UNIT 2

Transportation and Assignment Model

UNIT II
TRANSPOTATION AND ASSGNMENT MODEL

Transportation Model

A special class of linear programming problem is Transportation Problem, where the objective is to minimize the cost of distributing a product from a number of sources (e.g. factories) to a number of destinations (e.g. warehouses) while satisfying both the supply limits and the demand requirement. The transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

The problem has more constraints and more variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem.

Transportation Algorithm: The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

Step 2: Determine the optimal solution using the MODI (Modified Distribution Method) or UV Method.

North West Corner rule:

The method starts at the North West (upper left) corner cell of the table

Step -1: Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

Step -2: Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column become zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed-out row (column).

Step -3: If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step -1.

Problem1:

Determine basic feasible solution using North West Corner Method

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Solution:

The allocation is shown in the following tableau:

						Capacity
	1 **50**	9	13	36	51	~~50~~
	24 **50**	12 **50**	16	20	1	~~100~~ ~~50~~
	14	33 **10**	1 **50**	23 **50**	26 **40**	~~150~~ ~~140~~ ~~90~~ ~~40~~
Requirement	~~100~~ ~~50~~	~~60~~ ~~10~~	~~50~~	~~50~~	~~40~~	

The arrows show the order in which the allocated (bolded) amounts are generated.

The starting basic solution is given as

$X_{11} = 50$, $X_{21} = 50$, $X_{22} = 50$ $X32 = 10$, $X_{33} = 50$, $X_{34} = 50$, $X_{35} = 40$

The corresponding transportation cost is

$50 \times 1 + 50 \times 24 + 50 \times 12 + 10 \times 33 + 50 \times 1 + 50 \times 23 + 40 \times 26 = 4420$

It is clear that as soon as a value of X_{ij} is determined, a row (column) is eliminated from.

Further Consideration. The last value of X_{ij} eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most $m + n - 1$ positive X_{ij} if the transportation problem has m sources and n destinations

Least Cost Method:

The LCM is also known as matrix minimum method, for the row and the column corresponding to which C_{ij} is minimum.

This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column. Corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column.

Then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly.

If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until left at the end with exactly one uncrossed-out row or column.

Problem2:

Determine the initial basic feasible solution using Least Cost Method Problem

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Solution:

The Least Cost method is applied in the following manner, We observe that C_{11}= 1 is the minimum unit cost in the table. Hence X_{11}= 50 and the first row is crossed out since the row has no more capacity.

						Capacity
	1 / 50	9	13	36	51	~~50~~
	24	12 / 60	16	20	1 / 40	~~100~~ ~~60~~
	14 / 50	33	1 / 50	23 / 50	26	~~150~~ ~~100~~ ~~50~~
Requirement	~~100~~ ~~50~~	~~60~~	~~50~~	~~50~~	~~40~~	

Then the minimum unit cost in the uncrossed-out row and column is C_{25}=1, hence X_{25}=40 and the fifth column is crossed out.

Next C_{33}=1is the minimum unit cost, hence X_{33}=50 and the third column are crossed out.

Next C_{22}=12 is the minimum unit cost, hence X_{22}=60 and the second column are crossed out.

Next, the uncrossed-out row and column now C_{31}=14 is the minimum unit cost,hence X_{31}=50 and crossed out the first column since it was satisfied

Finally, C_{34}=23 is the minimum unit cost, hence X_{34}=50 and the fourth column are crossed out.

So that the basic feasible solution developed by the LCM transportation cost is

1 x 50 + 12 x 60 + 1 x 40 + 14 x 50 + 1 x 50 + 23 x 50 = 2710

Note: That the minimum transportation cost obtained by the least cost method is much lower than the corresponding cost of the solution developed by using the North-West Corner Method.

Vogel Approximation Method (VAM):

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

Step 1: For each row (column) with strictly positive capacity (requirement), determine a penalty by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column).

Step 2: Identify the row or column with the largest penalty among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal, select the topmost row and the extreme left column.

Step 3: Select X_{ij} as a basic variable if C_{ij} is the minimum cost in the row or column With largest penalty, the numerical value of X_{ij} as high as possible subject to the row and the column constraints. Depending upon whether ai or bj is the smaller of the two i_{th} row or j_{th} column is crossed out.

Step 4: The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

Problem:
Solve the following transportation problem

Origin	Destination				a_i
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240

Solution:
Note: a_i = capacity (supply) b_j = requirement (demand) Now, compute the penalty for various rows and columns which is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

The highest penalty in the row or column, the highest penalty occurs in the second column and the minimum unit cost i.e. C_{ij} in this column is C_{12}=22.
Hence assign 40 to this cell i.e. X_{12}=40 and cross out the second column (since second column was satisfied.

This is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4	80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	

The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the first row and the minimum unit cost in this row is $C_{14} = 4$, hence $X_{14} = 80$ and cross out the first row. The modified table is as follows

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 8 which occurs in the third column and the minimum cost in this column is $C_{23}=9$, hence $X_{23}=30$ and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7	40	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the second row and the smallest cost in this row is $C_{24}=15$, hence $X_{24}=30$ and cross out the fourth column with the adjusted capacity, requirement and penalty values.

The modified table is as follows:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7 **30**	10	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	

The transportation cost corresponding to this choice of basic variables is

22 x 40 + 4 x 80 + 9 x 30 + 7 x 30 + 24 x 10 + 32 x 50 = 3520

Modified Distribution Method:

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method

Step 1: Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

Step 2: Introduce dual variables corresponding to the row constraints and the column constraints. If there are m origins and n destinations then there will be m+n dual variables. The dual variables corresponding to the row constraints are represented by U_i, i = 1,2., m whereas the dual variables corresponding to the column constraints are represented by V_j, j = 1,2., n The values of the dual variables are calculated from the equation given below $U_i + V_j = C_{ij}$ if $X_{ij} > 0$

Step 3: Any basic feasible solution has m + n -1 and $X_{ij} > 0$. Thus, there will be m + n -1 equation to determine m + n dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

Step 4: If $X_{ij}=0$, the dual variables calculated in Step 3 are compared with the C ij values of this allocation as $C_{ij} – U_i – V_j$. If al $C_{ij} – U_i – V_j \geq 0$, then by the *theorem of complementary slackness* it can be shown that the corresponding solution of the transportation problem is optimum. If one or more $C_{ij} – Ui – Vj < 0$, we select the cell with the least value of $C_{ij} – U_i – V_j$ and allocate as much as possible subject to the row and column constraints. The allocations of the number of adjacent cells are adjusted so that a basic variable becomes non-basic.

Step 5: A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

Problem 3:
Solve the following transportation problem using Modified Distribution method

						Supply
	1	9	13	36	51	**50**
	24	12	16	20	1	**100**
	14	33	1	23	26	**150**
Demand	**100**	**70**	**50**	**40**	**40**	**300**

Solution:
Step 1: Determine the basic feasible solution. The basic feasible solution usingleast cost method is

$X_{11}=50$, $X_{22}=60$, $X_{25}=40$, $X_{31}=50$, $X_{32}=10$, $X_{33}=50$ and $X_{34}=40$

Step 2: The dual variables U_1, U_2, U_3 and V_1, V_2, V_3, V_4, V_5 can be calculated from the corresponding C_{ij} values, that is

$U_1 + V_1=1$ $\quad$ $U_2 + V_2 = 12$ $\quad$ $U_2 + V_5 =1$

$U_3 + V_1 =14$ $\quad$ $U_3 + V_2 = 33$ $\quad$ $U_3+V_3 = 1$ $\quad$ $U_3+V_4 =23$

Step 3: Choose one of the dual variables arbitrarily is zero that is u3=0 as it occurs most often in the above equations. The values of the variables calculated are
$U_1= -13$, $U_2= -21$, $U_3= 0$ $V_1= 14$, $V_2= 33$, $V_3= 1$, $V_4= 23$, $V_5= 22$

Step 4: Now we calculate $C_{ij} – U_i – V_j$ values for all the cells where $X_{ij}=0$ (i.e. un allocated cell by the basic feasible solution) That is
Cell (1,2) = $C_{12}-U_1-V_2 = 9+13-33 = -11$
Cell (1,3) = $C_{13}-U_1-V_3 = 13+13-1 = 25$
Cell (1,4) = $C_{14}-U_1-V_4 = 36+13-23 = 26$
Cell (1,5) = $C_{15}-U_1-V_5 = 51+13-22 = 42$
Cell (2,1) = $C_{21}-U_2-V_1 = 24+21-14 = 31$
Cell (2,3) = $C_{23}-U_2-V_3 = 16+21-1 = 36$
Cell (2,4) = $C_{24}-U_2-V_4 = 20+21-23 = 18$
Cell (3,5) = $C_{35}-U_3-V_5 = 26-0-22 = 41$

Note that in the above calculation all the $C_{ij} – U_i – V_j \geq 0$ except for cell (1, 2) *where* $C_{12} – U1 – V2 = 9+13-33 = -11$.

Thus, in the next iteration x12 will be a basic variable changing one of the present basic variables non-basic. For allocating one unit in cell (1, 2), have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net transportation cost for each unit of such reallocation is -33 -1 + 9 +14 = -11

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in the cell (3, 2) will be negative. Thus, the revised basic feasible solution is

$X_{11}= 40$, $X_{12}= 10$, $X_{22}= 60$, $X_{25}= 40$, $X_{31}= 60$, $X_{33}= 50$, $X_{34}= 40$

Unbalanced Transportation Problem:

The total supply (capacity) at the origins is equal to the total demand (requirement) at the destination it is called balanced transportation problem

when the total supply is not equal to the total demand, which are called as unbalanced transportation problem.

In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero.

Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost.

Problem:

Solve the following unbalanced transportation problem

Plant	W_1	W_2	W_3	**Supply**
X	20	17	25	400
Y	10	10	20	500
Demand	400	400	500	

Solution:

In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

		Warehouses		
Plant	W_1	W_2	W_3	**Supply**
X	20	17	25	400
Y	10	10	20	500
Unsatisfied Demand	0	0	0	400
Demand	400	400	500	1300

Now this problem is solved as a balanced problem

Degenerate Transportation Problem:

In a transportation problem, if a basic feasible solution with m origins and n destinations has less than m + n -1 positive X_{ij} i.e. occupied cells, then the problem is said to be a Degenerate transportation problem

The degeneracy problem does not cause any serious difficulty, but it can cause computational problem wile determining the optimal minimum solution.

Therefore, it is important to identify a degenerate problem as early as beginning and take the necessary action to avoid any computational difficulty.

The degeneracy can be identified through the following results:

In a transportation problem, a degenerate basic feasible solution exists if and only if some partial sum of supply (row) is equal to a partial sum of demand (column). For example, the following transportation problem is degenerate. Because in this problem

$$\mathbf{a_1 = 400 = b_1} \qquad \mathbf{a_2 + a_3 = 900 = b_2 + b_3}$$

Plant	Warehouses W1	W2	W3	Supply (a_i)
X	20	17	25	400
Y	10	10	20	500
Unsatisfied demand	0	0	0	400
Demand (b_j)	400	400	500	1300

There is a technique called perturbation, which helps to solve the degenerate problems.

Perturbation Technique: The degeneracy of the transportation problem can be avoided if we ensure that no partial sum of a_i (supply) and b_j (demand) is equal. Let

$$a_i = a_i + d_i \quad i = 1, 2, \ldots\ldots, m$$
$$b_j = b_j \quad j = 1, 2, \ldots\ldots, n -1$$
$$b_n = b_n + md \quad d > 0$$

This modified problem is constructed in such a way that no partial sum of ai is equal to the bj. Once the problem is solved, we substitute d = 0 leading to optimum solution of the original problem

Example:

Plant	Warehouses W1	W2	W3	Supply (a_i)
X	20	17	25	400 + d
Y	10	10	20	500 + d
Unsatisfied demand	0	0	0	400 + d
Demand (b_j)	400	400	500 + 3d	1300 + 3d

Now this modified problem can be solved by using any of the three methods viz. North-west Corner, or Least Cost, or VAM

Assignment Model

Given n facilities, n jobs and the effectiveness of each facility to each job, here the problem is to assign each facility to one and only one job so that the measure of effectiveness if optimized. Here the optimization means Maximized or Minimized

There are many management problems has a assignment problem structure.

For example, the head of the department may have 6 people available for assignment and 6 jobs to fill. Here the head may like to know which job should be assigned to which person so that all tasks can be accomplished in the shortest time possible.

Another example a container company may have an empty container in each of the location 1, 2,3,4,5 and requires an empty container in each of the locations 6, 7, 8,9,10. It would like to ascertain the assignments of containers to various locations so as to minimize the total distance.

The third example is, a marketing set up by making an estimate of sales performance for different salesmen as well as for different cities one could assign a particular salesman to a particular city with a view to maximize the overall sales.

Note that with n facilities and n jobs there are n! possible assignments.

The simplest way of finding an optimum assignment is to write all the n! possible arrangements, evaluate their total cost and select the assignment with minimum cost. But this method leads to a Calculation problem of formidable size even when the value of n is moderate.

<u>Assignment Problem Structure and Solution</u>

The structure of the Assignment problem is similar to a transportation problem, is as follows:

		Jobs				
		1	2	...	n	
	1	c_{11}	c_{12}	...	c_{1n}	1
	2	c_{21}	c_{21}	...	c_{2n}	1
Workers		.	.	.	.	.
		.	.	.	.	
		.	.	.	.	
	n	c_{n1}	c_{n2}	...	c_{nn}	1
		1	1	...	1	

The element C_{ij} represents the measure of effectiveness when i^{th} person is assigned j^{th} job. Assume that the overall measure of effectiveness is to be minimized. The element X_{ij} represents the number of i^{th} individuals assigned to the j^{th} job. Since i^{th} the following

$X_{i1} + X_{i2} + \ldots\ldots\ldots\ldots\ldots + X_{in} = 1$, where i = 1, 2, ... , n person can be assigned only one job and j^{th} job can be assigned to only one person.

we have $X_{1j} + X_{2j} + \ldots\ldots\ldots\ldots\ldots + X_{nj} = 1$, where j = 1, 2,., n and the objective function is formulated as Minimize

$C_{11}X_{11} + C_{12}X_{12} + \ldots\ldots\ldots\ldots + C_{nn}X_{nn}$ and $X_{ij} \geq 0$

The assignment problem is actually a special case of the transportation problem where m = n and ai = bj = 1.
However, it may be easily noted that any basic feasible solution of an assignment problem contains (2n – 1) variables of which (n – 1) variables are zero.

Because of this high degree of degeneracy the usual computation techniques of a transportation problem become very inefficient. So, that a separate computation Technique is necessary for the assignment problem.

"If a constant is added to every element of a row/column of the cost matrix of an Assignment problem the resulting assignment problem has the same optimum solution as theoriginal assignment problem and vice versa".

Hungarian Method: The Hungarian Method is discussed in the form of a series of computational steps as follows, when the objective function is that of minimization type.

Step 1: From the given problem, find out the cost table. Note that if the number of origins is not equal to the number of destinations then a dummy origin or destination must be added.

Step 2: In each row of the table find out the smallest cost element, subtract this smallest cost element from each element in that row. So, that there will be at least one zero in each row of the new table. This new table is known as First Reduced Cost Table.

Step 3: In each column of the table find out the smallest cost element, subtract this smallest cost element from each element in that column. As a result of this, each row and column has at least one zero element. This new table is known as Second Reduced Cost Table.

Step 4: Now determine an assignment as follows:
i. For each row or column with a single zero element cell that has not be assigned or eliminated, box that zero element as an assigned cell.
ii. For every zero that becomes assigned, cross out all other zeros in the same row and for column.
iii. If for a row and for a column there are two or more zero and one can't be chosen by inspection, choose the assigned zero cell arbitrarily.
iv. The above procedures may be repeated until every zero element cell is either assigned(boxed) or crossed out.

Step 5: An optimum assignment is found, if the number of assigned cells is equal to the number of rows (and columns). In case we had chosen a zero cell arbitrarily, there may be an alternate optimum. If no optimum solution is found i.e. some rows or columns without an assignment then go to Step 6.
Step 6: Draw a set of lines equal to the number of assignments which has been made in Step 4, covering all the zeros in the following manner

i. Mark check (√) to those rows where no assignment has been made
ii. Examine the checked (√) rows. If any zero element cell occurs in those rows, check (√) the respective columns that contains those zeros
iii. Examine the checked (√) columns. If any assigned zero element occurs in those columns, check (√) the respective rows that contain those assigned zeros.
iv. The process may be repeated until now more rows or column can be checked.
v. Draw lines through all unchecked rows and through all checked columns.

Step 7: Examine those elements that are not covered by a line. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them, add this smallest element to every element that lies at the intersection of two lines. Then the resulting matrix is a new revised cost table

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Problem1:

A work shop contains four persons available for work on the four jobs. Only one person can work on any one job. The following table shows the cost of assigning each person to each job. The objective is to assign person to jobs such that the total assignment cost is a minimum.

		Jobs			
		1	**2**	**3**	**4**
	A	20	25	22	28
Persons	**B**	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

Solution:

Step 1: From the given problem, find out the cost table. Note that if the number of origins is not equal to the number of destinations, then a dummy origin or destination must be added.
Problem has 4 row and 4 columns. So, it is balanced problem

Step 2: In each row of the table find out the smallest cost element, subtract this smallest cost element from each element in that row. So, that there will be at least one zero in each row of the new table. This new table is known as First Reduced Cost Table.

0	**5**	**2**	**8**
0	**3**	**8**	**2**
2	**0**	**4**	**7**
2	**0**	**1**	**1**

Step 3: In each column of the table find out the smallest cost element, subtract this smallest cost element from each element in that column. As a result of this, each row and column has at least one zero element. This new table is known as Second Reduced Cost Table.

		Jobs			
		1	**2**	**3**	**4**
	A	0	5	1	7
Persons	**B**	0	3	7	1
	C	2	0	3	6
	D	2	0	0	0

Step 4: Determine an Assignment

By examine row A of the table in Step 3, we find that it has only one zero (cell A_1) box this zero and cross out all other zeros in the boxed column. In this way we can eliminate cell B_1. Now examine row C, we find that it has one zero (cell C_2) box this zero and cross out (eliminate) the zeros in the boxed column. This is how cell D_2 gets eliminated. There is one zero in the column 3. Therefore, cell D_3 gets boxed and this enables us to eliminate cell D_4. Therefore, we can box (assign) or cross out (eliminate) all zero's. The resultant table is shown below

		Jobs			
		1	**2**	**3**	**4**
	A	$\boxed{0}$	5	1	7
Persons	**B**	0	3	7	1
	C	$\times$	$\boxed{0}$	3	6
	D	2	$\times$	$\boxed{0}$	$\times$

Step 5: The solution obtained in Step 4 is not optimal. Because it is able to make three assignments when four were required

Step 6: Cover all the zeros of the table shown in the Step 4 with three lines (since already we made three assignments). Check row B since it has no assignment. Note that row B has a zero in column 1, therefore check column 1.

Then we check row A since it has a zero in column 1. Note that no other rows and columns are checked. Now draw three lines through unchecked rows (row C and D) and the checked column (column 1). This is shown in the table given below:

		Jobs			
		1	2	3	4
	A	$\boxed{0}$	5	1	7
Persons	B	0	3	7	1
	C	2	$\boxed{0}$	3	6
	D	2	0	0	0

Step 7: Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6.

Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (C_1 and D_1) that lie at the intersection of two lines.

Finally, we get the new revised cost table, which is shown below

		Jobs			
		1	2	3	4
	A	0	4	0	6
Persons	B	0	2	6	0
	C	3	0	3	6
	D	3	0	0	0

Step 8: Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9: Determine an assignment Examine each of the four rows in the table given in Step 7, we may find that it is only row C which has only one zero box this cell C_2 and cross out D_2.

Note that all the remaining rows and columns have two zeros. Choose a zero arbitrarily, say A_1 and box this cell so that the cells A_3 and B_1 get eliminated.
Now row B (cell B_4) and column 3 (cell D_4) has one zero box these cells so that cell D_4 is eliminated. Thus, all the zeros are either boxed or eliminated. This is shown in the following table

		Jobs			
		1	2	3	4
	A	$\boxed{0}$	4	$\cancel{0}$	6
Persons	B	0	2	6	$\boxed{0}$
	C	$\cancel{3}$	$\boxed{0}$	3	6
	D	3	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.
The total cost of assignment is: 78
that is **A1 + B4 + C2 + D3 = 20 + 17 + 17 + 24 = 78**

<u>Unbalanced Assignment Problem:</u>

In the previous section we assumed that the number of persons to be assigned and the number of jobs were same. Such kind of assignment problem is called as balanced assignment problem.
Suppose if the number of person is different from the number of jobs then the assignment problem is called as unbalanced.
If the number of jobs is less than the number of persons, some of them can't be assigned any job. So that we have to introduce on or more dummy jobs of zero duration to make the unbalanced assignment problem into balanced assignment problem.
This balanced assignment problem can be solved by usingthe Hungarian Method as discussed in the previous section. The persons to whom the dummy jobs are assigned are left out of assignment. Similarly, if the number of persons is less than number of jobs then we have introduce one or more dummy persons with zero duration to modify the unbalanced into balanced and then the problem is solved using the Hungarian Method. Here the jobs assigned to the dummy persons are left out.

Problem :
Solve the following unbalanced assignment problem of minimizing the total time for performing all the jobs

		Jobs 1	2	3	4	5
	A	5	2	4	2	5
	B	2	4	7	6	6
Workers	C	6	7	5	8	7
	D	5	2	3	3	4
	E	8	3	7	8	6
	F	3	6	3	5	7

Solution :

In this problem the number of jobs is less than the number of workers so we have to introduce a dummy job with zero duration. The revised assignment problem is as follows:

		Jobs 1	2	3	4	5	6
	A	5	2	4	2	5	0
	B	2	4	7	6	6	0
Workers	C	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Now the problem becomes balanced one since the number of workers is equal to the number jobs. So that the problem can be solved using Hungarian Method.

Step 1: The cost table

Workers \ Jobs	1	2	3	4	5	6
A	5	2	4	2	5	0
B	2	4	7	6	6	0
C	6	7	5	8	7	0
D	5	2	3	3	4	0
E	8	3	7	8	6	0
F	3	6	3	5	7	0

Step 2: Find the First Reduced Cost Table

Workers \ Jobs	1	2	3	4	5	6
A	5	2	4	2	5	0
B	2	4	7	6	6	0
C	6	7	5	8	7	0
D	5	2	3	3	4	0
E	8	3	7	8	6	0
F	3	6	3	5	7	0

Step 3 Find the second Cost Reduced table:

Workers \ Jobs	1	2	3	4	5	6
A	3	0	1	0	1	0
B	0	2	4	4	2	0
C	4	5	2	6	3	0
D	3	0	0	1	0	0
E	6	1	4	6	2	0
F	1	4	0	3	3	0

Step 4 : Determine an Assignment By using the Hungarian Method the assignment is made as follows

		Jobs					
		1	2	3	4	5	6
Workers	A	3	~~0~~	1	$\boxed{0}$	1	~~0~~
	B		2	4	4	2	0
	C	$\boxed{0}$	5	2	6	3	
	D	3	~~0~~	~~0~~	1	$\boxed{0}$	~~0~~
	E	6	1	4	6	2	$\boxed{0}$
	F	1	4		3	3	0

Step 5: The solution obtained in Step 4 is not optimal. Because, able to make five assignments when six were required.

Step 6: Cover all the zeros of the table shown in the Step 4 with five lines

Step 7: Develop the new revised table. Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (A_6, B_6, D_6 and F_6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below

Finally, we get the new revised cost table, which is shownbelow

Step 8: Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

		Jobs					
		1	2	3	4	5	6
Workers	A	3	0	1	0	1	1
	B	0	2	4	4	2	1
	C	3	4	1	5	2	0
	D	3	0	0	1	0	1
	E	5	0	3	5	1	0
	F	1	4	0	3	3	1

Step 9: Determine an assignment

Workers		Jobs 1	2	3	4	5	6
	A	3	$\cancel{0}$	1	$\boxed{0}$	1	1
	B	$\boxed{0}$	2	4	4	2	1
Workers	C	$\cancel{0}$	4	1	5	2	$\boxed{0}$
	D	3	$\cancel{0}$	$\cancel{0}$	1	$\boxed{0}$	1
	E	5	$\boxed{0}$	3	5	1	0
	F	1	4	$\boxed{0}$	3	3	1

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.
Thus, the worker A is assigned to Job4, worker B is assigned to job 1, worker C is assigned to job 6, worker D is assigned to job 5, worker E is assigned to job 2, and worker F is assigned to job 3. Since the Job 6 is dummy so that worker C can't be assigned. The total Minimum time is 14, that is $\mathbf{A_4 + B_1 + D_5 + E_2 + F_3 = 2 + 2 + 4 + 3 + 3 = 14}$

Infeasible Assignment Problem:

Sometimes it is possible a particular person is incapable of performing certain job or a specific job can't be performed on a particular machine. In this case the solution of the problem takes into account of these restrictions so that the infeasible assignment can be avoided.
The infeasible assignment can be avoided by assigning a very high cost to the cells where assignments are restricted or prohibited.

Problem:
A computer centre has five jobs to be done and has five computer machines to perform them. The cost of processing of each job on any machine is shown in the table below.

Computer Machines	Jobs 1	2	3	4	5
1	70	30	X	60	30
2	X	70	50	30	30
3	60	X	50	70	60
4	60	70	20	40	X
5	30	30	40	X	70

Because of specific job requirement and machine configurations certain jobs can't be done on certain machines. These have been shown by X in the cost table. The assignment of jobs to the machines must be done on a one to one basis. The objective here is to assign the jobs to the available machines so as to minimize the total cost without violating the restrictions as mentioned above.

Solution :-

Step 1: The cost Table; Because certain jobs cannot be done on certain machines we assign a high cost say for example500 to these cells i.e. cells with X and modify the cost table. The revised assignment problem is as follows**:**

		Jobs 1	2	3	4	5
Computer Machines	1	70	30	500	60	30
	2	500	70	50	30	30
	3	60	500	50	70	60
	4	60	70	20	40	500
	5	30	30	40	500	70

Now we can Solve this Problem Using Hungarian Method

Step 2: Find the First Reduced Cost Table

		Jobs 1	2	3	4	5
Computer Machines	1	40	0	470	30	0
	2	470	40	20	0	0
	3	10	450	0	20	10
	4	40	50	0	20	480
	5	0	0	10	470	40

Step 3: Find the Second Reduced Cost Table

		Jobs				
		1	2	3	4	5
Computer Machines	1	40	0	470	30	0
	2	470	40	20	0	0
	3	10	450	0	20	10
	4	40	50	0	20	480
	5	0	0	10	470	40

Step 4: Determine an Assignment

		Jobs				
		1	2	3	4	5
Computer Machines	1	40	$\boxed{0}$	470	30	~~0~~
	2	470	40	20	0	0
	3	10	450	0	$\boxed{0}$	10
	4	40	50	~~0~~	20	480
	5	$\boxed{0}$	~~0~~	$\boxed{0}$	470	40

Step 5: The solution obtained in Step 4 is not optimal. Because to make four assignments when five were required.

Step 6: Cover all the zeros of the table shown in the Step 4 with four lines

Check row 4 since it has no assignment. Note that row 4 has a zero in column 3, therefore check column3. Then we check row 3 since it has a zero in column 3. Note that no other rows and columns are checked. Now, draw four lines through unchecked rows (row 1, 2, 3 and 5) and the checked column (column 3). This is shown in the table given below.

Step 7: Develop the new revised table. Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 10. Subtract this smallest element from the uncovered cells and add 1 to elements (A_6, B_6, D_6 and F_6) that lie at the intersection of two lines.

		Jobs 1	2	3	4	5
	1	40	$\boxed{0}$	470	30	0
Computer Machines	2	470	40	20		0
	3	10	450		$\boxed{0}$	10
	4	40	50	0	20	480
	5	$\boxed{0}$	0	$\boxed{0}$	470	40

Finally, we get the new revised cost table, which is shownbelow

Step 8: Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

		Jobs 1	2	3	4	5
	1	40	0	471	30	0
Computer Machines	2	470	40	21	0	0
	3	0	440	0	10	0
	4	30	40	0	10	470
	5	0	0	11	470	40

Step 9: Determine an assignment
Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.
Thus, the Machine1 is assigned to Job5, Machine 2 is assigned to job4, Machine3 is assigned tojob1, Machine 4 is assigned to job3 and Machine5 is assigned to job2.

The minimum assignment cost is: 170

Traveling Salesman problem:

Traveling salesman problem is similar to the assignment problem, but here two extra restrictions are imposed. The first restriction is that we cannot select the element in the leading diagonal as we do not follows **i** again by **i.** The second restriction is that we do not produce an item again until all the items are produced once. The second restriction means no city is visited twice until the tour of all the cities is completed

Mathematically a traveling salesman problem can be stated as follows: Optimize

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_{ij}$$

subject to

$$\left.\begin{array}{l} \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1,\dots,n \\ \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1,\dots,n \\ x_{ij} = 0 \;\; or \;\; 1, \;\; i = 1,\dots,n \;\; , \;\; j = 1,\dots n. \\ \textit{The first and second restriction.} \end{array}\right\}$$

Where d_{ij} is the distance from city i to city j, and x_{ij} is to be some positive integer or zero, and the only possible integer is one, so the condition of $x_{ij} = 0 \text{ or } 1$, is automatically satisfied.

Associated to each traveling salesman problem there is a matrix called distance matrix $[d_{ij}]$ where d_{ij} is the distance from city i to city j. In this paper we call it distance matrix, and represent it as follows:

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{bmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{11} & d_{22} & d_{23} & \cdots & d_{2n} \\ \vdots & & & & \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{nn} \end{bmatrix} \end{array}$$

which is always a square matrix, thus each city can be assigned to only one city.
In fact any solution of this problem will contain exactly m non-zero positive individual allocations.

Steps to Solve Travelling Sales Man Problems:

Step 1

In a minimization (maximization) case, find the minimum (maximum) element of each row in the distance matrix (say a_i) and write it on the right hand side of the matrix.

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{11} & d_{22} & d_{23} & \cdots & d_{2n} \\ \vdots & & & & \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{nn} \end{bmatrix} \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix}$$

Then divide each element of i th row of the matrix by a_i . These operations create at least one ones in each rows.

$$\begin{bmatrix} d_{11}/a_1 & d_{12}/a_1 & d_{13}/a_1 & \cdots & d_{1n}/a_1 \\ d_{11}/a_2 & d_{22}/a_2 & d_{23}/a_2 & \cdots & d_{2n}/a_2 \\ \vdots & & & & \\ d_{n1}/a_n & d_{n2}/a_n & d_{n3}/a_n & \cdots & d_{nn}/a_n \end{bmatrix} \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix}$$

In term of ones for each row and column do assignment, otherwise go to step 2 .

Step 2

Find the minimum (maximum) element of each column in distance matrix (say b_j), and write it below j th column. Then divide each element of j th column of the matrix by b_j .

These operations create at least a one in each column. Make assignment in terms of ones. If no feasible assignment can be achieved from step (1) and (2) then go to step 3.

$$\begin{array}{c} \begin{bmatrix} d_{11}/a_1b_1 & d_{12}/a_1b_2 & d_{13}/a_1b_3 & \cdots & d_{1n}/a_1b_n \\ d_{11}/a_2b_1 & d_{22}/a_2b_2 & d_{23}/a_2b_3 & \cdots & d_{2n}/a_2b_n \\ \vdots & & & & \\ d_{n1}/a_nb_1 & d_{n2}/a_nb_2 & d_{n3}/a_nb_3 & \cdots & d_{nn}/a_nb_n \end{bmatrix} \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix} \\ \begin{matrix} b_1 & \quad\quad b_2 & \quad\quad b_3 & \quad\quad \cdots & \quad\quad b_n \end{matrix} \end{array}$$

Note: In a maximization case, the end of step 2 we have a fuzzy matrix, which all elements are belong to $[0,1]$, and the greatest element is one [6].

Step 3

Draw the minimum number of lines to cover all the ones of the matrix. If the number of drowned lines less than n , then the complete solution is not possible, while if the number of lines is exactly equal to n , then the complete solution is obtained.

Step 4

If a complete solution is not possible in step 3, then select the smallest (largest) element (say d_{ij}) out of those which do not lie on any of the lines in the above matrix. Then divide by d_{ij} each element of the uncovered rows or columns, which d_{ij} lies on it. This operation create some new ones to this row or column.

If still a complete optimal solution is not achieved in this new matrix, then use step 4 and 3 iteratively. By repeating the same procedure the optimal solution will be obtained.

Priority plays an important role in this method, when we want to assign the ones.

Priority rule:

For maximization (minimization) traveling salesman problem, assign the ones on the rows which have greatest (smallest) element on the right hand side, respectively.

If a tour is not reached, so do the assignment that will make a tour. We note that if a tour does not occur, then assign the element immediately greater than one

One question arise here, what to do with non square matrix? To make square, a non square matrix, we add one artificial row or column which all elements are one. Thus we solve the problem with the new matrix, by using the new method.

The matrix after performing the steps reduces to a matrix which has ones in each row and each column. So, the optimal solution has been reached

Problem:

Consider the following traveling salesman problem. Design a tour to five cities to the salesman such that minimize the total distance. Distance between cities is shown in the following matrix.

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} - & 10 & 3 & 6 & 9 \\ 5 & - & 5 & 4 & 2 \\ 4 & 9 & - & 7 & 8 \\ 7 & 1 & 3 & - & 4 \\ 3 & 2 & 6 & 5 & - \end{bmatrix} \end{array}$$

Solution :

Find the minimum element of each row in the distance matrix (say a_i) and write it on the right hand side of the matrix, as follows:

$$\begin{array}{c c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} - & 10 & 3 & 6 & 9 \\ 5 & - & 5 & 4 & 2 \\ 4 & 9 & - & 7 & 8 \\ 7 & 1 & 3 & - & 4 \\ 3 & 2 & 6 & 5 & - \end{bmatrix} & \begin{array}{c} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{array} \end{array}$$

Then divide each element of i th row of the matrix by a_i .These operations create some ones to each row, and the matrix reduces to following matrix.

$$\begin{array}{c|c} \begin{bmatrix} - & 3.3 & 1 & 2 & 3 \\ 2.5 & - & 2.5 & 2 & 1 \\ 1 & 2.25 & - & 1.75 & 2 \\ 7 & 1 & 3 & - & 4 \\ 1.5 & 1 & 3 & 2.5 & - \end{bmatrix} & \begin{array}{c} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{array} \end{array}$$

Now find the minimum element of each column in distance matrix (say b_j), and divide each element of j th column of the matrix by b_j .This operation create some ones to each row and each column. This operation create some ones to each row and each column

$$\begin{array}{c|c} \begin{bmatrix} - & 3.3 & 1 & 1.14 & 3 \\ 2.5 & - & 2.5 & 1.14 & 1 \\ 1 & 2.25 & - & 1 & 2 \\ 7 & 1 & 3 & - & 4 \\ 1.5 & 1 & 3 & 1.428 & - \end{bmatrix} & \begin{array}{c} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{array} \end{array}$$

The minimum number of lines required to pass through all ones is 4, and the minimum element of the uncovered is 1.428 on 5th row, so divide each element of 5th row of the matrix by 1.428.

$$\begin{array}{c|c} \begin{bmatrix} - & 3.3 & 1 & 1.14 & 3 \\ 2.5 & - & 2.5 & 1.14 & 1 \\ 1 & 2.25 & - & 1 & 2 \\ 7 & 1 & 3 & - & 4 \\ 1.05 & 0.7 & 2.1 & 1 & - \end{bmatrix} & \begin{array}{c} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{array} \end{array}$$

Now, minimum number of lines required to pass through all the ones of the matrix is 5. So, the complete solution is possible, and we can assign the ones, it is based on priority rule. Priority rule is assigning one on the rows which have the smallest element on the right hand side, respectively.

The details of this program are as follows:

City 1 assigns to City 3	distance 3
City 2 assigns to City 5	distance 2
City 3 assigns to City 4	distance 7
City 4 assigns to City 2	distance 1
City 5 assigns to City 1	distance 3

so the optimal assignment has been reached, and the optimal path is $(1,3),(3,4),(4,2),(2,5),(5,1)$ and total distance according to this plan is 16.

Problem: Consider the following traveling salesman problem. Design a tour to five cities to the salesman such that minimize the total distance. Distance between cities is shown in the following matrix.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} - & 11 & 10 & 12 & 4 \\ 2 & - & 6 & 3 & 5 \\ 3 & 12 & - & 14 & 6 \\ 6 & 14 & 4 & - & 7 \\ 7 & 9 & 8 & 12 & - \end{bmatrix} \end{array}$$

Now the minimum element of second column is 1.28. Divide each element of second column by 1.28 and the minimum element of 4th column is 1.5. Divide each element of 4th column by 1.5. These operations create some ones on second and 4th column, and the reduced matrix is as follows:

$$\begin{bmatrix} - & 2.148 & 2.5 & 2 & 1 \\ 1 & - & 3 & 1 & 2.5 \\ 1 & 3.125 & - & 2.25 & 2 \\ 1.5 & 2.734 & 1 & - & 1.75 \\ 1 & 1 & 1.14 & 1.14 & - \end{bmatrix}\begin{array}{c} 4 \\ 2 \\ 3 \\ 4 \\ 7 \end{array}$$

The minimum number of lines required to pass through all the ones of the matrix is 5.

So, the complete solution is possible, and we can assign the ones, it is based on priority rule. Priority rule is assigning one on the rows which have the smallest element on the right hand side, respectively.

The details of this program are as follows:

City 1 assigns to City 5 distance 4

City 2 assigns to City 4 distance 3

City 3 assigns to City 1 distance 3

City 4 assigns to City 3 distance 4

City 5 assigns to City 2 distance 9.

So the optimal path has been reached, $2-4-3-1-5-2$, and total distance according to this plan is 23.