

MODULE-III

IIR FILTER DESIGN

INTRODUCTION:

In general, filters are used to extract required quantities and remove all other unwanted quantities. So far, we already studied filter concept in Regulated power supply and in Amplifiers. Regulated power supply is a cascading arrangement of transformer, rectifier, filter and voltage regulator, where the function of filter is to minimize unwanted components called ripples or ac components. In common emitter amplifier one capacitor is connected in between input and base of the transistor called coupling capacitor or blocking capacitor. Where the capacitor acts as a filter, it blocks DC and freely passes AC signals.

CLASSIFICATION OF FILTERS:

Based on frequency response, filters are classified into four types.

- Low Pass Filters (LPF)
- High Pass Filters (HPF)
- Band Pass Filters (BPF)
- Band Stop Filters (BSF)

Low pass filter allows only low frequency signals and attenuate all other high frequency signals. High pass filter allows only high frequency signals and attenuate all other low frequency signals. Band pass filter allows only a certain band of frequency signals and attenuate all other unwanted band of frequency signals. Band stop filter or Band reject filter or Band elimination filter allows entire band of frequency signals and attenuate a narrow band or unwanted band of frequency signals.

Based on type of input, type of output and type of components of a filter, filters are classified into two types.

- Analog Filters
- Digital Filters

ANALOG FILTERS:

- Analog filters produce only analog signals with an input of analog signal.



- Analog filters are described by differential equations, which involve only differentials. For example.

$y(t) = \frac{d}{dt} [x(t)]$ is a differential equation of a differentiator.

$x(t) = \frac{d}{dt} [y(t)]$ is a differential equation of an integrator.

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- Analog filters consists of only analog components like resistors, capacitors, inductors, etc.,
- Thermal stability of analog filter is poor because of all analog components are temperature sensitive.
- Analog filters are non-programmable

DIGITAL FILTERS:

- Digital filters produces only digital signals with an input of digital signal.
- Digital filters are described by difference equation, which does not involves differentials. Which involves shifts. For example

$$Y(n) = 2x(n) + 3x(n-1) + 4x(n-2)$$
 and

$$Y(n) = 2x(n) + 3x(n-1) + 4y(n-2).$$
- Digital filters consists of only discrete components like adders, constant multipliers and delay systems, etc.,
- Thermal stability of digital filter is high.
- Digital filters are programmable

Based on impulse response, filters are classified into two types

- Finite Impulse Response Filters (FIR) Filters
- (Infinite Impulse Response Filters) IIR Filters

Impulse response can be defined as the output of a discrete LTI system with an input of impulse sequence. If impulse response consists of

Finite number of samples we call it as FIR systems or FIR filters

Infinite number of samples we call it as IIR systems or IIR filters

Ex 1: $y(n) = 2x(n) + 3x(n-1) + 4x(n-2)$ is a FIR filter difference equation, because of impulse response contains only three samples $h(n) = \{2, 3, 4\}$ is a stable filter

Ex 2: $y(n) = 2x(n) + 3y(n-1)$ is a IIR filter difference equation, because of impulse response contains infinite number samples. $h(n) = \{2, 6, 18, 56, \dots\}$ is a un stable filter

Ex 3: $y(n) = 2x(n) + 1/3 y(n-1)$ is a IIR filter difference equation, because of impulse response contains infinite number samples. $h(n) = \{2, 2/3, 2/9, 2/27, \dots\}$ is a stable filter

DESIGN OF IIR DIGITAL FILTERS FROM ANALOG FILTERS:

IIR filter design involves, design of analog filter and then transforming analog filter to digital filter

DESIGN STEPS:

Following steps were required to design IIR Digital LPF / HPF / BPF / BSF.

Step 1 :

Take the specifications of a digital low pass filter

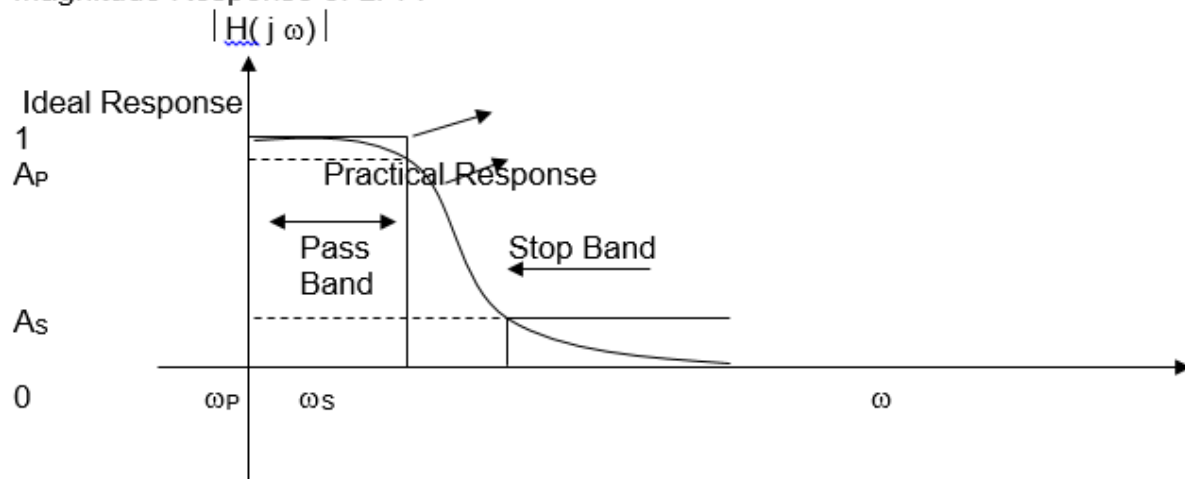
A_P : Gain at pass band digital frequency ω_P .

A_S : Gain at stop band digital frequency ω_S .

Ω_P : Pass band analog frequency corresponding to digital frequency ω_P

Ω_S : Stop band analog frequency corresponding to digital frequency ω_S

Magnitude Response of LPF:



Step 2:

Choose any one of the following transformations to design a digital filter

a) Impulse Invariant Transformation

b) Bilinear Transformation

Determine the analog filter frequency ratio Ω_S / Ω_P

Step 3:

Decide the order (N) of the filter may be from

a) BUTTERWORT Approximation

b) CHEBYSHEV Approximation

Step 4:

Determine analog filter cutoff frequency Ω_C .

Step 5:

Determine analog filter transfer function from order (N) and cutoff frequency (Ω_C) of the filter.

Step 6:

Now determine the desired filter transfer function from frequency transformation techniques.

Step 7:

Convert analog filter transfer function $H_a(s)$ into digital filter transfer function $H(z)$.

Step 8:

Finally realize the filter by a suitable structure (DF I / DF II / Cascade Form / Parallel Form)

ANALOG FILTER APPROXIMATIONS:

To design an analog filter numbers of approximation procedures were developed, among them popular approximations are

- BUTTERWORTH Approximation
- CHEBYSHEV Approximation

BUTTERWORTH APPROXIMATION:

Analog Butterworth Low Pass Filter (ABLPF) is designed by approximating the magnitude of frequency response $|H(j\Omega)|$ is selected such that the magnitude is maximally flat in the pass band and monotonically decreasing in the stop band. The approximated squared magnitude frequency response of low pass filter as given by

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

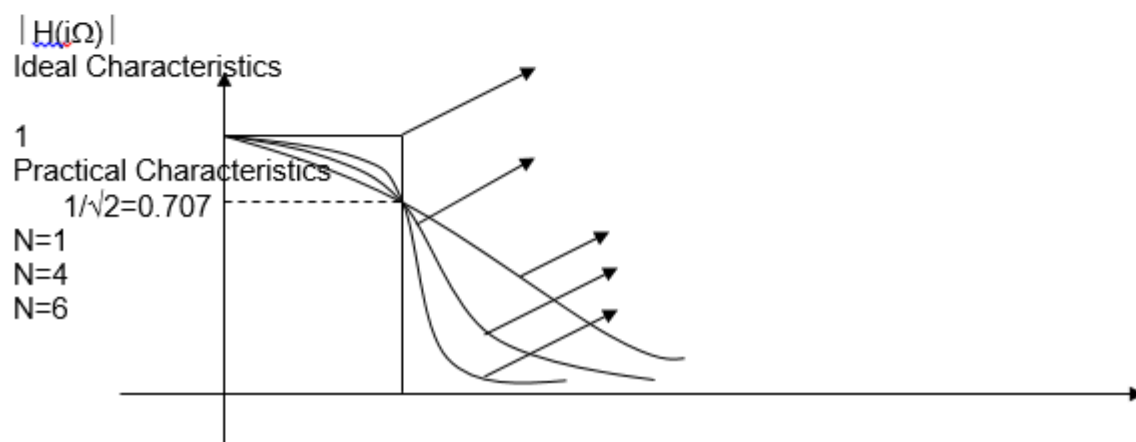
Where,

Ω_c : Cutoff frequency of an analog filter

N : Order of the filter (Number of poles)

MAGNITUDE RESPONSE OF ABLPF:

It is very clear from above $|H(j\Omega)|$, the approximate magnitude response approaches the ideal response if the N value increases as shown below.



PROPERTIES OF ABLPF:

- BUTTERWORTH Filter is a all pole design.
- At the cutoff frequency ($\Omega = \Omega_c$), the magnitude response $|H(j\Omega)| = 0.707$.
- Order (N) of the filter is depends on specifications of the desired filter
- Magnitude of frequency response is maximally flat over pass band
- Magnitude of frequency response is monotonically decreases with increase in N.
- The magnitude response approaches the ideal response as the order of the filter increases.

ORDER OF ABLPF

Order of the filter is nothing but number of poles of desired filter can be calculated from specifications of a digital low pass filter.

At $\Omega = \Omega_P$

$$\Rightarrow |H(j\Omega_P)|^2 = \frac{1}{1 + \left(\frac{\Omega_P}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow A_P^2 = \frac{1}{1 + \left(\frac{\Omega_P}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow \frac{1}{A_P^2} = 1 + \left(\frac{\Omega_P}{\Omega_C}\right)^{2N}$$

$$\Rightarrow \frac{1}{A_P^2} - 1 = \left(\frac{\Omega_P}{\Omega_C}\right)^{2N} \text{------(1)}$$

At $\Omega = \Omega_S$

$$\Rightarrow |H(j\Omega_S)|^2 = \frac{1}{1 + \left(\frac{\Omega_S}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow A_S^2 = \frac{1}{1 + \left(\frac{\Omega_S}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow \frac{1}{A_S^2} = 1 + \left(\frac{\Omega_S}{\Omega_C}\right)^{2N}$$

$$\Rightarrow \frac{1}{A_S^2} - 1 = \left(\frac{\Omega_S}{\Omega_C}\right)^{2N} \text{------(2)}$$

Divide equation 2 by 1

$$\begin{aligned} \Rightarrow \left(\frac{\Omega_S}{\Omega_C} \right)^{2N} / \left(\frac{\Omega_P}{\Omega_C} \right)^{2N} &= \left(\frac{1}{A_S^2} - 1 \right) / \left(\frac{1}{A_P^2} - 1 \right) \\ \Rightarrow \left(\frac{\Omega_S}{\Omega_P} \right)^{2N} &= \left(\frac{1}{A_S^2} - 1 \right) / \left(\frac{1}{A_P^2} - 1 \right) \\ \Rightarrow 2N \log \left(\frac{\Omega_S}{\Omega_P} \right) &= \log \left(\frac{1}{A_S^2} - 1 \right) / \left(\frac{1}{A_P^2} - 1 \right) \\ \Rightarrow N &= \frac{1}{2} \frac{\log \left[\left(\frac{1}{A_S^2} - 1 \right) / \left(\frac{1}{A_P^2} - 1 \right) \right]}{\log \left(\frac{\Omega_S}{\Omega_P} \right)} \end{aligned}$$

CUTOFF FREQUENCY OF ABLPF:

At $\Omega = \Omega_P$

$$\begin{aligned} \Rightarrow |H(j\Omega_P)|^2 &= \frac{1}{1 + \left(\frac{\Omega_P}{\Omega_C} \right)^{2N}} \\ \Rightarrow A_P^2 &= \frac{1}{1 + \left(\frac{\Omega_P}{\Omega_C} \right)^{2N}} \\ \Rightarrow \frac{1}{A_P^2} &= 1 + \left(\frac{\Omega_P}{\Omega_C} \right)^{2N} \\ \Rightarrow \frac{1}{A_P^2} - 1 &= \left(\frac{\Omega_P}{\Omega_C} \right)^{2N} \\ \Rightarrow \frac{\Omega_P}{\Omega_C} &= \left(\frac{1}{A_P^2} - 1 \right)^{\frac{1}{2N}} \\ \Rightarrow \Omega_C &= \frac{\Omega_P}{\left(\frac{1}{A_P^2} - 1 \right)^{\frac{1}{2N}}} \end{aligned}$$

TRANSFER FUNCTION OF ABLPF:

Transfer function of the analog filter is depends on order (N) and cutoff frequency (Ω_C).

If N is even, then $H(s) = \prod_{k=1}^{N/2} \left(\frac{\Omega_C^2}{s^2 + b_k \Omega_C s + \Omega_C^2} \right)$

If N is odd, then $H(s) = \left(\frac{\Omega_C}{s + \Omega_C} \right) \prod_{k=1}^{N-1} \left(\frac{\Omega_C^2}{s^2 + b_k \Omega_C s + \Omega_C^2} \right)$

Where, $b_k = 2 \sin \left(\frac{(2k-1)\pi}{2N} \right)$

Chebyshev Approximation:

There are two types of Chebyshev approximations.

- Chebyshev Type – I approximation
- Chebyshev Type – II approximation (Inverse Chebyshev)

Analog Chebyshev type – I Low Pass Filter (ACLPF) is designed by approximating the magnitude of frequency response $|H(j\Omega)|$ is selected such that the magnitude response is equiripple in the pass band and monotonic in the stop band.

Analog Chebyshev type – II Low Pass Filter (ACLPF) is designed by approximating the magnitude of frequency response $|H(j\Omega)|$ is selected such that the magnitude response is monotonic in the pass band and equiripple in the stop band. Here our syllabus is restricted to type – I approximation.

The approximated squared magnitude frequency response of type – I low pass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

Where,

$$\varepsilon = \sqrt{\frac{1}{A_p^2} - 1},$$

$$\text{For small values of } N, \quad C_N(x) = \begin{cases} \cos[N \cos^{-1}(x)], & \text{for } |x| \leq 1 \\ \cosh[N \cosh^{-1}(x)], & \text{for } |x| > 1 \end{cases}$$

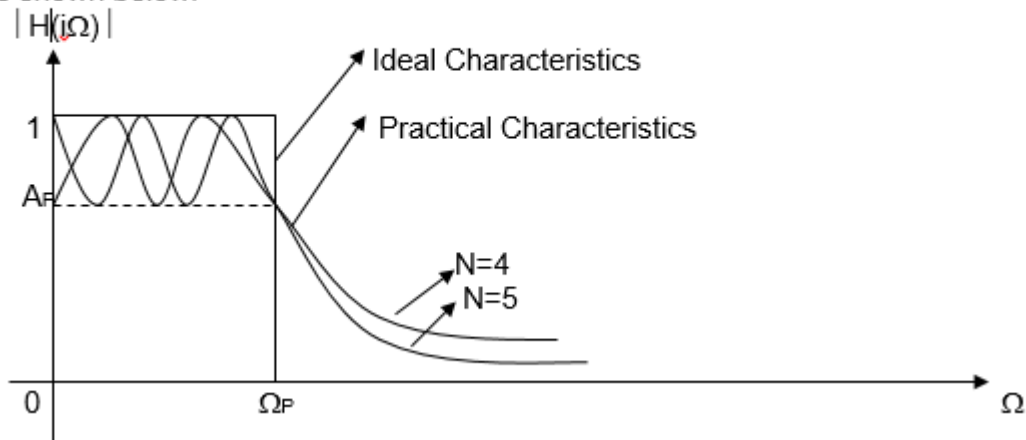
$$\text{For large values of } N, \quad C_N(x) = 2x C_{N-1}(x) - C_{N-2}(x),$$

with initial conditions $C_0(x) = 1$ and $C_1(x) = x$.

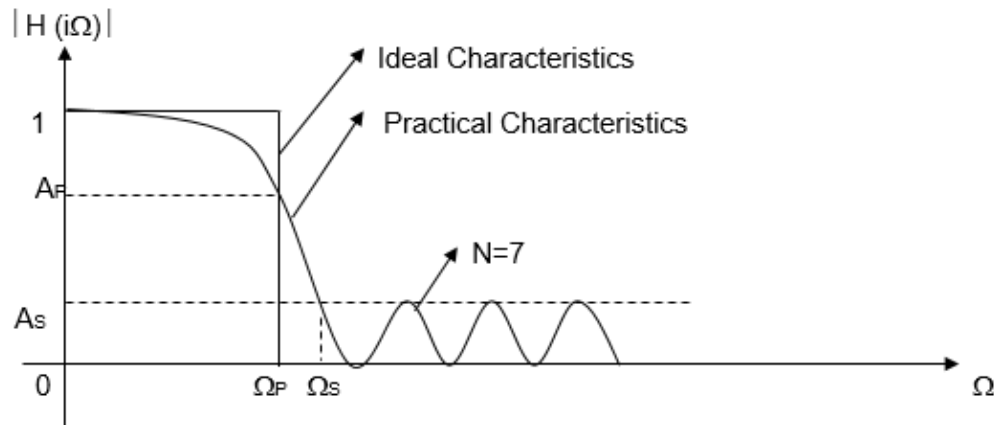
- Ω_c : Cutoff frequency of an analog filter
 N : Order of the filter (Number of poles)
 ε : Attenuation constant
 $C_N(\Omega / \Omega_c)$: Chebyshev polynomial

MAGNITUDE RESPONSE OF TYPE – I ACLPF:

Here also the approximate magnitude response approaches the ideal response if the N value increases as shown below.



MAGNITUDE RESPONSE OF TYPE – II ACLPF:



PROPERTIES OF TYPE ACLPF:

- CHEBYSHEV Filter is a all pole design.
- At the cutoff frequency ($\Omega = \Omega_c$), the magnitude response $|H(j\Omega)| = A_p$.
- Order (N) of the filter is depends on specifications of the desired filter
- Type – I Magnitude frequency response is oscillates between 1 and A_p within pass band.
- Type – I Magnitude frequency response is monotonic outside the pass band.
- Type – II Magnitude frequency response is monotonic within the pass band.
- Type – II Magnitude frequency response oscillates with in the stop band.
- The magnitude response approaches the ideal response as the order of the filter increases.

ORDER OF ACLPF

Order of the filter is nothing but number of poles of desired filter can be calculated from specifications of a digital low pass filter.

$$N = \frac{\cosh^{-1} \left[\sqrt{\left(\frac{1}{A_s^2} - 1 \right) / \left(\frac{1}{A_p^2} - 1 \right)} \right]}{\cosh^{-1} \left(\frac{\Omega_s}{\Omega_p} \right)}$$

The value of N is always nearest and maximum integer.

CUTOFF FREQUENCY OF ACLPF:

$$\Omega_C = \frac{\Omega_P}{\left(\frac{1}{A_P^2} - 1 \right)^{\frac{1}{2N}}}$$

TRANSFER FUNCTION OF ACLPF:

Transfer function of the analog filter is depends on order (N) and cutoff frequency (Ω_C).

If N is even, then $H(s) = \prod_{k=1}^{N/2} \left(\frac{B_k \Omega_C^2}{s^2 + b_k \Omega_C s + c_k \Omega_C^2} \right)$

If N is odd, then $H(s) = \left(\frac{B_0 \Omega_C}{s + c_0 \Omega_C} \right) \prod_{k=1}^{N/2} \left(\frac{B_k \Omega_C^2}{s^2 + b_k \Omega_C s + c_k \Omega_C^2} \right)$

Where, $b_k = 2 Y_N \sin \left(\frac{(2k-1)\pi}{2N} \right)$

$c_k = Y_N^2 + \cos^2 \left(\frac{(2k-1)\pi}{2N} \right)$ and $c_0 = Y_N$.

$$Y_N = \frac{1}{2} \left[\left(\sqrt{\frac{1}{\epsilon^2} + 1} + \frac{1}{\epsilon} \right)^{\frac{1}{N}} - \left(\sqrt{\frac{1}{\epsilon^2} + 1} + \frac{1}{\epsilon} \right)^{-\frac{1}{N}} \right]$$

To get B_k values, use $H(0) = A_P$ for even N and $H(0) = 1$ for odd N.

Assume $B_0 = B_1 = B_2 = B_3 = B_4 = \dots = B_k$

ANALOG TO DIGITAL TRANSFORMATIONS:

Analog to digital transformations are used to obtain digital filter transfer function $H(z)$ from analog filter transfer function $H(s)$. There are two methods to $H(z)$ from $H(s)$

- Impulse Invariant Transformation
- Bilinear Transformation

IMPULSE INVARIANT TRANSFORMATION:

Let us consider an analog filter having system function $H(s)$ contains N number of poles located at $s = p_1, p_2, p_3, \dots, p_N$, implies

$$H(s) = \frac{1}{(s-p_1)(s-p_2)(s-p_3)\dots\dots(s-p_N)}$$

Split into partial fractions

$$= \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3} + \dots\dots + \frac{A_N}{s-p_N}$$

$$= \sum_{k=1}^N \frac{A_k}{s-p_k} \dots\dots\dots (1)$$

Apply inverse laplace transform both side

$$L^{-1}[H(s)] = \sum_{k=1}^N L^{-1} \left(\frac{A_k}{s-p_k} \right)$$

$$h(t) = \sum_{k=1}^N A_k e^{-p_k t} \text{ is the impulse response of an analog filter}$$

Replace t by nT, to get impulse response of digital filter

$$h(nT) = \sum_{k=1}^N A_k e^{p_k nT}$$

$$= \sum_{k=1}^N A_k (e^{p_k T})^n$$

Apply z transform both side, to get H(z)

$$ZT[h(nT)] = \sum_{k=1}^N ZT[A_k (e^{p_k T})^n]$$

$$H(z) = \sum_{k=1}^N A_k \left(\frac{z}{z - e^{p_k T}} \right)$$

$$H(z) = \sum_{k=1}^N \left(\frac{A_k}{1 - e^{p_k T} z^{-1}} \right) \dots\dots\dots (2)$$

Compare equations 1 and 2

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1 - e^{p_k T} z^{-1}} \Rightarrow \frac{1}{s-a} \rightarrow \frac{1}{1 - e^{aT} z^{-1}}$$

It is very clear from above conversion formula is that the analog filter pole located at $s = a$ is transformed to digital filter pole located at $z = e^{aT}$. Therefore we can say that Impulse Invariant Transformation method is an indirect one is used to get $H(z)$ from $H(s)$ by replacing

$$\frac{1}{s-a} \rightarrow \frac{1}{1 - e^{aT} z^{-1}}$$

RELATION B/N ANALOG FREQUENCY (Ω) & DIGITAL FREQUENCY (ω):

Analog filter pole is located at $s = a$ is transformed to digital filter pole located at $z = e^{aT}$.

$$\Rightarrow z = e^{sT}$$

Replace z by $r e^{j\omega}$ and s by $\sigma + j\Omega$

$$\Rightarrow r e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$\Rightarrow r e^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

Compare magnitude and phase

$$\Rightarrow r = e^{\sigma T} \text{ and } \omega = \Omega T$$

$$\Rightarrow \Omega = \frac{\omega}{T}$$

$$\Rightarrow \Omega \propto \omega$$

$$\Rightarrow \frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p}$$

RELATION B/N ANALOG FILTER POLE & DIGITAL FILTER POLE:

Analog filter pole is located at $s = a$ is transformed to digital filter pole located at $z = e^{aT}$.

$$\Rightarrow z = e^{sT}$$

Take $s = \sigma + j\Omega$

$$\Rightarrow z = e^{(\sigma + j\Omega)T}$$

$$\Rightarrow z = e^{\sigma T} e^{j\Omega T} \text{ and } |z| = e^{\sigma T}$$

Case 1:

If $\sigma < 0$, then $|z| < 1$. That means if the analog filter pole is left sided, then it is transformed to digital filter pole of inside the unit circle. This case belongs to stable system.

Case 2:

If $\sigma > 0$, then $|z| > 1$. That means if the analog filter pole is right sided, then it is transformed to digital filter pole of outside the unit circle. This case belongs to unstable system.

Case 3:

If $\sigma = 0$, then $|z| = 1$. That means if the analog filter pole is on the imaginary axis, then it is transformed to digital filter pole located on the unit circle. Here the system is marginally stable.

Note:

$$\text{If } s = \sigma + j\Omega + j\frac{2\pi}{T}k, \text{ then } z = e^{(\sigma + j\Omega + j\frac{2\pi}{T}k)T} = e^{\sigma T} e^{j\Omega T}$$

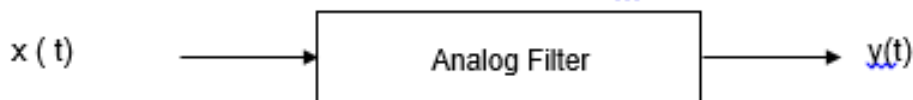
It is very clear from above assumption is that the digital filter pole is unique for

the different analog filter poles located at $s = \sigma + j\Omega + j\frac{2\pi}{T}k$, where $k = 0, \pm 1, \pm 2, \dots$

Impulse invariant transformation is a many to one type of conversion method, because of it maps many poles in s domain into single pole in z domain. Effect of getting single digital frequency for various analog frequencies is known as frequency aliasing.

BILINEAR TRANSFORMATION:

Let us consider an analog filter having input $x(t)$ and output $y(t)$



To obtain the digital filter transfer function, assume the relation between input $x(t)$ and output $y(t)$ of an analog filter may be

$$y(t) = \frac{d}{dt} [x(t)]$$

Apply Laplace transform both side to get analog filter transfer function $H(s)$

$$\Rightarrow \text{LT}[y(t)] = \text{LT}\left[\frac{d}{dt} [x(t)]\right]$$

$$\Rightarrow Y(s) = s X(s)$$

$$\Rightarrow \frac{Y(s)}{X(s)} = s$$

$$\Rightarrow H(s) = s \text{-----} (1)$$

Convert differential equation into difference equation by applying integration for differential equation over the range two successive discrete samples $(n-1)T$ and nT

$$\Rightarrow \int_{(n-1)T}^{nT} y(t) dt = \int_{(n-1)T}^{nT} \frac{d}{dt} [x(t)] dt$$

$$\Rightarrow \int_{(n-1)T}^{nT} y(t) dt = x(t) \Big|_{(n-1)T}^{nT}$$

Use trapezoidal rule for the numerical integration

$$\Rightarrow \frac{nT - (n-1)T}{2} [y(nT) + y((n-1)T)] = x(nT) - x((n-1)T)$$

$$\Rightarrow \frac{T}{2} [y(nT) + y((n-1)T)] = x(nT) - x((n-1)T)$$

Apply Z transform both side to get the digital filter transfer function $H(z)$

$$\Rightarrow \frac{T}{2} [Y(z) + z^{-1} Y(z)] = X(z) - z^{-1} X(z)$$

$$\Rightarrow \frac{T}{2} [1 + z^{-1}] Y(z) = [1 - z^{-1}] X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)$$

$$\Rightarrow H(z) = \frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \text{-----} (2)$$

Compare equations (1) and (2) to get the relation between s and z

$$\Rightarrow s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \text{ or } z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

Bilinear transformation is a direct method is used to get $H(z)$ from $H(s)$ by replacing

$$s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$$

RELATION B/N ANALOG FREQUENCY (Ω) & DIGITAL FREQUENCY (ω):

From Bilinear transformation conversion formula

$$s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$$

On the imaginary axis of s plane $\sigma = 0 \Rightarrow s = j\Omega$. Corresponding z plane is unit circle, where

$$\Rightarrow j\Omega = \frac{2}{T} \left(\frac{1-(e^{j\omega})^{-1}}{1+(e^{j\omega})^{-1}} \right)$$

$$\Rightarrow j\Omega = \frac{2}{T} \left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}} \right)$$

$$\Rightarrow j\Omega = \frac{2}{T} \frac{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})}{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}})}$$

$$\Rightarrow j\Omega = \frac{2}{T} \left(\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}} \right)$$

$$\Rightarrow j\Omega = \frac{2}{T} \left(\frac{2j \sin\left(\frac{\omega}{2}\right)}{2 \cos\left(\frac{\omega}{2}\right)} \right)$$

$$\Rightarrow \Omega = \frac{2}{T} \left(\frac{\sin\left(\frac{\omega}{2}\right)}{\cos\left(\frac{\omega}{2}\right)} \right)$$

$$\Rightarrow \Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right) \text{ or } \omega = 2 \tan^{-1}\left(\frac{\Omega T}{2}\right)$$

$$\Rightarrow \Omega \propto \tan\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \frac{\Omega_s}{\Omega_p} = \frac{\tan\left(\frac{\omega_s}{2}\right)}{\tan\left(\frac{\omega_p}{2}\right)}$$

RELATION B/N ANALOG FILTER POLE & DIGITAL FILTER POLE:

From Bilinear transformation conversion formula

$$\Rightarrow z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

Replace $s = \sigma + j\omega$

$$\Rightarrow z = \frac{1 + \frac{T}{2}(\sigma + j\omega)}{1 - \frac{T}{2}(\sigma + j\omega)}$$

$$\Rightarrow z = \frac{1 + \frac{T}{2}\sigma + j\frac{T}{2}\omega}{1 - \frac{T}{2}\sigma - j\frac{T}{2}\omega}$$

$$\Rightarrow |z| = \frac{\sqrt{\left(1 + \frac{T}{2}\sigma\right)^2 + \left(\frac{T}{2}\omega\right)^2}}{\sqrt{\left(1 - \frac{T}{2}\sigma\right)^2 + \left(\frac{T}{2}\omega\right)^2}}$$

Case 1:

If $\sigma < 0$, then $|z| < 1$. That means if the analog filter pole is left sided, then it is transformed to digital filter pole of inside the unit circle. This case belongs to stable system.

Case 2:

If $\sigma > 0$, then $|z| > 1$. That means if the analog filter pole is right sided, then it is transformed to digital filter pole of outside the unit circle. This case belongs to unstable system.

Case 3:

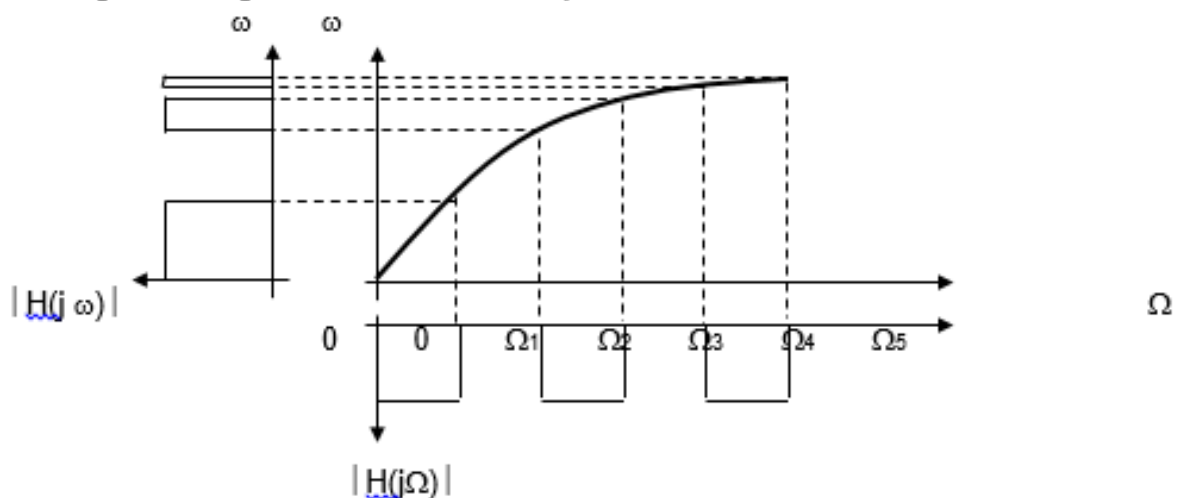
If $\sigma = 0$, then $|z| = 1$. That means if the analog filter pole is on the imaginary axis, then it is transformed to digital filter pole located on the unit circle. Here the system is marginally stable.

FREQUENCY WARPING:

Relation between analog frequency and digital frequency of bilinear transformation is given by

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right) \text{ or } \omega = 2 \tan^{-1}\left(\frac{\Omega T}{2}\right)$$

Here the relation between analog frequency and digital frequency is nonlinear. When the s plane is mapped into the z plane using bilinear transformation, this non-linear relationship introduces distortion in frequency axis called frequency warping. Warping effect can be explained by considering an analog filter with a number of pass bands as shown.



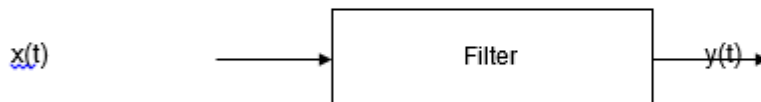
It is very clear from above magnitude response of both analog and digital filters is that the corresponding digital filter is also having same number of pass bands (like 0 to Ω_1 and Ω_2 to Ω_3 and Ω_4 to Ω_5), but with disproportionate bandwidth. This disproportionate bandwidth results in frequency axis is known as frequency warping.

CHARACTERISTICS OF COMMONLY USED ANALOG FILTERS:

Due to filtering

- Some frequency components may be boosted in strength
- Some frequency components may be attenuated
- Some frequency components may be un changed

For a distortion less filter, output wave shape is exact replica of input wave shape over specified band of frequencies. If we know the filter having input $x(t)$ and output $y(t)$ is said to be distortion less if and only if $y(t) = A x(t - t_0)$, where A is constant and t_0 is delay.



Apply Fourier transform both side, to obtain the frequency response

$$FT [y(t)] = FT [A x(t - t_0)]$$

$$Y(j\omega) = A FT [x(t - t_0)]$$

$$Y(j\omega) = A e^{-j\omega t_0} X(j\omega)$$

$$H(j\omega) = A e^{-j\omega t_0}$$

It is very clear from above frequency response, for a distortion less filter magnitude is constant over required band of frequencies and phase is linear.

FREQUENCY TRANSFORMATIONS:

Frequency transformation technique is used to obtain following analog filter transfer function from known low pass filter transfer function.

- Low pass filter of another cutoff frequency Ω_c .
- High pass filter with a cutoff frequency of Ω_c .
- Band pass filter with lower cutoff frequency Ω_L and upper cutoff frequency Ω_H .
- Band stop filter with lower cutoff frequency Ω_L and upper cutoff frequency Ω_H .

Use the following conversion formulas to get desired analog filter

For a new low pass filter, replace s by s / Ω_c .

For a high pass filter, replace s by Ω_c / s .

For a band pass filter, replace s by $Q(s^2 + \Omega_0^2) / \Omega_0 s$.

For a band stop filter, replace s by $\Omega_0 s / Q(s^2 + \Omega_0^2)$.

Where,

$$\Omega_0 = \sqrt{\Omega_L \Omega_H} \text{ and}$$

$$Q = \Omega_0 / (\Omega_H - \Omega_L)$$

DESIGN OF IIR DIGITAL FILTERS:

DESIGN OF DIGITAL BUTTERWORTH FILTER:

Following steps were required to design IIR Digital LPF / HPF / BPF / BSF.

Step 1:

Take the specifications of a digital low pass filter

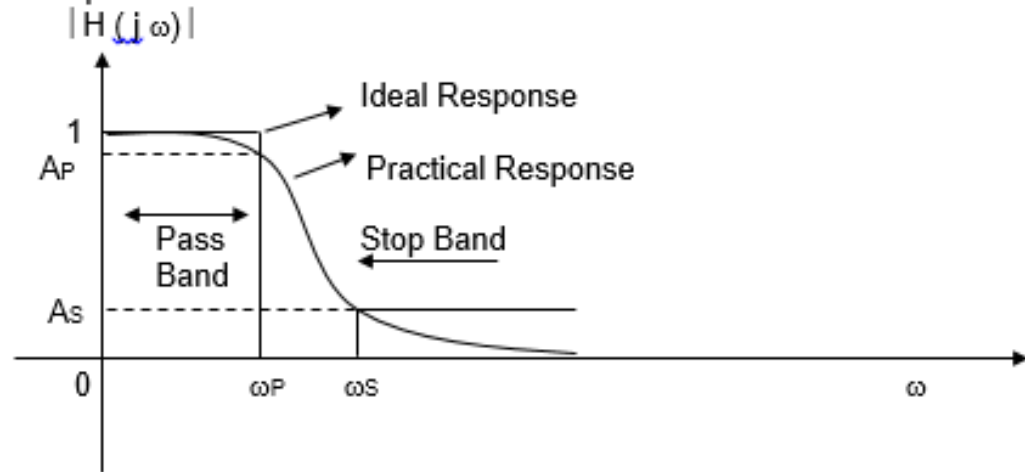
A_P : Gain at pass band digital frequency ω_P .

A_S : Gain at stop band digital frequency ω_S .

Ω_P : Pass band analog frequency corresponding to digital frequency ω_P

Ω_S : Stop band analog frequency corresponding to digital frequency ω_S

Magnitude Response of LPF:



Step 2:

Choose any one of the following transformations to design a digital filter

a. Impulse Invariant Transformation

b. Bilinear Transformation

Determine the analog filter frequency ratio Ω_S / Ω_P using

$$\text{IIT Method} \Rightarrow \frac{\Omega_S}{\Omega_P} = \frac{\omega_S}{\omega_P}, \text{ or}$$

$$\text{BLT Method} \Rightarrow \frac{\Omega_S}{\Omega_P} = \frac{\tan\left(\frac{\omega_S}{2}\right)}{\tan\left(\frac{\omega_P}{2}\right)}$$

Step 3:

Decide the order (N) of the BUTTERWORTH filter

$$N = \frac{1}{2} \frac{\log \left[\left(\frac{1}{A_S^2} - 1 \right) / \left(\frac{1}{A_P^2} - 1 \right) \right]}{\log \left(\frac{\Omega_S}{\Omega_P} \right)}, \text{ Take Integer } N (N \geq N)$$

Step 4:

Determine analog filter cutoff frequency Ω_c using

$$\text{IIT Method} \Rightarrow \Omega_c = \frac{\omega_p}{\left(\frac{1}{A_p^2} - 1 \right)^{\frac{1}{2N}}}, \text{ or}$$

$$\text{BLT Method} \Rightarrow \Omega_c = \frac{\frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)}{\left(\frac{1}{A_p^2} - 1 \right)^{\frac{1}{2N}}}$$

Step 5:

Determine analog filter transfer function from order (N) and cutoff frequency (Ω_c) of the filter

$$\text{If N is even, then } H(s) = \prod_{k=1}^{N/2} \left(\frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \right)$$

$$\text{If N is odd, then } H(s) = \left(\frac{\Omega_c}{s + \Omega_c} \right) \prod_{k=1}^{(N-1)/2} \left(\frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \right)$$

$$\text{Where, } b_k = 2 \sin\left(\frac{(2k-1)\pi}{2N}\right), K = 0, 1, 2, \dots$$

Step 6:

Now determine the desired filter transfer function from frequency transformation techniques.

For a new low pass filter, replace s by s / Ω_c .

For a high pass filter, replace s by Ω_c / s .

For a band pass filter, replace s by $Q (s^2 + \Omega_0^2) / \Omega_0 s$.

For a band stop filter, replace s by $\Omega_0 s / Q (s^2 + \Omega_0^2)$.

Where,

$$\Omega_0 = \sqrt{\Omega_L \Omega_H} \text{ and } Q = \Omega_0 / (\Omega_H - \Omega_L)$$

Step 7:

Convert analog filter transfer function $H_a(s)$ into digital filter transfer function $H(z)$ using

$$\text{IIT Method} \Rightarrow \frac{1}{s-a} \rightarrow \frac{1}{1 - e^{aT} z^{-1}}$$

$$\text{BLT Method} \Rightarrow s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$$

Step 8:

Finally realize the filter by a suitable structure (DF I / DF II / Cascade Form / Parallel Form)

DESIGN OF DIGITAL CHEBYSHEV FILTER:

Following steps were required to design IIR Digital LPF / HPF / BPF / BSF.

Step 1 :

Take the specifications of a digital low pass filter

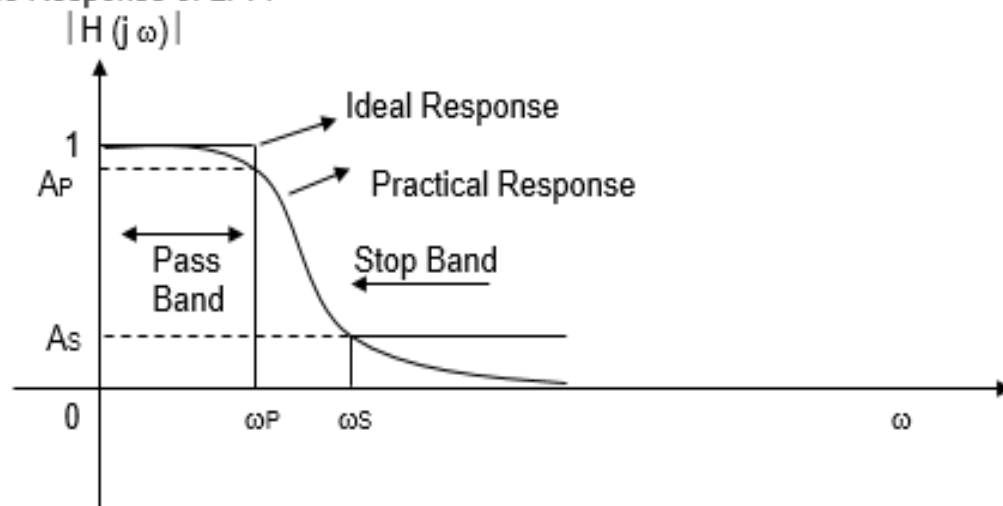
A_P : Gain at pass band digital frequency ω_P .

A_S : Gain at stop band digital frequency ω_S .

Ω_P : Pass band analog frequency corresponding to digital frequency ω_P

Ω_S : Stop band analog frequency corresponding to digital frequency ω_S

Magnitude Response of LPF:



Step 2:

Choose any one of the following transformations to design a digital filter

a. Impulse Invariant Transformation

b. Bilinear Transformation

Determine the analog filter frequency ratio Ω_S / Ω_P using

IIT Method $\Rightarrow \frac{\Omega_S}{\Omega_P} = \frac{\omega_S}{\omega_P}$, or

BLT Method $\Rightarrow \frac{\Omega_S}{\Omega_P} = \frac{\tan\left(\frac{\omega_S}{2}\right)}{\tan\left(\frac{\omega_P}{2}\right)}$

Step 3:

Decide the order (N) of the CHEBYSHEV filter

$$N = \frac{\text{Cosh}^{-1} \left[\sqrt{\left(\frac{1}{A_s^2} - 1 \right) / \left(\frac{1}{A_p^2} - 1 \right)} \right]}{\text{Cosh}^{-1} \left(\frac{\Omega_s}{\Omega_p} \right)}, \text{ Take Integer } N (N \geq N)$$

Step 4:
Determine analog filter cutoff frequency Ω_c using

$$\text{IIT Method} \Rightarrow \Omega_c = \frac{\omega_p}{\left(\frac{1}{A_p^2} - 1 \right)^{\frac{1}{2N}}}, \text{ or}$$

$$\text{BLT Method} \Rightarrow \Omega_c = \frac{\frac{2}{T} \tan \left(\frac{\omega_p}{2} \right)}{\left(\frac{1}{A_p^2} - 1 \right)^{\frac{1}{2N}}}$$

Step 5:
Determine analog filter transfer function from order (N) and cutoff frequency (Ω_c) of the filter.

$$\text{If } N \text{ is even, then } H(s) = \prod_{k=1}^{N/2} \left(\frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \right)$$

$$\text{If } N \text{ is odd, then } H(s) = \left(\frac{B_0 \Omega_c}{s + c_0 \Omega_c} \right) \prod_{k=1}^{(N-1)/2} \left(\frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \right)$$

$$\text{Where, } b_k = 2 Y_N \sin \left(\frac{(2k-1)\pi}{2N} \right),$$

$$c_k = Y_N^2 + \cos^2 \left(\frac{(2k-1)\pi}{2N} \right), \text{ and } c_0 = Y_N.$$

$$Y_N = \frac{1}{2} \left[\left(\sqrt{\frac{1}{\epsilon^2} + 1} + \frac{1}{\epsilon} \right)^{\frac{1}{N}} - \left(\sqrt{\frac{1}{\epsilon^2} + 1} + \frac{1}{\epsilon} \right)^{-\frac{1}{N}} \right]$$

To get B_k values, use $H(0) = A_p$ for even N and $H(0) = 1$ for odd N.
Assume $B_0 = B_1 = B_2 = B_3 = B_4 = \dots = B_k$

Step 6:
Now determine the desired filter transfer function from frequency transformation techniques.
For a new low pass filter, replace s by s / Ω_c .
For a high pass filter, replace s by Ω_c / s .

For a band pass filter, replace s by $Q (s^2 + \Omega_0^2) / \Omega_0 s$.

For a band stop filter, replace s by $\Omega_0 s / Q (s^2 + \Omega_0^2)$.

Where,

$$\Omega_0 = \sqrt{\Omega_L \Omega_H} \text{ and} \\ Q = \Omega_0 / (\Omega_H - \Omega_L)$$

Step 7:

Convert analog filter transfer function $H_a(s)$ into digital filter transfer function $H(z)$ using

IIT Method $\Rightarrow \frac{1}{s-a} \rightarrow \frac{1}{1-e^{aT} z^{-1}}$

BLT Method $\Rightarrow s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$

Step 8:

Finally realize the filter by a suitable structure (DF I / DF II / Cascade Form / Parallel Form)