UNIT-V

LAPLACE TRANSFORM

5.1 INTRODUCTION

A transformation is an operation which converts a mathematical expression to a different but equivalent form. The well known transformation logarithms reduce multiplication and division to a simpler process of addition subtraction.

The Laplace transform is a powerful mathematical technique which solves linear equations with given initial conditions by using algebra methods. The Laplace transform can also be used to solve systems of differential equations, Partial differential equations and integral equations. In this chapter, we will discuss about the definition, properties of Laplace transform and derive the transforms of some functions which usually occur in the solution of linear differential equations.

5.2 LAPLACE TRANSFORM

Let $f(t)$ be a function of t defined for all $t \geq 0$ .then the Laplace transform of$f(t)$, denoted by $L[\, f(t)]$ is defined by

$$L[\, f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Provided that the integral exists, "s" is a parameter which may be real or complex. Clearly $L[\, f(t)]$is a function of s and is briefly written as $F(s)$ $(i.e.)$ $L[\, f(t)] = F(s)$

Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous is an interval $a \leq t \leq b$, if the interval can be sub divided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

Exponential order

A function $f(t)$ is said to be exponential order if $\lim_{t\to\infty} e^{-st} f(t)$ is a finite quantity, where $s > 0$(exists).

Example: 5. 1 Show that the function $f(t) = e^{t^3}$ is not of exponential order.

Solution:

$$\lim_{t\to\infty} e^{-st}\, e^{t^3} = \lim_{t\to\infty} e^{-st+t^3} = \lim_{t\to\infty} e^{t^3-st}$$

$= e^{\infty} = \infty$, not a finite quantity.

Hence $f(t) = e^{t^3}$is not of exponential order.

Sufficient conditions for the existence of the Laplace transform

The Laplace transform of $f(t)$ exists if

i) $f(t)$ is piecewise continuous in the interval $a \leq t \leq b$

ii) $f(t)$ is of exponential order.

Note: The above conditions are only sufficient conditions and not a necessary condition.

Example: 5.2 Prove that Laplace transform of e^{t^2} does not exist.

Solution:

$$\lim_{t\to\infty} e^{-st}\, e^{t^2} = \lim_{t\to\infty} e^{-st+t^2} = \lim_{t\to\infty} e^{t^2-st}$$

$$= e^{\infty} = \infty \text{ ,not a finite quantity.}$$

$\therefore e^{t^2}$ is not of exponential order.

Hence Laplace transform of e^{t^2} does not exist.

5.3 PROPERTIES OF LAPLACE TRANSFORM

Property: 1 Linear property

$\boldsymbol{L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]}$ **, where a and b are constants.**

Proof:

$$L[af(t) \pm bg(t)] = \int_0^\infty [af(t) \pm bg(t)]\, e^{-st} dt$$

$$= a\int_0^\infty f(t) e^{-st} dt \pm b\int_0^\infty g(t)\, e^{-st} dt$$

$$\boxed{L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]}$$

Property: 2 Change of scale property.

If $\boldsymbol{L[f(t)] = F(s)}$**, then** $\boldsymbol{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$; $\boldsymbol{a} > 0$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$$

By the definition of Laplace transform, we have

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt \cdots\cdots (2)$$

Put at= x $ie.,\, t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$

$$(2) \Rightarrow L[f(at)] = \int_0^\infty e^{\frac{-sx}{a}} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{\frac{-sx}{a}} f(x) dx$$

Replace x by t, $L[f(at)] = \frac{1}{a} \int_0^\infty e^{\frac{-st}{a}} f(t) dt$

$$\boxed{\boldsymbol{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)} \ ;\ \boldsymbol{a} > 0}$$

Property: 3 First shifting property.

If $\boldsymbol{L[f(t)] = F(s)}$, **then i)** $\boldsymbol{L[e^{-at} f(t)] = F(s+a)}$

ii) $\boldsymbol{L[e^{at} f(t)] = F(s-a)}$

Proof:

(i) $L[e^{-at} f(t)] = F(s+a)$

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots (1)$$

By the definition of Laplace transform, we have

$$L[e^{-at} f(at)] = \int_0^\infty e^{-st}\, e^{-at} f(t)\, dt$$
$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$
$$= F(s+a) \quad \text{by (1)}$$

(ii) $L[e^{at} f(at)] = \int_0^\infty e^{-st}\, e^{at} f(t)\, dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$
$$= F(s-a) \quad \text{by (1)}$$

Property: 4 Laplace transforms of derivatives $L[f'(t)] = sL[f(t)] - f(0)$

Proof:

$$u = e^{-st}$$
$$\therefore du = -se^{-st} dt$$
$$dv = f'(t) dt$$
$$\therefore v = \int f'(t) dt$$
$$= f(t)$$

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt = \int_0^\infty u dv$$
$$= [uv]_0^\infty - \int u dv$$
$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty f(t)\, (-s) e^{-st}\, dt$$
$$= 0 - f(0) + sL[f(t)]$$
$$= sL[f(t)] - f(0)$$
$$\boldsymbol{L[f'(t)] = sL[f(t)] - f(0)}$$

Property: 5 Laplace transform of derivative of order n

$$\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)}$$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) \cdots\cdots (1)$

$$L[f^n(t)] = L[[f'(t)]']$$
$$= sL[f'(t)] - f'(0)$$
$$= s[sL[f(t)] - f(0)] - f'(0)$$
$$= s^2 L[f(t)] - sf(0) - f'(0)$$

Similarly, $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$

In general, $\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)}$

Laplace transform of integrals

Theorem: 1 If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t) dt$

$\therefore g'(t) = f(t)$

And $\quad g(0) = \int_0^0 f(t) dt = 0$

Now $L[g'(t)] = L[f(t)]$

$$sL[g(t)] - g(0) = L[f(t)]$$

$$sL[g(t)] = L[f(t)] \quad \therefore g(0) = 0$$

$$L[g(t)] = \frac{L[f(t)]}{s}$$

$$\therefore \boldsymbol{L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}}$$

Theorem: 2 If $L[f(t)] = F(s)$, then $L[tf(t)] = -\frac{d}{ds}F(s)$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$$

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)\, dt = \frac{d}{ds}F(s)$$

$$\int_0^\infty (-t)e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$$

$$-\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$$

$$-L[tf(t)] = \frac{d}{ds}F(s)$$

$$\therefore \boldsymbol{L[tf(t)] = -\frac{d}{ds}F(s)}$$

Note: In general $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$

Example: 5.3 If $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}$ then find $L[f(2t)]$.

Solution:

Given $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$

$$L[f(2t)] = \frac{1}{2}F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2}+1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$$

$$= \frac{1}{2}\frac{\left[\frac{s^2}{4} - \frac{s}{2}+4\right]}{(s+1)^2\left(\frac{s-2}{2}\right)}$$

$$= \frac{s^2-2s+1}{4(s+1)^2(s-2)}$$

Laplace transform of some Standard functions

Result: 1 Prove that $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

$$L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}}\, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$\therefore \boldsymbol{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}}$ $\qquad \because \int_0^\infty e^{-u} u^n\, du$

Let $st = u \cdots\cdots (1)$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

When $t \to 0 (1) => u \to 0$

,

$t \to \infty, (1) => u \to \infty$

Note: If n is an integer, then $\Gamma(n+1) = n!$

$\therefore L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer

If $n = 0$, then $L[1] = \frac{1}{s}$

If $n = 1$, then $L[t] = \frac{1}{s^2}$

Similarly $L[t^2] = \frac{2!}{s^3}$

$L[t^3] = \frac{3!}{s^4}$

Result: 2 Prove that $\boldsymbol{L(e^{at}) = \frac{1}{s-a}}$, $\boldsymbol{s > a}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s-a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s-a}$$

Result: 3 Prove that $\boldsymbol{L(e^{-at}) = \frac{1}{s+a}}$, $\boldsymbol{s > a}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s+a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s+a}$$

Result: 4 Prove that $L[sinat] = \frac{a}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[sinat] = \int_0^\infty e^{-st}\, sinat\, dt$$

$\therefore L[sinat] = \frac{a}{s^2+a^2}, s > |a|$ $\left[\because \int_0^\infty e^{-at}\, sinbt\, dt = \frac{b}{a^2+b^2}\right]$

Result: 5 Prove that $L[cosat] = \frac{s}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[cosat] = \int_0^\infty e^{-st}\, cosat\, dt$$

$\therefore L[cosat] = \frac{s}{s^2+a^2}, s > |a|$ $\because \int_0^\infty e^{-at}\, cosbt\, dt = \frac{a}{a^2+b^2}$

Result: 6 Prove that $L[sinhat] = \frac{a}{s^2-a^2}\ , s > |a|$

Proof:

We have $L[sinhat] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$\therefore L[sinhat] = \frac{a}{s^2-a^2}\ , s > |a|$

Result: 7 Prove that $L[coshat] = \frac{s}{s^2-a^2}\ , s > |a|$

Proof:

We have $L[coshat] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$\therefore L[coshat] = \frac{s}{s^2-a^2}\ , s > |a|$

Example: 5.4 Find $L\left[t^{\frac{1}{2}}\right]$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = \frac{1}{2}$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} \qquad \because \Gamma(n+1) = n\Gamma n$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}+1}} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\sqrt{\pi}}{2s\sqrt{s}}$$

Example: 5.5 Find the Laplace transform of $t^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{t}}$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = -\frac{1}{2}$

$$\therefore L\left[t^{-\frac{1}{2}}\right] = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} \qquad \because \Gamma(n+1) = n\Gamma n$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}$$

FORMULA

$L[f(t)] = F(s)$	$L[f(t)] = F(s)$
$L[1] = \frac{1}{s}$ $L[t] = \frac{1}{s^2}$ $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ **if n is not an integer** $L[t^n] = \frac{n!}{s^{n+1}}$ **if n is an integer** $L(e^{at}) = \frac{1}{s-a}$ $L(e^{at}) = \frac{1}{s+a}$	$L[sinat] = \frac{a}{s^2 + a^2}$ $L[cosat] = \frac{s}{s^2 + a^2}$ $L[coshat] = \frac{s}{s^2 - a^2}$ $L[sinhat] = \frac{a}{s^2 - a^2}$

Problems using Linear property

Example: 5.6 Find the Laplace transform for the following

i.	$3t^2 + 2t + 1$	v.	$sin\sqrt{2}\,t$	ix.	$sin^2 t$
ii.	$(t+2)^3$	vi.	$sin(at+b)$	x.	$cos^2 2t$
iii.	a^t	vii.	$cos^3 2t$	xi.	$cos5tcos4t$

iv. e^{2t+3}	**viii.** $sin^3 t$	

Solution:

(i) Given $f(t) = 3t^2 + 2t + 1$

$$L[f(t)] = L[3t^2 + 2t + 1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= 3L[t^2] + 2L[t] + L[1]$$
$$= 3\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$

$$\therefore L[3t^2 + 2t + 1] = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(ii) Given $f(t) = (t+2)^3 = t^3 + 3t^2(2) + 3t2^2 + 2^3$

$$L[f(t)] = L[t^3 + 3t^2(2) + 3t2^2 + 2^3]$$
$$= L[t^3] + L[6t^2] + L[12t] + L[8]$$
$$= L[t^3] + 6L[t^2] + 12L[t] + 8L[1]$$
$$= \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{12}{s}$$

(iii) Given $f(t) = a^t$

$$L[f(t)] = L[a^t] = L[e^{tloga}]$$
$$L[a^t] = \frac{1}{s - loga}$$

(iv)Given $f(t) = e^{2t+3}$

$$L[f(t)] = L[e^{2t+3}] = L[e^{2t}.e^3]$$
$$= e^3 L[e^{2t}]$$
$$= e^3\left[\frac{1}{s-2}\right]$$
$$\therefore L[e^{2t+3}] = e^3\left[\frac{1}{s-2}\right]$$

(v) $L\left[sin\sqrt{2}t\right] = \frac{\sqrt{2}}{s^2+2}$

(vi)Given $f(t) = \sin(at+b) = sinatcosb + cosatsinb$

$$L[f(t)] = L[\sin(at+b)]$$
$$= L[sinatcosb + cosatsinb]$$
$$= cosb\ L[\text{sin}at] + sinb\ L[\text{cos}at]$$
$$L[\sin(at+b)] = cosb\frac{s}{s^2+a^2} + sinb\ \frac{s}{s^2+a^2}$$

(vii) Given $f(t) = \cos^3 2t = \frac{1}{4}[3cos2t + cos6t]$

$$\because\ \cos^3\theta = \frac{3cos\theta + cos3\theta}{4}$$

$$L[f(t)] = \frac{1}{4}L[3cos2t + cos6t]$$
$$= \frac{1}{4}[3L(cos2t) + L(cos6t)]$$
$$= \frac{1}{4}\left[3\frac{s}{s^2+4} + \frac{s}{s^2+36}\right]$$

$$L[\cos^3 2t] = \frac{1}{4}\left[3\frac{s}{s^2+4} + \frac{s}{s^2+36}\right]$$

(viii) Given $f(t) = \sin^3 t = \frac{1}{4}[3sint - sin3t]$

$$L[f(t)] = \frac{1}{4}L[3sint - sin3t]$$

$$= \frac{1}{4}[3L(sint) - L(sin3t)]$$

$$= \frac{1}{4}\left[3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$L[\sin^3 t] = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

(ix) Given $f(t) = \sin^2 t = \frac{1-cos2t}{2}$

$$L[f(t)] = L\left[\frac{1-cos2t}{2}\right]$$

$$= \frac{1}{2}[L(1) - L(cos2t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

(x) Given $f(t) = \cos^2 2t = \frac{1+cos4t}{2}$

$$L[f(t)] = L\left[\frac{1+cos4t}{2}\right]$$

$$= \frac{1}{2}[L(1) + L(cos4t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

$$L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

(xi) Given $f(t) = cos5tcos4t$

$$L[f(t)] = L[cos5tcos4t]$$

$$= \frac{1}{2}[L(cos9t) + L(cost)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+81} + \frac{s}{s^2+1}\right]$$

Problems using First Shifting theorem

$$\boxed{\begin{gathered} \boldsymbol{L[e^{-at}f(t)] = L[f(t)]_{s\to s+a}} \\ \boldsymbol{L[e^{at}f(t)] = L[f(t)]_{s\to s-a}} \end{gathered}}$$

Example: 5.7 Find the Laplace transform for the following:

i. $\boldsymbol{te^{-3t}}$	**vii.** $\boldsymbol{t^2 2^t}$
ii. $\boldsymbol{t^3 e^{2t}}$	**viii.** $\boldsymbol{t^3 2^{-t}}$
iii. $\boldsymbol{e^{4t}sin2t}$	**ix.** $\boldsymbol{e^{-2t}sin3tcos2t}$
iv. $\boldsymbol{e^{-5t}cos3t}$	**x.** $\boldsymbol{e^{-3t}cos4tcos2t}$
v. $\boldsymbol{sinh2tcos3t}$	**xi.** $\boldsymbol{e^{4t}cos3tsin2t}$

vi. cosh3tsin2t	

(i) te^{-3t}

$$L[te^{-3t}] = L[t]_{s\to s+3}$$

$$= \left(\frac{1}{s^2}\right)_{s\to s+3} \qquad \because L(t) = \frac{1}{s^2}$$

$$\therefore L[te^{-3t}] = \frac{1}{(s+3)^2}$$

(ii) t^3e^{2t}

$$L[t^3e^{2t}] = L[t^3]_{s\to s-2}$$

$$= \left(\frac{3!}{s^4}\right)_{s\to s-2} \qquad \because L(t) = \frac{3!}{s^{3+1}}$$

$$\therefore L[t^3e^{2t}] = \frac{6}{(s-2)^4}$$

(iii) $e^{4t}sin2t$

$$L[e^{4t}sin2t] = L[sin2t]_{s\to s-4}$$

$$= \left(\frac{2}{s^2+2^2}\right)_{s\to s-4}$$

$$= \frac{2}{(s-4)^2+4}$$

$$= \frac{2}{s^2-8s+16+4}$$

$$\therefore\ L[e^{4t}sin2t] = \frac{2}{s^2-8s+20}$$

(iv) $L\left[e^{-5t}cos3t\right]$

$$L[e^{-5t}cos3t] = L[cos3t]_{s\to s+5}$$

$$= \left(\frac{s}{s^2+3^2}\right)_{s\to s+5}$$

$$= \frac{s+5}{(s+5)^2+9}$$

$$= \frac{s+5}{s^2+10s+25+9}$$

$$\therefore\ L[e^{-5t}cos3t] = \frac{s+5}{s^2+10s+34}$$

(v) $L[sinh2tcos3t]$

$$L[sinh2tcos3t] = L\left[\left(\frac{e^{2t}-e^{-2t}}{2}\right)cos3t\right]$$

$$= \frac{1}{2}[L(e^{2t}cos3t) - L(e^{-2t}cos3t)]$$

$$= \frac{1}{2}[L(cos3t)_{s\to s-2} - L(cos3t)_{s\to s+2}]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+3^2}\right)_{s\to s-2} - \left(\frac{s}{s^2+3^2}\right)_{s\to s+2}\right]$$

$$\therefore L[sinh2tcos3t] = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+9} - \frac{s+2}{(s+2)^2+9}\right]$$

(vi) $L[cosh3tsin2t]$

$$L[cosh3tsin2t] = L\left[\left(\frac{e^{3t}+e^{-3t}}{2}\right)sin2t\right]$$

$$= \frac{1}{2}[L(e^{3t}sin2t) + L(e^{-3t}sin2t)]$$

$$= \frac{1}{2}[L(sin2t)_{s\to s-3} + L(sin2t)_{s\to s+3}]$$

$$= \frac{1}{2}\left[\left(\frac{2}{s^2+2^2}\right)_{s\to s-3} + \left(\frac{2}{s^2+2^2}\right)_{s\to s+3}\right]$$

$$\therefore\ L[cosh3tsin2t] = \frac{1}{2}\left[\frac{2}{(s-3)^2+4} + \frac{2}{(s+3)^2+4}\right]$$

(vii) $\boldsymbol{t^2 2^t}$

$$L[t^2 2^t] = L\left[t^2 e^{log2^t}\right]$$

$$= L[t^2 e^{tlog2}] = L[t^2]_{s\to s-log2}$$

$$= \left(\frac{2!}{s^3}\right)_{s\to s-log2}$$

$$= \frac{2}{(s-log2)^3}$$

$$\therefore L[t^2 2^t] = \frac{2}{(s-log2)^3}$$

(viii) $\boldsymbol{t^3 2^{-t}}$

$$L[t^3 2^{-t}] = L\left[t^3 e^{log2^{-t}}\right]$$

$$= L[t^3 e^{-tlog2}] = L[t^3]_{s\to s+log2}$$

$$= \left(\frac{3!}{s^4}\right)_{s\to s+log2}$$

$$= \frac{6}{(s+log2)^4}$$

$$\therefore L[t^3 2^{-t}] = \frac{6}{(s+log2)^4}$$

(ix) $\boldsymbol{L[e^{-2t}sin3tcos2t]}$

$$L[e^{-2t}sin3tcos2t] = L[sin3tcos2t]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin(3t+2t) + \sin(3t-2t)]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin 5t + \sin t]_{s\to s+2}$$

$$= \frac{1}{2}[\mathrm{L}(\sin 5t) + L(\sin t)]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} + \frac{1}{s^2+1^2}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}\right]$$

$$\therefore L[e^{-2t}sin3tcos2t] == \frac{1}{2}\left[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}\right]$$

(x) $\boldsymbol{L[e^{-3t}cos4tcos2t]}$

$$L[e^{-3t}cos4tcos2t] = L[cos4tcos2t]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos(4t+2t) + \cos(4t-2t)]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos 6\mathrm{t} + \cos 2t]_{s\to s+3}$$

$$= \frac{1}{2}[\mathrm{L}(\cos 6\,t) + L(\cos 2\,t)]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+6^2} + \frac{s}{s^2+2^2}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}\right]$$

$$\therefore L[e^{-3t}cos4tcos2t] = \frac{1}{2}\left[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}\right]$$

(xi) $L[e^{4t}cos3tsin2t]$

$$L[e^{4t}cos3tsin2t] = L[cos3tsin2t]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin(3t+2t) - \sin(3t-2t)]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin 5t - \sin t]_{s\to s-4}$$

$$= \frac{1}{2}[\mathrm{L}(\sin 5t) - L(\sin t)]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1}\right]$$

$$\therefore L[e^{4t}cos3tsin2t] = \frac{1}{2}\left[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1}\right]$$

Exercise: 5.1

Find the Laplace transform for the following

1. $\cos^2 3t$ **Ans:** $\frac{1}{4}\left[\frac{3s}{s^2+9} + \frac{s}{s^2+81}\right]$
2. $sin3tcos4t$ **Ans:** $\frac{1}{4}\left[\frac{7}{s^2+49} - \frac{1}{s^2+1}\right]$
3. te^{2t} **Ans:** $\frac{1}{(s-2)^2}$
4. t^4e^{-3t} **Ans:** $\frac{4!}{(s-3)^5}$
5. $e^{4t}sin2t$ **Ans:** $\frac{2}{(s-4)^2+4}$
6. $e^{-5t}cos3t$ **Ans:** $\frac{s+5}{(s+5)^2+9}$
7. $t^3 3^t$ **Ans:** $\frac{3!}{(s-log3)^4}$
8. $t^5 4^{-t}$ **Ans:** $\frac{5!}{(s+log4)^6}$
9. $e^{-2t}sin3tcos2t$ **Ans:** $\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}$
10. $e^{-3t}cos4tcos2t$ **Ans:** $\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}$
11. $sinhtsin4t$ **Ans:** $\frac{4}{(s-1)^2+16} - \frac{4}{(s+1)^2+16}$
12. $cosh2tcos2t$ **Ans:** $\frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} - \frac{s+2}{(s+2)^2+4}\right]$

5.4 LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

Problems using the formula

$$\boldsymbol{L[tf(t)] = \frac{-d}{ds}L[f(t)]}$$

Example: 5.8 Find the Laplace transform for $\boldsymbol{tsin4t}$

Solution:

$$L[tsin4t] = \frac{-d}{ds}L[tsin4t]$$

$$= \frac{-d}{ds}\left[\frac{4}{s^2+4}\right]$$

$$= \frac{-[(s^2+16)0-4(2s)]}{(s^2+16)^2}$$

$$\therefore L[tsin4t] = \frac{8s}{(s^2+16)^2}$$

Example: 5.9 Find $\boldsymbol{L[tsin^2t]}$

Solution:

$$L[tsin^2t] = \frac{-d}{ds}L[\sin^2 t] = \frac{-d}{ds}L\left[\frac{(1-cos2t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) - L(cos2t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{\boldsymbol{s}}{\boldsymbol{s^2+4}}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{\boldsymbol{s^2+4-s^2}}{\boldsymbol{s(s^2+4)}}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{\boldsymbol{4}}{\boldsymbol{s(s^2+4)}}\right]$$

$$= -\frac{4}{2}\frac{d}{ds}\left[\frac{\boldsymbol{1}}{\boldsymbol{s(s^2+4)}}\right]$$

$$= -2\left[\frac{\boldsymbol{0-(3s^2+4)}}{\boldsymbol{(s^3+4s)^2}}\right]$$

$$\therefore L[tsin^2t] = \frac{\boldsymbol{2(3s^2+4)}}{\boldsymbol{(s^3+4s)^2}}$$

Example: 5.10 Find $\boldsymbol{L[tcos^2 2t]}$

Solution:

$$L[cos^2 2t] = \frac{-d}{ds}L[\cos^2 2t] = \frac{-d}{ds}L\left[\frac{(1+cos4t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) + L(cos4t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} + \frac{\boldsymbol{s}}{\boldsymbol{s^2+16}}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{\boldsymbol{(s^2+16)1-s.2s}}{\boldsymbol{(s^2+16)^2}}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{\boldsymbol{s^2+16-2s^2}}{\boldsymbol{(s^2+16)^2}}\right]$$

$$\therefore L[cos^2 2t] = \frac{1}{2}\left[\frac{1}{s^2} - \frac{\boldsymbol{16-s^2}}{\boldsymbol{(s^2+16)^2}}\right]$$

Example: 5.11 Find the Laplace transform for $tsinh2t$

Solution:

$$L[sinh2t] = \frac{-d}{ds}L[sinh2t]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2-4}\right]$$

$$= \frac{-[(s^2-4)0-2(2s)]}{(s^2-4)^2}$$

$$\therefore L[tsinh2t] = \frac{4s}{(s^2-4)^2}$$

Example: 5.12 Find the Laplace transform for $f(t) = sinat - atcosat$

Solution:

$$L[sinat - atcosat] = \mathrm{L}(\sin at) - a\,\mathrm{L(tcosat)}$$

$$= \frac{a}{s^2+a^2} - a\left(\frac{-d}{ds}L[cosat]\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{s^2+a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a(s^2+a^2)+a(a^2-s^2)}{(s^2+a^2)^2}$$

$$= \frac{as^2+a^3+a^3-as^2)}{(s^2+a^2)^2}$$

$$\therefore L[sinat - atcosat] = \frac{2a^3}{(s^2+a^2)^2}$$

Example: 5.13 Find the Laplace transform for the following

(i) $te^{-3t}sin2t$ (ii) $te^{-t}cosat$ 9iii) $tsinhtcos2t$

Solution:

(i) $L[te^{-3t}sin2t] = L[tsin2t]_{s\to s+3} = \frac{-d}{ds}L[sin2t]_{s\to s+3}$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+2^2}\right)_{s\to s+3}$$

$$= \left[\frac{(s^2+4)0-2(2s)}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s+3}$$

$$\therefore \quad L[te^{-3t}sin2t] = \frac{4(s+3)}{((s+3)^2+4\,)^2}$$

(ii) $L[te^{-t}cosat] = L[tcosat]_{s\to s+1} = \frac{-d}{ds}L[cosat]_{s\to s+1}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right)_{s\to s+1}$$

$$= -\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$\therefore\ L[te^{-t}cosat] = \frac{(s+1)^2-a^2}{((s+1)^2+a^2)^2}$$

(iii) $L[tsinhtcos2t]$

$$L[tsinhtcos2t] = L\left[t\left(\frac{e^t-e^{-t}}{2}\right)cos2t\right]$$

$$= \frac{1}{2}[L(te^tcos2t) - L(te^{-t}cos2t)]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L[cos2t]_{s\to s-1} + \frac{d}{ds}L[cos2t]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \frac{d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$\therefore L[tsinhtcos2t] = \frac{1}{2}\left[\frac{(s-1)^2-4}{((s-1)^2+4)^2} + \frac{4-(s+1)^2}{((s+1)^2+4)^2}\right]$$

Problems using the formula

$$\boldsymbol{L[t^2f(t)] = \frac{d^2}{ds^2}L[f(t)]}$$

Example: 5.14 Find the Laplace transform for (i) t^2sint (ii) t^2cos2t

Solution:

(i) $L[t^2sint] = \frac{d^2}{ds^2}L[sint]$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{d}{ds}\left(\frac{[(s^2+1)0-1(2s)]}{(s^2+1)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right)$$

$$= -2\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)$$

$$= \frac{-2\left[(s^2+1)^2(1)-s(2)(s^2+1)(2s)\right]}{(s^2+1)^4}$$

$$= \frac{-2(s^2+1)[(s^2+1)-4s^2]}{(s^2+1)^4}$$

$$= \frac{-2[1-3s^2]}{(s^2+1)^3}$$

$$\therefore L[t^2sint] = \frac{6s^2-2}{(s^2+1)^3}$$

(ii) $L[t^2cos2t] = \frac{d^2}{ds^2}L[cos2t]$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds}\left(\frac{[(s^2+4)1-s(2s)]}{(s^2+4)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)$$

$$= \frac{\left[(s^2+4)^2(-2s)-(4-s^2)2(s^2+4)(2s)\right]}{(s^2+4)^4}$$

$$= \frac{2s(s^2+4)[(s^2+4)(-1)-(4-s^2)2]}{(s^2+4)^4}$$

$$= \frac{2s[s^2-12]}{(s^2+4)^3}$$

$\therefore L[t^2 \cos 2t] = \frac{2s[s^2-12]}{(s^2+4)^3}$

Example: 5.15 Find the Laplace transform for (i) $t^2e^{-2t}cost$ (ii) $t^2e^{4t}sin3t$

Solution:

(i) $L[t^2e^{-2t}cost] = L[t^2cost]_{s\to s+2} = \frac{d^2}{ds^2}L[cost]_{s\to s+2}$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \left[\frac{\left[(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)(2s)\right]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= (s^2+1)\left[\frac{[(s^2+1)(-2s)-4s(1-s^2)]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= \left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right]_{s\to s+2}$$

$$= \left[\frac{2s^3-6s}{(s^2+1)^3}\right]_{s\to s+2}$$

$\therefore \quad L[t^2e^{-2t}cost] = \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}$

(ii) $L[t^2e^{4t}sin3t] = L[t^2sin3t]_{s\to s-4} = \frac{d^2}{ds^2}L[sin3t]_{s\to s-4}$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]_{s\to s-4} = -6\frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= -6\left[\frac{\left[(s^2+9)^2(1)-(s)2(s^2+9)(2s)\right]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6(s^2+9)\left[\frac{[(s^2+9)-4s^2]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]_{s\to s-4}$$

$$= \left[\frac{18s^2-54}{(s^2+9)^3}\right]_{s\to s-4}$$

$$\therefore\ L[t^2e^{4t}sin3t] = \frac{18(s-4)^2-54}{((s-4)^2+9)^3}$$

Exercise: 5.2

Find the Laplace transform for the following

1. $tsinat$ **Ans:** $\frac{2as}{(s^2+a^2)^2}$
2. $tcosat$ **Ans**: $\frac{s^2-a^2}{(s^2+a^2)^2}$
3. $te^{-4t}sin3t$ **Ans:** $\frac{6(s+4)}{(s+4)^2+9}$
4. $tcos2tsin6t$ **Ans:** $\frac{8s}{(s^2+64)^2} - \frac{4s}{(s^2+16)^2}$
5. $te^{-2t}cos2t$ **Ans:** $\frac{(s-2)^2-4}{((s+4)^2+4)^2}$

Problems using the formula

$$\boldsymbol{L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]ds}$$

This formula is valid if $\lim_{t\to 0}\frac{f(t)}{t}$ is finite.

The following formula is very useful in this section

$$\int \frac{ds}{s} = logs$$

$$\int \frac{ds}{s+a} = \log(s+a)$$

$$\int \frac{s\,ds}{s^2+a^2} = \frac{1}{2}\log(s^2+a^2)$$

$$\int \frac{a\,ds}{s^2+a^2} = \tan^{-1}\frac{s}{a}$$

Example: 5.16 Find $\boldsymbol{L\left[\frac{cosat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{cosat}{t} = \frac{cosa(0)}{0} = \frac{1}{0} = \infty$$

∴ Laplace transform does not exists.

Example: 5.17 Find $\boldsymbol{L\left[\frac{sinat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{sinat}{t} = \frac{sina(0)}{0} = \frac{0}{0}$$

$$= \lim_{t\to 0} acosat \qquad \text{(by applying L−Hospital rule)}$$

$$\lim_{t\to 0} acosat = acos0 = a$$, finite quantity.

Hence Laplace transform exists

$$L\left[\frac{sinat}{t}\right] = \int_s^\infty L[(sinat)]ds$$
$$= \int_s^\infty \frac{a}{s^2+a^2} ds$$
$$= \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$
$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{a}\right]$$
$$= \left[\frac{\pi}{2} - \tan^{-1}\frac{s}{a}\right]$$
$$\therefore L\left[\frac{sinat}{t}\right] = cot^{-1}\frac{s}{a}$$

Example: 5.18 Find $L\left[\frac{sin^3t}{t}\right]$

Solution:

$$\frac{sin^3t}{t} = \frac{3sint - sin3t}{4t}$$
$$\lim_{t\to 0}\frac{sin^3t}{t} = \lim_{t\to 0}\frac{3sint - sin3t}{4t}$$
$$= \frac{0-0}{0} = \frac{0}{0} \qquad \text{(by applying L}-\text{Hospital rule)}$$
$$= \lim_{t\to 0}\frac{3sint - sin3t}{4t} = 0$$

Hence Laplace transform exists

$$L\left[\frac{sin^3t}{t}\right] = L\left[\frac{3sint - sin3t}{4t}\right]$$
$$= \frac{1}{4}\int_s^\infty L[(3sint - sin3t)]ds$$
$$= \frac{1}{4}\int_s^\infty \left(3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right) ds$$
$$= \frac{1}{4}\left[3\tan^{-1}s - \tan^{-1}\frac{s}{3}\right]_s^\infty$$
$$= \frac{1}{4}\left[3(\tan^{-1}\infty - \tan^{-1}s) - \left(\tan^{-1}\infty - \tan^{-1}\frac{s}{3}\right)\right]$$
$$= \frac{1}{4}\left[(\frac{\pi}{2} - \tan^{-1}s) - \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right)\ \right]$$
$$= \frac{1}{4}\left[\cot^{-1}s - cot^{-1}\frac{s}{3}\right]$$

Example: 5.19 Find $L\left[e^{-2t}\frac{sin2tcos3t}{t}\right]$

Solution:

$$L\left[e^{-2t}\frac{\boldsymbol{sin2tcos3t}}{\boldsymbol{t}}\right] = L\left[\frac{\boldsymbol{sin2tcos3t}}{\boldsymbol{t}}\right]_{s\to s+2}$$
$$= \frac{1}{2}\left[\int_s^\infty L(\sin(3t+2t) - \sin(3t-2t))ds\right]_{s\to s+2}$$
$$= \frac{1}{2}\left[\int_s^\infty L((\sin 5t) - L(\sin t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\tan^{-1}\frac{s}{5} - \tan^{-1} s\right]_s^\infty\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\left(\tan^{-1}\infty - \tan^{-1}\frac{s}{5}\right) - (\tan^{-1}\infty - \tan^{-1} s)\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{5}\right) - \left(\frac{\pi}{2} - \tan^{-1} s\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{s}{5} - cot^{-1} s\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{(s+2)}{5} - cot^{-1}(s+2)\right]$$

Example: 5.20 Find the Laplace transform for $\frac{e^{-at}-e^{-bt}}{t}$

Solution:

$$\lim_{t\to 0}\frac{e^{-at}-e^{-bt}}{t} = \lim_{t\to 0}\frac{e^0-e^0}{0} = \frac{1-1}{0} = \frac{0}{0} \qquad \text{(use L– Hospital rule)}$$

$$= \lim_{t\to 0}\frac{-ae^{-at}+be^{-bt}}{1}$$

$$= -a + b = b - a = \text{a finite quantity}$$

Hence Laplace transform exists.

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty L[e^{-at} - e^{-bt}]ds$$

$$= \int_s^\infty [L(e^{-at}) - L(e^{-bt})]ds$$

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= [\log(s+a) - \log(s+b)]_s^\infty$$

$$= \left[log\frac{s+a}{s+b}\right]_s^\infty$$

$$= \left[log\frac{s\left(1+\frac{a}{s}\right)}{s\left(1+\frac{b}{s}\right)}\right]_s^\infty$$

$$= log 1 - log\frac{s+a}{s+b} = 0 - log\frac{s+a}{s+b} \qquad \because log 1 = 0$$

$$= log\frac{s+a}{s+b}$$

Example: 5.21 Find the Laplace transform of $\frac{1-cost}{t}$

Solution:

$$\lim_{t\to 0}\frac{\mathbf{1-cost}}{\mathbf{t}} = \frac{0}{0} \qquad \lim_{t\to 0}\frac{\mathbf{sint}}{\mathbf{1}} = \frac{0}{1} = 0 \qquad \text{(use L– Hospital rule)}$$

$L\left[\frac{1-cost}{t}\right]$ exists.

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[(1-cost)]ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2+1}\right]_s^\infty$$

$$= \left[log\,\frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= 0 - log\,\frac{s}{\sqrt{s^2+1}}$$

$$= log\,\frac{\sqrt{s^2+1}}{s}$$

Example: 5.22 Find the Laplace transform for $\frac{cosat-cosbt}{t}$

Solution:

$$\lim_{t\to 0}\frac{cosat-cosbt}{t} = \frac{1-1}{0} = \frac{0}{0} \qquad \text{(use L− Hospital rule)}$$

$$= \lim_{t\to 0}\frac{-asinat+bsinbt}{1} = 0 = \text{a finite quantity}$$

Hence Laplace transform exists.

$$L\left[\frac{cosat-cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty [L(cosat) - L(cosbt)]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[log\,\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[log\,\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[log\,\frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[log1 - log\,\frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{2}\left[log\,\frac{s^2+a^2}{s^2+b^2}\right] \qquad [\because log1 = 0]$$

$$= \frac{1}{2}\left[log\,\frac{s^2+b^2}{s^2+a^2}\right]$$

Example: 5.23 Find the Laplace transform of $\frac{sin^2t}{t}$

Solution:

$$\frac{sin^2t}{t} = \frac{1-cos2t}{2t}$$

$$\lim_{t\to 0}\frac{\mathbf{1-cos2t}}{\mathbf{2t}} = \frac{0}{0}$$

$$\lim_{t\to 0}\frac{\mathbf{2sin2t}}{\mathbf{2}} = \frac{0}{1} = 0 \qquad \text{(use L− Hospital rule)}$$

Laplace transform exists.

$$L\left[\frac{sin^2t}{t}\right] = L\left[\frac{1-cos2t}{2t}\right] = \frac{1}{2}\int_s^\infty L[(1-cos2t)]ds$$

$$= \frac{1}{2}\int_s^\infty [L(1) - L(cos2t)]ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - log\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}log\frac{\sqrt{s^2+4}}{s}$$

Example: 5.24 Find the Laplace transform for $\frac{sin2tsin5t}{t}$

Solution:

$$L\left[\frac{sin2tsin5t}{t}\right] = \int_s^\infty L[sin2tsin5t]ds$$

$$= \int_s^\infty \frac{1}{2}[L(\cos(-3t) - L(cos7t)]ds$$

$$= \frac{1}{2}\int_s^\infty [L(\cos(3t) - L(cos7t)]ds \qquad [\because \cos(-\theta) = cos\theta]$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{s}{s^2+9} - \frac{s}{s^2+49}\right)ds$$

$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2+9) - \frac{1}{2}\log(s^2+49)\right]_s^\infty$$

$$= \frac{1}{4}\left[log\frac{s^2+9}{s^2+49}\right]_s^\infty$$

$$= \frac{1}{4}\left[log\frac{s^2\left(1+\frac{9}{s^2}\right)}{s^2\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{4}\left[log\frac{\left(1+\frac{9}{s^2}\right)}{\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{4}\left[log1 - log\frac{s^2+9}{s^2+49}\right] = -\frac{1}{4}\left[log\frac{s^2+9}{s^2+49}\right] \qquad [\because log1 = 0]$$

$$= \frac{1}{4}\left[log\frac{s^2+49}{s^2+9}\right]$$

Problems using $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

Example: 5.25 Find the Laplace transform for (i) $\int_0^t e^{-2t}dt$ (ii) $\int_0^t cos2tdt$

(iii) $\int_0^t tsin3tdt$ (iv) $t\int_0^t costdt$

Solution:

(i) $L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s}L[e^{-2t}] = \frac{1}{s}\left(\frac{1}{s+2}\right)$

$$\therefore L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s(s+2)}$$

(ii) $L\left[\int_0^t cos2tdt\right] = \frac{1}{s}L[cos2t] = \frac{1}{s}\left(\frac{s}{s^2+4}\right)$

$$\therefore L\left[\int_0^t cos2tdt\right] = \frac{1}{s^2+4}$$

(iii) $L\left[\int_0^t tsin3tdt\right] = \frac{1}{s}L[tsin3t]$

$$= \frac{1}{s}\left[\frac{-d}{ds}\left[L[sin3t]\right]\right]$$

$$= \frac{-1}{s}\left[\frac{d}{ds}\left[\frac{3}{s^2+9}\right]\right]$$

$$= \frac{-1}{s}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{6}{(s^2+9)^2}$$

(iv) $L\left[t\int_0^t costdt\right] = \frac{-d}{ds}L\left[\int_0^t costdt\right]$

$$= \frac{-d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+1}\right)\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{2s}{(s^2+1)^2}$$

Example: 5.26 Find the Laplace transform for $e^{-t}\int_0^t tcos4tdt$

Solution:

$$L\left[e^{-t}\int_0^t tcos4tdt\right] = L\left[\int_0^t tcos4tdt\right]_{s\to s+1} = \left[\frac{-1}{s}\frac{d}{ds}L(cos4t)\right]_{s\to s+1}$$

$$= -\left(\frac{1}{s}\frac{d}{ds}\frac{s}{s^2+16}\right)_{s\to s+1}$$

$$= \left[\frac{-1}{s}\ \frac{(s^2+16)1-s(2s)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(s^2+16-2s^2)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(-s^2+16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\frac{(s^2-16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t tcos4tdt\right] = \frac{1}{s+1}\left[\frac{(s+1)^2-16}{((s+1)^2+16)^2}\right]$$

Example: 5.27 Find the Laplace transform of $e^{-t}\int_0^t \frac{sint}{t}dt$

Solution:

$$L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = L\left[\int_0^t \frac{sint}{t}dt\right]_{s\to s+1}$$

$$= \left[\frac{1}{s} L\left(\frac{sint}{t}\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty L(sint)ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}[\tan^{-1} s\,]_s^\infty\,\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\tan^{-1}\infty - \tan^{-1} s)\,\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\frac{\pi}{2} - \tan^{-1} s)\quad\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1} s\,\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = \frac{1}{s+1}cot^{-1}(s+1)$$

Exercise: 5.3

Find the Laplace transform of

1. $\frac{sint}{t}$ **Ans:** $\cot^{-1}\frac{s}{2}$
2. $e^{-2t}\frac{sint}{t}$ **Ans:** $cot^{-1}(s+2)$
3. $\frac{sinat-sinbt}{t}$ **Ans:** $\cot^{-1}\frac{s}{a} - \cot^{-1}\frac{s}{b}$
4. $\frac{e^{-at}-cosbt}{t}$ **Ans:** $log\frac{\sqrt{s^2+b^2}}{s+a}$
5. $\frac{1-e^{-t}}{t}$ **Ans:** $log\frac{s+1}{s}$
6. $e^{-t}\int_0^t \frac{sint}{t}dt$ **Ans:** $\frac{1}{s+1}cot^{-1}(s+1)$
7. $e^{-t}\int_0^t tcostdt$ **Ans:** $\frac{1}{s+1}\left[\frac{s^2+2s}{(s^2+2s+2\,)^2}\right]$
8. $e^{-t}\int_0^t te^{-t}sintdt$ **Ans:** $\frac{1}{s}\left[\frac{2(s+1)}{s^2+2s+2}\right]$

Evaluation of integrals using Laplace transform

Note: (i) $\int_0^\infty f(t)e^{-st}dt = L[f(t)]$

(ii) $\int_0^\infty f(t)e^{-at}dt = [L[f(t)]]_{s=a}$

(iii) $\int_0^\infty f(t)dt = [L[f(t)]]_{s=0}$

Example: 5.28 If $L[f(t)] = \frac{s+2}{s^2+4}$, then find the value of $\int_0^\infty f(t)dt$

Solution:

Given $L[f(t)] = \frac{s+2}{s^2+4}$

We know that $\int_0^\infty f(t)dt = [L[f(t)]]_{s=0}$

$$= \left[\frac{s+2}{s^2+4}\right]_{s=0} = \frac{2}{4}$$

$$\int_0^\infty f(t)dt = \frac{1}{2}$$

Example: 5.29 If $L[f(t)] = \frac{5s+4}{s^2-9}$, then find the value of $\int_0^\infty e^{-2t} f(t)dt$

Solution:

Given $L[f(t)] = \frac{5s+4}{s^2-9}$

We know that $\int_0^\infty e^{-2t} f(t)dt = [L[f(t)]]_{s=2}$

$$= \left[\frac{5s+4}{s^2-9}\right]_{s=2} = \frac{14}{-5}$$

$$\therefore \int_0^\infty e^{-2t} f(t)dt = \frac{-14}{5}$$

Example: 5.30 Find the values of the following integrals using Laplace transforms:

(i) $\int_0^\infty te^{-2t}cos2tdt$ **(ii)** $\int_0^\infty t^2 e^{-t} sintdt$ **(iii)** $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right) dt$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t}dt$ **(v)** $\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right) dt$

Solution:

(i) $\int_0^\infty te^{-2t}cos2tdt = L[tcos2t]_{s=2} = \left[\frac{-d}{ds} L(cos2t)\right]_{s=2}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s=2}$$

$$= -\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\left[\frac{(4-s^2)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\frac{(4-4)}{(4+4)^2} = 0$$

(ii) $\int_0^\infty t^2 e^{-t} sintdt = L[t^2 sint]_{s=1} = \frac{d^2}{ds^2} L[sint]_{s=1}$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)_{s=1}$$

$$= \frac{d}{ds}\left[\frac{-1(2s)}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\frac{d}{ds}\left[\frac{s}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)^2(1)-s.2(s^2+1)(2s)\right)]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{[(s^2+1)[(s^2+1)-4s^2)]]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{(1-3s^2)}{(s^2+1)^3}\right]_{s=1}$$

$$= \left[\frac{6s^3-2}{(s^2+1)^3}\right]_{s=1} = \frac{4}{8} = \frac{1}{2}$$

(iii) $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right) dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} = \int_s^\infty [L[e^{-t} - e^{-2t}]ds]_{s=0}$

$$= \int_s^\infty [[L(e^{-t}) - L(e^{-2t})]\square s]_{s=0}$$

$$= \int_s^\infty \left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds\right]_{s=0}$$

$$= \{[\log(s+1) - \log(s+2)]_s^\infty\}_{s=0}$$

$$= \left\{\left[log\,\frac{s+1}{s+2}\right]_s^\infty\right\}_{s=0}$$

$$= \left\{log\,\frac{s\left(1+\frac{1}{s}\right)}{s\left(1+\frac{2}{s}\right)}\Bigg|_s^\infty\right\}_{s=0}$$

$$= \left[0 - log\,\frac{s+1}{s+2}\right]_{s=0} \qquad \because log1 = 0$$

$$= \left[log\,\frac{s+2}{s+1}\right]_{s=0} = log2$$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt$

$$\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt = L\left[\frac{1-cost}{t}\right]_{S=1} = \int_s^\infty [L[(1-cost)]ds]_{S=1}$$

$$= \int_s^\infty [[L(1) - L(cost)]ds]_{S=1}$$

$$= \int_s^\infty \left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds\right]_{S=1}$$

$$= \left\{\left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty\right\}_{S=1}$$

$$= \left\{\left[\log s - \log\sqrt{s^2+1}\right]_s^\infty\right\}_{S=1}$$

$$= \left\{\left[log\,\frac{s}{\sqrt{s^2+1}}\right]_s^\infty\right\}_{S=1}$$

$$= \left[0 - log\,\frac{s}{\sqrt{s^2+1}}\right]_{S=1}$$

$$= \left[log\,\frac{\sqrt{s^2+1}}{s}\right]_{S=1}$$

$$= log\sqrt{2}$$

(v) $\int_0^\infty \left(\frac{e^{-at} - cosbt}{t}\right) dt$

$$\int_0^\infty \left(\frac{e^{-at} - cosbt}{t}\right) dt = L\left[\frac{e^{-at} - cosbt}{t}\right]_{S=0} = \int_s^\infty [L[(e^{-at} - cosbt)]ds]_{S=0}$$

$$= \int_s^\infty [[L(e^{-at}) - L(cosbt)]ds]_{S=0}$$

$$= \int_s^\infty \left[\left(\frac{1}{s+a} - \frac{s}{s^2+b^2}\right) ds\right]_{S=0}$$

$$= \left\{\left[\log(s+a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty\right\}_{S=0}$$

$$= \left\{\left[\log(s+a) - \log\sqrt{s^2+b^2}\right]_s^\infty\right\}_{S=0}$$

$$= \left\{\left[log\,\frac{s+a}{\sqrt{s^2+b^2}}\right]_s^\infty\right\}_{S=0}$$

$$= \left[0 - log\frac{s+a}{\sqrt{s^2+b^2}}\right]_{s=0}$$

$$= \left[log\frac{\sqrt{s^2+b^2}}{s+a}\right]_{s=0}$$

$$= log\frac{\sqrt{b^2}}{a}$$

$$= log\frac{b}{a}$$

Exercise: 5.4

Find the values of the following integrals using Laplace transforms

1. $\int_0^\infty te^{-2t}costdt$ **Ans:** $\frac{3}{25}$
2. $\int_0^\infty te^{-3t}sintdt$ **Ans:** $\frac{13}{250}$
3. $\int_0^\infty \left(\frac{e^{-at}-e^{-bt}}{t}\right)dt$ **Ans:** $log\frac{b}{a}$
4. $\int_0^\infty e^{-2t}\frac{\sin^2 t}{t}dt$ **Ans:** $\frac{1}{4}log2$
5. $\int_0^\infty \left(\frac{cosat-cosbt}{t}\right)dt$ **Ans:** $log\frac{a}{b}$

Laplace transform of Piecewise continuous functions

$$\int_0^\infty \boldsymbol{f(t)e^{-st}dt = L[f(t)]}$$

Example: 5.31 Find the Laplace transform of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{e^{-t}; 0} < t < \pi \\ \boldsymbol{0\,; t} > \pi \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\pi e^{-st}e^{-t}dt + \int_\pi^\infty e^{-st}0dt$$

$$= \int_0^\pi e^{-(s+1)t}dt$$

$$= \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^\pi = \frac{e^{-(s+1)\pi}-e^0}{-(s+1)}$$

$$\therefore L[f(t)] = \frac{1-e^{-(s+1)\pi}}{-(s+1)}$$

Example: 5.32 Find the Laplace transform of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{sint; 0} < t < \pi \\ \boldsymbol{0\,; t} > \pi \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_{\boldsymbol{0}}^{\boldsymbol{\pi}} e^{-st}sintdt + \int_\pi^\infty e^{-st}0dt$$

$$= \int_0^\pi e^{-st}sintdt$$

$$= \left[\frac{\boldsymbol{e^{-st}}}{\boldsymbol{(-s)^2+1}}(-ssint - cost)\right]_{\boldsymbol{0}}^{\boldsymbol{\pi}} = \frac{e^{-s\pi}}{s^2+1}[-ssin\pi - cos\pi] - \frac{e^0}{s^2+1}[-ssin0 - cos0]$$

$$= \frac{e^{-s\pi}}{s^2+1}(0+1) - \frac{1}{s^2+1}(-1) = \frac{e^{-s\pi}+1}{s^2+1}$$

$\therefore L[f(t)] = \frac{e^{-s\pi}+1}{s^2+1}$

Example: 5.33 Find the Laplace transform of $f(t) = \begin{cases} t; 0 < t < 1 \\ 0 ; t > 1 \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^1 e^{-st}tdt + \int_1^\infty e^{-st}0dt$$

$$= \int_0^1 te^{-st}dt$$

$$= \left[t\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{(-s)^2}\right]_0^1 = \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} - 0 + \frac{1}{s^2}$$

$$\therefore L[f(t)] = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

Exercise: 5.5

1. Find the Laplace transform of $f(t) = \begin{cases} 0; 0 < t < 2 \\ 3 ; t > 2 \end{cases}$ **Ans:** $\frac{3e^{-2s}}{s}$

2. Find the Laplace transform of $f(t) = \begin{cases} e^t; 0 < t < 1 \\ 0 ; t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-(s-1)}}{s-1}$

3. Find the Laplace transform of $f(t) = \begin{cases} 1; 0 < t < 1 \\ 0 ; t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-s}}{s}$

Unit step function

The unit step function $U(t-a)$ is defined as $U(t-a) = \begin{cases} 0; t < a \\ 0 ; t > a \end{cases}$

Example: 5.34 Find the Laplace transform of unit step functions.

Solution:

$$L[U(t-a)] = \int_0^\infty U(t-a)e^{-st}dt$$

$$= \int_0^a 0dt + \int_a^\infty (1)e^{-st}dt = \int_a^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = 0 - \frac{e^{-sa}}{-s} = \frac{e^{-sa}}{s}$$

$$L[U(t-a)] = \frac{e^{-sa}}{s}$$

Second Shifting theorem

Statement: If $L[f(t)] = F(s)$, then $L[f(t-a)U(t-a)] = e^{-as}F(s)$

Proof:

$$U(t-a)f(t-a) = \begin{cases} 0; t < a \\ f(t-a) ; t > a \end{cases}$$

By the definition of Laplace transform,

$$L[U(t-a)f(t-a)] = \int_0^\infty U(t-a)f(t-a)e^{-st}dt$$

$$= \int_0^a 0dt + \int_a^\infty f(t-a)e^{-st}dt$$

$$L[U(t-a)f(t-a)] = \int_0^\infty e^{-s(a+x)}f(x)dx$$

$$= \int_0^\infty e^{-sa} e^{-sx} f(x)dx$$

$$= e^{-sa} \int_0^\infty e^{-sx} f(x)dx$$

Replace x by t

$$L[U(t-a)f(t-a)] = e^{-sa} \int_0^\infty e^{-st} f(t)dt$$

$$= e^{-sa} L[f(t)] = e^{-sa} F(s)$$

$$L[U(t-a)f(t-a)] = e^{-sa} F(s)$$

Let $t - a = x \cdots (1)$

$t = a + x$

$dt = dx$

When $t = a, (1) => x = 0$

When $t = \infty, (1) => x = \infty$

5.5 PERIODIC FUNCTIONS

Definition: A function $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t+T) = f(t)$ for all t is called periodic function.

Example:

$sint = \sin(t + 2\pi) = \sin(t + 4\pi) \cdots$

$\therefore$ $sint$ is periodic function with period 2π.

Let $f(t)$ be a periodic function with period T. Then

$$\boxed{L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)dt}$$

Problems on Laplace transform of Periodic function

Example: 5.35 Find the Laplace transform of $f(t) = \begin{cases} sin\omega t; 0 < t < \frac{\pi}{\omega} \\ 0 ; \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$

Solution:

The given function is a periodic function with period $T = \frac{2\pi}{\omega}$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\frac{\pi}{\omega}} sin\omega t e^{-st} dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}(0)dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} sin\omega t e^{-st} dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-st}}{(-s)^2+\omega^2}(-ssin\omega t - \omega cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left\{\frac{e^{\frac{-s\pi}{\omega}}}{s^2+\omega^2}[-ssin\pi - \omega cos\pi] + \frac{\omega}{s^2+\omega^2}\right\}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{\frac{-s\pi}{\omega}}\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{1}{1^2 - \left(e^{\frac{-\pi s}{\omega}}\right)^2} \left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)}\left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2}\right]$$

$$\therefore L[f(t)] = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}$$

Example: 5.36 Find the Laplace transform of $\boldsymbol{f(t) = \begin{cases} E; 0 \le t \le a \\ -E\,; a \le t \le 2a \end{cases}}$ **given that** $\boldsymbol{f(t+2a) = f(t)}$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a Ee^{-st}dt + \int_a^{2a} -Ee^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[E\int_0^a e^{-st}dt - E\int_a^{2a} e^{-st}dt\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\frac{e^{-as}}{-s} + \frac{1}{s} - \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s}\right]$$

$$= \frac{E}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{s}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\therefore L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

Example: 5.37 Find the Laplace transform of $\boldsymbol{f(t) = \begin{cases} 1; 0 \le t \le \frac{a}{2} \\ -1\,; \frac{a}{2} \le t \le a \end{cases}}$ **given that** $\boldsymbol{f(t+a) = f(t)}$.

Solution:

The given function is a periodic function with period $T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}}(1)e^{-st}dt + \int_{\frac{a}{2}}^{a}(-1)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}dt - \int_{\frac{a}{2}}^{a} e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{2a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{\frac{-sa}{2}}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{\frac{-sa}{2}}}{s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{\frac{-sa}{2}}+e^{-as}}{s}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-sa}{2}}\right)^2}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-sa}{2}}\right)\left(1+e^{\frac{-sa}{2}}\right)}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{s}\frac{\left(1-e^{\frac{-sa}{2}}\right)}{\left(1+e^{\frac{-sa}{2}}\right)} \qquad \left[\because tanhx = \frac{(1-e^{-2x})}{(1+e^{-2x})}\right]$$

$$\therefore L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

Example: 5.38 Find the Laplace transform of $\boldsymbol{f(t) = \begin{cases} t;0 \le t \le a \\ 2a - t\,;a \le t \le 2a \end{cases}}$ **given that** $\boldsymbol{f(t+2a) = f(t)}$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a te^{-st}dt + \int_a^{2a}(2a-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^a - \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Exercise: 5.6

1. Find the Laplace transform of

$f(t) = \begin{cases} 1;0 \le t \le \frac{a}{2} \\ -1\,;\frac{a}{2} \le t \le a \end{cases}$ given that $f(t+a) = f(t)$. **Ans:** $\frac{k}{s}\tanh\left(\frac{as}{2}\right)$