Unit - I

NUMERICAL SOLUTIONS OF EQUATIONS AND INTERPOLATION

Polynomial function :- A function 'f(x)' is said to be a polynomial function, if f(x) is a polynomial in "x".

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n.$$

Algebraic & Transcendental equations :- The equation f(x)=0 is said to be algebraic if f(x) is purely polynomial in "x" and if f(x) contains trigonometric, logarithemic, exponential etc, then the equation f(x)=0 is called a Transcendental equation.

Ex :- (i) $f(x) = x^2 - 6x + 8 = 0$, $f(x) = x^3 - 6x^2 + 8x - 2 = 0$ are algebraic equations.

(ii) $f(x) = c_1 e^{x} + c_2 e^{x} = 0$, $f(x) = 2\log x - \pi/4 = 0$ are transcendental equations

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Intermediate value Theorem :-

If $f(x)$ is continuous in the $[a, b]$ and $f(a)$, $f(b)$ are of opposite signs, then the equation $f(x)=0$ has atleast one root between $x=a$ and $x=b$

Regula Faltimethod (or) False position method :-

Let $f(x)=0$ be the given equation. Let a and b are the two initial approximate values. So that $f(a)$ and $f(b)$ have opposite signs, then a root lies between a and b.

Step 1:- First order expression is

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

calculate $f(x_1)$.

Step 2:- If $f(x_1)$ is negative and $f(a)$ is positive then the root lies between x_1 and a.

$$\text{Then } x_2 = \frac{a f(x_1) - x_1 f(a)}{f(x_1) - f(a)}.$$

This process is continued until, we get the desired accuracy.

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Step 8:- If $f(x_1)$ is negative and $f(a)$ is positive then the root lies between x_1 and a

Then $x_2 = \dfrac{a f(x_1) - x_1 f(a)}{f(x_1) - f(a)}$.

This process is continued until, we get the desired accuracy.

P) By using ~~regula~~ Regula-Falsi method. Find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2? Carry out three approximations.

Sol:- Given equation

$f(x) = x^4 - x - 10$

and given that $x_0 = 1.8$ and $x_1 = 2$

$f(x_0) = f(1.8) = (1.8)^4 - (1.8) - 10$

$= -1.3024 < 0$

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$f(x_1) = 2^4 - 2 - 10$

$= 4 > 0$

$\therefore f(2) = 4 > 0$

$f(x_0)$ and $f(x_1)$ have opposite signs.

$\therefore$ The root lies between 1.8 and 2.

The first order approximation

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(0)}$$

$$= \frac{1.8(4) - 2(-1.3024)}{4 - (-1.3024)}$$

$$x_2 = 1.849$$

$f(x_2) = f(1.849)$

$= (1.849)^4 - (1.849) - 10$

$= -0.161 < 0$

$\therefore f(x_2)$ has the negative sign.

$\therefore$ The root lies between 1.849 and 2.

$\therefore$ The second order approximation

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

Here $x_1 = 2$ and $x_2 = 1.849$

$$= \frac{2 f(1.849) - 1.849 f(2)}{f(1.849) - f(2)}$$

$$= \frac{2(-0.161) - 1.849(4)}{-0.161 - 4}$$

$$x_3 = 1.8549.$$

$$f(x_3) = f(1.8549) = (1.8549)^4 - (1.8549) - 10$$

$$\cancel{f(x_3)} = -0.019 < 0$$

$\therefore$ $f(x_3)$ has the negative sign.

$\therefore$ The root lies between 1.8549 and 2.

Here $x_1 = 2$ and $x_3 = 1.8549$.

$\therefore$ The third order approximation

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The Third order approximation

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{2 f(1.8549) - 1.8549 f(2)}{f(1.8549) - f(2)}$$

$$= \frac{2 \times (-0.019) - 1.8549 (4)}{-0.019 - 4}$$

$$\doteq 1.8557 > 0$$

$\therefore$ By using Regula-Falsi method the root of the given equation is

1.8557.

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2) Find the root of the equation $x \log_{10} x = 1.2$ by using Regula - Falsi (or) False- Position method? Carry out three approximations.

Sol:- Given equation

$$f(x) = x \log_{10} x - 1.2$$

$$f(0) = 0 \times \log 0 - 1.2$$
$$= -1.2 < 0$$

$$f(1) = 1 \log 1 - 1.2$$
$$= -1.2 < 0$$

$$f(2) = 2 \log_{10}(2) - 1.2$$
$$= -0.5979 < 0$$

$$f(3) = 3 \log_{10}(1.2) - 1.2$$
$$= 0.23136 > 0$$

$\therefore$ $f(2)$ and $f(3)$ have the opposite signs.

$\therefore$ The root lies between 2 and 3.

∴ The first order approximation

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Here $x_0 = 2$ and $x_1 = 3$.

$$= \frac{2 f(3) - 3 f(2)}{f(3) - f(2)}$$

$$= \frac{2(0.23136) - 3(-0.5979)}{0.23136 - (-0.5979)}$$

$$x_2 = 2.7210$$

$$f(x_2) = (2.7210) \log_{10}(2.7210) - 1.2$$

$$= -0.0171 < 0$$

∴ $f(x_2)$ has the negative sign.

∴ The root lies between 2.7210 and 3.

∴ The second order approximation

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

Here $x_1 = 3$; $x_2 = 2.7210$

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$$x_3 = \frac{3\, f(2.7210) - 2.7210\, f(3)}{f(2.7210) - f(3)}$$

$$= \frac{3(-0.0171) - (2.7210)(0.23136)}{-0.0171 - 0.23136}$$

$$= \cancel{2.7410}\ 2.7406.$$

$$f(x_3) = f(2.7406) = 2.7406 \log_{10}(2.7406) - 1.2$$

$$= -0.00056 < 0$$

$\therefore$ $f(x_3)$ has the negative sign.

$\therefore$ The root lies between 2.7406 and 3.

$\therefore$ The third order approximation

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

Here $x_1 = 3$ and $x_3 = 2.7406$

$$= \frac{3(-0.00056) - 2.7406(0.23136)}{-0.00056 - 0.23136}$$

$$= 2.74122$$

$\therefore$ By using Regula-Falsi method, the root of the given equation is 2.74122

P) Find the real root of $xe^{x} = 2$ using Regula - Falsi method ?

Sol^n :- Let $f(x) = xe^{x} - 2 = 0$.

Then $f(0) = -2 < 0$, $f(1) = e - 2 = 2.7183 - 2$
$= 0.7183 > 0$.

$f(0)$ and $f(1)$ have the opposite signs.

$\therefore$ The root lies between 0 and 1.

Here $x_0 = 0$ and $x_1 = 1$.

By Regula - Falsi method.

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0 - (-2)}{0.7183 - (-2)}$$

$$= \frac{2}{2.7183} = 0.73575$$

$$f(x_2) = (0.73575)e^{0.73575} - 2 = -0.46445 < 0$$

$\therefore f(x_2)$ have the negative sign.

$\therefore$ The root (x_3) lies between x_1 and x_2.

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{(0.73575)(0.7183) - (1)(0.46445)}{0.7183 + 0.46445}$$

$$= \frac{0.992939}{1.18275} = 0.83951$$

$$f(x_3) = (0.83951)\, e^{0.83951} - 2 = -0.056339 < 0.$$

$f(x_3)$ has negative sign

$\therefore$ the root lies between x_1 and x_3

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$= \frac{(0.83951)(0.7183) + 0.056339}{0.7183 + 0.056339.}$$

$$= 0.851171.$$

$$f(x_4) = f(0.851171) = (0.851171)\, e^{0.851171} - 2$$

$$= -0.0062278 < 0.$$

$f(x_4)$ has negative sign.

Now the root lies between x_1 and x_4

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$= \frac{(0.851171)(0.87183) + 0.006227}{0.7183 + 0.006227}$$

$$= \frac{0.617623}{0.724527} = 0.85245$$

$$f(x_5) = f(0.85245) = (0.85245)\, e^{0.85245} - 2$$

$$= -0.0006756 < 0.$$

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$f(x_5)$ has the negative sign.

$\therefore$ The root lies between x_1 and x_5.

$$x_6 = \frac{x_5 f(x_1) - x_1 f(x_5)}{f(x_1) - f(x_5)}$$

$$= \frac{(0.85245)(0.7183) + 0.00067756}{0.7183 + 0.00067756}$$

$$= \frac{0.612990}{0.71897} = 0.85260$$

$$\cancel{f(x_6) = (0.85260)\, e^{0.85260} - 2 = 0.85260}$$

$\therefore$ The root of $x e^{x} - 2 = 0$ is 0.85260

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Newton – Raphson method :-

(i) Let $f(x)=0$ be the given equation.

Let $f(a)$ and $f(b)$ have the opposite signs for $x=a$, $x=b$.

(ii) Then find $x_0 = \frac{a+b}{2}$.

(iii) Then by using Newton-Raphson method, we can find the approximate root by using the following formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

By putting $n = 0, 1, 2, ---$. we get the approximate roots of the given equation.

P) Find a real root of $x^3 - x - 2 = 0$ by using Newton-Raphson method ?

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Solⁿ:- Given equation

$f(x) = x^3 - x - 2$

$f(0) = 0^3 - 0 - 2 = -2 < 0.$

$f(1) = 1^3 - 1 - 2 = -2 < 0.$

$f(2) = 2^3 - 2 - 2 = 8 - 2 - 2 = 4 > 0.$

$\therefore$ $f(1)$ and $f(2)$ have opposite signs.

$\therefore$ The root lies between 1 and 2.

$$\therefore x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$$

$\therefore$ By using Newton-Raphson method, the formula for approximate root is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow (1)$$

$f(x) = x^3 - x - 2$

$f'(x) = 3x^2 - 1$

Put $n = 0$ in eq (1) $\Rightarrow$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\cancel{f(0) = 0^3 - 0 - 2.}$

$\cancel{f'(0) = 3(0)^2 - 1 = 3 - 1 = -}$

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$f(x_0) = f(1.5) = (1.5)^3 - (1.5) - 2 = -0.125$

$f'(x_0) = f'(1.5) = 3(1.5)^2 - 1 = \cancel{3}5.75$

$\therefore x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 1.5 - \dfrac{(-0.125)}{5.75}$

$= 1.5 + \dfrac{0.125}{5.75}$

$x_1 = 1.5217$

put $n=1$ in eq (1).

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$f(x_1) = f(1.5217) = (1.5217)^3 - (1.5217) - 2 = 0.0019$

$f'(x_1) = \cancel{3(} f'(1.5217) = 3(1.5217)^2 - 1$

$= 5.9467$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1.5217 - \dfrac{0.0019}{5.9467}$

$x_2 = 1.5213$

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∴ the approximate root of the given equation is $\underline{\underline{1.521}}$

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2) Find a root of $e^x \sin x = 1$ by using Newton - Raphson method ?

Solⁿ :- Given $f(x) = e^x \sin x - 1.$

$f(0) = e^0 \sin 0 - 1 = -1 < 0.$

$f(1) = e^1 \sin(1) - 1.$

$f(1) = 1.2873 > 0$

$\therefore$ the root lies between 0 and 1.

$$\therefore x_0 = \frac{a+b}{2} = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

By using Newton-Raphson method, the formula for approximate is given by.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = e^x \sin x - 1.$

$f'(x) = e^x \sin x + e^x \cos x.$

put $n = 0$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x_0) = e^{0.5} \sin(0.5) - 1 = -0.2095$

$f'(x_0) = e^{0.5} \sin(0.5) + e^{0.5} \cos(0.5) = 2.2373$

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$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{(-0.2095)}{2.2373}$$

$$= 0.5 + \frac{0.2095}{2.2373}$$

$$x_1 = 0.59\cancel{8}36$$

put $n=1$

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\cancel{f(x_1) = e^1 \sin(1) - 1 = 0.0126}$$

$$\cancel{f'(x_1) = e^1 \sin(1) + e^1 \cos(1) = 2.5134}$$

$\cancel{x_{1+1}}$

$$f(x_1) = e^{0.5936} \sin(0.5936) - 1 = 0.0126.$$

$$f'(x_1) = e^{0.5936} \sin(0.5936) + e^{0.5936} \cos(0.5936)$$

$$= 2.5134$$

$$x_{1+1} = 0.5936 - \frac{0.0126}{2.5134}$$

$$x_2 = 0.5936 - 0.005013$$

$$x_2 = 0.5886$$

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Put $n=2$.

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= \cancel{0.58}$$

$$f(x_2) = e^{0.5886} \sin(0.5886) - 1 = \cancel{0.588} \; -0.000081$$

$$f'(x_2) = e^{0.5886} \sin(0.5886) + e^{0.5886} \cos(0.5886)$$

$$= 2.4982$$

$$x_{2+1} = x_3 = 0.5886 + \frac{0.000081}{2.4982}$$

$$= 0.5886.$$

$\therefore$ The approximate root of the given equation is 0.5886.

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1) Find the square root of 24 by using Newton-Raphson method ?

Let x be the square root of 24

Sol(1): then $f(x) = x^2 - 24 = 0$

$f(0) = 0^2 - 24 = -24 < 0$

$f(1) = 1^2 - 24 = -23 < 0$

$f(2) = 2^2 - 24 = -20 < 0$

$f(4) = 4^2 - 24 = -8 < 0$

$f(5) = 5^2 - 24 = 1 > 0$

$\therefore$ The root lies between 4 and 5

$\therefore x_0 = \frac{4+5}{2} = 4.5$

$\therefore$ By using Newton-Raphson method, the formula for approximate root is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (1)$$

$f(x) = x^2 - 24$

$f'(x) = 2x$

Put $n = 0$ in eq (1)

The first approximation is

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x_0) = 0^2 - 24 = \cancel{-24}$

$f'(\cancel{x_0}) = f'$

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$f(x_0) = f(4.5) = x_0^2 - 24 = (4.5)^2 - 24 = -3.75$

$f'(x_0) = f'(4.5) = 2x_0 = 2(4.5) = 9.$

$$\therefore x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 4.5 - \frac{(-3.75)}{9}$$

$$= 4.9166.$$

The second order approximation is

Put $n=1$ in eq (i) $\Rightarrow$

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(x_1) = f(4.9166) = x_1^2 - 24 = (4.9166)^2 - 24 = 4.8991$

$f'(x_1) = 2x_1 = 2(4.9166) = \cancel{4.8}\ 9.8322$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 4.9166 - \frac{4.8991}{9.8332}$$

$$= 4.8989.$$

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Put $n=2$ in eq (1) $\Rightarrow$

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$f(x_2) = x_2^2 - 24 = (4.8989)^2 - 24 = -0.00078$$

$$f'(x_2) = 2x_2 = 2(4.8989) = 9.7978.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 4.8989 - \frac{(-0.00078)}{9.7978}$$

$$= 4.8989$$

$\therefore$ The approximate ~~a~~ square root of 24 by using Newton - Raphson method is

$$\underline{\underline{4.8989}}$$

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Interpolation

Def:- The method of computing the value of the function $y = f(x)$ for any given value of x when a set of value of $y = f(x)$ for certain value of x are given

i.e., x : x_0 x_1 x_2 - - - - - - - - x_n

y : y_0 y_1 y_2 - - - - - - - - y_n

is the given values for $y = f(x)$.

The process of finding the value of $y = f(x)$ for any ~~value~~ value between $[x_0, x_n]$ is known as "interpolation".

In this technique, we discuss two types of differences of a function $y = f(x)$.

They are

(i) Forward differences.

(ii) Backward differences.

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i) Forward differences :-

Consider $y = f(x)$ is a function.

Let $y_0, y_1, y_2, ---- y_n$ be the values of y corresponding to $x_0, x_1, x_2, --- x_n$ of x respectively. Then $y_1 - y_0, y_2 - y_1, y_3 - y_2, --- y_n - y_{n-1}$ are the forward differences of y, denoted by $\Delta y_0, \Delta y_1, \Delta y_2, --- \Delta y_n$.

$\therefore \Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2 ----$

$--- \quad -- \quad \Delta y_n = y_n - y_{n-1}$.

The symbol Δ is called the forward difference operator.

The differences of the first order differences are called second order forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1 . ----$

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I) Forward difference table :-

The forward differences are arranged in a tabular column is called a forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
				$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$	
		$\Delta y_1 = y_2 - y_1$			$\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4 y_0$.
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		
				$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$	
		$\Delta y_2 = y_3 - y_2$			
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
x_4	y_4				

II) Backward differences :-

The first order backward differences are denoted by $\nabla y_1, \nabla y_2, \nabla y_3, ----$

$$\nabla y_1 = y_1 - y_0, \quad \nabla y_2 = y_2 - y_1, \quad \nabla y_3 = y_3 - y_2 ----$$

The symbol ∇ is called the backward difference operator.

The second order backward differences are denoted by $\nabla^2 y_2$, $\nabla^2 y_3$, $\nabla^2 y_4$.

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$, $\nabla^2 y_3 = \nabla y_3 - \nabla y_2$.

Backward difference Table :-

The backward differences are arranged in tabular column is called a backward difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$y_1 - y_0 = \nabla y_1$		
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	
		$y_3 - y_2 = \nabla y_3$		
x_3	y_3			

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Newton's forward Interpolation formula :-

$$y = f(x) = y_0 + P(\Delta y_0) + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$+ \cdots + \frac{P(P-1)(P-2)\cdots(P-(n-1))}{n!}\Delta^n y_0$$

where $P = \frac{x - x_0}{h}$

Newton's Backward Interpolation formula :-

$$y = f(x) = y_n + P(\nabla y_n) + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

$$+ \cdots + \frac{P(P+1)(P+2)\cdots(P+(n-1))}{n!}\nabla^n y_n$$

where $P = \frac{x - x_n}{h}$

(1) for $x = 0, 1, 2, 3, 4$, $f(x) = 1, 14, 15, 5, 6$ find $f(3)$ using forward difference table.

Sol:- Given

x	0	1	2	3	4
$y = f(x)$	1	14	15	5	6

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		13			
1	14		-12		
		1			21
2	15		-11		
		-10		22	
3	5		11		
4	6				

$P = \frac{x - x_0}{h} =$

$x = 3, x_0 = 0, h = 1$

$P = \frac{3-0}{1} = 3$

$$y = f(x) = y_0 + P(\Delta y_0) + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$y = f(3) = 1 + 3(13) + \frac{3(2)}{2}(-12) + \frac{3(2)(1)}{6} 1 + \frac{3(2)(1)(0)}{4!} \times (21)$$

$y = f(x) = f(3) = 5$

2) Applying Newton's forward Interpolation formula compute the value of $\sqrt{5.5}$. Given that $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, $\sqrt{7} = 2.646$, $\sqrt{8} = 2.828$ Correct upto three places of decimal.

Sol:- Given

x	5	6	7	8	
$f(x) = y = \sqrt{x}$	2.236	2.449	2.646	2.828	

$f(x) = \sqrt{x}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	2.236			
		0.213		
6	2.449		-0.016	
		0.197		0.001
7	2.646		-0.015	
		0.182		
8	2.828			

$x = 5.5$

$P = \frac{x - x_0}{h}$ $\quad x_0 = 5$

$h = 1$

$P = \frac{5.5 - 5}{1} = 0.5$

$$y = f(x) = y_0 + P(\Delta y_0) + \frac{P(P-1)}{2!}(\Delta^2 y_0) + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$$

$$y = f(x) = 2.236 + (0.5)(0.213) + \frac{(0.5)(-0.5)}{2}(-0.016)$$

$$+ \frac{0.5(-0.5)(-1.5)}{6}(0.001) + \cdots$$

$\doteq 2.345$

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2) Given $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$ and $\sin 60^\circ = 0.8660$, find $\sin 52^\circ$ using Newton's Interpolation formula.

Sol: Given

x	45	50	55	60
$f(x)=y=\sin x$	0.7071	0.7660	0.8192	0.8660

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
55	0.8192	0.0532		-0.0007
			-0.0064	
60	0.8660	0.0468		

$$p = \frac{x-x_0}{h}$$ $x = 52$, $x_0 = 45$, $h = 5$

$$p = \frac{52-45}{5} = 1.4$$

$$y = f(x) = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_0) + \cdots$$

$$y = f(x) = 0.7071 + (1.4)(0.0589) + \frac{(1.4)(0.4)}{2}(-0.0057) + \frac{(1.4)(0.4)(-0.6)}{6}(-0.0007) + \cdots$$

$$= 0.7071 + 0.08246 + -0.001596 + 0.0000392$$

$$= 0.788032$$

$\therefore \sin 52^\circ = 0.788032$

3) State appropriate interpolation formula which is to be used to calculate the value of $\exp(1.75)$ from the following data

x	1.7	1.8	1.9	2.0
$y = e^x$	5.474	6.050	6.686	7.389

Sol: Given

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.7	5.474			
		0.576		
			0.06	
1.8	6.050			0.007
		0.636		
			0.067	
1.9	6.686			
		0.703		
2.0	7.389			

$$P = \frac{x - x_0}{h} = \frac{1.75 - 1.7}{0.1} \qquad x = 1.75,\ x_0 = 1.7$$

$$= 0.5 \qquad h = 0.1$$

$$y = f(x) = y_0 + P(\Delta y_0) + \frac{P(P-1)}{2!}(\Delta^2 y_0) + \frac{P(P-1)(P-2)}{3!}(\Delta^3 y_0) + \cdots$$

$$= 5.474 + (0.5)(0.576) + \frac{(0.5)(-0.5)}{2}(0.06) + \frac{(0.5)(-0.5)(-1.5)}{6}(0.007)$$

$$= 5.474 + 0.288 - 0.0075 + 0.000437$$

$$= 5.75493 \qquad \therefore \boxed{e^{1.75} = 5.75493}$$

Q) Construct difference table for the following data

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$

Sol: Given

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.1	0.003						
		0.064					
			0.017				
0.3	0.067			0.002			
		0.081					
0.5	0.148		0.019		0.001		
		0.1		0.003		0	
0.7	0.248		0.022				0
					0.001		
		0.122		0.004		0	
0.9	0.370		0.026				
					0.001		
		0.148		0.005			
1.1	0.518		0.031				
		0.179					
1.3	0.697						

$p = \frac{x - x_0}{h}$ $x = 0.6$ $h = 0.2$

$x_0 = 0.1$

$p = \frac{0.6 - 0.1}{0.2} = 2.5$

$$y = f(x) = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0)$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_0)$$

$$y = f(x) = 0.003 + (2.5)(0.064) + \frac{(2.5)(1.5)}{2}(0.017) + \frac{(2.5)(1.5)(0.5)}{6}(0.002)$$

$$+ \frac{(2.5)(1.5)(0.5)(-0.5)}{24}(0.001)$$

$= 0.003 + 0.16 + 0.0318 + 0.000625 + 0.000039$

$= 0.1954$ $\boxed{f(0.6) = 0.1954}$

Q) Find Newton's forward difference interpolating polynomial for the data.

x	0	1	2	3
$f(x)$	1	3	7	13

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$

$p = x$

$$y = f(x) = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0)$$

$= 1 + 2x + \frac{x(x-1)}{2}(2)$

$= 2 + 4x + (x^2 - x)2$

$= 2x^2 + 2x + 2$

10) The following table gives corresponding values of x & y. Constructe the difference table and then express y as a function of x?

x	0	1	2	3	4
y	3	6	11	18	27

Sol:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

$$P = \frac{x - x_0}{h} = \frac{x-0}{1} = x$$

$$y = f(x) = y_0 + P(\Delta y_0) + \frac{P(P-1)}{2!}(\Delta^2 y_0) + \frac{P(P-1)(P-2)}{3!}(\Delta^3 y_0)$$

$$y = f(x) = 3 + 3x + \frac{x(x-1)}{2}(2)$$

$$= 6 + 6x + 2x^2 - 2x$$

$$= 2x^2 + 4x + 6$$

$$= x^2 + 2x + 3$$

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(19) The population of a town in the decimal census was given below. Estimate the population for the given year 1895 and 1925

year x:	1891	1901	1911	1921	1931
Population y: (thousands)	46	66	81	93	101

Sol:- (i) Given

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$$P = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

$x = 1895$
$x_0 = 1891$
$h = 10$

$$y = f(x) = y_0 + P(\Delta y_0) + \frac{P(P-1)}{2!}(\Delta^2 y_0) + \frac{P(P-1)(P-2)}{3!}(\Delta^3 y_0) + \frac{P(P-1)(P-3)(P-2)}{4!}(\Delta^4 y_0)$$

$$= 46 + (0.4)(20) + \frac{(0.4)(-0.6)}{2}(-5) + \frac{(0.4)(-0.6)(-1.6)}{6}(2) + \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24}(-3)$$

$$= 46 + 8 + 0.6 + 0.128 + 0.1248$$

$= 54.8528$ thousands.

$\therefore$ Estimated population for the year 1895 is

54.8528 thousands

(ii) Estimate the population of the year 1925 :-

Here interpolation is desired at the end of the table. Thus we use Newton's backward interpolation formula.

∵ Here $x = 1925$, $x_n = 1931$

$$P = \frac{x - x_n}{h}$$

$$= \frac{1925 - 1931}{10} = -0.6$$

∴ The Newton's Backward interpolation formula is given by

$$y = f(x) = y_n + P(\nabla y_n) + \frac{P(P+1)}{2!}(\nabla^2 y_n) +$$

$$\frac{P(P+1)(P+2)}{3!}(\nabla^3 y_n) + \frac{P(P+1)(P+2)(P+3)}{4!}(\nabla^4 y_n)$$

$+ \cdots$

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$$y = f(1925) = 101 + (-0.6)(8) + \frac{(-0.6)((-0.6)+1)}{2!}(-4)$$

$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-1) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)\cdot(-0.6+3)}{4!}(-3)$$

$$= 101 - 4.8 + 0.48 + 0.056 + 0.1008$$

$$\therefore y = f(1925) = 96.84$$

$\therefore$ Estimated population for the year 1925 is 96.84 thousands.

Interpolation with Unevenly spaced Points :-

In the Previous sections, we have derived interpolation formulas which are of great importance. But in those formulas, the disadvantage is that the values of the independent variables are to be equally spaced. we desire to have interpolation formula with unequally spaced values of the independent variables. we discuss Lagrange's Interpolation formula which uses unevenly spaced points and also for function values

Lagrange's Interpolation Formula :-

Let $x_0, x_1, x_2, \ldots x_n$ be $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2, \ldots y_n$ be the corresponding values of $y = f(x)$. Then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1) +$$

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$$\frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}f(x_2)+$$

$$\cdots+\frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}f(x_n)$$

This is known as Lagrange's Interpolation formula.

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1) Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ use lagrange's Interpolation formula.

Sol: Given

$x_0 = 1 \quad x_1 = 7 \quad x_2 = 15$

$y_0 = 168 \quad y_1 = 192 \quad y_2 = 336 \quad x = 10$

by lagrange's Interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$f(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)}(168) + \frac{(10-1)(10-15)}{(7-1)(7-15)}(192) + \frac{(10-1)(10-7)}{(15-1)(15-7)}(336)$$

$$= \frac{3\times(-5)}{(-6)(-14)}(168) + \frac{9\times(-5)}{6(-8)}(192) + \frac{(9)(3)}{14\times 8}(336)$$

$$= -30 + 180 + 81$$

$$= 231 \qquad f(10) = 231.$$

2) Using Lagrange's Interpolation formula find the value of $y(10)$ by following table

x	5	6	9	11
y	12	13	14	16

Sol: Given $x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11 \quad x = 10$

$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$

by lagrange's Interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

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$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$y(10)=\frac{\cancel{5}\times 4(-1)}{-1(-4)(-6)}(12)+\frac{(5)(\cancel{4})(-1)}{1(-3)(-5)}(13)+\frac{5(4)(-1)}{4(3)(-2)}(14)$$

$$+\frac{5(4)(+1)}{\cancel{4(3)(-2)}\ 6(5)(2)}(16)$$

$$=\frac{-4}{-24}(12)-\frac{20}{15}(13)+\frac{20}{+24}(14)+\frac{20}{60}(16)$$

$$=2-4.3333+11.666+5.3333$$

$$=14.6666$$

$$\boxed{\therefore y(10)=14.\cancel{6666}}$$

P) Find the unique polynomial $P(x)$ of degree 2 (or) less such that $P(1) = 1$, $P(3) = 27$, $P(4) = 64$ using Lagrange's interpolation formula.

Soln:- Given

$x_0 = 1$, $x_1 = 3$, $x_2 = 4$

$y_0 = f(x_0) = 1$, $y_1 = f(x_1) = 27$, $y_2 = f(x_2) = 64$

by Lagrange's interpolation formula.

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

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$$y = f(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$$

$$y = f(x) = \frac{x^2-7x+12}{(-2\times-3)} + \frac{x^2-4x-x+4}{(2\times-1)}(27) + \frac{x^2-3x-x+3}{(3\times1)}(64)$$

$$= \frac{x^2-7x+12}{6} + \frac{x^2-5x+4}{-2}(27) + \frac{x^2-4x+3}{3}(64)$$

$$= x^2\left(1/6 - 27/2 + 64/3\right) + x\left(-7/6 + 135/2 - 256/3\right) + \left(12/6 - 108/2 + 192/3\right)$$

$$= x^2(8) + x(-19) + 12$$

$$= 8x^2 - 19x + 12$$

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P) A Curve passes through the points (0, 18), (1, 10), ~~at~~ (3, −18) and (6, 90). Find the slope of the Curve at $x = 2$?

sol^n:- We are given

x	0	1	3	6
y	18	10	−18	90

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 6$

$y_0 = f(x_0) = 18$; $y_1 = f(x_1) = 10$; $y_2 = f(x_2) = -18$; $y_3 = f(x_3) = 90$

Since the arguments (x values) are not equally spaced, we will use Lagrange's formula.

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

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$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}\cdot(10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}\cdot(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}\cdot(90)$$

i.e., $f(x) = (x^2-4x+3)(x-6)(-1) + x(x^2-9x+18) +$

$x(x^2-7x+6) + x(x^2-4x+3)$

$= (-x^3+10x^2-27x+18) + (x^3-9x^2+18x) +$

$(x^3-7x^2+6x) + (x^3-4x^2+3x)$

$= 2x^3-10x^2+18$

$\therefore f(x) = 2x^3-10x^2+18$

$\therefore f'(x) = \frac{d}{dx}(f(x)) = 6x^2-20x$

Thus the slope of the curve at $x=2$ is given by.

$f'(2) = 6(2)^2 - 20(2)$

$= 6(4) - 20(2) = 24-40$

$= -16$

P.

Partial Fractions using

Lagrange's interpolation Formula

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Prob:- Using Lagrange's formula, Express the function (45)

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$$ as a sum of partial fractions.

Sol:- Let $y = 3x^2+x+1$ for $x=1,\ x=2;\ x=3$

Let

x :	$1\ (x_0)$	$x_1 = 2$	$x_2 = 3$
y :	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

The Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \cancel{\frac{(x-1)(x-2)}{(0-1)(0-2)}(5)} + \cancel{\frac{(x-1)(x-3)}{(2-1)(2-3)}(15)} + \cancel{\frac{(x-1)(x-2)}{(3-1)(3-1)}(31)}$$

Substituting the above values we get,

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

Thus $$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}$$ //

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