Magnetic dipole moment of a moving electric dipole

Group-2
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Abstract

This paper aims to resolve the inconsistency between different transformation equations by expressing the electric current created by a moving electric dipole as the sum of polarization and magnetization currents and calculating the resulting magnetic field. The analysis clarifies that the correct transformation involves both the magnetic dipole moment and the magnetic field created by the polarization current. The implications of these findings are discussed, highlighting the importance of considering both components in the description of a moving electric dipole. We also aim to highlight the difference in assumptions involved compared to other papers/sources which have worked/highlighted this problem (*G.M. Fisher*, *J.D. Jackson, 3rd Ed.*).

Introduction

The relativistic transformations of the polarization (electric moment density) **P** and magnetization (magnetic moment density) **M** of macroscopic electrodynamics[1] imply corresponding transformations of the electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$, respectively, of a particle. Thus, to first order in v/c,

$$\mathbf{p} = \mathbf{p}_0 + \frac{1}{c}\mathbf{v} \times \mathbf{m}_0, \tag{1}$$

$$\mathbf{m} = \mathbf{m}_0 - \frac{1}{c}\mathbf{v} \times \mathbf{p}_0. \tag{2}$$

Here, the subscript 0 denotes quantities in the particle's rest frame and $\mathbf{v}$ is the particle's velocity. According to Eq. (1), a moving rest-frame magnetic dipole $\mathbf{m}_0$ develops an electric dipole moment $\mathbf{p} = \mathbf{v} \times \mathbf{m}_0/c$. While this fact is well known and understood[2], the complementary effect that a moving electric dipole acquires a magnetic moment does not seem to be understood equally well. There does not appear to be a consensus in the literature as to the transformation (2). For example, Barut[3] and Vekstein[4] agree to first order in v/c with Eq. (2), but according to Fisher

$$\mathbf{m} = \mathbf{m}_0 - \frac{1}{2c}\mathbf{v} \times \mathbf{p}_0, \tag{3}$$

implying that a moving rest-frame electric dipole $\mathbf{p}_0$ acquires a magnetic dipole moment $\mathbf{m} = -\mathbf{v} \times \mathbf{p}_0/2c$, which differs by a factor of 1/2 from that of Eq. (2). The tasks of some problems involving a moving electric dipole in the authoritative text of Jackson[5] seem at first sight to be consistent with Fisher's transformation (3).

Here, we would like to introduce a bit of the background[6] leading up to the above-mentioned expressions (involves **covariant electrodynamics**, which is a different topic altogether, but let's not die on that hill today!).

The electromagnetic tensor, conventionally labelled as $\mathbf{F}$, is defined as the exterior derivative of the electromagnetic four-potential $\mathbf{A}$. Now, E and B are 2 different kinds of vectors. $\mathbf{E}$ is a polar vector , but $\mathbf{B}$ is an axial vector.

$\nabla \times \mathbf{A} = \mathbf{B}$ can be written as

$$\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = B_{\alpha\beta}, (\alpha \text{ and } \beta \text{ are } 1, 2, 3)$$

$\mathbf{E}$ can be derived from potentials by

$$E_\alpha = -\frac{\partial \phi}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial t}, (\alpha = 1, 2, 3)$$

We thus introduce a 4D anti-symmetric field tensor F_{ij} , which, as a function of the covariant four-vector potential

$$\phi_i = (-c\mathbf{A}, \phi)$$

is,

$$F_{ij} = \frac{\partial \phi_j}{\partial x^i} - \frac{\partial \phi_i}{\partial x^j}$$

The tensor is :

$$F_{ij} = \begin{bmatrix} 0 & -cB_z & cB_y & -E_x \\ cB_z & 0 & -cB_x & -E_y \\ -cB_y & cB_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{bmatrix}$$

Since it is possible to write the fields covariantly in terms of the potential, it should be possible to write Maxwell's equations themselves in a covariant form. We can verify the source equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0(\rho \mathbf{u} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

correspond to

$$\frac{\partial F^{ji}}{\partial x^j} = \frac{j^i}{\epsilon_0}$$

which is just the covariant divergence of the field tensor. The other field equations :

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

correspond to

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0$$

The tensor expression for the fields immediately permits a derivation of the Lorentz transformation of the fields. Using

$$Q^i_j = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

we can write

$$F'^{ij} = Q^i_k Q^j_l F^{kl}$$

and thence we can easily derive the relations

$$E'_{\parallel} = E_{\parallel}$$

$$B'_{\parallel} = B_{\parallel}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp})$$

$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{\mathbf{v} \times \mathbf{E}_{\perp}}{c^2})$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c$

Now, for material media, we can write the following equations

$$\frac{\partial F^{il}}{\partial x^i} = \frac{j^l + j^l_M}{\epsilon_0},$$

$$j^l_m = \frac{\partial M^{il}}{\partial x^i}$$

$$j^l_m = (\frac{j_M + j_p}{C}, \rho_P) = (\frac{\nabla \times \mathbf{M}}{c} + \frac{1}{c}\frac{\partial \mathbf{P}}{\partial t}, -\nabla \cdot \mathbf{P})$$

The components of M^{ij} are then given by

$$(M^{ij}) = \begin{bmatrix} 0 & -M_z/c & M_y/c & -P_x \\ M_z/c & 0 & -M_x/c & -P_y \\ -M_y/c & M_x/c & 0 & -P_z \\ P_x & P_y & P_z & 0 \end{bmatrix}$$

$$\frac{\mathbf{H}}{c\epsilon_0} = c\mathbf{B} - \frac{\mathbf{M}}{c\epsilon_0}$$

$$\frac{\mathbf{H}}{\epsilon_0} = \mathbf{E} + \frac{\mathbf{P}}{\epsilon_0}$$

We can introduce a new field H^{ij} defined by

$$H^{ij}/\epsilon_0 = F^{ij} - M^{ij}/\epsilon_0 \tag{18-65}$$

The source equations then become simply

$$\frac{\partial H^{ij}}{\partial x^i} = j^j, \tag{18-66}$$

where,

$$(H^{ij}) = \begin{bmatrix} 0 & -H_z/c & H_y/c & D_x \\ H_z/c & 0 & -H_x/c & D_y \\ -H_y/c & H_x/c & 0 & D_z \\ -D_x & -D_y & -D_z & 0 \end{bmatrix}$$

The signs in the defining equations of the auxiliary fields correspond to the way in which the equivalent currents and charges are derived from the moments.

These are the covariant Maxwell equations related bythe previously stated equation, which show the connection between the fields H^{ij} and F^{ij} and the moments M^{ij}.

The transformation properties of the moments follow directly from the covariant formulation. We obtain for the transformation from plain to primed system moving with relative velocity v,

$$P'_{\parallel} = P_{\parallel}$$
$$M'_{\parallel} = M_{\parallel}$$
$$\mathbf{P}'_{\perp} = \gamma(\mathbf{P}_{\perp} - \frac{\mathbf{v} \times \mathbf{M}_{\perp}}{c^2})$$
$$\mathbf{M}'_{\perp} = \gamma(\mathbf{M}_{\perp} + \mathbf{v} \times \mathbf{P}_{\perp})$$

Since we are dealing with the non-relativistic assumption, we'd have $\gamma to 1$. Thus, with this assumption, and keeping in mind that the equations are in Gaussian units, we can find the moments in the moving frame with respect to the rest frame, which turn out to be similar to the ones we'd try to prove now.

The aim of this note is to clear up the inconsistency of the differing transformations (2) and (3). To this end, we shall first express the electric current created by a moving electric dipole as the sum of polarization and magnetization currents, and then calculate the magnetic field of the moving dipole as the sum of the magnetic fields due to these currents.

Magnetic dipole moment due to a moving electric dipole

Here, we assume that we have a rest-frame electric dipole $\mathbf{p}_0$ (let's say that the rest frame in this case is $\boldsymbol{K}$, located at $\mathbf{r} = \mathbf{r}_0(t)$. Again, it's important to note that we deal with the problem under the non-relativistic assumption (i.e. $\gamma \to 1$). Also,for simplicity, we have assumed the dipole to be a point-like particle, so it produces a polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{p}_0 \delta\left(\mathbf{r} - \mathbf{r}_0(t)\right) \tag{4}$$

Here, an important thing to note is that the polarization $P(\mathbf{r}, t)$ is dependent on time, which essentially means that the charge distribution (or the charge density, in particular) is dependent on time. This is the point of discord that leads to a marked difference in the final results. We can make a hand-wavy argument as to why the charge distribution should explicitly depend on time. Since the dipole is moving, the charge distribution(i.e. bound charges due to the dipole) is not constant in space with respect to time, so the charge distribution should be dependent on time.

The other subtle piece of information to keep in mind is that $\mathbf{r}$ is the point in space where we want to make an observation and is **not** dependent on time. The initial position of the dipole $\mathbf{r_0}(t)$ on the other hand, is dependent on time (as is evident from its definition) as the dipole moves around in space.

The net charge on the dipole vanishes, but it is a carrier of a bound charge distribution

$$\begin{aligned}\rho_b(\mathbf{r}, t) &= -\nabla \cdot \mathbf{P}(\mathbf{r}, t) \\ &= -\nabla \cdot \left(\mathbf{p}_0 \delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)\right)\end{aligned}$$

We need to use the following identity -

$$\nabla \cdot (\psi \mathbf{A}) = \psi(\nabla \cdot \mathbf{A}) + (\nabla \psi) \cdot \mathbf{A}$$

where ψ is a scalar function. Thus, we get

$$\begin{aligned}\rho_b(\mathbf{r}, t) &= \delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)\left(\nabla \cdot \mathbf{p_0}\right) + \left(\nabla \delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)\right) \cdot \mathbf{p_0} \\ &= -\mathbf{p}_0 \cdot \nabla \delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)\end{aligned} \tag{5}$$

The first term *vanishes* as the divergence of the dipole moment is zero. This is because $\mathbf{p_0}$ is independent of r.

Also, the master equation holds true since

$$-Q_b = \oiint_{S(V)} \mathbf{P}.d\mathbf{A}$$

If we now use the **divergence theorem** on this known equation, we can easily get the master equation.

Since the dipole is moving, it has a velocity $\mathbf{v} = d\mathbf{r}_0(t)/dt$. Thus, it creates a current density

$$\begin{aligned}\mathbf{J}(\mathbf{r}, t) &= \mathbf{v}\rho_b(\mathbf{r}, t) \\ &= -\mathbf{v}\left(\mathbf{p}_0 \cdot \nabla\right)\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right) \qquad (6)\end{aligned}$$

This is true as the **current density** is defined in this way. The same current density can obviously be obtained by modeling the moving electric dipole as two equal and opposite point charges, separated by an infinitesimal displacement and moving with the same velocity $\mathbf{v}$. If we consider two point charges q and $-q$ at $(\mathbf{r_0}(t) - \mathbf{k}$ and $\mathbf{r_0}(t) + \mathbf{k}$ respectively, then we can obtain the polarization as-

$$\mathbf{P}(\mathbf{r}, t) = q\delta\left(\mathbf{r} - \mathbf{r}_0(t) - \mathbf{k}\right) - q\delta\left(\mathbf{r} - \mathbf{r}_0(t) + \mathbf{k}\right)$$

If we now try to do a **Taylor expansion** in a small sphere around $\mathbf{r_0}(t)$ (due to the *delta function*), then $\mathbf{k}$ comes out as $\mathbf{k} \to 0$, and we can club $q\mathbf{k}$ together to get p_0, thus getting back to the same expression for $\mathbf{P}(\mathbf{r}, t)$

On the other hand, the time-dependent polarization (4) produces a polarization current density $\mathbf{J}_p(\mathbf{r}, t)$ according to the definition

$$\begin{aligned}\mathbf{J}_p(\mathbf{r}, t) &= \frac{\partial \mathbf{P}(\mathbf{r}, t)}{\partial t} \\ &= -\mathbf{p}_0(\mathbf{v} \cdot \nabla)\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right) \qquad (7)\end{aligned}$$

We can prove this using the standard definition using apt assumptions. We know that

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{dt}} = lim_{\Delta t \to 0}\frac{\Delta \mathbf{P}}{\Delta t}$$

Thus, for a very short interval Δt, we can write

$$\Delta \mathbf{P} = \nabla \mathbf{P} \cdot (\mathbf{v}\Delta t)$$

Thus, we can arrive to the final form on acknowledging the fact that the direction of gradient of $\mathbf{P}(\mathbf{r}, t)$ is opposite to that of the gradient of $\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)$

(since polarization is due to the dipole, whose direction is opposite to that of $(\mathbf{r} - \mathbf{r_0}(t))$.

Here, we also assume that the dipole moment $\mathbf{p}_0$ itself does not depend on time (we'd be dealing with it, don't worry!). Now, if the moving electric dipole develops a magnetic dipole moment $\mathbf{m}$, then in addition to the polarization (4), there is also a magnetization

$$\mathbf{M}(\mathbf{r}, t) = \mathbf{m}\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right) \tag{8}$$

which stands true to its definition. This will now produce a magnetization current $\mathbf{J}_m(\mathbf{r}, t)$ according to

$$\begin{aligned}\mathbf{J}_m(\mathbf{r}, t) &= c\nabla \times \mathbf{M} \\ &= c\nabla \times \left[\mathbf{m}\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)\right]\end{aligned} \tag{9}$$

Let us now surmise that the magnetic dipole moment $\mathbf{m}$ generated by the motion of the electric dipole is given by

$$\mathbf{m} = -\frac{1}{c}\mathbf{v} \times \mathbf{p}_0 \tag{10}$$

(which we do want to show, eventually). Thus, the sum of the bound currents (7) and (9) equals the "convection" current (6). Indeed, we then have

$$\begin{aligned}\mathbf{J}_p(\mathbf{r}, t) + \mathbf{J}_m(\mathbf{r}, t) = &- \mathbf{p}_0(\mathbf{v} \cdot \nabla)\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right) \\ &- \nabla \times \left[\left(\mathbf{v} \times \mathbf{p}_0\right)\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)\right]\end{aligned}$$

Now, we can write the second term using the identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\nabla\mathbf{B}) - \mathbf{B} \cdot (\nabla\mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

Also, we need to use the identity

$$\nabla \cdot (\psi\mathbf{A}) = \psi(\nabla \cdot \mathbf{A}) + (\nabla\psi) \cdot \mathbf{A}$$

Thus, we get

$$\begin{aligned}\mathbf{J}_p(\mathbf{r},t)+\mathbf{J}_m(\mathbf{r},t) = &-\mathbf{v}\delta\left(\mathbf{r}-\mathbf{r}_0(t)\right)(\nabla\cdot\mathbf{p_0})-\mathbf{v}(\mathbf{p_0}\cdot\nabla)\delta\left(\mathbf{r}-\mathbf{r}_0(t)\right)\\ &+\delta\left(\mathbf{r}-\mathbf{r_0}(t)\right)\mathbf{p_0}(\nabla\cdot\mathbf{v})-\delta\left(\mathbf{r}-\mathbf{r}_0(t)\right)\mathbf{p_0}\cdot(\nabla\mathbf{v})\\ &+(\mathbf{v}\cdot\nabla)(\mathbf{p_0}\delta\left(\mathbf{r}-\mathbf{r}_0(t)\right))-\mathbf{p}_0(\mathbf{v}\cdot\nabla)\delta\left(\mathbf{r}-\mathbf{r}_0(t)\right)\end{aligned}$$

Now, $\mathbf{v}$ doesn't depend on r, it's magnitude remains the same in space. Thus, $\nabla\cdot\mathbf{v}=0$ and $\nabla\mathbf{v}=0$. Also, $\mathbf{p_0}$ is independent of spatial coordinates, so $\nabla\cdot\mathbf{p_0}=0$. Also, the penultimate term in the expression can be re-written as $\mathbf{p_0}(\mathbf{v}\cdot\nabla)\delta\left(\mathbf{r}-\mathbf{r}_0(t)\right)$. Thus, we do get

$$\begin{aligned}\mathbf{J}_p(\mathbf{r},t)+\mathbf{J}_m(\mathbf{r},t) &= -\mathbf{v}\left(\mathbf{p}_0\cdot\nabla\right)\delta\left(\mathbf{r}-\mathbf{r}_0(t)\right)\\ &= \mathbf{J}(\mathbf{r},t)\end{aligned}\tag{11}$$

Now, if we try to look it from a different perspective compared to the paper's approach, we can separate the rest-frame and lab-frame and try to prove the same using this formulation. In its rest-frame the point electric dipole $\mathbf{p}_0$ (located at the origin) has polarization density,

$$\mathbf{P}^\star=\mathbf{p}_0\delta^3\mathbf{r}^\star.$$

The associated charge density is,

$$\rho^\star=-\nabla^\star\cdot\mathbf{P}^\star=-\mathbf{p}_0\cdot\nabla^\star\delta^3\mathbf{r}^\star.$$

In the lab frame the charge density is,

$$\rho=\gamma\rho^\star\approx\rho^\star\approx-\mathbf{p}_0\cdot\boldsymbol{\nabla}\delta^3\mathbf{r}=-\boldsymbol{\nabla}\cdot\mathbf{P},$$

where the lab-frame polarization density is,

$$\mathbf{P}=\mathbf{p}_0\delta^3\mathbf{r},$$

at the instant when the dipole is at the origin, noting that $\mathbf{r}\approx\mathbf{r}^\star$ and $\boldsymbol{\nabla}^\star=\boldsymbol{\nabla}$ in the low-velocity approximation. [10] When the dipole is moving with a velocity $\mathbf{v}$ in the lab frame it has current density,

$$\begin{aligned}\mathbf{J} &= \gamma\rho^\star\mathbf{v}\approx\rho^\star\mathbf{v}=-\mathbf{v}\left(\mathbf{p}_0\cdot\boldsymbol{\nabla}\right)\delta^3\mathbf{r}\\ &= -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\delta^3(\mathbf{r})-\mathbf{v}\left(\boldsymbol{\nabla}\cdot\mathbf{p}_0\delta^3\mathbf{r}\right)+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\delta^3\mathbf{r}\\ &= -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\delta^3\mathbf{r}-\boldsymbol{\nabla}\times\left(\mathbf{v}\times\mathbf{p}_0\delta^3\mathbf{r}\right)\\ &\equiv \mathbf{J}_p+\mathbf{J}_m,\end{aligned}$$

noting that the operator $\boldsymbol{\nabla}$ does not act on the constant vectors $\mathbf{p}_0$ and $\mathbf{v}$. The current density $\mathbf{J}_p$ can be thought of as an "electric-polarization current",

$$\mathbf{J}_p = -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\delta^3\mathbf{r} \equiv -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{P} = \frac{d\mathbf{P}}{dt}.$$

Likewise, the current density $\mathbf{J}_m$ can be thought of as a "magnetic-polarization current",

$$\mathbf{J}_m = -\boldsymbol{\nabla}\times\left(\mathbf{v}\times\mathbf{p}_0\delta^3\mathbf{r}\right) = c\boldsymbol{\nabla}\times\mathbf{m}\delta^3\mathbf{r} \equiv c\boldsymbol{\nabla}\times\mathbf{M},$$

using the convention that we had adopted, $\mathbf{m} = -\mathbf{v}/c\times\mathbf{p}_0$ and

$$\mathbf{M} = \mathbf{m}\delta^3\mathbf{r}.$$

The circulating magnetization current (since it involves a **curl**) due to the magnetic moment (10) is modified by the currents arising from the *polarization current density* (7) so that the resulting net current is directed along the dipole's velocity $\mathbf{v}$.

In the non-relativistic (quasi-static) limit, the vector potential due to the polarization current (7) is given by

$$\begin{aligned}
\mathbf{A_p} &= \frac{1}{c}\int d^3r'\frac{\mathbf{J}_p\left(\mathbf{r}',t\right)}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\mathbf{p}_0}{c}\int d^3r'\frac{\left(\mathbf{v}\cdot\nabla'\right)\delta\left(\mathbf{r}'-\mathbf{r}_0(t)\right)}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\mathbf{p}_0}{c}[\int_{-\infty}^{\infty}\frac{\mathbf{v}}{|\mathbf{r}-\mathbf{r}'|}\cdot\nabla'\delta\left(\mathbf{r}'-\mathbf{r_0}(t)\right)d^3r' \\
&- \int_{-\infty}^{\infty}\nabla'\left(\frac{\mathbf{v}}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\left(\int_{-\infty}^{\infty}\nabla'\delta\left(\mathbf{r}'-\mathbf{r_0}(t)\right)d^3r'\right)] \\
&= \frac{1}{c}\frac{\mathbf{v}\cdot\left(\mathbf{r}-\mathbf{r}_0(t)\right)}{|\mathbf{r}-\mathbf{r}_0(t)|^3}\mathbf{p}_0
\end{aligned} \tag{12}$$

In Line 3, we use **integration by parts** so that the second term vanishes as the delta function goes to 0 at the limits. If we take the gradient in the second term, we get $\frac{\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$ and we set $\mathbf{r}'$ to $\mathbf{r_0}(t)$ as we integrate the delta function over the whole range and the value lies in the range.

The vector potential of the magnetization current (9) that is due to the magnetic moment (10) can be evaluated in a similar fashion -

$$
\begin{aligned}
\mathbf{A}_m(\mathbf{r},t) &= \frac{1}{c}\int d^3r' \frac{\mathbf{J}_m\left(\mathbf{r}',t\right)}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{1}{c}\int d^3r' \frac{\nabla' \times \left[(\mathbf{v}\times\mathbf{p}_0)\,\delta\left(\mathbf{r}'-\mathbf{r}_0(t)\right)\right]}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{1}{c}\frac{(\mathbf{v}\times\mathbf{p}_0)\times(\mathbf{r}-\mathbf{r}_0(t))}{|\mathbf{r}-\mathbf{r}_0(t)|^3}
\end{aligned} \tag{13}
$$

The magnetic field $\mathbf{B}$ of the moving electric dipole can be represented as the sum

$$
\mathbf{B} = \mathbf{B}_m + \mathbf{B}_p, \tag{14}
$$

where

$$
\begin{aligned}
\mathbf{B}_m &= \nabla \times \mathbf{A}_m(\mathbf{r},t) \\
&= -\frac{1}{c}\frac{3\left[\mathbf{n}\cdot(\mathbf{v}\times\mathbf{p}_0)\right]\mathbf{n} - \mathbf{v}\times\mathbf{p}_0}{|\mathbf{r}-\mathbf{r}_0(t)|^3}
\end{aligned} \tag{15}
$$

which is the magnetic field due to the vector potential (13), which is also the magnetic field of the magnetic moment (10), and

$$
\begin{aligned}
\mathbf{B}_p &= \nabla \times \mathbf{A}_p(\mathbf{r},t) \\
&= \frac{1}{c}\nabla \times \left(\frac{\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}_0(t))}{|\mathbf{r}-\mathbf{r}_0(t)|^3}\right)\mathbf{p}_0 \\
&= \frac{1}{c}\left[\frac{\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}_0(t))}{|\mathbf{r}-\mathbf{r}_0(t)|^3}\left(\nabla\times\mathbf{p}_0\right) + \nabla\left(\frac{\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}_0(t))}{|\mathbf{r}-\mathbf{r}_0(t)|^3}\right)\times\mathbf{p}_0\right] \\
&= -\frac{\mathbf{p}_0}{c}\times\frac{-\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}_0(t))\,3\,|\mathbf{r}-\mathbf{r}_0(t)|^2\,\mathbf{n} + |\mathbf{r}-\mathbf{r}_0(t)|^3\left[\mathbf{v}+0+0+0\right]}{|\mathbf{r}-\mathbf{r}_0(t)|^6} \\
&= -\frac{\mathbf{p}_0}{c}\times\frac{-3\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}_0(t))\,\mathbf{n} + |\mathbf{r}-\mathbf{r}_0(t)|\,\mathbf{v}}{|\mathbf{r}-\mathbf{r}_0(t)|^4} \\
&= \frac{\mathbf{p}_0}{c}\times\frac{3(\mathbf{v}\cdot\mathbf{n})\mathbf{n}-\mathbf{v}}{|\mathbf{r}-\mathbf{r}_0(t)|^3}
\end{aligned} \tag{16}
$$

where $\mathbf{n} = \frac{\mathbf{r}-\mathbf{r}_0(t)}{|\mathbf{r}-r_0(t)|}$

is the magnetic field due to the vector potential (12), created by the polarization current (7). Here,

$$\mathbf{n} = \frac{\mathbf{r} - \mathbf{r}_0(t)}{|\mathbf{r} - \mathbf{r}_0(t)|} \tag{17}$$

which is the unit vector along the direction from the dipole's location $\mathbf{r}_0$ to the field point $\mathbf{r}$. The sum of the expressions (15) and (16) reduces to

$$\mathbf{B} = \frac{1}{c}\mathbf{v} \times \mathbf{E} \tag{18}$$

where

$$\mathbf{E} = \frac{3\left(\mathbf{n} \cdot \mathbf{p}_0\right)\mathbf{n} - \mathbf{p}_0}{|\mathbf{r} - \mathbf{r}_0(t)|^3} \tag{19}$$

which is the well-known electric field of the moving dipole in the non-relativistic limit (for a point far from the location of the dipole). We note that the magnetic field (18) is the same as that obtained by Lorentz transforming to first order in v/c the dipole's rest-frame electromagnetic field to the laboratory frame.

The dipole transformation (2), implying that a moving rest-frame electric dipole $\mathbf{p}_0$ acquires a magnetic dipole moment $\mathbf{m} = -\mathbf{v} \times \mathbf{p}_0/c$, is thus correct; however, the magnetic field due to the dipole moment $\mathbf{m}$ is not the entire magnetic field of the moving electric dipole, which also includes the magnetic field created by the polarization current. [12] The magnetic dipole moment $\mathbf{m}/2 = -(1/2c)\mathbf{v} \times \mathbf{p}_0$ arises in a different decomposition of the moving dipole's magnetic field,

$$\mathbf{B} = \mathbf{B}_{m/2} - \frac{3}{2c}\mathbf{n} \times \frac{(\mathbf{v} \cdot \mathbf{n})\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{n})\,\mathbf{v}}{|\mathbf{r} - \mathbf{r}_0(t)|^3}, \tag{20}$$

where the first term is the magnetic field due to the dipole moment $\mathbf{m}/2$ and the second term, which is symmetric in $\mathbf{p}_0$ and $\mathbf{v}$, has a curl that is proportional to the displacement current of the electric quadrupole field that is created when the electric dipole's location is off the origin (i.e., $\mathbf{r}_0 \neq 0$).

Here, if we deal with Jackson's argument, the multiple moments of charge and charge densities depend on the choice of origin, but not the charge. Problems 6.21 and 6.22 deal with an *off-center* dipole. In this case, the vector potential $\mathbf{A}$ that we can obtain from the Lorenz transformations of electric and magnetic fields is of the form -

$$\mathbf{A} = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{v}}{cr^3} = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{v} - (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3} + \frac{(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3} = \mathbf{A}_a + \mathbf{A}_s$$

where,

$$\mathbf{A}_a = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{v} - (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3} = \frac{1}{2}\mathbf{A}_m, \quad \mathbf{A}_s = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3}$$

That is, Jackson argues that we can decompose the magnetic field in terms of a symmetric and an anti-symmetric component, as long as the sum of them remains the same. Here, we can evaluate the symmetric part as

$$\begin{aligned}
\mathbf{B}_s &= \boldsymbol{\nabla} \times \mathbf{A}_s = \boldsymbol{\nabla} \times \frac{(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{p}_0}{2cr^3} \\
&= (\mathbf{p}_0 \cdot \mathbf{r})\,\boldsymbol{\nabla} \times \frac{\mathbf{v}}{2cr^3} - \frac{\mathbf{v}}{2cr^3} \times \boldsymbol{\nabla}\,(\mathbf{p}_0 \cdot \mathbf{r}) + (\mathbf{v} \cdot \mathbf{r})\boldsymbol{\nabla} \times \frac{\mathbf{p}_0}{2cr^3} - \frac{\mathbf{p}_0}{2cr^3} \times \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{r}) \\
&= -\frac{3\,(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{r} \times \mathbf{v}}{2cr^5} - \frac{\mathbf{v} \times \mathbf{p}_0}{2cr^3} - \frac{3(\mathbf{v} \cdot \mathbf{r})\mathbf{r} \times \mathbf{p}_0}{2cr^5} - \frac{\mathbf{p}_0 \times \mathbf{v}}{2r^3} \\
&= \mathbf{v} \times \frac{3\,(\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{r}}{2cr^5} + \mathbf{p}_0 \times \frac{3(\mathbf{v} \cdot \mathbf{r})\mathbf{r}}{2cr^5} = -3\mathbf{r} \times \frac{(\mathbf{v} \cdot \mathbf{r})\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{r})\,\mathbf{v}}{2cr^5}
\end{aligned}$$

which is equal to the second term of the decomposition (20). The first term just comes out to be equal to $\frac{1}{2}$ of what we get from this paper, which we evaluate using Problem 11.27 from *J.D. Jackson, 3rd Ed.*

The transformation (3) of Fisher was derived using the standard definition $\mathbf{m} = (1/2c) \int d^3r \mathbf{r} \times \mathbf{J}$. The origin of this definition is in a multipole expansion of the vector potential of a localized current distribution that is assumed to be divergenceless ($\nabla \cdot \mathbf{J} = 0$), so that the value of $\mathbf{m}$ is independent of the choice of the reference point $\mathbf{r} = 0$.[14] However, the net current density (6) of a moving electric dipole is not divergenceless; by the continuity equation, $\nabla \cdot \mathbf{J} = -\partial\rho_b/\partial t$, and $\partial\rho_b/\partial t \neq 0$ for the moving dipole's charge density ρ_b [see Eq. (5)]. But as a curl, the magnetization component $\mathbf{J}_m$ of the net current density $\mathbf{J}$ is divergenceless. It is only the magnetization current $\mathbf{J}_m = -\nabla \times [(\mathbf{v} \times \mathbf{p}_0)\,\delta\,(\mathbf{r} - \mathbf{r}_0)]$ that, using $\mathbf{m} = (1/2c) \int d^3r \mathbf{r} \times \mathbf{J}_m$, determines the magnetic dipole moment that is appropriate for a moving electric dipole.

Here, we would be solving Problem 11.27 from Jackson to highlight the assumptions involved here which led him to conform him with Fisher's result.

Let K' frame be moving with velocity v w.r.t. K

$$\vec{v} = \vec{\beta} c$$

$$\therefore \vec{m} = \frac{\vec{p} \times \vec{\beta}}{2}$$

Assumption: ρ' independent of t' in k (rest frame)

$$\therefore \rho' = \rho\left(x'\right)$$

$$\therefore \int \rho' d^3 x' = q' = 0$$

And, $\quad \vec{p}' = \int \vec{x}' \rho' d^3 x'$ in rest frame

$$\begin{aligned}
&J'^{\mu} = (c\rho', 0) \\
&J^{\mu} = \left(\gamma c \rho', \gamma \vec{\beta} c \rho'\right) \\
&x^0 = \gamma\left(x'^0 + \overrightarrow{\beta_0} \cdot \vec{x}'\right) \\
&\vec{x} = \vec{x}' + \frac{\gamma - 1}{\beta^2}\left(\vec{\beta} \cdot \vec{x}'\right) + \gamma \beta x'^0 \\
&\therefore \rho\left(x^0, \vec{x}\right) = \gamma \rho'\left(\vec{x}'\right) \\
&\vec{J}\left(x^0, \vec{x}\right) = \gamma \vec{v} \rho'\left(\overrightarrow{x'}\right) \\
&\vec{x}' = \vec{x} + \frac{\gamma - 1}{\beta^2} \vec{\beta}(\vec{\beta} \cdot \vec{x}) - \gamma \beta x^0
\end{aligned}$$

Let $x^0 = 0$ (lab frame) $\Rightarrow x'^0 = -\vec{\beta} \cdot \vec{x}'$

$$\Rightarrow \vec{x} = \vec{x}' - \frac{\gamma}{\gamma + 1} \vec{\beta}\left(\vec{\beta} \cdot \vec{x}'\right)$$

$$\begin{aligned}
\vec{m} &= \frac{1}{2c} \int \vec{x} \times \vec{J} d^3 x \\
&= \frac{1}{2c} \int \left(\vec{x}' - \frac{\gamma}{\gamma + 1} \vec{\beta}\left(\vec{\beta} \cdot \vec{x}'\right)\right) \times \left(\gamma \vec{\beta} c \rho'\right) \frac{d^3 x'}{\gamma} \\
&= \frac{1}{2} \int \vec{x}' \times \vec{\beta} \rho' d^3 x' = \frac{1}{2} \vec{p}' \times \vec{\beta} \\
\vec{m} &= \frac{1}{2} \vec{p}' \times \frac{\vec{v}}{c}
\end{aligned}$$

Thus, it can be seen that the charge density has been assumed to be independent of time, which leads to a divergenceless convection current, which should be wrong. The full magnetic field of a moving electric dipole can be decomposed in more than one way. Decomposition (20) is not incorrect, but decomposition (14) is singled out by the clear-cut sources of its two terms: the divergenceless magnetization current due to the magnetization produced by the magnetic dipole moment $\mathbf{m} = -\mathbf{v} \times \mathbf{p}_0/c$ that is consistent with the transformation relations of special relativity, and the non-zero-divergence polarization current due to the polarization produced by a moving electric dipole. We can choose any decomposition as long as we get a non-zero divergence for the polarization current density.

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