5.1 INTRODUCTION

- The central starting point of Fourier analysis is **Fourier series**. They are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines.
- This trigonometric system is ***orthogonal***, allowing the computation of the coefficients of the Fourier series by use of the well-known Euler formulas, as shown. Fourier series are very important to the engineer and physicist because they allow the solution of linear differential equations and partial differential.
- Fourier series are, in a certain sense, more universal than the familiar Taylor series in calculus because many *discontinuous* periodic functions that come up in applications can be developed in Fourier series but do not have Taylor series expansions.

- The Fourier Transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. The Fourier Transform shows that any waveform can be re-written as the sum of sinusoidal functions.
- The Fourier transform is a mathematical function that decomposes a waveform, which is a function of time, into the frequencies that make it up. The result produced by the Fourier transform is a complex valued function of frequency.
- The absolute value of the Fourier transform represents the frequency value present in the original function and its complex argument represents the phase offset of the basic sinusoidal in that frequency.

- The Fourier transform is also called a generalization of the Fourier series. This term can also be applied to both the frequency domain representation and the mathematical function used.
- The Fourier transform helps in extending the Fourier series to non-periodic functions, which allows viewing any function as a sum of simple sinusoids.

5.2 OBJECTIVES

- After studying this chapter we will learn about how Fourier transforms is useful many physical applications, such as partial differential equations and heat transfer equations.
- With the use of different properties of Fourier transform along with Fourier sine transform and Fourier cosine transform, one can solve many important problems of physics with very simple way.
- Thus we will learn from this unit to use the Fourier transform for solving many physical application related partial differential equations.

5.3 FOURIER SERIES

- A function *f* (*x*) is called a **periodic function** if *f* (*x*) is defined for all real *x*, except possibly at some points, and if there is some positive number *p*, called a **period** of *f* (*x*) such that

$$f(x+p) = f(x) \qquad\qquad for\ all\ x$$

- Familiar periodic functions are the *cosine, sine, tangent*, and *cotangent*. Examples of functions that are not periodic are $x, x^2, x^3, e^x, \cos hx\ etc.$ to mention just a few.

If *f(x)* has a period of *p* then it has also a period of *2p*

$$f(x+2p) = f\{(x+p)+p\} = f(x+p) = f(x)$$

Or in general we can write

$$f(x+np) = f(x)$$

- A Fourier series is defined as an expansion of a real function or representation of a real function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0, a_n, and\ b_n$ are constants, called the **Fourier coefficients** of the series. We see that each term has the period of 2π Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π*.*

- The **Fourier coefficients** of *f(x)*, given by the **Euler formulas**

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n = 1, 2, 3, \dots.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, 3, \dots$$

The above Fourier series is given for period 2π. The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $= 2L$. Then we can introduce a new variable *v* such that ,$f(x)$ as a function of *v*, has period 2π.

- If we set

$$x = \frac{p}{2\pi} v \quad \Rightarrow v = \frac{2\pi}{p} x \quad \Rightarrow v = \frac{\pi}{L} x$$

This means $v = \pm\pi$ corresponds to $x = \pm L$. This represent f, as function of *v* has a period of 2π. Hence the Fourier series is

$$f(v) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nv + \sum_{n=1}^{\infty} b_n \sin nv$$

- Now using $v = \frac{\pi}{L} x$ Fourier series for the period of *(-L, L)* is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \frac{\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin n \frac{\pi}{L} x$$

This is Fourier series we obtain for a function of *f(x)* period 2*L* the Fourier series.

The coefficient is given by

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t) dt,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx,$$

- $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx,$

5.4 SOME IMPORTANT RESULTS

- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx$
- $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b\sin bx$
- $\int_0^{\infty} \frac{\sin ax}{x} \, dx = \frac{\pi}{2}$
- $\int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$
- $\int_{-\infty}^{\infty} \frac{\sin mx}{(x-b)^2+a^2} \, dx = \frac{\pi}{a} \, e^{-am} \sin bm \,, \qquad [m > 0]$

5.5 FOURIER INTEGRAL

- Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only.
- Since, of course, many problems involve functions that are ***nonperiodic*** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to “Fourier integrals.”

5.6 FOURIER INTEGRAL THEOREM

Fourier integral theorem states that $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)dt\,du$

Proof. We know that Fourier series of a function f (x) in (-c, c) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

Where a_0 , a_n and b_n are given by

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)dt,$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt,$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt,$$

Substituting the values of a_0 , a_n and b_n in above equation, we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt\cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt,\sin\frac{n\pi x}{c}$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c} + \sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\left(\frac{n\pi t}{c} - \frac{n\pi x}{c}\right)\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi}{c}(t-x)\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\left\{1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\}dt$$

Since cosine functions are even functions i.e., $\cos(-\theta) = \cos\theta$ the expression

$$\left\{1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\} = \sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{c}(t-x)$$

We now let the parameter *c* approach infinity, transforming the finite interval [-*c*, *c*] into the infinite interval (-∞ to +∞). We set

$$\frac{n\pi}{c} = \omega, and\ \frac{\pi}{c} = d\omega \qquad with\ c \to \infty$$

Then we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\ \left\{\int_{-\infty}^{\infty} d\omega \cos \omega(t-x)\right\} dt$$

On simplifying

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t) cos\ \omega(t-x)\, d\omega\, dt \qquad Proved$$

5.7. FOURIER SINE AND COSINE INTEGRALS

$$f(x) = \frac{2}{\pi}\int_0^{\infty} sin\,\omega x\, du\ \int_0^{\infty} f(t)\, sin\,\omega t\, dt \qquad \text{(Fourier Sine Integrals)}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} cos\,\omega x\, du\ \int_0^{\infty} f(t)\, cos\,\omega t\, dt \qquad \text{(Fourier Cosine Integrals)}$$

Proof: We can write

$$\cos\omega(t-x) = \cos(\omega t - \omega x) = \cos\omega t\cos\omega x + \sin\omega t\sin\omega x$$

Using this expansion in Fourier integral theorem, we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \cos\omega(t-x)\, d\omega\, dt$$

$$\Rightarrow\ f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)(cos\,\omega t\cos\omega x + \sin\omega t\sin\omega x) d\omega\, dt$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)(cos\,\omega t\cos\omega x\, d\omega\, dt + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega t\sin\omega x\, d\omega\, dt$$

Now to solve the above equation, we have two different cases, using the following conditions

$$\int_{-a}^{a} f(x)dx = 0 \qquad for\ odd\ function$$

And

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\, dx \qquad for\ even\ function$$

Case I: when *f(t)* is even function: this means

$$\Rightarrow f(t)\sin\omega t \qquad is\ odd\ function\ and$$
$$f(t)\cos\omega t \quad is\ even\ function$$

Hence

$$\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega t\sin\omega x\, d\omega\, dt\ = 0$$

And

$$\Rightarrow f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)(cos\,\omega t\cos\omega x\, d\omega\, dt = \frac{2}{\pi}\int_{0}^{\infty}\cos\omega x\ d\omega\int_{-\infty}^{\infty} f(t)cos\,\omega t\ dt$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} cos\,\omega x\, du\int_{0}^{\infty} f(t)\, cos\,\omega t\, dt$$

This is known as Fourier cosine integral.

Case II: If *f(t)* is odd function: this means

$$\Rightarrow f(t)\sin\omega t \qquad is\ even\ function\ and$$

$$f(t)\cos\omega t \quad is\ odd\ function$$

Hence

$$\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\omega t\cos\omega x\, d\omega\, dt \; = 0$$

And

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t) sin\,\omega t\ sin\,\omega x\ d\omega\ dt = \frac{2}{\pi}\int_0^\infty sin\,\omega x\ \ d\omega\int_{-\infty}^\infty f(t) sin\,\omega t\ \ dt$$

$$f(x) = \ \frac{2}{\pi}\int_0^\infty sin\,\omega x\ du\ \int_0^\infty f(t)\ sin\,\omega t\ dt$$

This is known as Fourier sine integral.

5.8. FOURIER'S COMPLEX INTEGRALS

We know from Fourier integral theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,d\omega\,dt$$

Now adding

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)dt\int_{-\infty}^{\infty} \sin\omega(t-x)\,d\omega = 0$$

Since

$$\int_{-\infty}^{\infty} \sin\ \omega(t-x)d\omega = 0 \qquad\qquad \textit{because of odd function}$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,d\omega\,dt + \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)dt\int_{-\infty}^{\infty} \sin\omega(t-x)\,d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\left[\int_{-\infty}^{\infty} \cos\omega(t-x) + i\sin\omega(t-x)\right]\ d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt[\int_{-\infty}^{\infty} e^{i\omega(t-x)}\ ]\ \ d\omega$$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\,d\omega\int_{-\infty}^{\infty} f(t)\,e^{i\omega t}\,dt}$$

This relation is known as Fourier's complex Integral.

Example 1. Express the following function

$$f(x) = \begin{cases} 1 & when \;\; x \le 1 \\ 0 & when \;\; x > 1 \end{cases}$$

as a Fourier integral. Hence evaluate

$$\int_0^\infty \frac{\sin u \cos ux}{u} du$$

Solution: we know the Fourier Integral theorem, the Fourier Integral of a function $f(x)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) cos\, \omega(t - x)\, d\omega\, dt$$

Using $\omega = u$ we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) cos\, u(t - x)\, du\, dt$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-1}^1 cos\, u(t - x)\, dt\, du \qquad\qquad since\; f(t) = 1$$

Now integrating w.r.t. *t* we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin u(t-x)}{u}\right]_{-1}^{1} du$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin u(1-x) + \sin u(1+x)}{u}\right] du$$

Now using $\sin C + sinD = 2\, sin\frac{C+D}{2} cos\frac{C-D}{2}$ and solving it we will get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin u \cos ux}{u} du$$

We can rewrite this

$$\int_0^\infty \frac{\sin u \cos ux}{u} du = \frac{\pi}{2} f(x)$$

$$\int_0^\infty \frac{\sin u \cos ux}{u} du = \begin{cases} \frac{\pi}{2} \times 1 = \frac{\pi}{2}, & for\ \ x < 1 \\ \frac{\pi}{2} \times 0 = 0, & for\ x > 1 \end{cases}$$

For *x=1,* which is a point of discontinuity of *f(x)*, value of integral $= \frac{\frac{\pi}{2}+0}{2} = \frac{\pi}{4}$

5.9. FOURIER TRANSFORMS

From the Fourier complex integral we know that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\, d\omega \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

We can rewrite the above expression as follows using $\omega = s$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, ds\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]$$

Now using $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = F(s)$ in above equation, we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, F(s) ds$$

Where $F(s)$ is called the Fourier Transform of $f(x)$.

And $f(x)$ is called the Inverse Fourier transform of $F(s)$.

Thus , we obtain the definition of Fourier transform is

$$\boxed{\begin{aligned} \boldsymbol{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt} \\ \boldsymbol{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}. F(s)ds} \end{aligned}}$$

5.10. FOURIER SINE TRANSFORMS

We know that from Fourier sine integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx \, ds \int_0^\infty f(t) \sin st \, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \, ds \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \, dt\right]$$

Now putting $F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \, dt$

We have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \, ds \, F(s)$$

In above equation $F(s)$ is called Fourier Sine transform of $f(x)$

$$\boldsymbol{F(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \, dt}$$

And $f(x)$ given below is known as inverse Fourier Sine transform of $F(s)$

$$\boldsymbol{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s) \sin sx \, ds}$$

5.11. FOURIER COSINE TRANSFORM

From Fourier cosine integral we know that

$$f(x) = \frac{2}{\pi}\int_0^\infty cos\,\omega x\,du \int_0^\infty f(t)\,cos\,\omega t\,dt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,ds\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt\right]$$

Now putting $F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,ds\,F(s)$$

In above equation $F(s)$ is called Fourier cosine transform of $f(x)$

$$\boldsymbol{F(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt}$$

And $f(x)$ given below is known as inverse Fourier cosine transform of $F(s)$

$$\boldsymbol{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,F(s)ds}$$

Example 2: Find the Fourier transform of e^{-ax^2} , where a>0.

Solution : The Fourier transform of f(x):

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

Hence

$$F\{e^{-ax^2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2+isx} dx$$

$$\Rightarrow F\{e^{-ax^2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2-\frac{s^2}{4a}+isx+\frac{s^2}{4a}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{is}{2\sqrt{a}}\right)^2-\frac{s^2}{4a}}\, dx$$

$$\Rightarrow F\{e^{-ax^2}\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{is}{2\sqrt{a}}\right)^2}\, dx$$

$Putting\ x\sqrt{a} - \frac{is}{2\sqrt{a}} = u \Rightarrow dx = \frac{du}{\sqrt{a}}$ in above expression we get,

$$\Rightarrow F\{e^{-ax^2}\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-u^2}\, du \qquad \left[since \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\right]$$

$$\Rightarrow F\{e^{-ax^2}\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi a}} \sqrt{\pi} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2a}} \qquad Ans.$$

Example 3: Find the Fourier transform of

$$f(x) = \begin{cases} 2 & for \;\; |x| < a \\ 0 & for \;\; |x| > a \end{cases}$$

Solution: We know that the Fourier transform of a function is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Using the given value of *f(x)* we get,

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 2e^{isx} dx = \frac{2}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx =$$

$$F\{f(x)\} = \frac{2}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{2}{\sqrt{2\pi}\, is} \left[e^{ias} - e^{-ias}\right] = \frac{4}{\sqrt{2\pi}\, s} \frac{\left[e^{ias} - e^{-ias}\right]}{2i}$$

$$F\{f(x)\} = \frac{4}{\sqrt{2\pi}\, s} \sin as = 2\sqrt{\frac{2}{\pi}} \frac{\sin sa}{s} \qquad Ans.$$

Example 4: Find Fourier Sine transform of $\frac{1}{x}$.

Solution: We have to find the Fourier sine transform of $f(x) = \frac{1}{x}$

We know that from Fourier sine transform

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

Now using the value of $f(x) = \frac{1}{x}$, we get,

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx \, dx$$

$$now\ using\ sx = t \Rightarrow dx = \frac{dt}{s}$$

We get $$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin t}{t}\, dt = \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2}\right) \qquad \Rightarrow since\ \int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

Hence $$F_s\,[f(x)] = \sqrt{\frac{\pi}{2}} \qquad Ans.$$

Example 5: Find the Fourier Sine Transform of e^{-ax}.

Solution: Here, $f(x) = e^{-ax}$.

The Fourier sine transform of $f(x)$:

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

On putting the value of $f(x)$ in (1), we get

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

On Integrating by parts, we get

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}[-a\, sinsx - s\, cos\, sx]\right]_0^\infty$$

$$using\ \left[\int_0^\infty e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - bcos\, bx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{a^2+s^2}\,(-s)\right] = \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right) \qquad Ans.$$

Example 6: Find the Fourier Cosine Transform of $f(x) = 5e^{-2x} + 2e^{-5x}$

Solution: The Fourier Cosine Transform of $f(x)$ is given by

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

Putting the value of $f(x)$, we get

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty (5e^{-2x} + 2e^{-5x}) \cos sx \, dx$$

$$= 5\int_0^\infty e^{-2x} \cos sx \, dx + 2\int_0^\infty e^{-5x} \cos sx \, dx$$

$$using \left[\int_0^\infty e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b\sin bx\right]$$

$$= 5\left[\frac{e^{-2x}}{(-2)^2+s^2}(-2\cos sx + s\sin sx\right]_0^\infty + 2\left[\frac{e^{-5x}}{(-5)^2+s^2}(-5\cos sx + s\sin sx\right]_0^\infty$$

$$= 5\left[0 - \frac{1}{4+s^2}(-2)\right] + 2\left[0 - \frac{1}{25+s^2}(-5)\right] = 5\left(\frac{2}{s^2+4}\right) + 2\left(\frac{5}{s^2+25}\right)$$

$$= 10\left(\frac{1}{s^2+4} + \frac{1}{s^2+25}\right) \qquad Ans.$$

5.12. PROPERTIES OF FOURIER TRANSFORMS

9.12.1 LINEAR PROPERTY: If $F_1(s)$ and $F_2(s)$ are Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively then

$$F[af_1(x) + b\, f_2(x)] = a\, F_1(s) + b\, F_2(s) \qquad \textit{where a and b are constants.}$$

Proof: we know from the definition of Fourier transform

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

We can write

$$F_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{isx}\, dx$$

And

$$F_2(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x) e^{isx}\, dx$$

Now

$$F[af_1(x) + b\, f_2(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af_1(x) + b\, f_2(x)] e^{isx}\, dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{isx}\, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x)] e^{isx}\, dx$$

$$\Rightarrow F[af_1(x) + b\, f_2(x)] = a\, F_1(s) + b\, F_2(s) \qquad \textit{Proved}$$

5.12.2. CHANGE OF SCALE PROPERTY

We know that Fourier transform equation is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Proof: we know

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\Rightarrow F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx \qquad \left[now\ put\ ax = t \Rightarrow dx = \frac{dt}{a}\right]$$

We have

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\frac{s}{a}t} \frac{dt}{a} = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} dt$$

$$\Rightarrow F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad Proved$$

5.12.3 SHIFTING PROPERTY

The Fourier transform equation is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Then

$$F\{f(x-a)\} = e^{isa}\ F(s)$$

Proof: Given

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

then

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

$$Put\ (x-a) = u\ \Rightarrow x = u + a\ \ and\ dx = du$$

We have

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{is(u+a)} du = e^{isa}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{isu} du$$

$$\Rightarrow F\{f(x-a)\} = e^{isa}\ F(s) \qquad\qquad Proved$$

5.13. FOURIER TRANSFORM OF DERIVATIVES

As we know from the properties of Fourier Transform

$$F\{f^n(x)\} = (-i\ s)^n\ F(s)$$

$$F\left(\frac{\partial^2 f}{dx^2}\right) = (-i\ s)^2\ F\{f(x)\} = -s^2\bar{f}\ \ \left[where\ \bar{f}\ \ is\ Forier\ Transform\ of\ f\right]$$

$$If\ F_c\ and\ F_s\ are\ cosine\ and\ sine\ Forier\ tranform\ f(x)\ then$$

$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}\ f(0) + sF_s(s)$$

Proof: From cosine Fourier transform we know that

$$F_c\left[f'(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos sx\ dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\ d\{f(x)\}$$

Now integrating by parts, we get

$$= \sqrt{\frac{2}{\pi}}\,[\cos sx \;\; f(x)]_0^\infty - \sqrt{\frac{2}{\pi}}\left[-s\int_0^\infty sin\, sx\, f(x)\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\,[0 - f(0)] + s\sqrt{\frac{2}{\pi}}\left[\int_0^\infty sin\, sx\, f(x)\, dx\right] \qquad \{assumimg\ f(x) \to 0\ as\ x \to \infty\}$$

Hence

$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}\; f(0) + sF_s(s) \qquad where\ \sqrt{\frac{2}{\pi}}\left[\int_0^\infty sin\, sx\, f(x)\, dx\right] = F_s(s)$$

5.14. FOURIER TRANSFORM OF PARTIAL DERIVATIVE OF A FUNCTION

The Fourier transform of the partial derivatives is given by

$$F\left[\frac{\partial^2 u}{\partial^2 x}\right] = -s^2\, F(u)$$

Where $F(u)$ is the Fourier transform of u.

The Fourier sine transform of the partial derivatives is given by

$$F_s\left[\frac{\partial^2 u}{\partial^2 x}\right] = s(u)_{x=0} - s^2\, F_s(u)$$

Where $F_s(u)$ is the Fourier sine transform of u

The Fourier cosine transform of the partial derivatives is given by

$$F_c\left[\frac{\partial^2 u}{\partial^2 x}\right] = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - s^2\, F_c(u)$$

Where $F_c(u)$ is the Fourier cosine transform of u.

5.15 TERMINAL QUESTIONS

1) Find the Fourier Transform of $f(x)$ if

$$f(x) = \begin{cases} x, |x| \leq a \\ 0, |x| > a \end{cases}$$

2) Show that the Fourier Transform of

$$f(x) = \begin{cases} \quad a - |x| \quad for\ |x| < a \\ 0 \qquad\qquad for\ |x| > a > 0 \end{cases}$$

Is $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$.

Hence show that $\int_0^\infty \left(\frac{sint}{t}\right)^2 dt = \frac{\pi}{2}$

3) Show that the Fourier Transform of

$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2a} & for\ |x| \leq a \\ 0 & for\ |x| > a \end{cases}$$

Is $\frac{\sin sa}{sa}$

4) Find the Fourier cosine Transform of e^{-ax}.

5) Find Fourier transform of

$$F(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$$

6) Find Fourier Sine Transform of

$$f(x) = \frac{1}{x(x^2 + a^2)}$$

7) Find the Fourier Sine and Cosine Transform of $ae^{-\alpha x} + be^{-\beta x}$, $\alpha, \beta > 0$

8)Find f(x) if its Fourier Sine transform is $\frac{s}{1+s^2}$

9)Find f(x) if its Fourier Sine Transform is $(2\pi s)^{\frac{1}{2}}$

5.16. Terminal Questions Answer

1) $\frac{1}{\sqrt{2\pi}}\frac{2i}{s^2}$

4) $F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)$

5) $\left(\frac{2a^2}{s} - \frac{4}{s^2}\right)\sin\ as + \frac{4a}{s^2}\cos as$

6) $\frac{\pi}{2a^2}(1 - e^{-ax})$

7) $\frac{as}{s^2+\alpha^2} + \frac{bs}{s^2+\beta^2}$, $\frac{a\alpha}{s^2+\alpha^2} + \frac{b\beta}{s^2+\beta^2}$

8) $\frac{2sin^2\,ax}{\pi^2x^2}$

9) $\frac{1}{x\sqrt{x}}$

THANKS