Module 3: Derived Rules and Theorems

Phil 150: Intro to Formal Logic

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Rules of Replacement(Logical Laws)

DeM: $\neg(A \wedge B) \equiv \neg A \vee \neg B$

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

Contra: $A \to B \equiv \neg B \to \neg A$

Material Implication: $A \to B \equiv \neg A \vee B$

- DeMorgan's (DeM)
- Contrapositive (Contra)
- Commutation (Com): $A \wedge B, A \vee B \equiv B \wedge A, B \vee A$
- Biconditional Exportation (B) $A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$
- Idempotent (I) $A \equiv A \wedge A, A \vee A$

$$A \to (\neg(A \wedge B))$$
$$A \to (\neg A \vee \neg B)$$

$$\cancel{\frac{\begin{array}{c} A \to (A \to B) \\ A \end{array}}{A \to B}}$$

Derived Rules of Inferences

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

- Weakening $B \vdash A \to B$
- Argument by Cases
- Hypothetical Syllogism $A \to B,\ B \to C \vdash A \to C$ (1.15)
- Modus Tollen $\neg B,\ A \to B \vdash \neg A$
- To sure to take a look at what exactly is meant by a *proof* of a derived rule.
- Also, remember which rules are derived, and which are primitive/basic.

Argument by Cases

- Also known as Constructive Dilemma.
- Intuitively: damned if you do; damned if you don't.
- 1. If you do it, you are damned.
  2. If you don't it you are damned.
  3. So...

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    3. So...
    4. You are damned.

More on Argument by Cases

$A \vee B$
$A \to C$
$B \to C$
C

$A \vee B$
A
C
B
C
$\to C$

- Two forms - one requires subproofs; one doesn't.
- Example: $A \vee (B \vee C) \vdash (A \vee B) \vee C$ ← Association

1. $A \vee (B \vee C)$ P
2. A Assu
3. $A \vee B$ 2 ∨I
4. $(A \vee B) \vee C$ 3 ∨I
5. $(A \vee B) \vee C$

2' $B \vee C$
3' B A
4' $A \vee B$ ∨I 3'
5' $(A \vee B) \vee C$ ∨I 4'
6' C A
$(A \vee B) \vee C$ ∨I 6'
$(A \vee B) \vee C$

Theorems: Derivations without Premises

- A derivation with no premises shows all its conclusions to be logical truths, which are also called theorems.
- More technically: A sentence P is a theorem if and only if $\{\} \vdash P$.
- $\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$

1	$(A \to B) \to A$	A
2.	$\neg A$	A
3	$\neg(A \to B)$	1, 2 MT
4	$A \land \neg B$	3 (NC)
5	A	$\land E$ 4
6	$\neg A$	2 R
7	A	
8	$((A \to B) \to A) \to A$	

Things to keep in mind

- A theorem almost always starts with a subderivation.
- two main options: $(RAA), (\rightarrow I)$
- (AC): advanced technique; requires the "excluded middle".
- $((A \rightarrow B) \rightarrow A) \rightarrow A$ using (RAA) and (AC).

1 $\neg(A \vee \neg A)$ A

2 $\neg A \wedge \neg\neg A$ Dem 1

3 $A \vee \neg A$ 1-2 RAA

4 A A

5 $((A \rightarrow B) \rightarrow A) \rightarrow A$ 4 W

6 $\neg A$

7 $(A \rightarrow B) \rightarrow A$

8 $\neg A$

9 $\neg(A \rightarrow B)$

10 $A \wedge \neg B$

$A \wedge \neg A$

A

$((A \rightarrow B) \rightarrow A) \rightarrow A$

$(\neg(G \vee Q) \wedge (K \to G)) \wedge \neg(P \vee \neg K)$

Problems

- Good for Argument by cases.
- 1. $(A \vee B) \wedge (B \to C) \vdash A \vee C$
  2. $(\neg H \vee M), \neg M \to \neg C \vdash (H \vee C) \to M$
  3. $(A \wedge B) \vee (A \wedge C) \vdash A \wedge (B \vee C)$
  4. $(S \wedge J) \vee (\neg S \wedge \neg J) \vdash (S \leftrightarrow J)$
  5. $K \to (F \vee C), J \to (C \vee D), \neg C \vdash \neg(F \vee D) \to \neg(K \vee J)$
- Theorems: Easier
- 1. $(A \vee B) \to (\neg B \to A)$
  2. $M \vee \neg(M \wedge N)$
  3. $[H \to (O \to N)] \to [(H \wedge O) \to N]$
- Theorems: Harder
  - $(D \to B) \to ((D \to T) \to (D \to (B \wedge T))$
    $(K \to F) \to (\neg F \to \neg(K \wedge P))$
  2. $(L \to (M \to N)) \to ((L \to M) \to (L \to N))$
- Theorems: Nightmare
- 1. $\neg((H \wedge F) \to C) \wedge \neg(H \to (F \to C))$
  2. $\neg(\neg(G \vee Q) \wedge (K \to G)) \wedge \neg(P \vee \neg K)$
  3. $\neg(A \leftrightarrow B) \leftrightarrow (\neg A \leftrightarrow B)$