Module 3

Eigenvalues, Eigenvectors and Diagonalization of Matrix

Out Lines

This present module will include the following topics:

(i) Preliminaries
(ii) Vector spaces and subspaces
(iii) Linear span and Basis
(iv) Dimension of vector space

Introduction

Linear algebra is the study of finite dimensional vector space. Dimension of vector spaces will be discussed in this text later. Basically, a vector space is a collection of vectors (a fancy word used to differ with the scalar or number) having sense of directions along with their values which are closed under addition and scalar multiplication. The subject has significant impacts on several branches of knowledge including mathematics and physics.

Preliminaries

Some well-known sets of numbers are as follows:

1. The set of natural number or positive integers, $N = \{1, 2, 3, \dots.\}$ generally used to count natural objects.
2. The set of integers $Z = \{\dots., -3, -2, -2, 0, 1, 2, \dots\}$ which includes both positive and negative integers and zero.
3. The of rational numbers $Q = \{\frac{p}{q}: p, q \in Z \text{ and } q \neq 0\}$ which has elements as terminating or recurring decimal representation. There are elements having non-terminating and non-recurring decimal representation. They are called the irrational numbers. The set of irrational numbers is denoted by $R\backslash Q$.
4. The set of real numbers R includes both the rational and irrational numbers.
5. The quadratic equation $x^2 + 1 = 0$ has roots, by fundamental theorem of classical algebra, $x = \pm\sqrt{-1} = \pm i$, which do not belong to R. So, the concept of complex number came into the picture. The set of complex number $C = \{a + ib: a, b \in R \text{ and } \sqrt{-1} = i\}$ includes the real number also.

Definition 3.1 (Binary operation) Let S be a non-empty set. A binary operation on S is mapping $(.): S \times S \to S$ given by $a.b \in S$, for all $a, b \in S$.

Addition and multiplication are binary operations on real numbers, rational numbers, integers and natural numbers. However, subtraction is not a binary operation on the set of natural numbers. Division is not a binary operation on the set of integers and so on.

Definition 3.2 (Group) Let G be a non-empty set. A binary operation $(.) : G \times G \rightarrow G$ satisfies the conditions as follows:

(i) Closure axiom: $a.b \in G$, for all $a, b \in G$.

(ii) Associative axiom: $(a.b).c = a.(b.c)$, for all $a, b, c \in G$.

(iii) Identity element: There exist $i \in G$ such that $a.i = i.a = a$, for all $a \in G$. In such case, i is called the identity element of the given set with respect to the binary operation.

(iv) Inverse element: For each $a \in G$, there exist an element a' such that $a.a' = a'.a = i$. In such case, a' is called the inverse of the element $a \in G$.

Then, $(G, .)$ is called a *group*. Furthermore, if $a.b = b.a$, for all $a, b \in G$, then $(G, .)$ is called a *commutative group*.

Exercise 3.1 Show that the set of all integers Zforms a commutative group under addition.

Exercise 3.2 Check whether the set of all natural numbers N forms a commutative group under addition.

Exercise 3.3 Check whether the set of all integers Z forms a commutative group under multiplication.

Vector spaces and subspaces

Definition 3.2 (Vector space) Let V be non-empty set. A binary operation (called addition of vectors) $+ : V \times V \rightarrow V$ and an external composition (called scalar multiplication) $. : F \times V \rightarrow V$ be defined in the following way:

(i) $x + y \in V$, for all $x, y \in V$ and (ii) $\mu x \in V$, for all $x \in V$ and $\mu \in F$, a field.

Now for these two operations if the following are satisfied:

1. $(V, +)$ is a commutative group.
2. $1x = x$ for all $x \in V$, and 1 be the multiplicative identity of the field F.
3. $\mu(x + y) = \mu x + \mu y$, for all $x, y \in V$ and $\mu \in F$.
4. $(\mu + \tau)x = \mu x + \tau x$, for all $x \in V$ and $\mu, \tau \in F$.
5. $\mu(\tau x) = (\mu\tau)x$, for all $x \in V$ and $\mu, \tau \in F$.

Then, V is said to be a *vector space* over the field F and sometimes it is written as $V(F)$or V_F.

Remark 3.1: When, $F = R$, the field of real numbers, then V is called a real vector space and when, $F = C$, the field of complex numbers, then V is called a complex vector space. The real and complex vector spaces are called the *linear spaces*.

Definition 3.3 (Null space) A vector space containing the zero vector only is called the *null space* or the *zero vector space*.

Example 3.1 Show that the set $C(0,1)$ of real valued continuous functions defined on $(0,1)$ forms a real vector space with respect to the addition and multiplication as follows:

$$(f+g)(x)=f(x)+g(x), \text{ for all } f,g\in C(0,1) \text{ and } x\in(0,1)$$

$$\text{and } (\mu f)(x)=\mu f(x) \text{ for all } f\in C(0,1),\ \mu\in R \text{ and } x\in(0,1).$$

Solution: Let, $f, g \in C(0,1)$. Then,

(i) $f+g\in C(0,1)$, since sum of two continuous function is again a continuous function on the common domain of definition

(ii) $\{(f+g)+h\}(x)=(f+g)(x)+h(x)=f(x)+g(x)+h(x)=f(x)+(g+h)(x)=\{f+(g+h)\}(x)$, for all $x\in(0,1)$. Therefore, $(f+g)+h=f+(g+h)$, for all $f,g,h\in C(0,1)$.

(iii) Zero function can be defined as $O(x)=0$, for all $x\in(0,1)$. And then, $(O+f)(x)=O(x)+f(x)=0+f(x)=f(x)$ for all $x\in(0,1)$. Therefore, $O+f=f$, for all $f\in C(0,1)$. That is O is the additive identity in $C(0,1)$.

(iv) $\{f+(-f)\}(x)=f(x)+\big(-f(x)\big)=f(x)-f(x)=0=O(x)$ for all $x\in(0,1)$. Therefore, $f+(-f)=O$, for all $f\in C(0,1)$. That is $-f$ is the additive inverse of f in $C(0,1$.

(v) Also, $(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x)$, for all $x\in(0,1)$. Therefore, $f+g=g+f$, for all $f,g\in C(0,1)$.

The above five points show that $C(0,1)$ is a commutative group under defined addition. Furthermore,

(vi) $(1f)(x)=f(x)$ for all $x\in(0,1)$. That is $1f=f$ for all $f\in C(0,1)$.

(vii) $\{\mu(f+g)\}(x)=\mu\{(f+g)(x)\}=\mu\{f(x)+g(x)\}=\mu f(x)+\mu g(x)=(\mu f+\mu g)(x)$, for all $x\in(0,1)$. Therefore, $\mu(f+g)=(\mu f+\mu g)$ for all $f,g\in C(0,1)$ and $\mu\in F$.

(viii) $\{(\mu+\tau)f\}(x)=(\mu+\tau)f(x)=\mu f(x)+\tau f(x)=(\mu f+\tau f)(x)$, for all $x\in(0,1)$. Therefore, $(\mu+\tau)f=\mu f+\tau f$, for all $f\in C(0,1)$ and $\mu,\tau\in R$.

(ix) $\{\mu(\tau f)\}(x)=\mu\{(\tau f)(x)\}=\mu\{\tau f(x)\}=\mu\tau f(x)=\{(\mu\tau)f\}(x)$ for all $x\in(0,1)$. Therefore, $\mu(\tau f)=(\mu\tau)f$, for all $f\in C(0,1)$ and $\mu,\tau\in R$.

Therefore, $C(0,1)$ is a vector space over the field of real numbers.

Exercise 3.1 Show that the set of all $m\times n$ real matrices form a real vector space with respect to the matrix addition and scaler multiplication of matrices.

Example 3.2 Check whether the set of integers Z with addition and scaler multiplication form a real vector space.

Solution: The set of integers Z with addition form a commutative group. However, if we take $x=1\in V=Z$ and $\mu=\frac{1}{3}\in F=R$, then $\mu x=\frac{1}{3}\notin V=Z$. So, the set of integers Z can not form a real vector space.

Example 3.3 Show that the set P_n of all polynomial of degree less or equal to n form a real vector space.

Solution: For the shake of simplicity, we will show the result for $n=2$. Therefore, we consider P_2, the set of all polynomial of degree less or equal to 2.

(i) Let us take two elements in $p, q \in P_2$ given by $p(x) = a_0x^2 + a_1x + a_2$ and $q(x) = b_0x^2 + b_1x + b_2$ and also $\mu \in R$. Then,

$$p(x) + q(x) = a_0x^2 + a_1x + a_2 + b_0x^2 + b_1x + b_2$$
$$= (a_0 + b_0)x^2 + (a_1 + b_1)x + (a_2 + b_2)$$

Therefore, $p + q \in P_2$.

(ii) Also, $\mu p(x) = (\mu a_0)x^2 + (\mu a_1)x + (\mu a_2)$ and therefore, $\mu p \in P_2$.

(iii) Let us take two elements in $p, q, r \in P_2$. Now, let by $p(x) = a_0x^2 + a_1x + a_2$, $q(x) = b_0x^2 + b_1x + b_2$ and $r(x) = c_0x^2 + c_1x + c_2$, then

$$\big(p(x) + q(x)\big) + r(x) = (a_0x^2 + a_1x + a_2 + b_0x^2 + b_1x + b_2) + c_0x^2 + c_1x + c_2$$
$$= (a_0 + b_0)x^2 + (a_1 + b_1)x + (a_2 + b_2) + c_0x^2 + c_1x + c_2$$
$$= \big((a_0 + b_0) + c_0\big)x^2 + \big((a_1 + b_1) + c_1\big)x + (a_2 + b_2) + c_2$$
$$= (a_0 + (b_0 + c_0))\,x^2 + (a_1 + (b_1 + c_1))x + a_2 + (b_2 + c_2)$$
$$= a_0x^2 + a_1x + a_2 + (b_0 + c_0)x^2 + (b_1 + c_1)x + (b_2 + c_2)$$
$$= p(x) + \big(q(x) + r(x)\big)$$

That is, $(p + q) + r = p + (q + r)$. So, the associative property is satisfied.

(iv) Zero function can be defined as $O(x) = 0 = 0x^2 + 0x + 0 \in P_2$ such that

$$p(x) + O(x) = O(x) + p(x) = p(x)$$

Therefore, O is the identity element in P_2under addition.

Let us take an element in $p \in P_2$. Now, let by $p(x) = a_0x^2 + a_1x + a_2$. Then, $(-p)(x) = -a_0x^2 - a_1x - a_2$. Therefore,

$$p(x) + (-p)(x) = (-p)(x) + p(x) = O(x)$$

That is

$$p + (-p) = (-p) + p = O$$

Therefore, −p is the additive inverse of the element p.

(v) Let us take two elements in $p, q \in P_2$ given by $p(x) = a_0x^2 + a_1x + a_2$ and $q(x) = b_0x^2 + b_1x + b_2$ and also $\mu \in R$. Then,

$$p(x) + q(x) = a_0x^2 + a_1x + a_2 + b_0x^2 + b_1x + b_2$$
$$= (a_0 + b_0)x^2 + (a_1 + b_1)x + (a_2 + b_2)$$
$$= (b_0 + a_0)x^2 + (b_1 + a_1)x + (b_2 + a_2)$$
$$= b_0x^2 + b_1x + b_2 + a_0x^2 + a_1x + a_2 = q(x) + p(x)$$

That is, $(p + q) = (q + p)$

Therefore, P_2 forms a commutative group under addition.

(vi) $\mu(p(x) + q(x)) = \mu(a_0x^2 + a_1x + a_2 + b_0x^2 + b_1x + b_2)$
$= \mu a_0x^2 + \mu a_1x + \mu a_2 + \mu b_0x^2 + \mu b_1x + \mu b_2$

$$= \mu(a_0x^2 + a_1x + a_2) + \mu(b_0x^2 + b_1x + b_2)$$
$$= \mu p(x) + \mu q(x)$$

That is $\mu(p + q) = \mu p + \mu q$.

(vii) $(\mu + \tau)p(x) = (\mu + \tau)(a_0x^2 + a_1x + a_2)$
$$= (\mu + \tau)a_0x^2 + (\mu + \tau)a_1x + (\mu + \tau)a_2$$
$$= \mu(a_0x^2 + a_1x + a_2) + \tau(a_0x^2 + a_1x + a_2)$$
$$= \mu p(x) + \tau p(x)$$

That is $(\mu + \tau)p = \mu p + \tau p$.

(viii) $\mu(\tau p(x)) = \mu(\tau a_0x^2 + \tau a_1x + \tau a_2) = \mu\tau a_0x^2 + \mu\tau a_1x + \mu\tau a_2$
$$= (\mu\tau)(a_0x^2 + a_1x + a_2) = (\mu\tau)p(x)$$

That is $\mu(\tau p) = (\mu\tau)p$.

(ix) $1p(x) = 1a_0x^2 + 1a_1x + 1a_2 = a_0x^2 + a_1x + a_2 = p(x)$

That is $1p = p$, where 1 is the multiplicative identity of R.

The set P_2 of all polynomial of degree less or equal to 2 form a real vector space. Similarly, for any $n \geq 0$, the set P_n of all polynomial of degree less or equal to n form a real vector space.

Definition 3.4 (Sub space) A non-empty subset W of a vector space V over a field F is called a subspace if W itself form a vector space under the same addition and scaler multiplication as of the vector space V.

Theorem 3.1 Let A W be a non-empty subset of a vector space V over a field F. Then, W is said to be a subspace if and only if the following conditions are satisfied:

(i) $x + y \in W$, for all $x, y \in W$ and (ii) $\mu x \in W$, for all $x \in W$ and $\mu \in F$, a field.

Example 3.4 Show that the set $W$of symmetric matrices form a subspace of the vector space V of all square matrices over the field R.

Solution: Clearly, W is a non-empty subset of the vector space V of all square matrices over the field R. Let, $A, B \in W$. That is, A and B are two symmetric square matrix. That is, $A^T = A$ and $B^T = B$ and we have $(A + B)^T = A^T + B^T = A + B$. So, $A + B$ is also a symmetric square matrix, i.e., $A + B \in W$. Again, for a real number μ , we get, $(\mu A)^T = \mu A^T = \mu A$. So, μA is also a symmetric square matrix, i.e., $\mu A \in W$. Therefore, by Theorem 3.1, we can conclude that the set $W$of symmetric matrices form a subspace of the vector space V of all square matrices over the field R.

Exercise 3.2 Show that the set W of all real matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a subspace of the vector space V of all square matrices of order 2 over the field R.

Exercise 3.4 Show that the set W of all polynomials of the form $ax^3 - bx^2 + bx + a$ is a subspace of the vector space $V = P_3$ of all polynomials of degree 3 over the field R.

Exercise 3.5 Show that the set W of all vectors of the form $\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$ is a subspace of the vector space $V = R^4$ over the field R

Example 3.5 Show that, $R^3 = \{(x, y, z): x, y, z \text{ are real numbers}\}$ is vector space over the field R. Is the set $W = \{(x, y, z): x, y, z \text{ are rational numbers}\}$ a subspace of R^3 ?

Solution: It is easy to check that $R^3 = \{(x, y, z): x, y, z \text{ are real numbers}\}$ is vector space over the field R. (Do it yourself).

Now set $W = \{(x, y, z): x, y, z \text{ are rational numbers}\}$ is a non-empty subset of $R^3 = \{(x, y, z): x, y, z \text{ are real numbers}\}$. Let, $x = (1, 2, 3)$ be an element in $W = \{(x, y, z): x, y, z \text{ are rational numbers}\}$ and $\mu\sqrt{2}$ be an element from the field R. But the scalar multiplication gives $\mu x = \sqrt{2}(1,2,3) = (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}) \notin W$. So, W is not a subspace of R^3.

Linear span and Basis

Definition 3.5 (Linear combination) Let V be a vector space over a field F. If, $x_1, x_2, \ldots x_n \in V$ and $\mu_1, \mu_2, \ldots \mu_n \in F$, then any vector $y = \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_n x_n$ is called a linear combination of the vectors $x_1, x_2, \ldots x_n \in V$.

Definition 3.6 (Linear span) Let V be a vector space over a field F. If, $S = \{x_1, x_2, \ldots x_k\} \subseteq V$, the set of all linear combination given by $L(S) = \{\mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_k x_k : \mu_i \in F\}$ is called the linear span of $x_1, x_2, \ldots x_k$.

In particular, if $V = L(S)$, then S is called the spanning set of the vector space V and V is said to be spanned by S.

Example 3.6 We consider, $S = \{x^2 - x + 1, -3x^2 + x + 2\} \subseteq P_2$, the vector space of all polynomial of degree less or equal to 2. Show that, the polynomial $r(x) = 6x^2 - 4x + 1$ is in $L(S)$.

Solution: Let, $p(x) = x^2 - x + 1$ and $q(x) = -3x^2 + x + 2$. If $r(x) = 6x^2 - 4x + 1$ is in $L(S)$, then there exists μ and τ such that

$$r(x) = \mu p(x) + \tau q(x)$$

This implies, $6x^2 - 4x + 1 = \mu(x^2 - x + 1\,) + \tau(-3x^2 + x + 2)$ which gives the following equations:

$$\mu - 3\tau = 6 \tag{3.1}$$

$$-\mu + \tau = -4 \tag{3.2}$$

$$\mu + 2\tau = 1 \tag{3.3}$$

Solving Equation (3.2) and Equation (3.2), we get $\tau = -1$ and $\mu = 3$. Also, the obtained values satisfy Equation (3.1). Therefore, the polynomial $r(x) = 6x^2 - 4x + 1$ is in $L(S)$.

Example 3.7 Let V be the vector space of all square matrices of order 2 over the field R. We consider $S = \{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\} \subseteq V$. Show that, the matrix $C = \begin{bmatrix} 3 & -5 \\ 5 & -1 \end{bmatrix}$ is in $L(S)$.

Solution: Let, $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. If $C = \begin{bmatrix} 3 & -5 \\ 5 & -1 \end{bmatrix}$ is in $L(S)$, then, there exists μ and τ such that

$\begin{bmatrix} 3 & -5 \\ 5 & -1 \end{bmatrix} = \mu \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + \tau \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ which gives the following equations:

$$\mu + \tau = 3 \tag{3.4}$$

$$\mu - \tau = -5 \tag{3.5}$$

$$-\mu + \tau = 5 \tag{3.6}$$

$$\mu = -1 \tag{3.7}$$

From Equation (3.7), $\mu = -1$. Then, from Equation (3.4), we get $\tau = 4$. The obtained values also satisfy Equation (3.5) and Equation (3.6). Therefore, the matrix $C = \begin{bmatrix} 3 & -5 \\ 5 & -1 \end{bmatrix}$ is in $L(S)$.

Exercise 3.6 Let V be the vector space of all square matrices of order 2 over the field R. We consider $S = \{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\} \subseteq V$. Is the matrix $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ in $L(S)$?

Definition 3.7 (Linearly dependent vectors) Let V be a vector space over a field F. The set of vectors $\{x_1, x_2, \ldots x_k\}$ in V is linearly dependent if there are scalars $\mu_1, \mu_2, \ldots \mu_k \in F$, at least one of them must be non-zero, such that $\mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_k x_k = 0$.

Definition 3.7 (Linearly independent vectors) Let V be a vector space over a field F. The set of vectors $\{x_1, x_2, \ldots x_k\}$ in V is linearly independent if $\mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_k x_k = 0$ for scalars $\mu_1, \mu_2, \ldots \mu_k \in F$, implies that $\mu_1 = \mu_2 = \ldots = \mu_k = 0$.

Example 3.8 Check whether the given set $\{1 + x,\ x + x^2,\ 1 + x^2\}$ is linearly independent in P_2.

Solution: Let us take three scalars c_1, c_2 and c_3 such that

$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = 0$ This implies $(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 = 0$. Comparing the coefficients of the above equation, we get

$$c_1 + c_3 = 0 \tag{3.8}$$

$$c_1 + c_2 = 0 \tag{3.9}$$

$$c_2 + c_3 = 0 \tag{3.10}$$

From Equation (3.8) and Equation (3.9), we get

$$c_1 + c_3 = c_1 + c_2 \Rightarrow c_3 = c_2$$

Putting $c_3 = c_2$ in Equation (3.10), we get $c_3 = c_2 = 0$. Therefore, from Equation (3.8), $c_1 = 0$.

This shows that the given set $\{1 + x,\ x + x^2,\ 1 + x^2\}$ is linearly independent

Exercise 3.8 Check whether the given set $\{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\}$ is linearly independent in M_{22}, the vector space of square matrices of order 2 over the field R.

Exercise 3.9 Check whether the given set $\{1 + x + x^2,\ 1 - x + 3x^2,\ 1 + 3x - x^2\}$ is linearly independent in P_2.

Exercise 3.10 Check whether the given set $\{x,\ 2x - x^2,\ 3x + 2x^2\}$ is linearly independent in P_2.

Example 3.9 Check whether the given set $\{Sin\, x,\ Cos\, x\}$ is linearly independent.

Solution: Let us take two scalars c_1 and c_2 such that

$$c_1\, Sin\, x + c_2\ \ Cos\, x = 0 \tag{3.11}$$

This equation (3.11) is true for all values of the angle $x \in [-\pi, \pi]$. So, in particular for $x = 0$ in Equation (3.11), $c_1 .0 + c_2 .1 = 0 \Rightarrow c_2 = 0$. Also, for $x = \frac{\pi}{2}$ in Equation (3.12), $c_1 .1 + c_2 .0 = 0 \Rightarrow c_1 = 0$.

Therefore, the given set $\{Sin\, x,\ Cos\, x\}$ is linearly independent.

Example 3.10 Check whether the given set $\{1,\ x,\ x^2, \dots ., x^n\}$ is linearly independent in P_n.

Solution: Let us take scalars c_1, c_2,….. and c_n such that

$$c_0 .1 + c_1 x + c_2 x^2 + \cdots .. + c_n x^n = 0$$

This can be written as

$$c_0 .1 + c_1 x + c_2 x^2 + \cdots .. + c_n x^n = 0.1 + 0x + 0x^2 + \cdots .. + 0x^n$$

Therefore, comparing the coefficients, we get $c_1 = c_2 = \cdots = c_n = 0$.

Hence, the given set $\{1,\ x,\ x^2, \dots ., x^n\}$ is linearly independent in P_n.

Exercise 3.11 Check whether the given set $\{1 + x,\ 1 + x^2, 1 - \ x + x^2\}$ is linearly independent in P_2.

Exercise 3.11 Check whether the given set $\{x,\ \ 1 +\ x\}$ is linearly independent in P_1.

Exercise 3.12 Check whether the given set $\{\begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}\}$ is linearly independent in M_{22}, the vector space of square matrices of order 2 over the field R.

Exercise 3.13 Check whether the given set $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is linearly independent.

Exercise 3.14 Check whether the given set $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is linearly independent.

Exercise 3.15 Check whether the given set $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\4\\-2\end{bmatrix}\right\}$ is linearly independent.

Exercise 3.16 Check whether the given set $\{2x,\ x-x^2,\ 1+x^3, 2-x^2+x^3\}$ is linearly independent in P_2.

Definition 3.8 (Basis of vector space) Let V be a vector space over a field F. A non-empty subset B of V is called a *basis* of V when

(i) The set B spans the vector space V. That is each element in V can be expressed as linear combination of elements from B. In mathematical notation, $L(B) = V$.

(ii) B is a linearly independent set.

Example 3.11 Check whether the given set $\{1+x,\ x+x^2,\ 1+x^2\}$ is a basis for P_2.

Solution: Let $ax^2 + bx + c$ be any polynomial in P_2. Let us take three scalars c_1, c_2 and c_3 such that

$$ax^2 + bx + c = c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) \tag{3.12}$$

Comparing the coefficients in Equation (3.12),

$$c_2 + c_3 = a \tag{3.13}$$

$$c_1 + c_2 = b \tag{3.14}$$

$$c_1 + c_3 = c \tag{3.15}$$

From Equation (3.14) and Equation (3.15), we get

$$c_2 - c_3 = b - c \tag{3.16}$$

From Equation (3.13) and Equation (3.16), we get

$c_2 = \frac{a+b-c}{2}$ and $c_3 = \frac{a-b+c}{2}$

So, form Equation (3.14), $c_1 = b - \frac{a+b-c}{2} = \frac{-a+b+c}{2}$

Therefore, Equation (3.12) can be rewritten as

$$ax^2 + bx + c = \left(\frac{-a+b+c}{2}\right)(1+x) + \left(\frac{a+b-c}{2}\right)(x+x^2) + \left(\frac{a-b+c}{2}\right)(1+x^2)$$

Hence, $\{1+x,\ x+x^2,\ 1+x^2\}$ spans the vector space P_2.

Let us take three scalars c_1, c_2 and c_3 such that

$c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = 0$ This implies $(c_1+c_3) + (c_1+c_2)x + (c_2+c_3)x^2 = 0$. Comparing the coefficients of the above equation, we get

$$c_1 + c_3 = 0 \tag{3.17}$$

$$c_1 + c_2 = 0 \tag{3.18}$$

$$c_2 + c_3 = 0 \tag{3.19}$$

From Equation (3.17) and Equation (3.18), we get

$$c_1 + c_3 = c_1 + c_2 \Rightarrow c_3 = c_2$$

Putting $c_3 = c_2$ in Equation (3.19), we get $c_3 = c_2 = 0$. Therefore, from Equation (3.17), $c_1 = 0$.

This shows that the given set $\{1 + x,\ x + x^2,\ 1 + x^2\}$ is linearly independent

Therefore, the given set $\{1 + x,\ x + x^2,\ 1 + x^2\}$ is a basis for P_2.

Exercise 3.17 Check whether the given set $\{(1,0,0),\ (1,1,0),\ (1,1,1)\}$ is a basis for $R^3(R)$.

Exercise 3.18 Check whether the given set $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right\}$ is a basis for M_{22}, the vector space of square matrices of order 2 over the field R.

Exercise 3.19 Check whether the given set $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is a basis for M_{22}, the vector space of square matrices of order 2 over the field R.

Exercise 3.20 Check whether the given set $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\right\}$ is a basis for M_{22}, the vector space of square matrices of order 2 over the field R.

Exercise 3.21 Check whether the given set $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}, \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right\}$ is a basis for M_{22}, the vector space of square matrices of order 2 over the field R.

Exercise 3.22 Check whether the given set $\{x,\ 1 + x, x - x^2\}$ is a basis for P_2.

Exercise 3.23 Check whether the given set $\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}-1\\5\\1\end{bmatrix}, \begin{bmatrix}1\\-3\\1\end{bmatrix}\right\}$ is a basis of $R^3(R)$.

Exercise 3.24 Check whether the given set $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}4\\9\\7\end{bmatrix}\right\}$ is a basis of $R^3(R)$.

Dimension of vector space

Definition 3.8 (Finite dimensional vector space) Let V be a vector space over a field F. If the basis B of V consists finitely many vectors, then the vector space V is called finite dimensional vector space. The number of vectors in basis is called the dimension of the vector space V and is denoted by $\dim(V)$.

Dimensions of some vector spaces:

1. The set $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ is a basis for $R^3(R)$.So, the dimension of $R^3(R)$ is 3.

2. The standard basis of P_n consists $(n+1)$ vectors. So, the dimension of P_n is $(n+1)$.
3. The standard basis of M_{mn} consists mn vectors. So, the dimension of M_{mn} is mn.

Definition 3.8 (Infinite dimensional vector space) Let V be a vector space over a field F. If the basis B of V consists infinitely many vectors, then the vector space V is called infinite dimensional vector space.

Four fundamental sub spaces regarding matrices

Let A be a $m \times n$ rectangular matrix, then four fundamental subspaces are as follows:

1. **Column space:** For the matrix A, the column space is denoted by $C(A)$, a linear combination of the columns in A. It is a subspace of R^m and has the dimension equal to the rank r of the matrix A.
2. **Row space:** For the matrix A, the row space is denoted by $C(A^T)$, a linear combination of the rows in A. It is a subspace of R^n and has the dimension equal to the rank r of the matrix A.
3. **Null space:** For the matrix A, the null space is denoted by $N(A)$. It is defined as $N(A) = \{X \in R^n : AX = O\}$. Its dimension is $(n-r)$.
4. **Left Null Space:** For the matrix A, the null space is the null space of A^T and is denoted by $N(A^T)$. Its dimension is $(m-r)$.

Example 3.12 Find the basis and the dimension for the four fundamental subspaces for the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Solution: Given matrix is

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Here, $\mathrm{m} = 3$ and $\mathrm{n} = 4$. Now applying elementary row operations, we have

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \qquad (R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 + R_1)$$

$$\sim \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to R_3 - 2R_2)$$

$$\sim \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_2 \to \frac{R_2}{3})$$

Here, the linear independent columns are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.Thus, basis of the column space $C(A)$ is $\{(1,2,-1),\ (3,9,3)\}$. So, $Dim\ C(A) = 2$.

To find null space, we have x_1 and x_3 pivot variables and x_2 and x_4 free variables. Also, for finding null space, $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This gives

$$x_1 + 3x_2 + 3x_3 + 2x_4 = 0 \tag{3.20}$$

$$x_3 + x_4 = 0 \tag{3.21}$$

From Equation (3.21), we get $x_3 = -x_4$

Then, from Equation (3.20), $x_1 = -3x_2 + x_4$

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

The basis of the null space $N(A)$ is $\{(1,2,-1),\ (3,9,3)\}$. So, $Dim\ N(A) = 2$.

Now the transpose matrix of $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ is

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \qquad (R_2 \to R_2 - 3R_1\ , R_3 \to R_3 - 3R_1 \text{and } R_4 \to R_4 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_4 \to R_4 - R_3)$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_2 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_2 \to \frac{R_2}{3})$$

Therefore, the basis of the row space $C(A^T)$ is $\{(1,3,3,2),(2,6,9,7)\}$. So, $Dim\ C(A^T) = 2$.

To find left null space, we have x_1 and x_2 pivot variables and x_3 free variables. Also, for finding null space, $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. This gives

$$x_1 + 2x_2 - x_3 = 0 \tag{3.22}$$

$$x_2 + 2x_3 = 0 \tag{3.23}$$

From Equation (3.23), we get $x_2 = -2x_3$. And then from Equation (3.22), we get $x_1 = 5x_3$.

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

The basis of the left null space $N(A^T)$ is $\{(5,-2,1),\}$. So, $Dim\ N(A^T) = 1$.

Exercise 3.25 Find the basis and the dimension for the four fundamental subspaces for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Exercise 3.26 Find the basis and the dimension for the four fundamental subspaces for the matrix

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 2 & 4 & 14 & 4 \\ 2 & 3 & 12 & 3 \end{bmatrix}$$