Module 2

MULTIVARIABLE CALCULUS:

1. Introduction to function of several variables
2. Jacobian and its properties
3. Functional dependence
4. Maxima and minima of functions with two variables
5. Lagrange's multiplier method.

Introduction to function of several variables:

The functions which depend on more than one independent variable are called functions o f several variables. Partial derivatives of a function of several variables are the derivative with respect to one of the variables when all the remaining variables are kept constant.

Partial derivatives

Let $z = f(x, y)$ be a function of two variables, keeping y constant and varying x only, the partial derivative of z with respect to x is defined as

$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$, which is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $z_x(x, y)$ or $f_x(x, y)$ or z_x or f_x

Similarly Partial Derivative of z w.r.t y can be defined as

$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$, which is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $z_y(x, y)$ or $f_y(x, y)$ or z_y or f_y

Partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ *or* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are known as first-order partial derivatives, while $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are second order partial derivatives.

Note: If f is a continuous function then $\frac{\partial^2 \mathrm{f}}{\partial \mathrm{x} \partial \mathrm{y}} = \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}\, \partial \mathrm{x}}$

Problems

1. Find the first order Partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the functions

i. $u = e^x \cos y$ ii. $u = \tan^{-1} \frac{y}{x}$

Solution:

i. $u = e^x \cos y$ --- (1)

Differentiate partially with respect to x equation (1)

$$\frac{\partial u}{\partial x} = e^x \cos y$$

Differentiate partially with respect to y equation (1)

$$\frac{\partial u}{\partial y} = -e^x \sin y$$

ii. $u = \tan^{-1} \frac{y}{x}$ --- (1)

Differentiate partially with respect to x equation (1)

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \left(\frac{-y}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2 + y^2}$$

Differentiate partially with respect to y equation (1)

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

2. If $u = \sin^{-1}\left(\frac{y}{x}\right)$ **then, show that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$**.**

Solution:

Given $u = \sin^{-1}\left(\frac{y}{x}\right)$ **--- (1)**

Differentiate the equation (1) partially u w.r.t. x**,** and y **.**

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{1}{y} = \frac{y}{\sqrt{y^2-x^2}}\ ;\ \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{-x}{y^2} = \frac{-x}{\sqrt{y^2-x^2}}$$

Now consider,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy}{\sqrt{y^2-x^2}} - \frac{xy}{\sqrt{y^2-x^2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

3. If $z = \log\sqrt{x^2+y^2}$ **then, show that** $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$**.**

Solution:

Given $z = \log\sqrt{x^2+y^2}$ **--- (1)**

Differentiate the equation (1) partially z w.r.t. x**,** and y **.**

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}2x\ ;\ \frac{\partial z}{\partial y} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}2y$$

Now consider,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2+y^2}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.$$

4. Given that $x = r\cos\theta$ **and** $y = r\sin\theta$ **then find** r_x**,** r_y**,** θ_x **and** θ_y **.**

$$\left(r^2 = x^2+y^2\ \&\ \theta = \tan^{-1}\frac{y}{x}\right)$$

Solution:

Given $x = r\cos\theta$ and $y = r\sin\theta$ can be written $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

($x = r\cos\theta$ and $y = r\sin\theta$ are the parametric equations of $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$)

$r = \sqrt{x^2 + y^2}$ --- (1) $\theta = \tan^{-1}\frac{y}{x}$ --- (2)

Differentiate (1) partially r w.r.t. x, and y .
Differentiate (2) partially θ w.r.t. x, and y .

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} 2x = \frac{r\cos\theta}{r} = \cos\theta; \qquad \frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{-y}{x^2} = \frac{-y}{x^2 + y^2};$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2}} 2y = \frac{r\sin\theta}{r} = \sin\theta; \qquad \frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{x}{x^2 + y^2};$$

5. If $z = f(ax + by)$ then, show that $b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$.

Solution:

Given $z = f(ax + by)$ **--- (1)**

Differentiate the equation (1) partially z w.r.t. x, and y .

$$\frac{\partial z}{\partial x} = f^1(ax + by)\,a; \qquad \frac{\partial z}{\partial y} = f^1(ax + by)b$$

Now consider,

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = abf^1(ax + by) - baf^1(ax + by)$$

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$$

Homogeneous functions and Euler's theorem

A polynomial in x and y , i.e. $f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_ny^n$ is said to be homogeneous if all its terms are of same degree.

In general, A function $f(x, y)$ is said to be homogenous of degree n if it can be expressed as $x^n\varphi\left(\frac{y}{x}\right)$ or $y^n\varphi\left(\frac{x}{y}\right)$ where 'n' can be positive, negative or zero.

Euler's Theorem for homogeneous function.

Statement

If u be a homogeneous function of degree n in x and y then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu$

Note:

If $z=f(x,y)$ is a homogeneous function of x, y of degree 'n', then

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

Problems

6. **Verify the Euler's theorem for the function** $u=\log(x^2+xy+y^2)$

Solution:

Given $u=\log(x^2+xy+y^2)$

u is a not homogeneous function.

So Euler's theorem can't be applied.

7. **Verify the Euler's theorem for the function** $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ **.**

Solution: Given $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$

--- (1)

$$u=\frac{1}{x\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}}$$

$$u=x^{-1}\ f\left(\frac{y}{x},\frac{z}{x}\right)$$

u is a homogeneous function in x with degree -1.

By Euler's theorem $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-u$.

--- (2)

Differentiate partially (1) u with respect to x, y and z, we get

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2x=\frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2y = \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2z = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Now consider,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{-y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{1}{\sqrt{(x^2+y^2+z^2)}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$$

--- (3)

From equations (2) and (3), Euler's theorem verified.

8. Verify the Euler's theorem for the function $u = ax^2 + 2hxy + b^2y^2$

Solution:

Given $u = ax^2 + 2hxy + b^2$

$$u = x^2\left(a + 2h\frac{y}{x} + b\frac{y^2}{x^2}\right) \qquad \text{--- (1)}$$

$$u = x^2\ f\left(\frac{y}{x}\right)$$

u is a homogeneous function in x with degree 2.

By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$.

--- (2)

Differentiate partially (1) u with respect to x and y , we get

$$\frac{\partial u}{\partial x} = 2ax + 2hy + 0$$

$$\frac{\partial u}{\partial y} = 0 + 2hx + 2by$$

Now consider, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + = 2ax^2 + 2hxy + 2ay^2 + 2hxy$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+=2(ax^2+2hxy+2ay^2)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u \qquad \text{--- (3)}$$

From equations (2) and (3), Euler's theorem verified.

9. Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$ **, where** $\log u=\frac{x^3+y^3}{3x+4y}$ **.**

Solution:

Let $z=\frac{x^3+y^3}{3x+4y}=x^2.\frac{1+\left(\frac{y}{x}\right)^3}{3+4\left(\frac{y}{x}\right)}$ where $z=\log u$ --- (1)

z is a homogenous function of degree 2 in x and y

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z \qquad \text{--- (2)}$$

Differentiate partially (1) w.r.t. x, and y .

$\frac{\partial z}{\partial x}=\frac{1}{u}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\frac{1}{u}\frac{\partial u}{\partial y}$

Substituting these in equation (2), we get

$$x\frac{1}{u}\frac{\partial u}{\partial x}+y\frac{1}{u}\frac{\partial u}{\partial y}=2\ln u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u.$$

10.If $u=\tan^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$ **then show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{4}\sin 2u$ **.**

Solution:

Given $u=\tan^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\tan u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} = \sqrt{x}\,\frac{\left(1+\frac{y}{x}\right)}{\left(1+\sqrt{\frac{y}{x}}\right)}$, where $z = \tan u$ --- (1)

z is a homogenous function in x with degree 1/2.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \qquad \text{--- (2)}$$

Differentiate partially (1) w.r.t. x, and y **.**

$\frac{\partial z}{\partial x} = \sec^2 u\,\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \sec^2 u\,\frac{\partial u}{\partial y}$

Substituting these in equation (2), we get

$$x\sec^2 u\,\frac{\partial u}{\partial x} + y\sec^2 u\,\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}\frac{1}{\sec^2 u}.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{2}{2}\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u.$$

11.If $u = \tan^{-1}\sqrt{x^4+y^4}$ **then prove that**

i. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ **and**

ii. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$

Solution:

Given $u = \tan^{-1}\sqrt{x^4+y^4}$

$\tan u = \sqrt{x^4+y^4}$

Let $z = \sqrt{x^4+y^4} = \sqrt{x^4\left(1+\left(\frac{y}{x}\right)^4\right)} = x^2\sqrt{\left(1+\left(\frac{y}{x}\right)^4\right)}$, where $z = \tan u$ --- (1)

Z is a homogenous function in x with degree 2.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z \qquad \text{---(2)}$$

Differentiate partially (1) w.r.t. x, and y .

$\frac{\partial z}{\partial x}=\sec^2 u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$

Substituting these in equation (2), we get

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\sin u}{\cos u}\frac{1}{\sec^2 u}.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u. \qquad \text{--- (3)}$$

Differentiate partially equation (3) w.r.t x and y again, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x}$$

Multiply x on both sides in above equation, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\ x\frac{\partial u}{\partial x} \qquad \text{--- (4)}$$

Similarly,

$$y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\frac{\partial u}{\partial y}$$

Multiply y on both sides in above equation, we get

$$y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial x}+yx\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\ y\frac{\partial u}{\partial y} \qquad \text{--- (5)}$$

Adding equations (4) and (5), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}\ y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial x}+yx\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\ y\frac{\partial u}{\partial y}+2\cos 2u\ x\frac{\partial u}{\partial x}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\ y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right)=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right)$$

---- (6)

Using equation (3) in (6)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}+\sin 2u=2\cos 2u\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}=2\cos 2u\sin 2u-\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$$

Total differentiation, differentiation of composite and implicit functions

Total Differentials

Consider a function $f=f(x,y)$ of two independent variable x and y then the total differential is defined as $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$ .Similarly, if a function $f=f(x,y,z)$ of three independent variable x, y and z then the total differential is defined as

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz.$$

Total Derivatives

Consider a function $f=f\big(x(t),y(t)\big)$ where x and y are functions of t, then total derivative of f with respect to t is defined as $\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

Similarly,if a function $f=f\big(x(t)y(t)z(t)\big)$ where x, y and z are functions of t then total derivative of f with respect to t is defined as $\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$

Problems

12. If $u=x^2+y^2+z^2$ and $x=e^{2t}$, $y=e^{2t}\cos 3t$, $z=e^{2t}\sin 3t$ then, Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution

Solution:

Given $u=x^2+y^2+z^2$, $x=e^{2t}$, $y=e^{2t}\cos 3t$ and $z=e^{2t}\sin 3t$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$=4x\,e^{2t}+2y\,(-e^{2t}3\sin 3t+2\cos 3t\,e^{2t})+2z(e^{2t}\cos 3t+\sin 3t\,2e^{2t})$$

$$=4x\,e^{2t}-6y\,e^{2t}\sin 3t+4y\cos 3t\,e^{2t})+6z\,e^{2t}\cos 3t+4z\,e^{2t}\sin 3t)$$

$$= 4xx - 6yz + 4yy + 6zy + 4zz$$

$$= 4x^2 - 6yz + 4y^2 + 6zy + 4z^2$$

$$= 4\left(x^2 + y^2 + z^2\right)$$

$$= 4u$$

$$\frac{du}{dt} = 4(e^{4t} + e^{4t}(\cos^2 t + \sin^2 t))$$

$$= 4(e^{4t} + e^{4t})$$

$$= 8e^{4t}$$

13.If $u = x\log xy$ **where** $x^3 + y^3 + 3xy = 1$ **then, find** $\frac{du}{dx}$ **.**

Solution:

Let $f(x, y) = x^3 + y^3 + 3xy - 1$

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{(3x^2 + 3y)}{(3y^2 + 3x)} = -\frac{x^2 + y}{y^2 + x}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = 1 + \log xy + \frac{\partial u}{\partial y}\left(-\frac{x^2 + y}{y^2 + x}\right)$$

$$\frac{du}{dx} = 1 + \log xy - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$$

Jacobians and their properties (without proof)

Jacobian is a functional determinant (whose elements are functions) which is useful in tra nsformation of variables from Cartesian to polar, cylindrical and spherical co-ordinates in multiple integrals.

The Jacobian of u , v with respect to x , y denoted $J\left(\frac{u, v}{x, y}\right)$ by or $\frac{\partial(u, v)}{\partial(x, y)}$ is a second orde r functional determinant defined as $J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Similarly the Jacobian of three functions u , v , w of three independent variables x , y , z i s defined as

$$J\left(\frac{u,v,w}{x,y,z}\right)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

Note: 1

If $J\left(\frac{u,v}{x,y}\right)=0$ then, u and v are not functionally independent. In fact they are mutually dependent

2

If $J\left(\frac{u,v}{x,y}\right)\neq 0$ then, we can express x and y in terms of u and v (explicitly) as i.e. x=x(u,v) and y=y(u,v).

3

Consequently, if $J\left(\frac{u,v}{x,y}\right)\neq 0$ then we can define the Jacobian of x and y w.r.t u and v as follows $J^*\left(\frac{x,y}{u,v}\right)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$.

Two Important Properties of Jacobians

1. ***If*** $J=\frac{\partial(u,v)}{\partial(x,y)}$ ***and*** $J^*=\frac{\partial(x,y)}{\partial(u,v)}$ ***then,*** $JJ^*=1$ ***i.e.*** $J=\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{J^*}=\frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$

2. ***If*** u ***,*** v ***are functions of*** r,s ***and*** r,s ***are functions of*** x,y ***then***

$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)}$ ***i.e.***

$J\left(\frac{u,v}{x,y}\right)=J\left(\frac{u,v}{r,s}\right).J\left(\frac{r,s}{x,y}\right)$ ***(Chain Rule for Jacobians)***

Problems

In each of the following cases , find the Jacobians $J=\frac{\partial(u,v)}{\partial(x,y)}$ **and** $J^1=\frac{\partial(x,y)}{\partial(u,v)}$ **also verify that** $JJ^1=1$**.**

14. $u=x+y$, $v=xy$

Solution:

Given that $u=x+y$, $v=xy$

$$\frac{\partial u}{\partial x}=1 \quad \frac{\partial u}{\partial y}=1$$

$$\frac{\partial v}{\partial x}=y \quad \frac{\partial v}{\partial y}=x$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 1\\ y & x\end{vmatrix}=x-y$$

Now, $u=x+y$

$$x=u-y$$

$$x=u-\frac{v}{x} \qquad \left(y=\frac{v}{x}\right)$$

$$x^2=ux-v$$

$$x^2-ux+v=0$$

$$x=\frac{u\pm\sqrt{u^2-4v}}{2} \qquad y=u-\frac{u\mp\sqrt{u^2-4v}}{2}. \qquad (y=u-x)$$

Consider, $x=\dfrac{u+\sqrt{u^2-4v}}{2}$ $\qquad y=u-\dfrac{u+\sqrt{u^2-4v}}{2}$

$$y=\frac{u-\sqrt{u^2-4v}}{2}$$

$$\frac{\partial x}{\partial u}=\frac{1}{2}\left(1+\frac{u}{\sqrt{u^2-4v}}\right) \quad \frac{\partial x}{\partial v}=\frac{-1}{\sqrt{u^2-4v}}$$

$$\frac{\partial y}{\partial u}=\frac{1}{2}\left(1-\frac{u}{\sqrt{u^2-4v}}\right) \quad \frac{\partial y}{\partial v}=\frac{1}{\sqrt{u^2-4v}}$$

$$J^1=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2}\left(1+\frac{u}{\sqrt{u^2-4v}}\right) & \frac{-1}{\sqrt{u^2-4v}}\\ \frac{1}{2}\left(1-\frac{u}{\sqrt{u^2-4v}}\right) & \frac{1}{\sqrt{u^2-4v}}\end{vmatrix}$$

$$=\frac{1}{2}\frac{1}{\sqrt{u^2-4v}}+\frac{1}{2}\frac{u}{\left(\sqrt{u^2-4v}\right)^2}+\frac{1}{2}\frac{1}{\sqrt{u^2-4v}}-\frac{1}{2}\frac{u}{\left(\sqrt{u^2-4v}\right)^2}$$

$$=\frac{u}{\sqrt{u^2-4v}}$$

$$JJ^1 = x - y\frac{1}{\sqrt{u^2-4v}} = \frac{\sqrt{u^2-4v}}{\sqrt{u^2-4v}} = 1$$

15. $u = x^2 - 2y$, $v = x + y$

Solution:

Given that $u = x + y$, $v = xy$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2$$

$$\frac{\partial v}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = 1$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2 \\ 1 & 1 \end{vmatrix} = 2(x+1)$$

Now, $u = x^2 - 2y$

$$u = x^2 - 2(v-x) \qquad (y = v - x)$$

$$x^2 + 2x - (2v+u) = 0$$

$$x = -1 \pm \sqrt{1+u+2v} \qquad y = v + 1 \mp \sqrt{1+u+2v}. \qquad (y = v - x)$$

Consider, $x = -1 + \sqrt{1+u+2v}$ $\qquad y = v + 1 - \sqrt{1+u+2v}$

$$\frac{\partial x}{\partial u} = \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} \qquad \frac{\partial x}{\partial v} = \frac{1}{\sqrt{1+u+2v}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} \qquad \frac{\partial y}{\partial v} = 1 + \frac{1}{\sqrt{1+u+2v}}$$

$$J^1 = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} & \frac{1}{\sqrt{1+u+2v}} \\ \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} & 1 + \frac{1}{\sqrt{1+u+2v}} \end{vmatrix}$$

$$= \frac{1}{2}\frac{1}{\sqrt{1+u+2v}} - \frac{1}{2}\frac{1}{\left(\sqrt{1+u+2v}\right)^2} + \frac{1}{2}\frac{1}{\left(\sqrt{1+u+2v}\right)^2}$$

$$= \frac{1}{2}\frac{1}{\sqrt{1+u+2v}}$$

$$JJ^1 = 2(x+1)\frac{1}{2}\frac{1}{\sqrt{1+u+2v}} = \frac{\sqrt{1+u+2v}}{\sqrt{1+u+2v}} = 1.$$

16. $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$

Solution:

Given that $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$

$$\frac{\partial u}{\partial x} = 1 - \frac{y^2}{x^2} \quad \frac{\partial u}{\partial y} = \frac{2y}{x}$$

$$\frac{\partial v}{\partial x} = -\frac{y^2}{x^2} \quad \frac{\partial v}{\partial y} = \frac{2y}{x}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-\frac{y^2}{x^2} & \frac{2y}{x} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{2y}{x}$$

Now, $u = x + v$ $\quad \left(v = \frac{y^2}{x} \right)$

$x = u - v$ $\quad y^2 = vx.$

$$y^2 = v(u-v)$$

$$y^2 = vu - v^2$$

$$y = \sqrt{vu - v^2}$$

Then,

$$\frac{\partial x}{\partial u} = 1 \quad \frac{\partial x}{\partial v} = -1$$

$$\frac{\partial y}{\partial u} = \frac{v}{2\sqrt{uv-v^2}} \quad \frac{\partial y}{\partial v} = \frac{u-2v}{2\sqrt{uv-v^2}}$$

$$J^1 = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ \frac{v}{2\sqrt{uv-v^2}} & \frac{u-2v}{2\sqrt{uv-v^2}} \end{vmatrix}$$

$$= \frac{u-2v}{2\sqrt{uv-v^2}} + \frac{v}{2\sqrt{uv-v^2}}$$

$$= \frac{u-v}{2\sqrt{uv-v^2}}$$

$$JJ^1 = \frac{2y}{x} \frac{u-v}{2\sqrt{uv-v^2}} = \frac{2y}{x} \frac{x}{2y} = 1. \quad \begin{pmatrix} u - v = x \\ \sqrt{uv-v^2} = y \end{pmatrix}$$

17. $u=\sqrt{xy}$, $v=\sqrt{\dfrac{y}{x}}$

Solution:

Given that $u=\sqrt{xy}$, $v=\sqrt{\dfrac{y}{x}}$

$$\frac{\partial u}{\partial x}=\frac{y}{2\sqrt{xy}} \quad \frac{\partial u}{\partial y}=\frac{x}{2\sqrt{xy}}$$

$$\frac{\partial v}{\partial x}=\frac{1}{2\sqrt{xy}} \quad \frac{\partial v}{\partial y}=-\frac{\sqrt{x}}{2y\sqrt{y}}$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\dfrac{y}{2\sqrt{xy}} & \dfrac{x}{2\sqrt{xy}}\\ \dfrac{1}{2\sqrt{xy}} & -\dfrac{\sqrt{x}}{2y\sqrt{y}}\end{vmatrix}=\left(\frac{-y}{1\sqrt{xy}}\right)\frac{\sqrt{x}}{2y\sqrt{y}}-\frac{x}{2\sqrt{xy}}\frac{1}{2\sqrt{xy}}=-\frac{1}{4y}-\frac{1}{4y}=-\frac{1}{2y}$$

Now, $u=\sqrt{xy}$ $\quad v=\sqrt{\dfrac{x}{y}}$

$$uv=\sqrt{x}\sqrt{y}\frac{\sqrt{x}}{\sqrt{y}} \qquad y^2=vx.$$

$$uv=x \qquad \sqrt{y}=\frac{\sqrt{x}}{v}$$

$$\sqrt{y}=\frac{\sqrt{uv}}{v}$$

$$\sqrt{y}=\frac{\sqrt{u}\sqrt{v}}{v}$$

$$\sqrt{y}=\frac{\sqrt{u}}{\sqrt{v}}$$

$$y=\frac{u}{v}$$

Then,

$$\frac{\partial x}{\partial u}=v \quad \frac{\partial x}{\partial v}=u$$

$$\frac{\partial y}{\partial u}=\frac{1}{v} \quad \frac{\partial y}{\partial v}=-\frac{u}{v^2}$$

$$J^{1}=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2}\end{vmatrix}$$

$$=-\frac{u}{v}-\frac{u}{v}$$

$$=-\frac{2u}{v}$$

$$JJ^1=\left(-\frac{1}{2y}\right)\left(-\frac{2u}{v}\right)=\frac{1}{y}y=1. \qquad \left(\frac{u}{v}=y\right)$$

18. $u=\sqrt{x^2+y^2}$, $v=\sqrt{x^2-y^2}$

Solution:

Given that $u=\sqrt{x^2+y^2}$, $v=\sqrt{x^2-y^2}$

$$\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial x}=\frac{x}{\sqrt{x^2-y^2}} \qquad \frac{\partial v}{\partial y}=-\frac{y}{\sqrt{x^2-y^2}}$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2-y^2}} & -\frac{y}{\sqrt{x^2-y^2}}\end{vmatrix}=-\frac{xy}{\sqrt{(x^2+y^2)(x^2-y^2)}}-\frac{xy}{\sqrt{(x^2+y^2)(x^2-y^2)}}$$

$$=-\frac{2xy}{\sqrt{x^4-y^4}}$$

Now, $u=\sqrt{x^2+y^2}$, $v=\sqrt{x^2-y^2}$

$$u^2+v^2=x^2+y^2+x^2-y^2 \qquad u^2-v^2=x^2+y^2-x^2+y^2.$$

$$x^2=\frac{u^2+v^2}{2} \qquad y^2=\frac{u^2-v^2}{2}$$

Then,

$$2x\frac{\partial x}{\partial u}=\frac{1}{2}2u \qquad 2x\frac{\partial x}{\partial v}=\frac{1}{2}2v$$

$$\frac{\partial x}{\partial u}=\frac{u}{\sqrt{2}\sqrt{u^2+v^2}} \qquad \frac{\partial x}{\partial v}=\frac{v}{\sqrt{2}\sqrt{u^2+v^2}}$$

$$2y\frac{\partial y}{\partial u}=\frac{1}{2}2u \qquad 2y\frac{\partial y}{\partial v}=\frac{1}{2}(-2v)$$

$$\frac{\partial y}{\partial u}=\frac{u}{\sqrt{2}\sqrt{u^2-v^2}} \qquad \frac{\partial y}{\partial u}=-\frac{v}{\sqrt{2}\sqrt{u^2-v^2}}$$

$$J^1=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \dfrac{u}{\sqrt{2}\sqrt{u^2+v^2}} & \dfrac{v}{\sqrt{2}\sqrt{u^2+v^2}} \\ \dfrac{u}{\sqrt{2}\sqrt{u^2-v^2}} & -\dfrac{v}{\sqrt{2}\sqrt{u^2-v^2}} \end{vmatrix}$$

$$=-\frac{uv}{2\sqrt{u^4-v^4}}-\frac{uv}{2\sqrt{u^4-v^4}}$$

$$=-\frac{uv}{\sqrt{u^4-v^4}}$$

$$JJ^1=\left(-\frac{2xy}{\sqrt{x^4-y^4}}\right)\left(-\frac{uv}{\sqrt{u^4-v^4}}\right)=\frac{2xy}{\sqrt{x^4-y^4}}\frac{\sqrt{x^4-y^4}}{2xy}=1.$$

$$\begin{pmatrix} uv=\sqrt{x^4-y^4} \\ 2xy=\sqrt{u^4-v^4} \end{pmatrix}$$

19. $x=e^u\cos v$, $y=e^u\sin v$

Solution:

Given that $x=e^u\cos v$, $y=e^u\sin v$

$$\frac{\partial x}{\partial u}=e^u\cos v \qquad \frac{\partial x}{\partial v}=-e^u\sin v$$

$$\frac{\partial y}{\partial u}=e^u\sin v \qquad \frac{\partial y}{\partial v}=e^u\cos v$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} e^u\cos v & -e^u\sin v \\ e^u\sin v & e^u\cos v \end{vmatrix}$$

$$=e^{2u}\cos^2 v+e^{2v}\sin^2 v$$

$$=e^{2u}$$

Now, $x=e^u\cos v$, $y=e^u\sin v$

$$\frac{y}{x}=\tan v, \qquad x^2+y^2=e^{2u}.$$

$v=\tan^{-1}\dfrac{y}{x}$ $\qquad\qquad u=\dfrac{1}{2}\ln(x^2+y^2)$

Then,

$$\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x}=-\frac{-y}{x^2+y^2} \qquad \frac{\partial v}{\partial y}=\frac{x}{x^2+y^2}$$

$$J^1=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{x}{x^2+y^2} & \dfrac{y}{x^2+y^2}\\ -\dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2}\end{vmatrix}$$

$$=\frac{x^2}{\left(x^2+y^2\right)^2}+\frac{y^2}{\left(x^2+y^2\right)^2}$$

$$=\frac{1}{x^2+y^2}$$

$$JJ^1=e^{2u}\,\frac{1}{x^2+y^2}=\frac{x^2+y^2}{x^2+y^2}=1. \qquad\qquad \left(x^2+y^2=e^{2u}\right)$$

Prove the following

20. **If** $x=u(1+v)$ **,** $y=v(1+u)$ **then** $\dfrac{\partial(u,v)}{\partial(x,y)}=1+u+v$ **.**

Solution:

$x=u+uv$, $\qquad\qquad y=v+uv$

$$\frac{\partial x}{\partial u}=1+v \qquad \frac{\partial x}{\partial v}=u$$

$$\frac{\partial y}{\partial u}=v \qquad \frac{\partial y}{\partial v}=1+u$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1+v & u\\ v & 1+u\end{vmatrix}$$

$$=(1+v)(1+u)-uv$$

$$=1+u+v+uv-uv$$

$$=1+u+v$$

21. **If** $x = \frac{u^2}{v}$ **,** $y = \frac{v^2}{u}$ **then** $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{3}$ **.**

Solution:

$$x = \frac{u^2}{v}, \qquad y = \frac{v^2}{u}$$

$$v = \frac{u^2}{x} \qquad u = \frac{v^2}{y}$$

$$v = \frac{v^4}{y^2 x} \qquad u = \frac{u^4}{x^2 y}$$

$$v^3 = xy^2 \qquad u^3 = x^2 y \qquad \begin{pmatrix} u^3v^3 = x^3y^3 \\ \Rightarrow uv = xy \end{pmatrix}$$

$$3u^2 \frac{\partial u}{\partial x} = 2xy \qquad 3u^2 \frac{\partial u}{\partial y} = x^2$$

$$\frac{\partial u}{\partial x} = \frac{2xy}{3u^2} \qquad \frac{\partial u}{\partial y} = \frac{x^2}{3u^2}$$

$$3v^2 \frac{\partial v}{\partial x} = y^2 \qquad 3v^2 \frac{\partial v}{\partial y} = 2xy$$

$$\frac{\partial v}{\partial x} = \frac{y^2}{3v^2} \qquad \frac{\partial v}{\partial y} = \frac{2xy}{3v^2}$$

$$J^1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2xy}{3u^2} & \frac{x^2}{3u^2} \\ \frac{y^2}{3v^2} & \frac{2xy}{3v^2} \end{vmatrix}$$

$$= \frac{4x^2y^2}{9u^2v^2} - \frac{x^2y^2}{9u^2v^2}$$

$$= \frac{3x^2y^2}{9u^2v^2}$$

$$= \frac{1}{3} \qquad (\because uv = xy)$$

Prove the following

22.

If $u = 2axy$ **and** $v = a(x^2 - y^2)$ **where** $x = r\cos\theta$ **and** $y = r\sin\theta$ **then** $\frac{\partial(u,v)}{\partial(r,\theta)} = -4a^2r^3$ **.**

Solution:

Given $u = 2axy$ and $v = a(x^2 - y^2)$

$u = 2ar^2 \sin\theta \cos\theta$ $u = a\left(r^2 \cos^2\theta - r^2 \sin^2\theta\right)$

$u = ar^2 \sin 2\theta$ $u = ar^2 \cos 2\theta$

$$\frac{\partial u}{\partial r} = 2ar \sin 2\theta \quad \frac{\partial u}{\partial \theta} = 2ar^2 \cos 2\theta$$

$$\frac{\partial v}{\partial x} = 2ar \cos 2\theta \quad \frac{\partial v}{\partial y} = -2ar^2 \sin 2\theta$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2ar\sin 2\theta & 2ar^2 \cos 2\theta \\ 2ar\cos 2\theta & -2ar^2 \sin 2\theta \end{vmatrix}$$

$$= -4a^2r^3 \sin^2 2\theta - 4a^2r^3 \cos^2 2\theta$$

$$= -4a^2r^3$$

23.

If $u = x^2 - 2y^2$ **and** $v = 2x^2 - y^2$ **where** $x = r\cos\theta$ **and** $y = r\sin\theta$ **then** $\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$ **.**

Solution:

Given $u = x^2 - 2y^2$ and $v = 2x^2 - y^2$

$u = r^2 \cos^2\theta - 2r^2 \sin^2\theta$ $u = 2r^2 \cos^2\theta - r^2 \sin^2\theta$

$u = ar^2 \sin 2\theta$ $u = ar^2 \cos 2\theta$

$$\frac{\partial u}{\partial r} = 2r\cos^2\theta - 4r\sin^2\theta \quad \frac{\partial u}{\partial \theta} = -2r^2 \cos\theta \sin\theta - 4r^2 \sin\theta\cos\theta$$

$$\frac{\partial v}{\partial x} = 4r\cos^2\theta - 2r\sin^2\theta \quad \frac{\partial v}{\partial y} = -4r^2 \cos\theta\sin\theta - 2r^2 \sin\theta\cos\theta$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2r\cos^2\theta - 4r\sin^2\theta & -2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta \\ 4r\cos^2\theta - 2r\sin^2\theta & -4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta \end{vmatrix}$$

$$= \left(2r\cos^2\theta - 4r\sin^2\theta\right)\left(-4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta\right)$$
$$-\left(-2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta\right)\left(4r\cos^2\theta - 2r\sin^2\theta\right)$$

$$= -8r^3\cos^3\theta\sin\theta + 16r^3\cos\theta\sin^3\theta - 4r^3\sin\theta\cos^3\theta + 8r^3\sin^3\theta\cos\theta$$
$$+8r^3\cos^3\theta\sin\theta - 4r^3\sin^3\theta\cos\theta + 16r^3\cos^3\theta\sin\theta - 8r^3\sin^3\theta\cos\theta$$

$$= 16r^3\cos\theta\sin\theta(\sin^2\theta + \cos^2\theta) - 4r^3\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$= 4r^3\sin\theta\cos\theta(4-1)$$

$$= 6r^3\sin 2\theta .$$

In each of the following cases, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

24. $u = xy^2$ **,** $v = yz^2$, $w = zx^2$

Solution:

Given $u = xy^2$ **,** $v = yz^2$, $w = zx^2$

$$\frac{\partial u}{\partial x} = y^2 \quad \frac{\partial u}{\partial y} = 2xy \quad \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = z^2 \quad \frac{\partial v}{\partial z} = 2yz$$

$$\frac{\partial w}{\partial x} = 2xz \quad \frac{\partial w}{\partial y} = 0 \quad \frac{\partial w}{\partial z} = x^2$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2xz & 0 & x^2 \end{vmatrix}$$

$$= y^2(x^2z^2) + 2xz(4xy^2z)$$

$$= 9x^2y^2z^2 .$$

$$= (3xyz)^2$$

25. $u = x(1-r^2)^{-\frac{1}{2}}$ **,** $v = y(1-r^2)^{-\frac{1}{2}}$ **,** $w = z(1-r^2)^{-\frac{1}{2}}$ **, where** $r^2 = x^2 + y^2 + z^2$ **.**

Solution:

Given $u = x(1-r^2)^{-\frac{1}{2}}$ **,** $v = y(1-r^2)^{-\frac{1}{2}}$, $w = z(1-r^2)^{-\frac{1}{2}}$

$u = x(1-x^2-y^2-z^2)^{-\frac{1}{2}}$ **,**

$$\frac{\partial u}{\partial x} = (1-x^2-y^2-z^2)^{-\frac{1}{2}} + x^2(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial x} = xy(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial w}{\partial x} = xz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$v = y(1-x^2-y^2-z^2)^{-\frac{1}{2}}$$

$$\frac{\partial u}{\partial y} = xy(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial y} = (1-x^2-y^2-z^2)^{-\frac{1}{2}} + y^2(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial w}{\partial y} = yz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$w = z(1-x^2-y^2-z^2)^{-\frac{1}{2}}$$

$$\frac{\partial u}{\partial z} = xz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial v}{\partial z} = yz(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$\frac{\partial w}{\partial z} = (1-x^2-y^2-z^2)^{-\frac{1}{2}} + z^2(1-x^2-y^2-z^2)^{-\frac{3}{2}}$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} (1-r^2)^{-\frac{1}{2}} + x^2(1-r^2)^{-\frac{3}{2}} & xy(1-r^2)^{-\frac{3}{2}} & xz(1-r^2)^{-\frac{3}{2}} \\ xy(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{1}{2}} + y^2(1-r^2)^{-\frac{3}{2}} & yz(1-r^2)^{-\frac{3}{2}} \\ xz(1-r^2)^{-\frac{3}{2}} & yz(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{1}{2}} + z^2(1-r^2)^{-\frac{3}{2}} \end{vmatrix}$$

$$= \begin{vmatrix} (1-r^2)^{-\frac{3}{2}}(1-y^2-z^2) & xy(1-r^2)^{-\frac{3}{2}} & xz(1-r^2)^{-\frac{3}{2}} \\ xy(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{3}{2}}(1-x^2-z^2) & yz(1-r^2)^{-\frac{3}{2}} \\ xz(1-r^2)^{-\frac{3}{2}} & yz(1-r^2)^{-\frac{3}{2}} & (1-r^2)^{-\frac{3}{2}}(1-y^2-x^2) \end{vmatrix}$$

$$=\frac{(1-r^2)^{-\frac{9}{2}}}{x}\begin{vmatrix} x(1-y^2-z^2) & xxy & xxz \\ xy & (1-x^2-z^2) & yz \\ xz & yz & (1-y^2-x^2) \end{vmatrix}$$

$$=\frac{(1-r^2)^{-\frac{9}{2}}}{x}\begin{vmatrix} x & y & z \\ xy & \dfrac{y(1-x^2-z^2)}{y} & yz \\ xz & yz & \dfrac{z(1-y^2-x^2)}{z} \end{vmatrix} \quad R_1^1 = R_1 + yR_2 + zR_3$$

$$=\frac{(1-r^2)^{-\frac{9}{2}}}{x}xyz\begin{vmatrix} 1 & 1 & 1 \\ y & \dfrac{(1-x^2-z^2)}{y} & z \\ z & z & \dfrac{(1-y^2-x^2)}{z} \end{vmatrix} \quad R_1^1 = R_1 + yR_2 + zR_3$$

$$=(1-r^2)^{-\frac{9}{2}}yz\begin{vmatrix} 1 & 0 & 0 \\ y & \dfrac{(1-x^2-z^2)}{y}-y & 0 \\ z & 0 & \dfrac{(1-y^2-x^2)}{z}-z \end{vmatrix}$$

$$=(1-r^2)^{-\frac{9}{2}}yz\left(\left(\frac{1-z^2-x^2-y^2}{y}\right)\left(\frac{1-z^2-x^2-y^2}{y}\right)\right)$$

$$=(1-r^2)^{-\frac{9}{2}}(1-r^2)(1-r^2)$$

$$=(1-r^2)^{-\frac{5}{2}}$$

In each of the following cases find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

26. $x=u$, $y=u\tan v$, $z=w$

Solution:

Given $x=u \quad y=u\tan v \quad z=w$

$$\frac{\partial x}{\partial u}=1 \qquad \frac{\partial x}{\partial v}=0 \qquad \frac{\partial x}{\partial w}=0$$

$$\frac{\partial y}{\partial u}=\tan v \qquad \frac{\partial y}{\partial v}=u\sec^2 v \qquad \frac{\partial y}{\partial w}=0$$

$$\frac{\partial z}{\partial u}=0 \qquad \frac{\partial z}{\partial v}=0 \qquad \frac{\partial z}{\partial w}=1$$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=u\sec^2 v$$

27. $x=u(1-v)$, $y=uv(1-w)$, $z=uvw$

Solution:

Given $x=u(1-v)$, $y=uv(1-w)$, $z=uvw$

$$\frac{\partial x}{\partial u}=1-v \qquad \frac{\partial x}{\partial v}=-u \qquad \frac{\partial x}{\partial w}=0$$

$$\frac{\partial y}{\partial u}=v(1-w) \qquad \frac{\partial y}{\partial v}=u(1-w) \qquad \frac{\partial y}{\partial w}=-uv$$

$$\frac{\partial z}{\partial u}=vw \qquad \frac{\partial z}{\partial v}=uw \qquad \frac{\partial z}{\partial w}=uv$$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$=(1-v)\left[u^2v(1-w)+u^2vw\right]+u\left[uv^2(1-w)+uv^2w\right]$$

$$=(1-v)\left[u^2v-u^2vw+u^2vw\right]+u^2v^2(1-w)+u^2v^2w$$

$$=(1-v)\left[u^2v-u^2vw+u^2vw\right]+u^2v^2(1-w)+u^2v^2w$$

$= u^2v - u^2v^2 + u^2v^2$

$= u^2v$

28. **If** $u = x+y+z$ **,** $uv = y+z$ **,** $z = uvw$ **then show that** $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$ **.**

Solution:

Given $u = x+y+z$ **,** $uv = y+z$ **,** $z = uvw$

$x = u - uv$ **,** $y = uv - uvw$ **,** $z = uvw$

$$\frac{\partial x}{\partial u} = 1-v \qquad \frac{\partial x}{\partial v} = -u \qquad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v(1-w) \qquad \frac{\partial y}{\partial v} = u(1-w) \qquad \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw \qquad \frac{\partial z}{\partial v} = uw \qquad \frac{\partial z}{\partial w} = uv$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[u^2v(1-w) + u^2vw\right] + u\left[uv^2(1-w) + uv^2w\right]$$

$$= (1-v)\left[u^2v - u^2vw + u^2vw\right] + u^2v^2(1-w) + u^2v^2w$$

$$= (1-v)\left[u^2v - u^2vw + u^2vw\right] + u^2v^2(1-w) + u^2v^2w$$

$$= u^2v - u^2v^2 + u^2v^2$$

$$= u^2v$$

Functional Dependence:

29. Using Jacobians, prove that $u = x+y$ and $v = \frac{1}{x+y}$ are functionally dependent.

Solution:

Given $u = x+y$ and $v = \frac{1}{x+y}$

$$\frac{\partial u}{\partial x}=1 \qquad \frac{\partial u}{\partial y}=1$$

$$\frac{\partial v}{\partial x}=-\frac{1}{(x+y)^2} \quad \frac{\partial v}{\partial y}=-\frac{1}{(x+y)^2}$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} 1 & 1 \\ -\frac{1}{(x+y)^2} & -\frac{1}{(x+y)^2} \end{vmatrix}$$

$$=-\frac{1}{(x+y)^2}+\frac{1}{(x+y)^2}$$

$$=0$$

Implies that u and v are functionally dependent.

$v=\frac{1}{u}$.

30. Prove that the functions $x=u^2-v^2$, $y=v^2-w^2$, $z=w^2-u^2$ are functionally depend ent.

Solution:

Given $x=u^2-v^2$, $y=v^2-w^2$, $z=w^2-u^2$

$$\frac{\partial x}{\partial u}=2u \quad \frac{\partial x}{\partial v}=-2v \quad \frac{\partial x}{\partial w}=0$$

$$\frac{\partial y}{\partial u}=0 \quad \frac{\partial y}{\partial v}=2v \quad \frac{\partial y}{\partial w}=-2w$$

$$\frac{\partial z}{\partial u}=-2u \quad \frac{\partial z}{\partial v}=0 \quad \frac{\partial z}{\partial w}=2w$$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 2u & -2v & 0 \\ 0 & 2v & -2w \\ -2u & 0 & 2w \end{vmatrix}$$

$$=2u(4vw)+2v(-4uw)$$

$$=8uvw-8uvw$$

$$=0$$

This implies that the functions are functionally dependent and is connected by the relation

$$x+y+z=u^2-v^2+v^2-w^2+w^2-u^2$$

$$x+y+z=0.$$

Maxima and Minima of Two Variables

1\.

If f(x, y) is a two variable function having extreme value. If it has minima (or) maximum for calculating the value we have to consider $p=\frac{\partial f}{\partial x}=0$, $q=\frac{\partial f}{\partial y}=0$ by this two partial derivatives we can get stationary points (a_1, b_1) (a_2, b_2), (a_3, b_3)….

2\.

At stationary points (a_1, b_1), (a_2, b_2), (a_3, b_3) we can calculate $r=\frac{\partial^2 f}{\partial x^2}$ $t=\frac{\partial^2 f}{\partial y^2}$ $s=\frac{\partial^2 f}{\partial x\partial y}$.

3\. From the above five partial derivatives we have to find $rt-s^2$ value.

4\. If $rt-s^2>0$ and $r<0$ given f has maximum value at (a,b).

5\. If $rt-s^2>0$ and $r>0$ given f has minimum value at (a,b).

6\. If $rt-s^2<0$ then f has no extreme value at this point saddle point exists.

7\.

If $rt-s^2=0$ then f has failed to have maximum or minimum. In this case it needs further investigation.

Problems:

31. Find the maximum and minimum values of the function $f(x,y)=x^3+y^3-3axy$

Sol: Let $f(x,y)=x^3+y^3-3axy$

$$\frac{\partial f}{\partial x}=3x^2-3ay=0 \Rightarrow 3x^2-3ay=0; \quad \frac{\partial f}{\partial y}=0 \Rightarrow 3y^2-3ax=0$$

From (1) $3x^2 = 3ay$ Sub in (2)

$$y=\frac{x^2}{a} \qquad 3\left(\frac{x^2}{a}\right)^2-3ax=0$$

$$\frac{3x^4}{a^2}-3ax=0 \qquad 3x^4-3a^3x=0 \qquad 3x(x^3-a^3)=0$$

$x = a$ $x = 0$

Corresponding values of y are $y = a$ $y = 0$

Stationary points are (0, 0), (a, a)

At the point (x, y) = (a, a) :

$$\frac{\partial^2 f}{\partial x^2}=6x \Rightarrow r=6x, \quad \frac{\partial^2 f}{\partial y^2}=6y \Rightarrow t=6y, \quad \frac{\partial^2 f}{\partial x\partial y}=-3a \Rightarrow s=-3a$$

at (a, a),

$$rt - s^2 = (6x)\ (6y)\ -\ (-3a)^2 = 36\ xy\ -\ 9a^2 = 36\ a^2 -\ 9a^2 = 27\ a^2 >\ 0$$

f has minimum value at (a, a)

At the point (x, y) = (0, 0)

$$\frac{\partial^2 f}{\partial x^2} = r = 0,\quad \frac{\partial^2 f}{\partial y^2} = 6(0) = 0,\quad \frac{\partial^2 f}{\partial x \partial y} = -3a$$

$rt - s^2 = 0 - (-3a)^2 = -9a^2 < 0 \Rightarrow r\,t - s^2 < 0$

f has no extreme value then there exist saddle point..

32. Find the maxima and minima values of $x^2 + y^2 + 6x + 12$

Sol: Let $f(x) = x^2 + y^2 + 6x + 12$

$$p = \frac{\partial f}{\partial x} = 2x + 6,\quad q = \frac{\partial f}{\partial y} = 2y$$

Consider $\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 6 = 0 \Rightarrow x = -3,\quad \frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

Stationary point is (-3, 0)

At (-3, 0) $\frac{\partial^2 f}{\partial x^2} = r = 2,\quad \frac{\partial^2 f}{\partial y^2} = 2 = t,\quad s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$$rt - s^2 = (2)\ ((2)\ -\ (0))^2 = 4 > 0, \text{ and } r = 2 > 0$$

f has minimum value at (-3, 0).

33.

Examine the functions for extreme values

$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2, (x > 0, y > 0)$.

Sol: $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$p = \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y,\quad q = \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

Consider

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0,\quad \frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0$$

By solving these two equations we have stationary points (0, 0), $(-\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$.

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 4 \quad t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

then $rt - s^2 = (12x^2 - 4)(12y^2 - 4) - (4)^2 = 144\,x^2y^2 - 48x^2 - 48y^2 + 16 - 16$

$rt - s^2 = 144\,x^2y^2 - 48x^2 - 48y^2$

Maxima or Minima at (0,0)

If $rt - s^2 = 0$

f has fail to have maxima (or) minima at (0, 0) in this case it needs further inves tigation.

Maxima or Minima at $\left(-\sqrt{2}, \sqrt{2}\right)$

$rt - s^2 = 144\left(-\sqrt{2}\right)^2 - 48\left(-\sqrt{2}\right)^2 - 48\left(-\sqrt{2}\right)^2 = 144(2)(2) - 48(2) - 48(2) = 384 > 0$

$r = 12x^2 - 4 = 12\left(-\sqrt{2}\right)^2 - 4 = 24 - 4 = 20 > 0$

$rt - s^2 > 0$ and $r > 0$

f has minima at $\left(-\sqrt{2}, \sqrt{2}\right)$.

Maxima or Minima at $\left(\sqrt{2}, -\sqrt{2}\right)$

$rt - s^2 = 144(\left(\sqrt{2}\right)^2 - \left(-\sqrt{2}\right)^2) - 48\left(-\sqrt{2}\right)^2 - 48\left(-\sqrt{2}\right)^2 = 144(4) - 48(2) - 48(2) = 384 > 0$

$r = 12x^2 - 4 = 12\left(\sqrt{2}\right)^2 - 4 = 20 > 0$

f has minima at $\left(\sqrt{2}, -\sqrt{2}\right)$.

34. Examine for minima and maxima values of Sin x + Sin y + Sin (x + y)

Sol:- Let f (x, y) = Sin x + Sin y + Sin (x + y)

$$\frac{\partial f}{\partial x} = cos\ x + cos\ (x+y), \qquad \frac{\partial f}{\partial y} = cos\ y + cos\ (x+y)$$

Let we consider $\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

Cos x + Cos (x+y) =0, Cos y + Cos (x+y) = 0

Cos x = Cos y

x = y

$\frac{\partial f}{\partial x} = \quad cos\, x + cos\,(x+x) = 0 \Rightarrow$ Cos x + Cos 2x = 0 $\Rightarrow$ 2 Cos

$$2\cos\left(\frac{3x}{2}\right) cos\left(\frac{x}{2}\right) = 0$$

$\Rightarrow 2\ \cos\left(\frac{3x}{2}\right)(or) \quad Cos\left(\frac{x}{2}\right) = 0 \Rightarrow cos\left(\frac{3x}{2}\right) = 0 \quad \frac{x}{2} = cos^{-1}(0)$

$\Rightarrow \frac{3x}{2} = cos^{-1}(0) \quad \frac{x}{2} = \pm\frac{\pi}{2} \Rightarrow \quad \frac{3x}{2} = \pm\frac{\pi}{2} \qquad x = \pm\pi, \quad x = \pm\frac{\pi}{3}$

$x = \pm\frac{\pi}{3}\ (or)\ x = \pm\pi \Rightarrow$ Corresponding values of $y = \pm\frac{\pi}{3}\ (or)\ y = \pm\pi$

Stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), \left(-\frac{\pi}{3}, -\frac{\pi}{3}\right), (\pi, \pi), (-\pi, -\pi)$

Maxima or Minima at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$r = \frac{\partial^2 f}{\partial x^2} = -sin\, x - sin(x+y), \qquad t = \frac{\partial^2 f}{\partial y^2} = -sin\, y - sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -sin(x+y)$$

$$r\,t - s^2 = -\left(sin\ x\ +\ sin\ (x+y)\right)\ \left[-\left(sin\, y\ +\ sin\ (x\ +\ y)\right)\right] = \ -\left(-sin\ (x+y)\right)^2$$

At the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$rt - s^2 = +\left[sin\left(\frac{\pi}{3}\right) + sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)\right]\left(sin\frac{\pi}{3} + \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)\right) - \left[-\left(sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)\right)\right]^2$$

$$= \left[\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right]\left[\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right] - \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\sqrt{3}\right)\left(\sqrt{3}\right) - \frac{3}{4} = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$rt - s^2 > 0$, and $r < 0$

f has maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Maxima (or) Minima at $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$

Here $rt - s^2 > 0$, and $r > 0$ f has minimum at $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$

Maxima (or) Minima at $(\pm\pi, \pm\pi)$

At $(\pm\pi, \pm\pi)$ $rt - s^2 = [\text{Sin}(\pi) + \text{Sin}(\pi+\pi)] = [\text{Sin}(\pi) + \text{Sin}(\pi+\pi) - (-(\text{Sin}(\pi+\pi)^2 = 0$

f has failed to have maxima or minima at $(\pm\pi, \pm\pi)$ it needs further inves tigation.

35.

A rectangular box open at the top is to have volume of 32 cubic feet, find the dime nsions of the box requiring least material for its construction (volume v = xyz, surface area S = xy + 2 yz = 2zx

Sol:- Let x feet, y feet, z feet are the dimensions of rectangular box

Volume of the rectangular box v = x y z = 32 cubic feets.

$\Rightarrow$ Surface of the rectangular box s = x y + 2 (z x + z y)

V = x y z = 32 $\Rightarrow z = \dfrac{32}{xy}$

$$S = xy + z\left(\frac{32}{xy}\right)x + 2\left(\frac{32}{xy}\right)y = xy + \frac{64}{y} + \frac{64}{x}$$

$$\frac{\partial S}{\partial x} = y + 64\left(-\frac{1}{x^2}\right) = y - \frac{64}{x^2} \Rightarrow p = 0 \Rightarrow y - \frac{64}{x^2} = 0$$

$$\frac{\partial S}{\partial y} = x + 64\left(-\frac{1}{y^2}\right) = x - \frac{64}{y^2} \Rightarrow q = 0 \Rightarrow x - \frac{64}{y^2} = 0.$$

Solve the above equations, we get Stationary point is (4, 4).

Maxima (or) Minima at (4, 4)

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 s}{\partial x^2} = -64\left(\frac{-2}{x^3}\right) = \frac{128}{x^3},\ t = \frac{\partial^2 s}{\partial y^2} = -64\left(-\frac{2}{x^3}\right) = \frac{128}{y^3},\ s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 s}{\partial x \partial y} = 1$$

At (4, 4) $rt - s^2 = \left(\frac{128}{x^3}\right)\left(\frac{128}{y^3}\right) - 1 = \left(\frac{128}{64}\right)\left(\frac{128}{64}\right) - 1$ = (2) (2) – 1= 3 > 0

$$r = \frac{128}{x^3} = \frac{128}{64} = 2 > 0\ .$$

If $rt - s^2 > 0$ and $r > 0$ then S has minimum at (4, 4)

$$x = 4, \quad y = 4 \;, \; z = \frac{32}{xy} = \frac{32}{16} = 2$$

The dimensions of the rectangular box are $x = 4\, ft,\; y = 4\, ft,\; z = 2\, ft$.

Exercise:

36. Find the minimum value of $x^2 + y^2 + z^2$ given that $xyz = a^3$.

37. Examine the function for extreme values
$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 \quad (x > 0,\ y > 0)$.

38. Show that the function $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$ is maximum at $(-7, 7)$ and minimum at $(3, 3)$.

Lagranges Method of Undetermined Multipliers

Let us take two functions $f(x, y, z) = 0 \rightarrow (1)$ and $\phi(x, y, z) = 0 \rightarrow (2)$ then the Lagr ange equation can be written as $F(x, y, z) = 0 \Rightarrow f(x, y, z) + \lambda\, \phi(x, y, z) = 0 \rightarrow (3)$ gives the maximum or minimum values of a given function. This equation is defined as Lagranges equation.

Hence, λ is denoted by Lagrange multiplier. For solving these equations we have t he following equations :

$$\frac{\partial F}{\partial x} = 0 \text{ i.e., } \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial y} = 0 \qquad (4)$$

$$\frac{\partial F}{\partial y} = 0 \text{ i.e., } \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \qquad (5)$$

$$\frac{\partial F}{\partial z} = 0 \text{ i.e., } \frac{\partial f}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0 \qquad (6)$$

From Equations (4), (5) and (6) we get the value of x,y,z

Then the function $f(x, y, z) = 0.$ gives the maximum or minimum value.

Problems:

39. Find the minimum value of the function $x^2 + y^2 + z^2$ if $xyz = a^3$.

Sol:-

Given that $f(x, y, z) = x^2 + y^2 + z^2$ Subject to the condition $\phi(x, y, z) = 0 \Rightarrow xyz - a^3 = 0$

Lagrange's function is $F = f(x, y, z) + \lambda\phi(x, y, z)$

$$F = x^2 + y^2 + z^2 + \lambda\,(xyz = -a^3) = 0$$

To find maxima or minima of the function we consider

$$\frac{\partial F}{\partial x} = 0\,, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial x} = 2x + \; yz = 0 \qquad (1)$$

$$\frac{\partial F}{\partial y} = 2y + \; xz = 0 \qquad (2)$$

$$\frac{\partial F}{\partial z} = 2z + \; xy = 0 \qquad (3)$$

To get extreme value set

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

From (1, 2, 3)

$$\frac{x}{yz} = -\frac{\lambda}{2}; \quad \frac{y}{xz} = -\frac{\lambda}{2}; \quad \frac{z}{xy} = -\frac{\lambda}{2} \Rightarrow \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}$$

$$\frac{x}{yz} = \frac{y}{xz} \Rightarrow \quad x^2 = y^2 \Rightarrow x^2 = z^2 \Rightarrow \quad x^2 = y^2 = z^2 \Rightarrow x = y = z$$

Substitute in $\phi(x, y, z) = x\,y\,z = a^3 \Rightarrow (x)\,(x)\,(x) = a^3 \Rightarrow x = a, \; y = a, \; z = a$

i.e. Minimum value of $f = a^2 + a^2 + a^2 = 3a^2$

40. Find the minimum value of the function $x^2 + y^2 + z^2$ if $x + y + z = 3a$.

Sol:-

Given that

$f(x, y, z) = x^2 + y^2 + z^2$ Subject to the condition $x + y + z - 3a = 0$

The Lagrangian function is $F = f(x, y, z) + \lambda\phi(x, y, z)$

$F = x^2 + y^2 + z^2 + \lambda(x + y + z - 3a)$

To get extreme value we consider

$$\frac{\partial F}{\partial x} = 0; \quad \frac{\partial F}{\partial y} = 0; \quad \frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial x} = 2x + \lambda = 0 \qquad (1)$$

$$\frac{\partial F}{\partial y} = 2y + \lambda = 0 \qquad (2)$$

$$\frac{\partial F}{\partial z} = 2z + \lambda = 0 \qquad (3)$$

Solving 1,2,3 we get

$x = -\lambda/2$ (4)

$y = -\lambda/2$ (5)

$z = -\lambda/2$ (6)

Sub in $\phi(x, y, z)$

$x + x + x = 3a$

$3x = 3a$

$x = a$

$y = a$

$z = a$

$x = y = z = a$

Minimum Value of $f(x, y, z) = x^2 + y^2 + z^2 = a^2 + a^2 + a^2 = 3a^2$

41. Find the maximum value of the function $x^2\, y^3\, z^4$ when $2x + 3y + 4z = -a$ by method of Lagranges method.

Sol:- Given that

$$f(x,y,z) = x^2y^3z^4$$

Subject to the condition $\phi(x,y,z) = 0$ i.e. $2x + 3y + 4z - a = 0$

The Lagrangian function is

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z) = 0$$

i.e. $F = x^2 y^3 z^4 + \lambda (2x + 3y + 4z - a) = 0$

To get extreme value we consider

$$\frac{\partial F}{\partial x} = 0; \quad \frac{\partial F}{\partial y} = 0; \quad \frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda = 0 \Rightarrow xy^3z^4 = -\lambda$$

$$\frac{\partial F}{\partial y} = 3y^2x^2z^4 + 3\lambda = 0 \Rightarrow x^2y^2z^4 = -\lambda$$

$$\frac{\partial F}{\partial z} = 4z^3x^2y^3 + 4\lambda = 0 \Rightarrow xy^3z^4 = -\lambda$$

From above equations we have $\Rightarrow xy^3z^4 = y^2x^2z^4 = z^3x^2y^3 = -\lambda$

Consider

$xy^3z^4 = y^2x^2z^4 \Rightarrow y = x$

From

$y^2x^2z^4 = z^3x^2y^3 \Rightarrow z = y$

$x = y = z = -\lambda \Rightarrow \quad x = y = z = \frac{a}{9}$

$f(x, y, z) = x^2y^3z^4$

Maximum value of $f_{\left(\frac{a}{9},\frac{a}{9},\frac{a}{9}\right)} = \left(\frac{a}{9}\right)^2 \left(\frac{a}{9}\right)^3 \left(\frac{a}{9}\right)^4 = \frac{a^9}{9^9}$

Exercise:

42. Find the minimum value of the function $x^2 + y^2 + z^2$ when $ax + by + cz = p$.

43. Find the maximum value of the function $x^m y^n z^p$ when $x + y + z = a$.