Mathematics for Rocket Simulation

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This document describes the mathematical model used for rigid body dynamics in the rocket simulation.

1 Rigid body

Definitions:

M is the mass of the rigid body.
i, j are indices representing particles in the rigid body.
$\mathbf{r_i}$ is the position vector of particle i in the rigid body.
m_i is the mass of particle i in the rigid body.
O_M is the center of mass of the rigid body. (1)
I is the inertia tensor of the rigid body about its center of mass. (6)
T is the kinetic energy of the rigid body. (12)
ω is the angular velocity of the rigid body. (22)
L is the angular momentum of the rigid body. (24)
τ is the torque applied to the rigid body.
$\rho(\mathbf{r})$ is the mass density at position $\mathbf{r}$.
V is the volume occupied by the rigid body.
a, b are indices representing the axes x, y, z.

A rigid body is an idealization of a solid body in which deformation is neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces or moments exerted on it.

1.1 Center of mass

The center of mass O_M of a system of particles is given by:

$$O_M = \frac{1}{M} \sum_i m_i \mathbf{r_i} \tag{1}$$

There may also be use for the infinite integral form:

$$O_M = \frac{1}{M} \int_V \mathbf{r} \rho(\mathbf{r}) \, dV \tag{2}$$

If the rigid body is made up of both point masses and continuous mass distributions, the total center of mass is given by:

$$O_M = O_{M,\text{discrete}} + O_{M,\text{continuous}} \tag{3}$$

1.2 Inertia tensor

The index notation form of the inertia tensor I of a rigid body about its center of mass is given by:

$$I_{ab} \equiv \sum_i m_i \left(r_i^2 \delta_{ab} - r_{i,a} r_{i,b} \right) \tag{4}$$

There may also be use for the infinite integral form:

$$I_{ab} \equiv \int_V \rho(\mathbf{r}) \left(r^2 \delta_{ab} - r_a r_b \right) dV \tag{5}$$

Where:

> δ_{ab} is the Kronecker delta defined as:
> $$\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

This gives simply the full matrix form as:

$$I = \sum_i m_i \begin{bmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & x_i^2 + z_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & x_i^2 + y_i^2 \end{bmatrix} \tag{6}$$

Where:

> x_i, y_i, z_i are the coordinates of particle i

Or for the continuous case:

$$I = \int_V \rho(\mathbf{r}) \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} dV \tag{7}$$

Note that I is a symmetric, i.e., $I_{ab} = I_{ba}$.

For a rigid body that is made up of point and continuous mass distributions, the total inertia tensor is simply the sum of the two:

$$I = I_{\text{discrete}} + I_{\text{continuous}} \tag{8}$$

To get the inertia tensor around an arbeitrary point P, we can use the parallel axis theorem:

$$I_{ab}^P = I_{ab}^{O_M} + M \left(d^2 \delta_{ab} - d_a d_b \right) \tag{9}$$

Where:

I_{ab}^{P} is the inertia tensor about point P.
$I_{ab}^{O_M}$ is the inertia tensor about the center of mass O_M.
$\mathbf{d} = \overrightarrow{O_M P}$ is the displacement vector from the center of mass to point P.

to get the inertia tensor in world coordinates, we can use the rotation matrix R derived from the orientation quaternion q:

$$I_{\text{world}} = RIR^T \tag{10}$$

if the inertia tensor is diagonalized, one can use a quaternion Q to rotate it to world coordinates as:

$$I_{\text{world}} = QIQ^{-1} \tag{11}$$

1.3 Kinetic energy

The rotational kinetic energy T_R of a rigid body is given by:

$$T_R = \frac{1}{2}\sum_{a,b} I_{ab}\omega_a\omega_b = \frac{1}{2}\omega L \tag{12}$$

1.4 Torque and angular acceleration

Torque τ can be calculated from a force $\mathbf{F}$ applied at point P as:

$$\tau = \overrightarrow{O_M P} \times \mathbf{F} \tag{13}$$

Where:

$\overrightarrow{O_M P}$ is the position vector from the center of mass O_M to point P.

The angular acceleration α of the rigid body is given by:

$$\alpha = I^{-1}(\tau - \omega \times (I\omega)) \tag{14}$$

Integrating using angular momentum L is often more stable. where:

$$\dot{L} = \tau \tag{15}$$

1.5 Conserved quanteties

In the absence of external forces and torques, the following quantities are conserved for a rigid body:

- Linear momentum $\mathbf{p} = M\mathbf{v}$
- Angular momentum $\mathbf{L} = I_{\text{world}}\omega$
- Kinetic energy $T = T_T + T_R = \frac{1}{2}Mv^2 + \frac{1}{2}\omega L$

- phase space volume
- Center of mass position O_M
- Mass M
- Inertia tensor in body space I
- Volume V
- Density distribution $\rho(\mathbf{r})$
- Total momentum $\mathbf{P} = \sum_i \mathbf{p_i}$
- Total angular momentum $\mathbf{L} = \sum_i \mathbf{r_i} \times \mathbf{p_i}$

1.6 Integration of position

The position of a rigid body is simply integrated as a point mass, using the center of mass O_M as the reference point.

$$\mathbf{O_M}(t) = \mathbf{O_M}(0) + \mathbf{v}(0)t + \int_0^t (t - t')\mathbf{a}(t')dt' \tag{16}$$

or in the finite difference form:

$$\mathbf{O_M}(t + \Delta t) = \mathbf{O_M}(t) + \mathbf{v}(t)\Delta t + \frac{1}{2}\mathbf{a}(t)(\Delta t)^2 \tag{17}$$

and the velocity as:

$$\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{a}(t')dt' \tag{18}$$

or in the finite difference form:

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \mathbf{a}(t)\Delta t \tag{19}$$

where:

$\mathbf{v}(t)$ is the velocity of the center of mass at time t.
$\mathbf{a}(t)$ is the acceleration of the center of mass at time t.

note that this only uses second order integration for position, and can be extended using the Taylor series if higher order accuracy is needed.

1.7 Integration of orientation

The orientation of the rigid body is reprecented using a quaternion q. The quaternion is updated using the angular velocity ω as follows:

$$\dot{q}(t) = \frac{1}{2}\Omega(\omega(t))q(t) \tag{20}$$

$$q(t + \Delta t) = q(t) + \dot{q}(t)\Delta t \tag{21}$$

where:

$\Omega(\omega(t))$ is a pure quaternion $(0, \omega)$

Note that the quaternion should be normalized often to prevent drift.
The angular velocity ω is updated using the angular acceleration α as follows:

$$\omega(t) = \omega(0) + \int_0^t \alpha(t')dt' \tag{22}$$

or in the finite difference form:

$$\omega(t + \Delta t) = \omega(t) + \alpha(t)\Delta t \tag{23}$$

where:

$\alpha(t)$ is the angular acceleration of the rigid body at time t, see (14)

But insted of integrating the angular velocity directly, it is often more stable to integrate the angular momentum L and then compute the angular velocity from it:

$$L(t) = L(0) + \int_0^t \tau(t')dt' \tag{24}$$

or in the finite difference form:

$$L(t + \Delta t) = L(t) + \tau(t)\Delta t \tag{25}$$

then compute the angular velocity as:

$$\omega(t) = I_{\mathrm{world}}^{-1} L(t) \tag{26}$$

2 References

https://ocw.mit.edu/courses/8-09-classical-mechanics-iii-fall-2014/6fe39e8d5ce4ce746ca256dfea665eda_MIT8_09F14_Chapter_2.pdf