(x)dx$$

So $< f, f > = \int_{-1}^{1} [f(x)]^2 dx = \int_{-1}^{1} x^4 dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$

Hence $||f|| = \sqrt{< f, f >} = \sqrt{\frac{2}{5}}$

3)Consider the inner product on $P[0,1]$. If f(x)=x then finds $||f||$. (Try this)

Unit Vector or Normalized: A vector u in an inner product space is called normalized or unit vector if $||u|| = 1$. So if v is any non-zero vector in an inner product space, then $u = \frac{v}{||v||}$.

Angle between two vectors: let u, v be two vectors in an inner product space. Then angle between u and v is defined by $\cos\theta = \frac{<u,v>}{||u||\ ||v||}$, $0 \leq \theta \leq \pi$.

Problems:

1) Find the angle between 1 and x^2 in $C[-1,1]$.

Solution:

Here $u = 1\ and\ v = x^2$

Then $< u, v > = < 1, x^2 > = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

Now $||u|| = \sqrt{< u, u >} = \sqrt{< 1,1 >} = \sqrt{2}$

As $< 1,1 > = \int_{-1}^{1} 1dx = 2$

Also $< x^2, x^2 > = \int_{-1}^{1} x^4 dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$

Then $||v|| = \sqrt{\frac{2}{5}}$

Now let θ be the angle between u and v.

So $\cos\theta = \frac{<u,v>}{||u||||v||} = \frac{\frac{2}{3}}{\sqrt{2}\times\sqrt{\frac{2}{5}}} = \frac{\sqrt{5}}{3}$

Hence $\theta = \cos^{-1}\left(\frac{\sqrt{5}}{3}\right) \cong 0.7297\ radians\ or\ 41.81°$

2) Find the angle between x and x^2 in $C[-1,1]$. (Try this)

3) If $\alpha = (4,1,8)\ and\ \beta = (1,0,-3)$ are two vectors in $\mathbb{R}^3$. Find the angle between $\alpha\ and\ \beta$.

Solution:

Here $\alpha = (4,1,8) and\ \beta = (1,0,-3)$

Then $||\alpha|| = \sqrt{4^2 + 1^2 + 8^2} = 9\ and\ ||\beta|| = \sqrt{1^2 + 0^2 + (-3)^2} = \sqrt{10}$

Now $< \alpha, \beta > = < (4,1,8), (1,0,-3) > = 4 - 24 = -20$

Hence $\cos\theta = -\frac{20}{9\times\sqrt{10}}$

Distance: Let u and v be vectors in an inner product space V. Then the distance between u and v is d (u, v) and is defined as $d(u, v) = ||u - v||$.

Orthogonal: Let u and v be two vectors in an inner product space V. If u and v are orthogonal then $< u, v > = 0$.

Problems:

1) Consider the inner product on $P[0,1]$. If $f(x) = x \; and \; g(x) = 3x - 2$ then find I)$d(f,g)$ II) $< f, g >$

Solution:

I) $d(f,g) = \|f - g\| = \sqrt{< f - g, f - g >}$

Here $f(x) = x \; and \; g(x) = 3x - 2$

So $f(x) - g(x) = -2x + 2 = 2(1 - x)$

Now $< f - g, f - g >= \int_0^1 (f(x) - g(x))^2 dx$

$$= 4\int_0^1 (1 - x)^2 dx$$

$$= 4\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{4}{3}$$

Hence $d(f,g) = \sqrt{\frac{4}{3}}$

II) $< f, g >= \int_0^1 x(3x - 2)dx$

$$= \int_0^1 (3x^2 - 2x)dx$$

$$= \left[\frac{3x^3}{3} - \frac{2x^2}{2}\right]_0^1 = 1 - 1 = 0$$

So, f and g are orthogonal.

2) Find a unit vector orthogonal to $(4, 2, 3)$ in $\mathbb{R}^3$.

Solution:

Let $u = (4,2,3)$ and $v = (x_1, x_2, x_3)$ be orthogonal to α.

Then $< u, v > = 0 \Rightarrow 4x_1 + 2x_2 + 3x_3 = 0$

Any solution of this equation gives a vector orthogonal to u .

By inspection a solution is $x_1 = 2, x_2 = -1 \; and \; x_3 = -2$.

So, we can take $v = (2, -1, -2)$ then $||v|| = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3.$

Hence unit vector orthogonal u $= \frac{v}{||v||} = \frac{1}{3}(2, -1, -2)$

3)Let $\boldsymbol{u} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{5} \end{bmatrix}$ $\boldsymbol{and}\ \boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{-1} \\ \mathbf{2} \end{bmatrix}$ then find I) d (u, v) II) $||\mathbf{u}||$ III) $||\mathbf{v}||$

Solution:

I) $d(u, v) = \left\| \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right\| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$

II) $||u|| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$

III) $||v|| = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$

4) Let $\boldsymbol{u} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-2} \end{bmatrix}$ $\boldsymbol{and}\ \boldsymbol{v} = \begin{bmatrix} \mathbf{4} \\ \mathbf{3} \end{bmatrix}$. Compute $\boldsymbol{I)}\ \boldsymbol{d(u, v)}\ \boldsymbol{II)} ||\boldsymbol{u}||\ \boldsymbol{III)}\ ||\boldsymbol{v}||$.
(Try this)

5) Are the vectors $\boldsymbol{u_1} = \begin{bmatrix} \mathbf{8} \\ \mathbf{-4} \end{bmatrix}$ $\boldsymbol{and}\ \boldsymbol{u_2} = \begin{bmatrix} \mathbf{3} \\ \mathbf{5} \end{bmatrix}$ orthogonal?

Solution:

Here $u_1 . u_2 = 24 - 20 = 4 \neq 0$, So the vectors are not orthogonal.

6) Are the vectors $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{-5} \end{bmatrix}$ $\boldsymbol{and}\ \boldsymbol{v_2} = \begin{bmatrix} \mathbf{2} \\ \mathbf{-4} \\ \mathbf{-2} \end{bmatrix}$ orthogonal? (Try this)

<u>Orthonormal Set:</u> Let S be a non-empty set of an inner product space V(F). The set S is called an orthogonal set if,

I) $||u_i|| = 1$ for each $u_i \in S$.

II)$< u_i, u_j >= 0$ for $u_i, u_j \in S, i \neq j$.

Note 1: $S \subseteq V$ is an orthonormal set ⇔ S contains mutually orthogonal unit vectors.

Note 2: An orthonormal set is an orthogonal set with the property that each vector is of length 1.

Note 3: An orthogonal set does not contain zero vector.

Example: The standard basis of the inner product space $\mathbb{R}^3$ or $V_3(\mathbb{R})$ is the set $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ then $||e_1|| = 1, ||e_2|| = 1$ and $||e_3|| = 1$ also $< e_1, e_2 >= 0, < e_2, e_3 >= 0$ and $< e_3, e_1 >= 0$. Thus the standard basis form an orthonormal set.

Problem 1: Prove that $S = \{ \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right), \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)\}$ is an orthonormal set in $\mathbb{R}^3$ with standard inner product.

Solution:

Let $u_1 = \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right), u_2 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ and $u_3 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.

So here $||u_1|| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = 1$

Similarly, $||u_2|| = 1$ and $||u_3|| = 1$.

Again $< u_1, u_2 > = \frac{1}{3}.\frac{2}{3} + \left(-\frac{2}{3}\right).\left(-\frac{1}{3}\right) + \left(-\frac{2}{3}\right).\left(\frac{2}{3}\right) = 0$.

Similarly, $< u_2, u_3 >= 0$ and $< u_3, u_1 >= 0$.

Hence S is an orthonormal set.

Problem 2: Consider $\mathbb{R}^3$ as an inner product space with usual dot product. For each of the following bases of $\mathbb{R}^3$, State whether it is orthonormal, orthogonal or neither.

a) $\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$

b) $\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\}$

c) $\{\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \end{bmatrix}\}$

Solution:

a) Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Therefore $||u_1|| = \sqrt{2}, ||u_2|| = \sqrt{2}$ and $||u_3|| = 1$.

Hence S is not an orthonormal set.

Now $< u_1, u_2 > = 1.0 + 0.1 + 1.1 = 1 \neq 0$

Similarly, $< u_2, u_3 >= 1 \neq 0$ and $< u_3, u_1 > = 1 \neq 0$.

Hence S is not an orthogonal set.

b) Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} and\ u_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

Here $||u_1|| = \sqrt{5}, ||u_2|| = 1\ and\ ||u_3|| = \sqrt{5}$.

Again $< u_1, u_2 > = 1.0 + 0.1 + 2.0 = 0$

Similarly, $< u_2, u_3 >= 0\ and < u_3, u_1 >= 0$.

Hence S is not an orthonormal, but it is orthogonal set.

c) Similarly try this. (Answer: S is both orthogonal and orthonormal)

Practice problem: Prove that the set

$$S = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) \right\}$$

is an orthonormal set in $\mathbb{R}^3(\mathbb{R})$ with standard inner product.

Orthonormal Basis: A basis of an inner product space V(F) which is also orthonormal is called orthonormal basis of the inner product space.

Example 1: The basis $S = \{(1,0), (0,1)\}$ of the inner product space $\mathbb{R}^2(\mathbb{R})$ is also orthonormal. So S is an orthonormal basis of $\mathbb{R}^2(\mathbb{R})$.

Example 2: The basis $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of the inner product space $\mathbb{R}^3(\mathbb{R})$ is also orthonormal. So S is an orthonormal basis of $\mathbb{R}^3(\mathbb{R})$.

Gram-Schmidt orthogonalization Process:

Working method for finding orthogonal basis:

Let $\{v_1, v_2, \dots, v_k\}$ be linearly independent basis of V(F). Define vectors $u_1, u_2, \dots u_k$ as follows.

$$u_1 = v_1, u_2 = v_2 - \frac{< u_1, v_2 >}{< u_1, u_1 >} u_1$$

$$u_3 = v_3 - \frac{< u_1, v_3 >}{< u_1, u_1 >} u_1 - \frac{< u_2, v_3 >}{< u_2, u_2 >} u_2$$

……

Then $\{u_1, u_2, \dots u_k\}$ is an orthogonal basis.

Working method for finding orthonormal basis:

Let $\{v_1, v_2, \dots, v_k\}$ be a given basis of a finite dimensional inner product space V(F). The vectors $u_1, u_2, \dots u_k$ of orthonormal basis of V(F) are given by,

$$u_1 = \frac{v_1}{||v_1||}, u_2 = \frac{w_2}{||w_2||} \; where \; w_2 = v_2 - < v_2, u_1 > u_1$$

$$u_3 = \frac{w_3}{||w_3||} \; where \; w_3 = v_3 - < v_3, u_1 > u_1 - < v_3, u_2 > u_2$$

…….

Problem 1: In $\mathbb{R}^3$ with the usual dot product, find an orthogonal basis for the span

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution:

Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} and \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Now $u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = v_2 - \frac{<u_1, v_2>}{<u_1, u_1>} u_1$

Now $< u_1, v_2 >= 1.0 + 1.1 + 1.0 = 1, < u_1, u_1 >= 1.1 + 1.1 + 0.0 = 2$

So $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Hence orthogonal basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

Problem 2: In $\mathbb{R}^4$ using Gram-Schmidt orthogonalization process, find an orthogonal basis for the span

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Solution:

Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \; and \; v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Now $u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = v_2 - \frac{<u_1,v_2>}{<u_1,u_1>} u_1$

Here $< u_1, v_2 >= 1.1 + 1.1 + 1.1 + 1.0 = 3$

And $< u_1, u_1 >= 1.1 + 1.1 + 1.1 + 1.1 = 4$

So $u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$.

Again $u_3 = v_3 - \frac{<u_1,v_3>}{<u_1,u_1>} u_1 - \frac{<u_2,v_3>}{<u_2,u_2>} u_2$

Now $< u_1, v_3 >= 1.1 + 1.1 + 1.0 + 1.0 = 2, < u_1, u_1 > = 4,$

$< u_2, v_3 >= \frac{1}{4}.1 + \frac{1}{4}.1 + \frac{1}{4}.0 + \left(-\frac{3}{4}\right).0 = \frac{1}{2}$ and

$$< u_2, u_2 >= \frac{1}{4}.\frac{1}{4} + \frac{1}{4}.\frac{1}{4} + \frac{1}{4}.\frac{1}{4} + \left(-\frac{3}{4}\right).\left(-\frac{3}{4}\right) = \frac{3}{4}$$

So $u_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{2}}{\frac{3}{4}} \times \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}.$

Problem 3: Consider the vector space P of polynomials with inner product

$$< p, q >= \int_{-1}^{1} p(x)q(x)dx$$

Using Gram-Schmidt procedure to find the orthogonal basis for the span $\{1, x , x^2, x^3\}$.

Solution:

Let$v_1 = 1 , v_2 = x , v_3 = x^2$ and $v_4 = x^3$.

Now by Gram-Schmidt process,

$$u_1 = v_1 = 1$$

Again, $u_2 = v_2 - \frac{<u_1,v_2>}{<u_1,u_1>} u_1$

$$< u_1, v_2 >= \int_{-1}^{1} 1.xdx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

$$< u_1, u_1 > = \int_{-1}^{1} 1.1dx = [x]_{-1}^{1} = 2$$

Hence $u_2 = x - \frac{0}{2}.1 = x$

Again $u_3 = v_3 - \frac{<u_1,v_3>}{<u_1,u_1>} u_1 - \frac{<u_2,v_3>}{<u_2,u_2>} u_2$

$$< u_1, v_3 >= \int_{-1}^{1} 1.x^2dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

$$< u_2, v_3 >= \int_{-1}^{1} x.x^2 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$< u_2, u_2 >= \int_{-1}^{1} x.x dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

So $u_3 = x^2 - \frac{\frac{2}{3}}{2}.1 - 0 = x^2 - \frac{1}{3}$

Again $u_4 = v_4 - \frac{<u_1,v_4>}{<u_1,u_1>} u_1 - \frac{<u_2,v_4>}{<u_2,u_2>} u_2 - \frac{<u_3,v_4>}{<u_3,u_3>} u_3$

$$< u_1, v_4 >= \int_{-1}^{1} 1.x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$< u_2, v_4 >= \int_{-1}^{1} x.x^3 dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$$

$$< u_3, v_4 \geq \int_{-1}^{1} (x^2 - \frac{1}{3}).x^3 dx = \left[\frac{x^6}{6} - \frac{1}{3}\left(\frac{x^4}{4}\right)\right]_{-1}^{1} = 0$$

$$< u_3, u_3 >= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2x^2}{3} + \frac{1}{9}\right) dx = \left[\frac{x^5}{5} - \frac{2}{3}\left(\frac{x^3}{3}\right) + \frac{1}{9}x\right]_{-1}^{1} = \frac{8}{45}$$

Hence $u_4 = x^3 - \frac{0}{2}.1 - \frac{\frac{2}{5}}{\frac{2}{3}}.x - 0 = x^3 - \frac{3}{5}x$

Thus, we obtain the orthogonal basis $\{u_1, u_2, u_3, u_4\}$ $i.e. \{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x\}$

<u>Problems on orthonormal basis:</u>

Problem 1: Given $\{(2,1,3), (1,2,3), (1,1,1)\}$ is a basis of $\mathbb{R}^3$. Use Gram-Schmidt procedure to find an orthonormal basis.

Solution:

Let $v_1 = (2,1,3), v_2 = (1,2,3) and\ v_3 = (1,1,1)$.

By Gram-Schmidt procedure an orthonormal basis $\{u_1, u_2, u_3\}$ is given by

$$u_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{14}}(2,1,3)$$

As $||v_1|| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$

Again

$$u_2 = \frac{w_2}{||w_2||} \; where \; w_2 = v_2 - < v_2, u_1 > u_1$$

Here $< v_2, u_1 >= 1.\frac{2}{\sqrt{14}} + 2.\frac{1}{\sqrt{14}} + 3.\left(\frac{3}{\sqrt{14}}\right) = \frac{13}{\sqrt{14}}$

So $w_2 = (1,2,3) - \frac{13}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}(2,1,3)\right) = \left(-\frac{12}{14}, \frac{15}{14}, \frac{3}{14}\right)$

Also $||w_2||^2 = \frac{378}{196}$

Hence $u_2 = \frac{14}{\sqrt{378}}\left(-\frac{12}{14}, \frac{15}{14}, \frac{3}{14}\right) = \frac{1}{\sqrt{378}} \; (-12, 15, 3)$

Again

$$u_3 = \frac{w_3}{||w_3||} \; where \; w_3 = v_3 - < v_3, u_1 > u_1 - < v_3, u_2 > u_2$$

Now $< v_3, u_1 > = 1.\frac{2}{\sqrt{14}} + 1.\frac{1}{\sqrt{14}} + 1.\left(\frac{3}{\sqrt{14}}\right) = \frac{6}{\sqrt{14}}$

And $< v_3, u_2 > = 1.\left(-\frac{12}{\sqrt{378}}\right) + 1.\left(\frac{15}{\sqrt{378}}\right) + 1\left(\frac{3}{\sqrt{378}}\right) = \frac{6}{\sqrt{378}}$

So $w_3 = (1,1,1) - \frac{6}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}(2,1,3)\right) - \frac{6}{\sqrt{378}}\left(\frac{1}{\sqrt{378}} \; (-12, 15, 3)\right)$

$= \left(\frac{126}{378}, \frac{126}{378}, -\frac{126}{378}\right) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$

So $||w_3||^2 = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$

Hence $u_3 = \frac{w_3}{||w_3||} = \sqrt{3}\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

Hence orthogonal basis is $\left\{\frac{1}{\sqrt{14}}(2,1,3), \frac{1}{\sqrt{378}} \; (-12, 15, 3), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\}$

Practice Problem: Applying Gram-Schmidt process to obtain an orthonormal basis of $\mathbb{R}^3(\mathbb{R})$ from the basis.

a) $\{(1,0,1),(1,0,-1),(0,3,4)\}$ $[Ans{:}\left\{\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right),(0,1,0)\right\}$

b) $\{(1,1,0),(-1,1,0),(1,2,1)\}$ $\left\{\frac{1}{\sqrt{2}}(1,1,0),\frac{1}{\sqrt{2}}(-1,1,0),(0,0,1)\right\}$

Singular Value Decomposition:

Let A be an $m \times n$ matrix with rank r. Then there exist an $m \times n$ matrix for which the diagonal entries in D are the first singular value of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that,

$$A = U\,E\,V^T \; where \; E = \begin{pmatrix} D_{(r\times r)} & 0_{(n-r)\times r} \\ 0_{r\times(n-r)} & 0_{(n-r)\times(n-r)} \end{pmatrix} \ldots \text{(1)}$$

Any factorization $A = U\,E\,V^T$ with U and V orthogonal, E as in equation (1) and positive diagonal entries in 'D' is called a Singular Value Decomposition (SVD) of A.

Problem 1: Find a singular value decomposition $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Solution:

Given matrix is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Let us compute $A^TA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Now the eigen value of A^TA is given by ,

$$|A^TA - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^3 - 1(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)(1-2\lambda+\lambda^2-1) = 0$$

$$\Rightarrow (1-\lambda)\lambda(\lambda-2) = 0$$

$$\Rightarrow \lambda = 0,1,2$$

So, the eigen values are 0,1 and 2.

To find the eigen vector corresponding to the eigen value $\lambda = 2$,

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -x_1 + x_2 = 0, x_1 - x_2 = 0 \; and \; x_3 = 0$$

$$\Rightarrow x_1 = x_2 = k \; (let) \;\; and \;\; x_3 = 0$$

Hence, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

So, the eigen vector corresponding to the eigen value $\lambda = 2$ spanned by

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Again, to find the eigen vector corresponding to the eigen value $\lambda = 1$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_2 = 0, x_1 = 0 \; and \; x_3 = k \; (let)$$

Hence, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So, the eigen vector corresponding to the eigen value $\lambda = 1$ spanned by

$$\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Again, to find the eigen vector corresponding to the eigen value $\lambda = 0$,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 + x_2 = 0, x_1 + x_2 = 0 \; and \; x_3 = 0$$

$$\Rightarrow -x_1 = x_2 = k(let)and \; x_3 = 0$$

Hence, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

So, the eigen vector corresponding to the eigen value $\lambda = 1$ spanned by

$$\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Now check these three vectors $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $X_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ are orthogonal.

$$X_1^T X_2 = (1 \quad 1 \quad 0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

Similarly, $X_2^T X_3 = 0$ and $X_3^T X_1 = 0.$

So, the vectors are orthogonal, so we normalize them to obtain,

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \; and \; v_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

So, the singular values of A are $\sigma_1 = \sqrt{2}, \sigma_2 = \sqrt{1} = 1$ and $\sigma_3 = \sqrt{0} = 0.$

Thus $V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$

The matrix is the same size as A with D in the upper left corner and 0^s elsewhere.

$$D = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

To find U, we compute

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These vectors already form an orthogonal basis (standard basis) for $\mathbb{R}^2, so$ we have

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, the singular value decomposition of A is

$$A = UEV^T$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Practice problem:

1) Find the singular value decomposition of $A = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$.

Ans: $\begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

2) Find the singular value decomposition of $A = \begin{pmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{2} \\ \mathbf{2} & \mathbf{-2} \end{pmatrix}$.

Ans: $\begin{pmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{3} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$