MODULE – II
RELATIONS

Introduction

The elements of a set may be related to one another. For example, in the set of natural numbers there is the l ess than' relation between the elements. The elements of one set may also be related to the elements anothe r set.

Binary Relation

A binary relation between two sets A and B is a rule R which decides, for any elements, whether a is in rela tion R to b. If so, then we write a R b. If a is not in relation R to b, then a /R b.

We can also consider a R b as the ordered pair (a, b) in which case we can define a binaryrelation from A to B as a subset of A X B. This subset is denoted by the relation R.

In general, any set of ordered pairs defines a binary relation.

For example, the relation of father to his child is F = {(a, b) / a is the father of b} In this relationF, the first member is the name of the father and the second is the name of the child.
The definition of relation permits any set of ordered pairs to define a relation.For exampl e, the set S given by

S = {(1, 2), (3, a), (b, a) ,(b, Joe)}

Definition

The domain D of a binary relation S is the set of all first elements of the ordered pairs in therelation.(i.e) D(S)= {a / $ b for which (a, b) Є S}

The range R of a binary relation S is the set of all second elements of theordered pairs in t he relation. (i.e) R(S) = {b / $ a for which (a, b) Є S}

For example

For the relation S = {(1, 2), (3, a), (b, a) ,(b, Joe)}D(S) = {1 , 3, b, b} and

R(S) = {2, a, a, Joe}

Let X and Y be any two sets. A subset of the Cartesian product X * Y defines a relation, say

C. For any such relation C, we have D(C) Í X and R(C) Í Y, and the relation C is said to from X to Y. If Y = X, then C is said to be a relation form X to X. In such case, c is called a relation in X. Thus any relation in X is a subset of X * X . The set X * X is called a *universalrelation* in X, while the empty se t which is also a subset of X * X is called a *void relation* in X.

For example: Let L denote the relation ―less than or equal to< and D denote the relation ―divides<where x

D y means — x divides y< . Both L and D are defined on the set {1, 2, 3, 4}

L = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4,4)}
D = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}
L Ç D = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)} = D

Properties of Binary Relations:

Definition: A binary relation R in a set X is reflexive if, for every x Є X, x R x, That is (x, x) Є R, or R is reflexive in X ó (x) (x Є X ® x R x).

For example:-

- The relation £ is reflexive in the set of real numbers.
- The set inclusion is reflexive in the family of all subsets of a universal set.
- The relation equality of set is also reflexive.
- The relation is parallel in the set lines in a plane.
- The relation of similarity in the set of triangles in a plane is reflexive.

Definition: A relation R in a set X is symmetric if for every x and y in X, whenever x R y, then yR x.(i.e) R is symmetric in X ó (x) (y) (x Є X ^ y Є X ^ x R y ® y R x}

For example:-

- The relation equality of set is symmetric.
- The relation of similarity in the set of triangles in a plane is symmetric.
- The relation of being a sister is not symmetric in the set of all people.
- However, in the set females it is symmetric.

Definition: A relation R in a set X is whenever x R y and y R z , then x R z. (i.e) transitive if, for everyx, y, and z are in X, R is transitive in X ó (x) (y) (z) (x Є X^ y Є X^ z Є X ^x R y ^ y R z ® x Rz)

For example:-

- The relations <, £, >, ³ and = are transitive in the set of real numbers
- The relations Í, Ì, Ê, É and equality are also transitive in the family of sets.

The relation of similarity in the set of triangles in a plane is transitive. Definition: A relatio n R in a set X is irreflexive if, for every x Є X , (x, x)ÏX.For example:-

< The relation < is irreflexive in the set of all real numbers.

<

The relation proper inclusion is irreflexive in the set of all nonempty subsets of auni versal set.

- Let X = {1, 2, 3} and S = {(1, 1), (1, 2), (3, 2), (2, 3), (3, 3)} is neither irreflexive norreflexive.

Definition: A relation R in a set x is anti symmetric if , for every x and y in X, whenever x R y and y R x, Then x = y. Symbolically,(x) (y) (x Є X ^ y Є X ^ x R y ^ y R x ® x = y)

For example

- The relation Í is anti symmetric in set of subsets.
- The relation —divides< is anti symmetric in set of real numbers.
- Consider the relation —is a son of< on the male children in a family. Evidently therelation i s not symmetric, transitive and reflexive.
- The relation — is a divisor of— is reflexive and transitive but not symmetric on the set of naturalnumbers.
- Consider the set H of all human beings. Let r be a relation — is married to —R is sym metric.
- Let I be the set of integers. R on I is defined as a R b if a – b is an even number. R is an reflexive,symmetric and transitive

Equivalence Relation:

Definition:A relation R in a set A is called an equivalence relation if

- a R a for every i.e. R is reflexive
- a R b => b R a for every a, b Є A i.e. R is symmetric
- a R b and b R c => a R c for every a, b, c Є A, i.e. R is transitive.For examp le

< The relation equality of numbers on set of real numbers.

< The relation being parallel on a set of lines in a plane.

Problem1: Let R in T as R= {(a, b) / (a, b Є T and a is similar to b} We have to show that relation R is an Equivalence relation

Solution :

A triangle a is similar to itself. a R a

If the triangle a is similar to the triangle b, then triangle b is similar to the triangle athena Rb => b R a

If a is similar to b and b is similar to c, then a is similar to c (i.e) a R b and b R c => a Rc. Hence R is an equivalence relation.

Problem 2: Let x = {1, 2, 3, … 7} and R = {(x, y) / x – y is divisible by 3} Show that R is anequivalence relation.

Solution: For any a Є X, a- a is divisible by 3, Hence a R a, R is reflexive

For any a, b Є X, if a – b is divisible by 3, then b – a is also divisibleby 3, R is symmetric.

For any a, b, c Є, if a R b and b R c, then a – b is divisible by 3 and b– c is divisible by 3. So that (a – b) + (b – c) is also divisible by3, hence a – c is also divisible by 3. Thus R is transitive.

Hence R is equivalence.

Problem3 .Let Z be the set of all integers. Let m be a fixed integer. Two integers a and b are said to be congruent modulo m if and only if m divides a-b, in which case we write a º b (mod m). This relation is called the relation of congruence modulo m and we can show that is an equivalence relation.

Solution :

< a - a=0 and m divides a – a (i.e) a R a, (a, a) Є R, R is reflexive .

< a R b = m divides a-b

m divides b - a b º a (mod m) b R a that is R is symmetric.

- a R b and b R c => a ºb (mod m) and bº c (mod m) O m divides a – b and m divides b-cO a – b = km and b – c = lm for some k ,l Є z

  O (a – b) + (b – c) = km+lmO a – c = (k +l) m

  O a R c

  O R is transitive

Hence the congruence relation is an equivalencerelation.

Definition:Let R be a relation from X to Y and S be a relation from Y to Z. Then the relation R o S is given by R o S = {(x, z) / xÎX Ù z Î Z Ù y Î Y such that (x, y) Î R Ù (y, z) Î S)} is called the composite relation of R and S.

The operation of obtaining R o S is called the composition of relations.

Example: Let R = {(1, 2), (3, 4), (2, 2)} and
S = {(4, 2), (2, 5), (3, 1),(1,3)}

Then R o S = {(1, 5), (3, 2), (2, 5)} and S o R = {(4, 2), (3, 2), (1, 4)}

It is to be noted that R o S ≠ S o R.Also Ro(S o T) = (R o S) o T = R o S o T

Note: We write R o R as R2; R o R o R as R3 and so on.Definition

Let R be a relation from X to Y, a relation R from Y to X is called the converse of R, where the ordered pairs of Ř are obtained by interchanging the numbers in each of the ordered pairs of R. This means for x Î X and y Î Y, that x R y ó y Ř x.

Then the relation Ř is given by R = {(x, y) / (y, x) Î R} is called the converseof R

Example:

Let R = {(1, 2),(3, 4),(2, 2)}

Then Ř = {(2, 1),(4, 3),(2, 2)}

Note: If R is an equivalence relation, then Ř is also an equivalence relation.

Definition

Let X be any finite set and R be a relation in X. The relation $R+ = R \cup R2 \cup R3\ldots$ in X. is called the *transitive closure* of R in X

Example: Let R = {(a, b), (b, c), (c, a)}.
Now R2 = R o R = {(a, c), (b, a), (c, b)}
R3 = R2 o R = {(a, a), (b, b), (c, c)}
R4 = R3 o R = {(a, b), (b, c), (c, a)} = R
R5= R3o R2 = R2 and so on.

Thus, R+ = R U R2 U R3 U R4 U…
= R U R2 U R3.
={(a, b),(b, c),(c, a),(a, c),(b, a),(c ,b),(a, b),(b, b),(c, c)}

We see that R+ is a transitive relation containing R. In fact, it is the smallesttransitive relation conta ining R.

Partial Ordering Relations:

Definition

A binary relation R in a set P is called *partial order relation* or *partial ordering* inP iff R is r eflexive, anti symmetric, and transitive.

A partial order relation is denoted by the symbol £., If £ is a partial ordering onP, then th e ordered pair (P, £) is called a *partially ordered set* or a *poset.*

< Let R be the set of real numbers. The relation ―less than or equal to < orO , is a partial ordering on R.

< Let X be a set and r(X) be its power set. The relation subset, Í on X is partial ordering.

< Let Sn be the set of divisors of n. The relation D means ―divides< on Sn ,is partialorderin g on Sn .

In a partially ordered set (P, £) , an element y Î P is said to cover an element x Î P if x <y and if there does not exist any element z Î P such that x £ z and z £ y; that is, y covers x Û (x < y Ù (x £ z £ y Þ x = z Ú z = y))

A partial order relation £ on a set P can be represented by means of a diagram known as a Hasse diagram or partial order set diagram of (P, £). In such a diagram, each element is represented by a small circle or a dot. The circle for x Î P is drawn below the circle for y Î P if x < y, and a line is drawn between x and y if y covers x.

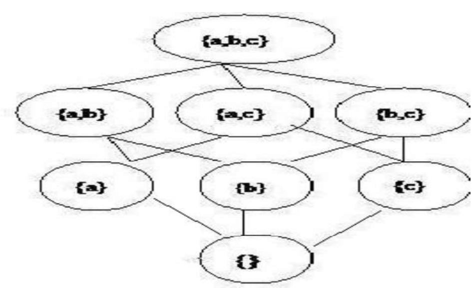
If x < y but y does not cover x, then x and y are not connected directly by a single line.However,they a re connected through one or more elements of P.

Hasse Diagram:

A Hasse diagram is a digraph for a poset which does not have loops and arcs implied by thetransitivity.

Example 10: For the relation {< a, a >, < a, b >, < a, c >, < b, b >, < b, c >, < c, c >} on set {a,b,c}, the Hass e diagram has the arcs {< a, b >, < b, c >} as shown below

HASSE DIAGRAM



Ex: Let A be a given finite set and r(A) its power set. Let Í be the subset relation onthe elements of r(A). Draw Hasse diagram of (r(A), Í) for A = {a, b, c}

LATTICES:

Lattices as Partially Ordered Set, Properties of Lattices, Lattices as Algebraic Systems, Sublattices, Direct Product and Homomorphism.

Definition: The relation f defined on a nonempty set X is called an anti-symmetric relation if and only if, ∀ x, y ∈ X, the property (x, y) ∈ f and (y, x) ∈ f implies that x = y.

It is possible to interpret an anti-symmetric relation using the arrow diagrams of relations.

In this context, a relation is called anti-symmetric if, whenever there is an arrow going from one element to an element different from it, ther e does not exist an arrow going back from the second element to the first.

Example: Let R1 = {(x, y) ∈ Z + × Z + | x divides y} and R2 = {(x, y) ∈ Z \ {0} × Z | x divides y}.

(a) Show that R1 is an anti-
symmetric relation on the set of positive integers. (b) Show that R2 isnot an anti-
symmetric relation on the set of integers by giving a counter example.

There are two relations which play a prominent role in mathematics. One of them is the equivalencer elation, which we have already seen is a relation which is reflexive, symmetric and transitive.

We now introduce the other relation called a partial order.

Definition: A relation f on a nonempty set X is called a partial order if f is reflexive, transitive and ant i-symmetric. Here (X, f) is a partially ordered set and is colloquially referred to as a poset.

The relation less than or equal to on the set of real numbers and the relation subset on the set of setsar e two fundamental partial orders. These can be thought of as models for the general partial order. It is common practice to use the symbol to denote a partial order.

Further, if (X,) is a poset and x y, then we read this as x is less than or equal to y.

Definition: Let (X,) be a poset. It there exist elements x and y in X, such that either (x, y) ∈ or (y, x) ∈ holds, then x and y are said to be comparable. In neither (x, y) nor (y, x) belongs to , then xand y are said to be incomparable.

Example1: Let X = {1, 2, 3, 4, 5}.

(a) The identity relation Id on X is reflexive, transitive and anti-
symmetric and is therefore a partialorder. However, no two elements of X are comparable.

(b) The relation Id ∪ {(1, 2)} is also a partial order on X. Here 1 and 2 are comparable.

(c) The relation = Id∪{(1, 2),(2, 1)} is both reflexive and transitive, but not anti-
symmetric.Observe that (1, 2), (2, 1) ∈and 1 6= 2.

(d) The relation Id ∪ {(1, 2),(3, 4)} is a partial order on X. Here, 1 and 2 are comparable and so are3 and 4.

Example: Let X = N. The relation = {(a, b) : a divides b} is a partial order on X.

Example: Let X be a nonempty collection of sets. Here, = {(A, B) : A, B ∈ X, A ⊆ B} is a partial orde r on X. 4. On R the set = {(a, b) : a ≤ b} is a partial order. It is called the usual partial order on R.

Definition: Let (X,) be a poset. 1. If any two elements in the poset (X,) are comparable, then is called a linear order and (X,) is called a linearly ordered set.

Often a linear order is also referred to as a total order or a complete order.

A subset, C of X, is called a chain if and only if induces a linear order on C. If C is a finite set, thenth e length of C is equal to the number of elements if C. If C is not a finite set, then the length of Cis sa id to be infinite.

A subset, A of X, is called an antichain if and only if no two elements of A are comparable. The lengt h of an antichain is defined in precisely the same manner as that of the chain.

The maximum of the lengths of the chains of X is called the height of X and the maximum of the leng ths of the antichains of X is called the width of X.

Let X be a nonempty set and let f be a relation on X. Then, recall from Definition, that f is reflexiveif $(x, x) \in f$ for all $x \in X$; f is transitive if $(x, y) \in f$ and $(y, z) \in f$ imply $(x, z) \in f$ for all $x, y, z \in X$; and f is anti-
symmetric if $(x, y) \in f$ and x 6= y implies (y, x) ∈/ f, i.e., for all distinct elements x, yof X both (x, y) and (y, x) cannot be in f. Relations which are simultaneously reflexive, transitive and anti-symmetric play an important role in mathematics; and we give a name to such relations. Definition: L et X be a nonempty set. A relation f on X is called a partial order if f is reflexive, transitive and anti-symmetric. Let f be a partial order on X and let $a, b \in X$. Then, a and b are said to be comparable (wit h respect to the partial order f) if either $(a, b) \in f$ or $(b, a) \in f$. When a partial order satisfies some othe r desirable properties, they are given different names. We fix some of thesein the following definition .

Definition: Let X be a nonempty set.

1. The pair (X, f) is called a partially ordered set (in short, poset) if f is a partial order on X.

2. A partial order f on X is called a linear order if either $(x, y) \in f$ or $(y, x) \in f$ for all $x, y \in X$, i.e., whe n any two elements of X are comparable. A linear order is also called a total order, or a complete orde r.

3. The poset (X, f) is said to be a linearly ordered set if f is a linear order on X.

4. A linearly ordered subset of a poset is called a chain in the poset. The maximum size of a chain in a poset is called the height of a poset.

5. Let (X, f) be a poset and let $A \subseteq X$. A is called an anti-
chain in the poset if no two elements of Aare comparable.

The maximum size of an anti-
chain in a poset is called the width of the poset. You may imagine theelements of a linearly ordered se t as points on a line. The height of a poset is the maximum of the cardinalities of all chains in the pos et. The width of a poset is the maximum of the cardinalities ofall anti-chains in the poset.

Examples:

1. The poset in Example1 has height 1 (size of the chain {1}) and width 5 (size of the anti-
chain {1,2, 3, 4, 5}).

2. The poset in Example1 has height 2 (respective chain is $\{1, 2\}$) and width 4 (respective anti-chains are $\{2, 3, 4, 5\}$ and $\{1, 3, 4, 5\}$).

3. The poset in Example1 has height 2 (respective chains are $\{1, 2\}$ and $\{3, 4\}$) and width 3 (are spective anti-chain is $\{1, 3, 5\}$).

4. The usual order (usual $\leq$) in N is a linear/complete/total order. The same holds for the usual orderin Z, Q and R.

5. If (X, f) is a finite linearly ordered set then the singleton subsets of X are the only anti-chains.In this case, the height of X is the number of elements in X and the width of X is 1.
6. The set N with the partial order f defined by "(a, b) ∈ f if a divides b" is not linearly ordered.H owever, the set $\{1, 2, 4, 8, 16\}$ is a chain. This is just a linearly ordered subset of the poset.

There are larger chains, for example, $\{2\ k : k = 0, 1, 2, \ldots\}$. The set of all primes is an anti-chain here. The poset (N, f) has infinite height and infinite width.

7. The poset $(P(\{1, 2, 3, 4, 5\}), \subseteq)$ is not linearly ordered. However, $\{\emptyset, \{1, 2\}, \{1, 2, 3, 4, 5\}\}$ is a ch ain in it. Also, $\{\emptyset, \{2\}, \{2, 3\}, \{2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$ is a chain. The height of thisposet is 6.

That is, if f is a partial order on a nonempty set X we write $x \leq y$ to mean that $(x, y) \in f$. Accordingly, t he poset (X, f) is written as $(X, \leq)$. Also, instead of writing '(X, f) is a poset' we will often write 'X is a poset with the partial order f'. Following custom, by $x \geq y$ we mean $y \leq x$; by $x < y$ we mean that $x \leq y$ and x 6= y; by $x > y$ we mean $y < x$. Also, we read $x \leq y$ as x is less than or equal to y; $x < y$ as x is le ss than y; $x \geq y$ as x is greater than or equal to y; and $x > y$ as x is larger than y.

Definition: Let $(\Sigma, \leq)$ be a finite linearly ordered set (like the English alphabet with $a < b < c < \cdots < z$) and let $\Sigma*$ be the collection of all words formed using the elements of Σ. For $a = a1a2 \cdots an$, $b = b1b2 \cdots bm \in \Sigma*$ for $m, n \in N$, define $a \leq b$ if (a) $a1 < b1$, or (b) $ai = bi$ for $i = 1, \ldots, k$ for some $k < \min\{m, n\}$ and $ak+1 < bk+1$, or (c) $ai = bi$ for $i = 1, 2, \ldots, n = \min\{m, n\}$. Then $(\Sigma*, \leq)$ is a linearly ordered set. This ordering is called the lexicographic or dictionary ordering. SometimesΣ is called th e alphabet and the linearly ordered set $\Sigma*$ is called the dictionary.

A directed graph representation of the poset $(A, \leq)$ with $A = \{1, 2, 3, 9, 18\}$Given a set, X, we can ord er the subsets of X by the subset relation: $A \subseteq B$, where A, B are any subsets of X.

For example, if $X = \{a, b, c\}$, we have $\{a\} \subseteq \{a, b\}$.

However, note that neither $\{a\}$ is a subset of $\{b, c\}$ nor $\{b, c\}$ is a subset of $\{a\}$.
We say that $\{a\}$ and $\{b, c\}$ are incomparable.
Definition:

A binary relation, $\leq$, on a set, X, is a partial order (or partial ordering) iff it is reflexive, transitive and antisymmetric,

that is: (1) (Reflexivity): $a \leq a$, for all $a \in X$;

(2) (Transitivity): If $a \leq b$ and $b \leq c$, then $a \leq c$, for all $a, b, c \in X$.

(3) (Antisymmetry): If a ≤ b and b ≤ a, then a = b, for all a, b ∈ X.

A partial order is a total order (ordering) (or linear order (ordering)) iff for all a, b ∈ X, either a ≤ bor b ≤ a. When neither a ≤ b nor b ≤ a, we say that a and b are incomparable.

A subset, C ⊆ X, is a chain iff ≤ induces a total order on C (so, for all a, b ∈ C, either a ≤ b orb ≤ a).
The strict order (ordering), < is the strict order associated with a partial order, ≤, then < is transitivean d antireflexive, which means that (4) a &< a, for all a ∈ X.

Conversely, let < be a relation on X and assume that < is transitive and anti-reflexive.

If confusion may arise, for example when we are dealing with several posets, we denote the partialor der on X by ≤X.

The trick is to draw a picture consisting of nodes and oriented edges, where the nodes are all theel ements of X and where we draw an oriented edge from a to b iff a is an immediate predecessor of b. Such a diagram is called a Hasse diagram for X, .

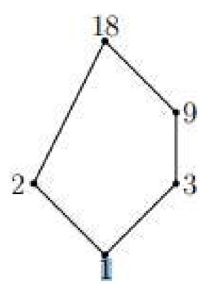
The Hasse diagram of a finite poset (X, ≤) is a picture drawn in the following way:

1. Each element of X is represented by a point and is labeled with the element.

2. If a ≤ b then the point labeled a must appear at a lower height than the point labeled b and further th e two points are joined by a line.

3. If a ≤ b and b ≤ c then the line between a and c is removed.

Example: Hasse diagram for the poset (A, ≤) with A = {1, 2, 3, 9, 18} and ≤ as the ‘divides’rel ation is given below.



Definition:

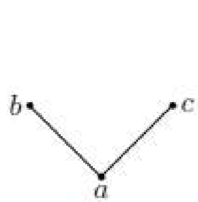
Let (X, ≤) be a poset and let A ⊆ X.

1. We say that an element x ∈ X is an upper bound of A if for each z ∈ A, z ≤ x; or equivalently, whe n each element of A is less than or equal to x. An element y ∈ X is called a lower bound of A if for eac h z ∈ A, y ≤ z; or equivalently, when y is less than or equal to each element of A.

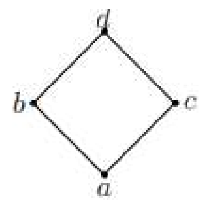
2. An element x ∈ A is called the maximum of A, if x is an upper bound of A. Thus, maximum of Ais an upper bound of A which is contained in A. Such an element is unique provided it exists. Inthis case, we denote x = max{z : z ∈ A}. Similarly, minimum of A is an element y ∈ A which is a lower b ound of A. If minimum of A exists, then it is unique; and we write y = min{z : z ∈ A}.

3. An element $x \in X$ is called the least upper bound (lub) of A in X if x is an upper bound of A and for each upper bound y of A, we have $x \leq y$; i.e., when x is the minimum (least) element of the setofall upper bounds of A. Similarly, the greatest lower bound (glb) of A is a lower bound of A which is greater than or equal to all upper bounds of A; it is the maximum (largest) of the set of all lower bounds of A.

4. An element $x \in A$ is a maximal element of A if $x \leq z$ for some $z \in A$ implies $x = z$; or equivalently, when no element in A is larger than x. An element $y \in A$ is called a minimal element of A if $z \leq y$ for some $z \in A$ implies $y = z$; or equivalently, when no element in A is less than y. Example: Consider the two posets X = {a, b, c} and Y = {a, b, c, d} described by the following Hasse diagrams:



X



Y

Let A = X. Then,

(a) the maximal elements of A are b and c,

(b) the only minimal element of A is a,

(c) a is the lower bound of A in X,

(d) A has no upper bound in X,

(e) A has no maximum element,

(f) a is the minimum element of A,

(g) no element of X is the lub of A, and

(h) a is the glb of A in X.

Example:

The following table illustrates the definitions by taking different subsets A of X, and also considering the same A as a subset of Y.

	$A=\{b,c\} \subseteq X$	$A=\{a,c\} \subseteq X$	$A=\{b,c\} \subseteq Y$
Maximal element(s) of A	b, c	c	b, c
Minimal element(s) of A	b, c	a	b, c
Lower bound(s) of A in X	a	a	a
Lower bound(s) of A in Y	a	a	a
Upper bound(s) of A in X	does not exist	c	d
Upper bound(s) of A in Y	does not exist	c	d
Maximum element of A	does not exist	c	does not exist
Minimum element of A	does not exist	a	does not exist
lub of A in X	does not exist	c	d
lub of A in Y	does not exist	c	d
glb of A in X	a	a	a
glb of A in Y	a	a	a

Definition: A linear order $\leq$ on a nonempty set X is said to be a well order if each nonempty subset of X has minimum. We call (X, $\leq$) a well ordered set to mean that $\leq$ is a well order on X.

Often we use the phrase 'X is a well ordered set with the ordering as $\leq$' to mean '(X, $\leq$) is a well order ed set'.

Lattice: A poset (L, $\leq$) is called a lattice if each pair x, y $\in$ L has an lub and also a glb. A lub of x,y is also written as x $\vee$ y (read as 'x or y' / 'join of x and y') and a glb of x, y as x $\wedge$ y (read as 'x and y' / ' meet of x and y'). A lattice is a poset in which any two elements have a meet and a join.

A complete lattice is a poset in which any subset has a greatest lower bound and a least upper bound.

It is easy to show that any finite lattice is a complete lattice and that a finite poset is a lattice iff it has a least element and a greatest element.

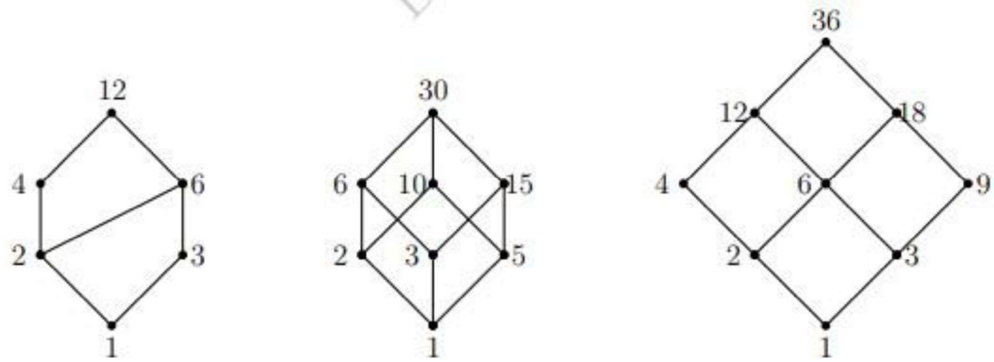
The poset N+ under the divisibility ordering is a lattice!

A lattice is called a distributive lattice if for all pairs of elements x, y the following conditions, called distributive laws, are satisfied: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Indeed, it turns out that the meet operation corresponds to greatest common divisor and the join opera tion corresponds to least common multiple.

However, it is not a complete lattice. The power set of any set, X, is a complete lattice under the subs et ordering.

Fix a positive integer n and let $D(n)$ denote the set of all divisors of n. For elements $x, y \in D(n)$, define $x \leq y$ if x divides y. Then $(D(n), \leq)$ is a distributive lattice, where $\vee = \mathsf{lcm}$ and $\wedge = \gcd$. For $n = 12, 30$ and 36, the corresponding lattices are shown below.



To check the first distributive law, let $a, b, c \in D(n)$, p a prime, and let $k \in \mathbb{N}$. Further, let $p^k | \mathsf{lcm}\{a, \gcd\{b, c\}\}$. Then, either $p^k | a$ or $p^k | b, c$. In that case, $p^k | \mathsf{lcm}\{a, b\}$ and $p^k | \mathsf{lcm}\{a, c\}$. So, $p^k | \gcd\{\mathsf{lcm}\{a, b\}, \mathsf{lcm}\{a, c\}\}$.

Now, let us assume that $p^k | \gcd\{\mathsf{lcm}\{a, b\}, \mathsf{lcm}\{a, c\}\}$. Then, $p^k | \mathsf{lcm}\{a, b\}$ and $p^k | \mathsf{lcm}\{a, c\}$. Then, either $p^k | a$ or ($p^k | b$ and $p^k | c$). So, $p^k | \mathsf{lcm}\{a, \gcd\{b, c\}\}$.

Thus, any power of a prime divides $a \vee (b \wedge c)$ if and only if it divides $(a \vee b) \wedge (a \vee c)$. Therefore, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Similarly, the second distributive law can be verified.

Proposition: If X is a lattice, then the following identities hold for all a, b, c ∈ X:L1
a ∨ b = b ∨ a, a ∧ b = b ∧ a
L2 (a ∨ b) ∨ c = a ∨ (b ∨ c), (a ∧ b) ∧ c = a ∧ (b ∧ c)L
3 a ∨ a = a, a ∧ a = a
L4 (a ∨ b) ∧ a = a, (a ∧ b) ∨ a = a.

Properties (L1) correspond to commutativity,
properties (L2) to associativity,
properties (L3) to idempotence andp
roperties (L4) to absorption.
Furthermore, for all a, b ∈ X, we have a ≤ b iff a ∨ b = b iff a ∧ b = a, called consistency.Pr
operties (L1)-(L4) are algebraic properties.
Properties: In a lattice (L, ≤), the following are true:

1. [Idempotence] : a ∨ a = a, a ∧ a = a

2. [Commutativity] : a ∨ b = b ∨ a, a ∧ b = b ∧ a

3. [Associativity] : a ∨ (b ∨ c) = (a ∨ b) ∨ c, a ∧ (b ∧ c) = (a ∧ b) ∧ c

4. $a \leq b \Leftrightarrow a \vee b = b$. Similarly, $a \leq b \Leftrightarrow a \wedge b = a$

5. [Absorption] : $a \vee (a \wedge b) = a = a \wedge (a \vee b)$

6. [Isotonicity] : $b \leq c \Rightarrow a \vee b \leq a \vee c$, $b \leq c \Rightarrow a \wedge b \leq a \wedge c$

7. $a \leq b, c \leq d \Rightarrow a \vee c \leq b \vee d$, $a \leq b, c \leq d \Rightarrow a \wedge c \leq b \wedge d$

8. [Distributive Inequality] : $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

9. [Modularity] : $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Proof. We prove only the first parts of all assertions; the second parts can be proved similarly.

(1) $a \vee a$ is an upper bound of $\{a, a\}$.

Hence $a \vee a \geq a$. On the other hand, a is an upper bound of $\{a, a\}$.

So, $a \vee a$ being the least of all upper bounds of $\{a, a\}$, is less than or equal to a.H
ence $a \vee a = a$.

(2) $a \leq b \vee a$, $b \leq b \vee a$.

So, $b \vee a$ is an upper bound of a, b.

Since $a \vee b$ is the least of all upper bounds of a, b, we have $a \vee b \leq b \vee a$.E
xchanging a and b, we get $b \vee a \leq a \vee b$.
Hence $a \vee b = b \vee a$.

(3) Let $d = a \vee (b \vee c)$.

Then, $d \geq a$, $d \geq b \vee c$ so that $d \geq a$, $d \geq b$ and $d \geq c$. So, $d \geq a \vee b$ and $d \geq c$.That i
s, $d \geq (a \vee b) \vee c$. Similarly, $e = (a \vee b) \vee c$ implies $e \geq a \vee (b \vee c)$.
Thus, the first part of the result follows.

(4) Let $a \leq b$. As b is an upper bound of $\{a, b\}$, and $a \vee b$ is the least of all upper bounds of $\{a,b\}$, we have $a \vee b \leq b$.

Also, $a \vee b$ is an upper bound of $\{a, b\}$ and hence $a \vee b \geq b$.S
o, we get $a \vee b = b$.
Conversely, let $a \vee b = b$.

As $a \vee b$ is an upper bound of $\{a, b\}$, we have $a \leq a \vee b = b$.T
herefore, $a \leq b \Leftrightarrow a \vee b = b$.

(5) By definition $a \wedge b \leq a$. So, $a \vee (a \wedge b) \leq a \vee a = a$ using (1).A
lso, by definition $a \vee (a \wedge b) \geq a$.
Hence, $a \vee (a \wedge b) = a$.

(6) Let $b \leq c$. Note that $a \vee c \geq a$ and $a \vee c \geq c \geq b$.S
o, $a \vee c$ is an upper bound of $\{a, b\}$.
Thus, $a \vee c \geq \text{lub}\{a, b\} = a \vee b$.

(7) Using (6), we have $a \vee c \leq b \vee c \leq b \vee d$. A
gain, using (6), we get $a \wedge c \leq b \wedge c \leq b \wedge d$.
(8) Note that $a \leq a \vee b$ and $a \leq a \vee c$.
Thus, $a = a \wedge a \leq (a \vee b) \wedge (a \vee c)$.
As $b \leq a \vee b$ and $c \leq a \vee c$, by (7), we get $b \wedge c \leq (a \vee b) \wedge (a \vee c)$.S
o, by definition $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.
(9) Let $a \leq c$. Then, $a \vee c = c$ and hence by (8), we have $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$. Conversely, let $a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

Then $a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$.

Theorem:

The direct product of two distributive lattices is a distributive lattice. Proof. Let (a1, b1), (a2, b2), (a3, b3) be elements in the direct product of two distributive lattices. Then $[(a1, b1) \vee (a2, b2)] \wedge (a3, b3) = (a1 \vee a2, b1 \vee b2) \wedge (a3, b3) = (a1 \vee a2) \wedge a3,(b1 \vee b2) \wedge b3 = (a1 \wedge a3) \vee (a2 \wedge a3), (b1 \wedge b3) \vee (b2 \wedge b3) = (a1 \wedge a3),(b1 \wedge b3) \vee (a2 \wedge a3),(b2 \wedge b3) = = (a1, b1) \wedge (a3,b3) \vee (a2, b2) \wedge (a3, b3)$.

This verifies one of the distributive laws. Similarly, the other one can be verified.

Definition: Let $(L1, \leq 1)$ and $(L2, \leq 2)$ be lattices. A function $f : L1 \rightarrow L2$ satisfying $f(a \vee 1 b) = f(a) \vee 2 f(b)$ and $f(a \wedge 1 b) = f(a) \wedge 2 f(b)$ is called a lattice homomorphism.

Further, if f is a bijection, then it is called a lattice isomorphism.

Definition:

Let $(L, \leq)$ be a lattice. It is called a bounded lattice if there exist elements $\alpha, \beta \in L$ such that for ea ch $x \in L$, we have $x \leq \alpha$ and $\beta \leq x$. Such an element α is called the largest element of L, and is denoted by 1. The element $\beta \in L$ satisfying $\beta \leq x$ for all $x \in L$ is called the smallest element of L, and is denot ed by 0.

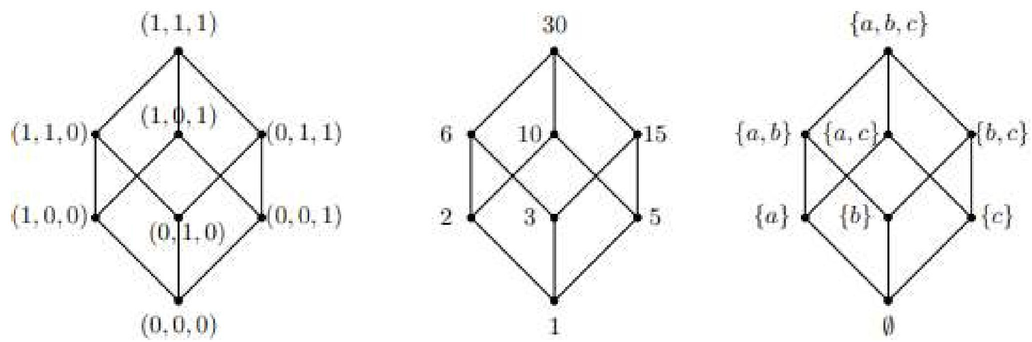
Notice that if a lattice is bounded, then 1 is the lub of the lattice and 0 is the glb of the lattice.

Definition: A lattice $(L, \leq)$ is said to be complete if each nonempty subset of L has lub and glbin L. For $A \subseteq L$, we write lub of A as $\vee A$, and glb of A, as $\wedge A$. It follows that each complete lattice is a b ounded lattice.

Examples:

1. The set [0, 5] with the usual order is a lattice which is both bounded and complete. So, is these t $[0, 1) \cup [2, 3]$.
2. The set (0, 5] with the usual order is a lattice which is neither bounded nor complete.

3. The set [0, 1) U (2, 3] with the usual order is a lattice which is bounded b ut not complete.
4. Every finite lattice is complete, and hence, bounded.
5. The set R with the usual order is a lattice. It is not a complete la ttice. Observe that the completeness property of R, i.e., “for every bounded nonempty subset a glb and an lubexist” is di fferent from the completeness in the lattice sense.



Definition:Let (L, ≤) be a bounded lattice. We say that (L, ≤) is a c omplemented lattice if for each x ∈ L, there exists y ∈ L such that $x \vee y = 1$ and $x \wedge y = 0$. Such an element y corresponding to the eleme nt x is called a complement of x, and is denoted by ¬x.

Theorem: Let (L, ≤) be a lattice and let a, b, c ∈ L. The following ta ble lists the properties that hold (make sense) in the specified type of lattices.

FUNCTIONS

A function is a special type of relation. It may be considered as a relat ion in which each element of the domain belongs to only one ordered pair i n the relation. Thus a function from A to B is a subset of A X B having the property that for each a ЄA, there is one and only oneb Є B such that (a, b) Î G.

Definition

Let A and B be any two sets. A relation f from A to B is called a functi on if for every a Є Athere is a unique b Є B such that (a, b) Є f .

Note that the definition of function requires that a relation must satisfy t wo additionalconditions in order to qualify as a function.

The first condition is that every a Є A must be related to some b Є B, (i.e) the domainof f must be A and not merely subset of A. The second r equirement of uniqueness can be expressed as (a, b) Є f ^ (b, c) Є f => b = c

Intuitively, a function from a set A to a set B is a rule which assigns t o every element of A,a unique element of B. If a ЄA, then the unique element of B assigned to a under f is denoted by f

(a).The usual notation for a function f from A to B is f: A® B defined by a ® f (a) where a ЄA, f(a) is called the image of a under f and a is called pre image of f(a).

< Let X = Y = R and f(x) = x2 + 2. Df = R and Rf Í R.

<

Let X be the set of all statements in lo gic and let Y = {True,False}. A mapping f: X®Y is a function.

<

A program written in high level language is mapped in to a machine language by acompiler. Similarly, the output f rom a compiler is a function of its input.

<

Let X = Y = R and f(x) = x2 is a function from X ® Y,and g(x2) = x is not a functionfrom X ® Y.

A mapping f: A ® B is called one-to-one (injective or 1 – 1) if distinct elements ofA are mapped into distinct elements of B. (i.e) f is one-to-one if

a1 = a2 => f (a1) = f(a2) or equivalently f(a1) ¹ f(a2) => a1 ¹ a2

For example, f: N ® N given by f(x) = x is 1-1 where N is the set of a natural numbers.

A mapping f: A® B is called onto (surjective) if for every b Є B th ere is an a Є A such that f (a) = B. i.e. if every element of B has a pre-image in A. Otherwise it is called into.

For example, f: Z®Z given by f(x) = x + 1 is an ontomapping. A mapping is both 1-1 and onto is called bijective

For example f: R®R given by f(x) = X + 1 is bijective.

Definition: A mapping f: R® b is called a constant mapp ing if, for all aÎA, f (a) =b, a fixed element.

For example f: Z®Z given by f(x) = 0, for all x ÎZ is a constant mapping.

Definition

A mapping f: A®A is called the identity mapping of A if f (a) = a, for all aÎA.Usually it is denoted by IA or simply I.

Composition of functions:

If f: A®B and g: B®C are two functions, then the composi tion of functions f and g,denoted by g o f, is the function is give n by g o f : A®C and is given by

g o f = {(a, c) / a Є A ^ c Є C ^ $bÎ B ': f(a)= b ^ g(b)
= c} and (g of)(a) = ((f(a))

Example 1: Consider the sets A = {1, 2, 3},B={a, b} and C = {x, y}. Let f: A® B be defined by f (1) = a ; f(2) = b and f(3)=b and Let g: B® C be defined by g(a) = x and g(b) = y

(i.e) f = {(1, a), (2, b), (3, b)} and g = {(a,x), (b, y)}.
Then g o f: A®C is defined by

(g of) (1) = g (f(1)) = g(a) = x
(g o f) (2) = g (f(2)) = g(b) = y
(g o f) (3) = g (f(3)) = g(b) = y

i.e., g o f = {(1, x), (2, y),(3, y)}

If f: A® A and g: A®A, where A = {1, 2, 3}, are given by f = {(1, 2), (2, 3), (3, 1)} and g = {(1, 3), (2, 2), (3, 1)}
Then g of = {(1, 2), (2, 1), (3, 3)}, fog= {(1, 1), (2, 3), (3, 2)}
f of = {(1, 3), (2, 1), (3, 2)} and gog= {(1, 1), (2, 2), (3, 3)}

Let f: A® B be a one-to-one and onto mapping. Then, its inverse, denoted by f -1 is givenby f - 1 = {(b, a) / (a, b) Î f} Clearly f-1: B® A is one-to-one and onto.

Also we observe that f o f -1 = IB and f -1o f =IA.If f -1 exists then f is called invertible.

*For example:*Let f: R ®R be defined by f(x) = x + 2
Then f -1: R® R is defined by f -1(x) = x - 2

Theorem: Let f: X ®Y and g: Y ® Z be two one to one and onto functions. Then gof is alsoone to one and onto function.

Proof

Let f:X ® Y g : Y ® Z be two one to one and onto functions. Let x1, x2 Î X

g o f (x1) = g o f(x2),
g (f(x1)) = g(f(x2)),

g(x1) = g(x2) since [f is 1-1]

x1 = x2 since [g is 1-1} gof is 1-1.

By the definition of composition, gof : X ® Z is a function.

We have to prove that every element of z Î Z an image el ement for some x Î Xunder gof.

Since g is onto $ y ÎY ': g(y) = z and f is onto from X to Y,

$ x ÎX ': f(x) = y.

Now, gof (x) = g (f (x))

= g(y) [since f(x) = y]

= z [since g(y) = z] which shows that gof is onto.

Theorem (g o f) -1 = f -1 o g - 1 (i.e) the inverse of a composite function can be expressed in terms ofthe comp osition of the inverses in the reverse order.

<u>Proof.</u> f: A ® B is one to one and onto. g: B ® C is one to one and onto.gof: A ® C is also one to one and onto. Þ (gof) -1: C ® A is one to one and onto.

Let a Î A, then there exists an element b Î b such that f (a) = b Þ a = f-1 (c). Now b Î B Þ there exists an element c Î C such t hat g (b) = c Þ b = g - 1(c).Then (gof)(a) = g[f(a)] = g(b) = c Þ a = (gof) - 1(c)... (1)

(f -1 o g-1) (c) = f -1(g -1 (c)) = f -1(b) = a Þ a = (f -1 o g -1)(c) ….(2) Combining (1) and (2), we have (gof) -1 = f -1 o g -1