CALCULUS AND TRANSFORMATION TECHNIQUES

Module 1

SINGLE VARIABLE CALCULUS:

1. Rolle's Theorem
2. Lagrange's mean value theorem
3. Cauchy's mean value theorem
4. Taylor's and Maclaurin's theorems with remainders (without proof) related problems.

Introduction

Closed Interval

An interval of the form $a \le x \le b$, that includes every point between 'a' & 'b' and also the end points, is called a closed interval and is denoted by $[a,b]$.

Open Interval

An interval of the form $a < x < b$, that includes every point between 'a' & 'b' but not the end points, is called an open interval and is denoted by (a,b).

Continuity

A real valued function $f(x)$ is said to be continuous at a point x_0, if $\underset{x \to 0}{Lim} f(x) = f(x_0)$. The function $f(x)$ is said to be continuous in an interval if it is continuous at every point in the interval.

Differentiability

A real valued function $f(x)$ is said to be differentiable at point x_0 if $\underset{x \to x_0}{Lim} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists uniquely and it is denoted by $f'(x_0)$. A real valued function f(x) is said to be differentiable in an interval if it is differentiable at every in the interval or if $\underset{h \to 0}{Lim} \dfrac{f(x+h) - f(x)}{h}$ exists uniquely. This is denoted by $f'(x)$. We say that either $f'(x)$ exists or $f(x)$ is differentiable.

Rolle's Theorem and its Geometrical Interpretation

Statement

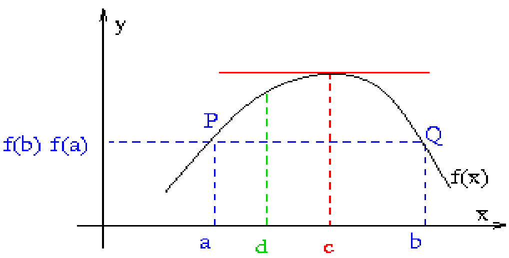
If a function $f(x)$ is defined in the interval $[a,b]$ such that

i. $f(x)$ is continuous in $[a,b]$
ii. $f(x)$ is differentiable in (a,b)
iii. $f(a) = f(b)$

Then there exists at least one point c in (a,b) that is $a<c<b$ such that $f'(c)=0$.

Geometrical Interpretation

- There are no breaks or gaps in between 'a' and 'b' for the given curve and including at the end points, hence the function is continuous in the $[a,b]$.
- Since a unique tangent can be drawn at each and every point in the interval except at the end points, the function is differentiable in the (a,b).
- At the end points 'a' and 'b' they are at the same height from the x-axis.



Conclusion

Therefore there exits at least one point 'c' (say) in between 'a' and 'b' such that the tangent at 'c' is parallel to . x.-axis.

Figure-1

Note: There may be exists more than one point in between $(a,\, b)$ at which $f'(x)$ vanishes.

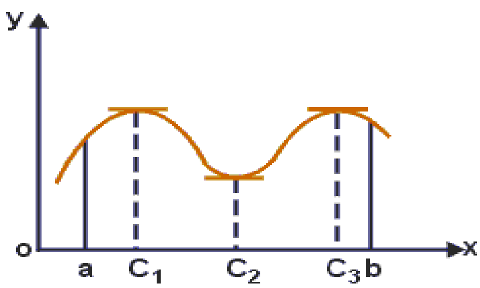


Figure-2

Problems on Rolle's Theorem

1. **Verify Rolle's theorem for the function** $f(x)=x^3-3x^2-x+3$ **in** $[1,3]$

Solution : Since $f(x)$ is a polynomial, it is continuous in $[1,3]$

$f'(x)=3x^2-6x-1$ is defined for all x in $(1,3)$

$\therefore f(x)$ is differentiable

$f(1)=f(3)=0$

All three conditions of Rolle's Theorem are satisfied,

Therefore $\exists\, c$ such that $f'(c)=0$

$$\Rightarrow 3c^2-6c-1=0 \Rightarrow c=\frac{6\pm\sqrt{36+12}}{6}=\frac{6\pm4\sqrt{3}}{6}=1\pm\frac{2\sqrt{3}}{3}=1\pm\frac{2}{\sqrt{3}}\approx 2.15\in(1,3)$$

$\therefore$ Rolle's Theorem is verified.

2. **Verify Rolles' Theorem for the function**

$$f(x) = \log\left\{\frac{(x^2+ab)}{x(a+b)}\right\} \; in \; [a,b]; \; b > a > 0.$$

Solution: Since $f(x)$ is a standard logarithmic function, it is continuous in $[a,b]$

$$f(x) = \log(x^2+ab) - \log(x) - \log(a+b),$$

$\Rightarrow f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$ is defined for all in (a,b)

$\therefore f(x)$ is differentiable

$$f(a) = \log\frac{(a^2+ab)}{a^2+ab} = \log 1 = 0 \text{ and } f(b) = \log\frac{(b^2+ab)}{ba+b^2} = \log 1 = 0$$

$\Rightarrow f(a) = f(b)$

All conditions of Rolles' Theorem are satisfied.

$\therefore \exists c$ such that $\Rightarrow f'(c) = 0 \; x$

$$\Rightarrow \frac{2c}{c^2+ab} - \frac{1}{c} = 0 \Rightarrow 2c^2 = c^2 + ab$$

$$\Rightarrow c = \pm\sqrt{ab}$$

But $c = +\sqrt{ab} \in (a,b)$

Hence Rolles' Theorem is verified.

3. **Verify the Rolle's theorem for the function** $f(x) = (x-a)^p (x-b)^q$ **in** $[a \; b]$

Solution : Since $f(x)$ is a polynomial, it is continuous in $[a,b]$

$f'(x) = (x-a)^{p-1}(x-b)^{q-1}\left[(q+p)x - (qa+pb)\right]$ is defined for all x in (a,b)

$\therefore f(x)$ Is differentiable, $f(a) = f(b) = 0$

Hence all the conditions of Rolle's Theorem are satisfied.

$\therefore \exists c$ such that $\Rightarrow f'(c) = 0$

$$\Rightarrow (c-a)^{p-1}(c-b)^{q-1}\left[(q+p)c - (qa+pb)\right] = 0$$

$$\Rightarrow \left[(q+p)c - (qa+pb)\right] = 0 = 0,$$

$$\Rightarrow c = \frac{pb+qa}{p+q} \in (a,b);$$

Thus the Rolle 's Theorem is verified.

4. **Verify Rolle's theorem for the function** $f(x) = e^x(\sin x - \cos x)$ **in** $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Solution :

Since $f(x)$ is a combination of standard functions, it is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$f'(x)=2e^{x}\sin x$ is defined for all x in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

$\therefore f(x)$ is differentiable in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

$$f\left(\frac{\pi}{4}\right)=e^{\frac{\pi}{4}}\left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)=e^{\pi/4}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)=0$$

$$f\left(\frac{5\pi}{4}\right)=e^{\frac{5\pi}{4}}\left(\sin\frac{5\pi}{4}-\cos\frac{5\pi}{4}\right)=e^{5\pi/4}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=0$$

Therefore $f\left(\frac{\pi}{4}\right)=0=f\left(\frac{5\pi}{4}\right)$

Hence all the conditions of Rolle's theorem as satisfied.

$\therefore\ \exists\, c$ such that $\Rightarrow f'(c)=0 \Rightarrow e^{c}\sin c=0\Rightarrow \sin c=0$

$\Rightarrow c=n\pi$, where $n=0,1,2,3.....$ $c=\pi\in\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$;

Thus Rolle's theorem is satisfied.

5. Verify Rolle's theorem for the function $f(x)=\dfrac{\sin 2x}{e^{2x}}$ in $\left[0,\frac{\pi}{2}\right]$

Solution :

Since $f(x)$ is a combination of standard functions, it is continuous in $\left[0,\frac{\pi}{2}\right]$

$f'(x)=\dfrac{e^{2x}2\cos 2x-\sin 2x\,2e^{2x}}{\left(e^{2x}\right)^2}$ is defined for all x in

$\therefore f(x)$ is differentiable

$f(0)=f\left(\frac{\pi}{2}\right)=0$

Hence all the conditions of Rolle's theorem as satisfied.

$\therefore\ \exists\, c$ such that $\Rightarrow f'(c)=0$

$$\Rightarrow\frac{2\left(\cos 2c-\sin 2c\right)}{\left(e^{2c}\right)}=0\quad\Rightarrow\cos 2c-\sin 2c=0\Rightarrow\cos 2c=\sin 2c$$

$$\Rightarrow\tan 2c=1\Rightarrow c=\frac{\pi}{8}$$

Thus Rolle's theorem is satisfied.

Lagrange's Mean Value Theorem

Statement

If a function $f(x)$ is defined in the interval $[a,b]$ such that

i. $f(x)$ is continuous in $[a,b]$

ii. $f(x)$ is differentiable in (a,b)

Then there exists at least one point c in (a,b) that is $a<c<b$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Proof: Let us construct a function $\varphi(x)=f(x)-kx$ ---- (1)

where k is a constant to be choosen suitably later. Since $f(x)$ and x are continuous in $[a,b]$,

differentiable in (a,b), and kx is also continous in [a,b], differentiable in (a,b). We can

Conclude that $\varphi(x)$ is also continous in [a, b] and differentiable in (a, b).

from (1) we have, $\varphi(a)=f(a)-ka$; $\varphi(b)=f(b)-kb$

$\therefore$ $\varphi(a)=\varphi(b)$ holds good if

$$f(a)-ka=f(b)-kb \Rightarrow k=\frac{f(b)-f(a)}{b-a} \quad \text{---- (2)}$$

Hence if k is choosen as given in (2), then $\varphi(x)$ satisfy all the condition of Rolls theorem.

Therefore by Rolles theorem there exist atleast one point c in (a,b) such that $\varphi'(c)=0$

Differentiating (1) w.r.t x,

we have $\varphi'(x)=f'(x)-k$ and $\varphi'(c)=0 \Rightarrow f'(c)-k=0$

i.e. $k=f'(c)$-------(3),

Equating R.H.S of (2) and (3)

we have $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Geometrical Interpretation of Lagrange's Mean Value Theorem

- There are no breaks or gaps in between 'a' and 'b' for the given curve and including at the end points, hence the given function is continuous in $[a,b]$
- Since a unique tangent can be drawn at each and every point in the interval except at the end points, the function is differentiable in the (a,b)

Conclusion: Therefore there exits at least one point 'c' (say) in between 'a' and 'b' at which the tangent at 'c' is parallel to the chord AB.

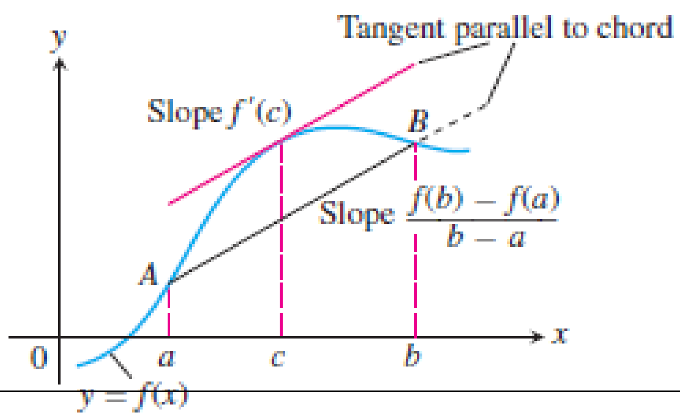


Figure-3

Note:

There may be exists more than one point in between (a, b) at which the tangents are parallel to the chord AB.

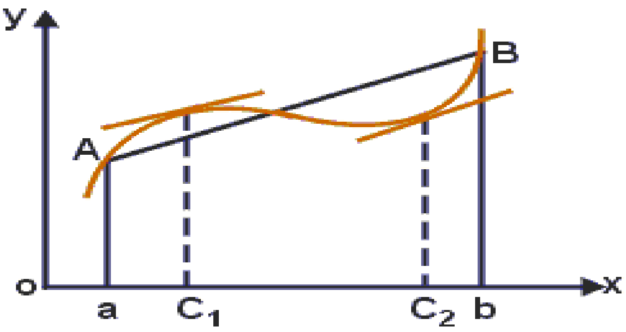


Figure-4

Lagrange's Mean Value Theorem

If a function $f(x)$ is defined in the interval $[a, a+h]$ such that

i. $f(x)$ is continuous in $[a, a+h]$

ii. $f(x)$ is differentiable in $(a, a+h)$

Then there exists at least one number θ, where $0<\theta<1$ such that $c=a+\theta h$ and $f'(a+\theta h)=\dfrac{f(a+h)-f(a)}{h} \Rightarrow f(a+h)=f(a)+hf'(a+\theta h)$ Where, $\theta=\dfrac{c-a}{h} \Rightarrow \theta=\dfrac{c-a}{b-a}$,

This leads the generalization of the Lagrange's Mean value as Taylor's series. This is very useful to express any function in terms of approximate polynomial function.

Problems on Lagrange's Mean Value theorem:

6. Verify Lagrange's Mean Value Theorem for the function $f(x)=\ln x$ **in** $[1,e]$.

Solution: $f(x)$ is continuous in $[1,e]$

$f'(x)=\dfrac{1}{x}$ is defined $\forall x \in (1,e)$ $f(x)$ is differentiable in $(1,e)$ $\therefore$

All conditions of Lagrange's Mean Value Theorem are satisfied, therefore there exist one point

c such that $f'(c)=\dfrac{f(e)-f(1)}{e-1} \Rightarrow \dfrac{1}{c}=\dfrac{1}{e-1} \Rightarrow c=e-1 \in (1\ e)$

Hence Lagrange's Mean Value Theorem is verified.

7. Verify Lagrange's Mean Value Theorem for the function $f(x)=(x-1)(x-2)(x-3)$ **in** $[0,4]$.

Solution : is continuous as it is an algebraic function in $\left[0,4\right]$.

$f(x)=x^3-6x^2+11x-6 \Rightarrow f'(x)=3x^2-12x+11$ is defined $\forall\, x\in(0,4)$

$\therefore\ f(x)$ is differentiable in $(0,4)$, All conditions of Lagrange's Mean Value Theorem are satisfied, therefore there exist on $f(x)$ e point c such that

$$f'(c)=\frac{f(4)-f(0)}{4-0}$$

$$\Rightarrow 3c^2-12c+11=\frac{6-(-6)}{4} \Rightarrow 3c^2-12c+8=0 \Rightarrow c=\frac{12\pm\sqrt{48}}{6}$$

$\therefore c=3.15$ and 0.85 both $\in(0,4)$

8. Prove that $\dfrac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\dfrac{b-a}{\sqrt{1-b^2}}$ **where** $a<c<b<1$ **using Lagrange's mean value theorem**

Solution :

$f(x)=\sin^{-1}x \therefore f'(x)=\dfrac{1}{\sqrt{1-x^2}}$, $f(x)$ is continous in [a,b] and differentiable in (a, b).

Applying Lagrange's mean value theorem, for $f(x)$ in $[a,b]$ we get when $a<c<b$

$$\frac{\sin^{-1}b-\sin^{-1}a}{b-a}=\frac{1}{\sqrt{1-c^2}}$$ We know that $a<c<b$

$$\Rightarrow a^2<c^2<b^2 \Rightarrow -a^2>-c^2>-b^2 \Rightarrow 1-a^2>1-c^2>1-b^2$$

$$\frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}} \Rightarrow \frac{1}{\sqrt{1-a^2}}<\frac{\sin^{-1}b-\sin^{-1}a}{b-a}<\frac{1}{\sqrt{1-b^2}}$$

on multiplying by $(b-a)$ which is positive, we have

$$\frac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{b-a}{\sqrt{1-b^2}}$$

9. If $f'(x)>0$ **for all the points in** $\left[a,b\right]$**, then prove that** $f(x)$ **is strictly increasing in the interval.**

Solution : Let x_1, x_2 be two numbers such that $a\le x_1<x_2\le b$.

Applying LMVT in $\left[x_1,x_2\right]$ $\exists\ c\in(x_1,x_2)$ such that $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)$

We have $f'(x)>0\ \forall\ x\in[a,b] \therefore \dfrac{f(x_2)-f(x_1)}{x_2-x_1}>0 \Rightarrow f(x_2)>f(x_1)$

$\therefore\ f(x)$ is an increasing function.

10. Show that for $x > 0$, $\log(1+x) > \dfrac{x}{1+x}$

Solution : Let $f(x) = \log(1+x) - \dfrac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \left\{\frac{(1+x).1 - x.1}{(1+x)^2}\right\}$$

$f'(x) = \dfrac{x}{(1+x)^2}$, clearly $f'(x) > 0$ since $x > 0$ and also $f(x)$ is continuous in [0,x] and differentiable in (0,x). Applying lagrange's mean value theorem for this $f(x)$ in [0, x] we have,

$f(x) = f(0) + (x-0)f'(c)$ but $f(0) = 0$

$\therefore\ f'(x) = xf'(c)$; $f'(x) > 0 \Rightarrow f'(c) = 0$ and hence $f(x) > 0$

i.e. $\log(1+x) - \dfrac{x}{1+x} > 0$ or $\log(1+x) > \dfrac{x}{1+x}$.

Cauchy's Mean Value Theorem

Statement :

If the functions $f(x)$ and $g(x)$ are defined in the interval $[a,b]$ such that

i. $f(x)$ and $g(x)$ are continuous in $[a,b]$

ii. $f(x)$ and $g(x)$ are differentiable in (a,b)

iii. $g'(x) \neq 0$ in (a,b)

Then there exists at least point c in (a,b) such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof : Let us define a function $\varphi(x) = f(x) - kg(x)$ ---(1)

where k is a constant to be choosen suitably later.from the given conditions it is evident that

$\varphi(x)$ is also continous in [a,b] and differentiable in (a,b).

from (1) we have, $\varphi(a) = f(a) - kg(a)$; $\varphi(b) = f(b) - kg(b)$

$\therefore\ \varphi(a) = \varphi(b)$ holds good if $f(a) - kg(a) = f(b) - kg(b) \Rightarrow k = \dfrac{f(b)-f(a)}{g(b)-g(a)}$ ----(2)

Here $g(b) \neq g(a)$, because if $g(b) = g(a)$ then $g(x)$ would satisfy all the conditions of Rolle's theorem and accordingly there must exist atleast one point c in (a,b) such that $g'(c) = 0$.

This contradicts the data that $g'(x) \neq 0$ for all x in (a,b). Hence if k is choosen as gi ven in (2),

then $\varphi(x)$ satisfy all the condition of Rolls theorem. Therefore by Rolles theorem th ere exist atleast one point c in (a,b) such that $\varphi'(c) = 0$

Differentiating (1) w.r.t x, we have $\varphi'(x) = f'(x) - kg'(x)$ and $\varphi'(c) = 0$

$\Rightarrow f'(c) - kg'(c) = 0$

i.e. $k = \dfrac{f'(c)}{g'(c)}$ -------

(3), Equating R.H.S of (2) and (3) we have $f'(c) = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

Problems on Cauchy Mean Value Theorems

11. Verify Cauchy's Mean Value Theorem for the function $f(x) = e^x; g(x) = e^{-x}$ **in** $[a,b]$

Solution : $f(x)$ and $g(x)$ are both continuous in $[a,b]$

$f'(x) = e^x$; $g'(x) = -e^{-x}$ are defined $\forall x \in (a,b)$

$\therefore f(x)$ and $g(x)$ are both differentiable in (a,b)

$g'(x) = -e^{-x} \neq 0, \forall x \in (a,b)$

$\exists c$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} \quad \Rightarrow e^{2c} = e^{a+b} \Rightarrow 2c = a + b$$

$\Rightarrow c = \dfrac{a+b}{2} \in (a,\ b)$ Hence the Cauchy's mean value theorem is verified.

12. Verify Cauchy's mean value theorem for the functions $\sqrt{x+9}$ **and** $\sqrt{x}$ **in** $[0,16]$

Solution : $f(x)$ and $g(x)$ are both continuous in $[0,16]$

$f'(x) = \dfrac{1}{2\sqrt{x+9}}$; $g'(x) = \dfrac{1}{2\sqrt{x}}$ are defined $\forall x \in (0,16)$

$\therefore f(x)$ and $g(x)$ are both differentiable in $(0,16)$

$g'(x) = \dfrac{1}{2\sqrt{x}} \neq 0, \forall x \in (a,b)$, $\exists c$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{f(16) - f(0)}{g(16) - g(0)} = \frac{1/2\sqrt{c+9}}{1/2\sqrt{c}} \Rightarrow \frac{\sqrt{25} - \sqrt{9}}{\sqrt{16} - \sqrt{0}} = \frac{1/\sqrt{c+9}}{1/\sqrt{c}}$$

On simplifying we get $c = 3 \in (0,16)$. Thus the theorem is verified.

13. Verify Cauchy's mean value theorem for the functions $f(x)$ **and** $f'(x)$ **in** $\left[1,e\right]$ **where** $f(x) = \log x$

Solution: $f(x) = \log x$, Let $g(x) = f'(x) = 1 \backslash x$

$f(x)$ and $g(x)$ are both continuous in $\left[1,e\right]$

$f'(x) = \dfrac{1}{x}$; $g'(x) = \dfrac{-1}{x^2}$ are defined $\forall x \in (1,e)$

$\therefore\ f(x)$ and are both differentiable in $(1,e)$

$g'(x) = \dfrac{-1}{x^2} \neq 0 \quad \forall x \in (1,e)$

$\exists c$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)} \Rightarrow \dfrac{1/c}{-1/c^2} = \dfrac{f(e) - f(1)}{g(e) - g(1)}$

$\Rightarrow -c = \dfrac{\log e - \log 1}{(1/e) - 1}$

on simplifying we get $c = 1.6 \in (1,e)$ since $e = 2.7$.

Thus the theorem is verified.

14. Show that the constant c **of Cauchy's mean valu** A **e theorem for the funct ions** $1/x^2$ **and** $1/x$ **in the interval** a,b **is the harmonic mean between** a **and** b $(0 < a < b)$

Solution : $f(x)$ and $g(x)$ are both continuous in $\left[a,b\right]$

$f'(x) = \dfrac{-2}{x^3}$; $g'(x) = \dfrac{-1}{x^2}$ are defined $\forall x \in (a,b)$

$\therefore\ f(x)$ and $g(x)$ are both differentiable in (a,b)

$g'(x) = \dfrac{-1}{x^2} \neq 0 \ \forall x \in (a,b)$ $\exists c$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$\Rightarrow \dfrac{-2/c^3}{-1/c^3} = \dfrac{1/b^2 - 1/a^2}{1/b - 1/a} \Rightarrow \dfrac{2}{c} = \dfrac{a^2 - b^2 / a^2 b^2}{a - b / ab} \Rightarrow \dfrac{2}{c} = \dfrac{(a-b)(a+b)ab}{(a-b)(a^2 b^2)}$

$c = \dfrac{2ab}{a+b}$ is the harmonic mean between a and b , $c \in (a,b)$.

Thus verified.

Taylor's theorem

Statement:

If $f:[a,b]\to R$ such that

i) f^{n-1} is continued on $[a,b]$

ii) f^{n-1} is differentiable on (a,b)

then exist f^{n} on (a,b) and $p\in z^{+}$ then there exist a point $c\in(a,b)$ such that

$$f(b)=f(a)+\frac{b-a}{1!}f^{1}(a)+\frac{(b-a)^{2}}{2!}f^{11}(a)+\frac{(b-a)^{3}3}{3!}f^{111}(a)+.....+\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)+R_{n}$$

where $R_{n}=\frac{(b-a)^{n}(b-a)^{n-p}}{(n-1)!}\cdot\frac{f^{n}(c)}{p}$ is defined as Roche's form of remainder.

Maclaurin's theorem:

In Taylor's series $[a,b]$ converts to $[0,x]$ then the result is defined as Maclaurin's series.

Statement:

If $f:[0,x]\to R$ such that

i) f^{n-1} is continuous on $[0,x]$

ii) f^{n-1} is differentiable on $[0,x]$

then there exist $p\in Z^{+}$ and $\theta\in(0,1)$ such that

$$f(x)=f(0)+\frac{x-0}{1!}f^{1}(0)+\frac{(x-0)^{2}}{2!}f^{11}(0)+\frac{(x-0)^{3}}{3!}f^{111}(0)+\frac{(x-0)^{4}}{4!}\overset{iv}{f}(0)+...$$

$$f(x)=f(0)+\frac{x}{1!}f^{1}(0)+\frac{x^{2}}{2!}f^{11}(0)+\frac{x^{3}}{3!}f^{111}(0)+\frac{x^{4}}{4!}\overset{iv}{f}(0)+......\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+R_{n}$$

where $R_n = \dfrac{x^n (1-\theta)^{n-p}}{(n-1)!\,p} f^n(\theta x)$ is defined as Roche's form of remainder.

***Note* :** (1) Taylor's series expansion of $f(x)$ in powers of $(x-a)$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} + \frac{(x-a)^2}{2!} f^{111}(a) + \frac{(x-a)^3}{3!} f^{111}(a) + \ldots.$$

Problems

15. Obtain the Mach Laurin's series expansions of the following functions.

(i) e^x ii) $\cos x$ iii) *sin x* iv) *cosh x* v) *sinh x*

i) Let the given function $f(x) = e^x$ $f(0) = 1$

$f^1(x) = e^x$ $f^1(0) = 1$

$f^{11}(x) = e^x$ $f^{11}(0) = 1$

$f^{111}(x) = e^x$ $f^{111}(0) = 1$

$f^{iv}(x) = e^x$ $f^{iv}(o) = 1$

by Maclaurin's series we have

$$f(x) = f(0) + \frac{x}{1!} f^1(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{111}(0) + \frac{x^4}{4!} f^{11}(0) + \ldots.$$

$$e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!} + \ldots..$$

$$e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots..\infty$$

ii) Given function $f(x) = \cos x$

by Maclaurin's series we have $f(x) = f(0) + \dfrac{x}{1!} f^1(0) + \dfrac{x^2}{2!} f^{11}(0) + \dfrac{x^3}{3!} f^{111}(0) + \ldots..$

$f(x) = \cos x$ $\qquad f(0) = 1$

$f^{1}(x) = \sin x$ $\qquad f^{1}(0) = 0$

$f^{11}(x) = -\cos x$ $\qquad f^{11}(0) = -1$

$f^{111}(x) = \sin x$ $\qquad f^{111}(0) = 0$

$f^{iv}(x) = \cos x$ $\qquad f^{iv}(0) = 1$

$f^{v}(x) = -\sin x$ $\qquad f^{v}(0) = 0$

Hence, $f(x) = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + ...$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

iii) Given that $f(x) = \sin x$

by Maclaurin's series we have $f(x) = f(0) + \frac{x}{1!} f^{1}(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{111}(0) + ...$

$f(x) = \sin x$ $\qquad f(0) = 0$

$f^{1}(x) = \cos x$ $\qquad f^{1}(0) = 1$

$f^{11}(x) = \sin x$ $\qquad f^{11}(0) = 0$

$f^{111}(x) = -\cos x$ $\qquad f^{111}(0) = -1$

$f^{iv}(x) = \sin x$ $\qquad f^{iv}(0) = 1$

$f^{v}(x) = \cos x$ $\qquad f^{v}(0) = 1$

$f^{vi}(x) = -\sin x$ $\qquad f^{vi}(0) = 0$

Hence

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \frac{x^6}{6!}(0) + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

iv) Let the given function $f(x) = \cosh x$

by Maclaurin's series we have $f(x) = f(0) + \frac{x}{1!} f^1(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{111}(0) + ...$

$f(0) = 1$ $\quad f^1(x) = \sinh x$ $\quad f^1(0) = 0$

$f^{11}(x) = \cosh x$ $\quad f^{11}(0) = 1$

$f^{111}(x) = \sinh x$ $\quad f^{111}(0) = 0$

$f^{iv}(x) = \cosh x$ $\quad f^{iv}(0) = 1$

$f^{v}(x) = \sinh x$ $\quad f^{v}(0) = 0$

$f^{vi}(x) = \cosh x$ $\quad f^{vi}(0) = 1$

$$f(x) = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(1) + ...$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$$

v) Let the given function is $f(x) = \sinh x$

by Maclaurin's series $f(x) = f(0) + \frac{x}{1!} f^1(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{111}(0) +$-----------

$f(0) = 0$

$f^1(x) = \cosh x$ $\quad f^1(0) = 1$

$f^{11}(x) = \sinh x \qquad f^{11}(0) = 0$

$f^{111}(x) = \cosh x \qquad f^{111}(0) = 1$

$f^{iv}(x) = \sinh x \qquad f^{iv}(0) = 0$

$f^{v}(x) = \cosh x \qquad f^{v}(0) = 1$

$f^{vi}(x) = \sinh x \qquad f^{vi}(0) = 0$

$$\sinh x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \frac{x^6}{6!}(0) + \ldots$$

$$\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

Taylor's series of expansion of $f(x)$ in powers of $(x-a)$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \frac{(x-a)^3}{3!} f^{111}(a) + \ldots.$$

Problems

16. Find the Taylor's series expansion of $\sin x$ about $x - \frac{\pi}{4}$

Sol. Given that $\qquad f(x) = \sin x$ at $x = a = \frac{\pi}{4}$

Taylor's series of expansion of $f(x)$ in powers of $(x-a)$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \frac{(x-a)^3}{3!} f^{111}(a) + \ldots.$$

$$f(a) = \sin\frac{\pi}{4} \quad = \frac{1}{\sqrt{2}}$$

$$f^1(x) = \cos x \Rightarrow f^1\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f^{11}(x) = -\sin x \Rightarrow f^{11}\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f^{111}(x) = -\cos x \Rightarrow f^{111}\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f^{iv}(x) = \sin x \Rightarrow f^{iv}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \dots$$

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x-\frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} + \frac{\left(x-\frac{\pi}{4}\right)^4}{4!} + \dots\right]$$

17. Find the Taylor's series expansion of $\sin 2x$ about $x = \frac{\pi}{4}$

Sol. Given that $f(x) = \sin 2x$ at $x - a = x - \frac{\pi}{4}$, $a = \frac{\pi}{4}$

Taylor's series of expansion of $f(x)$ is powers of $(x-a)$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f^1(a) + \frac{(x-a)^2}{2!} + f^{11}(a) + \frac{(x-a)^3}{3!} f^{111}(a) + \dots$$

$$f(a) = \sin 2\left(\frac{\pi}{4}\right) = 1$$

$$f^1(x) = 2\cos 2x \Rightarrow f^1\left(\frac{\pi}{4}\right) = 2\cos 2\left(\frac{\pi}{4}\right) = 0$$

$$f^{11}(x) = -4\sin 2x \Rightarrow f^{11}\left(\frac{\pi}{4}\right) = -4\sin 2\left(\frac{\pi}{4}\right) = -4$$

$$f^{111}(x) = -8\cos\sin 2x \Rightarrow f^{111}\left(\frac{\pi}{4}\right) = -8\cos 2\left(\frac{\pi}{4}\right) = 0$$

$$f^{iv}(x)=16\sin 2x \Rightarrow f^{iv}\left(\frac{\pi}{4}\right)=16\sin 2\left(\frac{\pi}{4}\right)=16$$

$$\sin 2x = 1+\frac{\left(x-\frac{\pi}{4}\right)}{1!}(0)+\frac{(x-a)^2}{2!}(-4)+\frac{(x-a)^3}{3!}(0)+\frac{(x-a)^4}{4!}(16)+\frac{(x-a)^5}{5!}(0)$$

$$+\frac{(x-a)^6}{6!}(-32)(-64)+\ldots$$

$$\sin 2x = 1-\frac{(x-a)^2}{2!}(4)+\frac{(x-a)^4}{4!}(16)-\frac{(x-a)^2}{6!}(64)+\ldots$$

Exercise Problems:

18. Verify Taylor's theorem for $f(x)=(1-x)^{\frac{5}{2}}$ with Lagrange's form of remainder up to 3 terms in the interval $[0,1]$
19. Show that $\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!}+\ldots+\infty$
20. Show that $\log\left(1+e^x\right)=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots.$ and hence deduce that $\frac{e^x}{e^{x+1}}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\ldots$
21. Find Maclaurin's theorem with Lagrange's form of remainder for $f(x)=\cos x$
22. Obtain the Taylor's series expansion of e^x and $x=-1$
23. Obtain the Maclaurin's series of expansion of $\log_e(1+x)$

The Taylor's series of Lagrange's form of remainder

24. Expand the function $f(x)=(1-x)^{\frac{5}{2}}$

$f(x),\ f^{1}(x),\ f^{11}(x)$ are continuous on $[0,1]$

$f^{111}(x)$ is differentiable on $(0,1)$, then satisfies the Taylor's series conditions.

$$f(x)= f(0)+\frac{x}{1!}f^{1}(0)+\frac{x^{2}}{2!}f^{11}(0)+\frac{x^{3}}{3!}f^{111}(0)+... \qquad \text{at } [0,x]=[0,1]$$

$$x=1, \ n=3$$

$$f(x)=(1-x)^{\frac{5}{2}} \qquad \Rightarrow f(0)=1$$

$$f^{1}(x)=\frac{5}{2}(1-x)^{\frac{5}{2}-1}(-1)=-\frac{5}{2}(1-x)^{\frac{3}{2}} \Rightarrow f^{1}(0)=-\frac{5}{2};$$

$$f^{11}(x)=-\frac{5}{2}\times\frac{3}{2}(1-x)^{\frac{3}{2}-1}(-1)=\frac{15}{4}(1-x)^{\frac{1}{2}} \ \Rightarrow f^{11}(0)=\frac{15}{4}$$

$$f^{111}(x)=\frac{15}{4}\times\frac{1}{2}(1-x)^{\frac{1}{2}-1}(-1)=\frac{-15}{8}(1-x)^{\frac{-1}{2}}$$

$$f^{111}(\theta x)=\frac{-15}{8}(1-\theta x)^{\frac{-1}{2}} \qquad \Rightarrow f^{111}(\theta)=\frac{-15}{8}(1-\theta)^{\frac{-1}{2}}$$

if $x=1$ then $f(1)= f(0)+ f^{1}(0)+ f^{11}(0)\frac{1}{2}+\frac{1}{3!}f^{111}(\theta)$

sub $f(1)$, $f(0)$, $f^{1}(0)$, $f^{11}(0)$, $f^{111}(\theta)$ in above form

$$0=1-\frac{5}{2}+\frac{1}{2}\left(\frac{15}{4}\right)-\frac{1}{6}\left(\frac{15}{8}\right)(1-\theta)^{\frac{-1}{2}}$$

$$0=\frac{-3}{2}+\frac{15}{8}-\frac{15}{48}(1-\theta)^{\frac{-1}{2}} \Rightarrow \frac{15}{48}(1-\theta)^{\frac{-1}{2}}=\frac{3}{8} \Rightarrow (1-\theta)^{\frac{-1}{2}}=\frac{3}{8}\times\frac{48}{15}=\frac{6}{5}$$

$$\Rightarrow (1-\theta)=\left(\frac{6}{5}\right)^{-2} \Rightarrow (1-\theta)=\left(\frac{5}{6}\right)^{2} \quad \Rightarrow \theta=1-\left(\frac{5}{6}\right)^{2} \Rightarrow \theta=0.3055\in(0,1)$$

$f(x)$ is applicable for Taylor's theorem.

25. Given function $f(x)=\dfrac{\sin^{-1}x}{\sqrt{1-x^{2}}}$

$$f(0)=0$$

$$f(x)\sqrt{1-x^2}=\sin^{-1}x$$

Differentiate w.r.t. x

$$\Rightarrow f(x)\frac{1}{2\sqrt{1-x^2}}(-2x)+\sqrt{1-x^2}\,f^1(x)=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow -x\,f(x)+\left(1-x^2\right)f^1(x)=1 \qquad \Rightarrow \left(1-x^2\right)f^1(x)=x\,f(x)=1 \qquad (1)$$

Put $x=0 \Rightarrow (1-0)\,f^1(0)-(0)\,f(0)=1 \Rightarrow f^1(0)=1$

d.w.r.t to 'x' in eq. (1) again

$$\Rightarrow \left(1-x^2\right)f^{11}(x)+f^1(x)(-2x)-\left[xf^1(x)+f(x)\cdot 1\right]=0$$

$$\Rightarrow \left(1-x^2\right)f^{11}(x)-3xf^1(x)-f(x)=0 \qquad ...(2)$$

For $x=0 \Rightarrow (1-0)\,f^{11}(0)-3(0)\,f(0)-f(0)=0 \Rightarrow f^{11}(0)=0$

D.w.r.t. 'x' in eq. (2), we get

$$\Rightarrow \left(1-x^2\right)f^{111}(x)+f^{11}(x)(-2x)-3\left[xf^{11}(x)+f^1(x)-f^1(x)=0\right]$$

$$\Rightarrow \left(1-x^2\right)f^{111}(x)-5xf^{11}-4f^1(x)=0$$

For $x=0 \Rightarrow (1-0)\,f^{111}(0)-5(0)\,f^{11}(0)-4f^1(0)=0$

$$f^{111}(0)-4=0 \Rightarrow f^{111}(0)=4$$

By Maclaurin's series we have, $f(x)=f(0)+\frac{x}{1!}f^1(0)+\frac{x^2}{2!}f^{11}(0)+\frac{x^3}{3!}f^{111}(0)+...$

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}}=\frac{x+x^3}{3!}(4)+...$$

26. Given that $\log\left(1+e^x\right)=f(x)$

$$f^{1}(x)=\frac{1}{1+e^{x}}e^{x}\Rightarrow f^{1}(x)=\frac{e^{x}}{1+e^{x}}$$

$$f^{1}(0)=\frac{e^{0}}{1+e^{0}}$$

Similarly, $f^{11}(0)=\frac{e^{0}}{\left(1+e^{0}\right)^{2}}=\frac{1}{(1+1)^{2}}\Rightarrow f^{11}(0)=\frac{1}{4}$

$$f^{111}(x)=\frac{\left(1+e^{x}\right)^{2}e^{x}-e^{x}\left[2\left(1+e^{x}\right)\cdot e^{x}\right]}{\left(1+e^{x}\right)^{4}}=\left(1+\cancel{e^{x}}\right)\left[\frac{e^{x}+e^{2x}-2e^{2x}}{\left(1+e^{x}\right)^{3}}\right]=\frac{e^{x}-e^{2x}}{\left(1+e^{x}\right)^{3}}$$

$$f^{111}(0)=\frac{e^{0}-e^{0}}{\left(1+e^{0}\right)^{3}}\Rightarrow f^{111}(0)=0$$

$$f^{iv}(x)=\frac{\left(1+e^{x}\right)^{3}\left[e^{x}-2e^{2x}\right]-\left[e^{x}-2e^{x}\right]\left[3\left(1+e^{x}\right)^{2}e^{x}\right]}{\left(1+e^{x}\right)^{6}}$$

$$=\frac{\left(1+e^{x}\right)2\left[\left(1+e^{x}\right)\left(e^{x}-2e^{2x}\right)-\left[3e^{x}\left(e^{x}-e^{2x}\right)\right]\right]}{\left(1+e^{x}\right)^{\cancel{6}\,4}}=\frac{e^{x}-2e^{2x}+e^{2x}-2e^{3x}-3e^{2x}+3e^{3x}}{\left(1+e^{x}\right)^{4}}$$

$$f^{iv}(x)=\frac{e^{x}-4e^{2x}+e^{3x}}{\left(1+e^{x}\right)^{4}}\Rightarrow f^{iv}(0)=\frac{e^{0}-4e^{0}+e^{0}}{\left(1+e^{0}\right)^{4}}\quad=\frac{1-4+1}{16}=\frac{-2}{16}=\frac{-1}{8}$$

By Maclaurin’s series we have

$$f(x)=f(0)+\frac{x}{1!}f^{1}(0)+\frac{x^{2}}{2!}f^{11}(0)+\frac{x^{3}}{3!}f^{111}(0)+\ldots.$$

$$=\log 2+\frac{x}{1!}\left(\frac{1}{2}\right)+\frac{x^{2}}{2}\left(\frac{1}{4}\right)+\frac{x^{3}}{6}(0)+\frac{x^{4}}{24}\left(\frac{-1}{8}\right)+\ldots$$

$$\log\left(1+e^{x}\right)=\log 2+\frac{x}{2}+\frac{x^{2}}{8}-\frac{x^{4}}{192}+\ldots$$

D.w.r.t. 'x' in above form, we get

$$\frac{e^x}{1+e^x}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\dots$$

27. Taylor series of expansion of $f(x)$ in powers of $(x-a)$ is given by

$$f(x)=f(a)+\frac{(x-a)}{1!}f^1(a)+\frac{(x-a)^2 f^{11}(a)}{2!}+\frac{(x-a)^3}{3!}f^{111}(a)+\dots$$

here $f(x)=e^x$ at $x-a=x+1$ $a=-1$

$$e^x=f(x) \qquad \Rightarrow f(a)=f(-1)=e^{-1}$$

$$f^1(x)\Rightarrow e^x \Rightarrow f^1(-1)=e^{-1}$$

$$f^{11}(x)\Rightarrow e^x \Rightarrow f^{11}(-1)=e^{-1}$$

$$f^{111}(x)\Rightarrow e^x \Rightarrow f^{111}(-1)=e^{-1}$$

$$f^{iv}(x)\Rightarrow e^x \Rightarrow f^{iv}(-1)=e^{-1}$$

Hence, $e^x=\frac{1}{e}\left(1+(x+1)+\frac{(x+1)^2}{2!}+\frac{(x+1)^3}{3!}+\dots\right)$

28. given $f(x)=\log_e(1+x)$

by Maclaurin's series

$$f(x)=f(0)+\frac{x}{1!}f^1(0)+\frac{x^2}{2!}+f^{11}(0)+\frac{x^3}{3!}f^{111}(0)+\dots$$

$$f(0)=\log_e 1$$

$$f(0)=0$$

$$f^1(x)=\frac{1}{1+x} \quad \Rightarrow f^1(0)=1$$

$$f^{11}(x)=-\frac{1}{(1+x)^2}\Rightarrow f^{11}(0)=-1$$

$$f^{111}(x)=\frac{-(-2)}{(1+x)^3}\Rightarrow f^{111}(0)=-1$$

$$f^{iv}(x)=\frac{2(-3)}{(1+x)^4}\Rightarrow f^{iv}(0)=\frac{-6}{1}$$

by Maclaurin’s series

$$f(x)=f(0)+\frac{x}{1!}f^{1}(0)+\frac{x^2}{2!}f^{11}(0)+\frac{x^3}{3!}f^{111}(0)+\ldots$$

$$\log_e(1+x)=0+\frac{x}{1!}(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(-6)+\ldots$$

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

29. $f(x)=\cos x$

by Maclaurin's theorem with Lagrange’s form of remainder

$$f(x)=f(0)+\frac{x}{1!}f^{1}(0)+\frac{x^2}{2!}f^{11}(0)+\frac{x^3}{3!}f^{111}(0)+\ldots+\frac{x^n}{n!}f^{n}(\theta x)$$

$$f(x)=\cos x$$

$$f(0)=1$$

$$f^{1}(x)=-\sin x\Rightarrow f^{1}(0)=0$$

$$f^{11}(x)=-\cos x\Rightarrow f^{11}(0)=-1$$

$$f^{111}(x)=\sin x\Rightarrow f^{111}(0)=0$$

$$f^{iv}(x) = \cos x \Rightarrow f^{1v}(0) = 1$$

$$f^{v}(x) = -\sin x \Rightarrow f^{v}(0) = 0$$

$$f^{vi}(x) = -\cos x \Rightarrow f^{vi}(0) = -1$$

by Maclaurin's theorem with Lagrange's form of remainder we get

$$f(x) = f(0) + \frac{x}{1!} f^{1}(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{111}(0) + \ldots + \frac{x^n}{n!} f^{n}(\theta x)$$

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \ldots + \frac{x^n}{n!} f^{n}(\theta x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots (-1)^n \frac{x^{2n}}{2n!} + \ldots$$

Taylor's Series Expansion of $f(x)$ about $x = a$

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots$ is called a Taylor's series expansion of $f(x)$ about $x = a$

Maclaurin's series expansion of f(x): Taylor's series expansion of f(x) about x=0 is called as Maclaurin's series expansion of $f(x)$.

i.e. $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$

30. Obtain Taylor's series expansion of e^x about $x = -1$

Solution: Given $f(x) = e^x$ and $a = -1$

By Taylor's series expansion $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots$

$f(x) = e^x \Rightarrow f(a) = f(-1) = e^{-1}$

$f'(x) = e^x \Rightarrow f'(a) = f'(-1) = e^{-1}$

$f''(x) = e^x \Rightarrow f''(a) = f''(-1) = e^{-1}$

$f^{iv}(x) = e^x \Rightarrow f^{iv}(x) = f^{iv}(-1) = e^{-1}$

$$e^x = e^{-1} + (x+1)e^{-1} + \frac{(x+1)^2}{2!}e^{-1} + \frac{(x+1)^3}{3!}e^{-1} + \frac{(x+1)^4}{4!}e^{-1}\ldots$$

$$e^x = \frac{1}{e}\left[1 + (x+1) + \frac{(x+1)^2}{2} + \frac{(x+1)^3}{6} + \frac{(x+1)^4}{24}\ldots\right]$$

31. Find Taylor's series expansion of $Sin2x$ about $x = \frac{\pi}{4}$

Solution: Given $f(x) = Sin2x$ and $a = \frac{\pi}{4}$

$$f(x) = Sin2x \Rightarrow f(a) = f(\tfrac{\pi}{4}) = Sin\frac{\pi}{2} = 1$$

$$f'(x) = 2Cos2x \Rightarrow f'(a) = f'(\tfrac{\pi}{4}) = 2Cos\frac{\pi}{2} = 0$$

$$f''(x) = -4Sin2x \Rightarrow f''(a) = f''(\tfrac{\pi}{4}) = -4Sin\frac{\pi}{2} = -4$$

$$f'''(x) = -8Cos2x \Rightarrow f'''(a) = f'''(\tfrac{\pi}{4}) = -8Cos\frac{\pi}{2} = 0$$

$$f^{iv}(x) = 16Sin2x \Rightarrow f^{iv}(a) = f^{iv}(\tfrac{\pi}{4}) = 16Sin\frac{\pi}{2} = 16$$

By Taylor's series expansion $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$

$$Sin2x = 1 - \frac{(x-\frac{\pi}{4})^2}{2}4 + \frac{(x-\frac{\pi}{4})^4}{24}16 + \ldots$$

$$Sin2x = 1 - 2(x-\tfrac{\pi}{4})^2 + \frac{2}{3}(x-\tfrac{\pi}{4})^4 + \ldots$$

32. Find the Taylor's Series expansion of $f(x) = (1-x)^{\frac{5}{2}}$ in powers of 'x'.

33. Obtain Taylor's series expansion of $Sinx$ about $x = \frac{\pi}{4}$

34. Expand $\log_e^x$ in powers of $x-1$ and hence evaluate $\log 1.1$ correct to four decimal place
s

Solution: Given $f(x) = \log x$ and here $a = 1$

$$f(x) = \log x \Rightarrow f(a) = f(1) = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(a) = f'(1) = 1$$

$$D^n\left[\log x\right] = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$f''(x) = \frac{-1}{x^2} \Rightarrow f''(a) = f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(a) = f'''(1) = 2$$

$f^{iv}(x)=\frac{-6}{x^4} \Rightarrow f^{iv}(a)=f^{iv}(1)=-6$

By Taylor's expansion $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$

$\log x=x-1-\frac{(x-1)^2}{2!}+\frac{2(x-1)^3}{3!}-\frac{6(x-1)^4}{4!}+\ldots$ Put $x=1.1 \Rightarrow \log_e^{1.1}=0.0953$

Problem: Obtain Maclaurin's series of the following functions

e^x, $Sinx$, $Cosx$, $Sinhx$, $Coshx$.

Problem: Obtain expansion of $\log(1+x)$ by Maclaurin's series

Problem: Show that $\log(1+e^x)=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$ and hence deduce that

$\frac{e^x}{e^x+1}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\ldots$

Solution: Given $f(x)=\log(1+e^x)$ and here $a=0$

$f(x)=\log(1+e^x) \Rightarrow f(a)=f(0)=\log 2$

$f'(x)=\frac{e^x}{(1+e^x)} \Rightarrow f'(a)=f'(0)=\frac{1}{2}$

$f''(x)=\frac{(1+e^x)e^x-e^x(0+e^x)}{(1+e^x)^2}=\frac{e^x}{(1+e^x)^2} \Rightarrow f''(a)=f''(0)=\frac{1}{4}$

$f'''(x)=\frac{(1+e^x)^2e^x-e^x2(1+e^x)(0+e^x)}{(1+e^x)^2}=\frac{(1+e^x)\left[e^x+e^{2x}-2e^{2x}\right]}{(1+e^x)^4}=\frac{e^x-e^{2x}}{(1+e^x)^3} \Rightarrow f'''(a)=f'''(0)=0$

$f^{iv}(x)=\frac{(1+e^x)^3(e^x-2e^{2x})-(e^x-e^{2x})3(1+e^x)^2(0+e^x)}{(1+e^x)^6} \Rightarrow f^{iv}(a)=f^{iv}(0)=-\frac{1}{8}$

By Maclaurin's series expansion $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$

$\log(1+e^x)=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$

Differentiate w.r.t. 'x' we get $\frac{e^x}{e^x+1}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\ldots$

35. Using Maclaurin's series expand $Tanx$ up to the fifth power of x and hence find series f or $\log\sec x$

Problem: Show that $\frac{Sin^{-1}x}{\sqrt{1-x^2}}=x+4\frac{x^3}{3!}+\ldots$ (or) Expand $\frac{Sin^{-1}x}{\sqrt{1-x^2}}$ in powers of 'x'.