Module 2

Eigenvalues, Eigenvectors and Diagonalization of Matrix

Out Lines

This present module will include the following topics:

(i) Eigenvalues and eigenvectors of matrices
(ii) Characteristic equations for matrices
(iii) Similarity between matrices
(iv) Diagonalization of matrices using the concept of eigenvalues
(v) Statement of Cayley-Hamilton theorem without proof and its application
(vi) Evaluation of the matrix's power using Cayley Hamilton theorem

Introduction

In the study of linear algebra, the theory of eigenvalues and eigenvectors are of much significant. An eigen vector (sometimes, it is called as characteristic vector) is a non-zero vector concerning to a linear transformation (like, matrix) which provides sense of stretching in particular direction by the linear transformation. There is a non-zero real value associated with eigenvector which is used as a factor (scale) for stretching vector. This is known as eigenvalue or characteristic value of the linear transformation corresponding to the eigenvector. Later, in our discussion, we will provide the mathematical definitions of eigenvector and eigenvalues in more precise sense.

Eigenvalues and eigenvectors

Let us consider a system of linear homogenous equations consisting n equations and n unknowns as follows:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0$$

The above system of linear equation can be written in a matrix notation as follows:

$AX = 0$, where $A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ \cdots & \cdots. & \cdots. \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$ is a $n \times n$ square matrix, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column matrix of $n \times 1$ order and 0 the zero matrix of $1 \times n$ order.

Similarly, for a scalar λ, we consider the following system of linear homogenous equations:

$$(a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n = 0$$

Here, also the system can be written in matrix notation as follows:

$(A - \lambda I)X = 0$, where λ is a scalar, $A = \begin{bmatrix} a_{11} & a_{12} \dots.. & a_{1n} \\ a_{21} & a_{22} \dots.. & a_{2n} \\ \dots & \dots. & \dots. \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$ is a $n \times n$ square matrix, $I = \begin{bmatrix} 1 & 0 \dots.. & 0 \\ 0 & 0 \dots.. & 0 \\ \dots & \dots. & \dots. \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix of $n \times n$ order, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column matrix of $n \times 1$ order and 0 the zero matrix of $1 \times n$ order.

Now, the above system $(A - \lambda I)X = 0$ has *trivial solution* $X = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, when the matrix $(A - \lambda I)$ is *non-singular*, i.e., $det\ (A - \lambda I) \neq 0$. So, for getting *non-trivial (non-zero)* solution X, the matrix $(A - \lambda I)$ should be *singular*, i.e., $det\ (A - \lambda I) = 0$. For such cases, $(A - \lambda I)$ is called the *characteristic matrix* and $det\ (A - \lambda I) = 0$ is called the *characteristic equation* corresponding to the linear transformation (matrix) A. More precisely, the characteristic equation can be written as follows:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \dots.. & a_{1n} \\ a_{21} & a_{22} - \lambda \dots.. & a_{2n} \\ \dots & \dots. & \dots. \\ a_{n1} & a_{n2} & a_{nn} - \lambda \end{vmatrix} = 0 \tag{2.1}$$

After expanding the determinant in the left-hand side of equation (2.1), we get a polynomial equation of λ in the following way:

$$(-1)^n \lambda^n + A_1 \lambda^{n-1} + A_2 \lambda^{n-2} + \cdots + A_n = 0 \tag{2.2}$$

Equation (2.2) represents the characteristic equation as a polynomial equation for λ. In equation (2.2), A_1, A_2, …., A_n are the functions of a_{ij} in Equation (2.1). Now, it is the time to define eigenvalue using the above equations.

Definition 2.1 (Eigenvalue) Suppose, A be a $n \times n$ square matrix. Then, the roots of the polynomial equation (2.2) (known as characteristic equation of the matrix A) is called the *eigenvalues* of the matrix A. In other terminologies, they are also said to be *latent roots* or *characteristic roots*.

Definition 2.2 (Eigenvector) Suppose, A be a $n \times n$ square matrix and λ be an eigenvalue of A. Then, corresponding to the eigen value λ, there exists a non-zero vector X such that $AX = \lambda X$. The nonzero vector X is called *eigenvector* or *characteristic vector* of the matrix A corresponding to the eigenvalue λ.

Properties of Eigenvalues and Eigenvectors

Property-1 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A. Then, the sum and product of eigenvalues are equal to trace and determinant of the matrix A. That is, $\sum_{i=1}^{n} \lambda_i = Trace(A) = \sum_{i=1}^{n} a_{ii}$ and $\prod_{i=1}^{n} \lambda_i = Det(A)$.

Example 2.1 Here is an application as the immediate application of Property-1. Let us take a matrix, $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 1 & 3 \end{bmatrix}$. Then, sum of the eigenvalues, $\sum_{i=1}^{n} \lambda_i = Trace(A) = 2 + 4 + 3 = 9$ and product of the eigenvalues, $\prod_{i=1}^{n} \lambda_i = Det(A) = 2(4 \times 3 - 5 \times 1) - 0(3 \times 3 - 2 \times 5) + 1(3 \times 1 - 2 \times 4) = 14 - 0 - 5 = 9$

Property-2 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ be an eigenvalue of A corresponding to the eigenvector X. Then, for k is an integer, λ^k will be an eigenvalue of A^k corresponding to the eigenvector X.

Example 2.2 Suppose, we take $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, a diagonal matrix. Then, its eigenvalues are 2, 4 and 3. Also, $A^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ has the eigenvalues 4, 16 and 9.

Property-3 The eigenvalues of a matrix $A = [a_{ij}]_{n\times n}$ and its transpose $A^T = [a_{ji}]_{n\times n}$ have same set of eigenvalues.

Property-4 Suppose, we take two square matrices, $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ and furthermore, $A = [a_{ij}]_{n\times n}$ is non-singular (hence, A^{-1} exists), then the matrices $A^{-1}B$ and BA^{-1} have same set of eigenvalues.

Property-5 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A. Then, for a scalar, $k \neq 0, k\lambda_i$, (for $i = 1,2,..,n$) are eigenvalues of the matrix kA.

Property-6 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ is an eigenvalue of A. Then, for a scalar, $k \neq 0$, $\lambda + k$ is an eigenvalue of the matrix $A + kI$, where I is the identity matrix of order $n \times n$.

Property-7 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A.Then, for a scalar, $k \neq 0$, $\lambda_i \pm k$, (for $i = 1,2,..,n$) are eigenvalues of the matrix $A \pm kI$, where I is the identity matrix of order $n \times n$.

Property-8 Suppose, $A = [a_{ij}]_{n \times n}$ be a non-singular $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A. Then, $\lambda_i{}^{-1}$, for $i = 1,2,..,n$, are the eigenvalues of the inverse matrix A^{-1}.

Property-9 Suppose, $A = [a_{ij}]_{n \times n}$ be a non-singular $n \times n$ square matrix and λ is an eigenvalue of A. Then, $\frac{Det\ (A)}{\lambda}$ is the eigenvalue of the matrix $Adj(A)$.

Proof: We know that, $Adj(A)\ A = Det\ (A)I$. Taking the eigenvector, we get $Adj(A)(AX) = Det\ (A)X$. That is, $Adj(A)(\lambda X) = Det\ (A)X$. That is, $Adj(A)X = \frac{Det\ (A)}{\lambda}X$. So, $\frac{Det\ (A)}{\lambda}$ is the eigenvalue of the matrix $Adj(A)$.

Property-9 Suppose, $A = [a_{ij}]_{n \times n}$ be a $n \times n$ square matrix and P be a non-singular $n \times n$ square matrix. Then, the matrices A and $P^{-1}AP$ have same set of eigenvalues.

Property-10 The eigenvalues of upper triangular, lower triangular, scalar and diagonal matrices are the elements on the principal diagonal of the corresponding matrices.

Example 2.3 Suppose, we take $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 1 \\ 0 & 10 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 6 & 0 \\ 2 & 5 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. A is a diagonal matrix which have eigenvalues 2, 4 and 3. B is an upper triangular matrix which have eigenvalues 4, 10 and 7. C is a lower triangular matrix which have eigenvalues 2, 6 and 4. D is a scalar matrix which have eigenvalues 5,5 and 5.

Property-11 Suppose, $A = [a_{ij}]_{n \times n}$ be a real symmetric matrix, i.e., $A = A^T$. Then, all the eigen values of the matrix A are real.

Property-12 The eigen vectors corresponding to the distinct eigenvalues of a matrix are linearly independent.

Property-13 Suppose, $A = [a_{ij}]_{n \times n}$ be a real symmetric matrix, i.e., $A = A^T$. Then, the eigen vectors corresponding to the distinct eigen values are orthogonal.

Method for finding Eigenvalues and Eigenvectors of matrices

Step 1: Find eigenvalues solving the characteristic equation $det\ (A - \lambda I) = 0$. If λ_i, for $i = 1,2,..,n$ are roots of the equations, they are called eigenvalues of A.

Step 2: When the eigenvalues λ_i are distinct, reduce the coefficient matrix in the matrix equation $(A - \lambda_i I)X = 0$ to the echelon form. The rank of the echelon matrix should be less than n, usually $r = (n - 1)$. So, there are $n - r = 1$ linearly independent eigen vectors corresponding to each distinct eigenvalue.

Step 3: Suppose, two eigen values $\lambda_i = \lambda_i$. Then, reduce the coefficient matrix in the matrix equation $(A - \lambda_i I)X = 0$ to the echelon form. The rank of the echelon matrix should be less than n, usually $r = (n-2)$. So, there are $n - r = 2$ linearly independent eigen vectors corresponding to that pair of eigenvalues.

Example 2.3 Find the eigenvalues and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 6 \\ 1 & 2 \end{bmatrix}$.

Solution: The characteristic equation is $det\ (A - \lambda I) = 0$

That is $\begin{vmatrix} 1-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0 \Rightarrow (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = 4, -1$

So, the eigenvalues are 4 and -1.

The eigenvalues are distinct.

Now to find eigenvectors corresponding to the eigenvalue 4, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then, we have $\begin{bmatrix} 1-4 & 6 \\ 1 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad R_2 \to R_2 + \tfrac{1}{3} R_1$$

Therefore, the rank of the coefficient matrix is $r = 1$. Also, the number of unknowns is $n = 2$. Therefore, there are $n - r = 1$ linearly independent eigenvectors corresponding to the eigenvalue 4. In such case, the above written matrix equation can be written as

$$-3x_1 + 6x_2 = 0 \Rightarrow x_1 = 2x_2$$

If we take $x_2 = k$, then $x_1 = 2k$ and hence the eigen vector can be written as $X = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Now to find eigenvectors corresponding to the eigenvalue -1, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then, we have, $\begin{bmatrix} 1+1 & 6 \\ 1 & 2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \to R_2 + \tfrac{1}{2} R_1$$

Therefore, the rank of the coefficient matrix is $r = 1$. Also, the number of unknowns is $n = 2$. Therefore, there are $n - r = 1$ linearly independent eigenvectors corresponding to the eigenvalue -1. In such case, the above written matrix equation can be written as

$$2x_1 + 6x_2 = 0 \Rightarrow x_1 = -3x_2$$

If we take $x_2 = k$, then $x_1 = -3k$ and hence the eigen vector can be written as $X = k \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Exercise 2.1 Find the eigenvalues and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 6 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 \\ 6 & 2 \end{bmatrix}$.

Example 2.4 Find the eigenvalues of the matrix, $3A^3 + 5A^2 - 6A + 2I$, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.

Solution: Since, $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ is an upper triangular matrix, the elements in its principal diagonal are the eigenvalues of this matrix. Therefore, the eigenvalues are 1, 3 and -2.

When 1 is the eigenvalue of A, the eigenvalue of $3A^3 + 5A^2 - 6A + 2I$ will be $3 \times 1^3 + 5 \times 1^2 - 6 \times 1 + 2 \times 1 = 4$.

When 3 is the eigenvalue of A, the eigenvalue of $3A^3 + 5A^2 - 6A + 2I$ will be $3 \times 3^3 + 5 \times 3^2 - 6 \times 3 + 2 \times 1 = 110$.

When -2 is the eigenvalue of A, the eigenvalue of $3A^3 + 5A^2 - 6A + 2I$ will be $3 \times (-2)^3 + 5 \times (-2)^2 - 6 \times (-2) + 2 \times 1 = 10$.

Then, eigenvalues of the matrix $3A^3 + 5A^2 - 6A + 2I$ are 4, 110 and 10.

Exercise 2.2 Find the eigenvalues of the matrices A^3 and A^{-1}, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 5 \\ 0 & 0 & -2 \end{bmatrix}$.

Exercise 2.3 Find the eigenvalues of the matrix A^{-1}, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Exercise 2.4 Find the eigenvalues and eigen vectors of the matrix $B = 2A^2 - \frac{1}{2}A + 3I$, where $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$.

Exercise 2.5 Find product and sum the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Exercise 2.6 Find the eigenvalues of the matrix A^{-1}, where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Example 2.5 Find the eigenvalues and eigen vectors of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution: The characteristic equation of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Longrightarrow \lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\Longrightarrow \lambda = 0, 15, 3$$

So, the eigenvalues are $0, 15, 3$.

Now to find eigenvectors corresponding to the eigenvalue 0, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 4R_2 + 3R_1 \text{ and } R_3 \to 4R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \to R_3 + R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$8x_1 - 6x_2 + 2x_3 = 0 \tag{2.3}$$

$$10x_2 - 10x_3 = 0 \tag{2.4}$$

From Equation (2.4), $x_2 = x_3$. Let, $x_2 = x_3 = k$. Then, from Equation (2.3), $x_1 = \frac{k}{2}$.

Therefore, the corresponding eigen vector can be written as $X = \begin{bmatrix} \frac{k}{2} \\ k \\ k \end{bmatrix} = \frac{k}{2} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Now to find eigenvectors corresponding to the eigenvalue 15, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ 0 & -20 & -40 \\ 0 & -40 & -80 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 7R_2 - 6R_1 \text{ and } R_3 \to 7R_3 + 2R_1$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_3 \to R_3 - 2R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-7x_1 - 6x_2 + 2x_3 = 0 \tag{2.5}$$

$$-20x_2 - 40x_3 = 0 \tag{2.6}$$

From Equation (2.6), $x_2 = -2x_3$. Let $x_3 = k$, then $x_2 = -2k$ and from Equation (2.5), $x_1 = 2k$. Hence, the corresponding eigen vector can be written as $X = k \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Now to find eigenvectors corresponding to the eigenvalue 3, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 5R_2 + 6R_1 \; and \; R_3 \to 5R_3 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \to 2R_3 - R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$5x_1 - 6x_2 + 2x_3 = 0 \tag{2.7}$$

$$-16x_2 - 8x_3 = 0 \tag{2.8}$$

From Equation (2.4), $x_2 = -\frac{1}{2}x_3$. Let, $x_3 = k$. Then, $x_2 = -\frac{k}{2}$ and from Equation (2.7), $x_1 = -k$. Therefore, the corresponding eigen vector can be written as $X = \frac{k}{2}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

Definition 2.3 (Orthogonal vector) Two vectors X_1 and X_2 are said to orthogonal, when ${X_1}^T X_2 = {X_2}^T X_1 = 0$, where ${X_1}^T$ and ${X_2}^T$ are transposes of the vectors X_1 and X_2, respectively.

Example 2.6 Let us consider two vectors $X_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $X_2 = \begin{bmatrix}0\\1\end{bmatrix}$. Then, ${X_1}^T X_2 = [1\ 0]\begin{bmatrix}0\\1\end{bmatrix} = 0$ and ${X_2}^T X_1 = [0\ 1]\begin{bmatrix}1\\0\end{bmatrix} = 0$. Therefore, X_1 and X_2 are orthogonal vectors.

Example 2.7 Let us consider three vectors $X_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $X_2 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$ and $X_2 = \begin{bmatrix}2\\-2\\1\end{bmatrix}$ Then,

$${X_1}^T X_2 = [1\ 2\ 2]\begin{bmatrix}2\\1\\-2\end{bmatrix} = 2 + 2 - 4 = 0$$

$${X_2}^T X_3 = [2\ 1 - 2]\begin{bmatrix}2\\-2\\1\end{bmatrix} = 4 - 2 - 2 = 0 \text{ and}$$

${X_3}^T X_1 = [2 - 2\ 1]\begin{bmatrix}1\\2\\2\end{bmatrix} = 2 - 4 + 2 = 0$. Therefore, X_1 , X_2 and X_3 are orthogonal vectors.

Remark 2.1 If two eigenvalues of a matrix are equal, we cannot conclude whether the corresponding eigen vectors are linearly independent or not.

Definition 2.4 (Linearly independent vectors) We consider a set of non-zero vectors $\{X_1, X_2, \dots, X_n\}$ such that $a_1X_1 + a_2X_2 + \ \dots + a_nX_n = 0$. If there exists only the trivial solution $a_1 = a_2 =..= a_n = 0$ of the equation, then the given vectors $X_1, X_2, \dots, X_n$ are called the *linearly independent* vectors. If there exists at least one non-zero a_i such that the equation holds, then the vectors are called linearly dependent vectors.

Example 2.8 Let us consider three vectors $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $X_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $X_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Then, we consider the equation $a_1X_1 + a_2X_2 + a_3X_3 = 0$

This implies $a_1\begin{bmatrix}1\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

That is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

Here the rank of the coefficient matrix is $r = 3 = n$, the number of unknowns. So, the system has only the trivial solution $a_1 = a_2 = a_3 = 0$. Therefore, $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $X_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $X_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ are linearly independent vectors.

Exercise 2.7 Show that, three vectors $X_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $X_2 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$ and $X_2 = \begin{bmatrix}2\\-2\\1\end{bmatrix}$ are linearly independent vectors.

(Hint: Try to make an upper triangular matrix from the coefficient matrix).

Example 2.9 Verify that the sum of the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is equal to the $Trace(A)$. Also, find the corresponding eigen vectors.

Solution: We have the characteristic equation

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$$

$$\Rightarrow \lambda = 2, 3, 6$$

Therefore, eigenvalues are 2, 3 and 6 and hence their sum is 11. Again, $Trace(A)$ is the sum of the elements in the principal diagonal of the matrix and is equal to 11. Therefore, the statement is verified.

Next, to find eigenvectors corresponding to the eigenvalue 2, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to R_2 + R_1\ and\ R_3 \to R_3 - R_1$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$x_1 - x_2 + x_3 = 0 \tag{2.9}$$

$$2x_2 = 0 \tag{2.10}$$

From Equation (2.10), $x_2 = 0$. Let, $x_3 = k$. Then, from Equation (2.9), $x_1 = -k$. Therefore, the corresponding eigen vector can be written as $X = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 3, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 3-3 & -1 & 1 \\ -1 & 5-3 & -1 \\ 1 & -1 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_3 \leftrightarrow R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to R_2 + R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \to R_3 + R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$x_1 - x_2 = 0 \tag{2.11}$$

$$x_2 - x_3 = 0 \tag{2.12}$$

Let us take $x_3 = k$. Then, from Equation (2.11) and (2.12), we get $x_2 = x_1 = k$. Therefore, the corresponding eigen vector can be written as $X = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 6, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 3-6 & -1 & 1 \\ -1 & 5-6 & -1 \\ 1 & -1 & 3-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 3R_2 - R_1$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_3 \to R_3 - 2R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-3x_1 - x_2 + x_3 = 0 \tag{2.13}$$

$$-2x_2 - 4x_3 = 0 \tag{2.14}$$

Let us take $x_3 = k$. Then, from Equation (2.14), we get $x_2 = -2k$ and from Equation (2.13), we get, $x_1 = k$. Therefore, the corresponding eigen vector can be written as $X = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Exercise 2.8 Find the eigenvalues and the corresponding eigen vectors for the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Exercise 2.9 Find the eigenvalues and the corresponding eigen vectors for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$.

Definition 2.5 (Similar matrices) Two matrices S and T are said to be *similar* to each other when there exists a non-singular matrix P such that $T = P^{-1}SP$.

In this context, it is to be noted that a matrix is said to be non-singular if it has non-zero determinant and thus its inverse exists.

Definition 2.4 (Diagonalizable matrix) A square matrix is said be *diagonalizable* if it is similar to diagonal matrix. So, a square matrix S is said to be diagonalizable, if there exists a non-singular matrix P and a diagonal matrix $D = Diag(d_{11}, d_{22}, \ldots . d_{nn})$ such that $D = P^{-1}SP$.

Remark 2.2 The eigenvalues of the diagonal matrix D are its diagonal entries $d_{11}, d_{22}, \ldots . d_{nn}$. The non-singular matrix P can be formed by corresponding eigen vectors.

Remark 2.3 In Definition 2.4, the non-singular matrix P is makes the matrix S to be diagonal. Here, P is called the *modal matrix* and resulting diagonal matrix $D = Diag(d_{11}, d_{22}, \ldots . d_{nn})$ is called the *spectral matrix*.

Remark 2.4 The transformation of the matrix S to $P^{-1}SP$ is called the similarity transformation.

Example 2.10 Diagonalize the square matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = \pm 1$$

Therefore, the eigenvalues are 1 and -1.

For the eigenvalue 1, let the eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have the matrix equation

$$\begin{bmatrix} 1-1 & 0 \\ 2 & -1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Clearly, the rank of the coefficient matrix in the above system is $r = 1 < 2 = n$, the number of unknown. So, there are $n - r = 2 - 1 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equation

$$2x_1 - 2x_2 = 0 \tag{2.15}$$

From Equation (2.15), we get $x_1 = x_2 = k$ and hence the corresponding eigen vector is

$$X = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For the eigenvalue -1, let the eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have the matrix equation

$$\begin{bmatrix} 1+1 & 0 \\ 2 & -1+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Clearly, the rank of the coefficient matrix in the above system is $r = 1 < 2 = n$, the number of unknown. So, there are $n - r = 2 - 1 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equation

$$2x_1 = 0 \tag{2.16}$$

From Equation (2.16), we get $x_1 = 0$. Here, $x_2 = k$ is arbitrary and hence the corresponding eigen vector is

$$X = k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using the eigen vectors, the matrix P can be formed as follows:

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Since, $\mathrm{Det}(P) = 1 \neq 0$, we can say that P^{-1} and is given by

$$P^{-1} = \frac{Adj(P)}{Det\ (P)} = \frac{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Therefore, $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ -1+2 & 0-1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 1-1 & 0-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$, a diagonal matrix.

Hence, diagonalization of the given matrix is completed.

Example 2.11 Diagonalize the square matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0$$

$$\Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda = -2, 3 \text{ and } 6$$

Therefore, eigenvalues are -2, 3 and 6.

Next, to find eigenvectors corresponding to the eigenvalue -2, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 3R_2 - R_1 \ and\ R_3 \to R_3 - R_1$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$3x_1 + x_2 + 3x_3 = 0 \tag{2.17}$$

$$20x_2 = 0 \tag{2.18}$$

From Equation (2.18), we get $x_2 = 0$. Let, $x_3 = k$. Then, from Equation (2.17), $x_1 = -k$. Hence, the corresponding eigen vector will be $X = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 3, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 2R_2 + R_1 \; and \; R_3 \to 2R_3 + 3R_1$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; R_3 \to R_3 - R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-2x_1 + x_2 + 3x_3 = 0 \tag{2.19}$$

$$5x_2 + 5x_3 = 0 \tag{2.20}$$

From Equation (2.20), we get $x_2 = -x_3$. Let, $x_3 = k$. Then, from Equation (2.20), $x_2 = -k$ and from Equation (2.19), $x_1 = k$. Hence, the corresponding eigen vector will be $X = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 6, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 8 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 5R_2 + R_1 \; and \; R_3 \to 5R_3 + 3R_1$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; R_3 \to R_3 \mp 2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-5x_1 + x_2 + 3x_3 = 0 \tag{2.21}$$

$$-4x_2 + 8x_3 = 0 \qquad (2.22)$$

From Equation (2.22), we get $x_2 = 2x_3$. Let, $x_3 = k$. Then, from Equation (2.22), $x_2 = 2k$ and from Equation (2.21), $x_1 = k$. Hence, the corresponding eigen vector will be $X = k\begin{bmatrix}1\\2\\1\end{bmatrix}$.

Using the eigen vector the modal matrix can be obtained as

$$P = \begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$$

Since, $\text{Det}(P) = -1(-1-2) - 1(0-2) + 1(0+1) = 6 \neq 0$, we can say that P^{-1} and is given by

$$P^{-1} = \frac{Adj(P)}{Det\ (P)} = \frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}$$

Therefore, $P^{-1}AP = \frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}2 & 3 & 6\\ 0 & -3 & 12\\ -2 & 3 & 6\end{bmatrix}$$

$= \frac{1}{6}\begin{bmatrix}-12 & 0 & 0\\ 0 & 18 & 0\\ 0 & 0 & 36\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{bmatrix} = D$, a diagonal matrix.

Hence, diagonalization of the given matrix is completed.

Exercise 2.10 Diagonalize the square matrix $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 3 & -1\\ 0 & -1 & 3\end{bmatrix}$.

Exercise 2.11 Find a non-singular matrix P which can diagonalize the square matrix $A = \begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$.

Exercise 2.12 Find a non-singular matrix P which can diagonalize the square matrix $A = \begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$.

Theorem 2.1 (Cayley-Hamilton) Every square matrix satisfies of its characteristic equation.

Example 2.12 Find the matrix A^3 and A^{-1} corresponding to the square matrix $A = \begin{bmatrix}2 & 4\\ 1 & 1\end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix}2 & 4\\ 1 & 1\end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 2 = 0 \tag{2.23}$$

This is the characteristic equation of the given matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$. Therefore, according the Cayley-Hamilton theorem,

$$A^2 - 3A - 2I = O \tag{2.24}$$

In Equation (2.24), $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity matrix and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the zero matrix. From Equation (2.24) we get

$$A^2 = 3A + 2I \tag{2.25}$$

Multiplying both side by the matrix A, we get

$$A^3 = 3A^2 + 2A$$

$$= 3\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= 3\begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 36 \\ 9 & 15 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 44 \\ 11 & 17 \end{bmatrix}$$

Again, from Equation (2.24), we can write

$$\frac{1}{2}(A - 3I)A = A\ \frac{1}{2}(A - 3I) = I$$

Therefore, $A^{-1} = \frac{1}{2}(A - 3I)$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$$

Exercise 2.13 Verify the state of the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Example 2.13 Find the matrix A^{-1} corresponding to the square matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 38\lambda - 40 = 0 \qquad (2.26)$$

Therefore, the characteristic equation of the given matrix is $\lambda^3 - 11\lambda^2 + 38\lambda - 40 = 0$. According to the Cayley-Hamilton theorem,

$$A^3 - 11A^2 + 38A - 40I = O \qquad (2.27)$$

In Equation (2.27), $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the identity matrix and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the zero matrix. From Equation (2.27) we get,

$$A\frac{1}{40}(A^2 - 11A + 38I) = \frac{1}{40}(A^2 - 11A + 38I)A = I$$

Hence,

$$A^{-1} = \frac{1}{40}(A^2 - 11A + 38I)$$

$$= \frac{1}{40}\left(\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - 11\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} + 38\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{40}\left(\begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix} - \begin{bmatrix} 33 & 11 & 11 \\ -11 & 55 & -11 \\ 11 & -11 & 33 \end{bmatrix} + \begin{bmatrix} 38 & 0 & 0 \\ 0 & 38 & 0 \\ 0 & 0 & 38 \end{bmatrix}\right)$$

$$= \frac{1}{40}\begin{bmatrix} 14 & -4 & -6 \\ 2 & 8 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

Example 2.14 Find the matrix A^3 corresponding to the square matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Example 2.15 Find the matrix A^{-1} corresponding to the square matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, after verifying the statement of Cayley-Hamilton theorem.

Example 2.16 Find the matrices A^{-1} and A^4 corresponding to the square matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Example 2.17 Verify the statement of Cayley-Hamilton theorem for the square matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.