# MODULE I
# LOGIC

Statements and Notations,Connectives,Negation,Conjunction,Disjunction, statement, Formulae a nd TruthTables ,Conditional and Bi-conditional,Well–formed Formulae, Tautologies, EquivalenceofFormulae,, Tautological Implications.

**Definition: Propositional Logic**

A proposition is a collection of declarative statements that has either a truth value "tr ue" or a truth value "false". A propositional consists of propositional variables and co nnectives. We denote the propositional variables by capital letters (A, B, C etc). The connectives connect the propositional variables.

Some examples of Propositions are given below:

- "Man is Mortal", it returns truth value"TRUE" as T.
- "12 + 9 = 3 − 2", it returns truth

value "FALSE" as F.

The following is not a Proposition

- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true orfalse.

**Connectives**

In propositional logic generally we use five connectives which are OR $(\vee)$, AND $(\wedge)$, Negation/ NOT (¬), Conditional or Implication / if-then (→), Bi conditional orIf and only if $(\leftrightarrow)$.

**Negation (¬)** − The negation of a proposition A (written as ¬A) is false when A istr ue and is true when A is false.

The truth table is as follows

| A | ¬A |
|---|---|
| True | False |

| | |
|---|---|
| False | True |

**AND (∧) −** The AND operation of two propositions A and B (written as A ∧ B) istr ue if both the propositional variable A and B is true.

The truth table is as follows −

| A | B | A∧B |
|---|---|---|
| True | True | False |
| True | False | False |
| False | True | False |
| False | False | True |

**OR (∨)−** The OR operation of two propositions A and B (written as A ∨ B) is trueif at least any of the propositional variable A or B is true.

The truth table is as follows −

| A | B | A ∨ B |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Implication / if-**

**then (→)** − An implication A→B is False if A is true and B is false. The restcases are true.

The truth table is as follows −

| A | B | A → B |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | True |
| False | False | True |

**If and only if (↔)**− A↔B is bi-conditional logical connective which is true whenp and q are both false or both are true.

The truth table is as follows −

| A | B | A ↔ B |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | False |
| False | False | True |

**Well Formed Formulas(WFFs)**

The well formed formulas(WFFs) or statement formulas or logic formulas are definedrecursively (or inductively) as below.

1. Propositional variables p,q,r,… and propositional constants F,T are well formedformulas. They are known as primitive WFFs.

2. If P and Q are WFFs then $\neg P, \neg Q, P \wedge Q, P \vee Q, P \rightarrow Q$ *and* $P \leftrightarrow Q$ are also WFFs.
3. All WFFs are obtained by the above procedures applied a finite number of times.
   For example, the following are WFFs
   $p,\ p \wedge q,\ p \rightarrow q,\ p \wedge (q \rightarrow r) \equiv (p \wedge q) \rightarrow r,\ (p \rightarrow q) \rightarrow (q \rightarrow p)$

**Note:** In order to avoid excessive use of parenthesis, we adopt an order of precedence forl ogical Operators.

$\neg, \wedge, \vee, \rightarrow$ *and* $\leftrightarrow$

**Tautologies**

A Tautology is a formula which is always true for every value of its propositional variables.

**Example** − Prove [(A → B) ∧ A] → B is a tautology

The truth table is as follows −

| A | B | A → B | (A → B) ∧ A | [(A → B) ∧ A] → B |
|---|---|---|---|---|
| True | True | True | True | True |
| True | False | False | False | True |
| False | True | True | False | True |
| False | False | True | False | True |

As we can see every value of [(A → B) ∧ A] → B is “True”, it is a tautology.

**Contradictions**

A Contradiction is a formula which is always false for every value of its propositionalv ariables.

**Example** − Prove (A VB) ∧ [(¬A) ∧ (¬B)] is a contradiction

The truth table is as follows −

| A | B | $A \vee B$ | $\neg A$ | $\neg B$ | $(\neg A) \wedge (\neg B)$ | $(A \vee B) \wedge [(\neg A) \wedge (\neg B)]$ |
|---|---|---|---|---|---|---|
| True | True | True | False | False | False | False |
| True | False | True | False | True | False | False |
| False | True | True | True | False | False | False |
| False | False | False | True | True | True | False |

As we can see every value of $(A \vee B) \wedge [(\neg A) \wedge (\neg B)]$ is "False", it is aContradictio n.

**Contingency**

A Contingency is a formula which has both some true and some false values fore very value of its propositional variables.

**Example** − Prove $(A \vee B) \wedge (\neg A)$ a contingencyThe

truth table is as follows −

| A | B | $A \vee B$ | $\neg A$ | $(A \vee B) \wedge (\neg A)$ |
|---|---|---|---|---|
| True | True | True | False | False |
| True | False | True | False | False |
| False | True | True | True | True |
| False | False | False | True | False |

As we can see every value of $(A \vee B) \wedge (\neg A)$ has both "True" and "False", itis a contingency.

**Propositional Equivalences**

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Two statements X and Y are logically equivalent if any of the following two conditions −

- The truth tables of each statement have the same truth values.
- The bi-conditional statement X ↔Y is a tautology.

**Example**−Prove¬(A∨B)and[(¬A)∧(¬B)]are equivalent

Testing by 1st method (Matching truth table)

| A | B | A ∨ B | ¬ (A ∨ B) | ¬A | ¬B | [(¬A) ∧ (¬B)] |
|---|---|---|---|---|---|---|
| True | True | True | False | False | False | False |
| True | False | True | False | False | True | False |
| False | True | True | False | True | False | False |
| False | False | False | True | True | True | True |

Here, we can see the truth values of ¬ (A ∨ B) and [(¬A) ∧ (¬B)] are same, hencethe statements are equivalent.

Testing by 2nd method (Bi-conditionality)

| A | B | ¬ (A ∨ B) | [(¬A) ∧ (¬B)] | [¬ (A ∨ B)] ↔ [(¬A) ∧ (¬B)] |
|---|---|---|---|---|
| True | True | False | False | True |
| True | False | False | False | True |
| False | True | False | False | True |
| False | False | True | True | True |

As [¬ (A ∨ B)] ↔ [(¬A) ∧ (¬B)] is a tautology, the statements are equivalent.

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**Laws of Propositional Logic:**

| S.No | Name of Laws | Primal Form | Dual Form |
|---|---|---|---|
| 1 | Idempotent Law | $p \vee p \equiv p$ | $p \wedge p \equiv p$ |
| 2 | Identity Law | $p \vee F \equiv p$ | $p \wedge T \equiv p$ |
| 3 | Dominant Law | $p \vee T \equiv T$ | $p \wedge F \equiv F$ |
| 4 | Complement Law | $p \vee \neg p \equiv T$ | $p \wedge \neg p \equiv F$ |
| 5 | Commutative Law | $p \vee q \equiv q \vee p$ | $p \wedge q \equiv q \wedge p$ |
| 6 | Associative Law | $p \vee (q \vee r) \equiv (p \vee q) \vee r$ | $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ |
| 7 | Distributive Law | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) \equiv (p \wedge q) \wedge (p \wedge r)$ |
| 8 | Absorption Law | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 9 | De Morgan's Law | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ |
| 10 | Double Negation Law | $\neg(\neg p) \equiv p$ | - |

**Logical Equivalences involving Conditional Statements**

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

**Logical Equivalences involving Biconditional Statements**

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

**Inverse,Converse,and Contra-positive**

A conditional statement has two parts − **Hypothesis** and **Conclusion**.

**Example of Conditional Statement** − "If you do your homework, you will notb e punished." Here, "you do your homework" is the hypothesis and "you will not be punished" is the conclusion.

**Inverse –**

An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is “If p, then q”, the inverse will be “If not p, then not q”. The inverse of “If you do your homework, you will notbe punished” is “If you do not do your homework, you will be punished.”

**Converse**–The converse of the conditional statement is computed by interchanging the hyp othesis and the conclusion. If the statement is “If p, then q”, the inverse willbe “If *q*, then*p*”. The converse of "If you do your homework, you will not be punished" is "If youw ill not be punished, you do not do your homework”.

**Contra-positive –**The contra-positive of the conditional is computed by interchanging the hypothesis and the c onclusion of the inverse statement. If the statement is “If p, then q”, the inverse wi ll be “If not *q*, then not *p*”. The Contra-positive of "If you do your homework, you will not be punished” is "If you will b e punished, you do your homework.

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# NORMAL FORMS

**Normal Forms, Disjunctive Normal Forms, Conjunctive Normal Forms, Principal Dis junctive Normal Forms, Principal Conjunctive Normal Forms, Rules of Inference, the Predicate Calculus, Predicates, Variables and Quantifiers, Predicate Formula, Free a nd Bound Variables.**

**Elementary Product**: A product of the variables and their negations in af ormula is called an elementary product. If p and q are any two atomic variables, then $p, \neg p \wedge q, \neg q \wedge p, \neg p \wedge \neg q$ are some examples of

elementary products.

**Elementary Sum**: A sum of the variables and their negations in a formula isc alled an elementary sum. If P and Q are any two atomic variables, then $p, \neg p \vee q, \neg q \vee p, \neg p \vee \neg q$ are some examples of elementary sums.

**Normal Forms:** We can convert any proposition in two normal forms −

1. Conjunctive Normal Form (CNF) 2.Disjunctive Normal Form (DNF)

**Conjunctive Normal Form**

A compound statement is in conjunctive normal form if it is obtained by operatin g AND among variables (negation of variables included) connectedwith OR s.

Examples: 1. $(p \vee q) \wedge (q \vee r)$

2. $(\neg p \vee q \vee r) \wedge (s \vee r)$

**Disjunctive Normal Form**

A compound statement is in disjunctive normal form if it is obtained by operatin g OR among variables (negation of variables included) connected withANDs.

Example: $(p \wedge q) \vee (\neg p \wedge \neg q) \vee (p \wedge \neg q \wedge \neg r)$

**Functionally Complete set**

A set of logical operators is called functionally complete if every compound p roposition is logically equivalent to a compound proposition involving onlyth is set of logical operators. $\wedge, \vee, \neg$ form a functionally complete set of operators.

**Minterms**: For two variables p and q there are 4 possible formulas whichc onsist of conjunctions of p,q or it's negation given by $p \wedge q, \neg p \wedge q, p \wedge \neg q, \neg p \wedge \neg q$

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**Maxterms**: For two variables p and q there are 4 possible formulas whichc onsist of disjunctions of p,q or its negation given by
$p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q$

**Principal Disjunctive Normal Form**: For a given formula an equivalent f ormula consisting of disjunctions of minterms only is known as principald isjunctive normal form (PDNF).

**Principal Conjunctive Normal Form:** For a given formula an equivalent f ormula consisting of conjunctions of maxterms only is known as principalc onjunctive normal form (PCNF).

**Problems:**

**Obtain DNF of $Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$.**

**Solution:**

$Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$

$\Leftrightarrow (Q \vee (P \wedge R)) \wedge (\neg((P \vee R) \wedge Q)$ (De morgan law)

$\Leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q)$ (De morgan law)

$\Leftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee ((P \wedge R) \wedge \neg P \wedge \neg R) \vee ((P \wedge R) \wedge \neg Q)$

(Extended distributed law)

$\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee F \vee (F \wedge R \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (Negation law)

$\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (Negation law)

**Obtain Pcnf and Pdnf of the formula $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$**

**Solution:**

Let S = $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

| P | Q | ¬ P | ¬ Q | $\neg P \vee \neg Q$ | $P \leftrightarrow \neg Q$ | S | Minterm | Maxterm |
|---|---|---|---|---|---|---|---|---|
| T | T | F | F | F | F | T | $P \wedge Q$ | |
| T | F | F | T | T | T | T | $P \wedge \neg Q$ | |
| F | T | T | F | T | T | T | $\neg P \wedge Q$ | |
| F | F | T | T | T | F | F | | $P \vee Q$ |

PCNF: $P \vee Q$ and PDNF: $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$