1.1 Per Unit System

Most power systems calculations are done with the values of voltage, current, impedance, and power normalized to a common power and voltage base. Using this technique reduces the complexity of the circuit calculations when transformers are involved. The voltage bases selected for the normalized calculation are usually the same as the rated voltage of the transformers in the system. Selecting these values effectively removes the need to multiply and divide the ohmic values of circuit impedances by the turns ratio of the transformers when computing the voltage and current in the system.Computing per-unit values (pu) requires selection of power and voltage bases. The base quantities of current and impedance are then computed from the power and voltage base. **The per unit value of any quantity is the ratio of the actual value of that quantity to the base value for that quantity.** To convert a pu value to a percentage, multiply the per unit value by 100. The nameplate of most power systems equipment gives the impedance of the equipment in either pu or percent. The base for this value is the rated power and voltage of the device.A single-phase model represents balanced three-phase systems in most power systems calculations. The load flow and short circuit models reduce the three-phase model to a single-phase representation of the system when balanced operation is assumed. Both of these problems traditionally represent the system data in pu to simplify the storage of the data and computation of the solution.

Per Unit Formulae for Single Phase Systems

For single-phase circuits, or three phase circuits analyzed on a per phase basis, the following formulae give the base quantities.

Define the power units used in the equations.

$$VA \equiv V \cdot A \qquad KVA \equiv 1000\ VA$$

To illustrate the per unit system calculations, let the single-phase power and voltage base be

$$S_{base} := 100000\ KVA$$

$$V_{base} := 69\ KV$$

The derived base quantities for single-phase or per-phase calculations are

$$I_{base} := \frac{S_{base}}{V_{base}}$$ Use for Sbase in KVA and Vbase in kV.

$$Z_{base} := \frac{V_{base}}{I}$$ Use for Ibase in KVA and Vbase in kV.

Base

$$Z_{base} := \frac{V_{base}^{\;2} \cdot 1000}{S_{base}}$$ Use for Sbase in KVA and Vbase in kV. **(1.1.3)**

$$Z_{base} := \frac{V_{base}^{\;2}}{S_{base}}$$ Use for Sbase in MVA and Vbase in kV. **(1.1.4)**

$$P_{base} := S_{base}$$

The following equations compute the pu values of voltage, current, impedance, and power from the corresponding actual values.

$$Z_{pu}\left(Z_{act}\right) := \frac{Z_{act}}{Z_{base}}$$ **(1.1.5)**

$$I_{pu}\left(I_{act}\right) := \frac{I_{act}}{I_{base}}$$ **(1.1.6)**

$$V_{pu}\left(V_{act}\right) := \frac{V_{act}}{V_{base}}$$ **(1.1.7)**

$$P_{pu}\left(P_{act}\right) := \frac{P_{act}}{S_{base}}$$ **(1.1.8)**

Per Unit Quantities for Three Phase Systems

Equations 1.1.1-1.1.8 establish the normalized circuit values for single-phase circuits. Using these equations on three-phase circuits requires that the current refer to line current, the voltages to line-to-neutral voltage, and the power to per phase power. Data in three-phase power systems are customarily given as total power and line-to-line voltage. Specifying the line-to-line voltage and the total power results in the same per unit quantities as the single phase relationships from above. Computations done in pu are valid for both single-phase and three-phase equivalents. Converting to actual circuit values requires the multiplication of the pu value by the proper base values to obtain either the three-phase or single-phase quantities. Redefining the equations above in terms of the three-phase power and the line-to-line voltage results in the following per unit values.

$$I_{base} := \frac{S_{base}}{\sqrt{3} \cdot V_{base}}$$ Sbase is total KVA and Vbase is the line-to-line voltage. **(1.1.9)**

$$Z_{base} := \frac{\left(\frac{V_{base}}{\sqrt{3}}\right)^{2} \cdot 1000}{S}$$ Sbase is total KVA and Vbase is line-to-line voltage. **(1.1.10)**

$$Z_{base} := \frac{V_{base}^{2} \cdot 1000}{S_{base}}$$ Sbase is total KVA and Vbase is the line-to-line voltage. **(1.1.10a)**

$$Z_{base} := \frac{V_{base}^{2}}{S_{base}}$$ Use for Sbase in total MVA and Vbase in kV line-to-line. **(1.1.11)**

The relationships for converting the actual circuit values to pu are the same as the single-phase relationships.

Changing Bases in the Per Unit System

For per unit calculations to be correct, all circuit variables must be converted using the same power and voltage bases. The rated line-to-line transformer voltage in a section of a system usually is the base voltage for that section of a system. The power base is usually selected to be a 100 MVA for most system studies on high voltage systems. Equipment connected to the system has a per unit or percent impedance value on the nameplate. This nameplate impedance is computed using the device power rating as the base power. The voltage base is the rated operating voltage of the equipment. It is often necessary to change the base of impedance values of equipment to match the base values used in analysis. Equation 1.1.12 gives the relationship for converting the equipment impedance in ohms to a pu impedance.

Suppose

$S_{base} := 100000\ \mathbf{KVA}$

$V_{base} := 69\ \mathbf{KV}$

$Z_{act} := 47.6\ \mathbf{\Omega}$

$$Z_{pu} := \frac{Z_{act} \cdot S_{base}}{V_{base}^{2}}$$ **(1.1.12)**

If the equipment impedance is given in per unit, Equation 1.1.13 allows the direct conversion of the per unit value to different voltage and power bases. The given values of voltage and power refer to the power and voltage ratings of the equipment. The new value of voltage and power refer to the base values used in the power system calculation.

For

$Z_{given} := 0.01$ given pu impedance of a piece of equipment

$V_{given} := 69\ \mathbf{KV}$ the given and new values of base voltages

$V_{new} := 34.5\ \mathbf{KV}$

$S_{given} := 10000\ \mathbf{KVA}$ the given and new values of base power

$S_{new} := 100000\ \mathbf{KVA}$

The new pu value of impedance is

PER UNIT SYSTEM

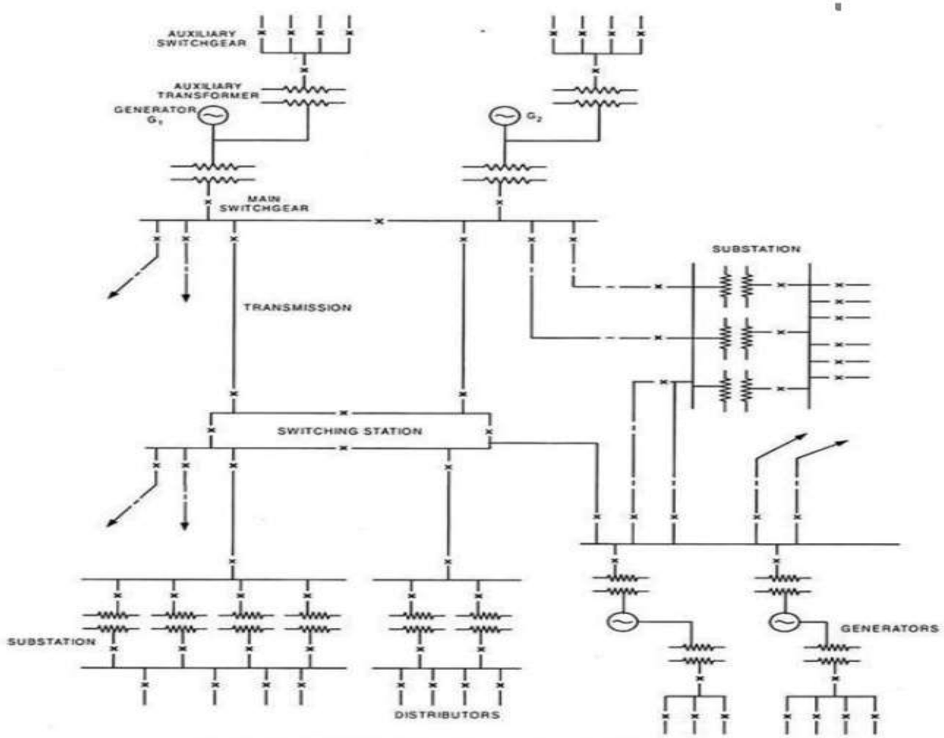


Fig. 1.7. *Simplified Single Line Diagram For a Typical Power System*

❖ Per Unit System

- In the power systems analysis, a **per-unit system** is the expression of system quantities as fractions of a defined base unit quantity.
- In a large interconnected power system with various voltage levels and various capacity equipments, it has been found quite convenient to work with per unit (p.u) system of quantities for analysis purpose rather than in absolute values of quantities.

The per unit quantity is defined as:

$$\text{Per unit quantity} = \frac{\text{Actual value of the quantity}}{\text{Base value of that quantity}}$$

❖ Per Unit System

To completely define a per unit system, minimum four base quantities are required.

Let us define:

$$Voltage\,(V)_{p.u} = \frac{Actual\ Voltage}{Base\ Voltage\,(V_B)}$$

$$Current\,(I)_{p.u} = \frac{Actual\ Curent}{Base\ Current\,(I_B)}$$

$$impedance\,(Z)_{p.u} = \frac{Actual\ impedance}{Base\ impedance\,(Z_B)}$$

$$Apparent\ Power\,(S)_{p.u} = \frac{Actual\ Apparent\ Power}{Base\ Apparent\ Power\,(S_B)}$$

❖ Base Quantities

- The selection of base quantities are also very important. Some of base quantities are chosen independently and arbitrarily while others automatically follow depending upon the fundamental relationships between system variables.

❖ How To Select Base Quantities

- **The rating of the equipment in a power system are given in terms of operating voltage and the capacity in *kVA*. Hence, universal practice is to use machine rating power (*kVA*) and voltage as base quantities** and the base values of current and impedance are calculated from both of them.

❖ Base Quantities

- In electrical engineering, the three basic quantities are voltage, current and impedance. If we choose any two of them as the base or reference quantity, the third one automatically will have a base or reference value depending upon the other two.
- E.g. if V and I are the base voltage and current in a system, the base impedance of the system is fixed and is given by:

$$\boxed{Base\ impedance(Z_B) = \frac{Base\ Voltage\ (V_B)}{Base\ Current\ (I_B)}}$$

$$\boxed{I_{base} = \frac{S_{base}}{V_{base}}}$$

❖ Base Quantities

$$Z_{base} = \frac{V_{base}}{I_{base}} = \frac{V_{base}}{S_{base}/V_{base}} = \frac{V^2{}_{base}}{S_{base}}$$

$$Z_{p.u} = \frac{Z_A}{Z_B} = \frac{Z_A}{V_B / I_B} = \frac{Z_A \times I_B}{V_B} = \frac{Z_A \times S_B}{V_B \times V_B} = \frac{Z_A \times S_B}{V^2{}_B}$$

- This means that the per unit impedance is directly proportional to the base kVA and inversely proportional to the square of base voltage.

❖ Base Quantities

- When all the quantities are converted in per unit values, the different voltage levels disappear and power network involving synchronous generators, transformers and line reduces to a system of simple impedances.
- When the problems to be solved are more complex, and particularly when transformers are involved, the advantages of calculations in per unit are more apparent.
- A well chosen per unit system can reduce the computational effort, simplify evaluation and facilitate the understanding of system characteristics.

❖ Importance of Per Unit System

- For an engineer, it is quite easy to remember the per unit values for all quantities rather than to remember actual values of all quantities.
- Look at the Table and realize how per unit system is easy to remember than actual value system.

Actual Voltage	V at 0.9 p.u	V at 0.95 p.u	V at 1.0 p.u	V at 1.05 p.u	V at 1.1 p.u
220 V	198V	209V	220V	231V	242V
440 V	396V	418V	440V	462V	484V
11kv	9.9kV	10.45kV	11kV	11.55kV	12.1 kV
33kv	29.7kV	31.35 kV	33kV	34.65kV	36.3kV
66kv	59.4kV	62.7kV	66kV	69.3kV	72.6kV
132kv	118.8kV	125.4kV	132kV	138.6kV	145.2kV
220kv	198kV	209kV	220kV	231kV	242kV
500kv	450kV	475kV	500kV	525kV	550kV

❖ Importance of Per Unit System

- It can be observed that only for voltage at different levels, it is quite difficult to remember all these limits. However, on the other hand, per unit is easy to remember.
- Furthermore, it is quite difficult to find the error in the actual values as compared to per unit system. For example, if voltage goes below 0.9 p.u. limit, it can be easily understood that voltage has gone below its safe limit; but in actual voltage values, it is difficult to know whether voltage has crossed the safe limit or not.
- The per unit representation of the impedance of an equipment is more meaningful than its absolute value.

❖ Effect of 1-ϕ and 3-ϕ on Per Unit System

- The per unit system has the advantage that base impedance expression remains same for single phase as well as three phase system. E.g. in single phase, we have the formula for Z_{base} as:

$$Z_{base} = \frac{V_{base}}{I_{base}} = \frac{V_{base}^2}{S_{base}}$$

- Now, for three phase, voltage and current are given by:

$$V_B = \frac{V_B}{\sqrt{3}} \qquad I_B = \frac{S_B}{\sqrt{3}V_B}$$

Now, the formula for Z_{base} will become:

$$Z_B = \frac{V_B}{I_B} = \frac{\frac{V_B}{\sqrt{3}}}{\frac{S_B}{V_B \times \sqrt{3}}} = \frac{V_B}{\sqrt{3}} \times \frac{V_B \times \sqrt{3}}{S_B} = \frac{V^2{}_B}{S_B} = \frac{V^2{}_{Base}}{S_{Base}}$$

- Hence, it can be seen that per unit system has no effect of single phase and three phase system for base impedance expression.

❖ Advantages of Per unit System

- The per-unit system was originally developed to simplify laborious hand calculations and while it is now not always necessary (due to the widespread use of computers), the per-unit system does still offer some distinct advantages over standard SI values:

- The per unit values of impedance, voltage, and current of a transformer are the same regardless of whether they are referred to the primary or the secondary side. This is a great advantage since the different voltage levels disappear and the entire system reduces to a system of simple impedance. This can be a pronounced advantage in power system analysis where large numbers of transformers may be encountered.

❖ Advantages of Per unit System

- Per-unit impedance values of equipment are normally found over a small range of values irrespective of the absolute size. On the other hand, ohmic values may have significant variation and are often proportional to nominal rating.

- Similar apparatus (generators, transformers, lines) will have similar per-unit impedances and losses expressed on their own rating, regardless of their absolute size. Because of this, per-unit data can be checked rapidly for gross errors. A per unit value out of normal range is worth looking into for potential errors.

❖ Advantages of Per unit System

- Manufacturers usually specify the impedance of apparatus in per unit values.
- The per unit value of the resistance of a machine furnishes almost at a glance its electrical losses in the percent of its rated power. For example, a transformer operating under rated conditions at **unity power factor** with a winding resistance of 0.01 per unit has a copper loss of 1%.
- $I^2R = (1.0)^2 \times 0.01 = 0.01\,p.u = 0.01 \times 100 = 1\%$
- This information is very useful to a power system engineer because he can estimate and locate the quantity of the various copper losses simply by looking at the one line per unit impedance diagram.

❖ Advantages of Per unit System

- The per unit system simplifies the analysis of problems that include star delta types of winding connections. The factor of $\sqrt{3}$ is not used for the per unit analysis. For example, consider the expression for the power:

$$\boxed{P = VI\cos\theta}$$

When the voltage and the current are expressed in per unit, this relationship gives the total power in per unit, regardless of a delta or star winding connection.

❖ Changing the Base of Per unit Quantities

- Some times the per unit impedance of a component of a system is expressed on a base other than the one selected as base for the part of the system in which the component is located.
- Since all impedances in any part of a system must be expressed on the same impedance base when making computations, it is necessary to have a means of converting per unit impedances from one base to another.

❖ Changing the Base of Per unit Quantities

❖ We know that $Z_{p.u} = \frac{Z_A \times S_B}{V^2_B}$

$$\frac{Z_{(p.u)new}}{Z_{(p.u)old}} = \frac{Z_{act} \times \frac{S_{(Base)new}}{V^2_{(Base)new}}}{Z_{act} \times \frac{S_{(Base)old}}{V^2_{(Base)old}}} = Z_{act} \times \frac{S_{(Base)new}}{V^2_{(Base)new}} \times \frac{V^2_{(Base)old}}{Z_{act} \times S_{(Base)old}}$$

$$\frac{Z_{(p.u)new}}{Z_{(p.u)old}} = \frac{V^2_{(Base)old}}{V^2_{(Base)new}} \times \frac{S_{(Base)new}}{S_{(Base)old}} = \left[\frac{V_{(Base)old}}{V_{(Base)new}}\right]^2 \times \frac{S_{(Base)new}}{S_{(Base)old}}$$

❖ Changing the Base of Per unit Quantities

$$Z_{(p.u)new} = Z_{(p.u)old} \times \left[\frac{V_{(Base)old}}{V_{(Base)new}}\right]^2 \times \frac{S_{(Base)new}}{S_{(Base)old}}$$

$$\boxed{Z_{(p.u)new} = Z_{(p.u)old} \times \left[\frac{base\,kV_{given}}{base\,kV_{new}}\right]^2 \times \frac{base\,kVA_{new}}{base\,kVA_{given}}}$$

- If the old base voltage and new base voltage are the same, then formula becomes:

$$\boxed{Z_{(p.u)new} = Z_{(p.u)old} \times \frac{base\,kVA_{new}}{base\,kVA_{given}}}$$

❖ Example:

- The reactance of a generator designated X'' is given as 0.25 per unit based on the generator's nameplate rating of 18 kV, 500 MVA. The base for calculations is 20 kV, and 100 MVA. Fins X'' on the new base.

❖ Data:

X''_{given} = 0.25 p.u, Base kV_{given} = 18 kV, Base kV_{New} = 20 kV,

Base kVA_{given} = 500 MVA, Base kVA_{New} = 100 MVA, $X''_{new} = ?$

❖ Solution:

- We know that the formula for finding the new impedance is given as below:

$$\boxed{Z_{(p.u)new} = Z_{(p.u)old} \times \left[\frac{base\,kV_{given}}{base\,kV_{new}}\right]^2 \times \frac{base\,kVA_{new}}{base\,kVA_{given}}}$$

❖ Solution:

❖ The above formula for X'' can be modified as below:

$$X''(p.u)_{new} = X''_{(p.u)old} \times \left[\frac{base\,kV_{given}}{base\,kV_{new}}\right]^2 \times \frac{base\,kVA_{new}}{base\,kVA_{given}}$$

❖ By putting the values in above equation we get:

$$X''(p.u)_{new} = 0.25 \times \left[\frac{18 \times 1000}{20 \times 1000}\right]^2 \times \frac{100 \times 10^6}{500 \times 10^6} = 0.25 \times \left[\frac{18}{20}\right]^2 \times \frac{1}{5}$$

$$X''(p.u)_{new} = 0.25 \times \frac{324}{400} \times \frac{1}{5} = 0.25 \times 0.81 \times 0.2$$

$$\boxed{X''(p.u)_{new} = 0.0405\,p.u}$$

Answer

❖ Example:

❖ A single phase 20 kVA, 480/120V, 60 Hz single phase transformer has a primary and secondary impedance of $Z_{primary} = 0.84 < 78.13$ degrees ohms and $Z_{secondary} = 0.0525 < 78.13$ degrees ohms.

❖ Determine the per unit transformer impedance referred to the LV winding and the HV winding.

❖ Solution:

According to our convention, the base values for this system are:

$$S_{base} = 20 \text{ kVA}, \quad V_{base1} = V_{base\ Primary} = 480\text{V}, \; V_{base2} = V_{base\ Secondary} = 120\text{V}$$

The Formula for per unit impedance is given by:

$$Z_{p.u\_primary} = \frac{Z_{Actual\_primary}}{Z_{base\_primary}} \qquad (1)$$

$$Z_{p.u\_secondary} = \frac{Z_{Actual\_secondary}}{Z_{base\_secondary}} \qquad (2)$$

It can be observed from eq. 1 and 2, that base impedance for primary and secondary are unknown.

❖Solution:

Now, the resulting base impedance for primary and secondary are:

$$Z_{base\_primary} = \frac{V^2{}_{base\_primary}}{S_{base}} = \frac{480^2}{20 \times 1000} = \frac{230400}{20000} = \frac{2304}{200}$$

$$Z_{base\_primary} = 11.52\ \Omega$$

$$Z_{base\_secondary} = \frac{V^2{}_{base\_secondary}}{S_{base}} = \frac{120^2}{20 \times 1000} = \frac{14400}{20000} = \frac{144}{200}$$

$$Z_{base\_secondary} = 0.72\ \Omega$$

❖Solution:

❖Now, the resulting per unit impedance at primary and secondary side of the transformer are:

$$\boxed{Z_{p.u\_primary} = \frac{Z_{Actual\_primary}}{Z_{base\_primary}} = \frac{0.84\angle 78.13^\circ}{11.52} = 0.0729\angle 78.13^\circ \; p.u}$$

$$\boxed{Z_{p.u\_secondary} = \frac{Z_{Actual\_secondary}}{Z_{base\_secondary}} = \frac{0.0525\angle 78.13^\circ}{0.72} = 0.0729\angle 78.13^\circ \; p.u}$$

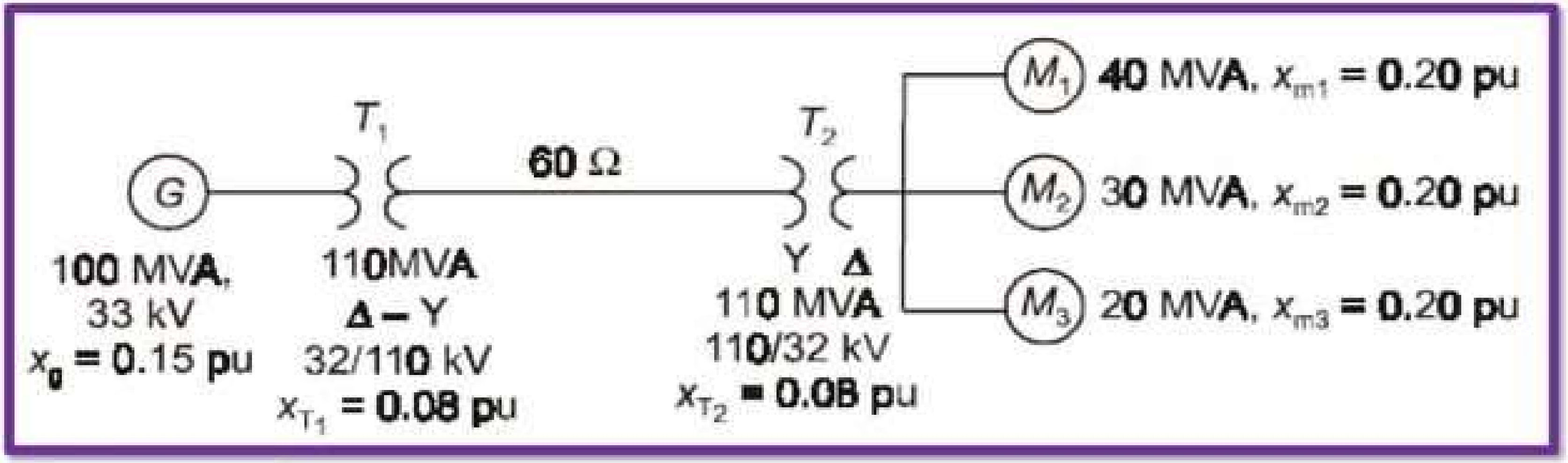
❖Hence, it can be observed that the per unit impedance are equal for both sides of the transformer. However, their actual values are different.

Answer

❖ Example:

This example is taken from the book Electrical Power System by D.Das, chapter 5, Example 5.4

❖ A 100 MVA, 33 kV, three phase generator has a reactance of 15%. The generator is connected to the motors through a transmission line and transformers as shown in Figure. Motors have rated inputs of 40 MVA, 30 MVA, and 20 MVA at 30 kV with 20% reactance each. Draw the per unit circuit diagram. Assume 100 MVA and 33 kV as common base values.



❖ Solution:

❖ We know that the formula for new per unit impedance is given by:

$$Z_{(p.u)new} = Z_{(p.u)old} \times \left[\frac{base\ kV_{given}}{base\ kV_{new}}\right]^2 \times \frac{base\ kVA_{new}}{base\ kVA_{given}}$$

❖Solution:

❖New Per unit Reactance of Generator *G:*

$$X_{G(p.u)_{new}} = j0.15 \times \left[\frac{33\times10^3}{33\times10^3}\right]^2 \times \frac{100\times10^6}{100\times10^6} = j0.15\times1\times1 = j0.15\ p.u$$

❖New Per unit Reactance of Transformer T*₁*:

$$X_{T_1(p.u)_{new}} = j0.08\times\left[\frac{32\times10^3}{33\times10^3}\right]^2 \times \frac{100\times10^6}{110\times10^6} = j0.08\times\left[\frac{32}{33}\right]^2 \times \frac{10}{11}$$

$$X_{T_1(p.u)_{new}} = j0.08\times\left[0.9696\right]_2 \times 0.90909 = j0.08\times0.94012\times0.90909 = j0.0683\ p.u$$

❖New Per unit Reactance of Transmission Line:

❖It can be observed that for transmission line the base voltage is changed. The new base voltage is determined by:

$$New\ Base\ Voltage = 33\ kV \times \frac{110\ kV}{32\ kV} = 33\ kV \times 3.4375 = 113.4375\ kV$$

❖Now, it can be noticed that the reactance of transmission line is given in ohms instead of per unit values. Hence, the new per unit reactance of transmission line is given by:

❖Solution:

$$X_{Line(p.u)} = j60 \times \frac{100 \times 10^6}{(113.4375 \times 10^3)^2} = (j60) \times \frac{100}{12868.066} = \frac{j6000}{12868.066} = j0.466\ p.u$$

❖New Per unit Reactance of Transformer T_2:

$$X_{T_2(p.u)_{new}} = j0.08 \times \left[\frac{110 \times 10^3}{113.4375 \times 10^3}\right]^2 \times \frac{100 \times 10^6}{110 \times 10^6} = j0.08 \times \left[\frac{110}{113.4375}\right]^2 \times \frac{10}{11}$$

$$X_{T_2(p.u)_{new}} = j0.08 \times [0.9696]_2 \times 0.90909 = j0.08 \times 0.94012 \times 0.90909 = j0.0683\ p.u$$

❖New Per unit Reactance of Motor M_1:

❖It can be observed that for motor, the base voltage is changed again. The new base voltage is calculated as below:

$$New\ Base\ Voltage = 113.4375\ kV \times \frac{32\ kV}{110\ kV} = 113.4375\ kV \times 0.2909 = 33\ kV$$

❖The per unit reactance of motor 1 is now calculated as below:

❖ Solution:

$$X_{M_1(p.u)_{new}} = j0.2\times\left[\frac{30\times10^3}{33\times10^3}\right]^2\times\frac{100\times10^6}{40\times10^6} = j0.2\times\left[\frac{30}{33}\right]^2\times\frac{10}{4}$$

$$X_{M_1(p.u)_{new}} = j0.2\times[0.90909]^2\times2.5 = j0.2\times0.8263\times2.5 = j0.413\ p.u$$

❖ New Per unit Reactance of Motor M_2:

$$X_{M_2(p.u)_{new}} = j0.2\times\left[\frac{30\times10^3}{33\times10^3}\right]^2\times\frac{100\times10^6}{30\times10^6} = j0.2\times\left[\frac{30}{33}\right]^2\times\frac{10}{3}$$

$$X_{M_2(p.u)_{new}} = j0.2\times[0.90909]^2\times3.333 = j0.2\times0.8263\times3.333 = j0.551\ p.u$$

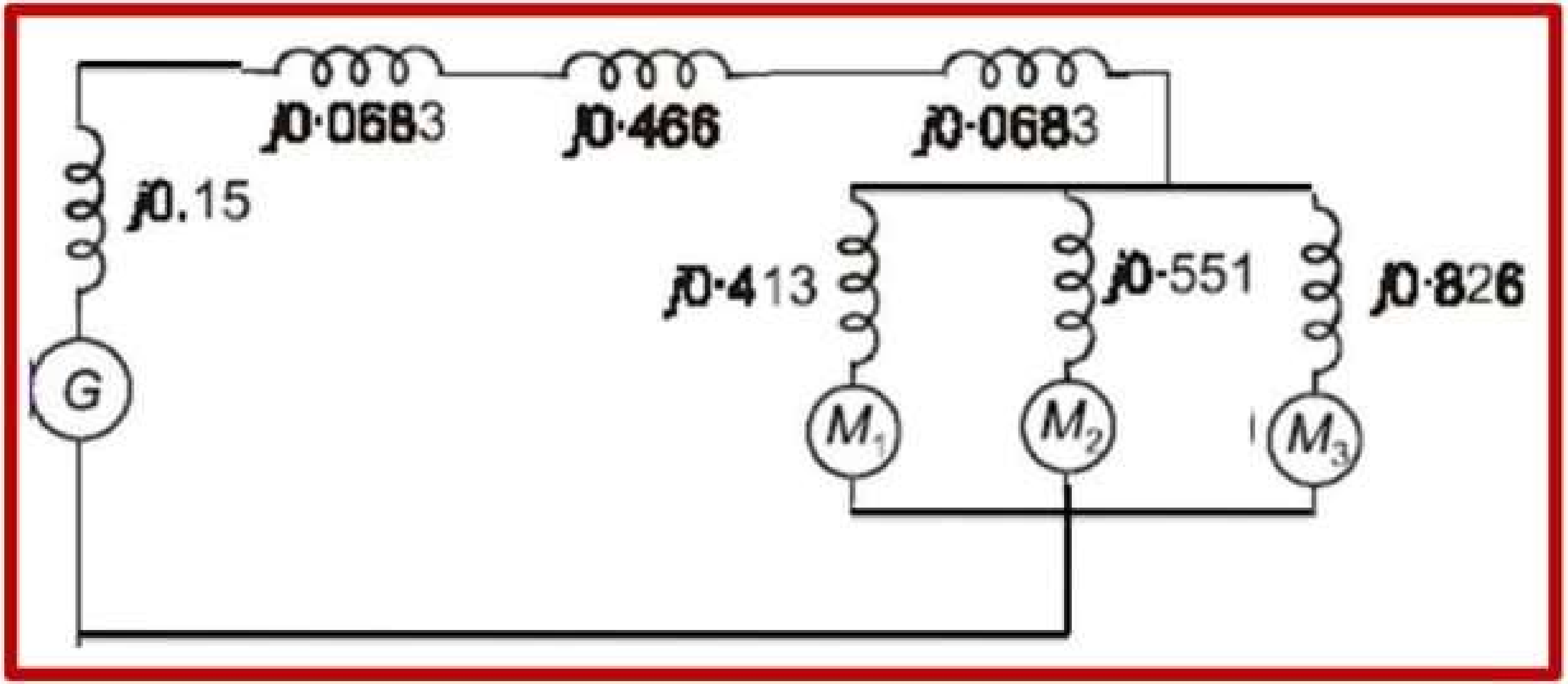
❖ New Per unit Reactance of Motor M_3:

$$X_{M_3(p.u)_{new}} = j0.2\times\left[\frac{30\times10^3}{33\times10^3}\right]^2\times\frac{100\times10^6}{20\times10^6} = j0.2\times\left[\frac{30}{33}\right]^2\times\frac{10}{2}$$

$$X_{M_3(p.u)_{new}} = j0.2\times[0.90909]_2\times5 = j0.2\times0.8263\times5 = j0.826\ p.u$$

❖ Solution:

The per unit circuit is now given as below:



Answer

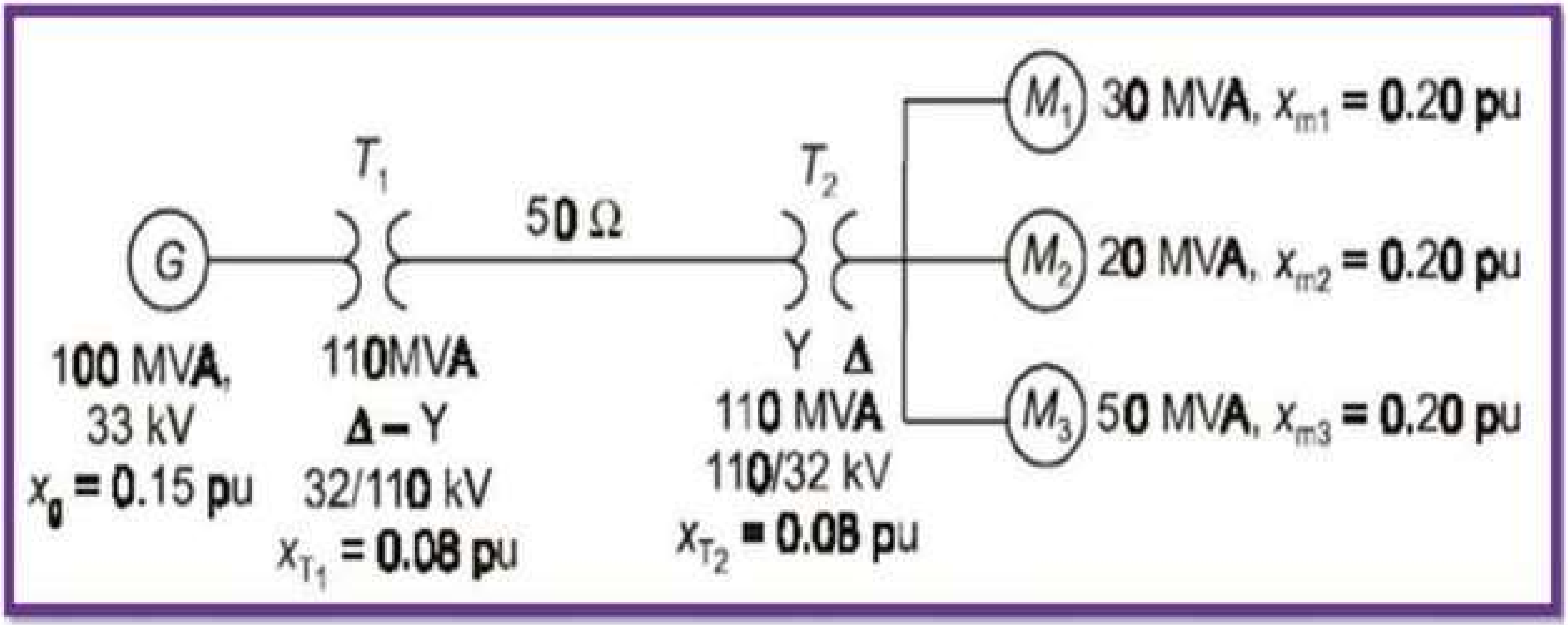
❖Example for Practice:

This example is taken from the book Electrical Power System by C. L. Wadhwa, chapter 1, Example 1.1

❖A 100 MVA, 33 kV, three phase generator has a sub transient reactance of 15%. The generator is connected to the motors through a transmission line and transformers as shown in Figure.

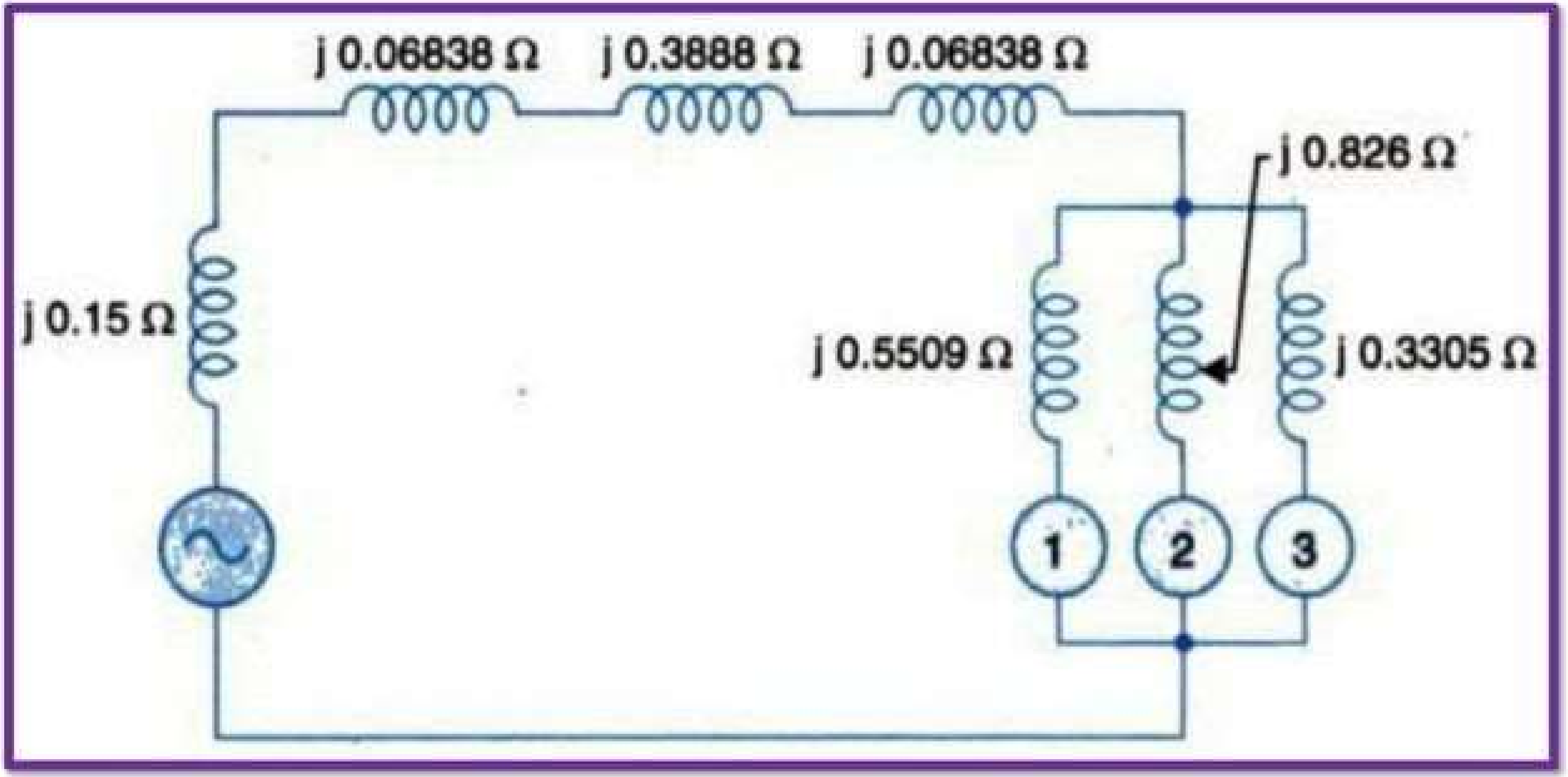
❖Motors have rated inputs of 30 MVA, 20 MVA, and 50 MVA at 30 kV with 20% sub transient reactance each. Selecting the generator rating as the base quantities in the generator circuit.

❖Draw the per unit circuit diagram.



❖ Solution:

Solving by the similar way, the new per unit diagram is as below:



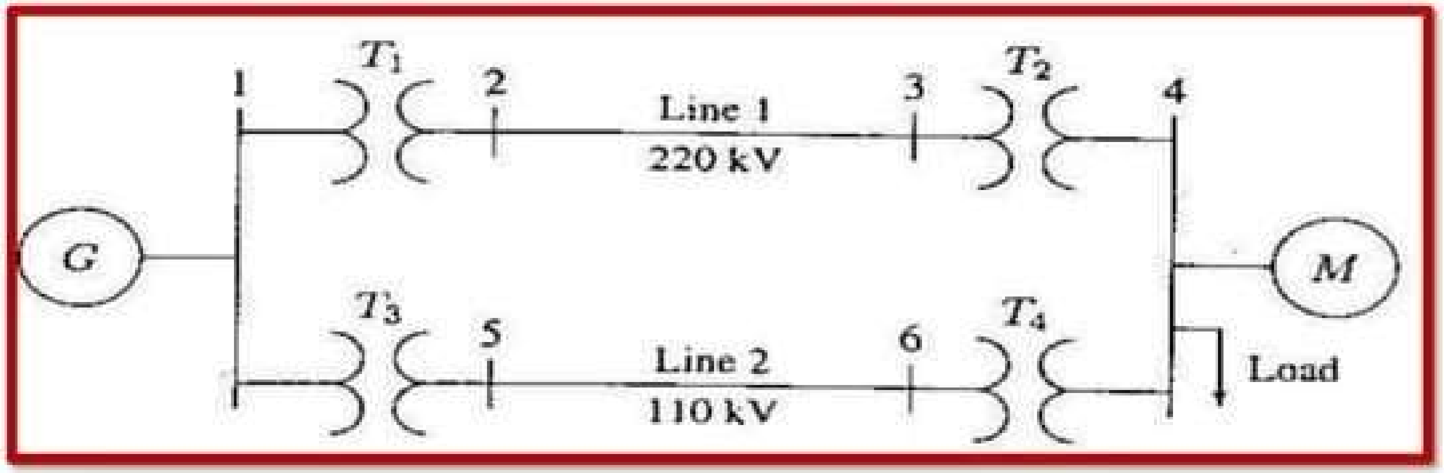
Answer

❖Example:

This example is taken from the book Power System Analysis by Hadi Sadat, chapter 3, Example 3.7.

❖The one line diagram of a three phase power system is shown in Figure. Select a common base of 100 MVA and 22 kV on the generator side. Drawn an impedance diagram with all impedances including the load impedance marked in per unit. The three phase load at bus 4 absorbs 57 MVA, 0.6 power factor lagging at 10.45 kV. Line 1 and 2 have reactance of 48.4 ohms and 65.3 ohms respectively. The manufacturer's data for each device is given as follow:

Name	*S*	*V*	*Xp.u*	*Name*	*S*	*V*	*Xp.u*
G	90MVA	22kV	X=18%	T_1	50MVA	22/220kV	X=10%
T_2	40MVA	220/11kV	X=6.0%	T_3	40MVA	22/110kV	X=6.4%
T_4	40MVA	110/11kV	X=8.0%	M	66.5MVA	10.45kV	X=18.5%



❖Solution:

❖The reactance is given in percent. Its per unit is obtained by dividing it by 100. such as 18% = 18/100 = 0.18p.u

❖We know that the formula for new per unit impedance is given by:

$$\boxed{Z_{(p.u)new} = Z_{(p.u)old} \times \left[\frac{base\,kV_{given}}{base\,kV_{new}}\right]^2 \times \frac{base\,kVA_{new}}{base\,kVA_{given}}}$$

❖New Per unit Reactance of Generator *G:*

$$X_{G(p.u)new} = j0.18 \times \left[\frac{22\times10^3}{22\times10^3}\right]^2 \times \frac{100\times10^6}{90\times10^6} = j0.18\times1\times\frac{10}{9} = \frac{j1.8}{9} = j0.2\ p.u$$

❖New Per unit Reactance of Transformer T_1:

$$X_{T_1(p.u)new} = j0.1 \times \left[\frac{22\times10^3}{22\times10^3}\right]^2 \times \frac{100\times10^6}{50\times10^6} = j0.1\times1\times\frac{10}{5} = \frac{j1.0}{5} = j0.2\ p.u$$

❖New Per unit Reactance of Transmission Line 1:

❖It can be observed that for transmission line the base voltage is changed. Hence, first new base voltage is required to determined, then its per unit reactance can be calculated. The formula for finding new base voltage is given by:

❖ Solution:

$$New\ Base Voltage = Old\ Base Voltage \times \frac{E_2}{E_1}$$

$$New\ Base Voltage = 22\ kV \times \frac{220\ kV}{22\ kV} = 220\ kV$$

❖Now, it can be noticed that the reactance of transmission line is given in ohms instead of per unit values. Hence, the formula to find per unit reactance of transmission line is given by:

$$Z_{p.u} = Z_{ohms} \times \frac{S_B}{V^2{}_B} = Z_{ohms} \times \frac{Base\ kVA}{(Base\ kV)^2}$$

$$X_{Line1(p.u)} = (j48.4) \times \frac{100 \times 10^6}{(220 \times 10^3)^2} = (j48.4) \times \frac{100}{48400} = \frac{j4840}{48400} = j0.1\ p.u$$

❖ New Per unit Reactance of Transformer T_2:

$$X_{T_2(p.u)_{new}} = j0.06 \times \left[\frac{220 \times 10^3}{220 \times 10^3}\right]^2 \times \frac{100 \times 10^6}{40 \times 10^6} = j0.06 \times 1 \times \frac{10}{4} = \frac{j0.6}{4} = j0.15\ p.u$$

❖Solution:

❖New Per unit Reactance of Transformer T_3:

$$X_{T_3(p.u)_{new}} = j0.064 \times \left[\frac{22\times10^3}{22\times10^3}\right]^2 \times \frac{100\times10^6}{40\times10^6} = j0.064\times1\times\frac{10}{4} = \frac{j0.64}{4} = j0.16\ p.u$$

❖New Per unit Reactance of Transmission Line 2:

❖It can be observed that for transmission line 2, the base voltage is changed again. The new base voltage is calculated as below:

$$New\ Base\ Voltage = 22\ kV \times \frac{110\ kV}{22\ kV} = 110\ kV$$

❖Now, it can be noticed that the reactance of transmission line is given in ohms instead of per unit values. Hence, the per unit reactance of transmission line is calculated as below:

$$X_{Line\,2\,(p.u)} = (j65.3)\times\frac{100\times10^6}{(110\times10^3)^2} = (j65.3)\times\frac{100}{12100} = \frac{j6530}{12100} = j0.54\ p.u$$

❖Solution:

❖New Per unit Reactance of Transformer T_4:

$$X_{T_4(p.u)_{new}} = j0.08 \times \left[\frac{110\times10^3}{110\times10^3}\right]^2 \times \frac{100\times10^6}{40\times10^6} = j0.08\times1\times\frac{10}{4} = \frac{j0.8}{4} = j0.2\ p.u$$

❖New Per unit Reactance of Motor M:

❖It can be observed that for motor, the base voltage is changed again from two points. One from transformer T_2 and other from transformer T_4. but, the new base voltage from both must have the same value. The new base voltage as calculated from Transformer T_2 is given as below:

$$New\ Base\ Voltage = 220\ kV \times \frac{11\ kV}{220\ kV} = 11\ kV$$

The new base voltage as calculated from Transformer T_4 is given as below:

$$New\ Base\ Voltage = 110\ kV \times \frac{11\ kV}{110\ kV} = 11\ kV$$

It can be noticed that both has the same voltage. The per unit reactance of motor is now calculated as below:

❖Solution:

$$X_{M(p.u)_{new}} = j0.185 \times \left[\frac{10.45\times10^3}{11\times10^3}\right]^2 \times \frac{100\times10^6}{66.5\times10^6} = j0.185\times(0.95)^2\times1.5037$$

$$X_{M\,(p.u)_{new}} = j0.185\times0.9025\times1.5037 = j0.25\ p.u$$

❖New Per unit Impedance of Load:

The load apparent power at 0.6 power factor lagging is 57 MVA. The angle for 0.6 power factor will be:

$$\cos(\theta) = 0.6; \quad \theta = \cos^{-1}(0.6) = 53.13°$$

Hence, the load is 57<53.13 degree *MVA*. To calculate the per unit impedance of the load, we need to first calculate actual impedance and base impedance of the load. The actual impedance of the load is calculated as below:

$$Z_{L(Actual)} = \frac{(V_{L-L})^2}{S^*_{L(3\phi)}} = \frac{(10.45\times10^3)^2}{57\times10^6\angle-53.13} = \frac{109.2025}{57\angle-53.13} = 1.91583\angle53.13$$

$$Z_{L(Actual)} = 1.91583\times(\cos53.13 + j\sin53.13) = 1.91583\times(0.6 + j0.8)$$

❖Solution:

$$Z_{L(Actual)} = (1.1495 + j1.53267)\ \Omega$$

The base impedance of the load is calculated as below:

$$Z_{L(Base)} = \frac{(V_{Base})^2}{S_{Base}} = \frac{(11\times10^3)^2}{100\times10^6} = \frac{121}{100} = 1.21\ \Omega$$

Now, the per unit impedance is calculated as below:

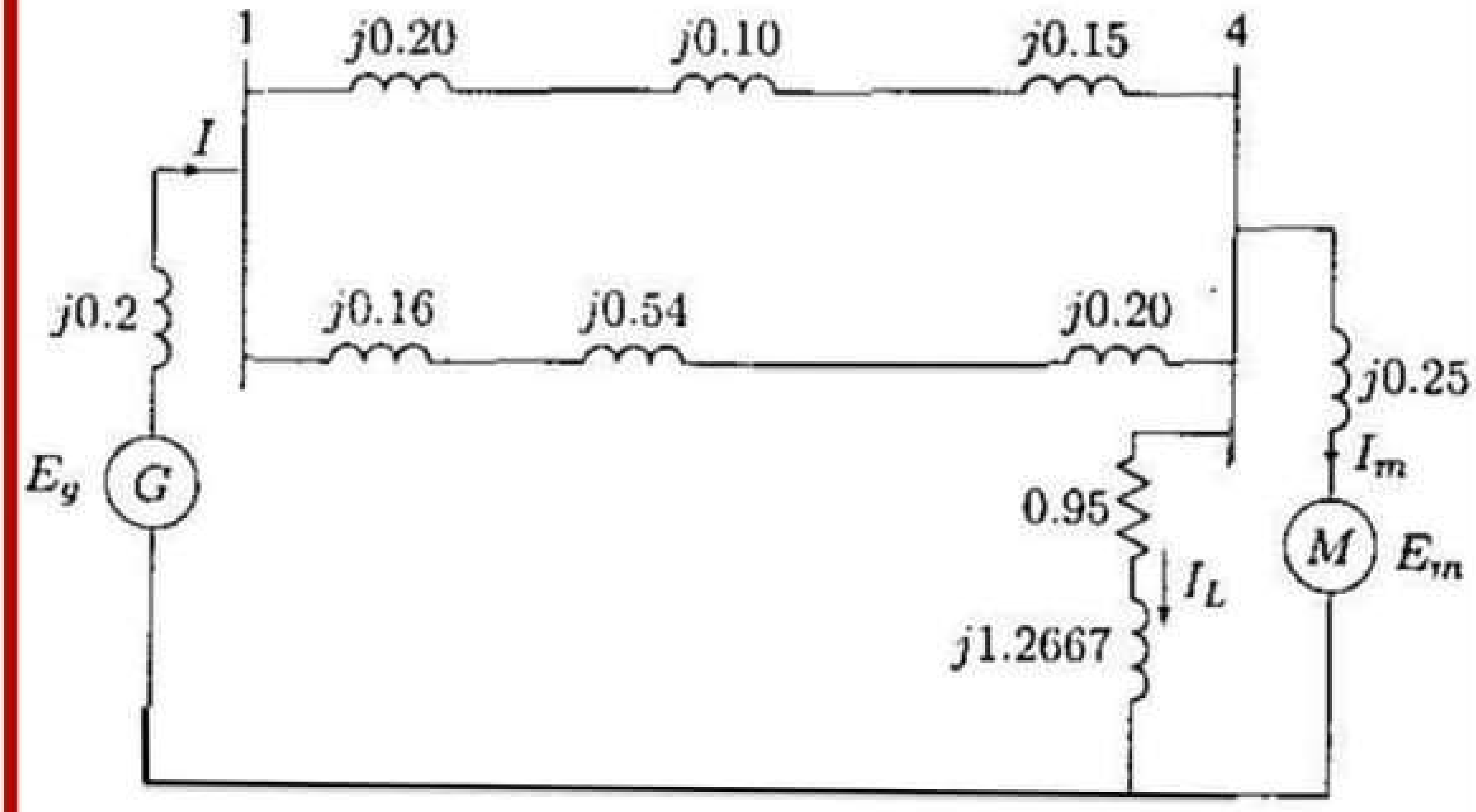
$$Z_{L(p.u)} = \frac{Z_{L(Actual)}}{Z_{L(Base)}}$$

$$Z_{L(p.u)} = \frac{1.1495 + j1.53267}{1.21}$$

$$Z_{L(p.u)} = (0.95 + j1.2667)\ p.u$$

❖Solution:

The per unit circuit is now given as below:



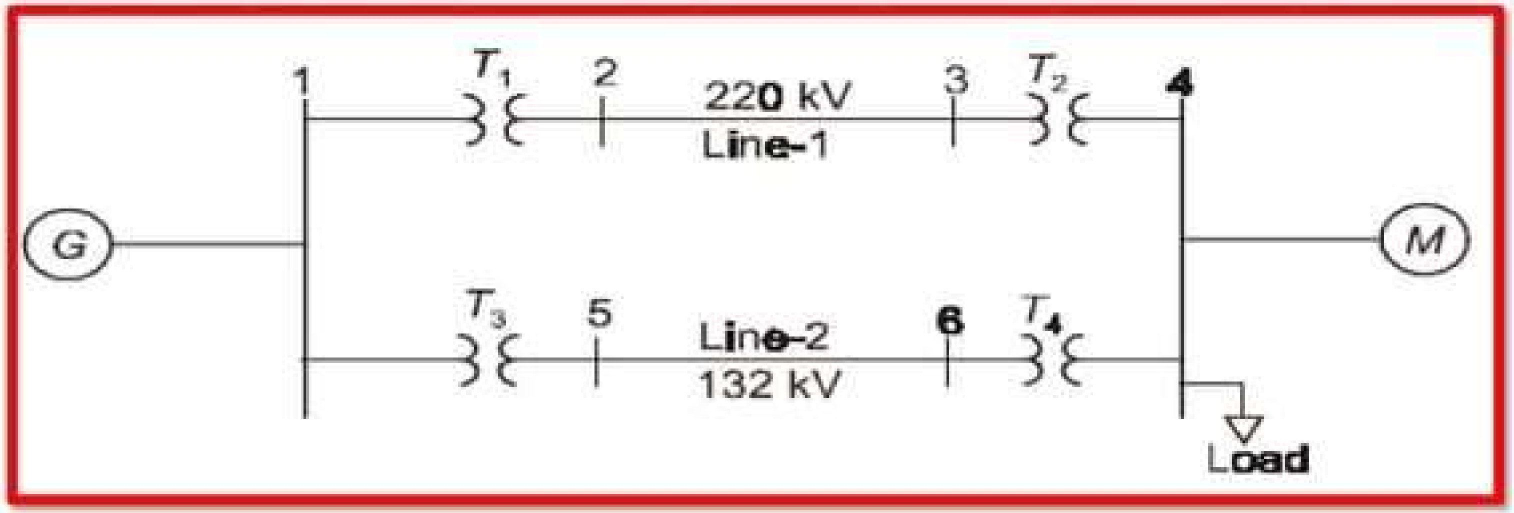
Answer

❖Example for Practice:

This example is taken from the book Electrical Power System D. Das, chapter 5, Example 5.8.

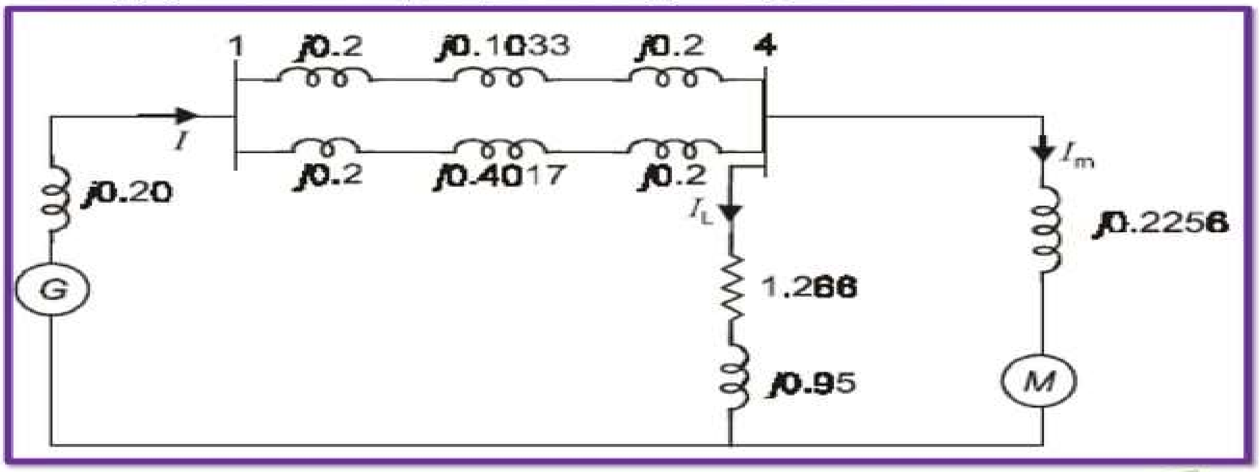
❖The one line diagram of a three phase power system is shown in Figure. Select a common base of 100 MVA and 13.8 kV on the generator side. Drawn an impedance diagram with all impedances including the load impedance marked in per unit. The three phase load at bus 4 absorbs 57 MVA, 0.8 power factor lagging at 10.45 kV. Line 1 and 2 have reactance of 50 ohms and 70 ohms respectively. The manufacturer's data for each device is given as follow:

Name	*S*	*V*	*Xp.u*	*Name*	*S*	*V*	*Xp.u*
G	$90MVA$	$13.8kV$	X=18%	T_1	$50MVA$	$13.8/220kV$	X=10%
T_2	$50MVA$	$220/11kV$	X=10.0%	T_3	$50MVA$	$13.8/132kV$	X=10.0%
T_4	$50MVA$	$132/11kV$	X=10.0%	M	$80MVA$	$10.45kV$	X=20%

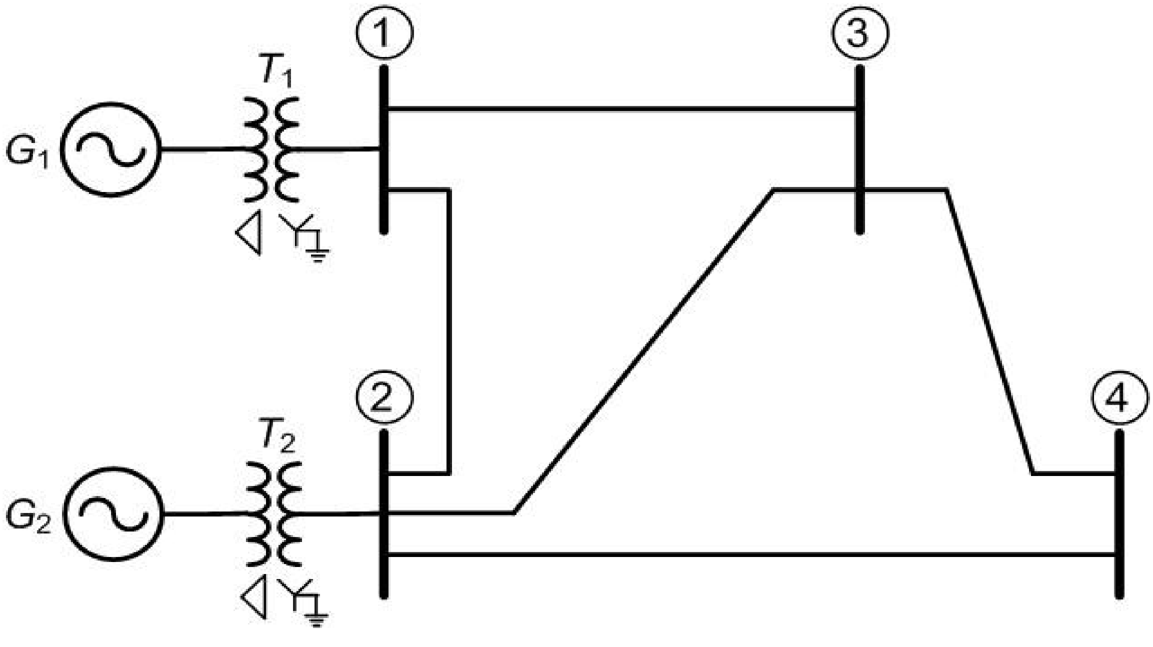


❖ Solution:

❖ Solving by the similar way, the per unit diagram is given below:



Answer



NETWORK ADMITTANCE AND IMPEDANCE MATRICES

As we have seen in Chapter 1 that a power system network can be converted into an equivalent impedance diagram. This diagram forms the basis of power flow (or load flow) studies and short circuit analysis. In this chapter we shall discuss the formation of ***bus admittance matrix*** (also known as Y_{bus} matrix) and ***bus impedance matrix*** (also known as Z_{bus} matrix). These two matrices are related by

$$Z_{bus} = Y_{bus}^{-1} \tag{3.1}$$

We shall discuss the formation of the Y_{bus} matrix first. This will be followed by the discussion of the formation of the Z_{bus} matrix.

3.1 FORMATION OF BUS ADMITTANCE MATRIX

Consider the voltage source V_S with a source (series) impedance of Z_S as shown in Fig. 3.1 (a). Using Norton's theorem this circuit can be replaced by a current source I_S with a parallel admittance of Y_S as shown in Fig. 3.1 (b). The relations between the original system and the Norton equivalent are

$$I_S = \frac{V_S}{Z_S} \text{ and } Y_S = \frac{1}{Z_S} \tag{3.2}$$

We shall use this Norton's theorem for the formulation of the Y_{bus} matrix.

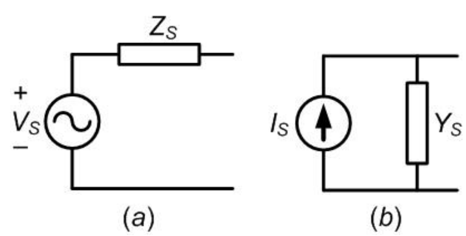


Fig. 3.1 (a) Voltage source with a source impedance and (b) its Norton equivalent.

For the time being we shall assume the short line approximation for the formulation of the bus admittance matrix. We shall thereafter relax this assumption and use the π-representation of the network for power flow studies. Consider the 4-bus power system shown in Fig. 3.2. This contains two generators G_1 and G_2 that are connected through transformers T_1 and T_2 to buses 1 and 2. Let us denote the synchronous reactances of G_1 and G_2 by X_{G1} and X_{G2} respectively and the leakage reactances of T_1 and T_2 by X_{T1} and X_{T2} respectively. Let Z_{ij}, $i = 1, \ldots, 4$ and $j = 1, \ldots, 4$ denote the line impedance between buses i and j.

Then the system impedance diagram is as shown in Fig. 3.3 where $Z_{11} = j(X_{G1} + X_{T1})$ and $Z_{22} = j(X_{G2} + X_{T2})$. In this figure the nodes with the node voltages of V_1 to V_4 indicate the buses 1 to 4 respectively. Bus 0 indicates the reference node that is usually the neutral of the Y-connected system. The impedance diagram is converted into an equivalent admittance diagram shown in Fig. 3.4. In this diagram $Y_{ij} = 1/Z_{ij}$, $i = 1, \ldots, 4$ and $j = 1, \ldots, 4$. The voltage sources E_{G1} and E_{G2} are converted into the equivalent current sources I_1 and I_2 respectively using the Norton's theorem discussed before.

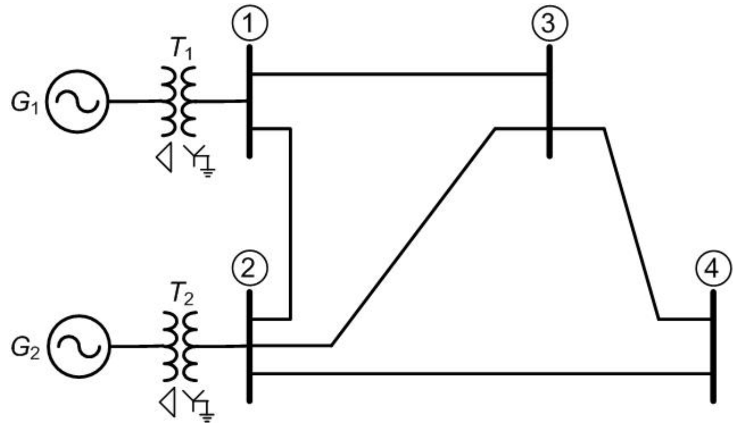


Fig. 3.2 Single-line diagram of a simple power network.

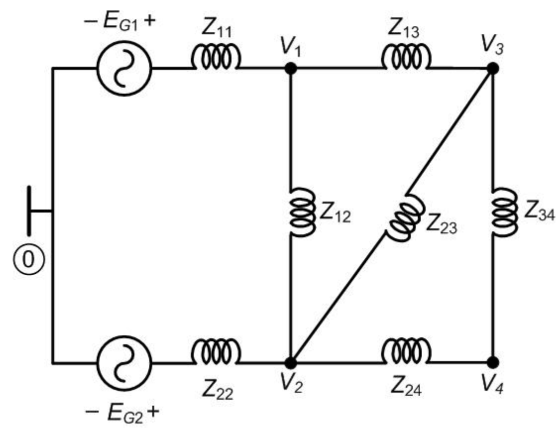


Fig. 3.3 Impedance diagram of the power network of Fig. 3.2.

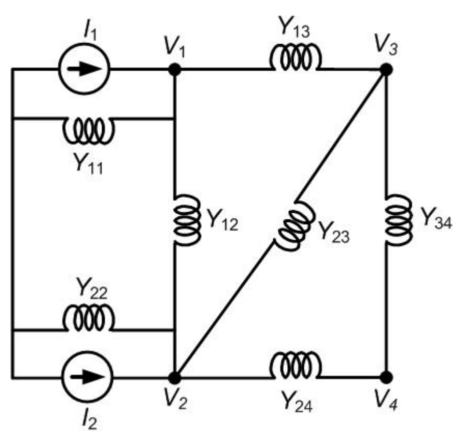


Fig. 3.4 Equivalent admittance diagram of the impedance of Fig. 3.3.

We would like to determine the voltage-current relationships of the network shown in Fig. 3.4. It is to be noted that this relation can be written in terms of the node (bus) voltages V_1 to V_4 and injected currents I_1 and I_2 as follows

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix} = Y_{bus} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \tag{3.3}$$

or

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = Z_{bus} \begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix} \tag{3.4}$$

It can be easily seen that we get (3.1) from (3.3) and (3.4).

Consider node (bus) 1 that is connected to the nodes 2 and 3. Then applying KCL at this node we get

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}(V_1 - V_2) + Y_{13}(V_1 - V_3) \\ &= (Y_{11} + Y_{12} + Y_{13})V_1 - Y_{12}V_2 - Y_{13}V_3 \end{aligned} \tag{3.5}$$

In a similar way application of KCL at nodes 2, 3 and 4 results in the following equations

$$\begin{aligned} I_2 &= Y_{22}V_2 + Y_{12}(V_2 - V_1) + Y_{23}(V_2 - V_3) + Y_{24}(V_2 - V_4) \\ &= -Y_{12}V_1 + (Y_{22} + Y_{12} + Y_{23} + Y_{24})V_2 - Y_{23}V_3 - Y_{24}V_4 \end{aligned} \tag{3.6}$$

$$\begin{aligned} 0 &= Y_{13}(V_3 - V_1) + Y_{23}(V_3 - V_2) + Y_{34}(V_3 - V_4) \\ &= -Y_{13}V_1 - Y_{23}V_2 + (Y_{13} + Y_{23} + Y_{34})V_3 - Y_{34}V_4 \end{aligned} \tag{3.7}$$

$$\begin{aligned} 0 &= Y_{24}(V_4 - V_2) + Y_{34}(V_4 - V_3) \\ &= -Y_{24}V_2 - Y_{34}V_3 + (Y_{24} + Y_{34})V_4 \end{aligned} \tag{3.8}$$

Combining (3.5) to (3.8) we get

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11} + Y_{12} + Y_{13} & -Y_{12} & -Y_{13} & 0 \\ -Y_{12} & Y_{22} + Y_{12} + Y_{23} + Y_{24} & -Y_{23} & -Y_{24} \\ -Y_{13} & -Y_{23} & Y_{13} + Y_{23} + Y_{34} & -Y_{34} \\ 0 & -Y_{24} & -Y_{34} & Y_{24} + Y_{34} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \tag{3.9}$$

Comparing (3.9) with (3.3) we can write

$$Y_{bus} = \begin{bmatrix} Y_{11}+Y_{12}+Y_{13} & -Y_{12} & -Y_{13} & 0 \\ -Y_{12} & Y_{22}+Y_{12}+Y_{23}+Y_{24} & -Y_{23} & -Y_{24} \\ -Y_{13} & -Y_{23} & Y_{13}+Y_{23}+Y_{34} & -Y_{34} \\ 0 & -Y_{24} & -Y_{34} & Y_{24}+Y_{34} \end{bmatrix} \tag{3.10}$$

In general the format of the Y_{bus} matrix for an n-bus power system is as follows

$$Y_{bus} = \begin{bmatrix} Y_1 & -Y_{12} & -Y_{13} & \cdots & -Y_{1n} \\ -Y_{12} & Y_2 & -Y_{23} & \cdots & -Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Y_{1n} & -Y_{2n} & -Y_{3n} & \cdots & Y_{nn} \end{bmatrix} \tag{3.11}$$

where

$$Y_k = \sum_{j=1}^{n} Y_{kj} \tag{3.12}$$

It is to be noted that Y_{bus} is a symmetric matrix in which the sum of all the elements of the k^{th} column is Y_{kk}.

Example 3.1: Consider the impedance diagram of Fig. 3.2 in which the system parameters are given in per unit by

$$Z_{11} = Z_{22} = j0.25,\ Z_{12} = j0.2,\ Z_{13} = j0.25,\ Z_{23} = Z_{34} = j0.4 \text{ and } Z_{24} = j0.5$$

The system admittance can then be written in per unit as

$$Y_{11} = Y_{22} = -j4,\ Y_{12} = -j5,\ Y_{13} = -j4,\ Y_{23} = Y_{34} = -j2.5 \text{ and } Y_{24} = -j2$$

The Y_{bus} is then given from (3.10) as

$$Y_{bus} = j\begin{bmatrix} -13 & 5 & 4 & 0 \\ 5 & -13.5 & 2.5 & 2 \\ 4 & 2.5 & -9 & 2.5 \\ 0 & 2 & 2.5 & -4.5 \end{bmatrix} \text{ per unit}$$

Consequently the bus impedance matrix is given by

$$Z_{bus} = j\begin{bmatrix} 0.1531 & 0.0969 & 0.1264 & 0.1133 \\ 0.0969 & 0.1531 & 0.1236 & 0.1367 \\ 0.1264 & 0.1236 & 0.2565 & 0.1974 \\ 0.1133 & 0.1367 & 0.1974 & 0.3926 \end{bmatrix} \text{ per unit}$$

It can be seen that like the Y_{bus} matrix the Z_{bus} matrix is also symmetric.

Let us now assume that the voltages E_{G1} and E_{G2} are given by

$$E_{G1} = 1\angle 30^\circ \text{ pu and } E_{G2} = 1\angle 0^\circ \text{ pu}$$

The current sources I_1 and I_2 are then given by

$$I_1 = \frac{E_{G1}}{Z_{11}} = \frac{1\angle 30^\circ}{0.25\angle 90^\circ} = 4\angle -60^\circ \text{ pu}$$
$$I_2 = \frac{E_{G2}}{Z_{22}} = \frac{1\angle 0^\circ}{0.25\angle 90^\circ} = 4\angle -90^\circ \text{ pu}$$

We then get the node voltages from (3.4) as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} j0.1531 & j0.0969 & j0.1264 & \text{j}0.1133 \\ 0.0969 & j0.1531 & j0.1236 & \text{j}0.1367 \\ j0.1264 & j0.1236 & j0.2565 & j0.1974 \\ j0.1133 & j0.1367 & j0.1974 & j0.3926 \end{bmatrix} \begin{bmatrix} 4\angle -60^\circ \\ 4\angle -90^\circ \\ 0 \\ 0 \end{bmatrix}$$

Solving the above equation we get the node voltages as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0.9677\angle 18.45^\circ \\ 0.9677\angle 11.55^\circ \\ 0.9659\angle 15.18^\circ \\ 0.9662\angle 13.56^\circ \end{bmatrix} \text{ per unit}$$

ΔΔΔ

3.1.1 Node Elimination by Matrix Partitioning

Sometimes it is desirable to reduce the network by eliminating the nodes in which the current do not enter or leave. Let (3.3) be written as

$$\begin{bmatrix} I_A \\ I_x \end{bmatrix} = \begin{bmatrix} K & L \\ L^T & M \end{bmatrix} \begin{bmatrix} V_A \\ V_x \end{bmatrix} \tag{3.13}$$

In the above equation I_A is a vector containing the currents that are injected, I_x is a null vector and the Y_{bus} matrix is portioned with the matrices K, L and M. Note that the Y_{bus} matrix contains both L and L^T due to its symmetric nature.

We get the following two sets of equations from (3.13)

$$I_A = KV_A + LV_x \tag{3.14}$$
$$0 = I_x = L^T V_A + MV_x \Rightarrow V_x = -M^{-1} L^T V_A \tag{3.15}$$

Substituting (3.15) in (3.14) we get

$$I_A = \left(K - LM^{-1}L^T\right)V_A \tag{3.16}$$

Therefore we obtain the following reduced bus admittance matrix

$$Y_{bus}^{reduced} = K - LM^{-1}L^T \tag{3.17}$$

Example 3.2: Let us consider the system of Example 3.1. Since there is no current injection in either bus 3 or bus 4, from the Y_{bus} computed we can write

$$K = \begin{bmatrix} -j13 & j5 \\ j5 & -j13.5 \end{bmatrix}, \quad L = \begin{bmatrix} j4 & 0 \\ j2.5 & j2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} -j9 & j2.5 \\ j2.5 & -j4.5 \end{bmatrix}$$

We then have

$$Y_{bus}^{reduced} = K - LM^{-1}L^T = \begin{bmatrix} -j10.8978 & j6.8978 \\ j6.8978 & -j10.8978 \end{bmatrix} \text{ per unit}$$

Substituting $I_1 = 4\angle{-60^\circ}$ per unit and $I_2 = 4\angle{-90^\circ}$ per unit we shall get the same values of V_1 and V_2 as given in Example 3.1.

Inspecting the reduced Y_{bus} matrix we can state that the admittance between buses 1 and 2 is $-j6.8978$. Therefore the self admittance (the admittance that is connected in shunt) of the buses 1 and 2 is $-j4$ per unit ($= -j10.8978 + j6.8978$). The reduced admittance diagram obtained by eliminating nodes 3 and 4 is shown in Fig. 3.5. It is to be noted that the impedance between buses 1 and 2 is the Thevenin impedance between these two buses. The value of this impedance is $1/(-j6.8978) = j0.145$ per unit.

ΔΔΔ

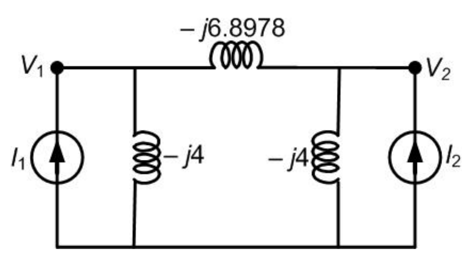


Fig. 3.5 Reduced admittance diagram after the elimination of buses 3 and 4.

3.1.2 Node Elimination by Kron Reduction

Consider an equation of the form

$$Ax = b \tag{3.18}$$

where A is an $(n \times n)$ real or complex valued matrix, x and b are vectors in either R^n or C^n. assume that the b vector has a zero element in the n^{th} row such that (3.18) is given as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ 0 \end{bmatrix} \tag{3.19}$$

We can then eliminate the k^{th} row and k^{th} column to obtain a reduced $(n - 1)$ number of equations of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix}^{new} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} \tag{3.20}$$

The elimination is performed using the following elementary operations

$$a_{kj}^{new} = a_{kj} - \frac{a_{kn}a_{nj}}{a_{nn}} \tag{3.21}$$

Example 3.3: Let us consider the same system of Example 3.1. We would like to eliminate the last two rows and columns. Let us first eliminate the last row and last column. Some of the values are given below

$$Y_{21}^{new} = Y_{21} - \frac{Y_{24}Y_{41}}{Y_{44}} = j5, \quad Y_{22}^{new} = Y_{22} - \frac{Y_{24}Y_{42}}{Y_{44}} = -j13.5 + j\frac{2 \times 2}{4.5} = -j12.6111$$

$$Y_{23}^{new} = Y_{23} - \frac{Y_{24}Y_{43}}{Y_{44}} = j2.5 + j\frac{2 \times 2.5}{4.5} = j3.6111, \quad Y_{4}^{new} = Y_{44} - \frac{Y_{44}Y_{44}}{Y_{44}} = 0$$

In a similar way we can calculate the other elements. Finally eliminating the last row and last column, as all these elements are zero, we get the new Y_{bus} matrix as

$$Y_{bus}^{new} = \begin{bmatrix} -j13 & j5 & j4 \\ j5 & -j12.6111 & j3.6111 \\ j4 & j3.6111 & -j7.6111 \end{bmatrix}$$

Further reducing the last row and the last column of the above matrix using (3.21), we obtain the reduced Y_{bus} matrix given in Example 3.2.

ΔΔΔ

3.1.3 Inclusion of Line Charging Capacitors

So far we have assumed that the transmission lines are modeled with lumped series impedances without the shunt capacitances. However in practice, the Y_{bus} matrix contains the shunt admittances for load flow analysis in which the transmission lines are represented by its π-equivalent. Note that whether the line is assumed to be of medium length or long length is irrelevant as we have seen in Chapter 2 how both of them can be represented in a π-equivalent.

Consider now the power system of Fig. 3.2. Let us assume that all the lines are represented in an equivalent-π with the shunt admittance between the line i and j being denoted by Y_{chij}. Then the equivalent admittance at the two end of this line will be $Y_{chij}/2$. For example the shunt capacitance at the two ends of the line joining buses 1 and 3 will be $Y_{ch13}/2$. We can then modify the admittance diagram Fig. 3.4 as shown in Fig. 3.6. The Y_{bus} matrix of (3.10) is then modified as

$$Y_{bus} = \begin{bmatrix} Y_{11}+Y_{12}+Y_{13}+Y_{ch1} & -Y_{12} & Y_{13} & 0 \\ -Y_{12} & Y_{22}+Y_{12}+Y_{23}+Y_{24}+Y_{ch2} & -Y_{23} & -Y_{24} \\ -Y_{13} & -Y_{23} & Y_{13}+Y_{23}+Y_{34}+Y_{ch3} & -Y_{34} \\ 0 & -Y_{24} & -Y_{34} & Y_{24}+Y_{34}+Y_{ch4} \end{bmatrix} \tag{3.22}$$

where

$$\begin{aligned} Y_{ch1} &= \frac{Y_{ch12}+Y_{ch13}}{2} \\ Y_{ch2} &= \frac{Y_{ch12}+Y_{ch23}+Y_{ch24}}{2} \\ Y_{ch3} &= \frac{Y_{ch13}+Y_{ch23}+Y_{ch34}}{2} \\ Y_{ch4} &= \frac{Y_{ch24}+Y_{ch34}}{2} \end{aligned} \tag{3.23}$$

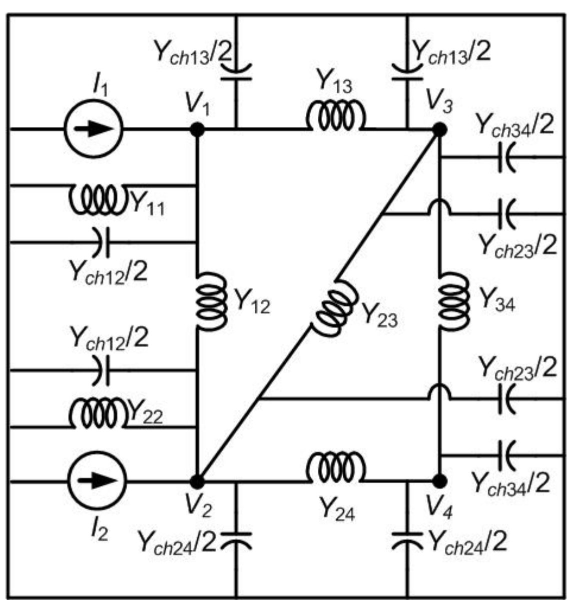


Fig. 3.6 Admittance diagram of the power system Fig. 3.2 with line charging capacitors.

3.2 ELEMENTS OF THE BUS IMPEDANCE AND ADMITTANCE MATRICES

Equation (3.1) indicates that the bus impedance and admittance matrices are inverses of each other. Also since Y_{bus} is a symmetric matrix, Z_{bus} is also a symmetric matrix. Consider a 4-bus system for which the voltage-current relations are given in terms of the Y_{bus} matrix as

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \tag{3.24}$$

We can then write

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = V_3 = V_4 = 0} \tag{3.25}$$

This implies that Y_{11} is the admittance measured at bus-1 when buses 2, 3 and 4 are short circuited. The admittance Y_{11} is defined as the *self admittance* at bus-1. In a similar way the self admittances of buses 2, 3 and 4 can also be defined that are the diagonal elements of the Y_{bus} matrix. The off diagonal elements are denoted as the *mutual admittances*. For example the mutual admittance between buses 1 and 2 is defined as

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = V_3 = V_4 = 0} \tag{3.26}$$

The mutual admittance Y_{12} is obtained as the ratio of the current injected in bus-1 to the voltage of bus-2 when buses 1, 3 and 4 are short circuited. This is obtained by applying a voltage at bus-2 while shorting the other three buses.

The voltage-current relation can be written in terms of the Z_{bus} matrix as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{3.27}$$

The *driving point impedance* at bus-1 is then defined as

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = I_3 = I_4 = 0} \tag{3.28}$$

i.e., the driving point impedance is obtained by injecting a current at bus-1 while keeping buses 2, 3 and 4 open-circuited. Comparing (3.26) and (3.28) we can conclude that Z_{11} is not the reciprocal of Y_{11}. The *transfer impedance* between buses 1 and 2 can be obtained by injecting a current at bus-2 while open-circuiting buses 1, 3 and 4 as

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = I_3 = I_4 = 0} \tag{3.29}$$

It can also be seen that Z_{12} is not the reciprocal of Y_{12}.

3.3 MODIFICATION OF BUS IMPEDANCE MATRIX

Equation (3.1) gives the relation between the bus impedance and admittance matrices. However it may be possible that the topology of the power system changes by the inclusion of a new bus or line. In that case it is not necessary to recompute the Y_{bus} matrix again for the formation of Z_{bus} matrix. We shall discuss four possible cases by which an existing bus impedance matrix can be modified.

Let us assume that an n-bus power system exists in which the voltage-current relations are given in terms of the bus impedance matrix as

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = Z_{orig} \begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix} \tag{3.30}$$

The aim is to modify the matrix Z_{orig} when a new bus or line is connected to the power system.

3.3.1 Adding a New Bus to the Reference Bus

It is assumed that a new bus p ($p > n$) is added to the reference bus through an impedance Z_p. The schematic diagram for this case is shown in Fig. 3.7. Since this bus is only connected to the reference bus, the voltage-current relations the new system are

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \\ V_p \end{bmatrix} = \begin{bmatrix} & Z_{orig} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & \cdots & 0 & Z_p \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix} = Z_{new} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix} \tag{3.31}$$

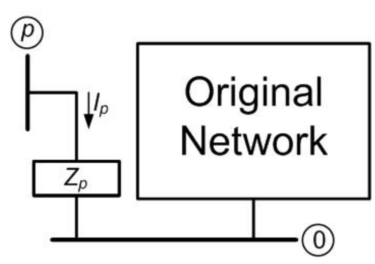


Fig. 3.7 A new bus is added to the reference bus.

3.3.2 Adding a New Bus to an Existing Bus through an Impedance

This is the case when a bus, which has not been a part of the original network, is added to an existing bus through a transmission line with an impedance of Z_b. Let us assume that p ($p > n$) is the new bus that is connected to bus k ($k < n$) through Z_b. Then the schematic diagram of the circuit is as shown in Fig. 3.8. Note from this figure that the current I_p flowing from bus p will alter the voltage of the bus k. We shall then have

$$V_k = Z_{k1}I_1 + Z_{k2}I_2 + \cdots + Z_{kk}\left(I_k + I_p\right) + \cdots + Z_{kn}I_n \tag{3.32}$$

In a similar way the current I_p will also alter the voltages of all the other buses as

$$V_i = Z_{i1}I_1 + Z_{i2}I_2 + \cdots + Z_{ik}\left(I_k + I_p\right) + \cdots + Z_{in}I_n, \quad i \neq k \tag{3.33}$$

Furthermore the voltage of the bus p is given by

$$\begin{aligned} V_p &= V_k + Z_b I_p \\ &= Z_{k1}I_1 + Z_{k2}I_2 + \cdots + Z_{kk}I_k + \cdots + Z_{kn}I_n + \left(Z_{kk} + Z_b\right)I_p \end{aligned} \tag{3.34}$$

Therefore the new voltage current relations are

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \\ V_p \end{bmatrix} = \begin{bmatrix} & & & Z_{k1} \\ & Z_{orig} & & \vdots \\ & & & Z_{kn} \\ Z_{k1} & \cdots & Z_{kn} & Z_{kk} + Z_b \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix} = Z_{new} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix} \tag{3.35}$$

It can be noticed that the new Z_{bus} matrix is also symmetric.

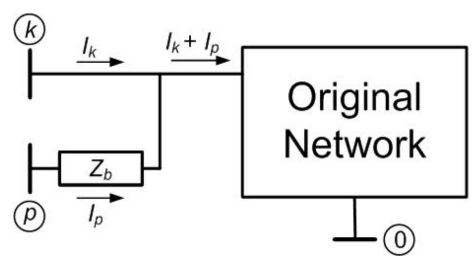


Fig. 3.8 A new bus is added to an existing bus through an impedance.

3.3.3 Adding an Impedance to the Reference Bus from an Existing Bus

To accomplish this we first assume that an impedance Z_b is added from a new bus p to an existing bus k. This can be accomplished using the method discussed in Section 3.3.2. Then to add this bus k to the reference bus through Z_b, we set the voltage V_p of the new bus to zero. However now we have an $(n + 1) \times (n + 1)$ Z_{bus} matrix instead of an $n \times n$ matrix. We can then remove the last row and last column of the new Z_{bus} matrix using the Kron's reduction given in (3.21).

3.3.4 Adding an Impedance between two Existing Buses

Let us assume that we add an impedance Z_b between two existing buses k and j as shown in Fig. 3.9. Therefore the current injected into the network from the bus k side will be $I_k - I_b$ instead of I_k. Similarly the current injected into the network from the bus j side will be $I_j + I_b$ instead of I_j. Consequently the voltage of the i^{th} bus will be

$$\begin{aligned} V_i &= Z_{i1}I_1 + Z_{i2}I_2 + \cdots + Z_{ij}\left(I_j + I_b\right) + Z_{ik}\left(I_k - I_b\right) + \cdots + Z_{in}I_n \\ &= Z_{i1}I_1 + Z_{i2}I_2 + \cdots + Z_{ij}I_j + Z_{ik}I_k + \cdots + Z_{in}I_n + \left(Z_{ij} - Z_{ik}\right)I_b \end{aligned} \tag{3.36}$$

Similarly we have

$$V_j = Z_{j1}I_1 + Z_{j2}I_2 + \cdots + Z_{jj}I_j + Z_{jk}I_k + \cdots + Z_{jn}I_n + \left(Z_{jj} - Z_{jk}\right)I_b \tag{3.37}$$

and

$$V_k = Z_{k1}I_1 + Z_{k2}I_2 + \cdots + Z_{kj}I_j + Z_{kk}I_k + \cdots + Z_{kn}I_n + \left(Z_{kj} - Z_{kk}\right)I_b \tag{3.38}$$

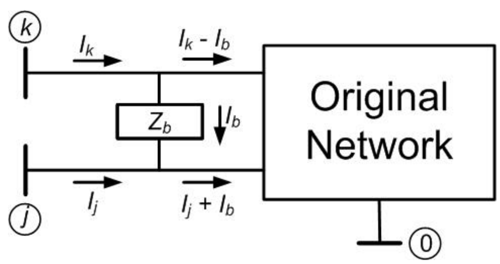


Fig. 3.9 An impedance is added between two existing buses.

We shall now have to eliminate I_b from the above equations. To do that we note from Fig. 3.9 that

$$V_k - V_j = Z_b I_b \Rightarrow 0 = Z_b I_b - V_k + V_j \tag{3.39}$$

Substituting (3.37) and (3.38) in (3.39) we get

$$\begin{aligned} 0 = Z_b I_b + \left(Z_{j1} - Z_{k1}\right)I_1 + \cdots + \left(Z_{jj} - Z_{kj}\right)I_j + \left(Z_{jk} - Z_{kk}\right)I_k + \cdots \\ + \left(Z_{jn} - Z_{kn}\right)I_n + \left(Z_{jj} - 2Z_{jk} + Z_{kk}\right)I_b \end{aligned} \tag{3.40}$$

We can then write the voltage current relations as

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \\ 0 \end{bmatrix} = \begin{bmatrix} & & & Z_{j1} - Z_{k1} \\ & Z_{orig} & & \vdots \\ & & & Z_{jn} - Z_{kn} \\ Z_{j1} - Z_{k1} & \cdots & Z_{jn} - Z_{kn} & Z_{bb} \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix} = Z_{new} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix} \tag{3.41}$$

where

$$Z_{bb} = Z_b + Z_{jj} - 2Z_{jk} + Z_{kk} \tag{3.42}$$

We can now eliminate the last row and last column using the Kron's reduction given in (3.21).

3.3.5 Direct Determination of Z_{bus} Matrix

We shall now use the methods given in Sections 3.3.1 to 3.3.4 for the direct determination of the Z_{bus} matrix without forming the Y_{bus} matrix first. To accomplish this we shall consider the system of Fig. 3.2 and shall use the system data given in Example 3.1. Note that for the construction of the Z_{bus} matrix we first eliminate all the voltage sources from the system.

<u>Step-1</u>: Start with bus-1. Assume that no other buses or lines exist in the system. We add this bus to the reference bus with the impedance of $j0.25$ per unit. Then the Z_{bus} matrix is

$$Z_{bus,1} = j0.25 \tag{3.43}$$

<u>Step-2</u>: We now add bus-2 to the reference bus using (3.31). The system impedance diagram is shown in Fig. 3.10. We then can modify (3.43) as

$$Z_{bus,2} = \begin{bmatrix} j0.25 & 0 \\ 0 & j0.25 \end{bmatrix} \tag{3.44}$$

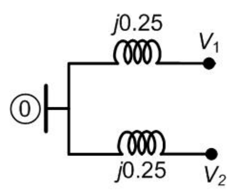


Fig. 3.10 Network of step-2.

<u>Step-3</u>: We now add an impedance of $j0.2$ per unit between buses 1 and 2 as shown in Fig. 3.11. The interim Z_{bus} matrix is then obtained by applying (3.41) on (3.44) as

$$Z_{bus,3}^{in} = \begin{bmatrix} j0.25 & 0 & j0.25 \\ 0 & j0.25 & -j0.25 \\ j0.25 & -j0.25 & j0.7 \end{bmatrix}$$

Eliminating the last row and last column using the Kron's reduction of (3.31) we get

$$Z_{bus,3} = \begin{bmatrix} j0.1607 & j0.0893 \\ j0.0893 & j0.1607 \end{bmatrix} \tag{3.45}$$

<u>Step-4</u>: We now add bus-3 to bus-1 through an impedance of $j0.25$ per unit as shown in Fig. 3.12. The application of (3.35) on (3.45) will then result in the following matrix

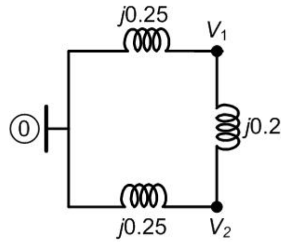


Fig. 3.11 Network of step-3.

$$Z_{bus,4} = \begin{bmatrix} j0.1607 & j0.0893 & j0.1607 \\ j0.0893 & j0.1607 & j0.0893 \\ j0.1607 & j0.0893 & j0.4107 \end{bmatrix} \tag{3.46}$$

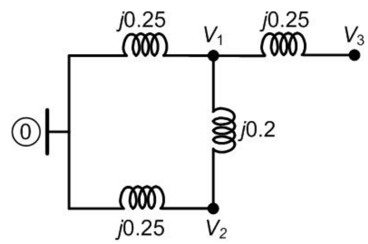


Fig. 3.12 Network of step-4.

<u>Step-5</u>: Connect buses 2 and 3 through an impedance of $j0.4$ per unit as shown in Fig. 3.13. The interim Z_{bus} matrix is then formed from (3.41) and (3.46) as

$$Z_{bus,5}^{in} = \begin{bmatrix} j0.1607 & j0.0893 & j0.1607 & -j0.0714 \\ j0.0893 & j0.1607 & j0.0893 & j0.0714 \\ j0.1607 & j0.0893 & j0.4107 & -j0.3214 \\ -j0.0714 & j0.0714 & -j0.3214 & j0.7928 \end{bmatrix}$$

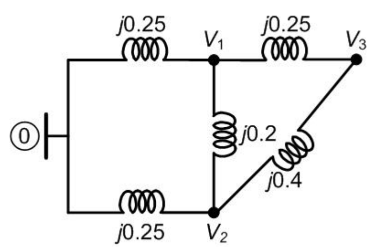


Fig. 3.13 Network of step-5.

Using the Kron's reduction we get the following matrix

$$Z_{bus,5} = \begin{bmatrix} j0.1543 & j0.0957 & j0.1318 \\ j0.0957 & j0.1543 & j0.1182 \\ j0.1318 & j0.1182 & j0.2804 \end{bmatrix} \tag{3.47}$$

Step-6: We now add a new bus-4 to bus-2 through an impedance of $j0.5$ as shown in Fig. 3.14. Then the application of (3.35) on (3.47) results in the following matrix

$$Z_{bus,6} = \begin{bmatrix} j0.1543 & j0.0957 & j0.1318 & j0.0957 \\ j0.0957 & j0.1543 & j0.1182 & j0.1543 \\ j0.1318 & j0.1182 & j0.2804 & j0.1182 \\ j0.0957 & j0.1543 & j0.1182 & j0.6543 \end{bmatrix} \tag{3.48}$$

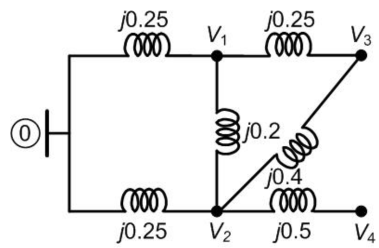


Fig. 3.14 Network of step-6.

Step-7: Finally we add buses 3 and 4 through an impedance of $j0.4$ to obtain the network of Fig. 3.3 minus the voltage sources. The application of (3.41) on (3.48) results in the interim Z_{bus} matrix of

$$Z_{bus,7}^{in} = \begin{bmatrix} j0.1543 & j0.0957 & j0.1318 & j0.0957 & j0.0360 \\ j0.0957 & j0.1543 & j0.1182 & j0.1543 & -j0.0360 \\ j0.1318 & j0.1182 & j0.2804 & j0.1182 & j0.1622 \\ j0.0957 & j0.1543 & j0.1182 & j0.6543 & j0.5360 \\ j0.0360 & -j0.0360 & j0.1622 & j0.5360 & j1.0982 \end{bmatrix}$$

Eliminating the 5th row and column through Kron's reduction we get the final Z_{bus} as

$$Z_{bus,7} = \begin{bmatrix} j0.1531 & j0.0969 & j0.1264 & \mathrm{j}0.1133 \\ 0.0969 & j0.1531 & j0.1236 & \mathrm{j}0.1367 \\ j0.1264 & j0.1236 & j0.2565 & j0.1974 \\ j0.1133 & j0.1367 & j0.1974 & j0.3926 \end{bmatrix} \tag{3.49}$$

The Z_{bus} matrix given in (3.49) is the as that given in Example 3.1 which is obtained by inverting the Y_{bus} matrix.

3.4 THEVENIN IMPEDANCE AND Z_{bus} MATRIX

To establish relationships between the elements of the Z_{bus} matrix and Thevenin equivalent, let us consider the following example.

Example 3.4: Consider the two bus power system shown in Fig. 3.15. It can be seen that the open-circuit voltages of buses a and b are V_a and V_b respectively. From (3.11) we can write the Y_{bus} matrix of the system as

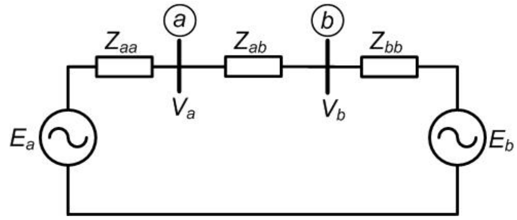


Fig. 3.15 Two-bus power system of Example 3.4.

$$Y_{bus} = \begin{bmatrix} \dfrac{1}{Z_{aa}} + \dfrac{1}{Z_{ab}} & -\dfrac{1}{Z_{ab}} \\ -\dfrac{1}{Z_{ab}} & \dfrac{1}{Z_{ab}} + \dfrac{1}{Z_{bb}} \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{aa} + Z_{ab}}{Z_{aa}Z_{ab}} & -\dfrac{1}{Z_{ab}} \\ -\dfrac{1}{Z_{ab}} & \dfrac{Z_{ab} + Z_{bb}}{Z_{ab}Z_{bb}} \end{bmatrix}$$

The determinant of the above matrix is

$$\left|Y_{bus}\right| = \frac{Z_{aa} + Z_{ab} + Z_{bb}}{Z_{aa}Z_{ab}Z_{bb}}$$

Therefore the Z_{bus} matrix is

$$Z_{bus} = Y_{bus}^{-1} = \frac{1}{\left|Y_{bus}\right|} \begin{bmatrix} \dfrac{Z_{ab} + Z_{bb}}{Z_{ab}Z_{bb}} & \dfrac{1}{Z_{ab}} \\ \dfrac{1}{Z_{ab}} & \dfrac{Z_{aa} + Z_{ab}}{Z_{aa}Z_{ab}} \end{bmatrix}$$

Solving the last two equations we get

$$Z_{bus} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{aa}\left(Z_{ab} + Z_{bb}\right)}{Z_{aa} + Z_{ab} + Z_{bb}} & \dfrac{Z_{aa}Z_{bb}}{Z_{aa} + Z_{ab} + Z_{bb}} \\ \dfrac{Z_{aa}Z_{bb}}{Z_{aa} + Z_{ab} + Z_{bb}} & \dfrac{Z_{bb}\left(Z_{aa} + Z_{ab}\right)}{Z_{aa} + Z_{ab} + Z_{bb}} \end{bmatrix} \tag{3.50}$$

Now consider the system of Fig. 3.15. The Thevenin impedance of looking into the system at bus-*a* is the parallel combination of Z_{aa} and $Z_{ab} + Z_{bb}$, i.e.,

$$Z_{th,a} = \frac{Z_{aa}\left(Z_{ab} + Z_{bb}\right)}{Z_{aa} + Z_{ab} + Z_{bb}} = Z_{11} \tag{3.51}$$

Similarly the Thevenin impedance obtained by looking into the system at bus-*b* is the parallel combination of Z_{bb} and $Z_{aa} + Z_{ab}$, i.e.,

$$Z_{th,b} = \frac{Z_{bb}\left(Z_{aa} + Z_{ab}\right)}{Z_{aa} + Z_{ab} + Z_{bb}} = Z_{22} \tag{3.52}$$

Hence the driving point impedances of the two buses are their Thevenin impedances.

Let us now consider the Thevenin impedance while looking at the system between the buses a and b. From Fig. 3.15 it is evident that this Thevenin impedance is the parallel combination of Z_{ab} and $Z_{aa} + Z_{bb}$, i.e.,

$$Z_{th,ab} = \frac{Z_{ab}\left(Z_{aa} + Z_{bb}\right)}{Z_{aa} + Z_{ab} + Z_{bb}}$$

With the values given in (3.50) we can write

$$\begin{aligned} Z_{11} + Z_{22} - 2Z_{12} &= \frac{Z_{aa}\left(Z_{ab} + Z_{bb}\right)}{Z_{aa} + Z_{ab} + Z_{bb}} + \frac{Z_{bb}\left(Z_{aa} + Z_{ab}\right)}{Z_{aa} + Z_{ab} + Z_{bb}} - 2\frac{Z_{aa}Z_{bb}}{Z_{aa} + Z_{ab} + Z_{bb}} \\ &= \frac{1}{Z_{aa} + Z_{ab} + Z_{bb}}\left[Z_{aa}Z_{ab} + Z_{ab}Z_{bb}\right] \end{aligned}$$

Comparing the last two equations we can write

$$Z_{th,ab} = Z_{11} + Z_{22} - 2Z_{12} \tag{3.53}$$

ΔΔΔ

As we have seen in the above example in the relation $V = Z_{bus}I$, the node or bus voltages V_i, $i = 1, \ldots, n$ are the open circuit voltages. Let us assume that the currents injected in buses $1, \ldots, k - 1$ and $k + 1, \ldots, n$ are zero when a short circuit occurs at bus k. Then Thevenin impedance at bus k is

$$Z_{th,k} = \frac{V_k}{I_k} = Z_{kk} \tag{3.54}$$

From (3.51), (3.52) and (3.54) we can surmise that the driving point impedance at each bus is the Thevenin impedance.

Let us now find the Thevenin impedance between two buses j and k of a power system. Let the open circuit voltages be defined by the voltage vector $V°$ and corresponding currents be defined by $I°$ such that

$$V° = Z_{bus}I° \tag{3.55}$$

Now suppose the currents are changed by ΔI such that the voltages are changed by ΔV. Then

$$V = V° + \Delta V = Z_{bus}\left(I° + \Delta I\right) \tag{3.56}$$

Comparing (3.55) and (3.56) we can write

$$\Delta V = Z_{bus}\Delta I \tag{3.57}$$

Let us now assume that additional currents ΔI_k and ΔI_j are injected at the buses k and j respectively while the currents injected at the other buses remain the same. Then from (3.57) we can write

$$\Delta V = Z_{bus}\begin{bmatrix} 0 \\ \vdots \\ \Delta I_j \\ \Delta I_k \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{1j}\Delta I_j + Z_{1k}\Delta I_k \\ \vdots \\ Z_{jj}\Delta I_j + Z_{jk}\Delta I_k \\ Z_{kj}\Delta I_j + Z_{kk}\Delta I_k \\ \vdots \\ Z_{nj}\Delta I_j + Z_{nk}\Delta I_k \end{bmatrix} \tag{3.58}$$

We can therefore write the following two equations form (3.58)

$$V_j = V_j^o + \Delta V_j = V_j^o + Z_{jj}\Delta I_j + Z_{jk}\Delta I_k$$

$$V_k = V_k^o + \Delta V_k = V_k^o + Z_{kj}\Delta I_j + Z_{kk}\Delta I_k$$

The above two equations can be rewritten as

$$V_j = V_j^o + \left(Z_{jj} - Z_{jk}\right)\Delta I_j + Z_{jk}\left(\Delta I_j + \Delta I_k\right) \tag{3.59}$$

$$V_k = V_k^o + Z_{kj}\left(\Delta I_j + \Delta I_k\right) + \left(Z_{kk} - Z_{kj}\right)\Delta I_k \tag{3.60}$$

Since $Z_{jk} = Z_{kj}$, the network can be drawn as shown in Fig. 3.16. By inspection we can see that the open circuit voltage between the buses k and j is

$$V_{oc,kj} = V_k^o - V_j^0 \tag{3.61}$$

and the short circuit current through these two buses is

$$I_{sc,kj} = \Delta I_j = -\Delta I_k \tag{3.62}$$

Also during the short circuit $V_k - V_j = 0$. Therefore combining (3.59) and (3.60) we get

$$V_k - V_j = \left(V_k^o - V_j^o\right) + \left(2Z_{kj} - Z_{jj} - Z_{kk}\right)I_{sc,kj} = 0 \tag{3.63}$$

Combining (3.61) to (3.63) we find the Thevenin impedance between the buses k and j as

$$Z_{th,kj} = \frac{V_{oc,kj}}{I_{sc,kj}} = Z_{jj} + Z_{kk} - 2Z_{kj} \tag{3.64}$$

The above equation agrees with our earlier derivation of the two bus network given in (3.53).

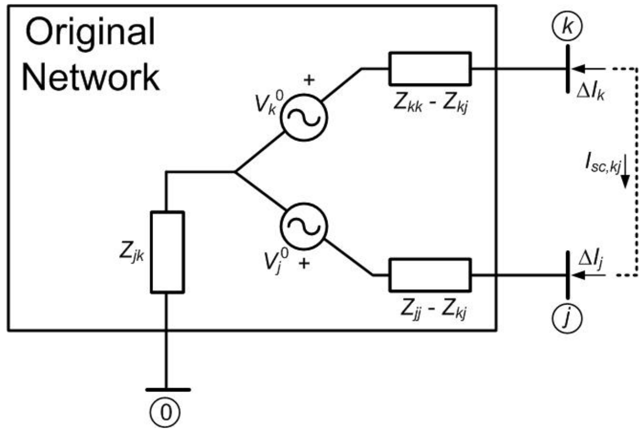


Fig. 3.16 Thevenin equivalent between buses k and j.