*Module 1: Matrices and Linear System of Equations*

## Contents:



## Definition of a matrix:

An order set of ‘mn’ numbers, real or complex arranged in a rectangular array with ‘m’ rows and ‘n’ columns written as,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Is called an $m \times n$ (read as m by n) matrix. These mn numbers are also called the elements of the matrix.

Thus, we write $A = [a_{ij}]_{m\times n}$ where $1 \leq i \leq m \; and \; 1 \leq j \leq n$ . The symbol $a_{ij}$ denotes the element in the i$^{th}$ row and j$^{th}$ column.

**Types of matrices:**

**1)** Row matrix:
A matrix which consists of a single row is called a row matrix.
i.e. here m=1, for example $[1\ 2\ 1\ 0]$.

**2)** Column matrix:
A matrix which consists of a single column is called a column matrix.

i.e. here n=1, for example $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

**3)** Square matrix:
A matrix in which number of rows and columns are equal is called a square matrix.

i.e. here m=n, for example $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & 3 & 3 \end{bmatrix}$

Note: The sum of the diagonal elements of a square matrix, A is called the trace of A.

**4)** Determinant of a matrix:

A determinant of a square matrix has same elements as the square matrix A and give a real or complex value. Determinant of a square matrix A is denoted by $|A|$.

For example, If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $then\ |A| =$ $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(45-48) - 2(36-42) + 3(32-35) = -3 + 12 - 9 = 0$

**Note a:**

1) If $|A| = 0$, then matrix A is called singular matrix.
2) If $|A| \neq 0$ , then matrix A is called Non-singular matrix.

**Note b:**

1) Difference between matrix and determinant is that in determinant the number of rows and columns must be equal whereas in a matrix, the number of rows and columns may not be equal.
2) On interchanging the rows and columns, a different matrix is formed whereas in a

determinant , an interchange of rows and columns does not change the absolute value of the determinant.

**5)** Diagonal matrix:

A square matrix in which all the elements except the diagonal elements are zero, is called a diagonal matrix.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**6)** Scalar matrix:

A diagonal matrix in which all diagonal elements are equal, is called a scalar matrix.

For example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**7)** Unit matrix or Identity matrix:

A diagonal matrix in which all the diagonal elements are equal to unity is called a unit matrix.

A unit matrix of order n is denoted by $I_n$.

For example, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**8)** Zero matrix or null matrix:

A matrix in which all the elements are zero is called a zero or null matrix. It is denoted by 0.

For example, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Note: Null matrix need not to be square matrix whereas unit matrix must be a square matrix.

**9)** Triangular matrix:

A square matrix in which every element either above or below the principal diagonal is zero, is called a triangular matrix.

For example, $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \quad and \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 8 & -5 & 1 \end{bmatrix}_{3\times 3}$ .

Here 1st one is called upper triangular matrix and 2nd one is called lower triangular matrix.

**10)** Symmetric matrix and skew-symmetric matrix:

A square matrix $A = [a_{ij}]$ is said to be symmetric iff $a_{ij} = a_{ji}$.

For example, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric iff $a_{ij} = -a_{ji}$.

For example, $\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$

**11)** Transpose of a matrix:

Matrix obtained by interchanging rows and columns is known as transpose of a matrix. The transpose of a matrix A is denoted by $A' or A^T$.

For example, If $A = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$ $then\ A^T =$

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

**12)** Idempotent matrix:

A matrix A such that $A^2 = A$ is called idempotent matrix.

For matrix, $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

**13)** Involutory matrix:

A matrix A such that $A^2 = I$ is called involutory matrix.

For matrix, $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

**Elementary Transformation:**

Following transformation are known as elementary transformation.

1) The interchange of any two rows (or columns).
   $R_i \leftrightarrow R_j$ stands for interchange of $i^{th}$ row and $j^{th}$ row.
   $C_i \leftrightarrow C_j$ stands for interchange of $i^{th}$ column and $j^{th}$ column.

2) The multifaction of elements of any row (or column) by any non-zero number.
   $R_i \rightarrow kR_i$ stands for the multifaction of $i^{th}$ row by k.

$C_i \rightarrow kC_i$ stands for the multifaction of i^th column by k.

3) The addition to the elements of any other row (or column) the corresponding elements of any other row (or column) multiplied by any number.
$R_i \rightarrow R_i + kR_j$ means add to the elements of $i^{th}$ row, k times the elements of $j^{th}$ column.

**Rank of a Matrix:**

A matrix is said to be of rank r if

i) It has at least one non-zero minor of order r.
ii) Every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the highest order of any non-vanishing minor of the matrix.

The rank of a matrix A is denoted by $\rho(A)$.

**Note:**

1) The rank of a null matrix is zero i.e. $\rho(A) = 0$.
2) Every matrix will have a rank.
3) Rank of a matrix is unique.
4) For a non-zero matrix, $\rho(A) \geq 1$.
5) The rank of a unit matrix of order n is n.
6) The rank of every non-singular matrix of order n is n and the rank of a singular matrix of order n is less than n.

## Echelon Form of a Matrix:

A matrix is said to be Echelon Form if

i) All non-zero rows, if any precede the zero rows.
ii) The number of zero preceding the first non-zero element in a row less than the number of such zeros in the succeeding row.
iii) The first non-zero element in each non-zero row is unity.

∴ The rank of a matrix in echelon form is the number of non-zero rows of the matrix.

Note: The condition (iii) is optional.

## Zero Row:

If all the elements in a row of a matrix are zeros, then it is called a zero row.

Example:

$$\begin{bmatrix} 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## Non-Zero Row:

If there is at least one non-zero element in a row, then it is called a non-zero row.

Example:

$$\begin{bmatrix} -1 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Problem 1:** Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

by reducing it to the echelon matrix.

Solution:

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

Applying row operations $R_2 \to R_2 - 3R_1 \; and \; R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \; [R_3 \to R_3 - 2R_2]$$

The last equivalent matrix is in echelon form.

∴ Rank of A = Number of non-zero rows=3

Hence $\rho(A) = 3$

**Problem 2:** Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ into echelon matrix form and hence find its rank.

Solution:

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

Applying row operations $R_2 \to R_2 + 2R_1 and\ R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} [R_3 \to 4R_3 + R_2]$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

This is echelon form. The number of non-zero rows is 3.

$$\therefore \rho(A) = 3$$

**Problem 3:** Find the rank of matrix

$$A=\begin{bmatrix}-1 & 2 & 1 & 8\\ 2 & 1 & -1 & 0\\ 3 & 2 & 1 & 7\end{bmatrix}$$

By reducing it to the echelon form. (Try it, Answer- $\rho(A)=3$ )

**Problem 4:** After reducing the matrix,

$$A=\begin{bmatrix}1 & 2 & 2 & 0\\ 2 & 4 & 3 & 2\\ 3 & 2 & 1 & 3\\ 6 & 8 & 7 & 5\end{bmatrix}$$

Into echelon matrix find its rank.

Solution:

Here the matrix is

$$A=\begin{bmatrix}1 & 2 & 2 & 0\\ 2 & 4 & 3 & 2\\ 3 & 2 & 1 & 3\\ 6 & 8 & 7 & 5\end{bmatrix}$$

$$[R_2\to R_2-2R_1, R_3\to R_3-3R_1 and\ R_4\to R_4-6R_1]$$

$$\sim\begin{bmatrix}1 & 2 & 2 & 0\\ 0 & 0 & -3 & 2\\ 0 & -4 & -8 & 3\\ 0 & -4 & -11 & 5\end{bmatrix}$$

$$R_2\leftrightarrow R_3$$

$$\sim\begin{bmatrix}1 & 2 & 2 & 0\\ 0 & -4 & -8 & 3\\ 0 & 0 & -3 & 2\\ 0 & -4 & -11 & 5\end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_2$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3$$

This is echelon form and the number of non-zero rows is 3.

Hence $\rho(A) = 3$

**Problem 5:** After reducing the matrix,

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Into echelon matrix find its rank. (Try it)

**Problem 6:** After reducing the matrix,

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

Into echelon matrix find its rank. (Try it)

*<u>System of linear simultaneous Equations:</u>*

An equation of the form,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where $x_1, x_2, \dots x_n$ are unknowns and $a_1, a_2, \dots, a_n, b$ are constants is called a linear equation in n unknowns.

**Non-homogeneous equations**:

Consider m linear non-homogeneous equations in n unknowns as given below,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The matrix form of the above system is $AX = B$, where A is the coefficient matrix formed by coefficients of unknowns.

i.e. $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b \\ \dots \\ b_n \end{bmatrix}$

The matrix $[A|B]$ is called the augmented matrix formed by the coefficient matrix together with the column formed by constants $b_1, b_2, \dots b_m$.

**Condition of consistency:**

The system of equations $AX = B$ is consistent iff the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A|B]$

$$\text{i.e. } \rho(A) = \rho([A|B])$$

**Nature of the solution for non-homogeneous system:**

The system of equations $AX = B$ is said to be

i) Consistent if $\rho(A) = \rho([A|B])$
ii) Consistent and a unique solution if if $\rho(A) = \rho([A|B]) = r = n$. Where r is the rank and n is the number of unknowns.
iii) Consistent and an infinite number of solutions is if $\rho(A) < \rho([A|B])\ i.e.\ r < n$
iv) Inconsistent if if $\rho(A) \neq \rho([A|B])$

**Problem 1:** Find if the following system is consistent or not. If the system is consistent then solve it.

$$x + y + 2z = 4, 2x - y + 3z = 9, 3x - y - z = 2$$

Solution:

The given system can be written as a matrix form $AX = B$.

Where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $and\ B = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$.

So here the augmented matrix is

$$[A|B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$$

$$[R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

$$[R_3 \to 3R_3 - 4R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

Here $\rho(A) = 3 \; and \; \rho([A|B]) = 3$

Since $\rho(A) = 3 = \rho([A|B])$, So the given equation is consistent with a unique solution.

Now the given system is equivalent to the system

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$$\Rightarrow x + y - 2z = 4, -3y - z = 1 \; and - 17z = -34$$

$$\Rightarrow z = 2, y = -1 \; and \; x = 1$$

So solution of the given system is $x = 1, y = -1 \; and \; z = 2$.

**Problem 2:** Prove that the following system is consistent and solve it.

$$3x + 3y + 2z = 1, x + 2y = 4, 10y + 3z = -2 \; and \; 2x - 3y - z = 5$$

Solution:

The given system can be written as $AX = B$, Where

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}.$$

Here the Augmented matrix is

$$\begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$R_2 \to 3R_2 - R_1, R_4 \to 3R_4 - 2R_1$$

$$\sim \begin{bmatrix} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 10 & 3 & -2 \\ 0 & -15 & -7 & 13 \end{bmatrix}$$

$$R_3 \to 3R_3 - 10R_2, R_4 \to R_4 + 5R_2$$

$$\sim \begin{bmatrix} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & -17 & 68 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{29}, R_4 \to \frac{R_4}{17}$$

$$\sim \begin{bmatrix} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

$$R_4 \to R_3 + R_4$$

$$\begin{bmatrix} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $\rho(A) = \rho([A|b]) = 3$ and number of unknowns are 3.

So, the given equation is consistent with a unique solution.

Now the given system can be rewritten as,

$$\begin{bmatrix} 3 & 3 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \\ -4 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x + 3y + 2z = 1, 3y - 2z = 11 \ and \ z = -4$$

$$\Rightarrow z = -4 \, , y = 1 \ and \ x = 2$$

Hence the solution of the system $x = 2, y = 1 \ and \ z = -4$.

**Assignment:**

i) Show that the system $x - 4y + 7z = 14, 3x + 8y - 2z = 13 \ and \ 7x - 8y + 26z = 5$ is not consistent.

ii) Solve the system $x + y + z = 9, 2x + 5y + 7z = 52 \ and \ 2x + y - z = 0$. $[Ans: x = 1, y = 3, z = 5]$

iii) Solve the system $x + y + z = 6, x - y + 2z = 5$ $and$ $2x - 2y + 3z = 7$. $[Ans: x = 1, y = 2, z = 3]$

iv) Show that the given system $x + 2y + z = 3, 2x + 3y + 2z = 5$ $and$ $2x - 2y + 3z = 7, 3x + 9y - z = 4$ are consistent and solve it. $[Ans: x = -1, y = 1, z = 2]$

v) Show that the system $x + y + z = 4, 2x + 5y - 2z = 3$ $and$ $x + 7y - 7z = 5$ is not consistent.

**Problem 3:** Show that the system $x + y + z = 6, x + 2y + 3z = 14$ $and$ $x + 4y + 7z = 30$ is consistent and solve it.

Solution:

The given system can be written as $AX = B$, Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $and$ $B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$.

Now Here, $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form.

$$\rho(A) = 2, \rho([A|B]) = 2$$

So, the system is consistent. Now rank of A is less than the number of unknowns, therefore the system will have infinite solution.

Now given system can be written as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + y + z = 6, y + 2z = 8$$

$$\Rightarrow y = 8 - 2z, x = 6 - y - z = 6 - 8 + z = z - 2$$

So, the solution are given by $z = k, y = 8 - 2k \ and \ x = k - 2$ , Where k is any constant.

**Problem 4:** For what values of $\lambda \ and \ \mu$ the system $x + y + z = 6, x + 2y + 3z = 10 \ and \ x + 2y + \lambda z = \mu$ have i) no solution ii) a unique solution iii) an infinite solution.

Solution:

Here augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case I: Let $\lambda = 3$ *and* $\mu \neq 10$, Then $\rho(A) \neq \rho([A|B])$.

Then the system is inconsistent, and the system has no solution.

Case-II: Let $\lambda \neq 3$ *and* $\mu \neq 10$ , Then $\rho(A) = \rho([A|B]) = 3$ also number of unknowns=3.

So, the system is consistent, and the system has unique solution.

Case III: Let $\lambda = 3$ *and* $\mu = 10$ ,Then $\rho(A) = \rho([A|B]) = 2$ also number of unknowns=3>2.

So, the system is consistent, and the system has an infinite number of solutions.

**Assignment**

**i)** For what values of $a$ *and* $b$ the system $x + y + z = 3, x + 2y + 2z = 6$ *and* $x + ay + 3z = b$ has i) no solution ii) a unique solution iii) an infinite solution. (Try It)

**ii)** For what values of $\lambda$ *and* $\mu$ the system $2x + 3y + 5z = 9, 7x + 3y - 2z = 8$ *and* $2x + 3y + \lambda z = \mu$

has i) no solution ii) a unique solution iii) an infinite solution.

**iii)** Test the consistency and solve the system $2x + 3y + 7z = 5, 3x + y - 3z = 12 \; and \; 2x + 19y - 47z = 32$.

**Homogeneous Linear Equations:**

Let us consider a system of m homogeneous equations in n unknowns $x_1, x_2, \dots, x_n$ as below:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

This system can be written as $AX = 0$, where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \; and \; 0 = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Now $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is always a solution of the homogeneous system. This is called trivial solution of the system $AX = 0$. It is also called zero solution.

**Working rule for finding the solution of the equation $\boldsymbol{AX = 0}$:**

1) If $r = n$ i.e. rank of A =number of variables, then the system has only trivial solution.
2) If $r < n$ then the system has an infinite number of nontrivial solutions, we will get $(n - r)$ linearly independent solutions.

**Problem 1:** Solve the system $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$.

Solution:

The given system can be written as $AX = 0$, Where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1 \; and \; R_3 \rightarrow R_3 - 7R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

So here rank of A = Number of variables =3.

Hence the system of equation has a trivial solution.

So $x = 0, y = 0 \; and \; z = 0$ is the only solution.

**Problem 2:** Solve the system of equation $x + 3y - 2z = 0\,, 2x - y - 4z = 0 \; and \; x - 11y + 14z = 0$.

Solution:

The given system can be written as $AX = 0$, Where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 and\; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the echelon matrix form. Number of nonzero rows are two.

So, rank of the matrix is 2. But number of variables is three. So, this will have $3 - 2 = 1$ non-zero solution.

Now the given system can be written as,

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 3y - 2z = 0, -7y + 8z = 0$$

Let $z = k$ then $y = \frac{8}{7}k \; and \; x = 2k - \frac{3.8}{7}k = -\frac{10}{7}k$

Hence the solutions are given by $x = -\frac{10}{7}k\,, y = \frac{8}{7}k \text{ and } z = k$.

**Problem 3:** Solve the system of equation $x + 3y + 2z = 0\,, 2x - y + 3z = 0{,}3x - 5y + 4z = 0 \text{ and } x + 17y + 4z = 0$. (Try it)

**Problem 4:** Solve the system $x + y - 2z + 3w = 0, x - 2y + z - w = 0\,, 4x + y - 5z + 8w = 0, and\ 5x - 7y + 2z - w = 0$.

Solution:

The given system can be written as $AX = 0$ , where

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - 4R_1 \text{ and } R_4 \to R_4 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \text{ and } R_4 \to R_4 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form. Here rank of A is two which is less than the number of variables (Four).

So, the given system has infinite number of nontrivial solutions. The number of independent solutions =4-2=2.

Now the given system can be written as,

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + y - 2z + 3w = 0 \ and - 3y + 3z - 4w = 0$$

Let $w = k_1 and\ z = k_2$

So $y = \frac{1}{3}(3z - 4w) = \frac{1}{3}(3k_2 - 4k_1)$

And $x = 2z - y - 3w = 2k_2 - \frac{1}{3}(3k_2 - 4k_1) - 3k_1 = \frac{1}{3}(3k_2 - 5k_1)$.

Hence the required solution is,

$x = \frac{1}{3}(3k_2 - 5k_1), y = \frac{1}{3}(3k_2 - 4k_1), z = k_2 and\ w = k_1$.

**Problem 5:** Show that the only real number $\lambda$ for which the system $x + 2y + 3z = \lambda x, 3x + y + 2z = \lambda y, 2x + 3y + z = \lambda z$ has non-zero solution is 6 and solve it when $\lambda = 6$.

Solution:

The given system can be written as $AX = 0$ where,

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

Here number of variables, $n = 3$. The given system will have a non-trivial solution if

Rank of A, less than number of unknowns. That is rank is less than three. For this we must have determinant of A is zero.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-2\lambda+\lambda^2-6) - 2(3-3\lambda-4)$$
$$+ 3(9-2+2\lambda) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-2\lambda-5) - 2(-3\lambda-1) + 3(2\lambda+7) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-2\lambda-5) + (12\lambda+23) = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 15\lambda + 18 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0$$

$$\Rightarrow (\lambda-6)(\lambda^2+3\lambda+3) = 0$$

So $\lambda = 6$ is the only real value and other values are complex for which this equation satisfies. (Proved)

Now when $\lambda = 6$ them,

$$A = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}$$

$$R_2 \to 5R_2 + 3R_1 \ and \ R_3 \to 5R_3 + 2R_1$$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}$$

Here rank of A is 2 which is less than the number of variables (Three). So, the system has infinite number of solutions.

So, the given system can be written as,

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x + 2y + 3z = 0 \; and - 19y + 19z = 0$$

$$\Rightarrow y = z \; and \; x = \frac{1}{5}.5z = z$$

Let $z = k$ then $x = y = z = k$

So, required solutions are given by $x = y = z = k$(k is arbitrary constant)

**Assignment:**

1) Solve the system $2x - y + 3z = 0, 3x + 2y + z = 0 \; and \; x - 4y + 5z = 0$. $[Ans: x = -k, y = z = k]$.
2) Solve the system $x + y + w = 0, y + z = 0, x + y + z + w = 0, x + y + 2z = 0$. $[Ans: x = y = z = w = 0]$.
3) Determine whether the following system will have a non-trivial solution if so, solve it.

$4x + 2y + z + 3w = 0, 6x + 3y + 4z + 7w = 0, 2x + y + w = 0$. $[x = k_1, y = -2k_1 - k_2, z = -k_2 \, and \, w = k_2]$

**LU Decomposition Method:**

Let given system of equation is,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system can be written as $AX = B$ where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Now let us consider $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \, and \, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$.

Again let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and then $LY = B$. Now solve for $y_1, y_2 \, and \, y_3$.

After that solve $x_1, x_2 \, and \, x_3$ from the equation $UX = Y$.

**Problem:** Solve the system $x + y + z = 1, 3x + y - 3z = 5 \ and \ x - 2y - 5z = 10$ by LU decomposition method.

Solution:

The given system can be written as, $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ and \ B = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

Let us consider $A = LU$ then

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

$$\Rightarrow u_{11} = 1, u_{12} = 1, u_{13} = 1, l_{21} = 3, u_{22} = -2, u_{23}$$
$$= -3 - 3 = -6, l_{31} = 1, l_{32} = \frac{1}{-2}(-2 - 1)$$
$$= \frac{3}{2}, u_{33} = -5 - 1 - \left(\frac{3}{2}\right).(-6) = -6 + 9 = 3$$

So $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{bmatrix} \ and \ U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

Now from the equation $LY = B$ we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

$$\Rightarrow y_1 = 1, 3y_1 + y_2 = 5 \; and \; y_1 + \frac{3}{2} y_2 + y_3 = 10$$

$$\Rightarrow y_1 = 1 \, , y_2 = 2 \; y_3 = 10 - 1 - 3 = 6$$

So $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

Now from the equation $UX = Y$ we get,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

$$\Rightarrow x + y + z = 1, -2y - 6z = 2 \, , 3z = 6$$

$$\Rightarrow z = 2, y = -7 \; and \; x = 1 - 2 + 7 = 6$$

Hence the required solution is $x = 6, y = -7 \; and \; z = 2$ .

**Assignment:**

**1)** Solve the system by factorization method;
$2x + 3y + z = 9, x + 2y + 3z = 6 \; and \; 3x + y + 2z = 8 \; \left[Ans: x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}\right]$

**2)** Solve the system by triangularization method;
$2x - 3y + 10z = 3, -x + 4y + 2z = 20 \; and \; 5x + 2y + z = -12 \; [Ans: x = -4, y = 3, z = 2]$

**3)** Solve the system by LU factorization method;

$-3x + 12y - 6z = -33, x - 2y + 2z = 7$ $and$ $y + z = -1$ $[Ans: x = 1, y = -2, z = 1]$

**4)** Solve the system by LU decomposition method;

$x + 2y = 7, x - 3y - z = 4$ $and$ $4y + 3z = 5$

$\left[Ans: x = \frac{69}{11}, y = \frac{4}{11}, z = \frac{13}{11}\right]$