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Chapter 1

Proofs in Quantified Logic

1.1 Rules for Quantifiers

For proofs in QL, we use all of the basic rules of SL plus four new basic rules: both introduction and elimination rules for each of the quantifiers.

Since all of the derived rules of SL are derived from the basic rules, they will also hold in QL. We will add another derived rule, a replacement rule called quantifier negation.

Universal elimination

If you have $\forall x Ax$, it is legitimate to infer that anything is an A. You can infer Aa, Ab, Az, Ad_3—in short, you can infer $A\mathcal{c}$ for any constant $\mathcal{c}$. This is the general form of the universal elimination rule (∀E):

m	$\forall\chi\mathcal{A}$	
	$\mathcal{A}[\mathcal{c}\|\chi]$	∀E m

$\mathcal{A}[\mathcal{c}|\chi]$ is a substitution instance of $\forall\chi\mathcal{A}$. The symbols for a substitution instance are not symbols of QL, so you cannot write them in a proof. Instead, you write the subsituted sentence with the constant $\mathcal{c}$ replacing all occurrences of the variable χ in $\mathcal{A}$. For example:

1	$\forall x(Mx \to Rxd)$	
2	$Ma \to Rad$	∀E 1
3	$Md \to Rdd$	∀E 1

Existential introduction

When is it legitimate to infer $\exists x Ax$? If you know that something is an A— for instance, if you have Aa available in the proof.

This is the existential introduction rule ($\exists$I):

m	$\mathcal{A}$	
	$\exists \chi \mathcal{A}[\chi \| \| c]$	$\exists$I m

It is important to notice that $\mathcal{A}[\chi||c]$ is not the same as a substitution instance. We write it with two bars to show that the variable χ does not need to replace all occurrences of the constant c. You can decide which occurrences to replace and which to leave in place. For example:

1	$Ma \to Rad$	
2	$\exists x(Ma \to Rax)$	$\exists$I 1
3	$\exists x(Mx \to Rxd)$	$\exists$I 1
4	$\exists x(Mx \to Rad)$	$\exists$I 1
5	$\exists y \exists x(Mx \to Ryd)$	$\exists$I 4
6	$\exists z \exists y \exists x(Mx \to Ryz)$	$\exists$I 5

Universal introduction

A universal claim like $\forall x Px$ would be proven if every substitution instance of it had been proven, if every sentence Pa, Pb, ... were available in a proof. Alas, there is no hope of proving *every* substitution instance. That would require proving Pa, Pb, ..., Pj_2, ..., Ps_7, ..., and so on to infinity. There are infinitely many constants in QL, and so this process would never come to an end.

Consider a simple argument: $\forall x Mx$, $\therefore \forall y My$

It makes no difference to the meaning of the sentence whether we use the variable x or the variable y, so this argument is obviously valid. Suppose we begin in this way:

1	$\forall x Mx$	want $\forall y My$
2	Ma	$\forall$E 1

We have derived Ma. Nothing stops us from using the same justification to derive Mb, ..., Mj_2, ..., Ms_7, ..., and so on until we run out of space or patience. We have effectively shown the way to prove Mc for any constant c. From this, $\forall x Mx$ follows.

$$\begin{array}{r|ll}
1 & \forall x Mx & \\ \cline{2-2}
2 & Ma & \forall\text{E } 1 \\
3 & \forall y My & \forall\text{I } 2
\end{array}$$

It is important here that a was just some arbitrary constant. We had not made any special assumptions about it. If Ma were a premise of the argument, then this would not show anything about *all* y. For example:

$$\begin{array}{r|ll}
1 & \forall x Rxa & \\ \cline{2-2}
2 & Raa & \forall\text{E } 1 \\
3 & \forall y Ryy & \text{not allowed!}
\end{array}$$

This is the schematic form of the universal introduction rule ($\forall$I):

$$\begin{array}{r|ll}
m & \mathcal{A} & \\
 & \forall \chi \mathcal{A}[\chi|c]^{*} & \forall\text{I } m
\end{array}$$

* c must not occur in any undischarged assumptions.

Note that we can do this for any constant that does not occur in an undischarged assumption and for any variable.

Note also that the constant may not occur in any *undischarged* assumption, but it may occur as the assumption of a subproof that we have already closed. For example, we can prove $\forall z(Dz \to Dz)$ without any premises.

$$\begin{array}{r|l|ll}
1 & & Df & \text{want } Df \\ \cline{3-3}
2 & & Df & \text{R } 1 \\
3 & \multicolumn{2}{l}{Df \to Df} & \to\text{I } 1\text{–}2 \\
4 & \multicolumn{2}{l}{\forall z(Dz \to Dz)} & \forall\text{I } 3
\end{array}$$

Existential elimination

A sentence with an existential quantifier tells us that there is *some* member of the UD that satisfies a formula. For example, $\exists x Sx$ tells us (roughly) that there is at least one S. It does not tell us *which* member of the UD satisfies S, however. We cannot immediately conclude Sa, Sf_{23}, or any other substitution instance of the sentence. What can we do?

Suppose that we knew both $\exists x Sx$ and $\forall x(Sx \to Tx)$. We could reason in this way:

> Since $\exists x Sx$, there is something that is an S. We do not know which constants refer to this thing, if any do, so call this thing Ω. From $\forall x(Sx \rightarrow Tx)$, it follows that if Ω is an S, then it is a T. Therefore Ω is a T. Because Ω is a T, we know that $\exists x Tx$.

In this paragraph, we introduced a name for the thing that is an S. We called it Ω, so that we could reason about it and derive some consequences from there being an S. Since Ω is just a bogus name introduced for the purpose of the proof and not a genuine constant, we could not mention it in the conclusion. Yet we could derive a sentence that does not mention Ω; namely, $\exists x Tx$. This sentence does follow from the two premises.

We want the existential elimination rule to work in a similar way. Yet since Greek letters like Ω are not symbols of QL, we cannot use them in formal proofs. Instead, we will use constants of QL which do not otherwise appear in the proof. A constant that is used to stand in for whatever it is that satisfies an existential claim is called a PROXY. Reasoning with the proxy must all occur inside a subproof, and the proxy cannot be a constant that is doing work elsewhere in the proof.

This is the schematic form of the existential elimination rule ($\exists$E):

$$
\begin{array}{r|l l}
m & \exists \chi \mathcal{A} & \\
n & \quad \begin{array}{|l} \underline{\mathcal{A}[c|\chi]^*} \end{array} & \\
p & \quad \begin{array}{|l} \mathcal{B} \end{array} & \\
 & \mathcal{B} & \exists\text{E } m,\ n\text{–}p
\end{array}
$$

* The constant c must not appear in $\exists \chi \mathcal{A}$, in $\mathcal{B}$, or in any undischarged assumption.

With this rule, we can give a formal proof that $\exists x Sx$ and $\forall x(Sx \rightarrow Tx)$ together entail $\exists x Tx$. The structure of the proof is effectively the same as the English-language argument with which we began, except that the subproof uses the constant "a" rather than the bogus name Ω.

$$
\begin{array}{r|l l}
1 & \exists x Sx & \\
2 & \underline{\forall x(Sx \rightarrow Tx)} & \text{want } \exists x Tx \\
3 & \quad \begin{array}{|l} \underline{Sa} \end{array} & \\
4 & \quad \begin{array}{|l} Sa \rightarrow Ta \end{array} & \forall\text{E } 2 \\
5 & \quad \begin{array}{|l} Ta \end{array} & \rightarrow\text{E } 3,\ 4 \\
6 & \quad \begin{array}{|l} \exists x Tx \end{array} & \exists\text{I } 5 \\
7 & \exists x Tx & \exists\text{E } 1,\ 3\text{–}6
\end{array}
$$

Quantifier negation

When translating from English to QL, we noted that $\neg\exists x\neg\mathcal{A}$ is logically equivalent to $\forall x\mathcal{A}$. In QL, they are provably equivalent. We can prove one half of the equivalence with a rather gruesome proof:

1	$\forall xAx$	want $\neg\exists x\neg Ax$
2	$\quad \exists x\neg Ax$	for reductio
3	$\quad\quad \neg Ac$	for $\exists$E
4	$\quad\quad\quad \forall xAx$	for reductio
5	$\quad\quad\quad Ac$	$\forall$E 1
6	$\quad\quad\quad \neg Ac$	R 3
7	$\quad\quad \neg\forall xAx$	$\neg$I 4–6
8	$\quad \neg\forall xAx$	$\exists$E 3–7
9	$\quad \forall xAx$	R 1
10	$\neg\exists x\neg Ax$	$\neg$I 2–8

In order to show that the two sentences are genuinely equivalent, we need a second proof that assumes $\neg\exists x\neg\mathcal{A}$ and derives $\forall x\mathcal{A}$. We leave that proof as an exercise for the reader.

It will often be useful to translate between quantifiers by adding or subtracting negations in this way, so we add two derived rules for this purpose. These rules are called quantifier negation (QN):

$$\neg\forall \chi\mathcal{A} \dashv\vdash \exists\chi\neg\mathcal{A}$$
$$\neg\exists \chi\mathcal{A} \dashv\vdash \forall\chi\neg\mathcal{A} \quad \text{QN}$$

Since QN is a replacement rule, it can be used on whole sentences or on subformulae.

Practice Exercises

Part 1 Provide a justification (rule and line numbers) for each line of proof that requires one.

1)

1	$\quad \neg(\exists xMx \vee \forall x\neg Mx)$
2	$\quad \neg\exists xMx \wedge \neg\forall x\neg Mx$
3	$\quad \neg\exists xMx$
4	$\quad \forall x\neg Mx$
5	$\quad \neg\forall x\neg Mx$
6	$\exists xMx \vee \forall x\neg Mx$

2)
$$\begin{array}{r|l}
1 & \forall x\exists y(Rxy \lor Ryx)\\
2 & \underline{\forall x\neg Rmx}\\
3 & \exists y(Rmy \lor Rym)\\
4 & \quad \mid \underline{Rma \lor Ram}\\
5 & \quad \mid \neg Rma\\
6 & \quad \mid Ram\\
7 & \quad \mid \exists x Rxm\\
8 & \exists x Rxm
\end{array}$$

3)
$$\begin{array}{r|l}
1 & \forall x(\exists y Lxy \to \forall z Lzx)\\
2 & \underline{Lab}\\
3 & \exists y Lay \to \forall z Lza\\
4 & \exists y Lay\\
5 & \forall z Lza\\
6 & Lca\\
7 & \exists y Lcy \to \forall z Lzc\\
8 & \exists y Lcy\\
9 & \forall z Lzc\\
10 & Lcc\\
11 & \forall x Lxx
\end{array}$$

4)
$$\begin{array}{r|l}
1 & \forall x(Jx \to Kx)\\
2 & \exists x\forall y Lxy\\
3 & \underline{\forall x Jx}\\
4 & Ja\\
5 & Ja \to Ka\\
6 & Ka\\
7 & \quad \mid \underline{\forall y Lay}\\
8 & \quad \mid Laa\\
9 & \quad \mid Ka \land Laa\\
10 & \quad \mid \exists x(Kx \land Lxx)\\
11 & \exists x(Kx \land Lxx)
\end{array}$$

Part 2 Without using the QN rule, prove $\neg\exists x\neg\mathcal{A} \vdash \forall x\mathcal{A}$

Part 3 Provide a proof of each claim.

1. $\vdash \forall xFx \lor \neg\forall xFx$
2. $\{\forall x(Mx \leftrightarrow Nx), Ma \land \exists xRxa\} \vdash \exists xNx$
3. $\{\forall x(\neg Mx \lor Ljx), \forall x(Bx \to Ljx), \forall x(Mx \lor Bx)\} \vdash \forall xLjx$
4. $\forall x(Cx \land Dt) \vdash \forall xCx \land Dt$
5. $\exists x(Cx \lor Dt) \vdash \exists xCx \lor Dt$

Part 4

Consider the following argument:

The hospital will only hire a skilled surgeon. All surgeons are greedy. Billy is a surgeon, but is not skilled. Therefore, Billy is greedy, but the hospital will not hire him.

UD: people
Gx: x is greedy.
Hx: The hospital will hire x.
Rx: x is a surgeon.
Kx: x is skilled.
b: Billy

$$\begin{array}{ll} & \forall x\big[\neg(Rx \wedge Kx) \rightarrow \neg Hx\big] \\ & \forall x(Rx \rightarrow Gx) \\ & Rb \wedge \neg Kb \\ \therefore & Gb \wedge \neg Hb \end{array}$$

Prove the symbolized argument.

Part 5 Our language QL is an expanded and more powerful way of dealing with the categorical syllogisms, we studied in Chapter 5. Here are a few of the valid forms. Translate them and prove them in QL. (The names are the ones given to them by medieval logicians. You don't have to know them.)

1) **Barbara:** All Bs are Cs. All As are Bs. $\therefore$ All As are Cs.
2) **Baroco:** All Cs are Bs. Some A is not B. $\therefore$ Some A is not C.
3) **Bocardo:** Some B is not C. All Bs are As. $\therefore$ Some A is not C.
4) **Celantes:** No Bs are Cs. All As are Bs. $\therefore$ No Cs are As.
5) **Celarent:** No Bs are Cs. All As are Bs. $\therefore$ No As are Cs.
6) **Campestres:** All Cs are Bs. No As are Bs. $\therefore$ No As are Cs.
7) **Cesare:** No Cs are Bs. All As are Bs. $\therefore$ No As are Cs.
8) **Dabitis:** All Bs are Cs. Some A is B. $\therefore$ Some C is A.
9) **Darii:** All Bs are Cs. Some A is B. $\therefore$ Some A is C.
10) **Disamis:** Some B is C. All Bs are As. $\therefore$ Some A is C.
11) **Ferison:** No Bs are Cs. Some B is A. $\therefore$ Some A is not C.
12) **Festino:** No Cs are Bs. Some A is B. $\therefore$ Some A is not C.
13) **Frisesomorum:** Some B is C. No As are Bs. $\therefore$ Some C is not A.

Part 6 Symbolize each of the following and add the additional assumptions "There is an A" and "There is a B." Then prove that the supplemented arguments forms are valid in QL.

1) **Barbari:** All Bs are Cs. All As are Bs. $\therefore$ Some A is C.
2) **Celaront:** No Bs are Cs. All As are Bs. $\therefore$ Some A is not C.
3) **Camestros:** All Cs are Bs. No As are Bs. $\therefore$ Some A is not C.
4) **Darapti:** All As are Bs. All As are Cs. $\therefore$ Some B is C.

5) **Felapton:** No Bs are Cs. All As are Bs. $\therefore$ Some A is not C.
6) **Baralipton:** All Bs are Cs. All As are Bs. $\therefore$ Some C is A.
7) **Fapesmo:** All Bs are Cs. No As are Bs. $\therefore$ Some C is not A.

Part 7 Provide a proof of each claim.

1) $\forall x \forall y Gxy \vdash \exists x Gxx$
2) $\forall x \forall y (Gxy \to Gyx) \vdash \forall x \forall y (Gxy \leftrightarrow Gyx)$
3) $\{\forall x(Ax \to Bx), \exists x Ax\} \vdash \exists x Bx$
4) $\{Na \to \forall x(Mx \leftrightarrow Ma), Ma, \neg Mb\} \vdash \neg Na$
5) $\vdash \forall z(Pz \vee \neg Pz)$
6) $\vdash \forall x Rxx \to \exists x \exists y Rxy$
7) $\vdash \forall y \exists x(Qy \to Qx)$

Part 8 Show that each pair of sentences is provably equivalent.

1) $\forall x(Ax \to \neg Bx) \dashv\vdash \neg\exists x(Ax \wedge Bx)$
2) $\forall x(\neg Ax \to Bd) \dashv\vdash \forall x Ax \vee Bd$
3) $\exists x Px \to Qc \dashv\vdash \forall x(Px \to Qc)$

Part 9 Show that each of the following is provably inconsistent.

1) $\{Sa \to Tm,\ Tm \to Sa,\ Tm \wedge \neg Sa\}$
2) $\{\neg\exists x \exists y Lxy,\ Laa\}$
3) $\{\forall x(Px \to Qx),\ \forall z(Pz \to Rz),\ \forall y Py,\ \neg Qa \wedge \neg Rb\}$

Part 10 Write a symbolization key for the following argument, translate it, and prove it:

> There is someone who likes everyone who likes everyone that he likes. Therefore, there is someone who likes himself.

Part 11 For each of the following pairs of sentences: If they are logically equivalent in QL, give proofs to show this. If they are not, construct a model to show this.

1) $\forall x Px \to Qc \dashv\vdash \forall x(Px \to Qc)$
2) $\forall x Px \wedge Qc \dashv\vdash \forall x(Px \wedge Qc)$
3) $Qc \vee \exists x Qx \dashv\vdash \exists x(Qc \vee Qx)$
4) $\forall x \forall y \forall z Bxyz \dashv\vdash \forall x Bxxx$
5) $\forall x \forall y Dxy \dashv\vdash \forall y \forall x Dxy$
6) $\exists x \forall y Dxy \dashv\vdash \forall y \exists x Dxy$

Part 12 For each of the following arguments: If it is valid in QL, give a proof. If it is invalid, construct a model to show that it is invalid.

1) $\forall x \exists y Rxy \vdash \exists y \forall x Rxy$
2) $\exists y \forall x Rxy \vdash \forall x \exists y Rxy$
3) $\exists x(Px \wedge \neg Qx) \vdash \forall x(Px \rightarrow \neg Qx)$
4) $\{\forall x(Sx \rightarrow Ta), Sd\} \vdash Ta$
5) $\{\forall x(Ax \rightarrow Bx), \forall x(Bx \rightarrow Cx)\} \vdash \forall x(Ax \rightarrow Cx)$
6) $\{\exists x(Dx \vee Ex), \forall x(Dx \rightarrow Fx)\} \vdash \exists x(Dx \wedge Fx)$
7) $\forall x \forall y(Rxy \vee Ryx) \vdash Rjj$
8) $\exists x \exists y(Rxy \vee Ryx) \vdash Rjj$
9) $\{\forall x Px \rightarrow \forall x Qx, \exists x \neg Px\} \vdash \exists x \neg Qx$
10) $\{\exists x Mx \rightarrow \exists x Nx, \neg \exists x Nx\} \vdash \forall x \neg Mx$