A Project Report/Thesis/Dissertation on

DYNAMICS OF QUANTUM SYSTEMS MODELED BY JAYNES CUMMINGS HAMILTONIAN UNDER THE INFLUENCE OF MEMORY CHANNELS

Submitted for the partial fulfillment of the requirement for the award of the Degree of

Bachelor of Science (Honors)

In

Chemistry

by

Lokesh

Under the Supervision/Guidance of

Dr. Manas Ranjan Dash and Dr. Natasha Awasthi

Designation and affiliating University/Organization

DIT UNIVERSITY, DEHRADUN, INDIA

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DECLARATION

This is to certify that the Project / Thesis / Dissertation entitled **"Dynamics of Quantum Systems Modeled by Jaynes Cummings Hamiltonian under the influence of Memory Channels**" in partial fulfillment of the requirement for the award of the degree of **Bachelor of Science (Honors)** in **Chemistry** submitted to **DIT University, Dehradun, Uttarakhand, India,** is an authentic record of bona fide work carried out by me, under the supervision /guidance of **Dr. Manas Ranjan Dash** and **Dr. Natasha Awasthi.**

The matter embodied in this Project/Thesis/Dissertation has not been submitted for the award of any other degree or diploma to any University/Institution.

Signature

Name of Candidate: Lokesh

Roll No: 211075007

Date: 24, May 2024

Place: Dehradun

CERTIFICATE

This is to certify that the Project / Thesis / Dissertation entitled **“Dynamics of Quantum Systems Modeled by Jaynes Cummings Hamiltonian under the influence of Memory Channels**” in partial fulfillment of the requirement for the award of the degree of **Bachelor of Science (Honors)** in **Chemistry** submitted to **DIT University, Dehradun, Uttarakhand, India,** is an authentic record of bona fide work carried out by **Mr. Lokesh, Roll No.** – **211075007** under my supervision/ guidance.

Signature

Name of Supervisor: Dr. Manas Ranjan Dash

Date: 24, May 2024

Place: Dehradun

Signature

Name of Supervisor: Dr. Natasha Awasthi

Date: 24, May 2024

Place: Dehradun

LIST OF ABBREVIATIONS

int	Interaction
TLS	Two-Level System
JCM	Jaynes-Cummings Model
ADC	Amplitude Damping Channel
PDC	Phase Damping Channel
PF	Phase Flip

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ABSTRACT

The study investigates the evolution of quantum systems described by the Jaynes Cummings Hamiltonian under the action of various decoherence channels with and without memory parameters. It probes into the effect of memory parameters on entanglement between a two-level atom and the photons of a single-mode radiation field.

Sustaining entanglement against noise is crucial to any kind of quantum technology. Thus, the effect of various noise channels on quantum systems has been widely studied and measures to prevent collapse of entanglement have been suggested. In my work, I have investigated the resilience offered to a quantum system by the application of memory parameters. Concurrence is taken as a measure of entanglement. The concurrence of the entangled system is measured against decoherence strength for various noise channels, once with memory parameters applied and once without memory parameters and the results are compared.

On application of memory, entanglement shows meagre signs of decay with increased decoherence strength. For memoryless channel, a sharp decay of entanglement was observed with increase in decoherence strength. The comparison confirmed that entanglement shows resilience against noise in memory channels than the memoryless channels.

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CHAPTER 1 - INTRODUCTION

1.1. General

The bizarre dynamics of the quantum realm have been studied for decades in the hopes of exploiting its vast potential. Quantum world operates very differently from our macroscopic monotonous world. The well-defined concept of position and momentum breaks down, objects start existing as a superposition of states and "spooky actions" like entanglement start taking hold. Every event is decided by probabilities, just like a coin toss or roll of some dice.

These uncanny phenomena that fuelled debates among luminaries like Bohr and Einstein are now seen as grounds for unprecedented scientific advancement. With extensive research and investment, quantum systems have been developed to apply principles of quantum mechanics to the fields of cryptography, communication, computation etc [1].

However, these systems, in comparison, are like the computers of 1960's that occupied a whole room and used vacuum tubes for transistors. Without sophisticated instruments and highly controlled conditions, operating a quantum system is still impossible.

1.2. Coherence and Decoherence

Quantum states are very fragile. Even minute interactions with environment can cause these states to collapse, a phenomenon the scientists call *decoherence*. Therefore, quantum systems need to be isolated and protected from the environment. Any factor that can cause decoherence in a quantum system constitutes *noise*.

On the other hand, the ability of quantum systems to maintain a definite phase relation between different states in a superposition is called *coherence*. In layman's terms, it is the ability of a quantum system to maintain a state. The duration for which a system sustains coherence is coined as *coherence time*. Longer coherence times are critical as they allow for more complex operations without errors. Different quantum systems show different coherence times [2].

1.3. System of Study

For this study, the system at hand is a quantum TLS comprised of a two-level atom in an optical cavity subjected to a single-mode radiation field. The atom-photon interactions in such systems are efficiently described by the Jaynes Cummings Model [3].

For coherent systems, the entanglement can be studied efficiently against various decoherence channels. Decoherence channels induce noise to the system. The effect of noise on coherence of the system can be studied by observing the entanglement between the bipartite system [4].

Various efforts have been made to sustain coherence for longer durations. Memory channels have shown promise of maintaining coherence and postponing the decay of quantum entanglement[5]. My work is one more of such endeavours.

1.4. Objective of the Thesis

This study is woven around discovering the effect of memory channels on evolution of systems described by JCM.

Specifically, the study deals with the evolution of the quantum state of such systems under the influence of decoherence channels, with and without memory. The concurrence of our system of study is measured against decoherence parameters and the results with and without memory channels are compared.

The objective is to demonstrate the resilience of quantum systems, described by JCM with memory channels over memoryless channels.

1.5. Organization of the Thesis

The study has been organized into 3 chapters.

Chapter 1- Introduction: The chapter introduces the foundational concepts of the study and the objective that guides it.

Chapter 2- Literature Review: The chapter sheds light on the literature and work published relevant to the motion.

Chapter 3- Calculations: The chapter deals with mathematics involved in our study and calculations of the result.

Chapter 4- Results and Discussion: The results of the study are presented and discussed in this chapter.

Chapter 5- Conclusion and Scope for Future Work: The study is concluded and recommendations for potential future work are discussed in this chapter.

2.1. The Two-level Atom

Quantum two level systems, better known as *Qubits*, are quantum systems that can exist as a superposition of two separate quantum states. A simple example is an atom with a single electron that can transit only between two electronic levels.

The atom has a low energy ground state, $|g\rangle$ and a high energy excited state $|e\rangle$. The energy difference between the two states becomes $\hbar\omega_0$ with a transition frequency of ω_0.

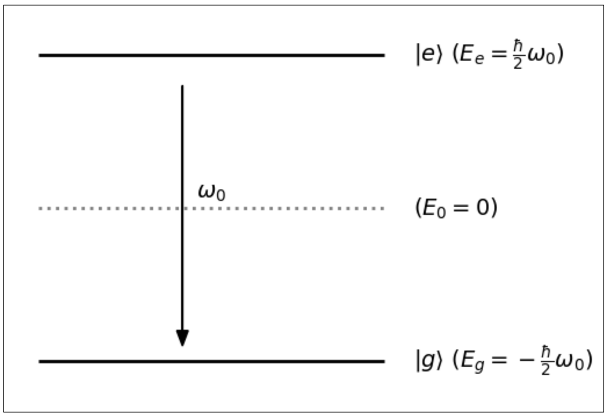

Figure 1:Energy states of a two-level atom.

We consider the zero-energy state E_0 to be midway between the two energy levels (Gerry & Knight, 2004). The energies of ground and excited states simply become $E_g = -\frac{1}{2}\hbar\omega_g$ and $E_e = \frac{1}{2}\hbar\omega_e$ where ω_g and ω_e are frequencies of ground and excited state respectively. Transitions to energy states higher than $|e\rangle$ are forbidden, making the atom a two-level system with only $|g\rangle$ and $|e\rangle$ energy levels [6].

The Hamiltonian of our two-level atom can now be written as:

$$H_{atom} = \frac{1}{2}\hbar\omega_0\sigma_z \quad \ldots(2.1)$$

where σ_z is the atomic inversion operator, also called Pauli-Z operator:

$$\sigma_z|g\rangle = -|g\rangle \qquad \sigma_z|e\rangle = |e\rangle$$

The eigenvalues of H_{atom} for eigenstates $|g\rangle$ and $|e\rangle$ give the energy of respective states:

$$H_{atom}|g\rangle = \frac{1}{2}\hbar\omega_0\sigma_z|g\rangle = -\frac{1}{2}\hbar\omega_0|g\rangle = \mathrm{E_g}|g\rangle$$

$$H_{atom}|e\rangle = \frac{1}{2}\hbar\omega_0\sigma_z|e\rangle = \frac{1}{2}\hbar\omega_0|e\rangle = \mathrm{E_e}|e\rangle$$

2.2. Quantized Radiation Field

The system in our study is subjected to a single-mode radiation field inside an optical cavity. The electromagnetic radiation field has a frequency of ω. The number of photons is denoted as n and the state of our single mode field can be written as $|n\rangle$.

The number of photons in a state are changed by the application of annihilation and creation operator. The photon annihilation operator, $\hat{a}$, decreases the number of photons by one. The photon creation operator, $\hat{a}^\dagger$, is the Hermitian adjoint of annihilation operator and increases the photon number by one [7].

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \ldots(2.2)$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \ldots(2.3)$$

The number operator, $\hat{n} = \hat{a}^\dagger\hat{a}$, gives the number of photons for a given state.

$$\hat{\mathrm{n}}|n\rangle = \hat{a}^\dagger\hat{a}|n\rangle = \mathrm{n}|\mathrm{n}\rangle \quad \ldots(2.4)$$

The Hamiltonian of our field can now be written as:

$$H_{field} = \hbar\omega\hat{a}^\dagger\hat{a} \quad \ldots(2.5)$$

For state $|n\rangle$, the eigenvalue of the H_{field} is:

$$H_{field}|n\rangle = \hbar\omega\hat{a}^\dagger\hat{a}|n\rangle = n\hbar\omega|n\rangle = E_n|n\rangle$$

2.3. Atom-Field Interaction

Our system is a bipartite system made up of a two-level atom and a single mode radiation field. The combined system can be expressed as a product state of field and the atom. Consider that- initially our atom is in the ground state $|g\rangle$ and the field is in state $|n\rangle$. The combined wavefunction of our initial state, $|\psi\rangle$ will be:

$$|\psi\rangle = |n\rangle|g\rangle = |n\rangle \otimes |g\rangle = |n, g\rangle \qquad \ldots(2.6)$$

On interaction with the applied field, the atom experiences transition from ground to excited state if $\omega \approx \omega_0$. On excitation, a photon is annihilated, and the atom jumps to the higher energy state. It returns to the ground by emission (creation) of a photon.

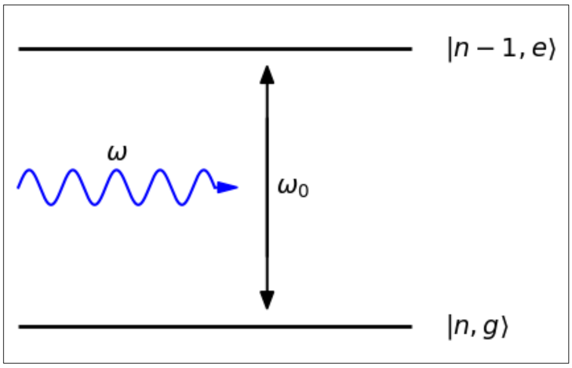

Figure 2: Atom-field interaction.

Mathematically, the interaction is implemented through the atomic transition operators σ_+ and σ_-.

$$\sigma_+ = |e\rangle\langle g| \qquad\qquad \sigma_- = |g\rangle\langle e|$$

Such that,

$$\sigma_+|g\rangle = |e\rangle\langle g|g\rangle = |e\rangle$$

$$\sigma_-|e\rangle = |g\rangle\langle e|e\rangle = |g\rangle$$

The Hamiltonian of atom field interaction is expressed as[7]:

$$H_{int} = \hbar\lambda(\hat{a}\sigma_+ + \hat{a}\sigma_-)$$

Where λ is the atom-field coupling constant [8]

Eigenvalues of H_{int} for state $|n, g\rangle$ and $|n, e\rangle$ will be:

$$H_{int}|n, g\rangle = \hbar\lambda(\hat{a}\sigma_+ + \hat{a}\sigma_-)|n, g\rangle = \hbar\lambda(\hat{a}\sigma_+|n, g\rangle + \hat{a}\sigma_-|n, g\rangle)$$

$$= \hbar\lambda\sqrt{n}|n-1, e\rangle$$

$$H_{int}|n,e\rangle = \hbar\lambda(\hat{a}\sigma_+ + \hat{a}\sigma_-)|n,e\rangle = \hbar\lambda(\hat{a}\sigma_+|n,e\rangle + \hat{a}\sigma_-|n,e\rangle)$$

$$= \hbar\lambda\sqrt{\mathrm{n}+1}|n+1,g\rangle$$

2.4. The Jaynes Cummings Hamiltonian

The Hamiltonian of systems such as ours are efficiently described by the Jaynes Cummings Model. The Jaynes Cummings Hamiltonian has three terms – Hamiltonian of the field, Hamiltonian of the atom and Hamiltonian of the atom field interaction. All these terms have been defined in the previous section.

Putting them together, the Jaynes Cummings Hamiltonian takes the form:

$$H_{JC} = H_{field} + H_{atom} + H_{int}$$

$$H_{JC} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\lambda(\hat{a}\sigma_+ + \hat{a}\sigma_-) \qquad \ldots(2.7)$$

2.5. Entanglement and its Measures

A bipartite system can be explained as a system made up of two constituent subsystems – like the atom and the field in our case. The state of a pure bipartite system can be expressed as [9]:

$$|\Psi_{af}\rangle \in \mathcal{H}_{af} = \mathcal{H}_a \otimes \mathcal{H}_f$$

Technically, entanglement in pure bipartite system is expressed as the inability to write them as a product of two states corresponding to the two subsystems [9].

$$|\Psi_{\mathrm{af}}\rangle = |\psi_{\mathrm{a}}\rangle|\phi_{\mathrm{f}}\rangle$$

In simple terms, it can be said that the state of one subsystem is not independent of another. The states of one subsystem determine the states of the other subsystem. For our system, it can be inferred that the absorption of a photon excites the atom to a higher energy state. Therefore, if the field was initially in the state $|n,g\rangle$, after absorption the state changes to $|n-1,e\rangle$. It can be inferred that the state of field being $|n\rangle$ or $|n-1\rangle$ decides whether the atom has $|g\rangle$ or $|e\rangle$.

This dependence easily reveals the entanglement between the atom and the field in our system.

Quantification of entanglement is done by defining measures of entanglement like Von Neumann Entropy, Negativity, Concurrence, Logarithmic Negativity etc. [10].

In our study, the measure of entanglement opted for is Concurrence. Concurrence was introduced by Wooters, 1998[11]. He gave a simple closed expression for calculating the concurrence in bipartite system can be written down as follows:

$$C(\rho) = max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \qquad \ldots(2.8)$$

Where, $\lambda_i's$ are the square roots of eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$ arranged in decreasing order. Here $\tilde{\rho}$ is derived by applying the spin-flip matrix to ρ^*, the complex conjugate of ρ derived in computational $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ [11]:

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho * (\sigma_2 \otimes \sigma_2)$$

In this study, we will later look at the density matrices with an X structure, where the formula for concurrence can be tweaked a little to make the calculations easier.

2.6. Evolution under Noisy Channels

Before introducing noise channels, I would like to discuss the evolution of a system through any quantum channel. It can be explained using the application of Kraus operators on the density operator of the system.

Evolution of the system, $\varepsilon(\rho)$, is given by:

$$\varepsilon(\rho) = \sum_i E_i \, \rho E_i^\dagger \qquad \ldots(2.9)$$

Where E_i and ${E_i}^\dagger$represent the Kraus operators and their complex conjugates. Kraus operators must satisfy the completeness relation:

$$\sum_i E_i^\dagger \, E_i \; = \; I$$

Where I is the Identity matrix of same dimensions as the Kraus operators. The evolution of the quantum systems through various noise channels calculated by the putting Kraus operators of each noise channel into the Eq. (2.9).

This is the case for a single qubit. For multiple qubits, the Kraus operators are applied on each qubit individually. Therefore, the overall Kraus operators for the system can be constructed as the tensor product of the operators for single qubit. We will investigate this with respect to our system of study in a while.

Noisy channels or Decoherence channels are basically the various sources of noise that affect our quantum system and induce errors. These channels can be classified into the various categories, the ones we are interested in are[12]:

1) *Amplitude Damping Channel* – This channel describes the energy dissipation of a qubit. A qubit in higher energy states loses energy to the environment and relaxes to the lower energy state with spontaneous emission of a photon.
2) *Phase Damping Channel* – This describes the dephasing of qubits without the loss of energy. Generally, it is due to random fluctuations in the environment of the qubits.
3) *Depolarizing Channel* – In this channel, the qubit loses its quantum information and becomes a mixed state with a certain probability. The state of the qubit becomes random.

The overall Kraus operators for each channel can be derived by taking tensor products of the individual Kraus operators of each channel:

$$E_{ij} = E_i^1 \otimes E_j^2 \qquad \ldots(2.10)$$

Where i, j represent the Kraus operators and the superscripts represent the qubits on which the operator is applied. Plugging in these operators into Eq. (2.9) gives the evolution of our system made up of two subsystems.

2.7. Memory Channels and Memoryless Channels

Open quantum systems can be broadly classified into two types – Markovian and non-Markovian Systems. In the case of Markovian systems, evolution depends only on the present state of the system. Previous operations on the state do not affect the future state of the system.

However, in a non-Markovian System, the evolution of the system depends on the history of operations. The effects of the previous operations are retained in the current state of the system, thus affecting the future states of the system. These remnant effects of previous events are termed as memory effects.

Quantum Channels where memory effects are not considered are called memoryless channels. The environmental correlation time in memoryless channels is smaller than the time between successive operations of channel over the qubits. Correlation time is nothing but the time till which the effects of previous operations persist in the system. This infers that memory effects from operation first qubit decay before the operations on the second qubits are performed. Therefore, no memory effects are observed.

Quantum channels which consider the effects of memory channels are called Memory channels. Here the environmental correlation time is larger than the time between successive operations. Therefore, the effects of previous operations remain during the second operation and affect the evolution of the system. [5]

In memory channels, when the same operations are applied to both the qubits with probability μ, the operations are termed as correlated. On the other hand, if operations are different, they are uncorrelated, and probability becomes $(1-\mu)$. Here, the probability μ gives us the memory parameter such that $0 \leq \mu \leq 1$. The evolution under the influence of memory channels is given by modifying Eq. (2.9) [13]; [14]:

$$\varepsilon(\rho) = (1-\mu)\sum_{i,j} E_{i,j}^{u}\,\rho E_{i,j}^{u\dagger} + \mu \sum_{k} E_{k,k}^{c}\,\rho E_{k,k}^{c\dagger} \qquad \ldots(2.11)$$

It is evident that $E_{i,j}^{u}$ describes the contribution from uncorrelated parts with probability $(1-\mu)$ and $E_{k,k}^{c}$ describes the contribution of correlated parts with probability μ. Kraus operators for uncorrelated and correlated parts are calculated separately.

3.1. Methodology

Now, after building a grasp of the necessary concepts, it is time to investigate the effects of memory channels on the systems described by Jaynes Cummings Hamiltonian.

The calculations start with defining the initial state of our system. We determine the time evolution of the system at time t. Further, we derive the density operator. Then, we use the Kraus operator approach to calculate the evolution of our system at $\mu = 0$ which represents memoryless evolution and agrees with the results of Ahadpour & Mirmasoudi, 2020 [8]. Further, for our objective, we calculate the evolution with different values of memory parameter and compare the results.

Throughout the course of our calculations, we will consider $\hbar = 1$ for the ease of our calculations. With this, the Jaynes Cummings Hamiltonian takes the form:

$$H = \omega \hat{a}^{\dagger}\hat{a} + \frac{1}{2}\omega_0 \sigma_z + \lambda(\hat{a}\sigma_+ + \hat{a}\sigma_-) \qquad \dots(3.1)$$

3.2. Initial State of the System

Initially, at time $t = 0$, we consider our state to be the following for the purpose of this study [8]:

$$|\psi(0)\rangle_{af} = \sum\nolimits_{n=0}^{\infty} P_n\,|n,e\rangle \qquad \dots(3.2)$$

The atom is in the excited state $|e\rangle$ with $P_n = exp\left(-\frac{\bar{n}}{2}\right)\sqrt{\frac{\bar{n}^{\mathrm{n}}}{n!}}$ for a coherent state with mean photon number, $\bar{n}$.

3.3. Time Evolution

The time evolution of the state at time t is given by:

$$|\psi(t)\rangle_{af} = U(t)|\psi(0)\rangle_{af} \qquad \dots(3.3)$$

Where $U(t)$ is the unitary time evolution operator such that, $U(t) = e^{-iHt}$. The atom and field Hamiltonian remain roughly constant throughout the interaction. We only consider the effects of interaction in our calculations i.e. only H_{int} is used for the calculation. For the interaction picture in case of resonance i.e. $\omega_0 \approx \omega$, $U(t)$ takes the matrix form [15] (See Appendix A):

$$U(t) = \begin{pmatrix} cos\left(\tau\sqrt{a^\dagger a}\right) & \frac{-ia^\dagger}{\sqrt{a^\dagger a+1}} sin\left(\tau\sqrt{a^\dagger a+1}\right) \\ \frac{-ia}{\sqrt{a^\dagger a}} sin\left(\tau\sqrt{a^\dagger a}\right) & cos\left(\tau\sqrt{a^\dagger a+1}\right) \end{pmatrix} \quad \ldots(3.4)$$

Here, $\tau = t\lambda$ is the dimensionless interaction time.

3.4. Density Operator

For any atom-field system, there are infinite photons in the field. Therefore, our atom-field space has dimensions $2 \times \infty$. We are only interested in a 2×2 system with atomic states $|e\rangle$, $|g\rangle$ and field states $|n\rangle$,$|n+1\rangle$. Therefore, we project this space into a 2×2 subspace. The density operator in this space becomes (See Appendix A):

$$\rho_{af}(t) = \kappa_1|n,g\rangle\langle n,g| + \kappa_2|n,e\rangle\langle n,e| + \kappa_3[|n,e\rangle\langle n+1,g| - |n+1,g\rangle\langle n,e|] + \kappa_4|n+1,g\rangle\langle n+1,g| + \kappa_5\rangle\langle n+1,e|\langle n+1,e| \quad \ldots(3.5)$$

In the matrix form it can be written as:

$$\rho_{af}(t) = \begin{pmatrix} \kappa_1 & 0 & 0 & 0 \\ 0 & \kappa_2 & \kappa_3 & 0 \\ 0 & -\kappa_3 & \kappa_4 & 0 \\ 0 & 0 & 0 & \kappa_5 \end{pmatrix} \quad \ldots(3.6)$$

where,

$$\kappa_1 = \frac{P_{n-1}^2}{N} sin^2\left(\tau\sqrt{n}\right)$$

$$\kappa_2 = \frac{P_n^2}{N} cos^2\left(\tau\sqrt{n+1}\right)$$

$$\kappa_3 = i\frac{P_n^2}{N} cos\left(\tau\sqrt{n+1}\right) sin\left(\tau\sqrt{n+1}\right)$$

$$\kappa_4 = \frac{P_n^2}{N} sin^2\left(\tau\sqrt{n+1}\right)$$

$$\kappa_5 = \frac{P_{n+1}^2}{N} cos^2\left(\tau\sqrt{n+2}\right)$$

And N is the normalization coefficient such that:

$$N = P_{n-1}^2 sin^2\left(\tau\sqrt{n}\right) + P_n^2 + P_{n+1}^2 cos^2\left(\tau\sqrt{n+2}\right)$$

In the next section, we shall use the density operator given in Eq. (3.5) and Eq. (3.6) to derive the evolution of our system under various noise channels.

3.5. Dynamics of Amplitude Damping Channel

For ADC, the Kraus operators for single qubits are as follows[16]:

$$\mathrm{E}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \qquad \mathrm{E}_1 = \begin{pmatrix} 0 & \sqrt{\lambda} \\ 0 & 0 \end{pmatrix}$$

The uncorrelated Kraus operators for two qubits have the form (See Appendix B):

$$E_{ij}^{\mathrm{u}} = E_i \otimes E_j \qquad (i = 0,1)$$

The correlated Kraus operators are[17]:

$$E_{00}^c = \begin{pmatrix} \sqrt{1-\lambda} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad E_{11}^c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\lambda} & 0 & 0 & 0 \end{pmatrix} \qquad \ldots(3.7)$$

For memoryless channel, using Eq. (2.9) we derive the evolved density matrix ρ_{AD} as:

$$\begin{pmatrix} \kappa_1 + \lambda(\kappa_5\lambda + \kappa_2 + \kappa_4) & 0 & 0 & 0 \\ 0 & (1-\lambda)(\kappa_5\lambda + \kappa_2) & \kappa_3(1-\lambda) & 0 \\ 0 & -\kappa_3(1-\lambda) & (1-\lambda)(\kappa_5\lambda + \kappa_4) & 0 \\ 0 & 0 & 0 & \kappa_5(\lambda-1)^2 \end{pmatrix}$$

Using Eq. (2.11) to derive the evolved density matrix with memory, $\rho_{AD}^{(m)}$:

$$\rho_{11} = (1-\lambda\mu)\kappa_1 + \lambda(1-\mu)(\kappa_2 + \kappa_4 + \lambda\kappa_5)$$

$$\rho_{22} = \mu\kappa_2 + (1-\lambda)(1-\mu)(\kappa_2 + \lambda\kappa_5)$$

$$\rho_{33} = \mu\kappa_4 + (1-\lambda)(1-\mu)(\kappa_4 + \lambda\kappa_5)$$

$$\rho_{44} = \lambda\mu\kappa_1 + (1-(2-\lambda)(1-\mu)\lambda)\kappa_5$$

$$\rho_{23} = \rho_{32}* = \big(1-\lambda(1-\mu)\big)\kappa_3$$

$$\rho_{14} = \rho_{41} = 0$$

3.6. Dynamics of Phase Damping Channel

For PDC, the Kraus Operators for single qubits are given using[8]: $E_0 = \sqrt{1-\lambda}I$ $\quad$ $E_1 = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 0 \end{pmatrix}$ $\quad$ $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$

The uncorrelated Kraus operators for two qubits have the form (See Appendix B):

$$E_{ij}^{u} = E_i \otimes E_j \quad (i,j = 0,1,2) \qquad \ldots(3.8)$$

The Kraus operators for the Correlated parts are given by[18]:

$$E_{k,k}^{u} = \sqrt{P_k}\sigma_k \otimes \sigma_k \quad (\, k \in 0,3) \qquad \ldots(3.9)$$

In the above equation, $P_0 = 1-\lambda$ and $P_3 = \lambda$. Also, $\sigma_0 = I$ i.e. identity matrix of dimensions 2×2 and σ_3 is the Pauli z operator.

The evolved density, ρ_{PD}, for memoryless evolution is:

$$\rho_{PD} = \begin{pmatrix} \kappa_1 & 0 & 0 & 0 \\ 0 & \kappa_2 & (1-\lambda)^2\kappa_3 & 0 \\ 0 & -(1-\lambda)^2\kappa_3 & \kappa_4 & 0 \\ 0 & 0 & 0 & \kappa_5 \end{pmatrix}$$

Using Eq. (2.11), the density matrix for PDC with memory, $\rho_{PD}^{(m)}$ becomes:

$$\begin{pmatrix} \kappa_1 & 0 & 0 & 0 \\ 0 & \kappa_2 & \big(1-\lambda(2-\lambda)(1-\mu)\big)\kappa_3 & 0 \\ 0 & -\big(1-\lambda(2-\lambda)(1-\mu)\big)\kappa_3 & \kappa_4 & 0 \\ 0 & 0 & 0 & \kappa_5 \end{pmatrix}$$

3.7. Dynamics of Depolarizing Channel

The single qubit Kraus operators for Depolarizing channel are as given as:

$E_i = \sqrt{P_i}\sigma_i \quad (i = 0,1,2,3)$

$P_1 = 1 - \lambda$ and $P_1 = P_2 = P_3 = \frac{\lambda}{3}$.

The uncorrelated Kraus operators will have the form (See Appendix B):

$$E_{ij}^{u} = \sqrt{P_i P_j}\sigma_i \otimes \sigma_j \quad (i = 0,1,2,3) \qquad \ldots(3.10)$$

The Kraus operators for the Correlated parts are given by[18]:

$$E_{k,k}^{u} = \sqrt{P_k}\sigma_k \otimes \sigma_k \quad (k = 0,1,2,3) \qquad \ldots(3.11)$$

Eq. (2.9), the density operator for memoryless Depolarizing Channel, ρ_{De} can be calculated. Various elements of the ρ_{De} are as follows.

$$\rho_{11} = (1-\lambda)^2\kappa_1 + \frac{\lambda}{3}\left((1-\lambda)(\kappa_2+\kappa_4) + \frac{1}{3}\kappa_5\right)$$

$$\rho_{22} = (1-\lambda)^2\kappa_2 + \lambda\left((1-\lambda)\left(\frac{\kappa_1}{3} + 3\kappa_5\right) + \lambda\kappa_4\right)$$

$$\rho_{33} = (1-\lambda)^2\kappa_4 + \lambda\left((1-\lambda)\left(\frac{\kappa_1}{3} + 3\kappa_5\right) + \lambda\kappa_2\right)$$

$$\rho_{44} = (1-\lambda)^2\kappa_5 + \frac{\lambda}{3}\left((1-\lambda)(\kappa_2+\kappa_4) + \frac{1}{3}\kappa_1\right)$$

$$\rho_{23} = \rho_{32} *= \left(1 - 2\lambda\left(1 - \frac{4}{9}\lambda\right)\right)\kappa_3$$

For Depolarizing channel with memory, $\rho_{De}^{(m)}$ has elements:

$$\rho_{11} = \left((1-\lambda)^2 + \frac{\lambda}{3}(4-3\lambda)\mu\right)\kappa_1 + \frac{\lambda}{3}(1-\lambda)(1-\mu)(\kappa_2+\kappa_4) + \frac{\lambda}{9}(\lambda+6\mu-\lambda\mu)\kappa_5$$

$$\rho_{22} = \kappa_2 + \lambda\left(\frac{1}{3}(1-\lambda)(1-\mu)(\kappa_1+\kappa_5) + \frac{1}{3}(3\lambda+4\mu-3\lambda\mu-6)\kappa_2 + \frac{1}{9}(\lambda+6\mu-\lambda\mu)\kappa_4\right)$$

$$\rho_{33} = \lambda\left(\frac{1}{3}(1-\lambda)(1-\mu)(\kappa_1+\kappa_5) + \frac{1}{9}(\lambda+6\mu-\lambda\mu)\kappa_2\right) + \left((1-\lambda)^2 + \frac{1}{3}(4-3\lambda)\lambda\mu\right)\kappa_4$$

$$\rho_{44} = \left((1-\lambda)^2 + \frac{\lambda}{3}(4-3\lambda)\mu\right)\kappa_5 + \frac{\lambda}{3}(1-\lambda)(1-\mu)(\kappa_2+\kappa_4) + \frac{\lambda}{9}(\lambda+6\mu-\lambda\mu)\kappa_1$$

In the next chapter, we use these definitions to calculate the results of our study.

4.1. Results of AD channel

Based on the definitions of various matrix elements, now we plot different results for each noise channel.

For ADC we have the following plots:

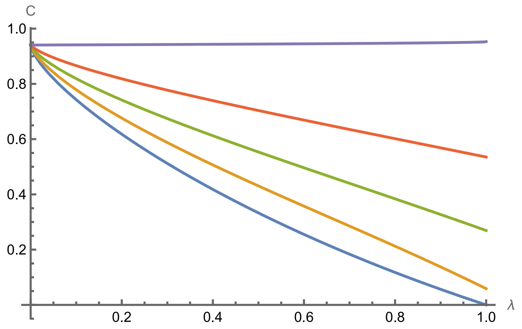

Figure 3: Concurrence vs decoherence parameter for different values of μ for ADC

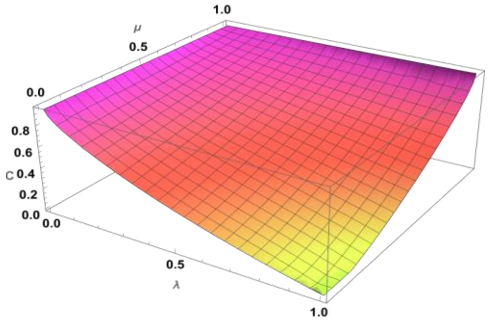

Figure 4: Concurrence vs decoherence parameter, λ and memory parameter, μ for ADC

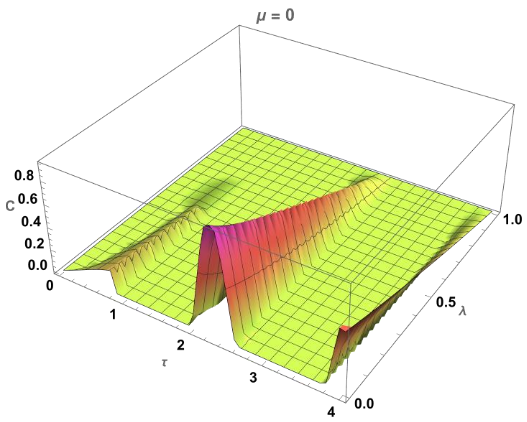

Figure 5.a: $\mu = 0$ (memoryless channel)

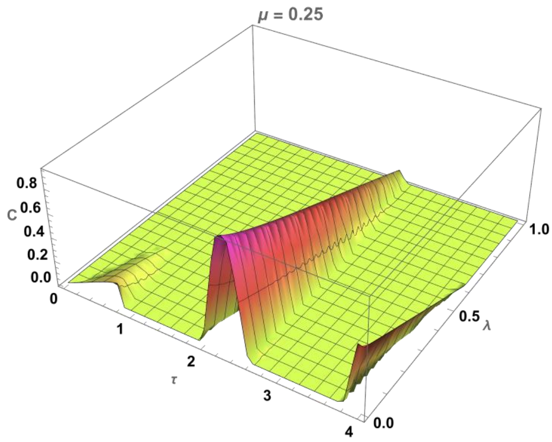

Figure 5.b: $\mu = 0.25$

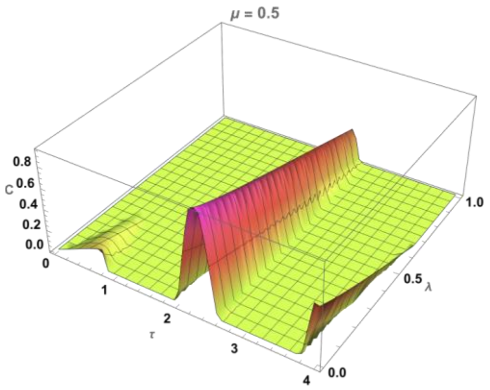

Figure 5.b: $\mu = 0.50$

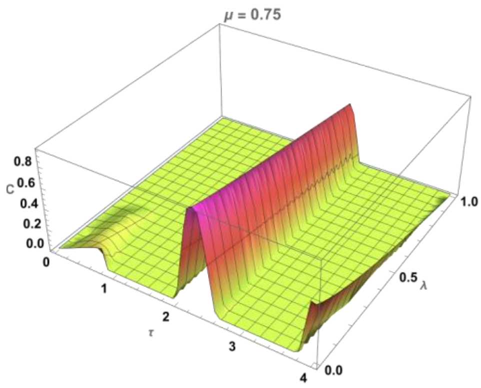

Figure 5.b: $\mu = 0.75$

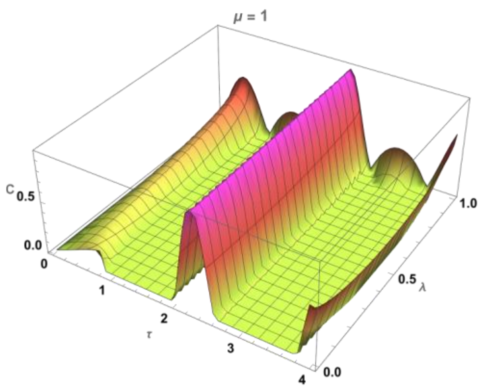

Figure 5.b: $\mu = 1$

Figure 5: Concurrence vs decoherence parameter, λ and time τ for different values of μ for ADC

Above results clearly indicate that the application of memory parameters offers resilience to our system against noise in ADC. Figure 3 shows that with same value of decoherence parameter, application of memory parameters sustain entanglement. Figure 4 shows how different values of memory parameter affect concurrence of our entangled system.

4.2. Results of Phase Damping

For PDC we have the following plots:

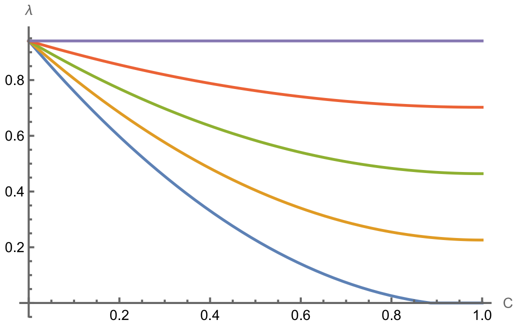

Figure 6: Concurrence vs decoherence parameter for different values of μ for PDC

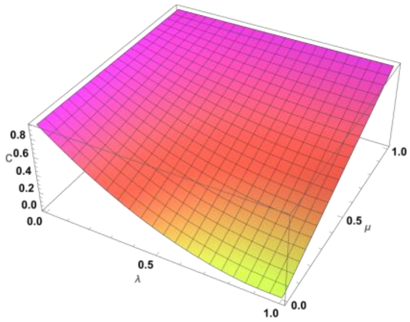

Figure 7: Concurrence vs decoherence parameter, λ and memory parameter, μ for PDC

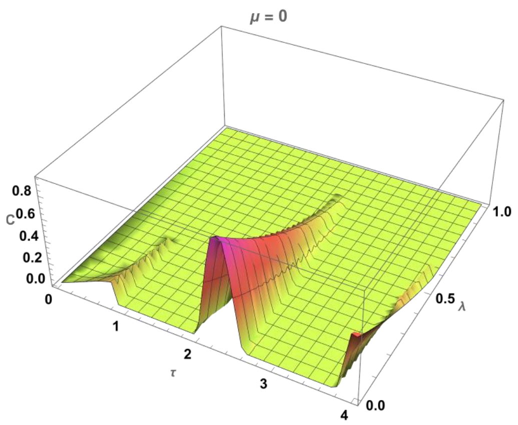

Figure 8.a: $\mu = 0$ (memoryless channel)

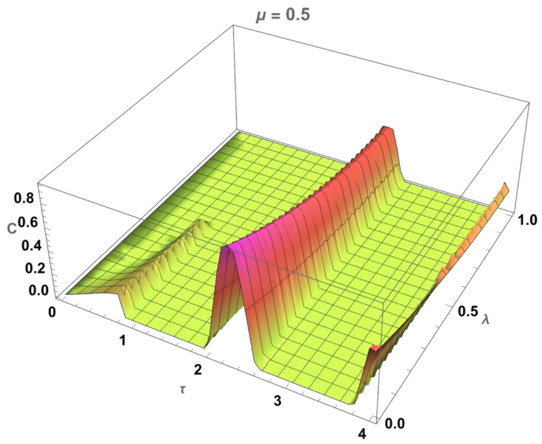

Figure 8.b: $\mu = 0.50$

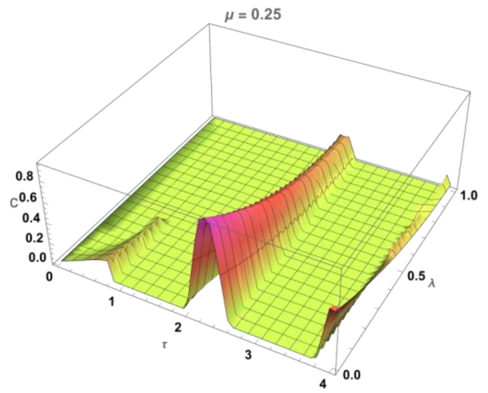

Figure 8.b: $\mu = 0.25$

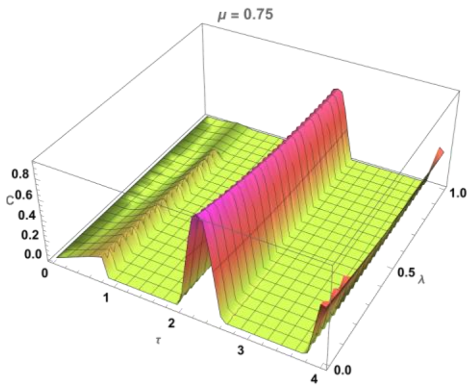

Figure 8.b: $\mu = 0.75$

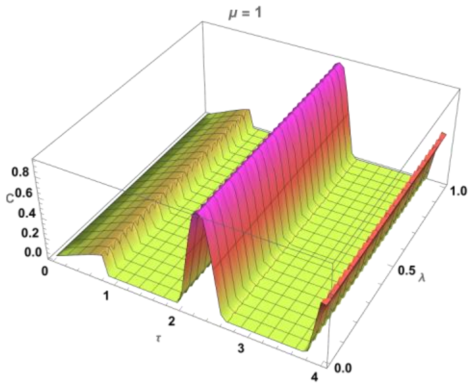

Figure 8.b: $\mu = 1$

Figure 8: Concurrence vs decoherence parameter, λ and time τ for different values of μ for PDC

From figure 7, we can see that the concurrence for same values of decoherence parameter is enhanced by the application of memory. Above results clearly indicate that the application of memory parameters offers resilience to our system against noise in PDC.

4.3. Results of Depolarizing channel

For Depolarizing we have the following plots:

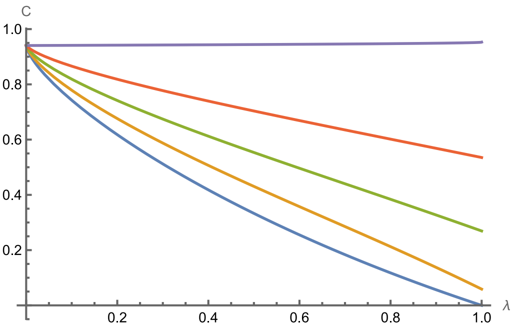

Figure 9: Concurrence vs decoherence parameter for different values of μ for Depolarizing

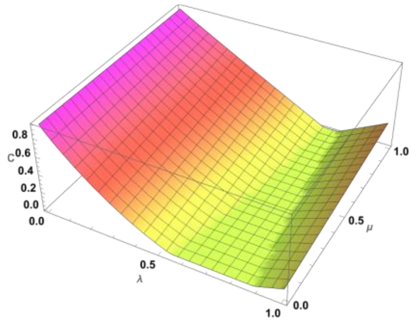

Figure 10: Concurrence vs decoherence parameter, λ and memory parameter, μ for Depolarizing

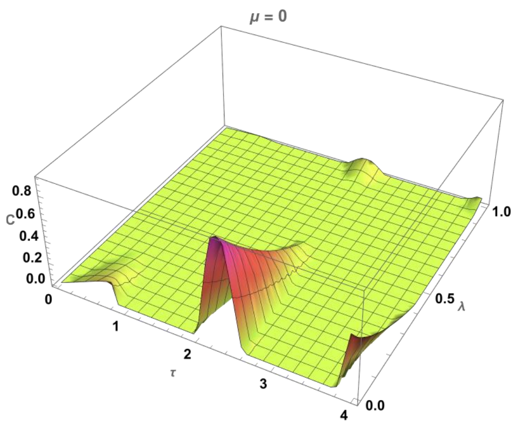

Figure 11.a: $\mu = 0$ (memoryless channel)

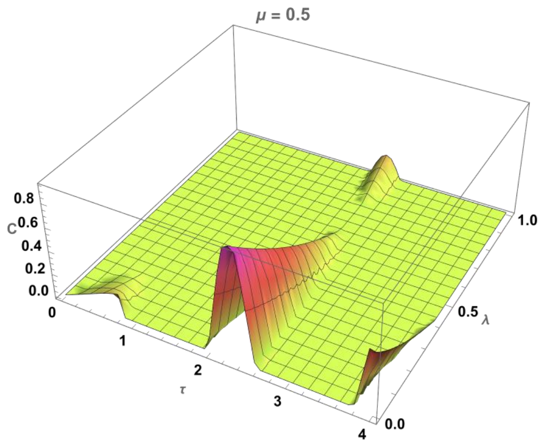

Figure 11.b: $\mu = 0.50$

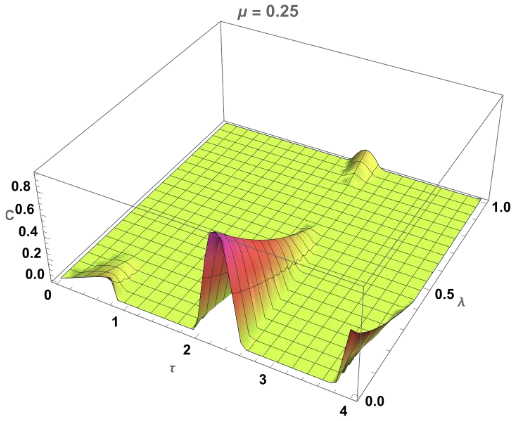

Figure 11.b: $\mu = 0.25$

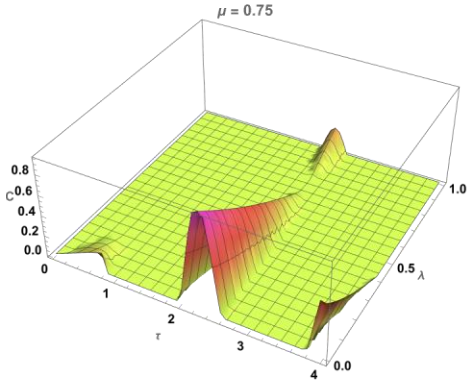

Figure 11.b: $\mu = 0.75$

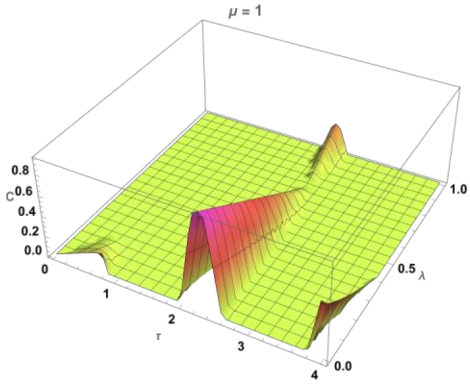

Figure 11.b: $\mu = 1$

Figure 11: Concurrence vs decoherence parameter, λ and time τ for different values of μ

Above results clearly indicate that the application of memory parameters offers resilience to our system against noise in Depolarizing. However, for depolarizing channel the effect is not very significant.

CHAPTER 5– CONCLUSION AND SCOPE FOR FUTURE WORK

5.1. Conclusion

We can conclude from the results of Chapter 4 that for systems described by Jaynes Cummings Model, memory effects play a constructive role in sustaining entanglement. The dynamics of our systems evolution through various noise channels with memory assert that memory channels can be prevent sudden death of entanglement and offer resilience against the effects of noise.

5.2. Future Scope

This work can further be applied to specific quantum systems like quantum dot qubits, Rydberg atom qubits etc.

Other measures of quantum entanglement can also be explored. Work can be done on the effects of memory on Quantum Discord and Super Quantum Discord.

Further, this work can also be extended to qutrit systems.

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Kraus Operators for two qubits:

Using Eq. (2.10), the Kraus operators of ADC for two qubits can be written as following:

$$E_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda} & 0 & 0 \\ 0 & 0 & \sqrt{1-\lambda} & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \qquad E_{11} = \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{10} = \begin{pmatrix} 0 & 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & 0 & \sqrt{(1-\lambda)\lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{01} = \begin{pmatrix} 0 & \sqrt{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{(1-\lambda)\lambda} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, for PDC, the Kraus operators are:

$$E_{00} = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \qquad E_{11} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{01} = \begin{pmatrix} \sqrt{(1-\lambda)\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{(1-\lambda)\lambda} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$E_{02} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{(1-\lambda)\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{(1-\lambda)\lambda} \end{pmatrix} \qquad E_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{20} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{(1-\lambda)\lambda} & 0 \\ 0 & 0 & 0 & \sqrt{(1-\lambda)\lambda} \end{pmatrix} \qquad E_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{10} = \begin{pmatrix} \sqrt{(1-\lambda)\lambda} & 0 & 0 & 0 \\ 0 & \sqrt{(1-\lambda)\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Kraus operators of PF channel for two qubits:

$$E_{00} = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \qquad E_{11} = \begin{pmatrix} 0 & 0 & 0 & -\lambda \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{pmatrix}$$

$$E_{01} = \begin{pmatrix} 0 & -i\sqrt{(1-\lambda)\lambda} & 0 & 0 \\ i\sqrt{(1-\lambda)\lambda} & 0 & 0 & -i\sqrt{(1-\lambda)\lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{(1-\lambda)\lambda} & 0 \end{pmatrix}$$

$$E_{10} = \begin{pmatrix} 0 & 0 & -i\sqrt{(1-\lambda)\lambda} & 0 \\ 0 & 0 & 0 & -i\sqrt{(1-\lambda)\lambda} \\ i\sqrt{(1-\lambda)\lambda} & 0 & 0 & 0 \\ 0 & i\sqrt{(1-\lambda)\lambda} & 0 & 0 \end{pmatrix}$$

Kraus operators of Depolarizing channel for two qubits:

$$E_{00} = \begin{pmatrix} 1-\frac{3\lambda}{4} & 0 & 0 & 0 \\ 0 & 1-\frac{3\lambda}{4} & 0 & 0 \\ 0 & 0 & 1-\frac{3\lambda}{4} & 0 \\ 0 & 0 & 0 & 1-\frac{3\lambda}{4} \end{pmatrix} \qquad E_{11} = \begin{pmatrix} 0 & 0 & 0 & \frac{\lambda}{4} \\ 0 & 0 & \frac{\lambda}{4} & 0 \\ 0 & \frac{\lambda}{4} & 0 & 0 \\ \frac{\lambda}{4} & 0 & 0 & 0 \end{pmatrix}$$

$$E_{22} = \begin{pmatrix} 0 & 0 & 0 & -\frac{\lambda}{4} \\ 0 & 0 & \frac{\lambda}{4} & 0 \\ 0 & \frac{\lambda}{4} & 0 & 0 \\ -\frac{\lambda}{4} & 0 & 0 & 0 \end{pmatrix} \qquad E_{33} = \begin{pmatrix} \frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & -\frac{\lambda}{4} & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{4} & 0 \\ 0 & 0 & 0 & \frac{\lambda}{4} \end{pmatrix}$$

$$E_{12} = \begin{pmatrix} 0 & 0 & 0 & -\frac{i\lambda}{4} \\ 0 & 0 & \frac{i\lambda}{4} & 0 \\ 0 & -\frac{i\lambda}{4} & 0 & 0 \\ \frac{i\lambda}{4} & 0 & 0 & 0 \end{pmatrix} \qquad E_{21} = \begin{pmatrix} 0 & 0 & 0 & -\frac{i\lambda}{4} \\ 0 & 0 & -\frac{i\lambda}{4} & 0 \\ 0 & \frac{i\lambda}{4} & 0 & 0 \\ \frac{i\lambda}{4} & 0 & 0 & 0 \end{pmatrix}$$

$$E_{13} = \begin{pmatrix} 0 & 0 & \frac{\lambda}{4} & 0 \\ 0 & 0 & 0 & -\frac{\lambda}{4} \\ \frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & -\frac{\lambda}{4} & 0 & 0 \end{pmatrix} \qquad E_{23} = \begin{pmatrix} 0 & 0 & -\frac{i\lambda}{4} & 0 \\ 0 & 0 & 0 & \frac{i\lambda}{4} \\ \frac{i\lambda}{4} & 0 & 0 & 0 \\ 0 & -\frac{i\lambda}{4} & 0 & 0 \end{pmatrix}$$

$$E_{31} = \begin{pmatrix} 0 & \frac{\lambda}{4} & 0 & 0 \\ \frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\lambda}{4} \\ 0 & 0 & -\frac{\lambda}{4} & 0 \end{pmatrix} \qquad E_{32} = \begin{pmatrix} 0 & -\frac{i\lambda}{4} & 0 & 0 \\ \frac{i\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i\lambda}{4} \\ 0 & 0 & -\frac{i\lambda}{4} & 0 \end{pmatrix}$$

$$E_{02} = \begin{pmatrix} 0 & -\frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 & 0 \\ \frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} \\ 0 & 0 & \frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 \end{pmatrix}$$

$$E_{20} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} \\ \frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{2} i \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 & 0 \end{pmatrix}$$

$$E_{10} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} \\ \frac{1}{2} \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{2} \sqrt{\left(1 - \frac{3\lambda}{4}\right)\lambda} & 0 & 0 \end{pmatrix}$$

$$E_{03} = \begin{pmatrix} \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} \end{pmatrix}$$

$$E_{01} = \begin{pmatrix} 0 & \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 & 0 \\ \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} \\ 0 & 0 & \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 \end{pmatrix}$$

$$E_{30} = \begin{pmatrix} \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\left(1-\frac{3\lambda}{4}\right)\lambda} \end{pmatrix}$$