Recurrence Relations and Generating Functions

1 Some number sequences

An **infinite sequence** (or just a **sequence** for short) is an ordered array

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

of countably many real or complex numbers, and is usually abbreviated as $(a_n; n \geq 0)$ or just (a_n). A sequence (a_n) can be viewed as a function f from the set of nonnegative integers to the set of real or complex numbers, i.e.,

$$f : \mathbb{Z}_{\geq 0} \to \mathbb{C}, \quad f(n) = a_n, \quad n = 0, 1, 2, \ldots$$

We call a sequence (a_n) an **arithmetic sequence** if it is of the form

$$a_0, \quad a_0 + q, \quad a_0 + 2q, \quad \ldots, \quad a_0 + nq, \quad \ldots$$

The general term of such sequences satisfies the recurrence relation

$$a_n = a_{n-1} + q, \quad n \geq 1.$$

A sequence (a_n) is called a **geometric sequence** if it is of the form

$$a_0, \quad a_0q, \quad a_0q^2, \quad \ldots, \quad a_0q^n, \quad \ldots$$

The general term of such sequences satisfies the recurrence relation

$$a_n = qa_{n-1}, \quad n \geq 1.$$

The **partial sums** of a sequence (a_n) are the summations:

$$\begin{aligned} s_0 &= a_0, \\ s_1 &= a_0 + a_1, \\ s_2 &= a_0 + a_1 + a_2, \\ &\;\vdots \\ s_n &= a_0 + a_1 + \cdots + a_n, \\ &\;\vdots \end{aligned}$$

The partial sums form a new sequence $(s_n; n \geq 0)$.

For an arithmetic sequence $a_n = a_0 + nq$ $(n \geq 0)$, we have the partial sum

$$s_n = \sum_{k=0}^{n} (a_0 + kq) = (n+1)a_0 + \frac{qn(n+1)}{2}.$$

For a geometric sequence $a_n = a_0 q^n$ $(n \geq 1)$, we have

$$s_n = \sum_{k=0}^{n} a_0 q^n = \begin{cases} \frac{q^{n+1}-1}{q-1} a_0 & \text{if } q \neq 1 \\ (n+1)a_0 & \text{if } q = 1. \end{cases}$$

Example 1.1. Determine the sequence a_n, defined as the number of regions which are created by n mutually overlapping circles in general position on the plane. (By **mutually overlapping** we mean that each pair of two circles intersect in two distinct points. Thus non-intersecting or tangent circles are notallowed. By **general position** we mean that there are no three circles through a common point.)

We easily see that the first few numbers are given as

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4, \quad a_3 = 8.$$

It seems that we might have $a_4 = 16$. However, by try-and-error we quickly see that $a_4 = 14$.

Assume that there are n circles in general position on a plane. When we takeo ne circle away, say the nth circle, there are $n-1$ circles in general position on th e same plane. By induction hypothesis the $n-1$ circles divide the plane into a_{n-1} regions. Note that the nth circle intersects each of the $n-1$ circles in $2(n-1)$ distinct points. Let the $2(n-1)$ points on the nth circle be ordered clockwise as $P_1, P_2, \ldots, P_{2(n-1)}$. Then each of the $2(n-1)$ arcs

$$P_1P_2, \quad P_2P_3, \quad P_3P_4, \quad \ldots, \quad P_{2(n-2)+1}P_{2(n-1)}, \quad P_{2(n-1)}P_1$$

intersects one and only one region in the case $n-1$ circles and separate the region into two regions. There are $2(n-1)$ more regions produced when the nth circle is drawn. We thus obtain the recurrence relation

$$a_n = a_{n-1} + 2(n-1), \quad n \geq 2.$$

Repeating the recurrence relation we have

$$\begin{aligned}
a_n &= a_{n-1} + 2(n-1) \\
&= a_{n-2} + 2(n-1) + 2(n-2) \\
&= a_{n-3} + 2(n-1) + 2(n-2) + 2(n-3) \\
&\quad \cdot \\
&= a_1 + 2(n-1) + 2(n-2) + 2(n-3) + \cdots + 2 \\
&= a_1 + 2 \cdot \frac{(n-1)n}{2} \\
&= 2 + n(n-1) \\
&= n^2 - n + 2, \qquad n \geq 2.
\end{aligned}$$

This formula is also valid for $n = 1$ (since $a_1 = 2$), although it doesn't hold for $n = 0$ (since $a_0 = 1$).

Example 1.2 (Fibonacci Sequence)**.** A pair of newly born rabbits of opposite sexes is placed in an enclosure at the beginning of a year. Baby rabbits need one month to grow mature; they become an adult pair on the first day of the second month. Beginning with the second month the female is pregnant, and gives exactly one birth of one pair of rabbits of opposite sexes on the first day ofthe third month, and gives exactly one such birth on the first day of each next month. Each new pair also gives such birth to a pair of rabbits on the first day of each month starting from the third month (from its birth). Find the number of pairs of rabbits in the enclosure after one year?

Let f_n denote the number of pairs of rabbits on the first day of the nth month. Some of these pairs are adult and some are babies. Let a_n and b_n denote the number of pairs of adult and baby rabbits respectively on the first day of the nth month. Then the total number of pairs of rabbits on the first day of the nth month is $f_n = a_n + b_n$, $n \geq 1$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n	0	1	1	2	3	5	8	13	21	34	55	89	144
b_n	1	0	1	1	2	3	5	8	13	21	34	55	89
f_n	1	1	2	3	5	8	13	21	34	55	89	144	233

If a pair is adult on the first day of the nth month, then it gives one birth of one pair on the first day of the next month. So $b_{n+1} = a_n$, $n \geq 1$. Note that each adult pair on the first day of the nth month is still an adult pair on first day of the next month and each baby pair on the first day of the nth month becomes an adult pair on the first day of the next month, we have $a_{n+1} = a_n + b_n = f_n$, $n \geq 1$. Thus

$$f_n = a_n + b_n = f_{n-1} + a_{n-1} = f_{n-1} + f_{n-2}, \qquad n \geq 3.$$

Let us define $f_0 = 0$.

The sequence $f_0, f_1, f_2, f_3, \ldots$ satisfying the recurrence relation

$$\begin{cases} f_n = f_{n-1} + f_{n-2}, \ n \geq 2 \\ f_0 = 0 \\ f_1 = 1 \end{cases} \tag{1}$$

is known as the **Fibonacci sequence**, and its terms are known as **Fibonacci numbers**.

Example 1.3. The partial sum of Fibonacci sequence is

$$s_n = f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1. \tag{2}$$

This can be verified by induction on n. For $n = 0$, we have $s_0 = f_2 - 1 = 0$. Now for $n \geq 1$, we assume that it is true for $n - 1$, i.e., $s_{n-1} = f_{n+1} - 1$. Then

$$\begin{aligned} s_n &= f_0 + f_1 + \cdots + f_n \\ &= s_{n-1} + f_n \end{aligned}$$

$$= f_{n+1} - 1 + f_n \qquad \text{(by the induction hypothesis)}$$
$$= f_{n+2} - 1. \qquad \text{(by the Fibonacci recurrence)}$$

Example 1.4. The Fibonacci number f_n is even if and only if n is a multiple of 3.

Note that $f_1 = f_2 = 1$ is odd and $f_3 = 2$ is even. Assume that f_{3k} is even, f_{3k-2} and f_{3k-1} are odd. Then $f_{3k+1} = f_{3k} + f_{3k-1}$ is odd ($even + odd = odd$), and subsequently, $f_{3k+2} = f_{3k+1} + f_{3k}$ is also odd ($odd + even = odd$). It follows that $f_{3(k+1)} = f_{3k+2} + f_{3k+1}$ is even ($odd + odd = even$).

Theorem 1.1. *The general term of the Fibonacci sequence* (f_n) *is given by*

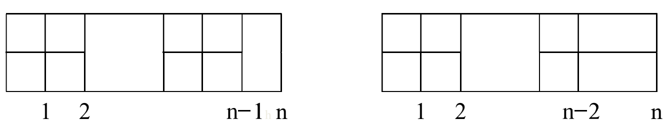
$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n \geq 0. \tag{3}$$

Example 1.5. Determine the number h_n of ways to perfectly cover a 2-by-n board with dominoes. (Symmetries are not counted in counting the number of coverings.)

We assume $h_0 = 1$ since a 2-by-0 board is empty and it has exactly one perfect cover, namely, the empty cover. Note that the first few terms can be easily obtained such as

$$h_0 = 1, \quad h_1 = 1, \quad h_2 = 2, \quad h_3 = 3, \quad h_4 = 5.$$

Now for $n \geq 3$, the 2-by-n board can be covered by dominoes in two types:



There are h_{n-1} ways in the first type and h_{n-2} ways in the second type. Thus

$$h_n = h_{n-1} + h_{n-2}, \qquad n \geq 2.$$

Therefore the sequence $(h_n; n \geq 0)$ is the Fibonacci sequence $(f_n; n \geq 0)$ with $f_0 = 0$ deleted, i.e.,

$$h_n = f_{n+1}, \qquad n \geq 0.$$

Example 1.6. Determine the number b_n of ways to perfectly cover a 1-by-n board by dominoes and monominoes.

$$b_n = b_{n-1} + b_{n-2}, \qquad b_0 = b_1 = 1,\ b_2 = 2.$$

Theorem 1.2. *The Fibonacci number f_n can be written as*

$$f_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \quad n \geq 0.$$

Proof. Let $g_0 = 0$ and

$$g_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \quad n \geq 1.$$

Note that $k > \lfloor \frac{n-1}{2} \rfloor$ is equivalent to $k > n - k - 1$. Since $\binom{m}{p} = 0$ for any integers m and p such that $p > m$, we may write g_n as

$$g_n = \sum_{k=0}^{n-1} \binom{n-k-1}{k}, \quad n \geq 1.$$

To prove the theorem, it suffices to show that the sequence (g_n) satisfies theFi bonacci recurrence relation with the same initial values. In fact, $g_0 = 0$,

$g_1 = \binom{0}{0} = 1$, and for $n \geq 0$,

$$
\begin{aligned}
g_{n+1} + g_n &= \sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=0}^{n-1} \binom{n-k-1}{k} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n-k}{k} + \sum_{k=1}^{n} \binom{n-k}{k-1} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \left[\binom{n-k}{k} + \binom{n-k}{k-1} \right] \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n-k+1}{k} \qquad \text{(By the Pascal formula)} \\
&= \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n-k+1}{k} + \binom{0}{n+1} \\
&= \sum_{k=0}^{n+1} \binom{(n+2)-k-1}{k} = g_{n+2}.
\end{aligned}
$$

We conclude that the sequence (g_n) is the Fibonacci sequence (f_n). □

2 Linear recurrence relations

Definition 2.1. A sequence $(x_n; n \geq 0)$ of numbers is said to satisfy a **linear recurrence relation of order** k if

$$x_n = \alpha_1(n)x_{n-1} + \alpha_2(n)x_{n-2} + \cdots + \alpha_k(n)x_{n-k} + \beta_n, \tag{4}$$

where $\alpha_k(n) \neq 0$, $n \geq k$, and the coefficients $\alpha_1(n), \alpha_2(n), \ldots, \alpha_k(n)$ and β_n are functions of n. The linear recurrence relation (4) is said to be **homogeneous** if $\beta_n = 0$ for all $n \geq k$, and is said to have **constant coefficients** if $\alpha_1(n), \alpha_2(n), \ldots, \alpha_k(n)$ are constants. The recurrence relation

$$x_n = \alpha_1(n)x_{n-1} + \alpha_2(n)x_{n-2} + \cdots + \alpha_k(n)x_{n-k}, \qquad (5)$$

where $\alpha_k(n)$ 0, $n \geq k$, is called the corresponding **homogeneous linear recurrence relation** of (4).

A **solution** of the recurrence relation (4) is a sequence (u_n) satisfying (4). The **general solution** of (4) is a solution

$$x_n = u_n(c_1, c_2, \dots, c_k) \tag{6}$$

with some parameters $c_1, c_2, \dots, c_k$, provided that for arbitrary initial values $u_0, u_1, \dots, u_{k-1}$ there exist constants $c_1, c_2, \dots, c_k$ such that (6) is the unique sequence satisfying both the recurrence relation (4) and the initial conditions.

Let S_∞ denote the set of all sequences $(a_n; n \geq 0)$. Then S_∞ is an infinite-dimensional vector space under the ordinary addition and scalar multiplication of sequences. Let N_k be the set all solutions of the nonhomogeneous linear recurrence relation (4), and H_k the set of all solutions of the homogeneous linearrecurrence relation (5). We shall see that H_k is a k-dimensional subspace of S_∞, and N_k a k-dimensional affine subspace of S_∞.

Theorem 2.2. (Structure Theorem for Linear Recurrence Relations)

(a) The solution space H_k *is a k-dimensional subspace of the vector space* S_∞ *of sequences. Thus, if* $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ *are linearly independent solutions of the homogeneous linear recurrence relation* (5)*, then the general solution of* (5) *is*

$$x_n = c_1 a_{n,1} + c_2 a_{n,2} + \cdots + c_k a_{n,k}, \qquad n \geq 0,$$

where $c_1, c_2, \dots, c_k$ *are arbitrary constants.*

(b) Let (a_n) *be a particular solution of the nonhomogeneous linear recurrence relation* (4)*. Then the general solution of* (4) *is*

$$x_n = a_n + h_n, \qquad n \geq 0,$$

where (h_n) *is the general solution of the corresponding homogeneous linearrecurrence relation* (5)*. In other words,* N_k *is a translate of* H_k *in* S_∞*, i.e.,*

$$N_k = (a_n) + H_k.$$

Proof. (a) To show that H_k is a vector subspace of S_∞, we need to show that H_k is closed under the addition and scalar multiplication of sequences. Let (a_n)

and (b_n) be solutions of (5). Then

$$\begin{aligned} a_n + b_n &= [\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}] \\ &\quad + [\alpha_1(n)b_{n-1} + \alpha_2(n)b_{n-2} + \cdots + \alpha_k(n)b_{n-k}] \\ &= \alpha_1(n)(a_{n-1} + b_{n-1}) + \alpha_2(n)(a_{n-2} + b_{n-2}) + \\ &\quad \cdots + \alpha_k(n)(a_{n-k} + b_{n-k}), \qquad n \geq k. \end{aligned}$$

For scalars c,

$$\begin{aligned} ca_n &= c[\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}] \\ &= \alpha_1(n)ca_{n-1} + \alpha_2(n)ca_{n-2} + \cdots + \alpha_k(n)ca_{n-k}, \qquad n \geq k. \end{aligned}$$

This means that H_k is closed under the addition and scalar multiplication ofsequences.

To show that H_k is k-dimensional, consider the projection $\pi : \mathrm{S}_\infty \to \mathbb{R}^k$, defined by

$$\pi(x_0, x_1, x_2, \ldots) = (x_0, x_1, \ldots, x_{k-1}).$$

We shall see that the restriction $\pi|\mathrm{H}_k : \mathrm{H}_k \to \mathbb{R}^k$ is a linear isomorphism. For each ($a_0, a_1, \ldots, a_{k-1}) \in \mathbb{R}^k$, define a_n inductively by

$$a_n = \alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}, \quad n \geq k.$$

Obviously, we have $\pi(a_0, a_1, a_2, \ldots) = (a_0, a_1, \ldots, a_{k-1})$. This means that $\pi|\mathrm{H}_k$ is surjective. Now let $(u_n) \in \mathrm{H}_k$ be such that $\pi(u_0, u_1, u_2, \ldots) = (0, 0, \ldots, 0)$, i.e.,

$$u_0 = u_1 = \cdots = u_{k-1} = 0.$$

Applying the recurrence relation (5) for $n = k$, we have $u_k = 0$. Again applying (5) for $n = k+1$, we obtain $u_{k+1} = 0$. Continuing to apply (5), we have $u_n = 0$for $n \geq k$. Thus (u_n) is the zero sequence. This means that π is injective. We have finished the proof that $\pi|\mathrm{H}_k$ is a linear isomorphism from H_k onto $\mathbb{R}^k$. So $\dim \mathrm{H}_k = k$.

(b) For each solution (v_n) of (4), we claim that the sequence $h_n := v_n - u_n$ $(n \geq 0)$ is a solution of (5). So

$$v_n = u_n + h_n, \quad n \geq 0.$$

In fact, since (u_n) and (v_n) are solutions of (4), applying the recurrence relation(4), we have

$$\begin{aligned} h_n &= [\alpha_1(n)v_{n-1} + \alpha_2(n)v_{n-2} + \cdots + \alpha_k(n)v_{n-k} + \beta_n] \\ &\quad - [\alpha_1(n)u_{n-1} + \alpha_2(n)u_{n-2} + \cdots + \alpha_k(n)u_{n-k} + \beta_n] \\ &= \alpha_1(n)(v_{n-1} - u_{n-1}) + \alpha_2(n)(v_{n-2} - u_{n-2}) + \cdots + \alpha_k(n)(v_{n-k} - u_{n-k}) \\ &= \alpha_1(n)h_{n-1} + \alpha_2(n)h_{n-2} + \cdots + \alpha_k(n)h_{n-k}, \qquad n \geq k. \end{aligned}$$

This means that (h_n) is a solution of (5). Conversely, for any solution (h_n) of(5), we have

$$\begin{aligned} u_n + h_n &= [\alpha_1(n)u_{n-1} + \alpha_2(n)u_{n-2} + \cdots + \alpha_k(n)u_{n-k} + \beta_n] \\ &\quad + [\alpha_1(n)h_{n-1} + \alpha_2(n)h_{n-2} + \cdots + \alpha_k(n)h_{n-k}] \\ &= \alpha_1(n)(u_{n-1} + h_{n-1}) + \alpha_2(n)(u_{n-2} + h_{n-2}) \\ &\quad + \cdots + \alpha_k(n)(u_{n-k} + h_{n-k}) + \beta_n \end{aligned}$$

for $n \geq k$. This means that the sequence $(u_n + h_n)$ is a solution of (4).

□

Definition 2.3. The **Wronskian** $W_k(n)$ of k solutions $(u_{n,1})$, $(u_{n,2})$, $\ldots$,$(u_{n,k})$ of the homogeneous linear recurrence relation (5) is the determinant

$$W_k(n) = \det \begin{bmatrix} u_{n,1} & u_{n,2} & \cdots & u_{n,k} \\ u_{n+1,1} & u_{n+1,2} & \cdots & u_{n+1,k} \\ \vdots & \vdots & & \vdots \\ u_{n+k-1,1} & u_{n+k-1,2} & \cdots & u_{n+j-1,k} \end{bmatrix}, \qquad n \geq 0.$$

The Wronskian of $(u_{n,1})$, $(u_{n,2})$, $\ldots$, $(u_{n,k})$ is also an infinite sequence

$$(W_k(n); n \geq 0).$$

Proposition 2.4. *If* $W_k(m) = 0$*, then* $W_k(n) = 0$ *for all* $n \geq m$.

Proof. Since $W_k(m) = 0$, there are constants $c_1, c_2, \ldots, c_k$, not all zero, suchthat for $i = 0, 1, \ldots, k-1$,

$$\sum_{j=1}^{k} c_j u_{m+i,j} = c_1 u_{m+i,1} + c_2 u_{m+i,2} + \cdots + c_k u_{m+i,k} = 0.$$

$j=1$

We claim that $W_k(m+1) = 0$. It is enough to show that for the same coefficients $c_1, c_2, \ldots, c_k$ above, the linear relation

$$\sum_{j=1}^{k} c_j u_{m+k,j} = c_1 u_{m+k,1} + c_2 u_{i,2} + \cdots + c_k u_{m+k,k} = 0$$

holds.

In fact, applying the recurrence relation (5) to $(u_{m+k,1}), (u_{m+k,2}), \ldots, (u_{m+k,k})$ respectively, we have

$$\sum_{j=1}^{k} c_j u_{m+k,j} = \sum_{j=1}^{k} c_j \sum_{i=0}^{k-1} \alpha_i(m+k) u_{m+i,j}$$

$$= \sum_{i=0}^{k-1} \alpha_i(m+k) \sum_{j=1}^{k} c_j u_{m+i,j} = 0.$$

This means that the vectors $(u_{m+i,1}), (u_{m+i,2}), \ldots, (u_{m+i,k})$, with $1 \leq i \leq k$, are linearly dependent. So $W_k(m+1) = 0$. Continue the procedure, we see that $W_k(n) = 0$ for $n \geq m$. □

Theorem 2.5. *The solutions $(u_{n,1}), (u_{n,2}), \ldots, (u_{n,k})$ of the homogeneous linear recurrence relation* (5) *are linearly independent if and only if there isa nonnegative integer n_0 such that the Wronskian*

$$W_k(n_0) \neq 0.$$

In other words, the solutions $(u_{n,1}), (u_{n,2}), \ldots, (u_{n,k})$ are linearly dependentif and only if $W_k(n) = 0$ for all $n \geq 0$.

Proof. We show that the sequences $(u_{n,1}), (u_{n,2}), \ldots, (u_{n,k})$ are linearly dependent if and only if $W_k(n) = 0$ for all $n \geq 0$. If $(u_{n,1}), (u_{n,2}), \ldots, (u_{n,k})$ are linearly dependent, then for $n \geq 0$ the columns of the matrix

$$\begin{matrix} u_{n,1} & u_{n,2} & \cdots & u_{n,k} \\ u_{n+1,1} & u_{n+1,2} & \cdots & u_{n+1,k} \end{matrix}$$

$$u_{n+k-1,1} \quad u_{n+k-1,2} \quad \cdots \quad u_{n+j-1,k}$$

are linearly dependent because the columns are part of the sequences $(u_{n,1})$, $(u_{n,2}), \ldots, (u_{n,k})$ respectively. It follows from linear algebra that the determinantof the matrix is zero, i.e., the Wronskian $W_k(n) = 0$ for all $n \geq 0$.

Conversely, if $W_k(n) = 0$ for all $n \geq 0$, in particular, $W_k(0) = 0$, then there are constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$\sum_{j=1}^{k} c_j u_{i,j} = c_1 u_{i,1} + c_2 u_{i,2} + \cdots + c_k u_{i,k} = 0, \qquad i = 0, 1, \ldots, k-1.$$

We claim that $\sum_{j=1}^{k} c_j u_{n,j} = 0$ for all n by induction. Assume it is true for $n \leq m-1$. Applying the recurrence relation (5) to the sequences $(u_{m,1})$, $(u_{m,2})$, $\ldots$, $(u_{m,k})$, we have

$$\sum_{j=1}^{k} c_j u_{m,j} = \sum_{j=1}^{k} c_j \sum_{i=1}^{k} \alpha_i(k) u_{m-i,j}$$

$$= \sum_{i=1}^{k} \alpha_i(k) \sum_{j=1}^{k} c_j u_{m-i,j} = 0.$$

This means that the sequences $(u_{n,1})$, $(u_{n,2})$, $\ldots$, $(u_{n,k})$ are linearly dependent.

□

3 Homogeneous linear recurrence relations with constant coefficients

In this section we only consider linear recurrence relations of the form

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \cdots + \alpha_k x_{n-k}, \qquad \alpha_k \quad 0, \quad n \geq k, \tag{7}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants. We call this kinds of recurrence relations as **homo geneous linear recurrence relations of order k with constant coefficients**. S ometimes it is convenient to write (7) as of the form

$$\alpha_0 x_n + \alpha_1 x_{n-1} + \cdots + \alpha_k x_{n-k} = 0, \qquad n \geq k \tag{8}$$

where $\alpha_0 /= 0$ and $\alpha_k \quad 0$. The following polynomial equation

$$\alpha_0 t^k + \alpha_1 t^{k-1} + \cdots + \alpha_{k-1} t + \alpha_k = 0, \tag{9}$$

is called the **characteristic equation** associated with the recurrence relation (8), and the polynomial on the left side of (9) is called the **characteristic polynomial**.

Example 3.1. The Fibonacci sequence (f_n; $n \geq 0$) satisfies the linear recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2$$

of order 2 with $\alpha_1 = \alpha_2 = 1$ in (7).

Example 3.2. The geometric sequence (x_n; $n \geq 0$), where $x_n = q^n$, satisfiesthe linear recurrence relation

$$x_n = qx_{n-1}, \qquad n \geq 1$$

of order 1 with $\alpha_1 = q$ in (7).

It is quite heuristic that solutions of the first order homogeneous linear recurrence relations are geometric sequences. This hints that the recurrence relation (7) may have solutions of the form $x_n = q^n$. The following theorem confirms thespeculation.

Theorem 3.1. (a) *For any number $q \neq 0$, the geometric sequence*

$$x_n = q^n$$

is a solution of the kth order homogeneous linear recurrence relation (8) *withconstant coefficients if and only if the number q is a root of the characteristicequation* (9).

(b) *If the characteristic equation* (9) *has k distinct roots $q_1, q_2, \ldots, q_k$, then the general solution of* (8) *is*

$$x_n = c_1q_1^n + c_2q_2^n + \cdots + c_kq_k^n, \qquad n \geq 0. \tag{10}$$

Proof. (a) Put $x_n = q^n$ into the recurrence relation (8). We obtain

$$\alpha_0q^n + \alpha_1q^{n-1} + \cdots + \alpha_kq^{n-k} = 0. \tag{11}$$

Since $q \neq 0$, dividing both sides of (11) by q^{n-k}, we have

$$\alpha_0q^k + \alpha_1q^{k-1} + \cdots + \alpha_{k-1}q + \alpha_k = 0 \tag{12}$$

This means that (11) and (12) are equivalent.

This finishes the proof of Part(a).

(b) Since $q_1, q_2, \ldots, q_k$ are roots of the characteristic equation (9), $x_n = q_i^n$ are solutions of the homogeneous linear recurrence relation (8) for all i ($1 \le i \le k$). Since the solution space of (8) is a vector space, the linear combination

$$x_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n, \qquad n \ge 0$$

are also solutions (8). Now given arbitrary values for $x_0, x_1, \ldots, x_{k-1}$, the sequence (x_n) is uniquely determined by the recurrence relation (8). Set

$$c_1 q_1^i + c_2 q_2^i + \cdots + c_k q_k^i = x_i, \qquad 0 \le i \le k-1.$$

The coefficients $c_1, c_2, \ldots, c_k$ are uniquely determined by Cramer's rule as fol-lows:

$$c_i = \frac{\det A_i}{\det A}, \qquad 1 \le i \le k$$

where A is the **Vandermonde matrix**

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix},$$

and A_i is the matrix obtained from A by replacing its ith column by the column $[x_0, x_1, \ldots, x_{k-1}]^T$. The determinant of A is given by

$$\det A = \prod_{1 \le i < j \le k} (q_j - q_i) \neq 0.$$

This finishes the proof of Part (b). □

Example 3.3. Find the sequence (x_n) satisfying the recurrence relation

$$x_n = 2x_{n-1} + x_{n-2} - 2x_{n-3},$$

$n \geq 3$

and the initial conditions $x_0 = 1$, $x_1 = 2$, and $x_2 = 0$.

Solution. The characteristic equation of the recurrence relation is

$$x^3 - 2x^2 - x + 2 = 0.$$

Factorizing the equation, we have

$$(x - 2)(x + 1)(x - 1) = 0.$$

There are three roots $x = 1, -1, 2$. By Theorem 3.1, we have the general solution

$$x_n = c_1(-1)^n + c_2 + c_3 2^n.$$

Applying the initial conditions,

$$\begin{cases} c_1 + c_2 + c_3 = 1 \\ c_1 - c_2 + 2c_3 = 2 \\ c_1 + c_2 + 4c_3 = 0 \end{cases}$$

Solving the linear system we have $c_1 = 2$, $c_2 = -2/3$, $c_3 = -1/3$. Thus

$$x_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}2^n.$$

Theorem 3.2. (a) *Let q be a root with multiplicity m of the characteristic equation* (9) *associated with the kth order homogeneous linear recurrence relation* (8) *with constant coefficients. Then the m sequences*

$$x_n = q^n, \quad nq^n, \quad \ldots, \quad n^{m-1}q^n$$

are linearly independent solutions of the recurrence relation (8).

(b) *Let $q_1, q_2, \ldots, q_s$ be distinct nonzero roots with the multiplicities*

$$m_1, \quad m_2, \quad \ldots, \quad m_s$$

respectively for the characteristic equation (9). *Then the sequences*

$$\begin{array}{rllll} x_n = & q_1^n, & nq_1^n, & \ldots, & n^{m_1-1}q_1^n; \\ & q_2^n, & nq_2^n, & \ldots, & n^{m_2-1}q_2^n; \\ & \cdot & \cdot & & \cdot \\ & q_s^n, & nq_s^n, & \ldots, & n^{m_s-1}q_s^n; \quad n \geq 0 \end{array}$$

are linearly independent solutions of the homogeneous linear recurrence relation (8). *Their linear combinations form the general solution of the recurrence relation* (8).

Proof. The **falling factorial polynomial** of degree i is the polynomial

$$[t]_i := t(t-1)\cdots(t-i+1).$$

Let q be a root of the characteristic polynomial

$$P(t) = \alpha_0 t^k + \alpha_1 t^{k-1} + \cdots + \alpha_{k-1} t + \alpha_k = (t-q)^m Q(t)$$

with multiplicity m. Then q is a root of the ith derivative $P^{(i)}(t)$ of $P(t)$, where $0 \le i \le m-1$. We claim that $x_n = [n]_i q^n$ is a solution of the recurrence relation for each $i = 0, 1, \dots, m-1$. In fact,

$$\begin{aligned}
\text{LHS of (8)} &= \alpha_0 [n]_i q^n + \alpha_1 [n-1]_i q^{n-1} + \cdots + \alpha_k [n-k]_i q^{n-k} \\
&= q^i \cdot \frac{d^i}{dt^i}\Big|_{t=q} \left(\alpha_0 t^n + \alpha_1 t^{n-1} + \cdots + \alpha_k t^{n-k}\right) \\
&= q^i \cdot \frac{d^i}{dt^i}\Big|_{t=q} \left(\alpha_0 t^k + \alpha_1 t^{k-1} + \cdots + \alpha_{k-1} t + \alpha_k\right) t^{n-k} \\
&= q^i \cdot \frac{d^i}{dt^i}\Big|_{t=q} (t-q)^m R(t), \quad \text{where } R(t) = Q(t) t^{n-k}.
\end{aligned}$$

Applying the Leibniz rule,

$$\begin{aligned}
\text{LHS of (8)} &= q^i \sum_{j=0}^{i} \binom{i}{j} \left(\frac{d^j}{dt^j}\Big|_{t=q} (t-q)^m\right) \left(\frac{d^{i-j}}{dt^{i-j}}\Big|_{t=q} R(t)\right) \\
&= q^i \sum_{j=0}^{i} \binom{i}{j} [m]_j (t-q)^{m-j}\Big|_{t=q} \frac{d^{i-j}}{dt^{i-j}}\Big|_{t=q} R(t) = 0.
\end{aligned}$$

So $x_n = [n]_i q^n$ is a solution. Note that

$$[t]_i = a_0 + a_1 t + \cdots + a_i t^i, \qquad (i = 0, 1, \dots, m-1)$$

for some integers $a_0, a-1, \dots, a_i$. It follows that there some integers $b_0, b_1, \dots, b_i$ such that

We see that

$$t^i = b_0[t]_0 + b_1[t]_1 + \cdots + b_i[t]_i, \qquad i = 0, 1, \ldots, m - 1.$$

$$x_n = n^i q^n = b_0[n]_0 q^n + b_1[n]_1 q^n + \cdots + b_i[n]_i q^n$$

is a solution, where $0 \le i \le m-1$. Next we show that $q^n, nq^n, \ldots, n^{m-1}q^n$ are linearly independent. Set

$$c_0 q^n + c_1 n q^n + \cdots + c_{m-1} n^{m-1} q^n = 0, \qquad \forall n \ge 0.$$

Since $q \neq 0$, we have

$$c_0 + c_1 n + \cdots + c_{m-1} n^{m-1} = 0 \qquad \forall n \ge 0.$$

Consider $c_0, c_1, \ldots, c_{m-1}$ as variables and let $n = 1, 2, \ldots, m$. We have a system of linear equations about $c_0, c_1, \ldots, c_{m-1}$. The coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 1^2 & \cdots & 1^{m-1} \\ 1 & 2 & 2^2 & \cdots & 2^{m-1} \\ 1 & 3 & 3^2 & \cdots & 3^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & m & m^2 & \cdots & m^{m-1} \end{pmatrix}$$

and $\det A = \prod_{1 \le i < j \le m} (j - i) \neq 0$. Hence $c_0 = c_1 = \cdots = c_{m-1} = 0$. □

Proof of General Linear Independence of Solutions: Consider the linear combination

$$\sum_{j=1}^{s} (c_{j,1} q_j^n + c_{j,2} n q_j^n + \cdots + c_{j,m_j} n^{m_j - 1} q_j^n) = 0 \tag{13}$$

with coefficients $c_{j,1}, c_{j,2}, \ldots, c_{j,m_j}$, where $j = 1, \ldots, s$. We assume the roots $q_1, \ldots, q_s$ are already arranged in increasing order of their modulus as $|q_1| \le |q_2| \le \cdots \le |q_s|$, and the roots of equal modulus are are already arranged in increasing order of their multiplicities. Let $q_r, \ldots, q_s$ be all roots of largest modulus with the same largest multiplicity $m = m_r = \cdots = m_s$. We may have $r = 1$. Assume $r > 1$. Then

$$|q_1| \le \cdots \le |q_{r-1}| \le |q_r| = \cdots = |q_s|.$$

We have either $|q_{r-1}| < |q_r|$ or $|q_{r-1}| = |q_r|$. If $|q_{r-1}| = |q_r|$, we must have $m_{r-1} < m_r$. Let $\omega_j = q_j / q_s$ for $j = 1, \ldots, s$. Then $\omega_r, \ldots, \omega_s$ are distinct roots of unity. Divide both sides of (13) by $n^{m-1} q_s^n$ and move the lower order terms to the right, we obtain

$$\sum_{j=r}^{s} c\,\omega_j^n = \varepsilon(n) \to 0 \ (n \to \infty).$$

Let $A(n)$ be the $(s-r+1)\times(s-r+1)$ matrix

$$A(n) = \begin{bmatrix} \omega_r^n & \omega_{r+1}^n & \cdots & \omega_s^n \\ \omega_r^{n+1} & \omega_{r+1}^{n+1} & \cdots & \omega_s^{n+1} \\ \vdots & \vdots & & \vdots \\ \omega_r^{n+s-r} & \omega_{r+1}^{n+s-r} & \cdots & \omega_s^{n+s-r} \end{bmatrix},$$

and $A_j(n)$ the matrix obtained from $A(n)$ by replacing its jth column with

$$[\varepsilon(n), \varepsilon(n+1), \ldots, \varepsilon(n+s-r)]^T.$$

Then $\det A(n) = \omega_r^n \omega_{r+1}^n \cdots \omega_s^n \prod_{r\le i<j\le s}(\omega_j - \omega_i) \ne 0$. By Cramer's rule, we have

$$c_{j,m} = \frac{\det A_j(n)}{\det A(n)} \to 0 \ (n \to \infty), \qquad j = r, \ldots, s.$$

It follows that $c_{j,m} = 0$ for $j = r, \ldots, s$. Likewise, applying the same method to other nonzero coefficients with highest order, we see that all coefficients in (13) are zero. $\square$

4 Matric representation of linear recurrence relations

Let us write $\boldsymbol{x}_0 = [x_0, x_1, \ldots, x_{k-1}]^T$, $\boldsymbol{x}_1 = [x_k, x_{k+1}, \ldots, x_{2k-1}]^T, \ldots,$

$$\boldsymbol{x}_n = [x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}]^T, \qquad n \ge 0.$$

Assume $x_m = a_1 x_{m-1} + a_2 x_{m-2} + \cdots + a_k x_{m-k}$, $m \ge k$. We see that

$$\boldsymbol{x}_n = A\boldsymbol{x}_{n-1}, \qquad n \ge 1.$$

Then

$$\boldsymbol{x}_n = A^n \boldsymbol{x}_0, \qquad n \ge 0.$$

Example 4.1. Find the sequence (x_n) satisfying the recurrence relation

$$x_n = 2x_{n-1} + x_{n-2} - 2x_{n-3},$$

$n \geq 3$

and the initial conditions $x_0 = 1$, $x_1 = 2$, and $x_2 = 0$.

The recurrence relation can be repeated as

$$\begin{aligned} x_3 &= 2x_2 + x_1 - 2x_0 = -2x_0 + x_1 + 2x_2 \\ x_4 &= 2x_3 + x_2 - 2x_1 = -4x_0 + 5x_2 \\ x_5 &= 2x_4 + x_3 - 2x_2 = -10x_0 + x_1 + 10x_2 \end{aligned}$$

Thus

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ -4 & 0 & 5 \\ -10 & 1 & 10 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad \text{i.e.,} \quad \boldsymbol{x}_1 = A\boldsymbol{x}_0.$$

It follows that $\boldsymbol{x}_n = A\boldsymbol{x}_{n-1} = A^2\boldsymbol{x}_{n-2} = \cdots = A^n\boldsymbol{x}_0$.

$$\boldsymbol{x}_n = \begin{bmatrix} x_{3n} \\ x_{3n+1} \\ x_{3n+2} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ -4 & 0 & 5 \\ -10 & 1 & 10 \end{bmatrix}^n \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad n \geq 0.$$

5 Nonhomogeneous linear recurrence relations with constant coefficients

Theorem 5.1. *Given a nonhomogeneous linear recurrence relation of the first order*

$$x_n = \alpha x_{n-1} + \beta_n, \qquad n \geq 1. \tag{14}$$

(a) *Let $\beta_n = cq^n$ be an exponential function of n. Then* (14) *has a particular solution of the following form.*

- *If $q \neq \alpha$, then $x_n = Aq^n$.*
- *If $q = \alpha$, then $x_n = Anq^n$.*

(b) *Let $\beta_n = \sum_{i=0}^{k} b_i n^i$ be a polynomial function of n with degree k.*

- *If* $\alpha \neq 1$, *then* (14) *has a particular solution of the form*

$$x_n = A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k,$$

where the coefficients $A_0, A_1, \ldots, A_k$ are recursively determined as

$$A_k = \frac{b_k}{1-\alpha},$$

$$A_i = \frac{1}{1-\alpha}\left(b_i + \alpha \sum_{j=i+1}^{k} (-1)^{j-i} \binom{j}{i} A_j\right), \qquad 0 \le i \le k-1.$$

- *If $\alpha = 1$, then the solution of* (14) *is given by*

$$x_n = x_0 + \sum_{i=1}^{n} \beta_i.$$

Proof. (a) We may assume $q \neq 0$; otherwise the recurrence (14) is homogeneous.

For the case $q \neq \alpha$, put $x_n = Aq^n$ in (14); we have

$$Aq^n = \alpha A q^{n-1} + cq^n.$$

The coefficient A is determined as $A = cq/(q-\alpha)$.

For the case $q = \alpha$, put $x_n = Anq^n$ in (14); we have

$$Anq^n = \alpha A(n-1)q^{n-1} + cq^n.$$

Since $q = \alpha$, then $\alpha A q^{n-1} = cq^n$. The coefficient A is determined as $A = cq/\alpha$.

(b) For the case $\alpha \neq 1$, put $x_n = \sum_{j=0}^{k} A_j n^j$ in (14); we obtain

$$\sum_{j=0}^{k} A_j n^j = \alpha \sum_{j=0}^{k} A_j (n-1)^j + \sum_{j=0}^{k} b_j n^j.$$

Then

$$\sum_{j=0}^{k} A_j n^j = \alpha \sum_{j=0}^{k} A_j \sum_{i=0}^{j} \binom{j}{i} n^i (-1)^{j-i} + \sum_{j=0}^{k} b_j n^j.$$

$$\sum_{i=0}^{k} A_i n^i = \alpha \sum_{i=0}^{k} n^i \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{i} A_j + \sum_{i=0}^{k} b_i n^i.$$

$i=0$

$$\sum_{i=0} \left(A_i - b_i - \alpha \sum_{j=i} (-1)^{j-i} \binom{j}{i} A_j \right) n^i = 0.$$

The coefficients $A_0, A_1, \ldots, A_k$ are determined recursively as

$$A_k = \frac{b_k}{1-\alpha},$$

$$A_i = \frac{1}{1-\alpha}\left(b_i + \alpha \sum_{j=i+1}^{k} (-1)^{j-i}\binom{j}{i} A_j\right), \qquad 0 \le i \le k-1.$$

As for the case $\alpha = 1$, iterate the recurrence relation (14); we have

$$\begin{aligned} x_n &= x_{n-1} + \beta_n = x_{n-2} + \beta_{n-1} + \beta_n \\ &= x_{n-1} + \beta_{n-2} + \beta_{n-1} + \beta_n = \cdots \\ &= x_0 + \beta_1 + \beta_2 + \cdots + \beta_n. \end{aligned}$$

□

Example 5.1. Solve the difference equation

$$\begin{aligned} x_n &= x_{n-1} + 3n^2 - 5n^3, \; n \ge 1 \\ x_0 &= 2. \end{aligned}$$

Solution.

$$\begin{aligned} x_n &= x_0 + \sum_{i=1}^{n} b_i = 2 + \sum_{i=1}^{n} \left(3i^2 - 5i^3\right) \\ &= 2 + 3\sum_{i=1}^{n} i^2 - 5\sum_{i=1}^{n} i^3 \\ &= 2 + 3 \times \frac{n(n+1)(2n+1)}{6} - 5 \times \left(\frac{n(n+1)}{2}\right)^2. \end{aligned}$$

We have applied the following identities

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Example 5.2. Solve the equation

$$
\begin{aligned}
x_n &= 3x_{n-1} - 4n, \; n \geq 1 \\
x_0 &= 2.
\end{aligned}
$$

Solution. Note that $x_n = 3^n c$ is the general solution of the corresponding homogeneous linear recurrence relation. Let $x_n = An + B$ be a particular solution. Then

$$An + B = 3[A(n-1) + B] - 4n$$

Comparing the coefficients of n^0 and n, it follows that $A = 2$ and $B = 3$. Thus the general solution is given by

$$x_n = 2n + 3 + 3^n c.$$

The initial condition $x_0 = 2$ implies that $c = -1$. Therefore the solution is

$$x_n = -3^n + 2n + 3.$$

Theorem 5.2. *Given a nonhomogeneous linear recurrence relation of the second order*

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + cq^n. \tag{15}$$

Let q_1 and q_2 be solutions of its characteristic equation

$$x^2 - \alpha_1 x - \alpha_2 = 0.$$

Then (15) *has a particular solution of the following forms, where A is a constant to be determined.*

(a) If $q \neq q_1, q \quad q_2$, then $x_n = Aq^n$.

(b) If $q = q_1, q_1 \neq q_2$, then $x_n = Anq^n$.

(c) If $q = q_1 = q_2$, then $x_n = An^2q^n$.

Proof. The homogeneous linear recurrence relation corresponding to (15) is

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2}, \qquad n \geq 2. \tag{16}$$

We may assume $q \neq 0$. Otherwise (15) is homogeneous.

(a) Put $x_n = Aq^n$ into (15); we have

$$Aq^n = \alpha_1 Aq^{n-1} + \alpha_2 Aq^{n-2} + cq^n.$$

Then

$$A(q^2 - \alpha_1 q - \alpha_2) = cq^2.$$

Since q is not a root of the characteristic equation $x^2 = \alpha_1 x + \alpha_2$, that is, $q^2 - \alpha_1 q - \alpha_2 \quad 0$, the coefficient A is determined as

$$A = \frac{cq^2}{q^2 - \alpha_1 q - \alpha_2}.$$

(b) Since $q = q_1 \quad q_2$, then $x_n = q^n$ is a solution of (16) but $x_n = nq^n$ is not, that is,

$$q^2 - \alpha_1 q - \alpha_2 = 0 \quad \text{and} \quad nq^n \quad \alpha_1(n-1)q^{n-1} + \alpha_2(n-2)q^{n-2}.$$

It follows that

$$\begin{aligned} nq^2 - \alpha_1(n-1)q - \alpha_2(n-2) &= n(q^2 - \alpha_1 q - \alpha_2) + \alpha_1 q + 2\alpha_2 \\ &= \alpha_1 q + 2\alpha_2 /= 0. \end{aligned}$$

Put $x_n = Anq^n$ into (15); we have

$$Anq^n = \alpha_1 A(n-1)q^{n-1} + \alpha_2 A(n-2)q^{n-2} + cq^n.$$

Then

$$A \quad nq^2 - \alpha_1(n-1)q - \alpha_2(n-2) \quad = cq^2.$$

Since $\alpha_1 q + 2\alpha_2 \quad 0$, the coefficient A is determined as

$$A = \frac{cq^2}{\alpha_1 q + 2\alpha_2}.$$

(c) Since $q = q_1 = q_2$, then both $x_n = q^n$ and $x_n = nq^n$ are solutions of (16), but $x_n = n^2 q^n$ is not. It then follows that

$$q^2 - \alpha_1 q - \alpha_2 = 0, \qquad \alpha_1 q + 2\alpha_2 = 0, \qquad \text{and}$$

$$n^2q^2 - \alpha_1(n-1)^2q - \alpha_2(n-2)^2 = n^2(q^2 - \alpha_1 q - \alpha_2) + 2n(\alpha_1 q + 2\alpha_2) - \alpha_1 q -$$

$$= -\alpha_1 q - 4\alpha_2 \neq 0.$$

Put $x_n = An^2q^n$ into (15); we have

$$Aq^{n-2}\left(n^2q^2 - \alpha_1(n-1)^2q - \alpha_2(n-2)^2\right) = cq^n.$$

The coefficient A is determined as

$$A = -\frac{cq^2}{\alpha_1 q + 4\alpha_2}.$$

□

Example 5.3. Solve the equation

$$\begin{aligned} x_n &= 10x_{n-1} - 25x_{n-2} + 5^{n+1}, \ n \geq 2 \\ x_0 &= 5 \\ x_1 &= 15. \end{aligned}$$

Put $x_n = An^2 \times 5^n$ into the recurrence relation. We have

$$An^2 \times 5^n = 10A(n-1)^2 \times 5^{n-1} - 25A(n-2)^2 \times 5^{n-2} + 5^{n+1}.$$

Dividing both sides we further have

$$An^2 = 2A(n-1)^2 - A(n-2)^2 + 5.$$

Thus $A = 5/2$. The general solution is given by

$$x_n = \frac{5}{2}n^2 5^n + c_1 5^n + c_2 n 5^n.$$

Applying the initial conditions $x_0 = 5$ and $x_1 = 15$, we have $c_1 = 5$ and $c_2 = -9/2$. Hence

$$x_n = \left(\frac{5}{2}n^2 - \frac{9}{2}n + 5\right)5^n.$$

Theorem 5.3. *Given a nonhomogeneous linear recurrence relation of thes econd order*

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \beta_n, \qquad n \geq 2, \tag{17}$$

where β_n is a polynomial function of n with degree k.

(a) *If $\alpha_1+\alpha_2 \neq 1$, then* (17) *has a particular solution of the form*

$$x_n = A_0 + A_1 n + \cdots + A_k n^k,$$

where $A_0, A_1, \ldots, A_k$ are constants to be determined. If $k \le 2$, then a particular solution has the form

$$x_n = A_0 + A_1 n + A_2 n^2.$$

(b) If $\alpha_1+\alpha_2 = 1$, then (17) *can be reduced to a first order recurrencerel ation*

$$y_n = (\alpha_1 - 1) y_{n-1} + \beta_n, \qquad n \ge 2,$$

where $y_n = x_n - x_{n-1}$ for $n \ge 1$.

Proof. (a) Let $\beta_n = \sum_{j=0}^{k} b_j n^j$. Put $x_n = \sum_{j=0}^{k} A_j n^j$ into the recurrence relation (17); we obtain

$$\sum_{j=0}^{k} A_j n^j = \alpha_1 \sum_{j=0}^{k} A_j (n-1)^j + \alpha_2 \sum_{j=0}^{k} A_j (n-2)^j + \sum_{j=0}^{k} b_j n^j$$

$$= \alpha_1 \sum_{j=0}^{k} A_j \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} n^i$$

$$+\alpha_2 \sum_{j=0}^{k} A_j \sum_{i=0}^{j} (-2)^{j-i} \binom{j}{i} n^i + \sum_{j=0}^{k} b_j n^j$$

$$= \alpha_1 \sum_{i=0}^{k} n^i \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{i} A_j$$

$$+\alpha_2 \sum_{i=0}^{k} n^i \sum_{j=i}^{k} (-2)^{j-i} \binom{j}{i} A_j + \sum_{j=0}^{k} b_j n^j.$$

Collecting the coefficients of n^i, we have

$$\sum_{i=0}^{k} \left[A_i - \alpha_1 \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{i} A_j - \alpha_2 \sum_{j=i}^{k} (-2)^{j-i} \binom{j}{i} A_j - \sum_{i=0}^{k} b_i \right] n^i = 0.$$

Since $\alpha_1 + \alpha_2 \neq 1$, the coefficients $A_0, A_1, \ldots, A_k$ are determined as

$$A_k = \frac{b_k}{1 - \alpha_1 - \alpha_2},$$

$$A_i = \frac{1}{1 - \alpha_1 - \alpha_2}\left(b_i + \sum_{j=i+1}^{k} (-1)^{j-i} \binom{j}{i} \left(\alpha_1 + 2^{j-i}\alpha_2\right) A_j \right),$$

where $0 \le i \le k - 1$.

(b) The recurrence relation (17) becomes

$$x_n = \alpha_1 x_{n-1} + (1 - \alpha_1) x_{n-2} + \beta_n, \qquad n \ge 2.$$

Set $y_n = x_n - x_{n-1}$ for $n \ge 1$; the recurrence (17) reduces to a first orderrec urrence relation. □

Example 5.4. Solve the following recurrence relation

$$\begin{aligned} x_n &= 6x_{n-1} - 9x_{n-2} + 8n^2 - 24n \\ x_0 &= 5, \\ x_1 &= 5. \end{aligned}$$

Solution. Put $x_n = A_0 + A_1 n + A_2 n^2$ into the recurrence relation; we obtain

$$A_0 + A_1 n + A_2 n^2 = 6[A_0 + A_1(n-1) + A_2(n-1)^2] - 9[A_0 + A_1(n-2) + A_2(n-2)^2] + 8n^2 -$$

Collecting the coefficients of n^2, n, and the constant, we have

$$(4A_2 - 8)n^2 + (4A_1 - 24A_2 + 24)n + (4A_0 - 12A_1 + 30A_2) = 0.$$

We conclude that $A_2 = 2$, $A_1 = 6$, and $A_0 = 3$. So $x_n = 2n^2 + 6n + 3$ is ap articular solution. Then the general solution of the recurrence is

$$x_n = 2n^2 + 6n + 3 + 3^n c_1 + 3^n n c_2.$$

Applying the initial condition $x_0 = x_1 = 5$, we have $c_1 = 2$, $c_2 = -4$. The sequence is finally obtained as

$$x_n = 2n^2 + 6n + 3 + 2 \times 3^n - 4n \times 3^n.$$

6 Generating functions

The (ordinary) **generating function** of an infinite sequence

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

is the infinite series

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots.$$

A finite sequence

$$a_0, a_1, a_2, \ldots, a_n$$

can be regarded as the infinite sequence

$$a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots$$

and its generating function

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

is a polynomial.

Example 6.1. The generating function of the constant infinite sequence

$$1, 1, \ldots, 1, \ldots$$

is the function

$$A(x) = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

Example 6.2. For any positive integer n, the generating function for the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, \ldots$$

is the function

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

Example 6.3. For any real number α, the generating function for the infinitesequence of binomial coefficients

$$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \ldots, \binom{\alpha}{n}, \ldots$$

is the function

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha.$$

Example 6.4. Let k be a positive integer and let

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

be the infinite sequence whose general term a_n is the number of nonnegativeintegral solutions of the equation

$$x_1+x_2+\cdots+x_k=n.$$

Then the generating function of the sequence (a_n) is

$$A(x) = \sum_{n=0}^{\infty} \left(\sum_{i_1+\cdots+i_k=n} 1 \right) x^n = \sum_{n=0}^{\infty} \sum_{i_1+\cdots+i_k=n} x^{i_1+\cdots+i_k}$$

$$= \left(\sum_{i_1=0}^{\infty} x^{i_1} \right) \cdots \left(\sum_{i_k=0}^{\infty} x^{i_k} \right) = \frac{1}{(1-x)^k}$$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{-k}{n} x^n = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n.$$

Example 6.5. Let a_n be the number of integral solutions of the equation

$$x_1+x_2+x_3+x_4=n,$$

where $0 \le x_1 \le 3$, $0 \le x_2 \le 2$, $x_3 \ge 2$, and $3 \le x_4 \le 5$.

The generating function of the sequence (a_n) is

$$A(x) = (1+x+x^2+x^3)(1+x+x^2)(x^2+x^3+ \quad)(x^3+x^4+x^5)$$

$$= \frac{x^5(1+x+x^2+x^3)(1+x+x^2)^2}{1-x}.$$

Example 6.6. Determine the generating function for the number of n-combinationsof apples, bananas, oranges, and pears where in each n-combination the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and the number of pears is at least two.

The required generating function is

$$A(x) = \left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{2i+1}\right)\left(\sum_{i=0}^{4} x^{i}\right)\left(\sum_{i=2}^{\infty} x^{i}\right)$$

$$= \frac{x^3(1-x^5)}{(1-x^2)^2(1-x)^2}.$$

Example 6.7. Determine the number a_n of bags with n pieces of fruit (apples,bananas, oranges, and pears) such that the number of apples is even, the number bananas is a multiple of 5, the number oranges is at most 4, and the number ofpears is either one or zero.

The generating function of the sequence (a_n) is

$$A(x) = \left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{5i}\right)\left(\sum_{i=0}^{4} x^{i}\right)\left(\sum_{i=0}^{1} x^{i}\right)$$

$$= \frac{(1+x+x^2+x^3+x^4)(1+x)}{(1-x^2)(1-x^5)}$$

$$= \frac{(1+x)(1-x^5)/(1-x)}{(1+x)(1-x)(1-x^5)}$$

$$= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (-1)^n \binom{-2}{n} x^n$$

$$= \sum_{n=0}^{\infty} \binom{n+1}{n} x^n = \sum_{n=0}^{\infty} (n+1)x^n.$$

Thus $a_n = n+1$.

Example 6.8. Find a formula for the number $a_{n,k}$ of integer solutions $(i_1, i_2, \ldots, i_k)$ of the equation

$$x_1 + x_2 + \cdots + x_k = n$$

such that $i_1, i_2, \ldots, i_k$ are nonnegative odd numbers.

The generating function of the sequence (a_n) is

$$A(x) = \left(\sum_{i=0}^{\infty} x^{2i+1}\right) \cdots \left(\sum_{i=0}^{\infty} x^{2i+1}\right) = \frac{x^k}{(1-x^2)^k}$$

$$= x^k \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^{2i} = \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^{2i+k}$$

$$= \begin{cases} \sum_{j=r}^{\infty} \binom{j+r-1}{j-r} x^{2j} & \text{for } k=2r \\ \sum_{j=r}^{\infty} \binom{j+r}{j-r} x^{2j+1} & \text{for } k=2r+1. \end{cases}$$

We then conclude that $a_{2s,2r} = \binom{s+r-1}{s-r}$, $a_{2s+1,2r+1} = \binom{s+r}{s-r}$, and $a_{n,k} = 0$ otherwise. We may combine the three cases into

$$a_{n,k} = \begin{cases} \binom{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor} & \text{if } n-k = \text{even}, \\ 0 & \text{if } n-k = \text{odd}. \end{cases}$$

Example 6.9. Let a_n denote the number of nonnegative integral solutions oft he equation

$$2x_1 + 3x_2 + 4x_3 + 5x_4 = n.$$

Then the generating function of the sequence (a_n) is

$$A(x) = \sum_{n=0}^{\infty} \left(\sum_{\substack{i,j,k,l \geq 0 \\ 2i+3j+4k+5l=n}} 1 \right) x^n$$

$$= \left(\sum_{i=0}^{\infty} x^{2i}\right) \left(\sum_{j=0}^{\infty} x^{3j}\right) \left(\sum_{k=0}^{\infty} x^{4k}\right) \left(\sum_{l=0}^{\infty} x^{5l}\right)$$

$$= \frac{1}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}.$$

Theorem 6.1. *Let s_n be the number of nonnegative integral solutions of the equation*

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = n.$$

Then the generating function of the sequence (s_n) is

$$A(x) = \frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_k})}.$$

7 Recurrence and generating functions

Since

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k, \qquad |x| < 1;$$

then

$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-ax)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k x^k, \qquad |x| < \frac{1}{|a|}.$$

Example 7.1. Determine the generating function of the sequence

$$0, 1, 2^2, \ldots, n^2, \ldots.$$

Since $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, then

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \sum_{k=0}^{\infty} \frac{d}{dx} x^k = \sum_{k=0}^{\infty} k x^{k-1}.$$

Thus $\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} k x^k$. Taking the derivative with respect to x we have

$$\frac{1+x}{(1-x)^3} = \sum_{k=0}^{\infty} k^2 x^{k-1}.$$

Therefore the desired generating function is

$$A(x) = \frac{x(1+x)}{(1-x)^3}.$$

Example 7.2. Solve the recurrence relation

$$
\begin{aligned}
a_n &= 5a_{n-1} - 6a_{n-2}, \qquad n \geq 2 \\
a_0 &= 1 \\
a_1 &= -2
\end{aligned}
$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Applying the recurrence relation, we have

$$A(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} (5a_{n-1} - 6a_{n-2})\, x^n$$

$$= a_0 + a_1 x - 5xa_0 + 5xA(x) - 6x^2 A(x).$$

Applying the initial values and collecting the coefficient functions of $A(x)$, wefu rther have

$$\left(1 - 5x + 6x^2\right) A(x) = 1 - 7x.$$

Thus the function $g(x)$ is solved as

$$A(x) = \frac{1 - 7x}{1 - 5x + 6x^2}.$$

Observing that $1 - 5x + 6x^2 = (1 - 2x)(1 - 3x)$ and applying partial fraction,

$$\frac{1 - 7x}{1 - 5x + 6x^2} = \frac{A}{1 - 2x} + \frac{B}{1 - 3x}.$$

The constants A and B can be determined by

$$A(1 - 3x) + B(1 - 2x) = 1 - 7x.$$

Then

$$\begin{aligned} A \quad + B &= \quad 1 \\ -3A \; - 2B &= -7 \end{aligned}$$

Thus $A = 5$, $B = -4$. Hence

$$A(x) = \frac{1 - 7x}{1 - 5x + 6x^2} = \frac{5}{1 - 2x} - \frac{4}{1 - 3x}.$$

Since

$$\frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n \quad \text{and} \quad \frac{1}{1 - 3x} = \sum_{n=0}^{\infty} 3^n x^n$$

We obtain the sequence

$$a_n = 5 \times 2^n - 4 \times 3^n, \qquad n \geq 0.$$

Theorem 7.1. *Let $(a_n; n \geq 0)$ be a sequence satisfying the homogeneous linear recurrence relation of order k with constant coefficients, i.e.,*

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_k a_{n-k}, \tag{18}$$

where $\alpha_k \neq 0$, $n \geq k$. Then its generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is a rational function of the form

$$A(x) = \frac{P(x)}{Q(x)}, \tag{19}$$

where $Q(x)$ is a polynomial of degree k with a nonzero constant term and $P(x)$ is a polynomial of degree strictly less than k.

Conversely, given such polynomials $P(x)$ and $Q(x)$, there exists a unique sequence (a_n) satisfying the linear homogeneous recurrence relation (18)*, and its generating function is the rational function in* (19).

Proof. The generating function $A(x)$ of the sequence (a_n) can be written as

$$A(x) = \sum_{i=0}^{k-1} a_i x^i + \sum_{n=k}^{\infty} a_n x^n = \sum_{i=0}^{k-1} a_i x^i + \sum_{n=k}^{\infty} \left(\sum_{i=1}^{k} \alpha_i a_{n-i} \right) x^n$$

$$= \sum_{i=0}^{k-1} a_i x^i + \sum_{i=1}^{k} \alpha_i \sum_{n=k}^{\infty} a_{n-i} x^n = \sum_{i=0}^{k-1} a_i x^i + \sum_{i=1}^{k} \alpha_i \sum_{n=k-i}^{\infty} a_n x^{n+i}$$

$$= \sum_{i=0}^{k-1} a_i x^i + \alpha_k x^k \sum_{n=0}^{\infty} a_n x^n + \sum_{i=1}^{k-1} \alpha_i x^i \left(\sum_{n=0}^{\infty} a_n x^n - \sum_{j=0}^{k-i-1} a_j x^j \right)$$

$$= \sum_{i=0}^{k-1} a_i x^i + A(x) \sum_{i=1}^{k} \alpha_i x^i - \sum_{i=1}^{k-1} \alpha_i x^i \sum_{j=0}^{k-i-1} a_j x^j$$

$$= A(x)\sum_{i=1}^{h} \alpha_i x^i + \sum_{i=0}^{h} a_i x^i - \sum_{l=1}^{h} x^l \sum_{i=1}^{h} \alpha_i a_{l-i}.$$

Then

$$\left(1-\sum_{i=1}^{k}\alpha_i x^i\right)A(x) = \sum_{i=0}^{k-1} a_i x^i - \sum_{l=1}^{k-1} x^l\left(\sum_{i=1}^{l}\alpha_i a_{l-i}\right)$$

$$= a_0 + \sum_{l=1}^{k-1}\left(a_l - \sum_{i=1}^{l}\alpha_i a_{l-i}\right)x^l.$$

Thus

$$P(x) = a_0 + \sum_{l=1}^{k-1}\left(a_l - \sum_{i=1}^{l}\alpha_i a_{l-i}\right)x^l,$$

$$Q(x) = 1 - \sum_{i=1}^{k}\alpha_i x^i.$$

Conversely, let (a_n) be the sequence whose generating function is $A(x) = P(x)/Q(x)$. Write

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad P(x) = \sum_{i=0}^{k} b_i x^i, \qquad Q(x) = 1 - \sum_{i=1}^{k}\alpha_i x^i.$$

Then $A(x) = \frac{P(x)}{Q(x)}$ is equivalent to

$$\left(1-\sum_{i=1}^{k}\alpha_i x^i\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{i=0}^{k} b_i x^i.$$

The polynomial $Q(x)$ can be viewed as an infinite series with $\alpha_i = 0$ for $i > k$. Thus

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty}\left(\sum_{i=1}^{n}\alpha_i a_{n-i}\right)x^n = \sum_{i=0}^{k} b_i x^i.$$

Equating the coefficients of x^n, we have the recurrence relation

$$a_n = \sum_{i=1}^{k} \alpha_i a_{n-i}, \qquad n \geq k.$$

□

Proposition 7.2 (Partial Fractions)**.** (a) *If $P(x)$ is a polynomial of degree less than k, then*

$$\frac{P(x)}{(1-ax)^k} = \frac{A_1}{1-ax} + \frac{A_2}{(1-ax)^2} + \cdots + \frac{A_k}{(1-ax)^k},$$

where $A_1, A_2, \ldots, A_k$ are constants to be determined.

(b) *If $P(x)$ is a polynomial of degree less than $p+q+r$, then*

$$\frac{P(x)}{(1-ax)^p(1-bx)^q(1-cx)^r} = \frac{A_1(x)}{(1-ax)^p} + \frac{A_2(x)}{(1-bx)^q} + \frac{A_3(x)}{(1-cx)^r},$$

where $A_1(x)$, $A_2(x)$, and $A_3(x)$ are polynomials of degree $q+r$, $p+r$, and $p+q$, respectively.

8 A geometry example

A polygon P in $\mathbb{R}^2$ is called **convex** if the segment joining any two points in P is also contained in P. Let C_n denote the number of ways to divide a labeled convex polygon with $n+2$ sides into triangles. The first a few such numbers are $C_1 = 1$, $C_2 = 2$, $C_3 = 5$.

We first establish a recurrence relation for C_{n+1} in terms of $C_0, C_1, \ldots, C_n$. Let $P_{v_1v_2\ldots v_{n+3}}$ denote a convex $(n+3)$-polygon with vertices $v_1, v_2, \ldots, v_{n+3}$. In each triangular decomposition of $P_{v_1v_2\ldots v_{n+3}}$ into triangles, the segment v_1v_{n+3} is one side of a triangle Δ in the decomposition; the third vertex of the triangle Δ isone of the vertices $v_2, v_3, \ldots, v_{n+2}$. Let v_{k+2} be the third vertex of Δ other than v_1 and v_{n+3} $(0 \le k \le n)$; see Figure 1 below. Then we have a convex $(k+2)$-

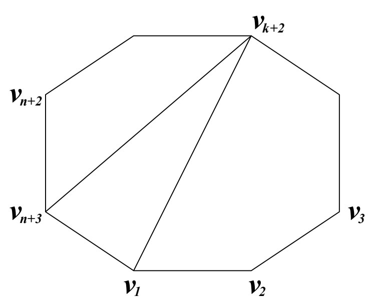


Figure 1: v_{k+2} is the third vertex of the triangle with the side v_1v_{n+3}

polygon $P_{v_1 v_2 \ldots v_{k+2}}$ and another convex $(n-k+2)$-polygon $P_{v_{k+2} v_{k+3} \ldots v_{n+3}}$. Then by induction there are C_k ways to divide $P_{v_1v_2\ldots v_{k+2}}$ into triangles, and C_{n-k} ways to divide $P_{v_{k+2}v_{k+3}\ldots v_{n+3}}$ into triangles. We thus have the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad \text{with} \quad C_0 = 1.$$

Consider the generating function $F(x) = \sum_{n=0}^{\infty} C_n x^n$. Then

$$\begin{aligned}
F(x)F(x) &= \left(\sum_{n=0}^{\infty} C_n x^n\right)\left(\sum_{n=0}^{\infty} C_n x^n\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} C_k C_{n-k}\right) x^n \\
&= \sum_{n=0}^{\infty} C_{n+1} x^n = \frac{1}{x}\sum_{n=1}^{\infty} C_n x^n \\
&= \frac{F(x)}{x} - \frac{1}{x}.
\end{aligned}$$

We thus obtain the equation

$$xF(x)^2 - F(x) + 1 = 0.$$

Solving for $F(x)$, we obtain

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Note that

$$\sqrt{1-4x} = 1 + \sum_{n=1}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} 4^n x^n$$

$$= 1 + \qquad a_n x^n,$$

$$n=1$$

where

$$
\begin{aligned}
a_n &= (-1)^n \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} \cdot 2^n \cdot 2^n \\
&= (-1)^n \frac{(-1)(-3)(-5)\cdots(-2(n-1)+1)}{n!} \cdot 2^n \\
&= -\frac{1\cdot 3\cdot 5\cdots(2(n-1)-1)}{n!} \cdot 2^n \\
&= -2 \cdot \frac{(2(n-1))!}{n!(n-1)!}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\sqrt{1-4x} &= 1 - 2\sum_{n=1}^{\infty} \frac{(2(n-1))!}{n!(n-1)!} x^n \\
&= 1 - 2\sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^{n+1}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
F(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n.
\end{aligned}
$$

Hence the sequence (C_n) is given by the binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0.$$

The sequence (C_n) is known as the **Catalan sequence** and the numbers C_n as the **Catalan numbers**.

Example 8.1. Let C_n be the number of ways to evaluate a matrix product

$$A_1 A_2 \cdots A_{n+1}, \qquad n \geq 0$$

by adding various parentheses. For instance, $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, and $C_3 = 5$. In general the formula is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Note that each way of evaluating the matrix product $A_1A_2 \cdots A_{n+2}$ will be finished by multiplying of two matrices at the end. There are exactly $n+1$ ways of multiplying the two matrices at the end:

$$A_1A_2 \cdots A_{n+2} = (A_1 \cdots A_{k+1})(A_{k+2} \cdots A_{n+2}), \qquad 0 \le k \le n.$$

This yields the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

Thus $C_n = \frac{1}{n+1}\binom{2n}{n}$, $n \ge 0$.

9 Exponential generating functions

The ordinary generating function method is a powerful algebraic tool for finding unknown sequences, especially when the sequences are certain binomial coefficients or the order is immaterial. However, when the sequences are not binomial type or the order is material in defining the sequences, we may need to consider a different type of generating functions. For example, the sequence $a_n = n!$ is the counting of the number of permutations of n distinct objects; its ordinary generating function

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n! x^n$$

cannot be easily figure out as a closed known expression. However, the generating function

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is obvious.

The **exponential generating function** of a sequence $(a_n; n \geq 0)$ is the infinite series

$$E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Example 9.1. The exponential generating function of the sequence

$$P(n,0),\ P(n,1),\ \ldots,\ P(n,n),\ 0,\ \ldots$$

is given by

$$E(x) = \sum_{k=0}^{\infty} \frac{P(n,k)}{k!} x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= (1+x)^n.$$

Example 9.2. The exponential generating function of the constant sequence $(a_n = 1; n \geq 0)$ is

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

The exponential generating function of the geometric sequence $(a_n = a^n; n \geq 0)$ is

$$E(x) = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} = e^{ax}.$$

Theorem 9.1. *Let $M = \{n_1\alpha_1, n_2\alpha_2, \ldots, n_k\alpha_k\}$ be a multiset over the set $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ with n_1 many α_1's, n_2 many α_2's, ..., n_k many α_k's. Let a_n be the number of n-permutations of the multiset M. Then the exponential generating function of the sequence $(a_n; n \geq 0)$ is given by*

$$E(x) = \left(\sum_{i=0}^{n_1} \frac{x^i}{i!}\right)\left(\sum_{i=0}^{n_2} \frac{x^i}{i!}\right) \cdots \left(\sum_{i=0}^{n_k} \frac{x^i}{i!}\right)$$

.   (20)

Proof. Note that $a_n = 0$ for $n > n_1+\cdots+n_k$. Thus $E(x)$ is a polynomial. The right side of (20) can be expanded to the form

$$\sum_{i_1,i_2,\dots,i_k=0}^{n_1,n_2,\dots,n_k} \frac{x^{i_1+i_2+\cdots+i_k}}{i_1!i_2!\cdots i_k!} = \sum_{n=0}^{n_1+n_2+\cdots+n_k} \frac{x^n}{n!} \sum_{\substack{i_1+i_2+\cdots+i_k=n\\ 0\le i_1\le n_1,\dots,0\le i_k\le n_k}} \frac{n!}{i_1!i_2!\cdots i_k!}.$$

Note that the number of permutation of M with exactly i_1 α_1's, i_2 α_2's, ..., and i_k α_k's such that

$$i_1 + i_2 + \cdots + i_k = n$$

is the multinomial coefficient

$$\binom{n}{i_1, i_2, \dots, i_k} = \frac{n!}{i_1!i_2!\cdots i_k!}.$$

It turns out that the sequence (a_n) is given by

$$a_n = \sum_{\substack{i_1+i_2+\cdots+i_k=n\\ 0\le i_1\le n_1,\dots,0\le i_k\le n_k}} \binom{n}{i_1, i_2, \dots, i_k}, \qquad n \ge 0.$$

We proved the closed form of the required exponential generating function. □

Example 9.3. Determine the number of ways to color the squares of a 1-by-n chessboard using the colors, red, white, and blue, if an even number of squares are colored red.

Let a_n denote the number of ways of such colorings and set $a_0 = 1$. Each such coloring can be considered as a permutation of three objects r (for red), w (for white), and b (for blue) with repetition allowed, and the element r appears even number of times. The exponential generating function of the sequence (a_n) is

$$E(x) = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2$$

$$= \frac{e^x + e^{-x}}{2} e^{2x} = \frac{1}{2}\left(e^{3x} + e^x\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (3^n + 1) \cdot \frac{x^n}{n!}.$$

Thus the sequence is given by

$$a_n = \frac{3^n + 1}{2}, \quad n \geq 0.$$

Example 9.4. Determine the number a_n of n digit (under base 10) numberswith each digit odd where the digit 1 and 3 occur an even number of times.

Assume $a_0 = 1$. The number a_n equals the number of n-permutations of the multiset $M = \{\infty 1, \infty 3, \infty 5, \infty 7, \infty 9\}$, in which 1 and 3 occur an even number of times. The exponential generating function of the sequence a_n is

$$E(x) = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)^2 \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^3$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 e^{3x}$$

$$= \frac{1}{4}\left(e^{5x} + 2e^{3x} + e^x\right)$$

$$= \frac{1}{4}\left(\sum_{n=0}^{\infty} \frac{5^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$= \sum^{\infty} \frac{5^n + 2 \times 3^n + 1}{} \frac{x^n}{}.$$

$n=0$ $\quad 4 \quad n!$

Thus

$$a_n = \frac{5^n + 2 \times 3^n + 1}{4}, \quad n \geq 0.$$

Example 9.5. Determine the number of ways to color the squares of a 1-by-n board with the colors, red, blue, and white, where the number of red squares is even and there is at least one blue square.

The exponential generating function for the sequence is

$$E(x) = \left(\sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!}\right)\left(\sum_{i=0}^{\infty} \frac{x^{i}}{i!}\right)\left(\sum_{i=1}^{\infty} \frac{x^{i}}{i!}\right)$$

$$= \frac{e^{x}+e^{-x}}{2}\, e^{x}(e^{x}-1)$$

$$= \frac{1}{2}\left(e^{3x}-e^{2x}+e^{x}-1\right)$$

$$= -\frac{1}{2} + \sum_{n=0}^{\infty} \frac{3^n-2^n+1}{2}\cdot\frac{x^n}{n!}$$

Thus

$$a_n = \frac{3^n - 2^n + 1}{2}, \quad n \geq 1$$

and

$$a_0 = 0.$$