Module -4: Linear Transformation

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Introduction:

In many branches of mathematics, we use functions with particular properties. For example, single variable calculus is largely concerned with studying functions of one variable that are differentiable. In linear algebra, this sort of function that we study is called linear transformation and the goal of this handout is to explain what a linear transformation is.

In linear algebra we will study functions where the input and output are both vectors. Such functions are often called a linear transformation. Our functions will often be named as T (for “Transformation”), and we will write $T: \mathbb{R}^m \to \mathbb{R}^n$ to mean that the inputs of T are vectors in $\mathbb{R}^m$ and outputs of T are vectors in $\mathbb{R}^n$. We will call $\mathbb{R}^m$, the domain and $\mathbb{R}^n$, then codomain.

<u>Linear Transformation</u>: A linear transformation from a vector space V to another vector space W is a mapping $T: V \to W$ such that for all u and v in V and for all scalars c,

$$\text{I)} T(u+v) = T(u) + T(v)$$
$$\text{II)} T(cu) = cT(u)$$

Note: $T: V \to W$ is a linear transformation if and only if,

$$T(c_1v_1 + c_2v_2 + \cdots + c_kv_k)$$
$$= c_1T(v_1) + c_2T(v_2) + \cdots c_kT(v_k)$$

For all $v_1, v_2, \dots, v_k$ in V and scalar $c_1, c_2, \dots, c_k$

Example 1: Define $T: M_{n\times n} \to M_{n\times n}$ by $T(A) = A^T$. Show that T is a linear transformation.

Solution:

Given that, $T: M_{n\times n} \to M_{n\times n}$ by $T(A) = A^T$.
We have to show that T is a linear transformation.
Suppose for A and B in $M_{n\times n}$ and scalar c.

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

Also $T(cu) = (cu)^T = u^T c^T = cu^T$
Hence T is a linear transformation.

Example 2: Define $S: \tau[\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ by,

$$S(f) = \int_a^b f(x)dx$$

Show that S is a linear transformation.

Solution:

Let f and g be in $\tau[a, b]$ then,

$$S(f+g) = \int_a^b (f+g)(x)dx$$

$$= \int_a^b \big(f(x) + g(x)\big)dx$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$= S(f) + s(g)$$

$$S(cf) = \int_a^b (cf)(x)dx$$

$$= \int_a^b cf(x)dx$$

$$= c\int_a^b f(x)dx$$

$$= cS(f)$$

Hence S is a linear transformation.

Example 3: Show that none of the following transformation is linear.

a) $T: M_{2\times 2} \to \mathbb{R}$ defined by $T(A) = \det A$
b)$T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 2^x$
c) $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$

Solution:

In each case, we give a specific counter example to show that one of the properties of the linear transformation fails to hold.

a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $and\ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Now $T(A + B) = det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$
Again, $T(A) + T(B) = detA + detB = 0 + 0 = 0$
Hence $T(A + B) \neq T(A) + T(B)$
So, T is not a linear transformation.

b) Let $x = 1\ and\ y = 2\ then\ T(x + y) = T(3) = 2^3 = 8$
Now $T(x) + T(y) = T(1) + T(2) = 2^1 + 2^2 = 6$
Therefore, $T(A + B) \neq T(A) + T(B)$
So, T is not an linear transformation.

c) Try yourself

Zero Transformation: For any vector space V, the transformation $T_0: V \to W$ that maps every vector in V to the zero vector in W is called a zero transformation.

i.e. $T_0(v) = 0\ \forall v \in V$

Identity Transformation: For any vector space V, the transformation $I: V \to V$ that maps every vector in V to itself is called the identity transformation.

$$\text{i.e. } I(v) = v, \forall v \in V$$

Properties of Linear Transformation:

Let $T: v \to w$ be a linear transformation then,

$$\text{a) } T(0) = 0$$
$$\text{b)} T(-v) = -T(v), \forall v \in V$$
$$\text{c)} T(u - v) = T(u) - T(v), \forall u, v \in V$$

Example: Suppose T is a linear transformation from $\mathbb{R}^2$ to P_2 such that $T\begin{bmatrix}1\\1\end{bmatrix} = 2 - 3x + x^2 and\ T\begin{bmatrix}2\\3\end{bmatrix} = 1 - x^2$. Find $T\begin{bmatrix}-1\\2\end{bmatrix} and\ T\begin{bmatrix}a\\b\end{bmatrix}$.

Solution:

Given that $T\begin{bmatrix}1\\1\end{bmatrix} = 2 - 3x + x^2 and\ T\begin{bmatrix}2\\3\end{bmatrix} = 1 - x^2$.

$$\text{Let } B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$$

Since B is a basis of $\mathbb{R}^2$,every vector in $\mathbb{R}^2$ is a span of B, now solving,

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1 + 2c_2\\c_1 + 3c_2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

$$\Rightarrow c_1 + 2c_2 = -1\ and\ c_1 + 3c_2 = 2$$

Subtracting the equations we get,

$$-c_2 = -3$$

$$\Rightarrow c_2 = 3$$

So $c_1 = -1 - 2c_2 = -1 - 6 = -7$

Therefore, we get,

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -7\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = T\left\{-7\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$$

$$\Rightarrow T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -7T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3T\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -7(2 - 3x + x^2) + 3(1 - x^2)$$

$$\Rightarrow T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -11 + 21x - 10x^2$$

Again,

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + 2c_2 \\ c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow c_1 + 2c_2 = a \text{ and } c_1 + 3c_2 = b$$

Subtracting the equations we get,

$$-c_2 = a - b$$

$$\Rightarrow c_2 = b - a$$

$$\text{So, } c_1 = a - 2c_2 = a - 2(b - a) = 3a - 2b$$

Therefore, we get,

$$\begin{bmatrix} a \\ b \end{bmatrix} = (3a - 2b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b - a)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix} a \\ b \end{bmatrix} = T\left\{(3a - 2b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b - a)\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$$

$$\Rightarrow T\begin{bmatrix} a \\ b \end{bmatrix} = (3a - 2b)T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b - a)T\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}a\\b\end{bmatrix} = (3a-2b)(2-3x+x^2)+(b-a)(1-x^2)$$

$$\Rightarrow T\begin{bmatrix}a\\b\end{bmatrix} = 5a-3b+(-9a+6b)x+(4a-3b)x^2$$

Practice Problem: Suppose T is a linear transformation from $\mathbb{R}^2$ to P_2 such that $T\begin{bmatrix}1\\1\end{bmatrix} = 1-2x \; and \; T\begin{bmatrix}3\\-1\end{bmatrix} = x+2x^2$. Find $T\begin{bmatrix}-7\\9\end{bmatrix} \, and \, T\begin{bmatrix}a\\b\end{bmatrix}$.

Ans: $T\begin{bmatrix}-7\\9\end{bmatrix} = 5-14x-8x^2 and \; T\begin{bmatrix}a\\b\end{bmatrix} = \frac{a+3b}{4} - \frac{a+7b}{4}x + \frac{a-b}{2}x^2$

Example 2: Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation for which $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\0\\4\end{bmatrix}$. Find $T\begin{bmatrix}5\\2\end{bmatrix} \, and \, T\begin{bmatrix}a\\b\end{bmatrix}$.

Solution:

Given that, $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by,

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix} \text{ and } T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\0\\4\end{bmatrix}$$

Since here $B = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a standard basis of $\mathbb{R}^2$,

every vector in $\mathbb{R}^2$ is a span of B.

$$\text{Here,}\begin{bmatrix}5\\2\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}5\\2\end{bmatrix} = T\left\{5\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$$\Rightarrow T\begin{bmatrix}5\\2\end{bmatrix} = 5T\begin{bmatrix}1\\0\end{bmatrix} + 2T\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}5\\2\end{bmatrix} = 5\begin{bmatrix}1\\2\\-1\end{bmatrix} + 2\begin{bmatrix}3\\0\\4\end{bmatrix} = \begin{bmatrix}11\\10\\3\end{bmatrix}$$

Again let,

$$c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$
$$\Rightarrow\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$
$$\Rightarrow c_1=a\ and\ c_2=b$$

So,

$$\begin{bmatrix}a\\b\end{bmatrix}=a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}$$
$$\Rightarrow T\begin{bmatrix}a\\b\end{bmatrix}=aT\begin{bmatrix}1\\0\end{bmatrix}+bT\begin{bmatrix}0\\1\end{bmatrix}$$
$$\Rightarrow T\begin{bmatrix}a\\b\end{bmatrix}=a\begin{bmatrix}1\\2\\-1\end{bmatrix}+b\begin{bmatrix}3\\0\\4\end{bmatrix}=\begin{bmatrix}a+3b\\2a\\-a+4b\end{bmatrix}$$

Example 3: If $T:\mathbb{R}^2\to\mathbb{R}^3$defined by, $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\2x-y\\3x+4y\end{bmatrix}$, then verify that T is a linear transformation or not.

Ans: (Hints: Check whether $T(u+v)=T(u)+T(v)\ and\ T(cu)=cT(u),\forall\ u,v\in\mathbb{R}^2 and\ c\in\mathbb{R}$)

Example 4: Let $T: P_2\to P_2$ be a linear transformation for which $T(1+x)=1+x^2, T(x+x^2)=x-x^2 and\ T(1+x^2)=1+x+x^2$. Find $T(4-x+3x^2) and\ T(a+bx+cx^2)$.

Solution:

Given that, $T: P_2\to P_2$ be a linear transformation for which $T(1+x)=1+x^2, T(x+x^2)=x-x^2 and\ T(1+x^2)=1+x+x^2$.

Since here,

$$B = \{1 + x, x + x^2, 1 + x^2\}$$

Is a basis of P_2. So, every vector in P_2 is a span of B.

Let,

$$4 - x + 3x^2 = c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2)$$

Now comparing the coefficients of each power of x we get,

$$c_1 + c_3 = 4 \quad \dots (1)$$

$$c_1 + c_2 = -1 \quad \dots (2)$$

$$c_2 + c_3 = 3 \quad \dots (3)$$

Subtracting 1st and 3rd equation we get,

$$c_1 - c_2 = 1 \quad \dots (4)$$

Adding 2nd and 4th equation we get,

$$2c_1 = 0 \Rightarrow c_1 = 0$$

Form equation (4) we get,

$$c_2 = c_1 - 1 = -1$$

Form equation (1) we get,

$$c_3 = 4 - c_1 = 4$$

So,

$$4 - x + 3x^2 = 0.(1 + x) - 1.(x + x^2) + 4.(1 + x^2)$$

$$\Rightarrow T(4 - x + 3x^2) = T\,\{0.(1 + x) - 1.(x + x^2) + 4.(1 + x^2)\}$$

$$\Rightarrow T(4 - x + 3x^2) = 0.T(1 + x) - 1.T(x + x^2) + 4.T(1 + x^2)$$

$$\Rightarrow T(4 - x + 3x^2) = -1(x - x^2) + 4(1 + x + x^2) = 4 + 3x + 5x^2$$

Again,

$$a + bx + cx^2 = c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2)$$

Now comparing the coefficients of each power of x we get,

$$c_1 + c_3 = a \quad \dots(1)$$

$$c_1 + c_2 = b \quad \dots(2)$$

$$c_2 + c_3 = c \quad \dots(3)$$

Subtracting 1st and 3rd equation we get,

$$c_1 - c_2 = a - c \quad \dots(4)$$

Adding 2nd and 4th equation we get,

$$2c_1 = a + b - c \Rightarrow c_1 = \frac{1}{2}(a + b - c)$$

Form equation (4) we get,

$$c_2 = c_1 - a + c = \frac{1}{2}(a + b - c) - a + c = \frac{1}{2}(b + c - a)$$

Form equation (1) we get,

$$c_3 = a - c_1 = a - \frac{1}{2}(a + b - c) = \frac{1}{2}(c + a - b)$$

So,

$$\begin{aligned} &a + bx + cx^2 \\ &\qquad = \frac{1}{2}(a + b - c).(1 + x) + \frac{1}{2}(b + c - a).(x + x^2) \\ &\qquad + \frac{1}{2}(c + a - b).(1 + x^2) \end{aligned}$$

$$\begin{aligned} &\Rightarrow T(a + bx + cx^2) \\ &\qquad = T\left\{\frac{1}{2}(a + b - c).(1 + x) + \frac{1}{2}(b + c - a).(x + x^2) \right. \\ &\qquad \left. + \frac{1}{2}(c + a - b).(1 + x^2)\right\} \end{aligned}$$

$$\begin{aligned} &\Rightarrow T(a + bx + cx^2) \\ &\qquad = \frac{1}{2}(a + b - c).T(1 + x) + \frac{1}{2}(b + c - a).T(x + x^2) \\ &\qquad + \frac{1}{2}(c + a - b).T(1 + x^2) \end{aligned}$$

$$\Rightarrow T(a+bx+cx^2)$$
$$= \frac{1}{2}(a+b-c).(1+x^2) + \frac{1}{2}(b+c-a).(x-x^2)$$
$$+ \frac{1}{2}(c+a-b).(1+x+x^2)$$

$$\Rightarrow T(a+bx+cx^2)$$
$$= \frac{1}{2}(a+b-c+c+a-b)$$
$$+ \frac{1}{2}(b+c-a+c+a-b)x$$
$$+ \frac{1}{2}(a+b-c+a-b-c+c+a-b)x^2$$

$$\Rightarrow T(a+bx+cx^2) = a+cx+\frac{1}{2}(3a-b-c)x^2$$

Practice Problem: Let $T: M_{2\times2} \to \mathbb{R}$ be a linear transformation for which $T\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 1, T\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} = 2, T\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = 3 \; and \; T\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = 4.$

Find $T\begin{bmatrix}1 & 3\\4 & 2\end{bmatrix} \; and \; T\begin{bmatrix}a & b\\c & d\end{bmatrix}$.

Composition of Linear Transformation:

If $T: U \to V$ and $S: V \to W$ are linear transformations, then the composition of S with T is the mapping $S \circ T$, defined by $(S \circ T)(u) = S\big(T(u)\big)$ where $u \in U$.

Example 1: Let $T: \mathbb{R}^2 \to P_1$ and $S: P_1 \to P_2$ be linear transformations defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$ and $S\big(p(x)\big) = xp(x)$.

Find

$$I)(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$II)\ (S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}.$$

Solution:

I) $(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix} = S\left(T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)\right) = S(3 + (3-2)x) = S(3+x) = x(3+x) = 3x + x^2$

II)$(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = S\left(T\begin{bmatrix} a \\ b \end{bmatrix}\right) \quad = S(a + (a+b)x)$

$$= x(a + (a+b)x)$$

$$= ax + (a+b)x^2$$

Example 2: If $T: U \to V$ and $S: V \to W$ are linear transformation then $S \circ T: U \to W$ is a linear transformation.

Solution:

Let u and $v \in U$ and c be any scalar then,

$$(S \circ T)(u+v) = S(T(u+v))$$

$$= S\big(T(u) + T(v)\big)$$

$$= S\big(T(u)\big) + S\big(T(v)\big)$$

$$= (S \circ T)(u) + (S \circ T)(v)$$

Hence, $(S \circ T)(u+v) = (S \circ T)(u) + (S \circ T)(v)$.

Again,

$$(S \circ T)(cu) = c(S \circ T)(u)$$

Hence $S \circ T$ is a linear transformation.

Example: Define linear transformation $S: \mathbb{R}^2 \to M_{2\times2}$ *and* $T: \mathbb{R}^2 \to \mathbb{R}^2$, by $S\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b & b \\ 0 & a-b \end{bmatrix}$ *and* $T\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d \\ -d \end{bmatrix}$. Compute $(S \circ T)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ *and* $(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix}$. Can you compute $(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix}$? If so compute it.

Solution:

Given that, $S: \mathbb{R}^2 \to M_{2\times2}$ *and* $T: \mathbb{R}^2 \to \mathbb{R}^2$, by

$S\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b & b \\ 0 & a-b \end{bmatrix}$ *and* $T\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d \\ -d \end{bmatrix}$.

$$(S \circ T)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = S\left(T\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$= S\left(\begin{bmatrix} 5 \\ -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 4 & -1 \\ 0 & 6 \end{bmatrix}$$

Again,

$$(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= S\left(\begin{bmatrix} 2x+y \\ -y \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2x & -y \\ 0 & 2x+2y \end{bmatrix}$$

Also,

$$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} = T\left(S\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= T\begin{bmatrix} x+y & y \\ 0 & x-y \end{bmatrix}$$

$T\begin{bmatrix} x+y & y \\ 0 & x-y \end{bmatrix}$ is not defined.

So, we cannot compute

$$T\begin{bmatrix} x+y & y \\ 0 & x-y \end{bmatrix}$$

Inverse of a linear transformation:

A linear transformation $T: V \rightarrow W$ is invertible if there is a linear transformation$T: W \rightarrow V$ such that $T' \circ T = I_V \; and \; T \circ T' = I_W$. In this case, T' is called an inverse T.

Example 1: Verify that the mapping $T: \mathbb{R}^2 \rightarrow P_1 \; and \; T': P_1 \rightarrow \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$ and $T(c+dx) = \begin{bmatrix} c \\ d-c \end{bmatrix}$ are inverse.

Solution:

$$(T' \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = T\left(T\begin{bmatrix} a \\ b \end{bmatrix}\right)$$

$$= T'(a + (a+b)x)$$

$$= \begin{bmatrix} a \\ a+b-a \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \end{bmatrix}$$

Again,

$$(T \circ T')(c+dx) = T\big(T'(c+dx)\big)$$

$$= T\left(\begin{bmatrix} c \\ d-c \end{bmatrix}\right)$$

$$= c + (c + (d-c))x$$

$$= c + dx$$

Hence, we get, $T' \circ T = I_{\mathbb{R}^2}\ and\ T \circ T' = I_{P_1}$.

So $T\ and\ T'$ are inverses of each other.

Example 2: Verify that S and T are inverses,

$$S: \mathbb{R}^2 \to \mathbb{R} \text{ defined by } S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x + y \\ 3x + y \end{bmatrix}$$

$$\text{And } T: \mathbb{R}^2 \to \mathbb{R} \text{ defined by } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ -3x + 4y \end{bmatrix}$$

Solution:

Given that,

$$S: \mathbb{R}^2 \to \mathbb{R} \text{ defined by } S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x + y \\ 3x + y \end{bmatrix}$$

$$\text{And } T: \mathbb{R}^2 \to \mathbb{R} \text{ defined by } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ -3x + 4y \end{bmatrix}.$$

Now,

$$S \circ T\begin{bmatrix} x \\ y \end{bmatrix} = S\left[T\begin{bmatrix} x \\ y \end{bmatrix}\right]$$

$$= S\begin{bmatrix} x - y \\ -3x + 4y \end{bmatrix}$$

$$= \begin{bmatrix} 4(x - y) + (-3x + 4y) \\ 3(x - y) + (-3x + 4y) \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}$$

Also,

$$T \circ S\begin{bmatrix} x \\ y \end{bmatrix} = T\left[S\begin{bmatrix} x \\ y \end{bmatrix}\right]$$

$$= T\begin{bmatrix} 4x + y \\ 3x + y \end{bmatrix}$$

$$= \begin{bmatrix} 4x + y - 3x - y \\ -3(4x + y) + 4(3x + y) \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 3: Let $S: P_1 \to P_1$ defined by $S(a + bx) = (-4a + b) + 2ax$

And $T: P_1 \to P_1$ defined by $T(a + bx) = \frac{b}{2} + (a + 2b)x$.

Solution:

Given that,

$S: P_1 \to P_1$ defined by $S(a + bx) = (-4a + b) + 2ax$

And $T: P_1 \to P_1$ defined by $T(a + bx) = \frac{b}{2} + (a + 2b)x$.

Now,

$$(S \circ T)(a + bx) = S\big(T(a + bx)\big)$$

$$= S\left[\frac{b}{2} + (a + 2b)x\right]$$

$$= \left(-4.\frac{b}{2} + a + 2b\right) + 2.\frac{b}{2}x$$

$$= a + bx$$

And

$$(T \circ S)(a + bx) = T\big(S(a + bx)\big)$$

$$= T[(-4a + b) + 2ax]$$

$$= \frac{2a}{2} + \big((-4a + b) + 2(2a)\big)x$$

$$= a + bx$$

Hence, we get,

$$(S \circ T)(a + bx) = I_{P_1} and\ (T \circ S)(a + bx) = I_{P_1}$$

Hence S and T are inverse of each other.

Assignment:

I) Let $T: P_1 \to P_1$ be a linear transformation for which $T(1) = 3 - 2x, T(x) = 4x - x^2\ and\ T(x^2) = 2 + 2x^2$
Find
a)$T(6 + x - 4x^2)$
b)$T(a + bx + cx^2)$

II) Define linear transformation $S: P_1 \to P_2$ and $T: P_2 \to P_2$ by $S(a + bx) = a + (a + b)x + 2bx^2\ and\ T(a + bx + cx^2) = b + 2cx$. Compute $(S \circ T)(3 + 2x - x^2)\ and\ (S \circ T)(a + bx + cx^2)$. Can you compute $(T \circ S)(a + bx)$? If so compute it.

Definition: Let $T: V \to W$ be linear transformation from a vector space V to W. We will say V as domain of T and W as a codomain of T.

Kernel of a Linear Transformation: Let V and W be vector spaces. Let $T: V \to W$ be a linear transformation. The set of all vectors $v \in V$ for which $Tv = 0$ is a subspace of V. It is called the kernel of T and denoted by $Ker(T)$,

i.e. $Ker(T) = \{T(v) = 0: v \in V\}$

Range of Linear Transformation: Let V and W be vector spaces and let $T: V \to W$ be a linear transformation. The set of all vectors $w \in W$ such that $w = Tv$ for some $v \in V$ is called range of T. It is a subspace of W and is denoted by $range(T)$.

i.e., $range\ (T) = \{w = T(v): v \in V\}$

Example 1: Find the kernel and range of the differential operator $D: P_3 \to P_2$ defined by $D\big(p(x)\big) = p'(x)$.

Solution:

Given that,

$D: P_3 \to P_2$ defined by $D\big(p(x)\big) = p'(x)$.

Since $p(x)$ is in P_3 i.e., $a + bx + cx^2 + dx^3$.

$$\therefore D(a + bx + cx^2 + dx^3) = 0 + b + 2cx + 3dx^2$$
$$= b + 2cx + 3dx^2$$

Now we have,

$$Ker(D) = \{a + bx + cx^2 + dx^3: b + 2cx + 3dx^2 = 0\}$$

Now, $b + 2cx + 3dx^2 = 0$ if and only if $b = c = d = 0$.

Hence,

$$Ker(D) = \{a + bx + cx^2 + dx^3: b = c = d = 0\}$$
$$= \{a: a \in \mathbb{R}\}$$

In other words, the kernel of D is the set of constant polynomials.

Now the range of D is all of P_2.

Since every polynomial in P_2 is the image under D (i.e., derivative) of some polynomial in P_3 if $a + bx + cx^2 \in P_2$.

$$\therefore a + bx + cx^2 = D\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right).$$

Example: Let $S: P_1 \to \mathbb{R}$ be the linear transformation defined by,

$$S(p(x)) = \int_0^1 p(x)dx$$

Find the kernel and range of S.

Solution:

Given that, $S: P_1 \to \mathbb{R}$ be the linear transformation defined by,

$$S(p(x)) = \int_0^1 p(x)dx$$

Let $p(x) = a + bx \in P_1$.

$$S(a+bx) = \int_0^1 (a+bx)dx = \left[a + \frac{b}{2}x^2\right]_0^1 = a + \frac{b}{2}$$

$$\therefore S(p(x) = S(a+bx) = a + \frac{b}{2}$$

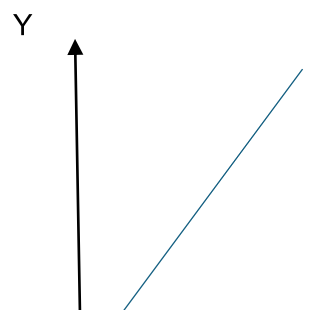
Now,

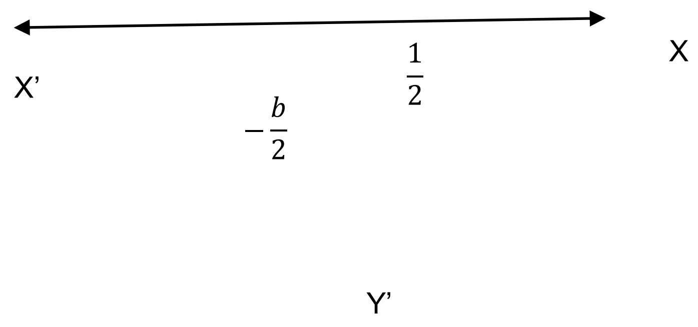
$$\begin{aligned} Ker(S) &= \{p(x) \in P_1 : S(p(x)) = 0\} \\ &= \{a + bx \in P_1 : S(a+bx) = 0\} \\ &= \left\{a + bx \in P_1 : a + \frac{b}{2} = 0\right\} \\ &= \left\{a + bx \in P_1 : a = -\frac{b}{2}\right\} \end{aligned}$$

Hence,

$$Ker(S) = \left\{-\frac{b}{2} + bx\right\}$$

Geometrically, $Ker(S)$ consists of all those linear polynomials whose graphs have the property that the area between the line and the x-axis is equally distributed above and below the axis on the interval $[0,1]$. (See the figure)





If $y = -\frac{b}{2} + bx$, then

$$\int_0^1 y dx = \int_0^1 \left(-\frac{b}{2} + bx\right) dx$$

$$= \left[-\frac{b}{2}x + b.\frac{x^2}{2}\right]_0^1 = 0$$

The range of S is $\mathbb{R}$, since every real number can be obtained as the image under S of some polynomial in P_1.

For example, if 'a' is an arbitrary real number, then

$$\int_0^1 a dx = [ax]_0^1 = a$$

$$\therefore S(a) = a$$

Example 3: Let $T: M_{2\times2} \to M_{2\times2}$ be the linear transformation defined by $T(A) = A^T$. Find the kernel and range of T.

Solution:

Given $T: M_{2\times2} \to M_{2\times2}$ be the linear transformation defined by $T(A) = A^T$.

We have,

$$Ker(T) = \{A \in M_{2\times2}: T(A) = 0\}$$
$$= \{A \in M_{2\times2}: A^T = 0\}$$

But if $A^T = 0$ then $A = (A^T)^T = 0^T = 0$.

It shows that, $Ker(T) = \{0\}$.

Also, for any matrix $A \in M_{2\times2}$, we have,

$$A = (A^T)^T = T(A^T) \text{ (Since } A^T \in M_{2\times2})$$

Hence, $range(T) = M_{2\times2}$.

Practice Problem: Let $T: P_1 \to P_1$ defined by $T(ax + b) = 2bx - a$ is linear. Describe its kernel and range.

<u>Matrix of Linear Transformation</u>

Theorem 1: Let V and W be two finite-dimensional vector spaces with bases B and C respectively, where $B = \{v_1, v_2, \dots, v_3\}$. If $T: V \to W$ is a linear transformation, then the $m \times n$ matrix A, defined by

$$A = [[T(v_1)]_c\ [T(v_2)]_c \dots [T(v_n)]_c]$$

Satisfies

$$A[v]_B = [T(v)]_c\ \forall v \in V$$

The matrix A is called the matrix of T with respect to the bases B and C.

Note: The matrix of a linear transformation T with respect to bases B and C is sometimes denoted by $[T]_{C\leftarrow B}$ (Note that the direction of the arrow right to left) (not left to right as for $T: V \to W$)

Example 1: Let $T: P_2 \to P_2$ be the linear transformation defined by $T(p(x)) = p(2x-1)$.

a) Find the matrix of T with respect to the $E = \{1, x, x^2\}$.

b) Compute $T(3 + 2x - x^2)$ indirectly using part (a).

Solution:

a) Given that $T: P_2 \to P_2$ defined by,

$$T(p(x)) = p(2x-1)$$

Now $T(1) = 1, T(x) = 2x - 1, T(x^2) = (2x-1)^2 = 1 - 4x + 4x^2$

So, the coordinate vectors are,

$$[T(1)]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [T(x)]_E = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \; and \; [T(x^2)]_E = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

$$\therefore [T]_E = [[T(1)]_E \; [T(x)]_E [T(x^2)]_E]$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

b) We apply theorem 1 as follows:

The coordinate vector of $p(x) = 3 + 2x - x^2$ with respect to E.

$$[p(x)]_E = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

By theorem 1,

$$[T(3 + 2x - x^2)] = [[T(p(x)]_E]$$

$$= [T]_E [p(x)]_E$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}$$

It follows that,
$T(3 + 2x - x^2) = 0.1 + 8.x - 4.x^2 = 8x - 4x^2$.

[Directly method: $T(3 + 2x - x^2) = 3 + 2(2x - 1) - (2x - 1)^2 = 8x - 4x^2$]

Example 2: Let $D: P_3 \to P_2$ be the differential operator $D(p(x)) = p'(x)$. Let $B = \{1, x, x^2, x^3\}$ and $C = \{1, x, x^2\}$ be bases for $P_3 and\ P_2$ respectively.

a) Find the matrix A of D with respect to B and C.

b) Find the matrix A' of D with respect to $B' and\ C$, where$B' = \{1, x, x^2, x^3\}$

c) Using part (a) compute $D(5 - x + 2x^2)$ and $D(a + bx + cx^2 + dx^3)$ to verify theorem 1.

Solution:

Given that, $D: P_3 \to P_2$ be the differential operator

$$D(p(x)) = \quad p'(x)$$

Let $p(x) = a + bx + cx^2 + dx^3 \in P_3$.

So,

$$\therefore D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$$

a) Since the images of the basis B under D are,

$$D(1) = 0, D(x) = 1, D(x^2) = 2x \ and\ D(x^3) = 3x^2$$

Therefore, the coordinate vectors with respect to C are

$$[D(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, [D(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [D(x^2)] = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \ and\ [D(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$A = [D]_{C \leftarrow B} = [[D(1)]_C [D(x)]_C [D(x^2)]_C [D(x^3)]_C]$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

b) Since the basis B' is just B in the reverse order, we see that
$$A = [D]_{C \leftarrow B'} = [[D(x^3)]_C [D(x^2)]_C [D(x)]_C [D(1)]_C]$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

c) First, we compute $D(5 - x + 2x^3) = -1 + 6x^2$.
Directly, getting the coordinate vector,
$$[D(5 - x + 2x^3) = [-1 + 6x^2]_C = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$$

On other hand,
$$[5 - x + 2x^2]_B = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

So, $A[5 - x + 2x^2]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix}$
$$= \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = [D(5 - x + 2x^3)]_C$$

Example 3 : Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$ and let $B = \{e_1, e_2, e_3\}$ and $C =$

$\{e_2, e_1\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^3$ respectively. Find the matrix T with respect to B and C and verify theorem (1) for $v = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

Solution: $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by,

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

First, we compute

$$T(e_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(e_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, T(e_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Next, we need their coordinate vectors with respect to C.

Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e_1 + e_2, \begin{bmatrix} -2 \\ 1 \end{bmatrix} = e_2 - 2e_1, \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3. e_2 + 0. e_1.$

We have $[T(e_1)]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $[T(e_2)]_C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $[T(e_3)]_C = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

∴ The matrix of T with respect to B and C is

$$A = [T]_{C \leftarrow B} = [[T(e_1)]_c [T(e_2)]_c [T(e_3)]_c]$$
$$= \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

To verify theorem (1) for v, we first compute,

$$T(v) = T\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

Then,

$$[v]_B = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

And

$$[T(v)]_C = \begin{bmatrix} -5 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

Also $A[v]_B = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} = [T(v)]_C$

Hence it is verified.