Module-4

LINEAR ALGEBRA –MATRICES

- Rank of a matrix: Echelon form
- Linear systems of equations: solving system of Homogeneous
- Non-Homogeneous equations
- Eigen values and Eigen vectors of a matrix and properties (without proofs)
- Diagonalization of a matrix by orthogonal transformation
- Cayley-Hamilton Theorem.

Definition of a matrix:

An order set of 'mn' numbers, real or complex arranged in a rectangular array with 'm' rows and 'n' columns written as,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Is called an $m \times n$(read as m by n) matrix. These mn numbers are also called the elements of the matrix.

Thus, we write $A = [a_{ij}]_{m\times n}$ where $1 \le i \le m \; and \; 1 \le j \le n$. The symbol a_{ij} denotes the element in the i^{th} row and j^{th} column.

Elementary Transformation:

Following transformation are known as elementary transformation.

1) The interchange of any two rows (or columns). $R_i \leftrightarrow R_j$ stands for interchange of i^{th} row and j^{th} row.
   $C_i \leftrightarrow C_j$ stands for interchange of i^{th} column and j^{th}column.

2) The multiplication of elements of any row (or column) by any non-zero number.
   $R_i \to kR_i$ stands for the multiplication of i^{th} row by k.
   $C_i \to kC_i$ stands for the multiplication of i^{th} column by k.

3) The addition to the elements of any other row (or column) the corresponding elements of any other row (or column) multiplied by any number.
   $R_i \to R_i + kR_j$ means add to the elements of i^{th} row, k times the elements of j^{th} column.

Rank of a Matrix:

A matrix is said to be of rank r if

i) It has at least one non-zero minor of order r.
ii) Every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the highest order of any non-vanishing minor of the matrix.

The rank of a matrix A is denoted by $\rho(A)$.

Note:

1) The rank of a null matrix is zero i.e.$\rho(A) = 0$.
2) Every matrix will have a rank.
3) Rank of a matrix is unique.
4) For a non-zero matrix, $\rho(A) \geq 1$.
5) The rank of a unit matrix of order n is n.
6) The rank of every non-singular matrix of order n is n and the rank of a singular matrix of order n is less than n.

Echelon Form of a Matrix:

A matrix is said to be Echelon Form if

i) All non-zero rows, if any precede the zero rows.

ii) The number of zero preceding the first non-zero element in a row less than the number of such zeros in the succeeding row.

iii) The first non-zero element in each non-zero row is unity.

∴ The rank of a matrix in echelon form is the number of non-zero rows of the matrix.

Note: The condition (iii) is optional.

Zero Row:

If all the elements in a row of a matrix are zeros, then it is called a zero row.

Example:

$$\begin{bmatrix} 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-Zero Row:

If there is at least one non-zero element in a row, then it is called a non-zero row.

Example:

$$\begin{bmatrix} -1 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 1: Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

by reducing it to the echelon matrix.

Solution:

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

Applying row operations $R_2 \to R_2 - 3R_1 \; and \; R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} [R_3 \to R_3 - 2R_2]$$

The last equivalent matrix is in echelon form.

∴ Rank of A = Number of non-zero rows=3

Hence $\rho(A) = 3$

Problem 2: Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ into echelon matrix form and hence find its rank.

Solution:

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

Applying row operations $R_2 \to R_2 + 2R_1 and\ R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} [R_3 \to 4R_3 + R_2]$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

This is echelon form. The number of non-zero rows is 3.

$$\therefore \rho(A) = 3$$

Problem 3: Find the rank of matrix

$$A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$$

By reducing it to the echelon form. (Try it, Answer-$\rho(A) = 3$)

Problem 4: After reducing the matrix,

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Into echelon matrix find its rank.

Solution:

Here the matrix is

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$$[R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1 and\ R_4 \to R_4 - 6R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_4 \to R_4 - R_2$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - R_3$$

This is echelon form and the number of non-zero rows is 3.

Hence $\rho(A) = 3$

Problem 5: After reducing the matrix,

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Into echelon matrix find its rank. (Try it)

Problem 6: After reducing the matrix,

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

Into echelon matrix find its rank. (Try it)

System of linear simultaneous Equations:

An equation of the form, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

Where $x_1, x_2, \dots x_n$ are unknowns and $a_1, a_2, \dots, a_n, b$ are constants is called a linear equation in n unknowns.

Non-homogeneous equations:

Consider m linear non-homogeneous equations in n unknowns as given below,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The matrix form of the above system is $AX = B$, where A is the coefficient matrix formed by coefficients of unknowns.

i.e. $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b \\ \dots \\ b_n \end{bmatrix}$

The matrix $[A|B]$ is called the augmented matrix formed by the coefficient matrix together with the column formed by constants $b_1, b_2, \dots b_m$.

Condition of consistency:

The system of equations $AX = B$ is consistent iff the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A|B]$

i.e. $\rho(A) = \rho([A|B])$

Nature of the solution for non-homogeneous system:

The system of equations $AX = B$ is said to be

i) Consistent if $\rho(A) = \rho([A|B])$

ii) Consistent and a unique solution if if $\rho(A) = \rho([A|B]) = r = n$. Where r is the rank and n is the number of unknowns.

iii) Consistent and an infinite number of solutions is if $\rho(A) < \rho([A|B])$ $i.e.\ r < n$

iv) Inconsistent if if $\rho(A) \neq \rho([A|B])$

Problem 1: Find if the following system is consistent or not. If the system is consistent then solve it.

$$x + y + 2z = 4, 2x - y + 3z = 9, 3x - y - z = 2$$

Solution:

The given system can be written as a matrix form $AX = B$.

Where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $and\ B = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$.

So here the augmented matrix is

$$[A|B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$$

$$[R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

$$[R_3 \to 3R_3 - 4R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

Here $\rho(A) = 3\ and\ \rho([A|B]) = 3$

Since $\rho(A) = 3 = \rho([A|B])$, So the given equation is consistent with a unique solution.

Now the given system is equivalent to the system

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$$\Rightarrow x + y - 2z = 4, -3y - z = 1 \; and - 17z = -34$$

$$\Rightarrow z = 2, y = -1 \; and \; x = 1$$

So solution of the given system is $x = 1, y = -1 \; and \; z = 2$.

Problem 2: Prove that the following system is consistent and solve it.

$$3x + 3y + 2z = 1, x + 2y = 4, 10y + 3z = -2 \; and \; 2x - 3y - z = 5$$

Solution:

The given system can be written as $AX = B$, Where

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; and \; B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}.$$

Here the Augmented matrix is

$$\begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$R_2 \to 3R_2 - R_1, R_4 \to 3R_4 - 2R_1$$

$$\sim \begin{bmatrix} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 10 & 3 & -2 \\ 0 & -15 & -7 & 13 \end{bmatrix}$$

$$R_3 \to 3R_3 - 10R_2, R_4 \to R_4 + 5R_2$$

$$\sim \begin{bmatrix} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & -17 & 68 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{29}, R_4 \to \frac{R_4}{17}$$

$$\sim\begin{bmatrix}3 & 3 & 2 & 1\\ 0 & 3 & -2 & 11\\ 0 & 0 & 1 & -4\\ 0 & 0 & -1 & 4\end{bmatrix}$$

$$R_4 \to R_3 + R_4$$

$$\begin{bmatrix}3 & 3 & 2 & 1\\ 0 & 3 & -2 & 11\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Here $\rho(A) = \rho([A|b]) = 3$ and number of unknowns are 3.

So, the given equation is consistent with a unique solution.

Now the given system can be rewritten as,

$$\begin{bmatrix}3 & 3 & 2\\ 0 & 3 & -2\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 11\\ -4\\ 0\end{bmatrix}$$

$$\Rightarrow 3x + 3y + 2z = 1, 3y - 2z = 11 \; and \; z = -4$$

$$\Rightarrow z = -4\,, y = 1 \; and \; x = 2$$

Hence the solution of the system $x = 2, y = 1 \; and \; z = -4$.

Problem 3: Show that the system $x + y + z = 6, x + 2y + 3z = 14 \; and \; x + 4y + 7z = 30$ is consistent and solve it.

Solution:

The given system can be written as $AX = B$, Where

$$A = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 7\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} \; and \; B = \begin{bmatrix}6\\ 14\\ 30\end{bmatrix}.$$

Now Here, $[A|B] = \begin{bmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 14\\ 1 & 4 & 7 & 30\end{bmatrix}$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & 8\\0 & 3 & 6 & 24\end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & 8\\0 & 0 & 0 & 0\end{bmatrix}$$

This is in echelon form.

$$\rho(A) = 2, \rho([A|B]) = 2$$

So, the system is consistent. Now rank of A is less than the number of unknowns, therefore the system will have infinite solution.

Now given system can be written as,

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\8\\0\end{bmatrix}$$

$$\Rightarrow x + y + z = 6, y + 2z = 8$$

$$\Rightarrow y = 8 - 2z, x = 6 - y - z = 6 - 8 + z = z - 2$$

So, the solution are given by $= k, y = 8 - 2k \text{ and } x = k - 2$, Where k is any constant.

Problem 4: For what values of $\lambda \text{ and } \mu$ the system $x + y + z = 6,$ $x + 2y + 3z = 10 \text{ and } x + 2y + \lambda z = \mu$ have i) no solution ii) a unique solution iii) an infinite solution.

Solution:

Here augmented matrix is

$$\begin{bmatrix}1 & 1 & 1 & 6\\1 & 2 & 3 & 10\\1 & 2 & \lambda & \mu\end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\begin{bmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 1 & \lambda - 1 & \mu - 6\end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case I: Let $\lambda = 3\ and\ \mu \neq 10$, Then $\rho(A) \neq \rho([A|B])$.

Then the system is inconsistent, and the system has no solution.

Case-II: Let $\lambda \neq 3\ and\ \mu \neq 10$, Then $\rho(A) = \rho([A|B]) = 3$ also number of unknowns=3.

So, the system is consistent, and the system has unique solution.

Case III: Let $\lambda = 3\ and\ \mu = 10$,Then $\rho(A) = \rho([A|B]) = 2$ also number of unknowns=3>2.

So, the system is consistent, and the system has an infinite number of solutions.

Homogeneous Linear Equations:

Let us consider a system of m homogeneous equations in n unknowns $x_1, x_2, \dots, x_n$ as below:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

This system can be written as $AX = 0$, where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}\ and\ 0 = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Now $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is always a solution of the homogeneous system. This is called trivial solution of the system $AX = 0$. It is also called zero solution.

Working rule for finding the solution of the equation $AX = 0$:

1) If $r = n$ i.e. rank of A =number of variables, then the system has only trivial solution.

2) If $r < n$ then the system has an infinite number of nontrivial solutions, we will get $(n - r)$ linearly independent solutions.

Problem 1: Solve the system $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$.

Solution:

The given system can be written as $AX = 0$, Where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1 \text{ and } R_3 \to R_3 - 7R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

So here rank of A = Number of variables =3.

Hence the system of equation has a trivial solution.

So $x = 0, y = 0 \text{ and } z = 0$ is the only solution.

Problem 2: Solve the system of equation $x + 3y - 2z = 0, 2x - y - 4z = 0 \text{ and } x - 11y + 14z = 0$.

Solution:

The given system can be written as $AX = 0$, Where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the echelon matrix form. Number of nonzero rows are two.

So, rank of the matrix is 2. But number of variables is three. So, this will have $3 - 2 = 1$ non-zero solution.

Now the given system can be written as,

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 3y - 2z = 0, -7y + 8z = 0$$

Let $z = k$ then $y = \frac{8}{7}k \; and \; x = 2k - \frac{3.8}{7}k = -\frac{10}{7}k$

Hence the solutions are given by $x = -\frac{10}{7}k \, , y = \frac{8}{7}k \; and \; z = k.$

Problem 3: Show that the only real number λ for which the system $x + 2y + 3z = \lambda x, 3x + y + 2z = \lambda y, 2x + 3y + z = \lambda z$ has non-zero solution is 6 and solve it when $\lambda = 6$.

Solution:

The given system can be written as $AX = 0$where,

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

Here number of variables, $n = 3$. The given system will have a non-trivial solution if

Rank of A, less than number of unknowns. That is rank is less than three. For this we must have determinant of A is zero.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1 - 2\lambda + \lambda^2 - 6) - 2(3 - 3\lambda - 4) + 3(9 - 2 + 2\lambda) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-2\lambda-5)-2(-3\lambda-1)+3(2\lambda+7)=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-2\lambda-5)+(12\lambda+23)=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+15\lambda+18=0$$

$$\Rightarrow \lambda^3-3\lambda^2-15\lambda-18=0$$

$$\Rightarrow (\lambda-6)(\lambda^2+3\lambda+3)=0$$

So $\lambda = 6$ is the only real value and other values are complex for which this equation satisfies. (Proved)

Now when $\lambda = 6$ them,

$$A = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}$$

$$R_2 \to 5R_2 + 3R_1 \; and \; R_3 \to 5R_3 + 2R_1$$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}$$

Here rank of A is 2 which is less than the number of variables (Three). So, the system has infinite number of solutions.

So, the given system can be written as,

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x+2y+3z=0 \; and -19y+19z=0$$

$$\Rightarrow y = z \; and \; x = \frac{1}{5}.5z = z$$

Let $z = k$ then $x = y = z = k$

So, required solutions are given by $x = y = z = k$(k is arbitrary constant)

Eigenvalues, Eigenvectors and Diagonalization of Matrix

Out Lines

This present module will include the following topics:

(i) Eigenvalues and eigenvectors of matrices
(ii) Characteristic equations for matrices
(iii) Similarity between matrices
(iv) Diagonalization of matrices using the concept of eigenvalues
(v) Statement of Cayley-Hamilton theorem without proof and its application
(vi) Evaluation of the matrix's power using Cayley Hamilton theorem

Introduction

In the study of linear algebra, the theory of eigenvalues and eigenvectors are of much significant. An eigen vector (sometimes, it is called as characteristic vector) is a non-zero vector concerning to a linear transformation (like, matrix) which provides sense of stretching in particular direction by the linear transformation. There is a non-zero real value associated with eigenvector which is used as a factor (scale) for stretching vector. This is known as eigenvalue or characteristic value of the linear transformation corresponding to the eigenvector. Later, in our discussion, we will provide the mathematical definitions of eigenvector and eigenvalues in more precise sense.

Eigenvalues and eigenvectors

Let us consider a system of linear homogenous equations consisting n equations and n unknowns as follows:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0$$

The above system of linear equation can be written in a matrix notation as follows:

$AX = 0$, where $A = \begin{bmatrix} a_{11} & a_{12} \dots\dots & a_{1n} \\ a_{21} & a_{22} \dots\dots & a_{2n} \\ \dots & \dots. & \dots. \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$ is a $n \times n$ square matrix, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column matrix of $n \times 1$ order and 0 the zero matrix of $1 \times n$ order.

Similarly, for a scalar λ, we consider the following system of linear homogenous equations:

$$(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0$$

Here, also the system can be written in matrix notation as follows:

$(A-\lambda I)X = 0$, where λ is a scalar, $A = \begin{bmatrix} a_{11} & a_{12} \ldots.. & a_{1n} \\ a_{21} & a_{22} \ldots.. & a_{2n} \\ \cdots & \ldots. & \ldots. \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$ is a $n \times n$ square matrix, $I = \begin{bmatrix} 1 & 0 \ldots.. & 0 \\ 0 & 0 \ldots.. & 0 \\ \cdots & \ldots. & \ldots. \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix of $n \times n$ order, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column matrix of $n \times 1$ order and 0 the zero matrix of $1 \times n$ order.

Now, the above system $(A-\lambda I)X = 0$ has *trivial solution* $X = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, when the matrix $(A-\lambda I)$ is *non-singular*, i.e., $det\,(A-\lambda I) \neq 0$. So, for getting *non-trivial (non-zero)* solution X, the matrix $(A-\lambda I)$ should be *singular*, i.e., $det\,(A-\lambda I) = 0$. For such cases, $(A-\lambda I)$ is called the *characteristic matrix* and $det\,(A-\lambda I) = 0$ is called the *characteristic equation* corresponding to the linear transformation (matrix) A. More precisely, the characteristic equation can be written as follows:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \ldots.. & a_{1n} \\ a_{21} & a_{22}-\lambda \ldots.. & a_{2n} \\ \cdots & \ldots. & \ldots. \\ a_{n1} & a_{n2} & a_{nn}-\lambda \end{vmatrix} = 0 \qquad (2.1)$$

After expanding the determinant in the left-hand side of equation (2.1), we get a polynomial equation of λ in the following way:

$$(-1)^n\lambda^n + A_1\lambda^{n-1} + A_2\lambda^{n-2} + \cdots + A_n = 0 \qquad (2.2)$$

Equation (2.2) represents the characteristic equation as a polynomial equation for λ. In equation (2.2), A_1, A_2, ..., A_n are the functions of a_{ij} in Equation (2.1). Now, it is the time to define eigenvalue using the above equations.

Definition 2.1 (Eigenvalue) Suppose, A be a $n \times n$ square matrix. Then, the roots of the polynomial equation (2.2) (known as characteristic equation of the matrix A) is called the *eigenvalues* of the matrix A. In other terminologies, they are also said to be *latent roots* or *characteristic roots*.

Definition 2.2 (Eigenvector) Suppose, A be a $n \times n$ square matrix and λ be an eigenvalue of A. Then, corresponding to the eigen value λ, there exists a non-zero vector X such that $AX = \lambda X$. The nonzero vector X is called *eigenvector* or *characteristic vector* of the matrix A corresponding to the eigenvalue λ.

Properties of Eigenvalues and Eigenvectors

Property-1 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A. Then, the sum and product of eigenvalues are equal to trace and determinant of the matrix A. That is, $\sum_{i=1}^{n} \lambda_i = Trace(A) = \sum_{i=1}^{n} a_{ii}$ and $\prod_{i=1}^{n} \lambda_i = Det(A)$.

Example 2.1 Here is an application as the immediate application of Property-1. Let us take a matrix, $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 1 & 3 \end{bmatrix}$. Then, sum of the eigenvalues, $\sum_{i=1}^{n} \lambda_i = Trace(A) = 2 + 4 + 3 = 9$ and product of the eigenvalues, $\prod_{i=1}^{n} \lambda_i = Det(A) = 2(4 \times 3 - 5 \times 1) - 0(3 \times 3 - 2 \times 5) + 1(3 \times 1 - 2 \times 4) = 14 - 0 - 5 = 9$

Property-2 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ be an eigenvalue of A corresponding to the eigenvector X. Then, for k is an integer, λ^k will be an eigenvalue of A^k corresponding to the eigenvector X.

Example 2.2 Suppose, we take $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, a diagonal matrix. Then, its eigenvalues are 2, 4 and 3. Also, $A^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ has the eigenvalues 4, 16 and 9.

Property-3 The eigenvalues of a matrix $A = [a_{ij}]_{n\times n}$ and its transpose $A^T = [a_{ji}]_{n\times n}$ have same set of eigenvalues.

Property-4 Suppose, we take two square matrices, $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ and furthermore, $A = [a_{ij}]_{n\times n}$ is non-singular (hence, A^{-1} exists), then the matrices $A^{-1}B$ and BA^{-1} have same set of eigenvalues.

Property-5 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A. Then, for a scalar, $k \neq 0$, $k\lambda_i$, (for $i = 1,2,..,n$) are eigenvalues of the matrix kA.

Property-6 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ is an eigenvalue of A. Then, for a scalar, $k \neq 0$, $\lambda + k$ is an eigenvalue of the matrix $A + kI$, where I is the identity matrix of order $n \times n$.

Property-7 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A.Then, for a scalar, $k \neq 0$, $\lambda_i \pm k$, (for $i = 1,2,..,n$) are eigenvalues of the matrix $A \pm kI$, where I is the identity matrix of order $n \times n$.

Property-8 Suppose, $A = [a_{ij}]_{n\times n}$ be a non-singular $n \times n$ square matrix and λ_i, for $i = 1,2,..,n$ are eigenvalues of A. Then, ${\lambda_i}^{-1}$, for $i = 1,2,..,n$, are the eigenvalues of the inverse matrix A^{-1}.

Property-9 Suppose, $A = [a_{ij}]_{n\times n}$ be a non-singular $n \times n$ square matrix and λ is an eigenvalue of A. Then, $\frac{Det\ (A)}{\lambda}$ is the eigenvalue of the matrix $Adj(A)$.

Proof: We know that, $Adj(A)\ A = Det\ (A)I$. Taking the eigenvector, we get $Adj(A)(AX) = Det\ (A)X$. That is, $Adj(A)(\lambda X) = Det\ (A)X$. That is, $Adj(A)X = \frac{Det\ (A)}{\lambda}X$. So, $\frac{Det\ (A)}{\lambda}$ is the eigenvalue of the matrix $Adj(A)$.

Property-9 Suppose, $A = [a_{ij}]_{n\times n}$ be a $n \times n$ square matrix and P be a non-singular $n \times n$ square matrix. Then, the matrices A and $P^{-1}AP$ have same set of eigenvalues.

Property-10 The eigenvalues of upper triangular, lower triangular, scalar and diagonal matrices are the elements on the principal diagonal of the corresponding matrices.

Example 2.3 Suppose, we take $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 1 \\ 0 & 10 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 6 & 0 \\ 2 & 5 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. A is a diagonal matrix which have eigenvalues 2, 4 and 3. B is an upper triangular matrix which have eigenvalues 4, 10 and 7. C is a lower triangular matrix which have eigenvalues 2, 6 and 4. D is a scalar matrix which have eigenvalues 5,5 and 5.

Property-11 Suppose, $A = [a_{ij}]_{n\times n}$ be a real symmetric matrix, i.e., $A = A^T$. Then, all the eigen values of the matrix A are real.

Property-12 The eigen vectors corresponding to the distinct eigenvalues of a matrix are linearly independent.

Property-13 Suppose, $A = [a_{ij}]_{n\times n}$ be a real symmetric matrix, i.e., $A = A^T$. Then, the eigen vectors corresponding to the distinct eigen values are orthogonal.

Method for finding Eigenvalues and Eigenvectors of matrices

Step 1: Find eigenvalues solving the characteristic equation $det\ (A - \lambda I) = 0$. If λ_i, for $i = 1,2,..,n$ are roots of the equations, they are called eigenvalues of A.

Step 2: When the eigenvalues λ_i are distinct, reduce the coefficient matrix in the matrix equation $(A - \lambda_i I)X = 0$ to the echelon form. The rank of the echelon matrix should be less than n, usually $r = (n - 1)$. So, there are $n - r = 1$ linearly independent eigen vectors corresponding to each distinct eigenvalue.

Step 3: Suppose, two eigen values $\lambda_i = \lambda_i$. Then, reduce the coefficient matrix in the matrix equation $(A - \lambda_i I)X = 0$ to the echelon form. The rank of the echelon matrix should be less than n, usually $r = (n-2)$. So, there are $n - r = 2$ linearly independent eigen vectors corresponding to that pair of eigenvalues.

Example 2.3 Find the eigenvalues and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 6 \\ 1 & 2 \end{bmatrix}$.

Solution: The characteristic equation is $det\,(A - \lambda I) = 0$

That is $\begin{vmatrix} 1-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0 \Rightarrow (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = 4, -1$

So, the eigenvalues are 4 and -1.

The eigenvalues are distinct.

Now to find eigenvectors corresponding to the eigenvalue 4, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then, we have $\begin{bmatrix} 1-4 & 6 \\ 1 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad , \qquad R_2 \to R_2 + \tfrac{1}{3}R_1$$

Therefore, the rank of the coefficient matrix is $r = 1$. Also, the number of unknowns is $n = 2$. Therefore, there are $n - r = 1$ linearly independent eigenvectors corresponding to the eigenvalue 4. In such case, the above written matrix equation can be written as

$$-3x_1 + 6x_2 = 0 \Rightarrow x_1 = 2x_2$$

If we take $x_2 = k$, then $x_1 = 2k$ and hence the eigen vector can be written as $X = k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Now to find eigenvectors corresponding to the eigenvalue -1, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then, we have, $\begin{bmatrix} 1+1 & 6 \\ 1 & 2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad R_2 \to R_2 + \tfrac{1}{2}R_1$$

Therefore, the rank of the coefficient matrix is $r = 1$. Also, the number of unknowns is $n = 2$. Therefore, there are $n - r = 1$ linearly independent eigenvectors corresponding to the eigenvalue -1. In such case, the above written matrix equation can be written as

$$2x_1 + 6x_2 = 0 \Rightarrow x_1 = -3x_2$$

If we take $x_2 = k$, then $x_1 = -3k$ and hence the eigen vector can be written as $X = k\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Exercise 2.1 Find the eigenvalues and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 6 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 \\ 6 & 2 \end{bmatrix}$.

Example 2.4 Find the eigenvalues of the matrix, $3A^3 + 5A^2 - 6A + 2I$, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.

Solution: Since, $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ is an upper triangular matrix, the elements in its principal diagonal are the eigenvalues of this matrix. Therefore, the eigenvalues are 1, 3 and -2.

When 1 is the eigenvalue of A, the eigenvalue of $3A^3 + 5A^2 - 6A + 2I$ will be $3 \times 1^3 + 5 \times 1^2 - 6 \times 1 + 2 \times 1 = 4$.

When 3 is the eigenvalue of A, the eigenvalue of $3A^3 + 5A^2 - 6A + 2I$ will be $3 \times 3^3 + 5 \times 3^2 - 6 \times 3 + 2 \times 1 = 110$.

When -2 is the eigenvalue of A, the eigenvalue of $3A^3 + 5A^2 - 6A + 2I$ will be $3 \times (-2)^3 + 5 \times (-2)^2 - 6 \times (-2) + 2 \times 1 = 10$.

Then, eigenvalues of the matrix $3A^3 + 5A^2 - 6A + 2I$ are 4, 110 and 10.

Exercise 2.2 Find the eigenvalues of the matrices A^3 and A^{-1}, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 5 \\ 0 & 0 & -2 \end{bmatrix}$.

Exercise 2.3 Find the eigenvalues of the matrix A^{-1}, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Exercise 2.4 Find the eigenvalues and eigen vectors of the matrix $B = 2A^2 - \frac{1}{2}A + 3I$, where $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$.

Exercise 2.5 Find product and sum the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Exercise 2.6 Find the eigenvalues of the matrix A^{-1}, where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Example 2.5 Find the eigenvalues and eigen vectors of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution: The characteristic equation of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ can be written as

$$det\,(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Longrightarrow \lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\Longrightarrow \lambda = 0, 15, 3$$

So, the eigenvalues are $0, 15, 3$.

Now to find eigenvectors corresponding to the eigenvalue 0, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 4R_2 + 3R_1 \ and \ R_3 \to 4R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \to R_3 + R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$8x_1 - 6x_2 + 2x_3 = 0 \tag{2.3}$$

$$10x_2 - 10x_3 = 0 \tag{2.4}$$

From Equation (2.4), $x_2 = x_3$. Let, $x_2 = x_3 = k$. Then, from Equation (2.3), $x_1 = \frac{k}{2}$.

Therefore, the corresponding eigen vector can be written as $X = \begin{bmatrix} \frac{k}{2} \\ k \\ k \end{bmatrix} = \frac{k}{2} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Now to find eigenvectors corresponding to the eigenvalue 15, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ 0 & -20 & -40 \\ 0 & -40 & -80 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 7R_2 - 6R_1 \ and \ R_3 \to 7R_3 + 2R_1$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_3 \to R_3 - 2R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-7x_1 - 6x_2 + 2x_3 = 0 \tag{2.5}$$

$$-20x_2 - 40x_3 = 0 \tag{2.6}$$

From Equation (2.6), $x_2 = -2x_3$. Let $x_3 = k$, then $x_2 = -2k$ and from Equation (2.5), $x_1 = 2k$. Hence, the corresponding eigen vector can be written as $X = k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Now to find eigenvectors corresponding to the eigenvalue 3, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 5R_2 + 6R_1 \; and \; R_3 \to 5R_3 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \to 2R_3 - R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$5x_1 - 6x_2 + 2x_3 = 0 \tag{2.7}$$

$$-16x_2 - 8x_3 = 0 \tag{2.8}$$

From Equation (2.4), $x_2 = -\frac{1}{2}x_3$. Let, $x_3 = k$. Then, $x_2 = -\frac{k}{2}$ and from Equation (2.7), $x_1 = -k$. Therefore, the corresponding eigen vector can be written as $X = \frac{k}{2}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

Definition 2.3 (Orthogonal vector) Two vectors X_1 and X_2 are said to orthogonal, when ${X_1}^T X_2 = {X_2}^T X_1 = 0$, where ${X_1}^T$ and ${X_2}^T$ are transposes of the vectors X_1 and X_2, respectively.

Example 2.6 Let us consider two vectors $X_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $X_2 = \begin{bmatrix}0\\1\end{bmatrix}$. Then, ${X_1}^T X_2 = [1\ 0]\begin{bmatrix}0\\1\end{bmatrix} = 0$ and ${X_2}^T X_1 = [0\ 1]\begin{bmatrix}1\\0\end{bmatrix} = 0$. Therefore, X_1 and X_2 are orthogonal vectors.

Example 2.7 Let us consider three vectors $X_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $X_2 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$ and $X_2 = \begin{bmatrix}2\\-2\\1\end{bmatrix}$ Then,

${X_1}^T X_2 = [1\ 2\ 2]\begin{bmatrix}2\\1\\-2\end{bmatrix} = 2 + 2 - 4 = 0$

${X_2}^T X_3 = [2\ 1 - 2]\begin{bmatrix}2\\-2\\1\end{bmatrix} = 4 - 2 - 2 = 0$ and

${X_3}^T X_1 = [2 - 2\ 1]\begin{bmatrix}1\\2\\2\end{bmatrix} = 2 - 4 + 2 = 0$. Therefore, X_1 , X_2 and X_3 are orthogonal vectors.

Remark 2.1 If two eigenvalues of a matrix are equal, we cannot conclude whether the corresponding eigen vectors are linearly independent or not.

Definition 2.4 (Linearly independent vectors) We consider a set of non-zero vectors $\{X_1, X_2, \dots, X_n\}$ such that $a_1X_1 + a_2X_2 + \ \dots + a_nX_n = 0$. If there exists only the trivial solution $a_1 = a_2 =..= a_n = 0$ of the equation, then the given vectors $X_1, X_2, \dots, X_n$ are called the *linearly independent* vectors. If there exists at least one non-zero a_i such that the equation holds, then the vectors are called linearly dependent vectors.

Example 2.8 Let us consider three vectors $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $X_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $X_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Then, we consider the equation $a_1X_1 + a_2X_2 + a_3X_3 = 0$

This implies $a_1\begin{bmatrix}1\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

That is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

Here the rank of the coefficient matrix is $r = 3 = n$, the number of unknowns. So, the system has only the trivial solution $a_1 = a_2 = a_3 = 0$. Therefore, $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $X_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $X_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ are linearly independent vectors.

Exercise 2.7 Show that, three vectors $X_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $X_2 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$ and $X_2 = \begin{bmatrix}2\\-2\\1\end{bmatrix}$ are linearly independent vectors.

(Hint: Try to make an upper triangular matrix from the coefficient matrix).

Example 2.9 Verify that the sum of the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is equal to the $Trace(A)$. Also, find the corresponding eigen vectors.

Solution: We have the characteristic equation

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$$

$$\Rightarrow \lambda = 2, 3, 6$$

Therefore, eigenvalues are 2, 3 and 6 and hence their sum is 11. Again, $Trace(A)$ is the sum of the elements in the principal diagonal of the matrix and is equal to 11. Therefore, the statement is verified.

Next, to find eigenvectors corresponding to the eigenvalue 2, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to R_2 + R_1\ and\ R_3 \to R_3 - R_1$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$x_1 - x_2 + x_3 = 0 \tag{2.9}$$

$$2x_2 = 0 \tag{2.10}$$

From Equation (2.10), $x_2 = 0$. Let, $x_3 = k$. Then, from Equation (2.9), $x_1 = -k$. Therefore, the corresponding eigen vector can be written as $X = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 3, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 3-3 & -1 & 1 \\ -1 & 5-3 & -1 \\ 1 & -1 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_3 \leftrightarrow R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to R_2 + R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \to R_3 + R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$x_1 - x_2 = 0 \tag{2.11}$$

$$x_2 - x_3 = 0 \tag{2.12}$$

Let us take $x_3 = k$. Then, from Equation (2.11) and (2.12), we get $x_2 = x_1 = k$. Therefore, the corresponding eigen vector can be written as $X = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 6, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 3-6 & -1 & 1 \\ -1 & 5-6 & -1 \\ 1 & -1 & 3-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 3R_2 - R_1$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_3 \to R_3 - 2R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-3x_1 - x_2 + x_3 = 0 \tag{2.13}$$

$$-2x_2 - 4x_3 = 0 \tag{2.14}$$

Let us take $x_3 = k$. Then, from Equation (2.14), we get $x_2 = -2k$ and from Equation (2.13), we get, $x_1 = k$. Therefore, the corresponding eigen vector can be written as $X = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Exercise 2.8 Find the eigenvalues and the corresponding eigen vectors for the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Exercise 2.9 Find the eigenvalues and the corresponding eigen vectors for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$.

Definition 2.5 (Similar matrices) Two matrices S and T are said to be *similar* to each other when there exists a non-singular matrix P such that $T = P^{-1}SP$.

In this context, it is to be noted that a matrix is said to be non-singular if it has non-zero determinant and thus its inverse exists.

Definition 2.4 (Diagonalizable matrix) A square matrix is said be *diagonalizable* if it is similar to diagonal matrix. So, a square matrix S is said to be diagonalizable, if there exists a non-singular matrix P and a diagonal matrix $D = Diag(d_{11}, d_{22}, \ldots . d_{nn})$ such that $D = P^{-1}SP$.

Remark 2.2 The eigenvalues of the diagonal matrix D are its diagonal entries $d_{11}, d_{22}, \ldots . d_{nn}$. The non-singular matrix P can be formed by corresponding eigen vectors.

Remark 2.3 In Definition 2.4, the non-singular matrix P is makes the matrix S to be diagonal. Here, P is called the *modal matrix* and resulting diagonal matrix $D = Diag(d_{11}, d_{22}, \ldots . d_{nn})$ is called the *spectral matrix*.

Remark 2.4 The transformation of the matrix S to $P^{-1}SP$ is called the similarity transformation.

Example 2.10 Diagonalize the square matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 0 \\ 2 & -1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = \pm 1$$

Therefore, the eigenvalues are 1 and -1.

For the eigenvalue 1, let the eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have the matrix equation

$$\begin{bmatrix} 1-1 & 0 \\ 2 & -1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Clearly, the rank of the coefficient matrix in the above system is $r = 1 < 2 = n$, the number of unknown. So, there are $n - r = 2 - 1 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equation

$$2x_1 - 2x_2 = 0 \tag{2.15}$$

From Equation (2.15), we get $x_1 = x_2 = k$ and hence the corresponding eigen vector is

$$X = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For the eigenvalue -1, let the eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have the matrix equation

$$\begin{bmatrix} 1+1 & 0 \\ 2 & -1+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Clearly, the rank of the coefficient matrix in the above system is $r = 1 < 2 = n$, the number of unknown. So, there are $n - r = 2 - 1 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equation

$$2x_1 = 0 \tag{2.16}$$

From Equation (2.16), we get $x_1 = 0$. Here, $x_2 = k$ is arbitrary and hence the corresponding eigen vector is

$$X = k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using the eigen vectors, the matrix P can be formed as follows:

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Since, $\text{Det}(P) = 1 \neq 0$, we can say that P^{-1} and is given by

$$P^{-1} = \frac{Adj(P)}{Det\ (P)} = \frac{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Therefore, $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ -1+2 & 0-1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 1-1 & 0-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$, a diagonal matrix.

Hence, diagonalization of the given matrix is completed.

Example 2.11 Diagonalize the square matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0$$

$$\Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda = -2, 3 \text{ and } 6$$

Therefore, eigenvalues are -2, 3 and 6.

Next, to find eigenvectors corresponding to the eigenvalue -2, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 3R_2 - R_1 \ and\ R_3 \to R_3 - R_1$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$3x_1 + x_2 + 3x_3 = 0 \tag{2.17}$$

$$20x_2 = 0 \tag{2.18}$$

From Equation (2.18), we get $x_2 = 0$. Let, $x_3 = k$. Then, from Equation (2.17), $x_1 = -k$. Hence, the corresponding eigen vector will be $X = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 3, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 2R_2 + R_1 \ and\ R_3 \to 2R_3 + 3R_1$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ R_3 \to R_3 - R_2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-2x_1 + x_2 + 3x_3 = 0 \tag{2.19}$$

$$5x_2 + 5x_3 = 0 \tag{2.20}$$

From Equation (2.20), we get $x_2 = -x_3$. Let, $x_3 = k$. Then, from Equation (2.20), $x_2 = -k$ and from Equation (2.19), $x_1 = k$. Hence, the corresponding eigen vector will be $X = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Next, to find eigenvectors corresponding to the eigenvalue 6, let the eigenvector be $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 8 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \to 5R_2 + R_1 \ and\ R_3 \to 5R_3 + 3R_1$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ R_3 \to R_3 \mp 2$$

Clearly, the rank of the coefficient matrix in the above system is $r = 2 < 3 = n$, the number of unknown. So, there are $n - r = 3 - 2 = 1$ linearly independent solutions. Then solution can be derived by solving the system of equations

$$-5x_1 + x_2 + 3x_3 = 0 \tag{2.21}$$

$$-4x_2 + 8x_3 = 0 \tag{2.22}$$

From Equation (2.22), we get $x_2 = 2x_3$. Let, $x_3 = k$. Then, from Equation (2.22), $x_2 = 2k$ and from Equation (2.21), $x_1 = k$. Hence, the corresponding eigen vector will be $X = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Using the eigen vector the modal matrix can be obtained as

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Since, $\text{Det}(P) = -1(-1-2) - 1(0-2) + 1(0+1) = 6 \neq 0$, we can say that P^{-1} and is given by

$$P^{-1} = \frac{Adj(P)}{Det\ (P)} = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Therefore, $P^{-1}AP = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix}$$

$= \frac{1}{6}\begin{bmatrix} -12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$, a diagonal matrix.

Hence, diagonalization of the given matrix is completed.

Exercise 2.10 Diagonalize the square matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

Exercise 2.11 Find a non-singular matrix P which can diagonalize the square matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Exercise 2.12 Find a non-singular matrix P which can diagonalize the square matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Theorem 2.1 (Cayley-Hamilton) Every square matrix satisfies of its characteristic equation.

Example 2.12 Find the matrix A^3 and A^{-1} corresponding to the square matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ can be written as

$$det\ (A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 2 = 0 \tag{2.23}$$

This is the characteristic equation of the given matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$. Therefore, according the Cayley-Hamilton theorem,

$$A^2 - 3A - 2I = O \tag{2.24}$$

In Equation (2.24), $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity matrix and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the zero matrix. From Equation (2.24) we get

$$A^2 = 3A + 2I \tag{2.25}$$

Multiplying both side by the matrix A, we get

$$A^3 = 3A^2 + 2A$$

$$= 3\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= 3\begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 36 \\ 9 & 15 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 44 \\ 11 & 17 \end{bmatrix}$$

Again, from Equation (2.24), we can write

$$\frac{1}{2}(A - 3I)A = A\ \frac{1}{2}(A - 3I) = I$$

Therefore, $A^{-1} = \frac{1}{2}(A - 3I)$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$$

Exercise 2.13 Verify the state of the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Example 2.13 Find the matrix A^{-1} corresponding to the square matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Solution: The characteristic equation of the given matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ can be written as

$$det\,(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 38\lambda - 40 = 0 \tag{2.26}$$

Therefore, the characteristic equation of the given matrix is $\lambda^3 - 11\lambda^2 + 38\lambda - 40 = 0$. According to the Cayley-Hamilton theorem,

$$A^3 - 11A^2 + 38A - 40I = O \tag{2.27}$$

In Equation (2.27), $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the identity matrix and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the zero matrix. From Equation (2.27) we get,

$$A\frac{1}{40}(A^2 - 11A + 38I) = \frac{1}{40}(A^2 - 11A + 38I)A = I$$

Hence,

$$A^{-1} = \frac{1}{40}(A^2 - 11A + 38I)$$

$$= \frac{1}{40}\left(\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - 11\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} + 38\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{40}\left(\begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix} - \begin{bmatrix} 33 & 11 & 11 \\ -11 & 55 & -11 \\ 11 & -11 & 33 \end{bmatrix} + \begin{bmatrix} 38 & 0 & 0 \\ 0 & 38 & 0 \\ 0 & 0 & 38 \end{bmatrix}\right)$$

$$= \frac{1}{40}\begin{bmatrix} 14 & -4 & -6 \\ 2 & 8 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

Example 2.14 Find the matrix A^3 corresponding to the square matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Example 2.15 Find the matrix A^{-1} corresponding to the square matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, after verifying the statement of Cayley-Hamilton theorem.

Example 2.16 Find the matrices A^{-1} and A^4 corresponding to the square matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, using the statement of Cayley-Hamilton theorem.

Example 2.17 Verify the statement of Cayley-Hamilton theorem for the square matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.