MODULE - I

FREQUENCY ANALYSIS OF DISCRETE TIME SIGNALS

INTRODUCTION TO DIGITAL SIGNAL PROCESSING

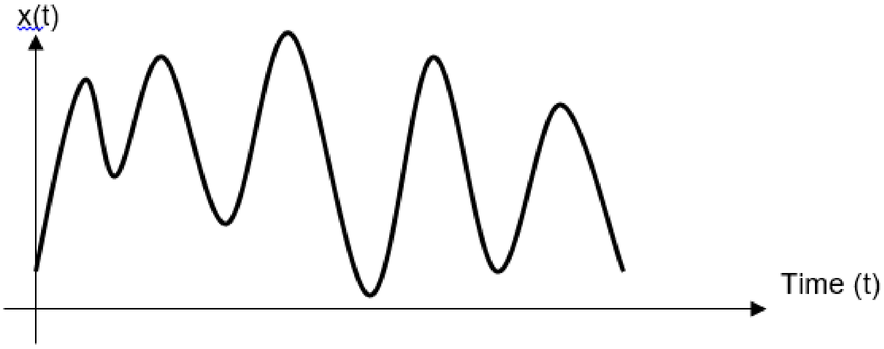
SIGNAL:

Signal can be defined as a function or physical quantity, which conveys some information and its amplitude varies with respect to single independent variable (time) or multi-independent variables. Based on variation in amplitude, signals are classified into mainly two types.

Continuous Time Signals
Discrete Time Signals

CONTINUOUS TIME SIGNALS:

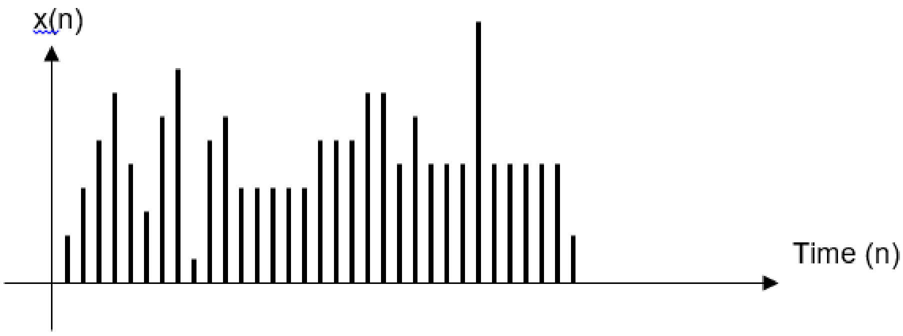
Continuous time signals are those for which the amplitude varies continuously according to continuous variation in time (t).



Amplitude restricted version of continuous time signals are known as analog signals, for which the amplitude is continuous. All analog signals are comes under continuous time signals.

DISCRETE TIME SIGNALS:

Discrete time signals are those for which the amplitude varies discreetly according to discrete variation in time (n). Discrete time signal x(n) can be represented as sequence of numbers, that's why these are also known as sequences.



Amplitude restricted version of discrete time signals are known as digital signals, for which the different number of amplitudes are restricted to finite number (Two). All digital signals are comes under discrete time signals.

DISCRETE-TIME SYSTEMS DESCRIBED BY DIFFERENCE EQUATIONS

Solutions of Linear Constant Coefficient Difference Equation:

Relation between input x(n) and output y(n) of a discrete time system is known as difference equation (DE). If it is the linear combination of input and output and it involves only constant coefficients known as linear constant coefficients difference equation(LCCDE).

General form of LCCDE is

$$y(n) + \sum_{k=1}^{N} a_K\, y(n-k) = \sum_{k=0}^{M} b_K\, x(n-k)$$

Solution of LCCDE contains two parts

Homogeneous solution [$y_h(n)$]
Particular solution [$y_P(n)$]
$y(n) = y_h(n) + y_P(n)$

Homogeneous solution is the response of a system to the given initial conditions, assuming the input $x(n) = 0$.
Particular solution is the response of a system to the given input $x(n)$, assuming zero initial conditions.

PROCEDURE FOR HOMOGENEOUS SOLUTION

1. Assume the input $x(n) = 0$ & output $y(n) = r^n$.
2. By substituting $x(n)$ & $y(n)$ in LCCDE, obtain the polynomial in r.
3. Determine the roots of polynomial (r_1, r_2, r_3).
4. Write the homogeneous solution
   (a) If all the roots are different (r_1, r_2, r_3), then
   $y_h(n) = A\, r_1^n + B\, r_2^n + C\, r_3^n$
   (b) If two roots are equal ($r_1 = r_2$), then
   $y_h(n) = (A + n\, B)\, r_1^n + C\, r_3^n$

PROCEDURE FOR PARTICULAR SOLUTION

1. Based on input, write the particular solution
   (a) If $x(n) = \delta(n)$, then $y_P(n) = 0$.
   (b) If $x(n) = u(n)$, then $y_P(n) = K$.
   (c) If $x(n) = a^n u(n)$, then $y_P(n) = K\, a^n$.
2. By substituting $x(n)$ & $y(n) = y_P(n)$ in LCCDE, obtain the value K.

PROCEDURE FOR THE SOLUTION OF LCCDE

1. Add homogeneous solution $y_h(n)$ and particular solution $y_P(n)$
   $y(n) = y_h(n) + y_P(n)$
2. From initial conditions, frame required number of equations in terms of A, B, C...
3. By solving equations, obtain A, B, C...
4. Compute $y(n)$.

Frequency analysis of Discrete Time signals:

Fourier series for DT periodic signal

Suppose that we are given a periodic sequence $x(n)$ with period N, that is, $x(n) = x(n+N)$ for all n. The Fourier series representation for $x(n)$ consists of N harmonically related exponential functions

$$e^{j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

And is expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

Synthesis equation	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$
Analysis equation	$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

Power density spectrum

The average power of discrete time periodic signal with period N is defined as

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} x(n)x^*(n)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\left(\sum_{k=0}^{N-1} c_k^* e^{-j2\pi kn/N}\right)$$

$$P_x = \sum_{k=0}^{N-1} c_k^* \left[\frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}\right]$$

$$= \sum_{k=0}^{N-1} |c_k|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

DTFT is a mathematical method or transformation is used to obtain frequency domain representation of a given discrete time signal x(n).
DTFT of a sequence can be defined as,

$$\text{DTFT}[\,x(n)\,] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

In general $X(e^{j\omega})$ is complex form and can be expressed in terms of its real and imaginary parts as

To obtain the graphical representation of $X(e^{j\omega})$, we can go for Magnitude Spectrum and Phase Spectrum. Spectrum is a Latin word for image.

Graphical representation of magnitude of $X(e^{j\omega})$ is known as Magnitude Spectrum.

$$|X(e^{j\omega})| = \sqrt{[\,X_R(e^{j\omega})\,]^2 + [\,X_I(e^{j\omega})\,]^2}$$

Graphical representation of phase of $X(e^{j\omega})$ is known as Phase Spectrum.

$$\angle X(e^{j\omega}) = Tan^{-1}\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right)$$

Inverse DTFT is used to compute required discrete sequence from known frequency domain representation. It is the reverse part of DTFT.
Inverse DTFT of $X(e^{j\omega})$ over the period 2π can be defined as

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

FREQUENCY DOMAIN REPRESENTATION OF A SYSTEM:
Response y(n) of a discrete time system can be defined as the convolution of input x(n) and impulse response h(n).

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \otimes h(n)$$

$$y(n) = x(n) \otimes h(n)$$

$$= \sum_{m=-\infty}^{\infty} x(m)\ h(n-m)$$

Apply DTFT both side

$$\mathrm{DTFT}\ [\,y(n)\,] = \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x(m)\ h(n-m) \right] e^{-j\omega n}$$

Change the order of summation

$$= \sum_{m=-\infty}^{\infty} x(m) \left[\sum_{n=-\infty}^{\infty} h(n-m)\ e^{-j\omega n} \right]$$

$$= \sum_{m=-\infty}^{\infty} x(m)\ \mathrm{DTFT}\ [\,h(n-m)\,]$$

$$= \sum_{m=-\infty}^{\infty} x(m)\ [e^{-j\omega m}\, H(e^{j\omega})]$$

$$= H(e^{j\omega}) \sum_{m=-\infty}^{\infty} x(m)\ [e^{-j\omega m}]$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Frequency Response ($H(e^{j\omega})$):

The ratio of output to input in frequency domain representation is known as System Function or Transfe Function or Frequency Response. in general, the frequency response $H(e^{j\omega})$ is complex form and can b expressed in terms of its real and imaginary parts as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

To obtain the graphical representation of $H(e^{j\omega})$, we can go for Magnitude Response and Phase Respons

Magnitude of Frequency Response $H(e^{j\omega})$ is known as Magnitude Response.

$$|H(e^{j\omega})| = \sqrt{[H_R(e^{j\omega})]^2 + [H_I(e^{j\omega})]^2}$$

Phase of Frequency Response $H(e^{j\omega})$ is known as Phase Response.

$$\angle H(e^{j\omega}) = Tan^{-1}\left(\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right)$$

FOURIER TRANSFORM OF DT APERIODIC SIGNALS:

Consider a discrete time signal x(n), then the fourier transform of x(n) is

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

Frequency Analysis of Discrete-Time Aperiodic Signals

Synthesis equation (inverse transform)	$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$
Analysis equation (direct transform)	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

CONVERGENCE OF FOURIER TRANSFORM:

Consider a discrete time signal x (n), then the Fourier transform of x (n) is

CONVERGENCE OF FOURIER TRANSFORM:

Consider a discrete time signal x(n), then the fourier transform of x(n) is

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

Uniform convergence is guaranteed if x(n) is absolutely summable. Indeed if

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

then

$$|X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

ENERGY DENSITY SPECTRUM

Let x(n) be a discrete time sequence and
DTFT[x(n)] = X($e^{j\omega}$), then the Parsevalls theorem provides the relation between x(n) and X($e^{j\omega}$) and gives total energy of a signal as follows.

$$\sum_{n=-\infty}^{\infty} \left| x(n) \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega$$

Proof:

$$\sum_{n=-\infty}^{\infty}\left| x(n)\right|^2 = \sum_{n=-\infty}^{\infty} x(n)\ \ x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\ [\,x(n)\,]^*$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega\right]^*$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\left(X(e^{j\omega})\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left| X(e^{j\omega})\right|^2\ d\omega$$

REVIEW OF Z-TRANSFORMS:

Z TRANSFORM OF A SEQUENCE:

Z Transform is mathematical tool is used to obtain required Z domain representation of a given discrete time domain signal or sequence. Z Transform of a sequence x(n) can be defined as

$$ZT[\,x(n)\,] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Where, x(n) is both sided sequence and ZT[x(n)] = X(z) is known as bi-directional or both sided Z Transform.

If x(n) is causal or right sided, then its Z transform can be defined as

$$ZT[\,x(n)\,] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

If x(n) is anti-causal or left sided, then its Z transform can be defined as

$$ZT[\,x(n)\,] = X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

Z Transform of right sided(causal) or left sided(anti-causal) sequence is known as uni-directional or one sided Z transform.

Z PLANE:

z is a complex variable,

$z = r\, e^{j\omega}$, Where r is magnitude of z and ω is phase of z.

$= r\cos(\omega) + j\, r\sin(\omega)$

$= \mathrm{Re}\{ z \} + j\, \mathrm{Im}\{ z \}$

REGION OF CONVERGENCE (ROC):
The range of values of z for which the basic definition of z transform will converges is referred to as Region of Convergence. For this we have to restrict the range of z to get finite X(z) from given discrete time sequence x(n).

RELATION B/N ZT & DTFT:
From basic definition of Z Transform

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, r^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [\, r^{-n}\, x(n)\,]\; e^{-j\omega n}$$

$$ZT[\, x(n) = DTFT[\, r^{-n}\, x(n)\,]$$

If r = 1, then ZT [x(n)] = DTFT [x(n)].

Z TRANSFORM OF A CAUSAL SEQUENCE:
A sequence x(n) is said to be causal if and only if x(n) = 0, for n < 0. That means causal sequences are right sided.

Example $x(n) = a^n u(n)$, $|a| < 1$
From basic definition of z transform

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Replace x(n) by $a^n u(n)$

$$ZT[\, a^n u(n)\,] = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots\dots + \left(\frac{a}{z}\right)^\infty$$

$$= \frac{1}{1-\frac{a}{z}}\ , \text{ if } \left|\frac{a}{z}\right| < 1$$

$$= \frac{1}{\frac{z-a}{z}}\ , \quad |a| < |z|$$

$$= \frac{z}{z-a}\ , \quad |z| > |a|$$

ROC of a causal (right sided) sequence is outside the circle of outer most pole.

Z TRANSFORM OF A ANTI-CAUSAL SEQUENCE:
A sequence x(n) is said to be anti-causal if and only if x(n) = 0, for n > 0. That means anti-causal sequences are left sided.

Example $x(n) = -a^n u(-n-1)$, $|a| > 1$
From basic definition of z transform

$$ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Replace x(n) by $-a^n u(-n-1)$

$$ZT[-a^n u(-n-1)] = \sum_{n=-\infty}^{\infty} [-a^n u(-n-1)] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^n$$

$$= -\sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n$$

$$= -\left[\left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \dots\dots + \left(\frac{z}{a}\right)^\infty\right]$$

$$= -\left(\frac{z}{a}\right)\left[1 + \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \dots\dots + \left(\frac{z}{a}\right)^\infty\right]$$

$$= -\left(\frac{z}{a}\right)\left(\frac{1}{1-\frac{z}{a}}\right), \text{ if } \left|\frac{z}{a}\right| < 1$$

$$= -\left(\frac{z}{a}\right)\left(\frac{a}{a-z}\right), \text{ if } |z| < |a|$$

$$= \frac{z}{z-a}, \quad |z| < |a|$$

ROC of an anti-causal (left sided) sequence is inside the circle of innermost pole.

ROC of a non-causal (both sided) sequence is ring located between two poles.

PROPERTIES OF ROC:

- ROC does not contain any pole.
- ROC is independent of zeroes.
- If x(n) is finite duration sequence, then the ROC is entire z plane possibly except z=0 and / or z=∞
- If x(n) is right sided sequence, i.e causal, then the ROC is outside the circle of outermost pole.
- If x(n) is left sided sequence, i.e anti-causal, then the ROC is inside the circle of innermost pole.
- If x(n) is both sided sequence, i.e non-causal, then the ROC is a circular ring located between two pole.

INVERSE Z TRANSFORM:

It is the reverse part of z transform, that means to obtain discrete time domain sequence x(n), from z domain X(z) and its ROC, inverse z transform is used. There are three methods used to determine discrete time sequence x(n).

- Partial Fractions Method
- Power Series Method or Long Division Method
- Residue Method or Contour Integral Method

PARTIAL FRACTIONS METHOD:

Partial fraction method is used only when the factors of denominator polynomial is known, for example,

if $X(z) = \dfrac{N(z)}{(z-p_1)(z-p_2)(z-p_3)}$, then to obtain x(n) take X(z) / z

$$\Rightarrow \frac{X(z)}{z} = \frac{N(z)}{z(z-p_1)(z-p_2)(z-p_3)},$$

If the degree of denominator is greater than the degree of numerator, then

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-p_1} + \frac{C}{z-p_2} + \frac{D}{z-p_3}$$

$$X(z) = A + B\left(\frac{z}{z-p_1}\right) + C\left(\frac{z}{z-p_2}\right) + D\left(\frac{z}{z-p_3}\right)$$

$$x(n) = Z^{-1}\left[A + B\left(\frac{z}{z-p_1}\right) + C\left(\frac{z}{z-p_2}\right) + D\left(\frac{z}{z-p_3}\right)\right]$$

$$x(n) = A\,Z^{-1}(1) + B\,Z^{-1}\left(\frac{z}{z-p_1}\right) + C\,Z^{-1}\left(\frac{z}{z-p_2}\right) + D\,Z^{-1}\left(\frac{z}{z-p_3}\right)$$

FORMULAE:

- $Z^{-1}[\,1\,] = \delta\,(n)$
- $Z^{-1}[\,z^k\,] = \delta\,(n + k)$
- $Z^{-1}[\,z^k X(z)] = x\,(n + k)$, if $Z^{-1}[\,X(z)] = x\,(n)$
- $Z^{-1}\left(\dfrac{z}{z-a}\right) = a^n\,u(n),\ \text{if}\ |z| > a$

  $= -a^n\,u(-n-1),\ \text{if}\ |z| < a$
- $Z^{-1}\left(\dfrac{az}{(z-a)^2}\right) = n\,a^n\,u(n),\ \text{if}\ |z| > a$
- $Z^{-1}\left(\dfrac{a^2z}{(z-a)^3}\right) = n(n-1)\,a^n\,u(n),\ \text{if}\ |z| > a$

APPLICATIONS OF Z TRANSFORM:

Z Transform is used

To obtain causality of a discrete LTI system
To obtain stability of a discrete LTI system
To solve difference equations
To realize discrete LTI system

CAUSALITY OF A SYSTEM:

- A sequence x(n) is said to be causal if and only if x(n) = 0, for n < 0.
  EX: $x(n) = (½)^n u(n)$.
- A sequence x(n) having z domain X(z) is said to be causal if and only if, the ROC of X(z) should be outside the circle of outer most pole.
  EX: $X(z) = \frac{z(z+1)}{(z-1/2)(z-1/3)(z-1/4)(z-1/8)}, \ |z| > 1/2$
- A discrete LTI system T(.) having input x(n) and output y(n) is said to be causal if and only if the present output only depends on present input and / or past inputs and / or past outputs, does not depends on feature inputs and outputs.
  EX: y(n) = 2 x(n) + 3 x(n – 1) + 4 x(n – 2) + 5 y(n – 1) + 6 y(n – 2)
- A discrete LTI system T(.) having impulse response h(n) is said to be causal if and only if h(n) = 0, for n < 0.
  $h(n) = (½)^n u(n)$.
- A discrete LTI system having rational system function H(z) is said be causal if and only if, the degree of numerator can't be greater than degree of denominator and ROC of H(z) should be outside the circle of outer most pole.
  EX: $H(z) = \frac{z(z+1)}{(z-1/2)(z-1/3)(z-1/4)(z-1/8)}, \ |z| > 1/2$

STABILITY OF A SYSTEM

- A sequence x(n) is said to be bounded if and only if magnitude of x(n) is finite for all values of n, i.e $|x(n)| < \infty$.
  EX: $x(n) = (½)^n u(n)$.
- A discrete LTI system T(.) having input x(n) and output y(n) is said to be stable if and only if the system response should be bounded, with a bounded input called BIBO system. i.e, if $|x(n)| <\infty$, then $|y(n)| <\infty$, for all values of n.
  EX: input: $x(n) = (½)^n u(n)$ and output: $y(n) = (1/4)^n u(n)$.
- A discrete LTI system T(.) having impulse response h(n) is said to be stable if and only if h(n) should be absolutely summable. i.e $\sum |h(n)| < \infty$.
  EX: $h(n) = (½)^n u(n)$
- A discrete LTI system T(.) having rational system function H(z) is said to be stable if and only if the ROC includes the unit circle.
  EX 1: $X(z) = \frac{z(z+1)}{(z-1/2)(z-1/3)(z-1/4)(z-1/8)}, \ |z| > 1/2$
  EX 2: $X(z) = \frac{z(z+1)}{(z-1/2)(z-2)(z-1/4)(z-1/8)}, \ 1/2 < |z| < 2$
- A discrete LTI causal system T(.) having rational system function H(z) is said to be stable if and only if all of the poles of H(z) should be inside the unit circle
  EX : $X(z) = \frac{z(z+1)}{(z-1/2)(z-1/3)(z-1/4)(z-1/8)}, \ |z| > 1/2$

SOLUTION FOR DIFFERENCE EQUATIONS OF DIGITAL FILTERS USING Z-TRANSFORMS

SYSTEM FUNCTION:

System function or Transfer function can be defined as the ratio of output to input in z domain representation. Let us consider a system T(.) having input sequence x(n), output sequence y(n) and assume

ZT[x(n)] = X(z) and

ZT[y(n)] = Y(z), then

System function $H(z) = \dfrac{Y(z)}{X(z)}$

POLE ZERO PLOT:

Determine the roots of numerator and denominator of rational system function

$$H(z) = \frac{(z-z_1)(z-z_2)(z-z_3)\ldots\ldots\ldots(z-z_M)}{(z-p_1)(z-p_2)(z-p_3)\ldots\ldots\ldots(z-z_N)}$$

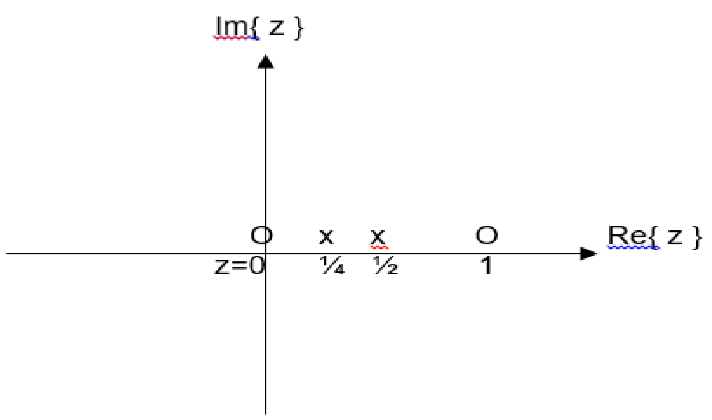
Where,

M: Number of zeroes (z_1, z_2, z_3.........z_M)

N: Number of poles (p_1, p_2, p_3.........p_N)

The roots of numerator polynomial is referred to as zeroes of H(z) and the roots of denominator polynomial is referred to as poles of H(z). After representing poles and zeroes on a z-plane, then z plane becomes pole zero plot. To represent zero, use small circle (O) and for poles, use small cross mark (x).

EX: $H(z) = \dfrac{z(z-1)}{(z-1/2)(z-1/4)}$



Example:

Determine the step response of the system

$$y(n) = \alpha y(n-1) + x(n), \qquad -1 < \alpha < 1$$

when the initial condition is $y(-1) = 1$.

Solution. By taking the one-sided z-transform of both sides of (3.6.11), we obtain

$$Y^+(z) = \alpha[z^{-1}Y^+(z) + y(-1)] + X^+(z)$$

Upon substitution for $y(-1)$ and $X^+(z)$ and solving for $Y^+(z)$, we obtain the result

$$Y^+(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{(1-\alpha z^{-1})(1-z^{-1})}$$

By performing a partial-fraction expansion and inverse transforming the result, we h

$$\begin{aligned} y(n) &= \alpha^{n+1}u(n) + \frac{1-\alpha^{n+1}}{1-\alpha}u(n) \\ &= \frac{1}{1-\alpha}(1-\alpha^{n+2})u(n) \end{aligned}$$