

# Review of Probability

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## Part I: Basic Equations and Notions

In all cases it is very important to identify the random experiment and clearly identify its outcomes. It is always a good idea to have a few examples of outcomes in mind. You can often choose what you consider to be an ‘outcome’ of the experiment. A prudent choice can make the problem much easier.

*Sample space:* This is the set of all possible outcomes of the experiment.

*Event:* This is a subset of the sample space. We find the probability of an event occurring.

*Mutually exclusive:* Two events,  $E$  and  $F$ , are mutually exclusive if  $E \cap F = \emptyset$ . Intuitively this means that if one of them occurred then we know that the other did not occur.

*Probability:* The probability of an event is the fraction of times the event occurs when the experiment is repeated many many times. If the outcomes in a sample space are all equally likely then:

$$P(E) = \frac{n(E)}{n(SS)}$$

*Conditional Probability:* Intuitively, the conditional probability of  $E$  given  $F$  is the probability of  $E$  having occurred if we know that  $F$  has occurred. In other words,  $F$  becomes our whole sample space. In general, to calculate a conditional probability we can use the formula:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

or, if the outcomes in the sample space are all equally likely we can use:

$$P(E|F) = \frac{n(E \cap F)}{n(F)}$$

or, if the experiment is a ‘compound’ experiment we might be able to reason in another way (see the section on compound experiments below).

*Independence:* Two events  $E$  and  $F$  are independent if one of the following is true (if any one of them is true then the other two are automatically true as well).

$$\begin{aligned} P(E|F) &= P(E) \\ P(F|E) &= P(F) \\ P(E \cap F) &= P(E)P(F) \end{aligned}$$

Intuitively,  $E$  and  $F$  are independent if knowing that  $E$  has occurred makes it neither more nor less likely that  $F$  has occurred. Sometimes we *recognise* that two events are independent and we can *use* this to find the probability of the intersection of the two events (using the third equation above). The example below illustrates this.

- i) A die is rolled twice. Find the probability that a 5 is rolled both times.

*Solution:* To find this probability, we recognise that the events {1st roll is a 5} and {2nd roll is a 5} are independent. Thus,

$$\begin{aligned}
 P(5 \text{ is rolled both times}) &= P(\{1\text{st is a } 5\} \cap \{2\text{nd is a } 5\}) \\
 &= P(1\text{st is a } 5)P(2\text{nd is a } 5) \\
 &= \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \\
 &= \frac{1}{36}.
 \end{aligned}$$

■

Other times it might not be obvious if two events are independent, but it would be interesting to ascertain if they are or not. To determine if two events are independent evaluate the left and right hand sides of any one of the three equations above. If they are equal then the two events are independent. The example below illustrates this.

- ii) The table below shows the employees at a company. They are classified according to sex and according to whether or not they were promoted in the previous year.

	Male	Female	<i>Total</i>
Promoted	6	2	8
Not promoted	18	6	24
<i>Total</i>	24	8	32

A person is selected at random. Determine if the events {Promoted} and {Male} are independent.

*Solution:* Looking at the table we get

$$\begin{aligned}
 P(\text{Promoted}) &= \frac{8}{32} = \frac{1}{4} \\
 P(\text{Promoted}|\text{Male}) &= \frac{6}{24} = \frac{1}{4}
 \end{aligned}$$

Since these two are equal we conclude that the two events are indeed independent. (The unspoken conclusion being that there is no evidence for sex discrimination in this company since the fraction of men that were promoted is the same as the overall fraction of employees that were promoted.)

■

*Addition rule:*

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

If  $E$  and  $F$  are mutually exclusive then this becomes

$$P(E \cup F) = P(E) + P(F)$$

*Multiplicative rule:*

$$P(E \cap F) = P(E)P(F|E)$$

If  $E$  and  $F$  are independent then this becomes

$$P(E \cap F) = P(E)P(F)$$

*Using the addition and multiplicative rules:*

- a) Suppose you want to find  $P(E \cap F)$ .

- \* If you know  $P(E)$ ,  $P(F)$  and  $P(E \cup F)$  then you can use the addition rule.
  - \* If you know  $P(E)$  and  $P(F|E)$  then you can use the multiplicative rule.
  - \* If you know  $P(E)$  and  $P(F)$  and you know that  $E$  and  $F$  are independent then you can use the multiplicative rule.
- b) Suppose you want to find  $P(E \cup F)$ .
- \* If you know  $P(E)$ ,  $P(F)$  and  $P(E \cap F)$  then you can use the addition rule.
  - \* If you know  $P(E)$  and  $P(F)$  and you know that  $E$  and  $F$  are mutually exclusive then you can use the addition rule.
  - \* If you know  $P(E)$ ,  $P(F)$  and  $P(F|E)$  then you can use the multiplicative rule to find  $P(E \cap F)$  and then use the addition rule.

*Complement rule:* If you cannot find the probability of an event  $E$  directly then try to identify what the complement of  $E$  is and see if you can find the probability of that. If you can then you can use the equation below to find the probability of  $E$ .

$$P(E) = 1 - P(E')$$

## Part II: Approaches to Solving Probability Problems

I see basically three different approaches to finding a probability. Often a problem can be solved in more than one way.

**1. Draw a picture of the sample space:** Here are the kinds of pictures we use most often.

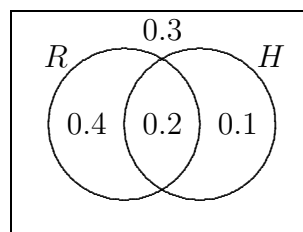
**a) Venn Diagrams:** These are used when there are at most three events of interest (along with their complements). You may need to use the addition rule and/or the multiplicative rule to fill in the probabilities of some of the basic regions. The examples below illustrate this.

- iii) Mary is particularly partial to raspberries and hazelnuts in chocolate candies. 70% of the candies in a box of assorted chocolates contain either raspberries or hazelnuts (or both). 30% contain hazelnuts and 60% contain raspberries. Mary selects a chocolate at random from the box. It contains raspberries. What is the probability that it also contains hazelnuts?

*Solution:* The sample space is the box of chocolates. We draw a Venn Diagram to represent it. To find  $P(R \cap H)$  we use the addition rule:

$$0.7 = 0.6 + 0.3 - P(R \cap H).$$

Solving yields  $P(R \cap H) = 0.2$ . It is then easy to fill in the probabilities of the other basic regions.



We are asked to find  $P(H|R)$ . Using the definition of conditional probability and reading off the Venn Diagram we get:

$$P(H|R) = \frac{P(H \cap R)}{P(R)} = \frac{0.2}{0.6} = \frac{1}{3}.$$

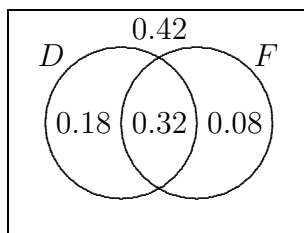
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- iv) In a certain company 50% of all employees and 80% of the female employees have one or more dependents. 40% of all employees are female. An employee is selected at random. What is the probability that a man with one or more dependents is selected?

*Solution:* The sample space is the set of all employees. We draw a Venn diagram to represent it. Let  $D$  denote the set of employees that have one or more dependents and let  $F$  denote the set of all female employees. We are told that  $P(D) = .5$ ,  $P(D|F) = .8$  and  $P(F) = .4$ . To find  $P(D \cap F)$  we use the multiplicative rule to get:

$$P(D \cap F) = P(F)P(D|F) = (.4)(.8) = .32.$$

Now it is easy to fill in the rest of the Venn diagram.



We are asked to find  $P(F' \cap D)$ . We can read this off the Venn Diagram. It is 0.18.

■

**b) Contingency Tables:** Contingency tables are usually used when there are two *variables* (as opposed to events) of interest. Each variable may take on two or more possible values. Usually the contingency table is given to us. The example below illustrates this.

- v) The table below describes voters in a certain district.

	Democrat	Republican	Independent	Total
Male	400	700	300	1400
Female	600	300	200	1100
Total	1000	1000	500	2500

A voter is selected at random.

- What is the probability that a male Democrat is selected?
- What is the probability that the person selected is Independent?
- If the person is female what is the probability that she is Independent?
- If the person is Independent what is the probability that they are female?
- Are the events {Independent} and {Female} independent?

*Solution:* The sample space of the experiment is simply the set of all voters and this is what the contingency table represents. The two variables of interest are sex that takes on the values Male/Female and party preference that takes on the values Democrat/Republican/Independent. Since a voter is selected ‘at random’ each individual is as likely to be chosen as any other. In other words the outcomes are all equally likely. Thus we can easily read probabilities off the table.

- $P(\text{Male} \cap \text{Democrat}) = \frac{400}{2500} = \frac{4}{25}.$

- b)  $P(\text{Independent}) = \frac{500}{2500} = \frac{1}{5}$ .
- c)  $P(\text{Independent}|\text{Female}) = \frac{200}{1100} = \frac{2}{11}$ .
- d)  $P(\text{Female}|\text{Independent}) = \frac{200}{500} = \frac{2}{5}$ .
- e) The answers to b) and c) are different so the two events are not independent (i.e. they are dependent).

■

Notice that any sample space that is depicted as a Venn diagram with two events of interest can also easily be depicted as a contingency table. The variables are whether or not the outcome lies in the first event and whether or not it lies in the second event. We illustrate with example iii) above.

- iii) *Solution:* The rows of the contingency table correspond to  $R$  and  $R'$  and the columns correspond to  $H$  and  $H'$ . The bottom right corner represents the whole sample space (i.e. the whole box of chocolates). Since we don't know the total number of chocolates in a box we'll use fractions instead of numbers of candies. This means that the bottom right corner will be 1. The  $R$ -Total box is 0.6 and the  $H$ -Total box is 0.3. We use the additive rule just as above to find the  $R \cap H$  box. We can then fill in the rest of the entries by making the rows and columns sum. We get the contingency table below.

	$H$	$H'$	Total
$R$	.2	.4	.6
$R'$	.1	.3	.4
Total	.3	.7	1

We then read off the probability from the table:

$$P(H|R) = \frac{P(H \cap R)}{P(R)} = \frac{.2}{.6} = \frac{1}{3}.$$

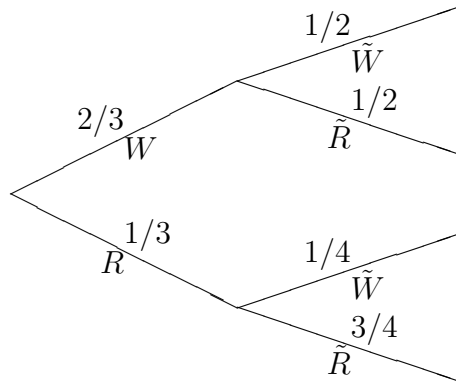
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**c) Tree Diagrams:** Tree diagrams are good to use when the experiment has two or more stages or when there are two or more variables of interest and the *conditional* probabilities of one variable given the other are given. The examples below illustrate these two cases.

- vi) Suppose we have a white urn containing two white balls and one red ball and we have a red urn containing one white ball and three red balls. An experiment consists of selecting at random a ball from the white urn and then (without replacing the first ball) selecting at random a ball from the urn having the same color as the ball.
- What is the probability that both balls that were drawn are red?
  - Find the probability that the second ball is red.
  - The experiment was performed and the second ball was red. What is the probability that it was drawn from the red urn?

*Solution:* This experiment consists of two stages: drawing the first ball and drawing the second ball so we draw a tree diagram to represent its sample space. Notice that a single outcome of the complete experiment consists of an outcome of the first stage followed by an outcome of the second stage. The first set of leaves correspond to the possible outcomes of the first stage: white and red. Let  $W$  denote all those outcomes of the complete experiment where a white ball was obtained on the first draw. Let  $R$

denote all those outcomes of the complete experiment where a red ball was obtained on the first draw. Notice that  $W$  and  $R$  are mutually exclusive and that  $W \cup R$  is the whole sample space. The first ball is drawn from the white urn that contains 2 white balls and one red ball so  $P(W) = 2/3$  and  $P(R) = 1/3$  respectively. The second set of leaves correspond to the possible outcomes of the second stage. Let  $\tilde{W}$  denote all those outcomes of the complete experiment where a white ball is obtained on the second draw and let  $\tilde{R}$  denote all those where a red ball is obtained on the second draw. Notice that  $\tilde{W}$  and  $\tilde{R}$  are mutually exclusive and  $\tilde{W} \cup \tilde{R}$  is the whole sample space. It is not immediately obvious how to find  $P(\tilde{W})$  and  $P(\tilde{R})$ . However, it is easy to find the *conditional* probabilities of  $\tilde{W}$  and  $\tilde{R}$  given  $W$  and  $R$  and that is what we need to fill in our tree diagram. If the first ball is white then the second ball is drawn from the white urn that now contains one white ball and one red ball. Thus,  $P(\tilde{W}|W) = 1/2$  and  $P(\tilde{R}|W) = 1/2$ . On the other hand, if the first ball is red then the second ball is drawn from the red urn.  $P(\tilde{W}|R) = 1/4$  and  $P(\tilde{R}|R) = 3/4$ . Thus, we get the tree diagram below.



Notice that each final leaf of the tree diagram represents a subset of the sample space. For instance, the top leaf is the subset  $W \cap \tilde{W}$  consisting of all those outcomes in which both the first and the second balls that are drawn are white. To find the probability of this event we use the multiplicative rule:

$$P(W \cap \tilde{W}) = P(W)P(\tilde{W}|W) = \left(\frac{2}{3}\right) \left(\frac{1}{2}\right) = \frac{1}{3}$$

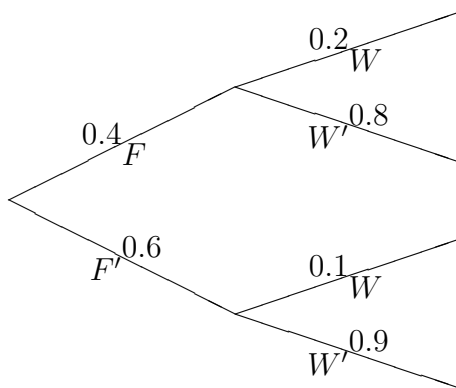
Thus, to find the probability of each final leaf we simply start at the root of the tree and multiply the numbers on the leaves that we traverse to get there. Notice also that the events represented by different final leaves are mutually exclusive. Thus, by the addition rule, if we want to find the probability of a union of two final leaves we simply add their probabilities. Reading off the tree diagram we can answer the questions. For your information we write in all the details using the addition and multiplicative rules, though of course one wouldn't normally bother to answer in such detail.

- $P(\text{both balls are red}) = P(R \cap \tilde{R}) = P(R)P(\tilde{R}|R) = \left(\frac{1}{3}\right) \left(\frac{3}{4}\right) = \frac{1}{4}.$
- $P(\text{second ball is red}) = P(\tilde{R}) = P((W \cap \tilde{R}) \cup (R \cap \tilde{R}))$   
 $= P(W \cap \tilde{R}) + P(R \cap \tilde{R}) = \left(\frac{2}{3}\right) \left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) \left(\frac{3}{4}\right) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$
- $P(\text{first ball is red}|\text{second ball is red}) = \frac{P(\text{both balls are red})}{P(\text{second ball is red})} = \frac{1/4}{7/12} = \frac{3}{7}.$

■

- vii) In the fiction section of a library 20% of the books are worn and need replacement. In the non-fiction 10% are worn and need replacement. The library's holdings are 40% fiction and 60% non-fiction. Find the probability that a book chosen at random from the library is worn and needs replacement.

*Solution:* The sample space of this experiment is the collection of books in the library's holdings. There are two variables of interest: whether or not a book is fiction and whether or not it is worn and needs replacement. Let's use  $F$  to denote fiction and  $W$  to denote worn. We are told that  $P(F) = .4$  and  $P(F') = .6$ . Also, we are told the following *conditional* probabilities:  $P(W|F) = .2$  and  $P(W|F') = .1$ . It follows that  $P(W'|F) = .8$  and  $P(W'|F') = .9$  (this is the same as saying that the probabilities on every group of leaves in a tree diagram add up to 1). Since we are told the *conditional* probabilities of  $W/W'$  given  $F/F'$  it is natural to draw a tree diagram. The  $W/W'$  leaves must come second since we know conditional probabilities for them. We get the tree diagram below.

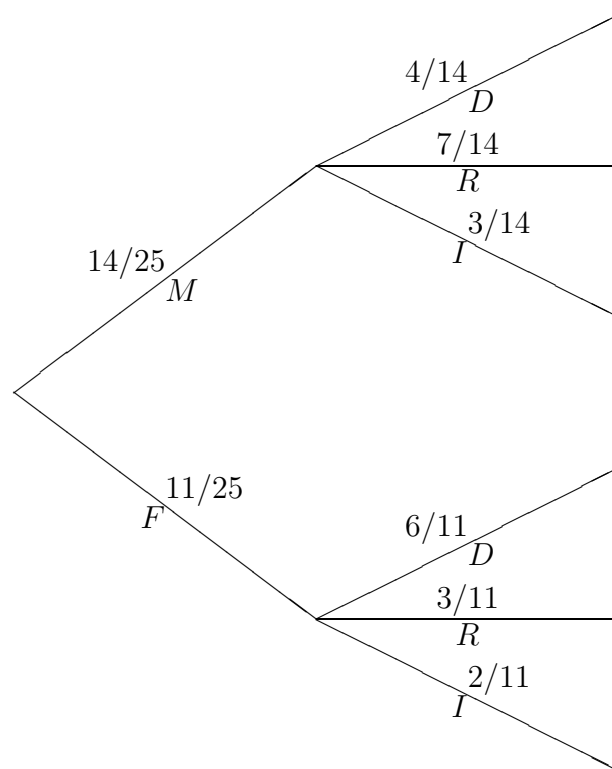


We are asked to find  $P(W)$ . There are two final nodes that correspond to the event  $W$ . Thus, reading off the tree we get  $P(W) = (0.4)(0.2) + (0.6)(0.1) = 0.14$ . ■

Any sample space that is depicted as a contingency table can also easily be depicted as a tree diagram. Remember, it is *conditional* probabilities that are written on the second set of leaves. We illustrate this with example v) above.

- v) *Solution:* The sample space is the set of all voters. There are two variables that we are interested in here: sex and party preference. We choose one of them for the first set of leaves and the other for the second set of leaves. We could do it either way here. Let's have sex for the first set of leaves and party preference for the second set of leaves. On the first set of leaves we write *unconditional* probabilities,  $P(M) = 1400/2500 = 14/25$  and  $P(F) = 1100/2500 = 11/25$ , and on the second set of leaves we write *conditional* probabilities conditioned on M or F as appropriate. For instance, on the leaf labelled I (for Independent) that emanates from the leaf labelled F we write  $P(I|F) = 2/11$ . The picture we get is shown below.





a)  $P(M \cap D) = \left(\frac{14}{25}\right) \left(\frac{4}{14}\right) = \frac{4}{25}$

b)  $P(I) = \left(\frac{14}{25}\right) \left(\frac{3}{14}\right) + \left(\frac{11}{25}\right) \left(\frac{2}{11}\right) = \frac{5}{25} = \frac{1}{5}$

c) We don't need any calculation for this one since we already did it when we constructed the tree. We simply read the answer off the tree:  $P(I|F) = \frac{2}{11}$ .

d)  $P(F|I) = \frac{P(F \cap I)}{P(I)} = \frac{\left(\frac{11}{25}\right) \left(\frac{2}{11}\right)}{\frac{1}{5}} = \frac{2}{5}$

e) The answers to b) and c) are different so the two events are not independent. ■

**d) A list or a picture that is specific to the experiment:** If the sample space is small enough we can simply list its elements or there may be some easy way to represent it as a picture. We have seen the example below many times in class.

viii) You roll a pair of dice. Find the conditional probability that one of them is a 5 given that the sum is 6.

*Solution:* The sample space is all possible rolls of a pair of dice. The picture below represents this.

	1	2	3	4	5	6
1					**	
2				*	*	
3			*		*	
4		*			*	
5	**	*	*	*	*	*
6					*	

The entries with stars ( $\star$ ) represent the event {sum is 6}. There are 5 outcomes in this event. The entries with asterisks ( $*$ ) represent the event {at least one of the dice is a 5}. We see that the intersection of the two events consists of 2 outcomes ((5,1) and (1,5)). Thus, the conditional probability that at least one of the dice is a 5 given that the sum is 6 is  $2/5$ .

■

**2. Counting:** If we choose the sample space in such a way that the outcomes are all equally likely to occur then

$$P(E) = \frac{n(E)}{n(SS)}$$

$$\text{and } P(E|F) = \frac{n(E \cap F)}{n(F)}.$$

We can use counting techniques to find the numerator and denominator in these expressions. When using this method you must decide in a very clear and unambiguous way what constitutes an ‘outcome’ of the experiment. There may be some freedom here. In particular, you may choose to have order matter or you may choose to not have it matter. If the event of interest depends on the order then you must have order matter. If the event of interest does not depend on the order then you can have the order matter or not as you choose. Whatever you choose, though, you must be consistent in the numerator and in the denominator. Here are two examples to illustrate the technique.

- ix) A man has 5 different pairs of socks colored red, orange, yellow, green and blue. He selects 2 socks at random.
- What is the probability that the 2 socks match?
  - What is the probability that the first sock that is selected is green and the second one is red?

*Solution:* The experiment is select 2 socks from a collection of 10 socks. Let’s call the socks, r1, r2, o1, o2, y1, y2, g1, g2, b1, and b2. An outcome consists of any two socks. For instance, one outcome is (y2, b1). Another outcome is (g1, r1). We need to decide if we want order to matter or not. In other words we need to decide if we want to consider (g1, r1) to be the same as (r1, g1) or if we want to consider these to be different. The event in part a) does not depend on the order in which the socks are drawn so we are free to do the problem either way. The event in part b), on the other hand, does depend on the order in which the socks are drawn so we must have order matter when we do part b). If order matters then  $n(SS) = (10)(9)$ . If order does not matter then  $n(SS) = C_2^{10} = \frac{(10)(9)}{2}$ .

- a) Let  $E = \{\text{two socks match}\}$ . If we consider outcomes so that order does not matter then an outcome of  $E$  is determined by its color of which there are 5. Thus  $n(E) = 5$ . Thus,

$$P(E) = \frac{n(E)}{n(SS)} = \frac{5}{\frac{(10)(9)}{2}} = \frac{1}{9}.$$

If we consider outcomes so that order does matter then we can construct an example of an outcome by first choosing the first sock (10 choices) and then choosing its matching pair (1 choice). Thus  $n(E) = (10)(1) = 10$ . Thus,

$$P(E) = \frac{n(E)}{n(SS)} = \frac{10}{(10)(9)} = \frac{1}{9}.$$

Of course we get the same answer either way.

- b) Let  $F = \{\text{first sock is green and second is red}\}$ . In this case we *must* consider order to matter since  $F$  depends on the order of the two socks. To construct an example of an outcome that lies in  $F$  we select a green sock (2 choices) and then we select a red sock (2 choices). Thus  $n(F) = (2)(2) = 4$ . Thus,

$$P(F) = \frac{n(F)}{n(SS)} = \frac{4}{(10)(9)} = \frac{2}{45}.$$

■

- x) An airport limousine has four passengers and stops at six different hotels. What is the probability that two or more people will be staying at the same hotel? (Assume that each person is just as likely to stay in one hotel as another.)

*Solution:* The experiment here is collect four passengers and drop them off at their hotels. Various things could happen when we do this. For instance all four passengers could be dropped off at the same hotel. Or, one passenger might be dropped off at one of the hotels, two at another and the last one at yet another. These two examples both describe things that could happen, but they are still somewhat imprecise. In particular it is not clear if we are thinking of the hotels as distinct from each other and as the passengers as distinct from each other. Let's consider both the hotels and the passengers to be distinct from each other. More concretely, let's call the hotels a, b, c, d, e and f and let's number the passengers 1, 2, 3, and 4. An example of an outcome might be aaaa (in other words all four passengers are dropped off at a). Another example is abbe (the first passenger is dropped off at a, the second and third are dropped off at b and the fourth is dropped off at e). Another example is babe and we should consider this to be different from abbe since the passengers get off at different places. Thinking of outcomes in this way we see that 'an outcome' is any string of letters taken from the letters a, b, c, d, e, and f (where repeats are allowed) of length 4. These outcomes are indeed all equally likely since we are told to assume that each passenger is as likely to get off at any hotel as any other independent of the other passengers. To count the total number of outcomes think of how we construct them. We start by writing down the first letter (there are 6 choices). Then we write down the second letter (6 choices again), etc. Thus, using the multiplication principle we see that  $n(SS) = 6^4$ . We are interested in the event  $E = \{\text{two or more people are dropped off at the same hotel}\}$ . In our notation this is the same as  $\{\text{all outcomes that have some letter appearing more than once}\}$ . It is not clear how to count the number of outcomes in this event directly. Let's think about the complement. The complement is  $E' = \{\text{all outcomes that have no repeated letters}\}$ . To construct an example of an outcome that lies in  $E'$  we choose the first letter (6 choices) then we choose the next letter (5 choices) etc. Thus, by the multiplication principle  $n(E') = (6)(5)(4)(3)$ . Thus

$$P(E) = 1 - P(E') = 1 - \frac{(6)(5)(4)(3)}{6^4} = 1 - \frac{5}{18} = \frac{13}{18}.$$

■

**3. Compound Experiments:** By a compound experiment I mean an experiment that consists of two or more activities or where there are two or more variables of interest. Since problems involving compound experiments occur so frequently and there are unique ways to approach them, they deserve a section to themselves. We have already seen one approach to take and that is to draw a tree diagram. However, if the experiment consists of more than

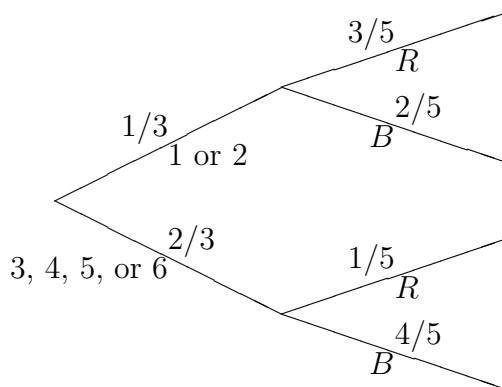
two or three activities or if there are more than two or three variables then the tree can be very large and unwieldy. In this section we look at other ways of approaching these problems. We distinguish two different types of compound experiments: those with dependent trials and those with independent trials. We further subdivide these into those where the trials are identical and those where the trials are not identical.

**a) Dependent Trials/Sampling without Replacement:**

• **Trials are not identical:** The example below illustrates the kinds of experiment considered here. Usually these experiments do not have more than two or three activities so a tree diagram probably is the best way to go about finding probabilities.

- xi) A die is rolled. If a 1 or a 2 is rolled then a ball is picked from an urn containing 3 red balls and 2 blue balls. Otherwise a ball is picked from an urn containing 1 red ball and 4 blue balls.
- What is the probability that a red ball is selected?
  - Given that a red ball is selected, what is the (conditional) probability that a 1 or a 2 was rolled?

*Solution:* This experiment consists of two activities or trials: roll a die and pick a ball out of an urn. Clearly these activities are not identical. Also, they are dependent, since what we do on the second trial depends on what happened on the first trial. To find the probabilities we draw a tree diagram of the sample space. We can then read off the probabilities.



$$\begin{aligned} \text{a) } P(R) &= \left(\frac{1}{3}\right) \left(\frac{3}{5}\right) + \left(\frac{2}{3}\right) \left(\frac{1}{5}\right) = \frac{1}{3} \\ \text{b) } P(1 \text{ or } 2 | R) &= \frac{P(\{1 \text{ or } 2\} \cap R)}{P(R)} = \frac{(1/3)(3/5)}{1/3} = \frac{3}{5} \end{aligned}$$

• **Trials are identical:** Examples of the kinds of experiments we are talking about here include dealing 5 cards from a deck of cards or choosing a committee of 3 people at random from a group of 20 people. The trials are identical in the sense that each card (respectively person) is a randomly selected card from the deck (respectively from the group of 20 people). The trials are dependent since what happens on later trials depends on what happens in earlier trials. For instance, if the King of Club is selected as the first card then it cannot be selected as the 2nd card.

There are two approaches we can use to find probabilities: counting techniques or multiplicative rule. Every problem can be solved using either technique. However, it's usually easier to use counting techniques when the event doesn't specify what happens on each trial and the multiplicative rule when the event does specify what happens on each trial. For

instance, if you are asked to find the probability of getting 2 aces and 3 kings when 5 cards are dealt from a deck of cards, then it's easier to use counting techniques. On the other hand, if you are asked to find the probability that the first two cards are aces and the next 3 are kings, then it's easier to use the multiplicative rule approach.

*Counting Techniques:* We illustrate with two examples.

xii) Four cards are dealt from a deck of cards.

- a) Find the probability that all the cards are hearts.
- b) Find the probability that exactly two of the cards are hearts.
- c) Find the probability that two of the cards are hearts, one of them is a spade, and one of them is a club.
- d) Find the probability that at least three of the cards are hearts.
- e) Find the probability that at least one of the cards is a heart.

*Solution:* The experiment is deal 4 cards from a deck. Notice that none of the events depend on the order in which the cards are dealt, they simply depend on the collection of cards that are dealt. So, this problem is probably easiest done using counting techniques. An outcome is any collection of 4 cards. Do we want the order of the cards to matter? Since none of the events depend on the order in which the cards are dealt we are free to choose either way. Generally it's simpler conceptually if we have it not matter and in this case we have  $n(SS) = C_4^{52} = \frac{(52)(51)(50)(49)}{(4)(3)(2)(1)}$ . If we decide to have order matter then  $n(SS) = P_4^{52} = (52)(51)(50)(49)$ .

- a) For illustration purposes let's do this problem both ways: where order matters and where it doesn't matter. First let's have order not matter. Then the number of collections of 4 cards that consist solely of hearts is  $C_4^{13} = \frac{(13)(12)(11)(10)}{(4)(3)(2)(1)}$ . Thus,

$$\begin{aligned}
 P(\text{all 4 cards are hearts}) &= \frac{\binom{13}{4}}{\binom{52}{4}} \\
 &= \frac{\frac{(13)(12)(11)(10)}{(4)(3)(2)(1)}}{\frac{(52)(51)(50)(49)}{(4)(3)(2)(1)}} \\
 &= \frac{(13)(12)(11)(10)}{(52)(51)(50)(49)} \\
 &= \frac{11}{4165}.
 \end{aligned}$$

If we choose instead to have order matter then the number of collections of 4 cards that consist solely of hearts is  $P_4^{13} = (13)(12)(11)(10)$ . Thus,

$$\begin{aligned}
 P(\text{all 4 cards are hearts}) &= \frac{(13)(12)(11)(10)}{(52)(51)(50)(49)} \\
 &= \frac{11}{4165}.
 \end{aligned}$$

Notice that we get the same answer either way (of course!). We'll do the rest of the parts of the problem having order not matter.

- b) We are trying to find the probability of getting exactly 2 hearts, so we need to calculate the number of outcomes that have exactly 2 hearts. To construct an example of one such outcome we first choose 2 hearts from the 13 hearts in the

deck ( $C_2^{13}$  ways) and then we choose two cards that aren't hearts ( $C_2^{39}$  ways). Thus the number of outcomes with exactly 2 hearts is  $(C_2^{13})(C_2^{39})$ . It follows that

$$\begin{aligned}
 P(\text{exactly two cards are hearts}) &= \frac{\binom{13}{2} \binom{39}{2}}{\binom{52}{4}} \\
 &= \frac{\left(\frac{(13)(12)}{(2)(1)}\right) \left(\frac{(39)(38)}{(2)(1)}\right)}{\frac{(52)(51)(50)(49)}{(4)(3)(2)(1)}} \\
 &= 6 \frac{(13)(12)(39)(38)}{(52)(51)(50)(49)} \\
 &= \frac{4446}{20825}.
 \end{aligned}$$

- c) The number of outcomes in the event {2 cards are hearts, one is a spade and one is a club} is  $(C_2^{13})(C_1^{13})(C_1^{13})$ . Thus,

$$\begin{aligned}
 &P(2 \text{ cards are hearts, one is a spade and one is a club}) \\
 &= \frac{\binom{13}{2} \binom{13}{1} \binom{13}{1}}{\binom{52}{4}} \\
 &= \frac{\left(\frac{(13)(12)}{(2)(1)}\right) (13)(13)}{\left(\frac{(52)(51)(50)(49)}{(4)(3)(2)(1)}\right)} \\
 &= (4)(3) \frac{(13)(12)(13)(13)}{(52)(51)(50)(49)} \\
 &= \frac{1014}{20825}.
 \end{aligned}$$

- d) We need to count the number of outcomes in the event {at least three cards are hearts}. To construct an example of such an outcome you might think to first choose 2 hearts ( $C_2^{13}$  ways) and then choose 2 other cards ( $C_2^{50}$  ways). The problem is, if you count like this, you will be counting some things twice. For instance if you chose the Ace and King of hearts and the 3 of spades and the 2 of hearts then that would be counted as different from choosing the Ace and 2 of hearts and the 3 of spades and the King of hearts when in fact these are the same. So this method of counting doesn't work. Instead, we think of the event {at least three cards are hearts} as the union of the events {exactly three cards are hearts} and {all four cards are hearts}. The number of outcomes in

the former is  $(C_3^{13})(C_1^{39})$  and the number in the latter is  $C_4^{13}$ . Thus,

$$\begin{aligned}
 & P(\text{at least three cards are hearts}) \\
 &= \frac{\binom{13}{3} \binom{39}{1}}{\binom{52}{4}} \\
 &= \frac{\left(\frac{(13)(12)(11)}{(3)(2)(1)}\right) \left(\frac{39}{1}\right) + \left(\frac{(13)(12)(11)(10)}{(4)(3)(2)(1)}\right)}{\left(\frac{(52)(51)(50)(49)}{(4)(3)(2)(1)}\right)} \\
 &= \frac{913}{20825}.
 \end{aligned}$$

- e) The event {at least one card is a heart} is the complement of {none of the cards is a heart}. Thus,

$$\begin{aligned}
 & P(\text{at least one card is a heart}) \\
 &= 1 - P(\text{none of the cards is a heart}) \\
 &= 1 - \frac{\binom{39}{4}}{\binom{52}{4}} \\
 &= 1 - \frac{\left(\frac{(39)(38)(37)(36)}{(4)(3)(2)(1)}\right)}{\left(\frac{(52)(51)(50)(49)}{(4)(3)(2)(1)}\right)} \\
 &= 1 - \frac{6327}{20825} \\
 &= \frac{14498}{20825}.
 \end{aligned}$$

■

xiii) Four cards are dealt from a deck of cards.

- Find the probability that the first card is a heart.
- Find the probability that the second card is a heart.
- Find the probability that the first two cards are both hearts.
- Find the probability that the first two cards are both hearts and the last two cards are both not hearts.
- Find the conditional probability that the second card is a heart given that the first is a heart.

*Solution:* The events here deal with the order in which the cards are dealt. These probabilities would be more easily calculated using the multiplicative rule approach. However, we will use counting techniques in order to illustrate them in this context. Since the events depend upon the order in which the cards are dealt we must count in such a way that order matters. In particular  $n(SS) = P_4^{52} = (52)(51)(50)(49)$ .

- We need to find the number of outcomes in the event {1st card is a heart}. To construct an example of such an outcome we choose the first card (13 choices) and then we choose the 2nd, 3rd and 4th cards (51, 50 and 49 choices respectively).

Thus,

$$\begin{aligned}P(\text{1st card is a heart}) &= \frac{(13)(51)(50)(49)}{(52)(51)(50)(49)} \\&= \frac{13}{52} \\&= \frac{1}{4}.\end{aligned}$$

- b) Here we need to count the number of outcomes in the event {2nd card is a heart}. To construct an example of such an outcome we first choose the second card (13 choices), and then we choose the first, third and fourth cards (51, 50 and 49 choices respectively). Thus,

$$\begin{aligned}P(\text{2nd card is a heart}) &= \frac{(13)(51)(50)(49)}{(52)(51)(50)(49)} \\&= \frac{13}{52} \\&= \frac{1}{4}.\end{aligned}$$

Notice that the answers to a) and b) are the same.

c)

$$\begin{aligned}P(\text{1st two cards are both hearts}) &= \frac{(13)(12)(50)(49)}{(52)(51)(50)(49)} \\&= \frac{(13)(12)}{(52)(51)} \\&= \frac{1}{17}.\end{aligned}$$

d)

$$\begin{aligned}P(\text{1st two cards are both hearts and last two are not hearts}) \\&= \frac{(13)(12)(39)(38)}{(52)(51)(50)(49)} \\&= \frac{741}{20825}.\end{aligned}$$

- e) We are being asked to find a conditional probability so our denominator will not be  $n(SS)$ . Instead we'll use

$$\begin{aligned}P(\text{2nd card is a heart} | \text{1st card is a heart}) \\&= \frac{n(\text{1st and 2nd cards are both hearts})}{n(\text{1st card is a heart})}.\end{aligned}$$



The numerator has the value  $(13)(12)(50)(49)$ . The denominator is  $(13)(51)(50)(49)$ . Thus,

$$\begin{aligned} P(\text{2nd card is a heart} | \text{1st card is a heart}) &= \frac{(13)(12)(50)(49)}{(13)(51)(50)(49)} \\ &= \frac{12}{51} \\ &= \frac{4}{17}. \end{aligned}$$

■

*Multiplicative Rule Approach:* When the event specifies what happens on each trial then it's usually easier to use multiplicative rule techniques. Multiplicative rule techniques are essentially the same as drawing a tree diagram except you don't actually draw the tree diagram. I call them multiplicative rule techniques because they use the multiplicative rule:

$$\begin{aligned} P(E_1 \cap E_2 \cap \dots \cap E_n) \\ = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2) \dots P(E_n|E_1 \cap E_2 \cap \dots \cap E_{n-1}) \end{aligned}$$

We apply this rule when  $E_1$  is an event that depends only on the outcome of the first trial,  $E_2$  is an event that depends only on the outcome of the second trial etc. Notice that the probabilities on the right hand side are *conditional* probabilities. This technique is easy to use because these conditional probabilities are easy to determine. For instance, consider example xiii) above where 4 cards are dealt from a deck of cards. In part a) we showed that

$$P(\text{1st card is a heart}) = \frac{13}{52}.$$

We can obtain this answer more easily by reasoning as follows. The event {1st card is a heart} depends only on the outcome of the first trial. So as far as that event is concerned we may as well be dealing only one card instead of four cards. When we deal a single card from a deck, the probability that it is a heart is  $13/52$  since there are 52 cards in the deck of which 13 are hearts. In part e) we showed that

$$P(\text{2nd card is a heart} | \text{1st card is a heart}) = \frac{12}{51}.$$

Again, we can obtain this answer more easily by reasoning as follows. If we know the first card is a heart then when we go to select the second card there are 51 cards left in the deck of which 12 are hearts. Thus, the conditional probability that the second card is a heart given that the first card is a heart is  $12/51$ . In part b) we showed that

$$P(\text{2nd card is a heart}) = \frac{13}{52}.$$

Once again, we can obtain this answer more easily by reasoning as follows. The second card can be any card in the deck and it is as likely to be any one card as any other card. There are 52 cards in the deck of which 13 are hearts. Thus, the (unconditional) probability that the second card is a heart is  $13/52$ . Using these alternative reasonings we see that these probabilities are easy to find. Now let's see how we can use our ability to easily calculate such probabilities along with the multiplicative rule to find other probabilities. Let's start by finding the probability that both the first and second cards are hearts. We did this in c) above and got  $\frac{(13)(12)}{(52)(51)} = \frac{1}{17}$ . Let's see how to calculate the same thing using the multiplicative

rule:

$$\begin{aligned} &P(\text{1st and 2nd cards are both hearts}) \\ &= P(\text{1st card is a heart})P(\text{2nd card is a heart}|\text{1st card is a heart}) \\ &= \left(\frac{13}{52}\right)\left(\frac{12}{51}\right) \\ &= \frac{1}{17}. \end{aligned}$$

Now let's use this method to calculate the probability that the the first two cards are both hearts and the last two are both not hearts. In part d) above we saw that this came to  $\frac{(13)(12)(39)(38)}{(52)(51)(50)(49)} = \frac{741}{20825}$ . Doing the same calculation using the multiplicative rule approach we get:

$$\begin{aligned} &P(\text{1st two cards are hearts and the last two are not hearts}) \\ &= P(\text{1st card is a heart}) \times P(\text{2nd card is a heart}|\text{1st card is a heart}) \\ &\quad \times P(\text{3rd card is not a heart}|\text{1st two cards are hearts}) \\ &\quad \times P(\text{4th card is not a heart}|\text{1st two are hearts and third is not a heart}) \\ &= \left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{39}{50}\right)\left(\frac{38}{49}\right) \\ &= \frac{741}{20825}. \end{aligned}$$

We illustrate all of this with another example.

- xiv) A board of directors has 12 members of which 8 are women and 4 are men. Three of them are selected at random to serve as president, vice-president and secretary.
- What is the probability that the president is a woman?
  - What is the probability that the vice-president is a woman?
  - Given that the president is a woman what is the conditional probability that the vice-president is a woman?
  - What is the probability that the president, vice-president and secretary are all women?
  - What is the probability that the president and vice-presidents are both women?
  - What is the probability that the president and vice-presidents are both women and the secretray is a man?
  - Consider the president and the vice-president. What is the probability that at least one of them is a woman?

*Solution:*

- $\frac{8}{12}$ .
- $\frac{8}{12}$ .
- $\frac{7}{11}$ .
- $\left(\frac{8}{12}\right)\left(\frac{7}{11}\right)\left(\frac{6}{10}\right) = \frac{14}{55}$ .
- $\left(\frac{8}{12}\right)\left(\frac{7}{11}\right)(1) = \frac{14}{33}$ .

$$f) \left(\frac{8}{12}\right) \left(\frac{7}{11}\right) \left(\frac{4}{10}\right) = \frac{28}{165}.$$

g)

$$\begin{aligned} &P(\text{at least one is a woman}) \\ &= P(\{\text{president is a woman}\} \cup \{\text{vice-president is a woman}\}) \\ &= P(\text{president is a woman}) + P(\text{vice-president is a woman}) \\ &\quad - P(\text{both the president and the vice-president are women}) \\ &= \frac{8}{12} + \frac{8}{12} - \left(\frac{8}{12}\right) \left(\frac{7}{11}\right) \\ &= \frac{10}{11}. \end{aligned}$$

Another way to find this probability is to use the complement rule. The complement of  $\{\text{at least one of the president and vice-president is a woman}\}$  is  $\{\text{both the president and vice-president are men}\}$ . Thus

$$\begin{aligned} &P(\text{at least one is a woman}) \\ &= 1 - P(\text{both are men}) \\ &= 1 - \left(\frac{4}{12}\right) \left(\frac{3}{11}\right) \\ &= \frac{10}{11}. \end{aligned}$$

■

## b) Independent Trials/Sampling with Replacement:

• **Trials are not identical:** When the trials are not identical it's best to use the multiplicative rule approach. Notice that if  $E_1$  is an event that depends only on the first trial,  $E_2$  is an event that depends only on the second trial etc. then  $P(E_2|E_1) = P(E_2)$  etc. so the multiplicative rule takes on the very simple form:

$$P(E_1 \cap E_2 \cap \dots \cap E_n) = P(E_1)P(E_2) \dots P(E_n).$$

We illustrate with an example.

- xv) There are three power lines from an island to the shore. In a fierce storm power line a fails with probability 0.04, line b fails with probability 0.02 and line c fails with probability 0.01. Assuming that the failure of any line is independent of the failure of the others what's the probability that in a fierce storm
- all three powerlines fail;
  - lines a and b fail;
  - lines a and b fail and line c functions;
  - exactly one line functions;
  - at least one line functions?

*Solution:* This experiment consists of three trials: observing line a, observing line b, and observing line c in a fierce storm. An outcome of the experiment consists of recording all three observations. Thus, for example,  $\{\text{line a fails}\}$  is an *event* (not an outcome) that consists of four outcomes: a fails and b and c function, a fails and b functions and c fails, a and b fail and c functions, and a, b, and c all fail. The power lines are different from each other so the trials are not identical. We are told

to assume that the failure of any one line is independent of the failure of the others so the trials are independent. Notice that power line a functions with probability  $1 - .04 = .96$ , line b functions with probability  $1 - .02 = .98$  and line c functions with probability  $1 - .01 = .99$ .

a)

$$\begin{aligned} &P(\text{all three lines fail}) \\ &= P(\{\text{a fails}\} \cap \{\text{b fails}\} \cap \{\text{c fails}\}) \\ &= P(\text{a fails})P(\text{b fails})P(\text{c fails}) \\ &= (.04)(.02)(.01) \\ &= .000008 \end{aligned}$$

b)

$$\begin{aligned} &P(\text{a and b fail}) \\ &= P(\{\text{a fails}\} \cap \{\text{b fails}\}) \\ &= P(\text{a fails})P(\text{b fails}) \\ &= (.04)(.02) \\ &= .0008 \end{aligned}$$

c)

$$\begin{aligned} &P(\text{a and b fail and c functions}) \\ &= P(\{\text{a fails}\} \cap \{\text{b fails}\} \cap \{\text{c functions}\}) \\ &= P(\text{a fails})P(\text{b fails})P(\text{c functions}) \\ &= (.04)(.02)(.99) \\ &= .000792 \end{aligned}$$

d) The event {exactly one line functions} consists of three outcomes: a functions and b and c fail, a fails and b functions and c fails, and a and b fail and c functions. We can easily find the probabilities of these outcomes using the multiplicative rule approach. Thus, we get:

$$\begin{aligned} &P(\text{exactly one line functions}) \\ &= P(\text{a functions and b and c fail}) \\ &\quad + P(\text{a fails and b functions and c fails}) \\ &\quad + P(\text{a and b fail and c functions}) \\ &= (.96)(.02)(.01) + (.04)(.98)(.01) + (.04)(.02)(.99) \\ &= .000192 + .000392 + .000792 \\ &= .001376 \end{aligned}$$

e)  $1 - P(\text{none function}) = 1 - .000008 = .999992$ .

■

• **Trials are identical:** When the trials are identical you can either use counting techniques or you can use the multiplicative rule approach. The multiplicative rule approach is probably easier but we illustrate both here.

*Counting Techniques:* We illustrate with an example.

xvi) A die is rolled 3 times.

- a) What is the probability that a 5 is rolled each time?
- b) What is the probability that the first and second rolls are both 5's?
- c) What is the probability that the first and second rolls are both 5's and the third roll is not a 5?
- d) What is the probability that exactly 2 5's are rolled?
- e) What's the probability that at least 2 5's are rolled?
- f) What's the probability that at least one 5 is rolled?

*Solution:* An outcome of the experiment consists of three rolls of the dice. Thus, an outcome is a set of three numbers. In some parts the event depends on what happens on each of the three trials so for these we certainly need to have order matter. Generally it's easiest to have order matter even for the other parts. So to construct an example of an outcome we select a number for the first roll (6 choices), then we select a number for the second roll (6 choices) and then we select a number for the third roll (6 choices). Thus,  $n(SS) = 6^3$ .

- a) There is only one outcome where a 5 is rolled each time, namely (5,5,5). Thus the probability is

$$\frac{1}{6^3} = \frac{1}{216}.$$

- b) To construct an outcome where the first and second rolls are both 5's we need only choose what the third roll is to be and there are 6 choices for this. Thus the probability is

$$\frac{6}{6^3} = \frac{1}{36}.$$

- c) To construct an outcome where the first and second rolls are both 5's and the third roll is not a 5 we need only choose what the third roll is to be and there are 5 choices for this. Thus the probability is

$$\frac{5}{6^3} = \frac{5}{216}.$$

- d) To construct an outcome that has exactly 2 5's we choose which 2 rolls will have the 5's ( $C_2^3 = 3$  choices) and then we choose what will be rolled on the remaining roll (5 choices). Thus the probability is

$$\frac{(3)(5)}{6^3} = \frac{5}{72}.$$

- e) The event {at least 2 5's are rolled} is the union of the two events {exactly 2 5's are rolled} and {5 is rolled each time} and these two events are mutually exclusive. The former has  $(3)(5) = 15$  outcomes in it. The latter has 1 outcome in it. Thus the probability is

$$\frac{15 + 1}{6^3} = \frac{2}{27}.$$

- f) The complement of {at least one 5 is rolled} is {no 5's are rolled}. To construct an example of an outcome in which no 5's are rolled we choose what the first roll will be (5 choices) then choose the next roll (5 choices) and then the next roll (5

choices). Thus, the number of outcomes in {no 5's are rolled} is  $5^3 = 125$ . Thus

$$\begin{aligned} P(\text{at least one 5 is rolled}) &= 1 - P(\text{no 5's are rolled}) \\ &= 1 - \frac{5^3}{6^3} \\ &= \frac{91}{216}. \end{aligned}$$

■

*Multiplicative Rule Approach:* As we observed above, if  $E_1$  is an event that depends only on the first trial,  $E_2$  depends only on the second trial etc. then since the trials are independent, the multiplicative rule takes the form:

$$P(E_1 \cap E_2 \cap \dots \cap E_n) = P(E_1)P(E_2) \dots P(E_n)$$

When using this technique it is important to understand what the outcomes of the experiment are and to understand the relationships between events. We illustrate with example xvi) above.

xvi) *Solution:* The experiment is to roll a die 3 times. So a single outcome consists of 3 numbers: what is rolled each of the 3 times. For example (3, 5, 3) is an outcome. To understand how events are related to each other consider the following events:  $A = \{\text{first roll is a 5 and the second and third rolls are not 5's}\}$ ,  $B = \{\text{first roll is a 5}\}$ ,  $C = \{\text{5 is rolled exactly once}\}$ ,  $D = \{\text{5 is rolled exactly twice}\}$ ,  $E = \{\text{5 is rolled at least once}\}$ . Notice that  $A \subset B$  and  $A \subset C$ . Indeed,  $A = B \cap C$ . Notice also that  $C$  and  $D$  are mutually exclusive and that  $A$ ,  $B$ ,  $C$  and  $D$  are all subsets of  $E$ .

Notice that each time we roll the die the probability of getting a 5 is  $1/6$  and the probability of not getting a 5 is  $5/6$ . Moreover, these are independent of any of the other rolls. So, for instance

$$\begin{aligned} P(\text{1st roll is a 5}) &= \frac{1}{6}, \\ P(\text{2nd roll is a 5} | \text{1st roll is a 5}) &= \frac{1}{6}, \\ \text{and } P(\text{2nd roll is a 5}) &= \frac{1}{6} \end{aligned}$$

Now let's find the required probabilities.

a)

$$\begin{aligned} &P(\text{5 is rolled each time}) \\ &= P(\{\text{1st roll is a 5}\} \cap \{\text{2nd roll is a 5}\} \cap \{\text{3rd roll is a 5}\}) \\ &= P(\text{1st roll is a 5})P(\text{2nd roll is a 5})P(\text{3rd roll is a 5}) \\ &= \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \\ &= \frac{1}{216}. \end{aligned}$$

b)

$$\begin{aligned} &P(\text{1st and second rolls are both 5's}) \\ &= P(\{\text{1st roll is a 5}\} \cap \{\text{2nd roll is a 5}\}) \\ &= P(\text{1st roll is a 5})P(\text{2nd roll is a 5}) \\ &= \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \\ &= \frac{1}{36}. \end{aligned}$$

c)

$$\begin{aligned} &P(\text{1st and second rolls are both 5's and 3rd roll is not a 5}) \\ &= P(\{\text{1st roll is a 5}\} \cap \{\text{2nd roll is a 5}\} \cap \{\text{3rd roll is not a 5}\}) \\ &= P(\text{1st roll is a 5})P(\text{2nd roll is a 5})P(\text{3rd roll is not a 5}) \\ &= \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \left(\frac{5}{6}\right) \\ &= \frac{5}{216}. \end{aligned}$$

d) The event here, {exactly 2 5's are rolled}, does not specify what happens on each roll. To do this problem we need to rewrite this event in terms of what happens on each roll. We write it as the union of  $C_2^3 = 3$  other events each of which specifies what happens on each roll. We get:

$$\begin{aligned} &\{\text{exactly 2 5's are rolled}\} \\ &= \{\text{1st and 2nd rolls are 5's and 3rd roll is not a 5}\} \\ &\quad \cup \{\text{1st and 3rd rolls are 5's and 2nd roll is not a 5}\} \\ &\quad \cup \{\text{2nd and 3rd rolls are 5's and 1st roll is not a 5}\} \end{aligned}$$

These three events are all mutually exclusive and they all have the same probability, namely  $\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)$ . Thus

$$\begin{aligned} &P(\text{exactly 2 5's are rolled}) \\ &= P(\text{1st and 2nd rolls are 5's and 3rd roll is not a 5}) \\ &\quad + P(\text{1st and 3rd rolls are 5's and 2nd roll is not a 5}) \\ &\quad + P(\text{2nd and 3rd rolls are 5's and 1st roll is not a 5}) \\ &= 3 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) \\ &= \frac{5}{72}. \end{aligned}$$

Notice, this is the binomial distribution we are using here. If we let  $X$  be the random variable that counts the number of 5's that are rolled then  $X$  has a binomial distribution with  $n = 3$  and  $p = 1/6$ . What we are trying to find is  $P(X = 2)$ . Using the formula for the binomial distribution we get

$$P(X = 2) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) = 3 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) = \frac{5}{72}.$$

e) Here we are being asked to find  $P(X \geq 2)$ . We use binomial distribution formula as follows

$$P(X \geq 2) = P(X = 2) + P(X = 3) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^3 = \frac{2}{27}.$$

f) Here we are being asked to find  $P(X \geq 1)$ . It's easiest to use the complement as follows.

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}.$$

■