# Sampling Patterns

**Dr. Cengiz Öztireli**

Computer Graphics Laboratory, ETH Zürich

ETH zürich

Disney Research

cgl

# Aliasing

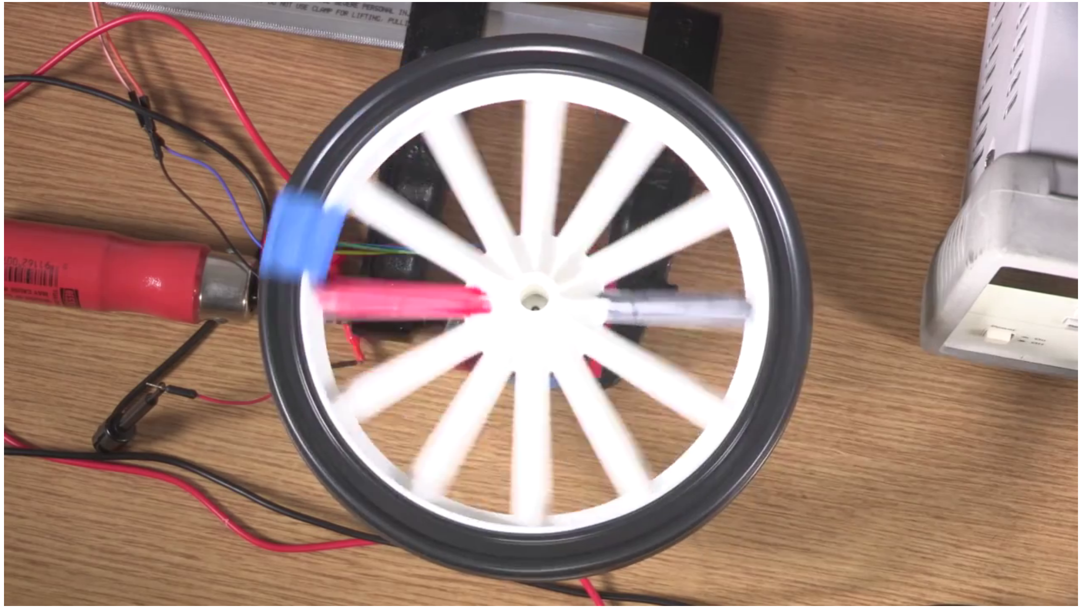
- Conversion from continuous to discrete

# Aliasing

- Aliasing artifacts – image plane sampling

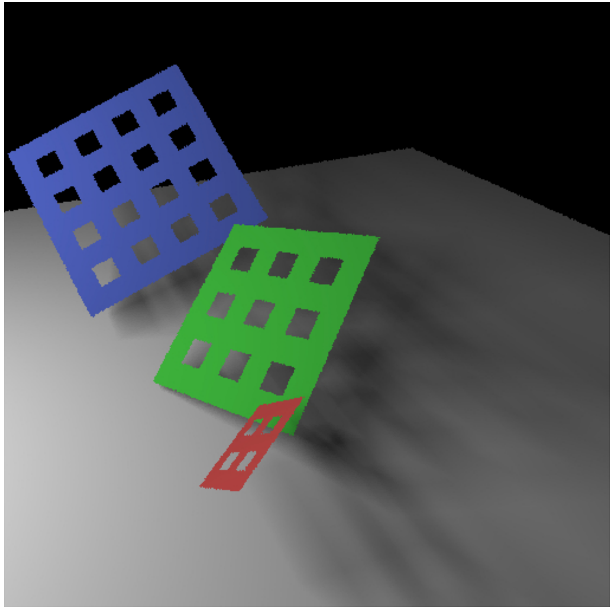
# Aliasing

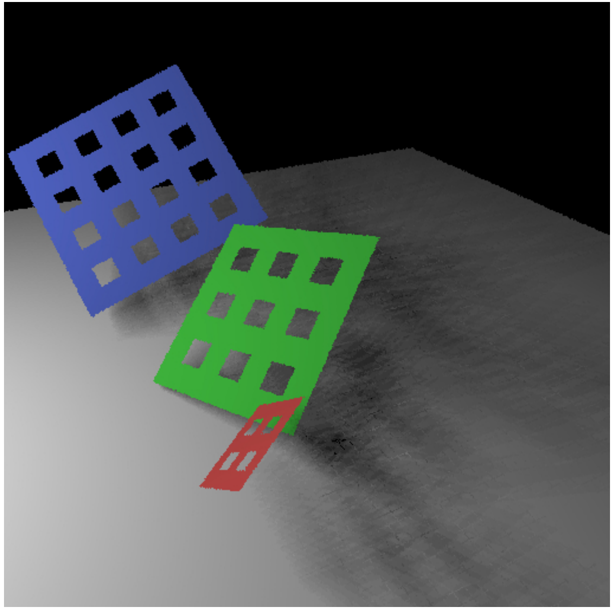
- Aliasing artifacts – temporal sampling



# Aliasing

- Aliasing artifacts in rendering – quadrature





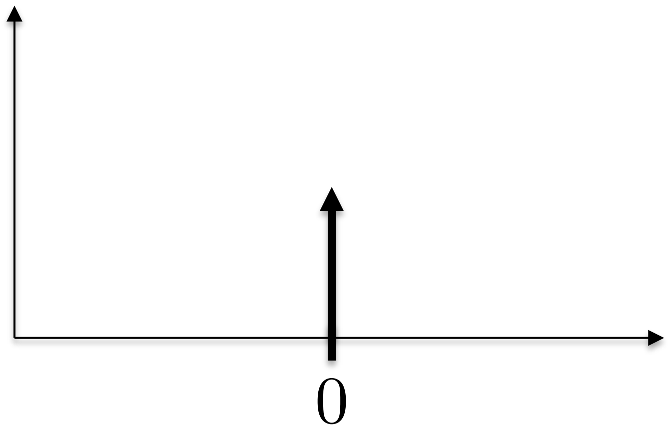
# Function Spaces

- Dirac delta

$$\delta(x)$$

$$\int_{-\infty}^{\infty} \delta(x) = 1$$

$$\delta(x) = 0 \;\; \forall x \neq 0$$

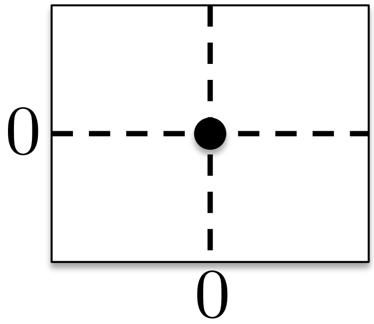


# Function Spaces

- Dirac delta

$$\delta(\mathbf{x}) = \delta(x)\delta(y)\cdots$$

$$\int \delta(\mathbf{x})d\mathbf{x} = \int \delta(x)\delta(y)\cdots d\mathbf{x} = \int \delta(x)dx \int \delta(y)dy \cdots = 1$$

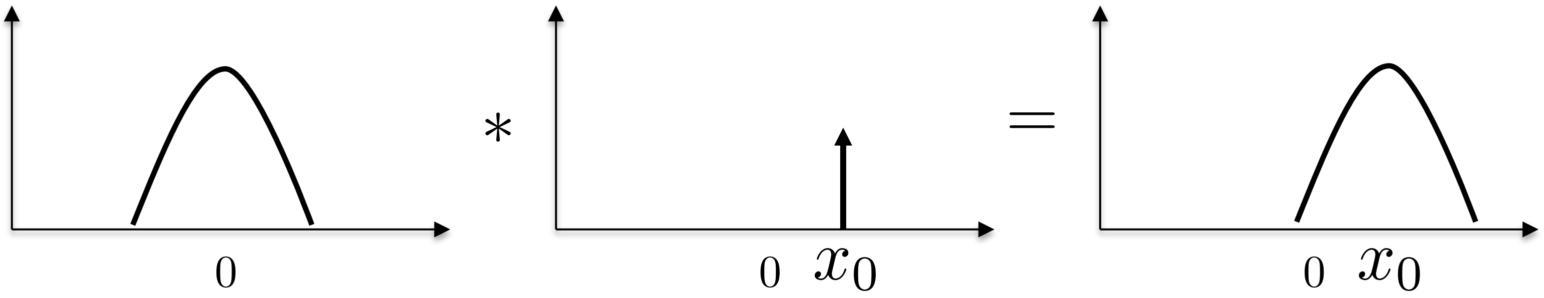


# Function Spaces

- Dirac delta – properties

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$$

$$f(x) * \delta(x - x_0) = f(x - x_0)$$



# Function Spaces

- Functions as vectors

| | |
|---|---|
| $f(x)$ | $\mathbf{f}$ |
| $\int f(x)g(x)dx$ | $\mathbf{f}^T\mathbf{g}$ |
| $\int f^2(x)dx$ | $\|\|\mathbf{f}\|\|^2$ |
| $b_i(x)$ | $\mathbf{b}_i$ |

# Function Spaces

- Functions in different bases

$$\mathbf{f} = \sum \mathbf{b}_i (\mathbf{f}^T \mathbf{b}_i) \qquad c_i = \mathbf{f}^T \mathbf{b}_i$$

Standard basis: $\mathbf{e}_i = [\cdots 0 \underset{i}{1} 0 \cdots]^T \qquad c_i = f_i$

$$c(x) = \int f(y) b(x,y) dy$$

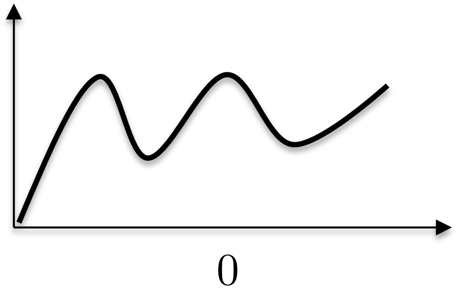
Standard basis: $b(x,y) = \delta(x-y)$

$$c(x) = \int f(y)\delta(x-y)dy = f(x)$$

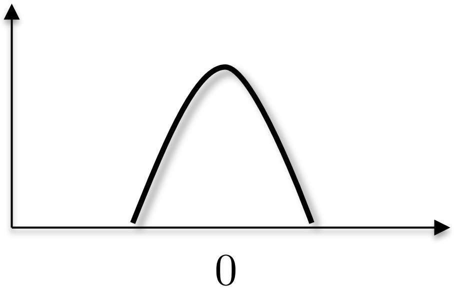
# Sampling Theorem

- Main idea

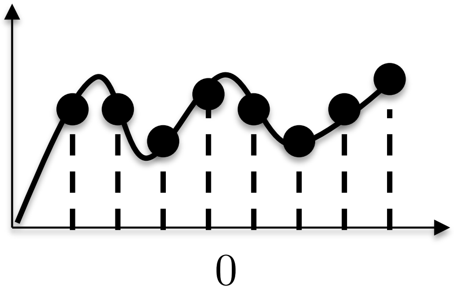
$f(x)$




$c(x)$



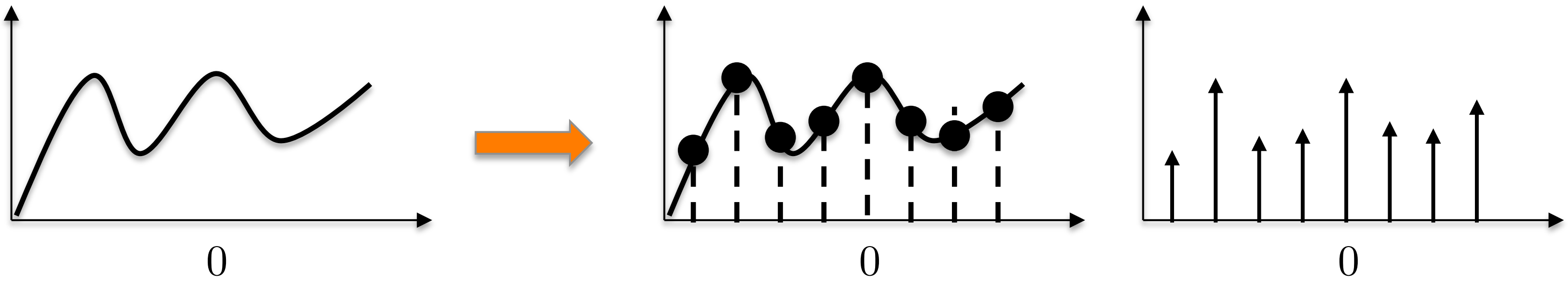

$f_i$



# Sampling Theorem

- Sampling

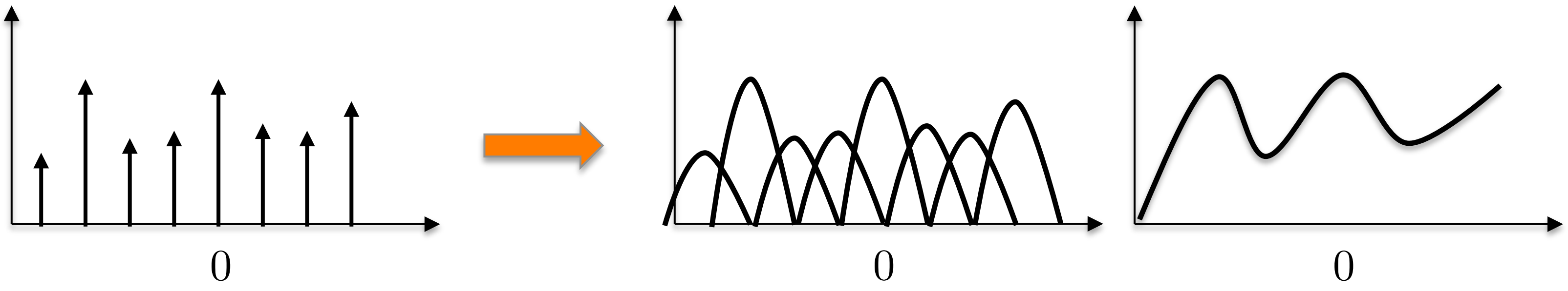
$$f(\mathbf{x})\frac{1}{n}\sum_{i=1}^{n}\delta(\mathbf{x}-\mathbf{x}_i) = \frac{1}{n}\sum_{i=1}^{n}f(\mathbf{x}_i)\delta(\mathbf{x}-\mathbf{x}_i)$$



# Sampling Theorem

- Reconstruction

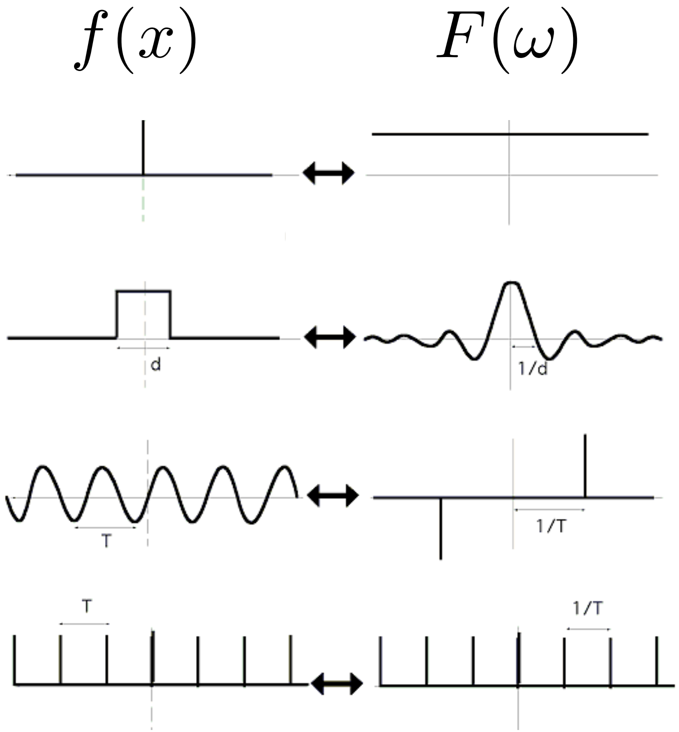
$$k(\mathbf{x}) * \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{x}_i)\delta(\mathbf{x}-\mathbf{x}_i) = \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{x}_i)k(\mathbf{x}-\mathbf{x}_i)$$



# Sampling Theorem

- Fourier transform

$$\underbrace{F(\omega)}_{\text{Fourier transform}} = \int_{-\infty}^{\infty} f(x)\underbrace{e^{-i2\pi\omega x}}_{\text{Orthogonal basis}} dx$$

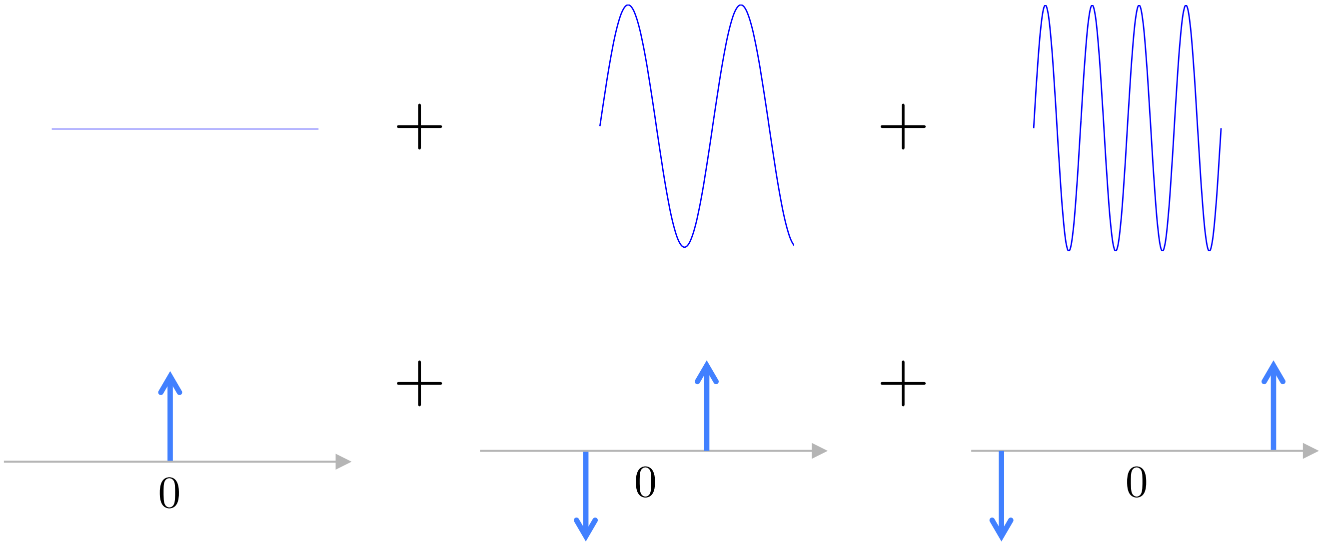


# Sampling Theorem

- Fourier transform

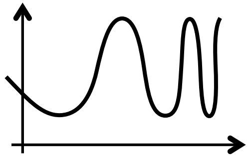
$$f(x) =$$

$$F(\omega) =$$

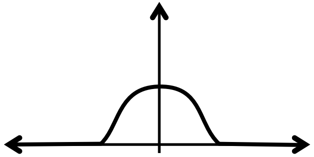


# Sampling Theorem

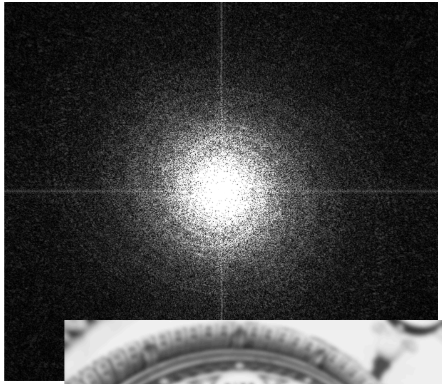
- Fourier transform – band-limited functions

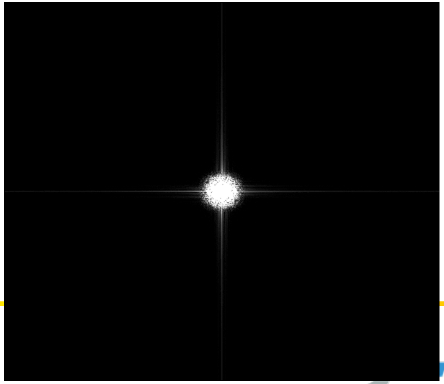


$$f(x)$$



$$|F(\omega)|$$



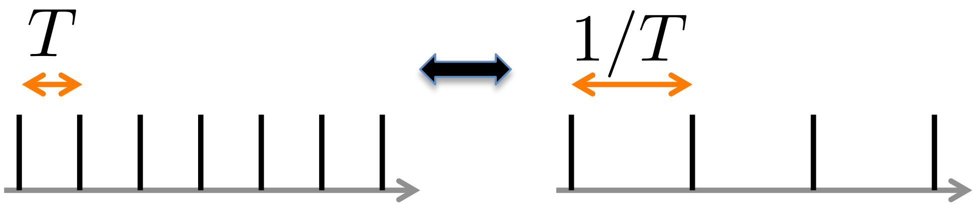


# Sampling Theorem

- Fourier transform – some properties

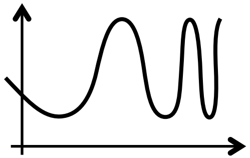
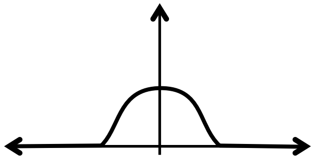
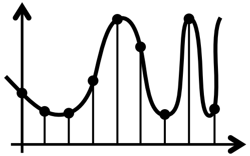
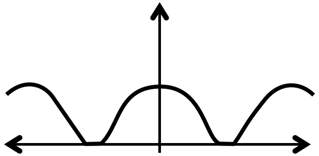
$$f(x)g(x) \longleftrightarrow F(\omega) * G(\omega)$$

$$f(x) * g(x) \longleftrightarrow F(\omega)G(\omega)$$



# Sampling Theorem

- Sampling

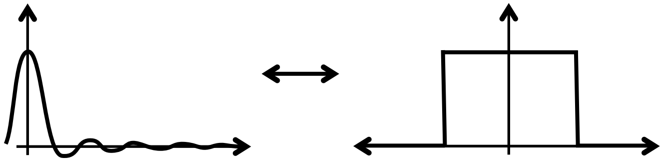
| Original function | | Sampled function | |
|---|---|---|---|
| Spatial Domain | Frequency Domain | Spatial Domain | Frequency Domain |
|  |  |  |  |
| $f(x)$ | $F(\omega)$ | $f(x)s(x)$ | $F(\omega) * S(\omega)$ |

$$s(x) = \frac{1}{n}\sum_{i=1}^{n} \delta(x - x_i) \qquad S(\omega) = \delta(\omega) + \sum_{w_i \neq 0} \delta(\omega - \omega_i)$$

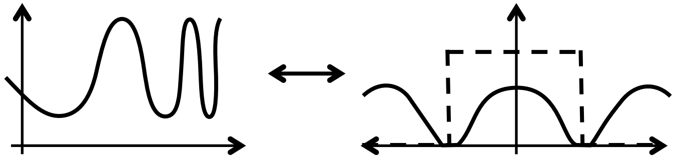
# Sampling Theorem

- Reconstruction

| Filter function | | Reconstruction | |
|---|---|---|---|
| Spatial Domain | Frequency Domain | Spatial Domain | Frequency Domain |



$$m(x) \qquad M(\omega)$$



$$m(x) * (f(x)s(x))$$

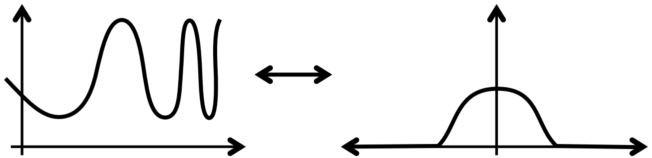
$$M(\omega)\,(F(\omega) * S(\omega))$$

# Sampling Theorem

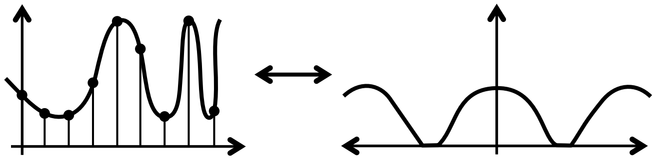
- Sampling & Reconstruction

Sampling

Spatial Domain Frequency Domain



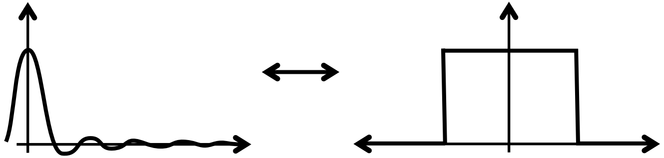
Original function



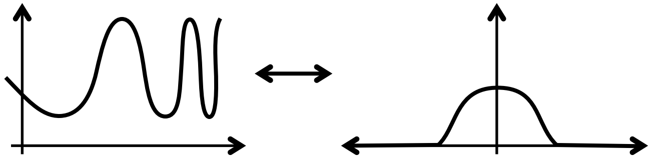
Sampled function

Reconstruction

Spatial Domain Frequency Domain



Filter function

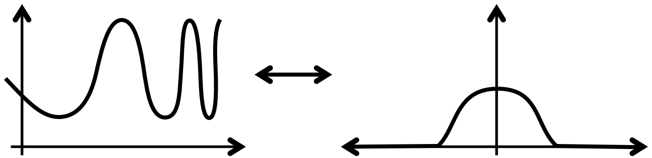


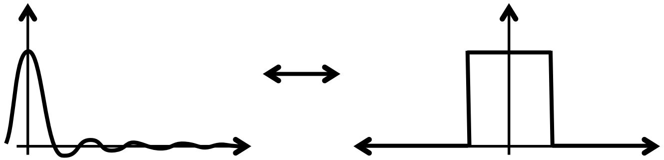
Reconstructed function

# Sampling Theorem

- Aliasing

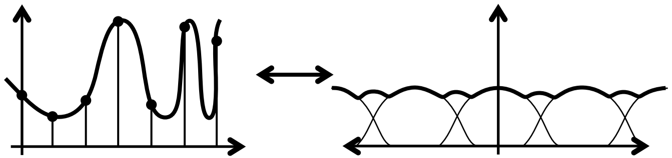
| Sampling | | Reconstruction | |
|---|---|---|---|
| Spatial Domain | Frequency Domain | Spatial Domain | Frequency Domain |

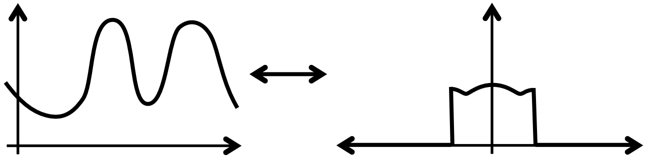




Original function

Filter function



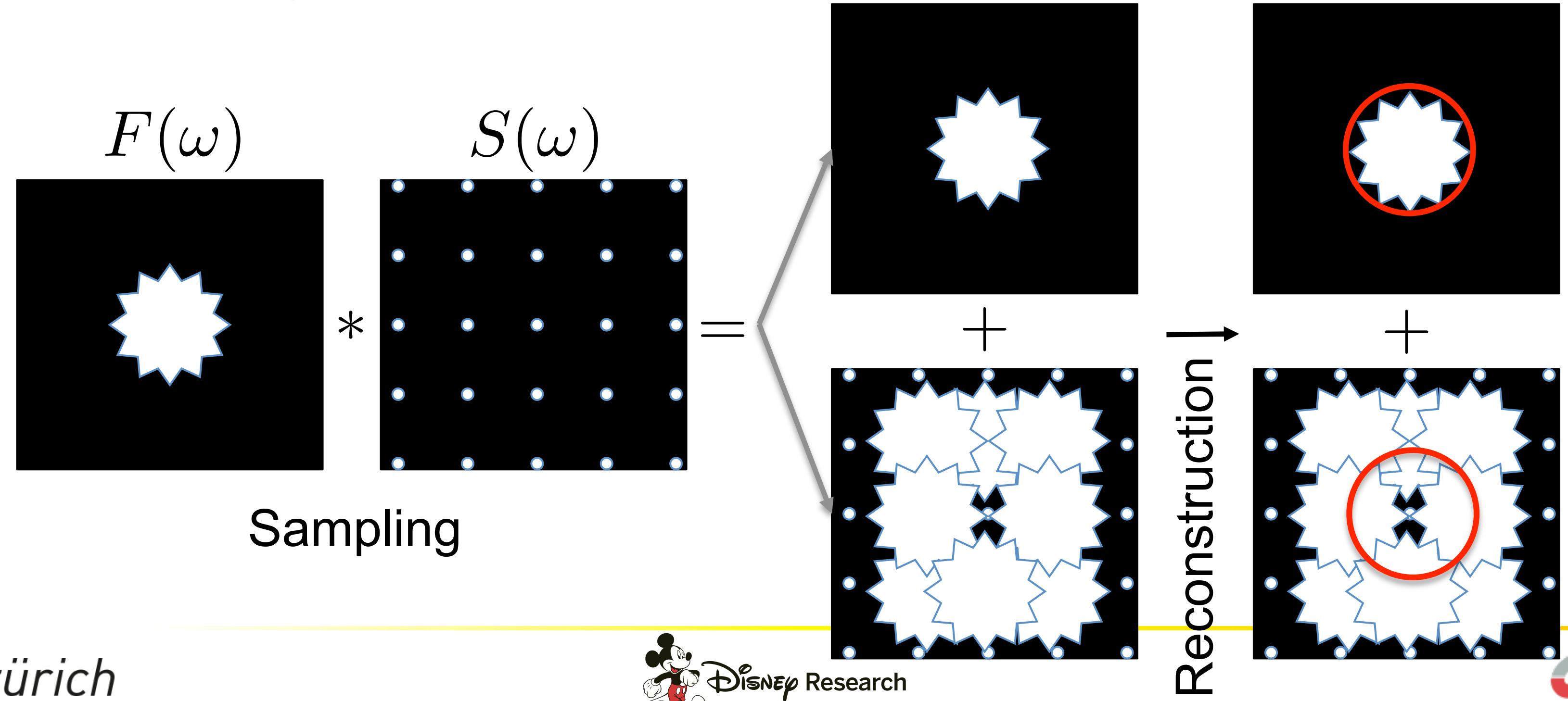


Sampled function

Reconstructed function

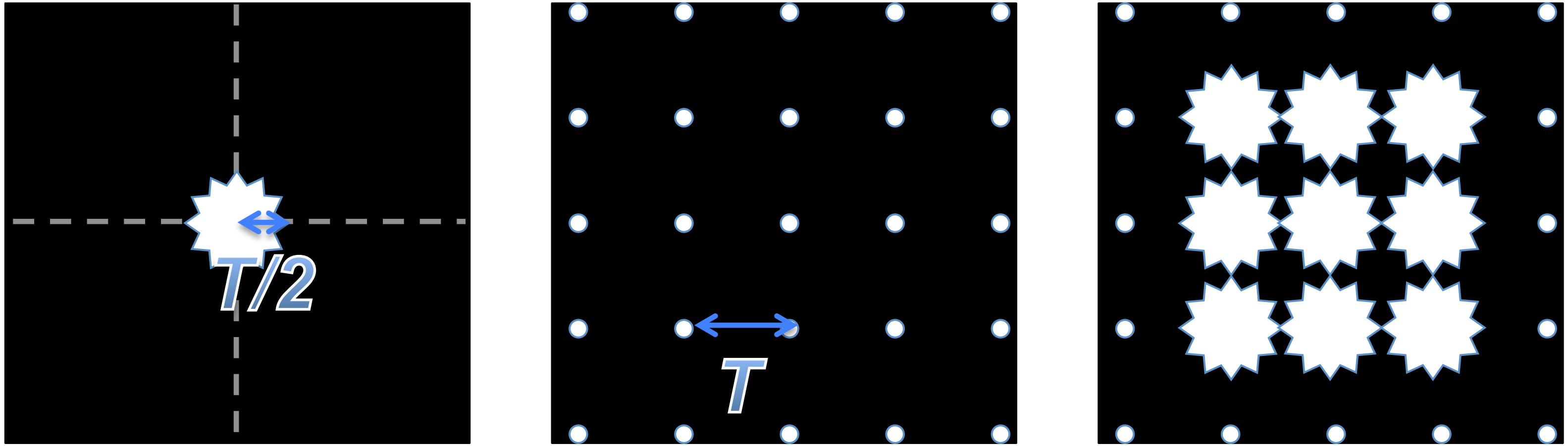
# Sampling Theorem

- Aliasing



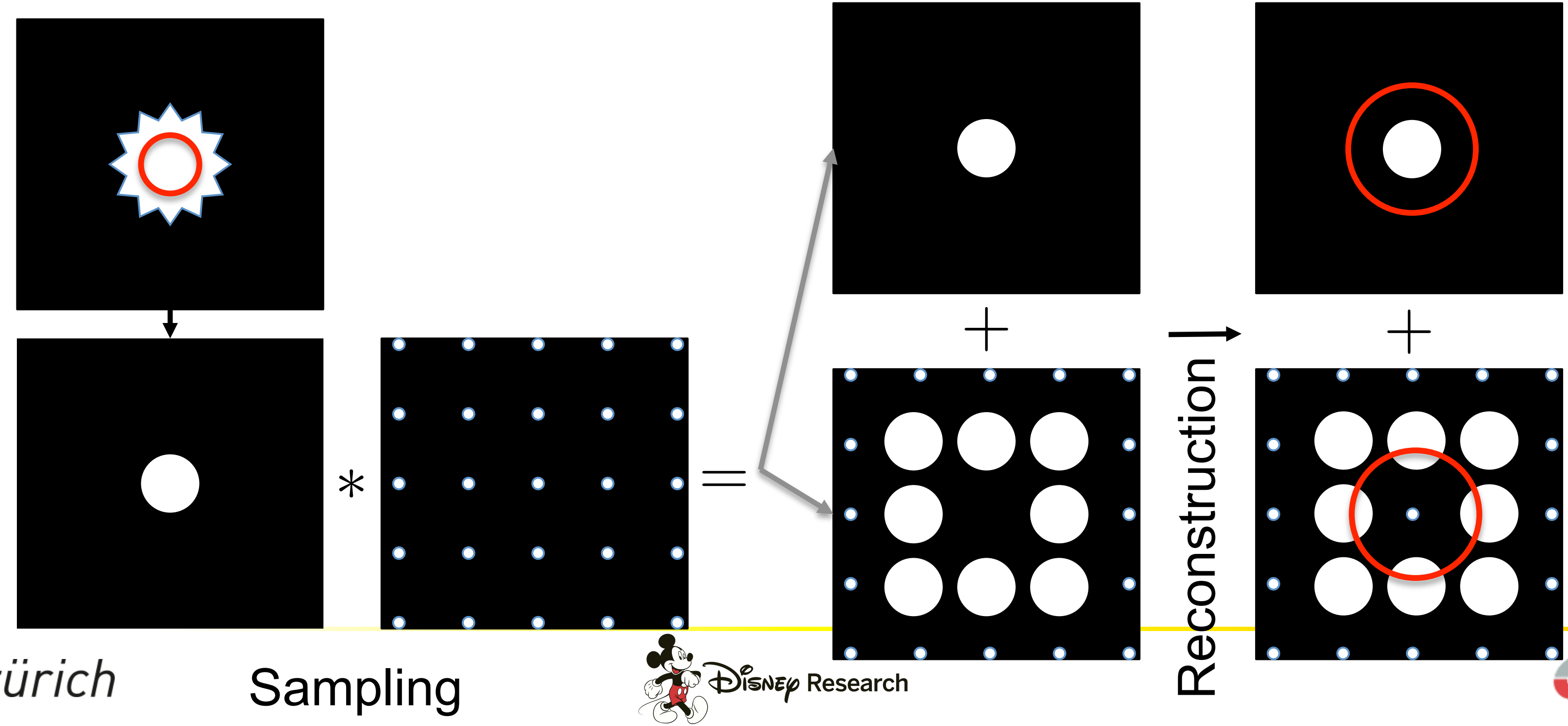
# Sampling Theorem

- The function should be bandlimited by T/2



# Anti-aliasing

- Pre-filtering for anti-aliasing



# Anti-aliasing

- Pre-filtering for anti-aliasing

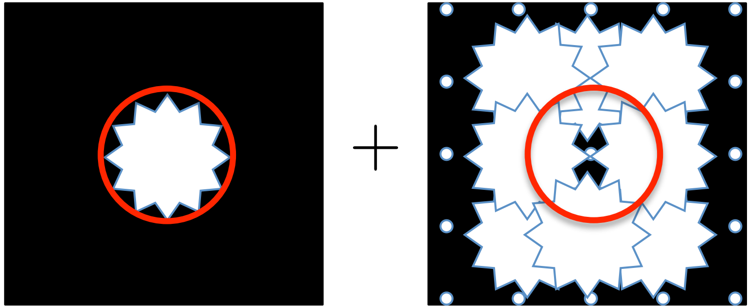
Original

Aliased

Pre-filtered

# Anti-aliasing

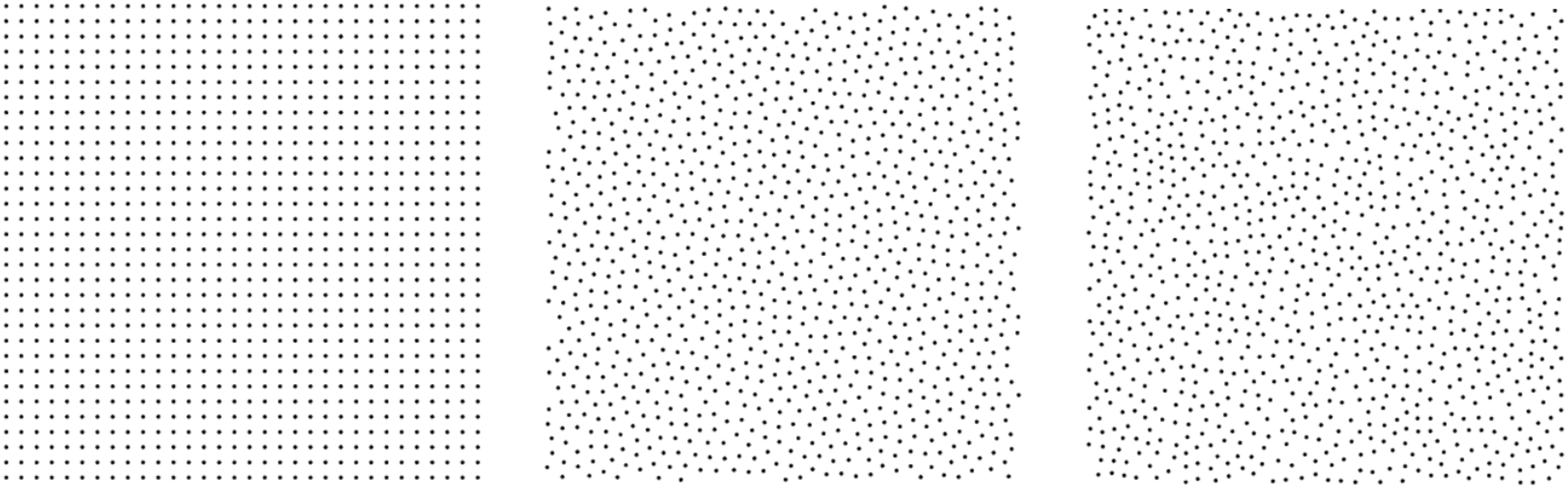
- Pre-filtering for anti-aliasing images
  - Problem: often not possible



  - Alternative: switch to irregular sampling

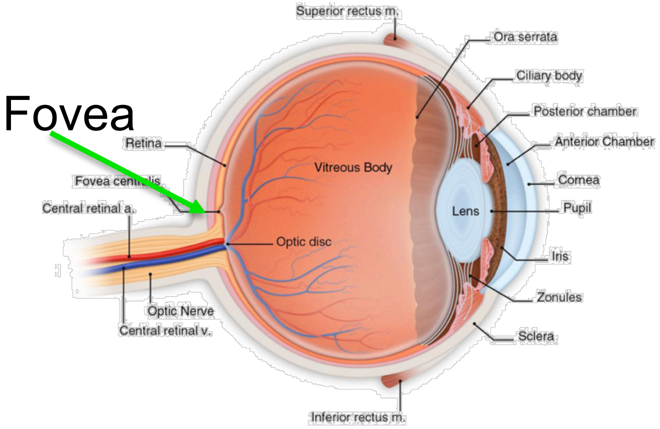
# Anti-aliasing

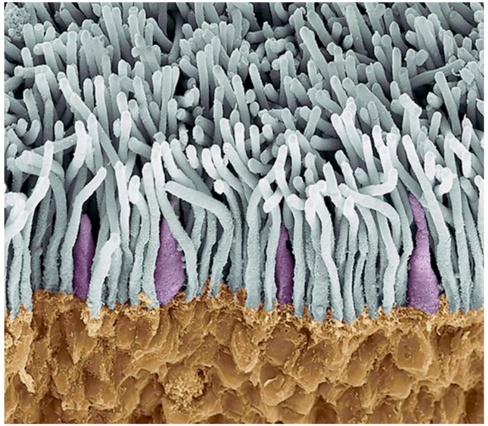
- Sampling patterns for anti-aliasing
  - Blue noise

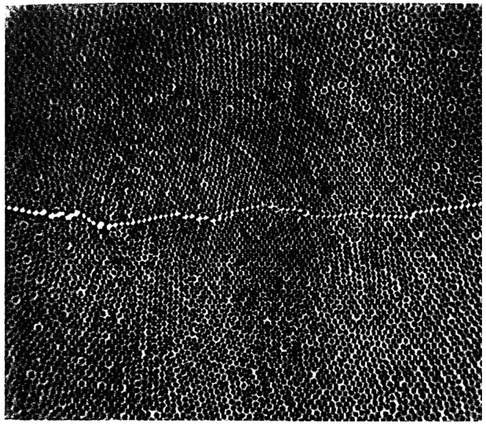


# Anti-aliasing

- Sampling patterns for anti-aliasing
  - Blue noise – in the human eye



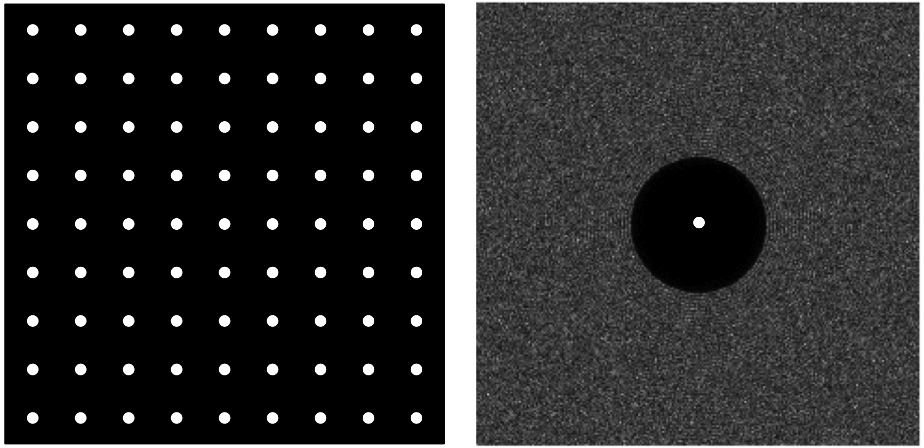





# Anti-aliasing

- Sampling patterns for anti-aliasing
  - Blue noise in the Fourier domain

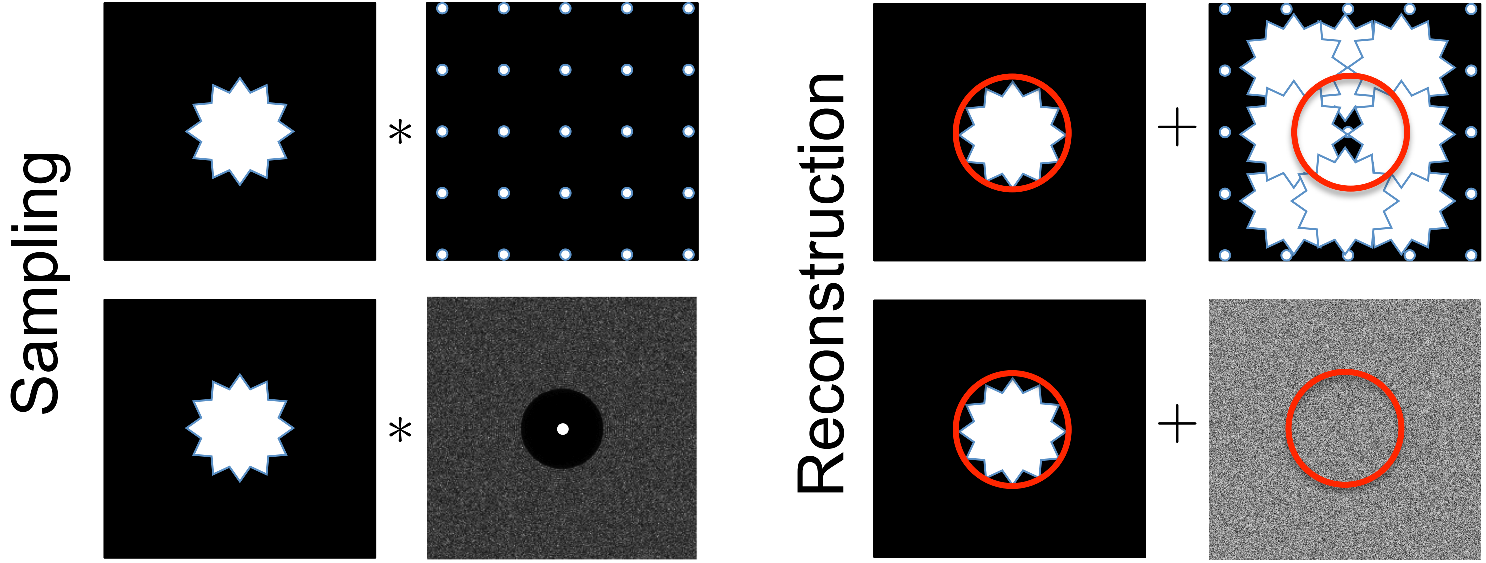
$S(\omega)$



Regular    Blue noise

# Anti-aliasing

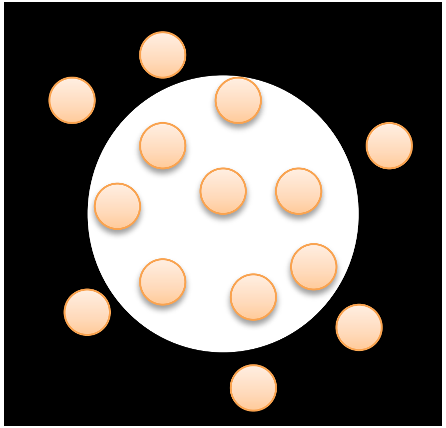
- Sampling patterns for anti-aliasing



# Quadrature

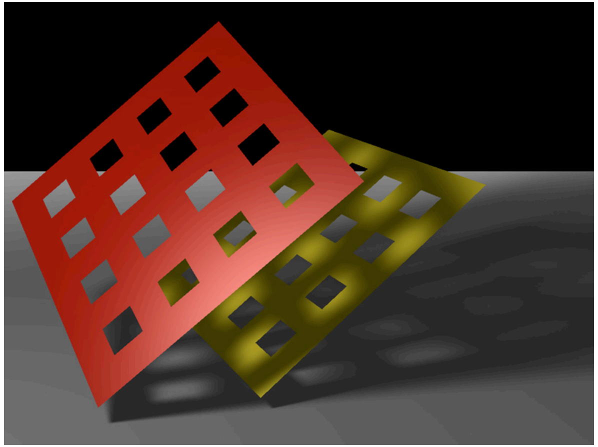
- Estimating integrals

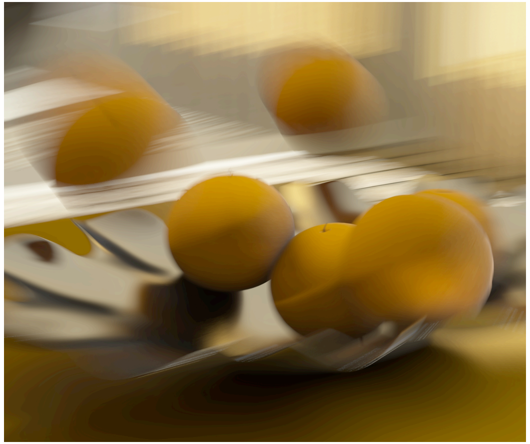
$$\frac{1}{|R|}\int_R f(\mathbf{x})d\mathbf{x} \approx \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{x}_i)$$



# Quadrature

- Estimating integrals
  - Application in rendering





# Quadrature

- Estimating integrals
  - Application in rendering
  - Over the pixel area
  - On the lens
  - Over area light sources
  - In general: over paths of light

# Quadrature

- The Fourier domain view

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\omega x} dx \qquad \boxed{\int f(x)dx = F(0)}$$

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{n}\sum_{i=1}^{n}\int f(x)\delta(x - x_i)dx$$

$$= \int f(x)\frac{1}{n}\sum_{i=1}^{n}\delta(x - x_i)dx \;= \left[F(\omega) * S(\omega)\right](0)$$

# Quadrature

- The Fourier domain view

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} f(x_i) &= \left[F(\omega) * S(\omega)\right](0) \\
&= \int F(\omega) S(-\omega) d\omega \\
&= \int F(\omega)\delta(\omega) d\omega + \int F(\omega)\bar{S}(-\omega) d\omega \\
&= \boxed{F(0) + \int F(\omega)\bar{S}(-\omega) d\omega}
\end{aligned}
$$

# Quadrature

- The Fourier domain view

$$\int f(x)dx - \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

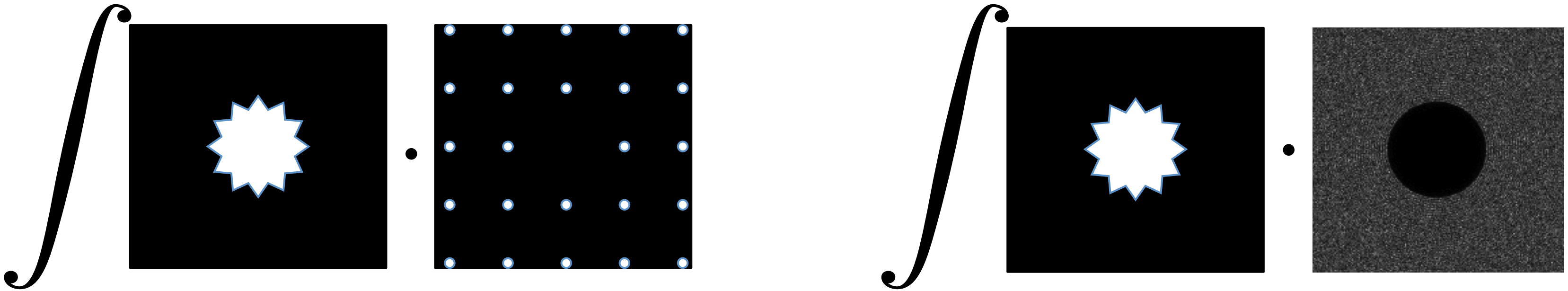
$$= F(0) - F(0) + \int F(\omega)\bar{S}(-\omega)d\omega$$

$$= \int F(\omega)\bar{S}(-\omega)d\omega$$

# Quadrature

- The Fourier domain view

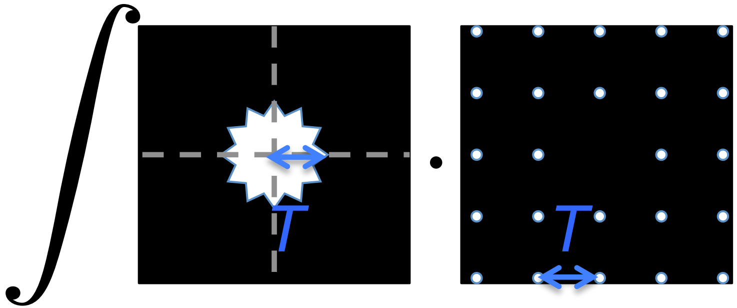
$$\int F(\omega)\bar{S}(-\omega)d\omega$$



# Quadrature

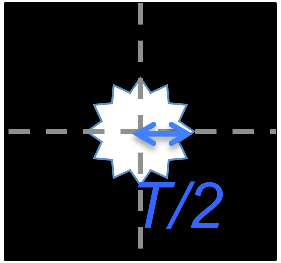
- The Fourier domain view

Integral sampling theorem




$$= 0$$

Classical sampling theorem

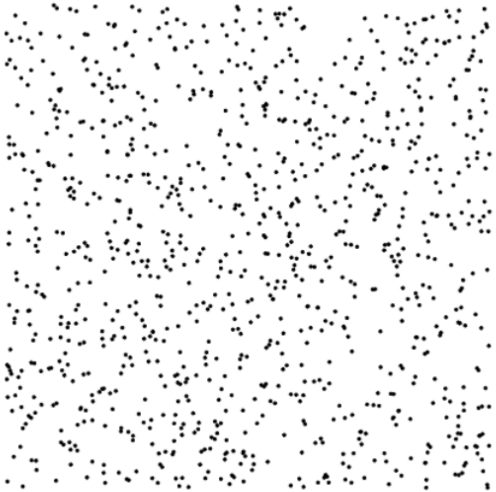


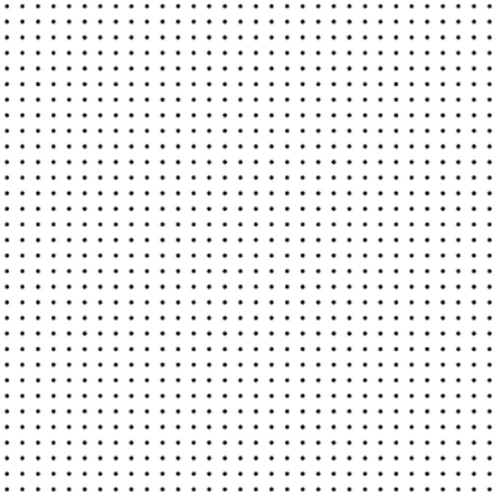
# Sampling Patterns

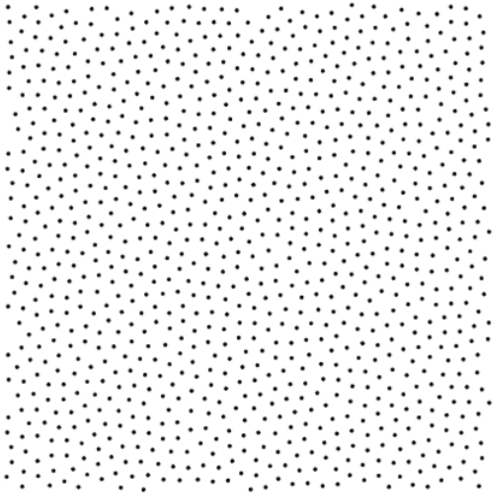
- Summary: point distributions are essential
  - Representation and reconstruction
  - Quadrature (integration)
- What are common patterns, how do we analyze and synthesize them?

# Sampling Patterns

- Point distributions
  - Random, regular, blue noise

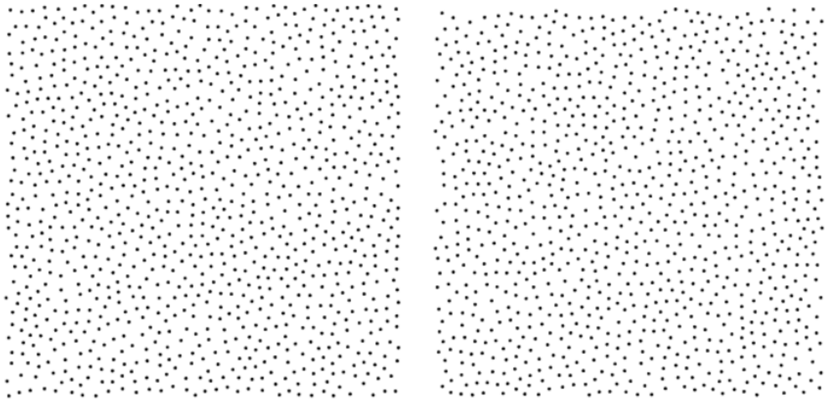




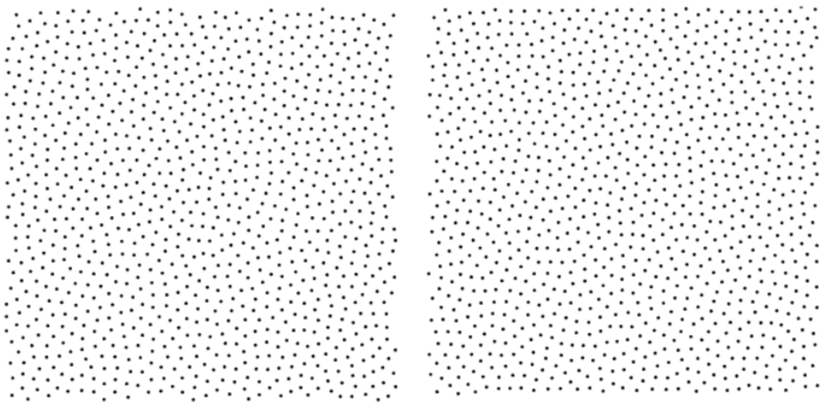


# Sampling Patterns

- Different distributions, same characteristics
  - Differentiate between distribution and pattern



Sampling Pattern 1

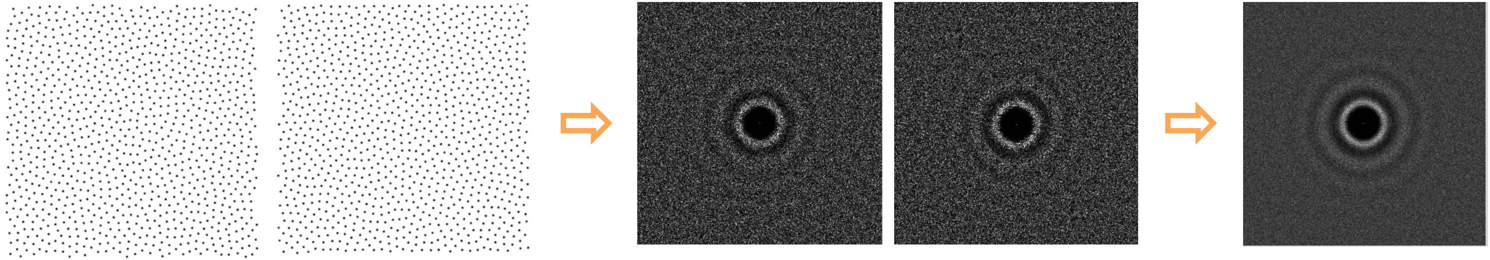


Sampling Pattern 2

# Sampling Patterns

- Characterization of sampling patterns
  - Periodogram

$$s(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\delta(\mathbf{x}-\mathbf{x}_i) \qquad P(\omega) = \mathbb{E}[|S(\omega)|^2]$$

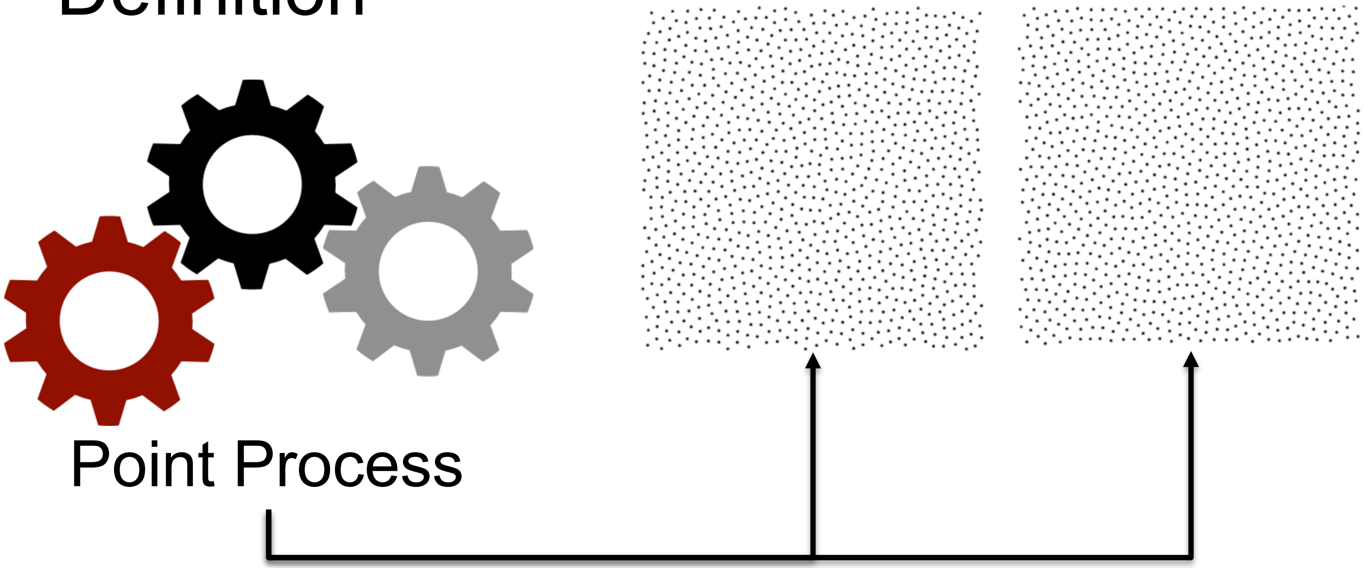


# Sampling Patterns

- Characterization of sampling patterns
  - Point process statistics
  - Describes the correlations of point locations
  - A generating point process
  - Point distributions as instances

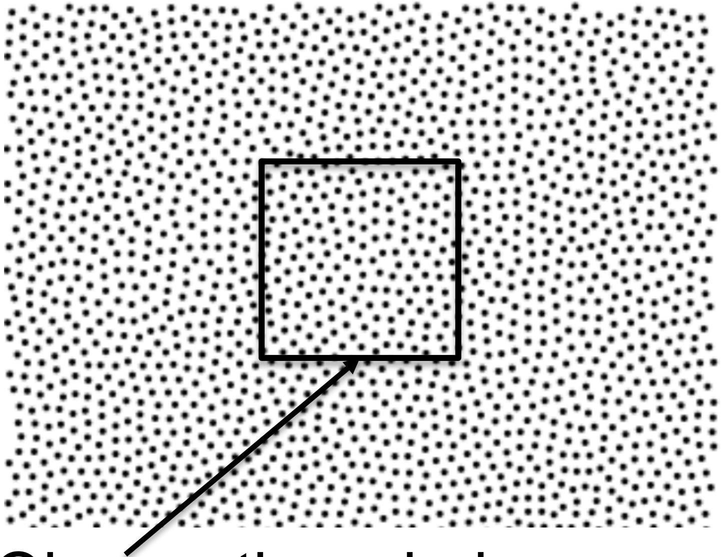
# Sampling Patterns

- Point processes
  - Definition

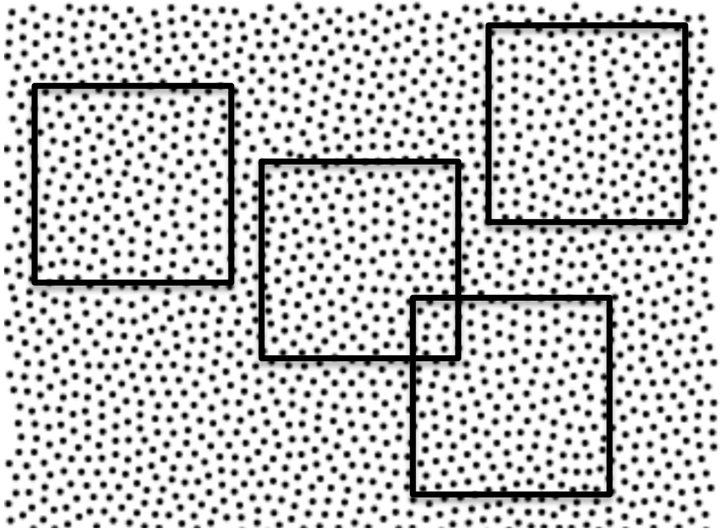


# Sampling Patterns

- Infinite point processes



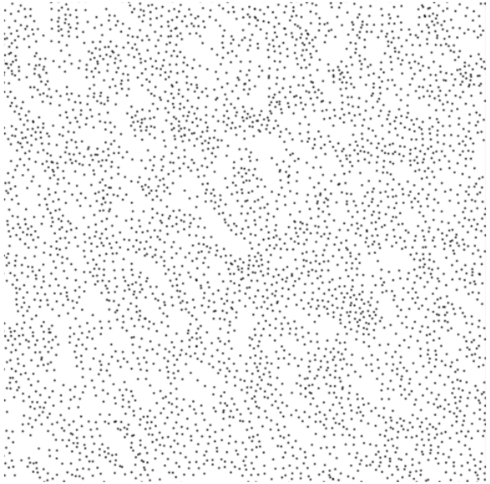
Observation window



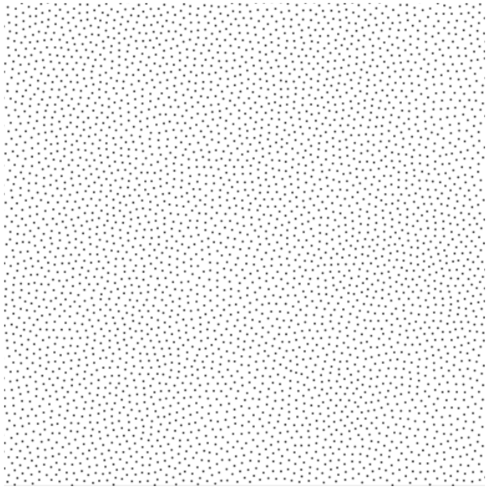
Ergodic processes

# Sampling Patterns

- Point processes



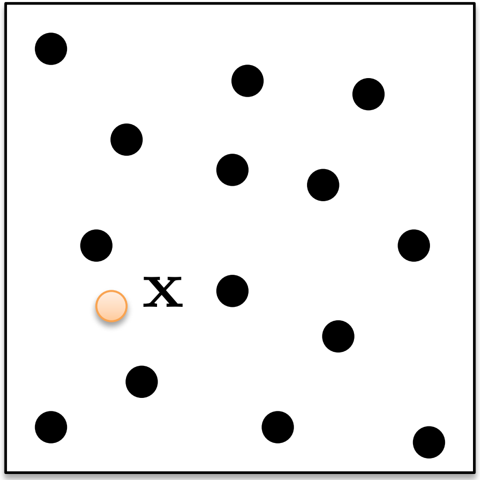
Stationary
(translation invariant)



Isotropic
(translation & rotation invariant)

# Sampling Patterns

- Point process statistics
  - First order product density




$\lambda(\mathbf{x})$

Expected number of points around x

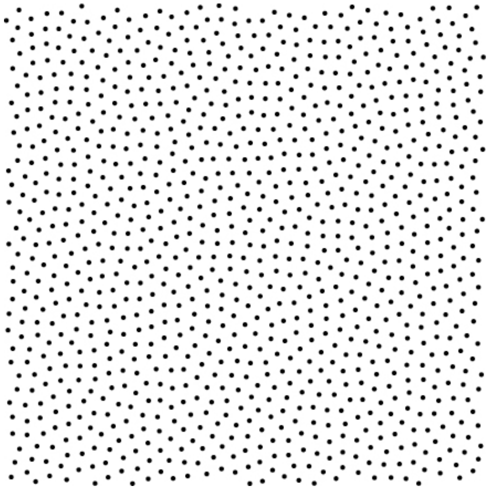
Measures local density

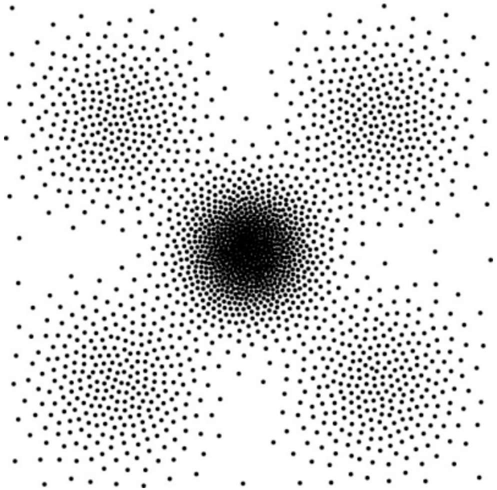
Stationary point process: constant

# Sampling Patterns

- Point process statistics
  - First order product density

$$\lambda(\mathbf{x})$$



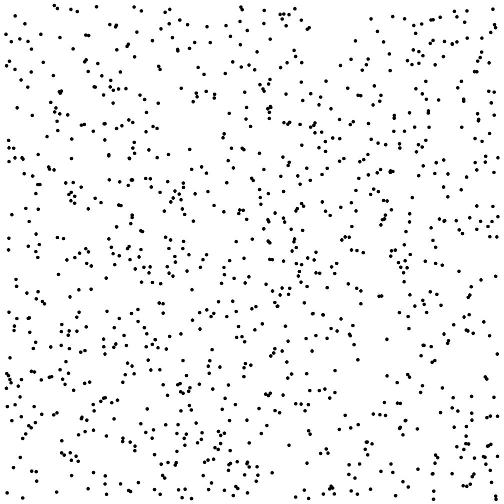


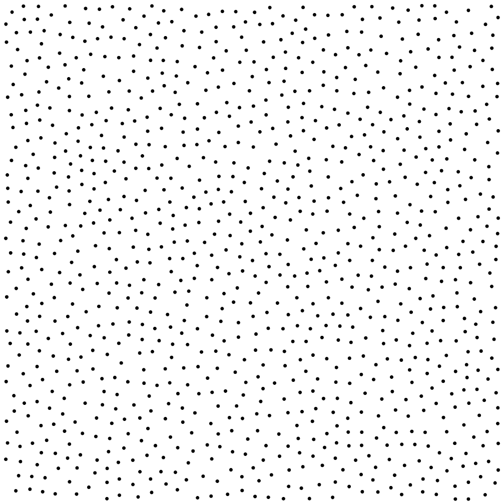
# Sampling Patterns

- Point process statistics
  - First order product density

$\lambda(\mathbf{x})$

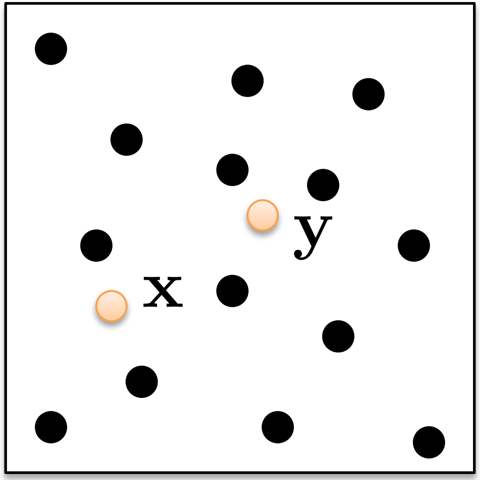
Constant





# Sampling Patterns

- Point process statistics
  - Second order product density



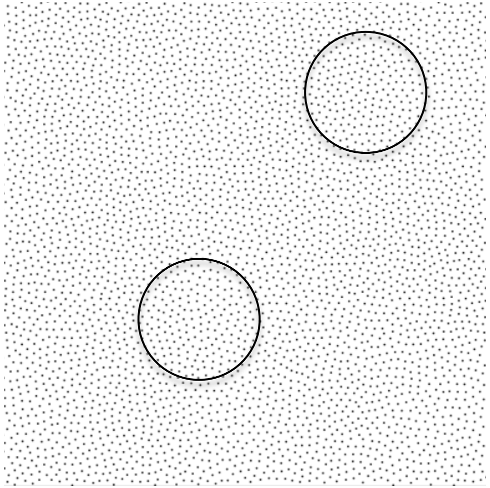

$\varrho(\mathbf{x}, \mathbf{y})$

Expected number of points around x & y

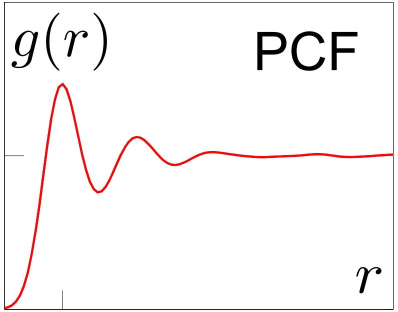
Measures the joint probability $p(\mathbf{x}, \mathbf{y})$

# Sampling Patterns

- Point process statistics
  - Isotropic point processes



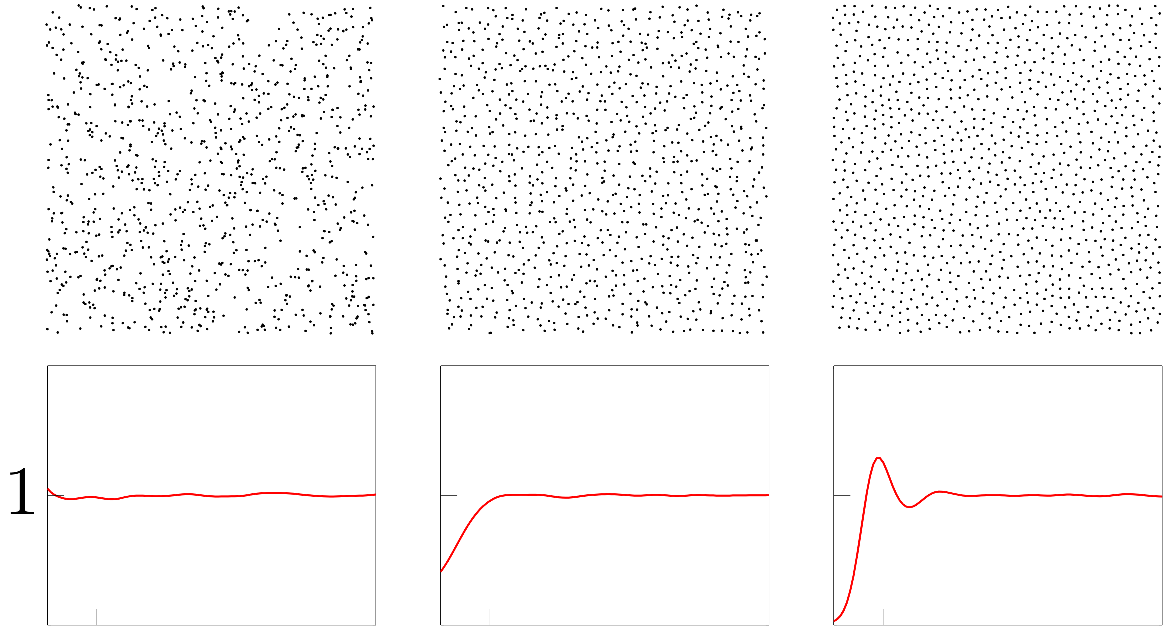
$$\lambda(\mathbf{x}) = \lambda$$
$$\varrho(\mathbf{x}, \mathbf{y}) = \lambda^2 g(||\mathbf{x} - \mathbf{y}||)$$




The Pair Correlation Function (PCF)

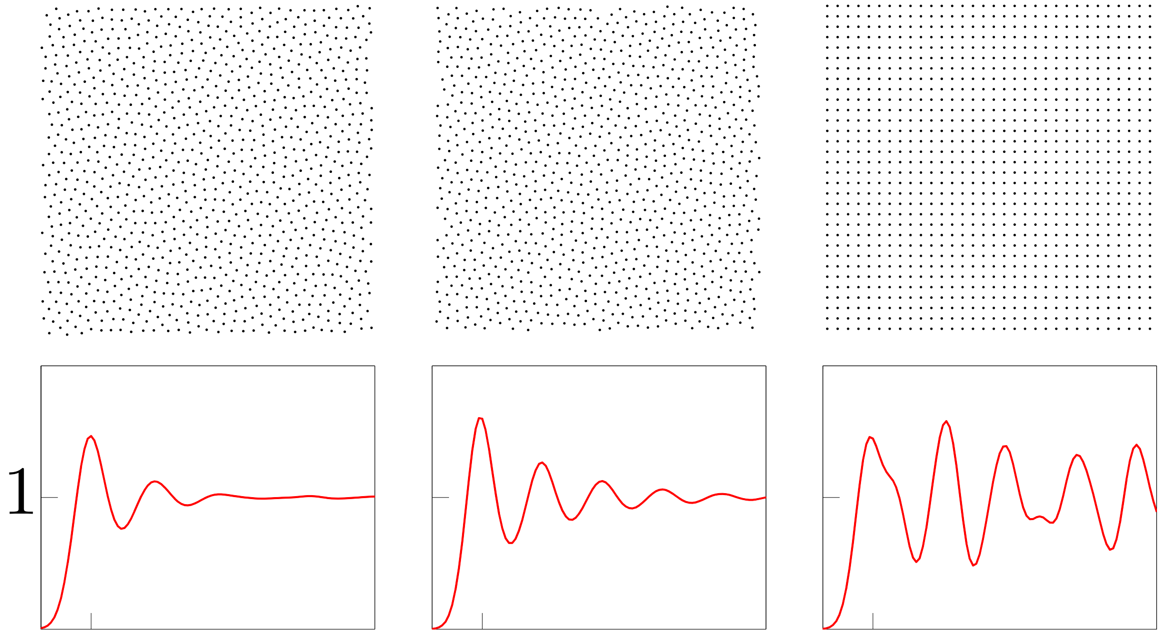
# Sampling Patterns

- Point process statistics - PCFs



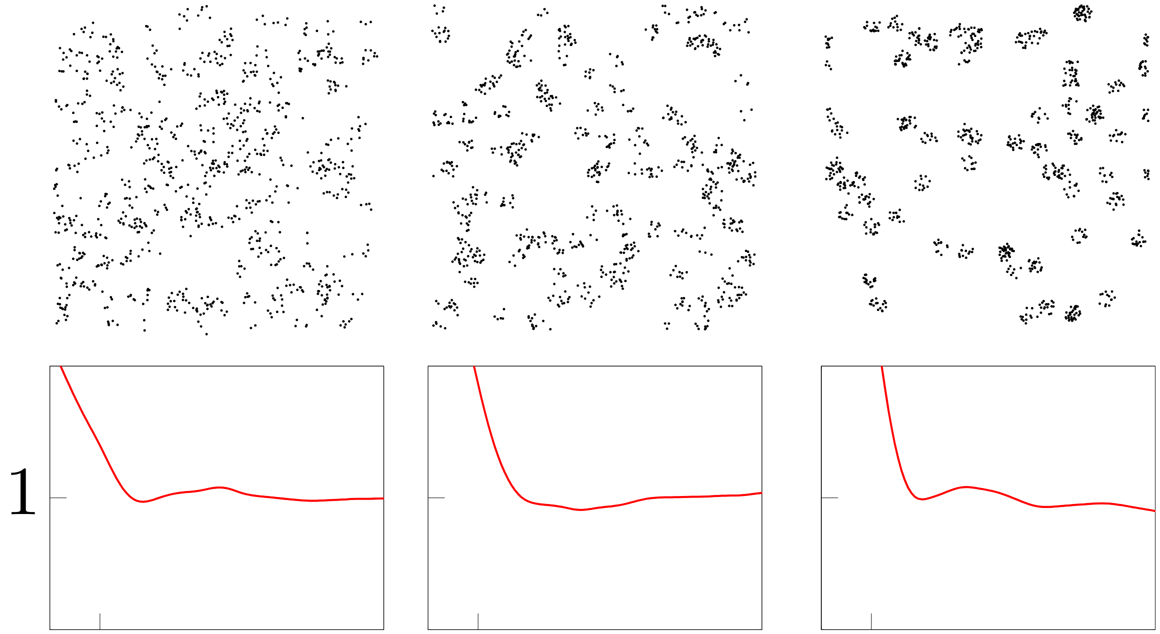
# Sampling Patterns

- Point process statistics - PCFs



# Sampling Patterns

- Point process statistics - PCFs



# Sampling Patterns

- Point process statistics
  - Pair correlation function - estimator

$$g(r) \approx \frac{1}{|\partial V_d| r^{d-1} \lambda^2} \sum_{i \neq j} k(r - d(\mathbf{x}_i, \mathbf{x}_j))$$

Volume of the unit hypercube in d dimensions

Density

Kernel, e.g. Gaussian

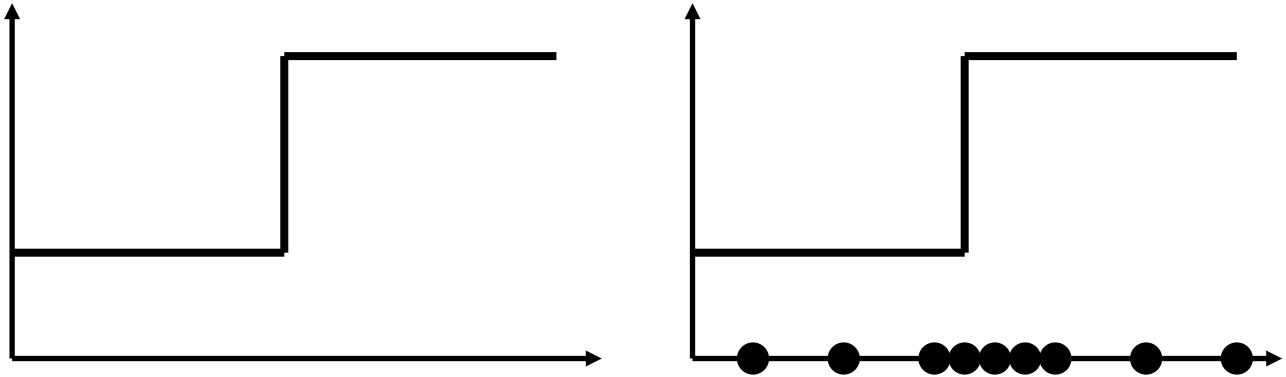
Distance measure

# Sampling Patterns

- Point process statistics
  - Pair correlation function
  - Smooth & robust estimator
  - Spatial characteristic
  - Relation to periodogram via Fourier transform

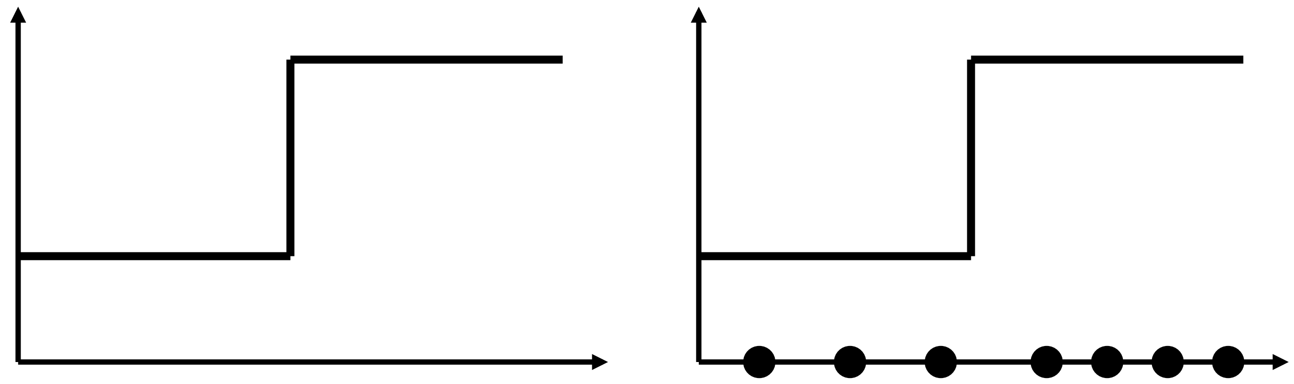
# Adaptive Sampling

- Sampling & reconstruction
  - Sample when local change is high



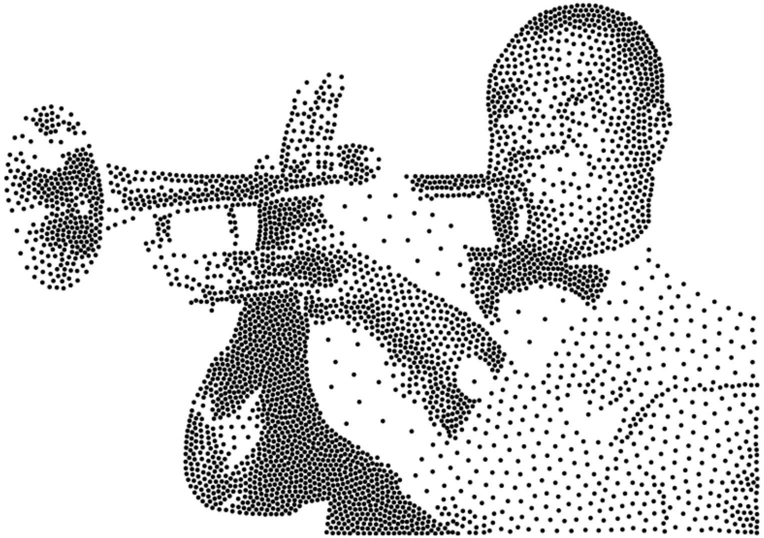
# Adaptive Sampling

- Sampling & quadrature
  - Importance sampling



# Adaptive Sampling

- Point process statistics for adaptive sampling



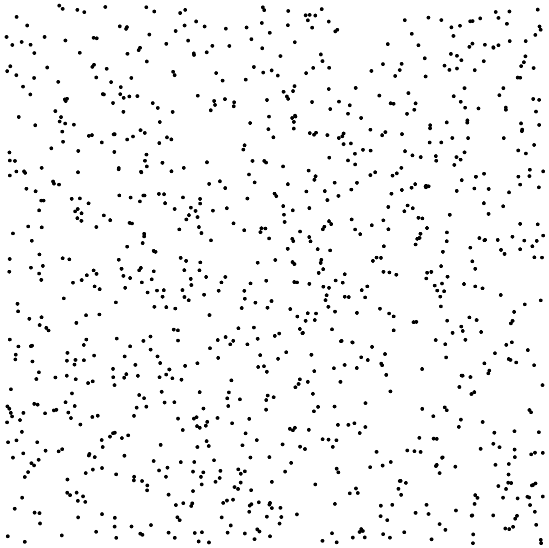
Density captured by $\lambda(\mathbf{x})$

Pair-wise statistics by $\varrho(\mathbf{x}, \mathbf{y})$

Higher-order statistics?

# Sampling Algorithms

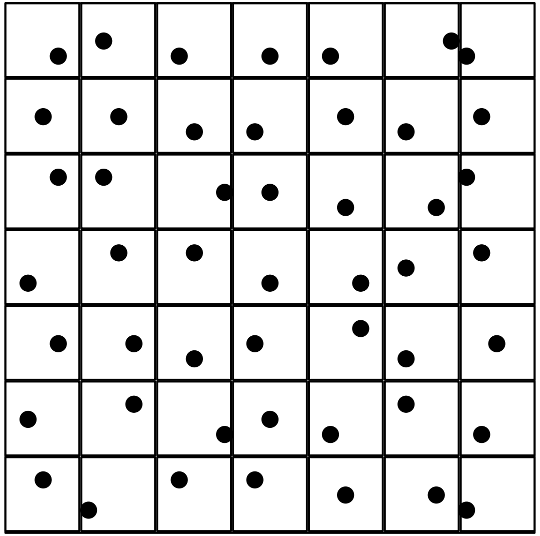
- Random Sampling



Iterate:
Randomly pick a point

# Sampling Algorithms

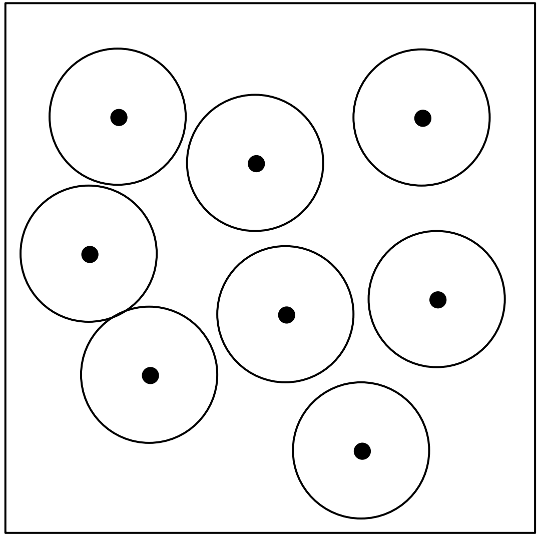
- Stratified sampling



For each stratum:
Randomly pick a point

# Sampling Algorithms

- Dart throwing



Iterate:

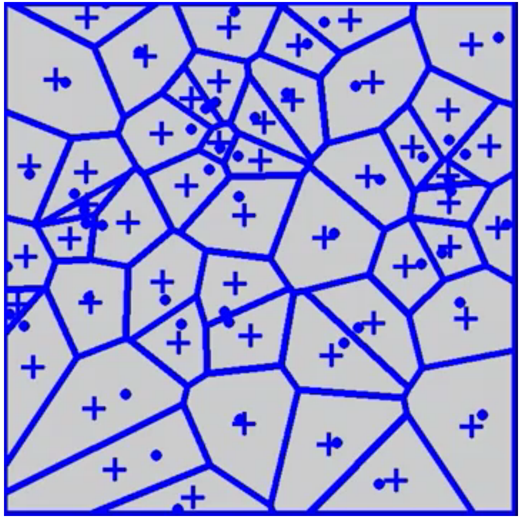
Randomly pick a point

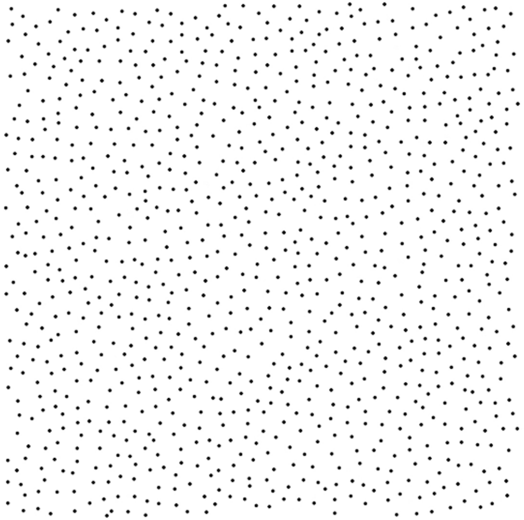
If it is not within the region of others

Add the point

# Sampling Algorithms

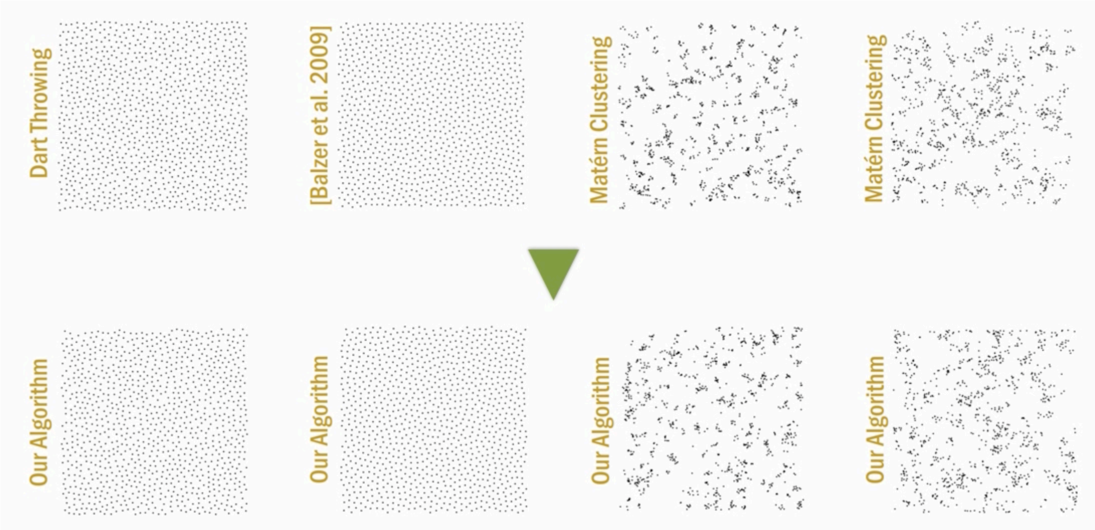
- Relaxation algorithms





# Sampling Algorithms

- General sampling
  - Fit PCF utilizing the smooth PCF approximation



# Some References

- Sampling theorem
  - Communication in the presence of noise
- Anti-aliasing
  - Spectrally optimal sampling for distribution ray tracing
- Quadrature
  - Monte Carlo theory, methods and examples (http://statweb.stanford.edu/~owen/mc/)
- Point patterns
  - Analysis and synthesis of point distributions based on pair correlation
  - Statistical Analysis and Modelling of Spatial Point Patterns
  - Digital halftoning (Ulichney et al. 1987)
- Blue noise distributions
  - A comparison of methods for generating Poisson disk distributions
  - Blue noise sampling with controlled aliasing
  - Anisotropic blue noise sampling