Решения задач

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Задача 22

$$\int \frac{dx}{\cos^4 x} = \int \frac{\cos^2 x + \sin^2 x}{\cos^4 x}\,dx = \int \frac{dx}{\cos^2 x} + \int \frac{\operatorname{tg}^2 x}{\cos^2 x}\,dx = \operatorname{tg} x + \int \frac{\operatorname{tg}^2 x}{\cos^2 x}\,dx = \begin{bmatrix} t = \operatorname{tg} x \\ dt = \dfrac{dx}{\cos^2 x} \end{bmatrix} =$$
$$= \operatorname{tg} x + \int t^2\,dt = \operatorname{tg} x + \frac{t^3}{3} + C = \operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + C$$

Ответ: $\operatorname{tg} x + \dfrac{\operatorname{tg}^3 x}{3} + C$

Задача 27

$$\int \sin^3 x \cos^2 x\,dx = \int \sin^3 x\,dx - \int \sin^5 x\,dx$$

$$\int \sin^5 x\,dx = \begin{bmatrix} u = \sin^4 x, & du = 4\sin^3 x \cos x\,dx \\ dv = \sin x\,dx, & v = -\cos x \end{bmatrix} = uv - \int v\,du = -\sin^4 x \cos x + 4\int \sin^3 x \cos^2 x\,dx$$

$$\int \sin^3 x \cos^2 x\,dx = \int \sin^3 x\,dx + \sin^4 x \cos x - 4\int \sin^3 x \cos^2 x\,dx$$

$$5\int \sin^3 x \cos^2 x\,dx = \int \sin^3 x\,dx + \sin^4 x \cos x; \qquad \int \sin^3 x \cos^2 x\,dx = \frac{\int \sin^3 x\,dx + \sin^4 x \cos x}{5}$$

$$\int \sin^3 x\,dx = \begin{bmatrix} u = \sin^2 x, & du = 2\sin x \cos x\,dx \\ dv = \sin x\,dx, & v = -\cos x \end{bmatrix} = uv - \int v\,du = -\sin^2 x \cos x + 2\int \sin x \cos^2 x\,dx =$$
$$= -\sin^2 x \cos x + 2\int \sin x\,dx - 2\int \sin^3 x\,dx = -\sin^2 x \cos x - 2\cos x - 2\int \sin^3 x\,dx$$

$$3\int \sin^3 x\,dx = -\sin^2 x \cos x - 2\cos x; \qquad \int \sin^3 x\,dx = -\frac{\sin^2 x \cos x + 2\cos x}{3}$$

$$\int \sin^3 x \cos^2 x\,dx = \frac{-\dfrac{\sin^2 x \cos x + 2\cos x}{3} + \sin^4 x \cos x}{5} = \cos x \cdot \frac{3\sin^4 x - \sin^2 x - 2}{15} + C$$

Ответ: $\cos x \cdot \dfrac{3\sin^4 x - \sin^2 x - 2}{15} + C$

Задача 28

$$\int \frac{dx}{\operatorname{tg}^8 x} = \int \frac{\cos^8 x}{\sin^8 x}\,dx = \int \frac{(1 - sin^2 x)^4}{\sin^8 x}\,dx = \int \frac{sin^8 x - 4sin^6 x + 6sin^4 x - 4sin^2 x + 1}{\sin^8 x}\,dx = \int dx - 4\int \frac{dx}{\sin^2 x} +$$
$$+6\int \frac{dx}{\sin^4 x} - 4\int \frac{dx}{\sin^6 x} + \int \frac{dx}{\sin^8 x}$$

$$\int \frac{dx}{\sin^{2n} x} = \begin{bmatrix} u = \dfrac{1}{\sin^{2n-2} x}, \; du = \dfrac{-(2n-3)\sin^{2n-3} x \cos x}{\sin^{4n-4} x} dx = -(2n-3)\dfrac{\cos x}{\sin^{2n-1} x} \\ dv = \dfrac{dx}{\sin^2 x}, \quad v = \int dv = \int \dfrac{dx}{\sin^2 x} = -\operatorname{ctg} x = \dfrac{\cos x}{\sin x} \end{bmatrix} = \int u\,dv = uv - \int v\,du =$$
$$= -\frac{\cos x}{\sin^{2n-1} x} - (2n-3)\int \frac{\cos^2 x}{\sin^{2n} x} dx = -\frac{\cos x}{\sin^{2n-1} x} - (2n-3)\int \frac{dx}{\sin^{2n} x} + (2n-3)\int \frac{dx}{\sin^{2n-2} x}$$

$$(2n-2)\int \frac{dx}{\sin^{2n} x} = -\frac{\cos x}{\sin^{2n-1} x} + (2n-3)\int \frac{dx}{\sin^{2n-2} x}$$

$$\int \frac{dx}{\sin^{2n} x} = \frac{-\dfrac{\cos x}{\sin^{2n-1} x} + (2n-3)\int \dfrac{dx}{\sin^{2n-2} x}}{(2n-2)} = \frac{2n-3}{2n-2}\int \frac{dx}{\sin^{2n-2} x} - \frac{\cos x}{(2n-2)\sin^{2n-1} x}$$

$$\int \frac{dx}{\operatorname{tg}^8 x} = \int dx - 4\int \frac{dx}{\sin^2 x} + 6\int \frac{dx}{\sin^4 x} - 4\int \frac{dx}{\sin^6 x} + \int \frac{dx}{\sin^8 x} = \int dx - 4\int \frac{dx}{\sin^2 x} + 6\int \frac{dx}{\sin^4 x} -$$
$$-4\int \frac{dx}{\sin^6 x} + \frac{5}{6}\int \frac{dx}{\sin^6 x} - \frac{\cos x}{6\sin^7 x} = \int dx - 4\int \frac{dx}{\sin^2 x} + 6\int \frac{dx}{\sin^4 x} - \frac{19}{6}\int \frac{dx}{\sin^6 x} - \frac{\cos x}{6\sin^7 x} =$$
$$= \int dx - 4\int \frac{dx}{\sin^2 x} + 6\int \frac{dx}{\sin^4 x} - \frac{19}{6}(\frac{3}{4}\int \frac{dx}{\sin^4 x} - \frac{\cos x}{4\sin^5 x}) - \frac{\cos x}{6\sin^7 x} = \int dx - 4\int \frac{dx}{\sin^2 x} +$$
$$+6\int \frac{dx}{\sin^4 x} - \frac{19}{8}\int \frac{dx}{\sin^4 x} + \frac{19}{24}\frac{\cos x}{\sin^5 x} - \frac{\cos x}{6\sin^7 x} = \int dx - 4\int \frac{dx}{\sin^2 x} + \frac{29}{8}\int \frac{dx}{\sin^4 x} + \frac{19}{24}\frac{\cos x}{\sin^5 x} -$$
$$-\frac{\cos x}{6\sin^7 x} = \int dx - 4\int \frac{dx}{\sin^2 x} + \frac{29}{8}(\frac{1}{2}\int \frac{dx}{\sin^2 x} - \frac{\cos x}{2\sin^3 x}) + \frac{19}{24}\frac{\cos x}{\sin^5 x} - \frac{\cos x}{6\sin^7 x} = \int dx -$$
$$-4\int \frac{dx}{\sin^2 x} + \frac{29}{16}\int \frac{dx}{\sin^2 x} - \frac{29}{16}\frac{\cos x}{\sin^3 x} + \frac{19}{24}\frac{\cos x}{\sin^5 x} - \frac{\cos x}{6\sin^7 x} = \int dx - \frac{35}{16}\int \frac{dx}{\sin^2 x} - \frac{29}{16}\frac{\cos x}{\sin^3 x} +$$
$$+\frac{19}{24}\frac{\cos x}{\sin^5 x} - \frac{\cos x}{6\sin^7 x} = x + \frac{35}{16}\operatorname{ctg} x - \frac{29}{16}\frac{\cos x}{\sin^3 x} + \frac{19}{24}\frac{\cos x}{\sin^5 x} - \frac{\cos x}{6\sin^7 x} + C$$

Ответ: $x + \frac{35}{16}\operatorname{ctg} x - \frac{29}{16}\frac{\cos x}{\sin^3 x} + \frac{19}{24}\frac{\cos x}{\sin^5 x} - \frac{\cos x}{6\sin^7 x} + C$

Задача 29

$$\int \frac{dx}{\sin^3 x} = \begin{bmatrix} u = \frac{1}{\sin x}, \quad du = \frac{-\cos x}{\sin^2 x}dx \\ dv = \frac{dx}{\sin^2 x}, \ v = \int dv = \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x = \frac{\cos x}{\sin x} \end{bmatrix} = \int u\,dv = uv - \int v\,du = -\frac{\cos x}{\sin^2 x} -$$
$$-\int \frac{\cos^2 x}{\sin^3 x}dx = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} +$$
$$+\int \frac{1}{2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\frac{1}{\cos^2\frac{x}{2}}dx = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \int \frac{1}{2}\frac{1}{tg\frac{x}{2}}\frac{1}{\cos^2\frac{x}{2}}dx = \begin{bmatrix} t = \operatorname{tg}\frac{x}{2} \\ dt = \frac{1}{2}\frac{1}{cos^2\frac{x}{2}}dx \end{bmatrix} =$$
$$= -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} - \int \frac{dt}{t} = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \ln|t| = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \ln|\operatorname{tg}\frac{x}{2}|$$
$$2\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{\sin^2 x} + \ln|\operatorname{tg}\frac{x}{2}| + 2C$$
$$\int \frac{dx}{\sin^3 x} = \frac{1}{2}\ln|\operatorname{tg}\frac{x}{2}| - \frac{1}{2}\frac{\cos x}{\sin^2 x} + C$$

Ответ: $\frac{1}{2}\ln|\operatorname{tg}\frac{x}{2}| - \frac{1}{2}\frac{\cos x}{\sin^2 x} + C$

Задача 30

$$\int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}}dx = \int \frac{1}{1+2\operatorname{tg}^2 x}\frac{1}{\cos^2 x}dx = \begin{bmatrix} t = \sqrt{2}\operatorname{tg} x \\ dt = \sqrt{2}\frac{1}{\cos^2 x}dx \end{bmatrix} = \frac{1}{\sqrt{2}}\int \frac{dt}{1+t^2} =$$
$$= \frac{\operatorname{arctg} t}{\sqrt{2}} + C = \frac{\operatorname{arctg}(\sqrt{2}\operatorname{tg} x)}{\sqrt{2}} + C$$

Ответ: $\frac{\operatorname{arctg}(\sqrt{2}\operatorname{tg} x)}{\sqrt{2}} + C$

Задача 31

$$\int \frac{dx}{5-4\sin x+3\cos x} = \frac{1}{5}\int \frac{dx}{1-\frac{4}{5}\sin x+\frac{3}{5}\cos x} = \frac{1}{5}\int \frac{dx}{1+\cos(\arccos(\frac{3}{5})+x)} = \begin{bmatrix} t = \arccos(\frac{3}{5})+x \\ dt = dx \end{bmatrix} =$$
$$= \frac{1}{5}\int \frac{dt}{1+\cos t} = \frac{1}{10}\int \frac{dt}{\cos^2\frac{t}{2}} = \begin{bmatrix} y = \frac{t}{2} \\ dy = \frac{1}{2}dt \end{bmatrix} = \frac{1}{5}\int \frac{dy}{\cos^2 y} = \frac{1}{5}\operatorname{tg} y + C = \frac{1}{5}\operatorname{tg}\frac{t}{2} + C =$$
$$= \frac{1}{5}\operatorname{tg}\frac{\arccos(\frac{3}{5})+x}{2} + C$$

Ответ: $\frac{1}{5}\operatorname{tg}\frac{\arccos(\frac{3}{5})+x}{2}+C$

Задача 32

$$\int(x^2+3x+5)\cos 2x\,dx=\begin{bmatrix} u=x^2+3x+5,\ du=(2x+3)\,dx \\ dv=\cos 2x\,dx \qquad v=\int dv=\int\cos 2x\,dx=\frac{1}{2}\sin 2x\end{bmatrix}=\int u\,dv=uv-\int v\,du=$$

$$=\frac{1}{2}(x^2+3x+5)\sin 2x-\frac{1}{2}\int(2x+3)\sin 2x\,dx=\begin{bmatrix} u=2x+3, \qquad du=2\,dx \\ dv=\sin 2x\,dx\ \ v=\int dv=\int\sin 2x\,dx=-\frac{1}{2}\cos 2x\end{bmatrix}=$$

$$=\frac{1}{2}(x^2+3x+5)\sin 2x+\frac{1}{4}(2x+3)\cos 2x-\frac{1}{2}\int\cos 2x\,dx=$$

$$=\frac{1}{2}(x^2+3x+5)\sin 2x+\frac{1}{4}(2x+3)\cos 2x-\frac{1}{4}\sin 2x+C$$

Ответ: $\frac{1}{2}(x^2+3x+5)\sin 2x+\frac{1}{4}(2x+3)\cos 2x-\frac{1}{4}\sin 2x+C$

Задача 33

$$\int xe^{x^{1/3}}\,dx=\begin{bmatrix} t=x^{1/3} \\ x=t^3 \\ dx=3t^2\,dt\end{bmatrix}=3\int t^5e^t\,dt=\begin{bmatrix} u=t^5, \qquad du=5t^4\,dt \\ dv=e^t\,dt\ \ v=e^t\end{bmatrix}=\int v\,du=uv-\int v\,du=$$

$$=3t^5e^t-15\int t^4e^t\,dt=3t^5e^t-15t^4e^t+60\int t^3e^t\,dt=3t^5e^t-15t^4e^t+60t^3e^t-180\int t^2e^t\,dt=$$

$$=3t^5e^t-15t^4e^t+60t^3e^t-180t^2e^t+360\int te^t\,dt=3t^5e^t-15t^4e^t+60t^3e^t-180t^2e^t+360te^t-$$

$$-360\int e^t\,dt=3t^5e^t-15t^4e^t+60t^3e^t-180t^2e^t+360te^t-360e^t+C=$$

$$=e^t(3t^5-15t^4+60t^3-180t^2+360t-360)+C=$$

$$=e^{x^{1/3}}(3x^{5/3}-15x^{4/3}+60x^{3/3}-180x^{2/3}+360x^{1/3}-360)+C$$

Ответ: $e^{x^{1/3}}(3x^{5/3}-15x^{4/3}+60x^{3/3}-180x^{2/3}+360x^{1/3}-360)+C$

Задача 34

$$\int\frac{dx}{x\sqrt{2+x-x^2}}=\int\frac{dx}{x\sqrt{(2-x)(1+x)}}=\begin{bmatrix} t=\sqrt{\frac{x+1}{2-x}},\ x=\frac{2t^2-1}{t^2+1} \\ dx=\frac{4t(t^2+1)-2t(2t^2-1)}{(t^2+1)^2}\,dt=\frac{6t}{(t^2+1)^2}\,dt\end{bmatrix}=$$

$$=\int\frac{\frac{6t}{(t^2+1)^2}}{\frac{2t^2-1}{t^2+1}t(2-\frac{2t^2-1}{t^2+1})}\,dt=2\int\frac{dt}{(2t^2-1)}=2\int\frac{dt}{(\sqrt{2}t-1)(\sqrt{2}t+1)}=$$

$$=2\int\frac{dt}{\sqrt{2}t-1}-2\int\frac{dt}{\sqrt{2}t+1}=\begin{bmatrix} y=\sqrt{2}t-1 & z=\sqrt{2}t+1 \\ dy=\sqrt{2}\,dt & dz=\sqrt{2}\,dt\end{bmatrix}=$$

$$=\frac{1}{\sqrt{2}}\int\frac{dy}{y}-\frac{1}{\sqrt{2}}\int\frac{dz}{z}=\frac{1}{\sqrt{2}}(\ln|y|-\ln|z|)+C=\frac{1}{\sqrt{2}}(\ln|\sqrt{2}t-1|-\ln|\sqrt{2}t+1|)+C=$$

$$=\frac{1}{\sqrt{2}}(\ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}}-1|-\ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}}+1|)+C$$

Ответ: $\frac{1}{\sqrt{2}}(\ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}}-1|-\ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}}+1|)+C$

Задача 35

$$\int\sin^8 x\,dx$$

$$\int\sin^{2n}x\,dx=\begin{bmatrix} u=\sin^{2n-1}x,\ du=(2n-1)\sin^{2n-2}xcosx\,dx \\ dv=\sin xdx,\quad v=\int dv=\int\sin x\,dx=-\cos x\end{bmatrix}=\int u\,dv=uv-\int v\,du=-\cos x\sin^{2n-1}x+$$

$$+(2n-1)\int\sin^{2n-2}x\cos^2 x\,dx=-\cos x\sin^{2n-1}x-(2n-1)\int\sin^{2n-2}x(1-\sin^2 x)\,dx=-\cos x\sin^{2n-1}x+$$

$$+(2n-1)\int\sin^{2n-2}x\,dx-(2n-1)\int\sin^{2n}x\,dx$$

$$2n\int \sin^{2n}x\,dx = -\cos x\sin^{2n-1}x + (2n-1)\int \sin^{2n-2}x\,dx$$

$$\int \sin^{2n}x\,dx = -\frac{1}{2n}\cos x\sin^{2n-1}x + \frac{(2n-1)}{2n}\int \sin^{2n-2}x\,dx$$

$$\int \sin^{8}x\,dx = -\frac{1}{8}\cos x\sin^{7}x + \frac{7}{8}\int \sin^{6}x\,dx = -\frac{1}{8}\cos x\sin^{7}x - \frac{7}{48}\cos x\sin^{5}x + \frac{35}{48}\int \sin^{4}x\,dx = -\frac{1}{8}\cos x\sin^{7}x -$$
$$-\frac{7}{48}\cos x\sin^{5}x - \frac{35}{192}\cos x\sin^{3}x + \frac{105}{192}\int \sin^{2}x\,dx = -\frac{1}{8}\cos x\sin^{7}x - \frac{7}{48}\cos x\sin^{5}x - \frac{35}{192}\cos x\sin^{3}x -$$
$$-\frac{105}{384}\cos x\sin x + \frac{105}{384}\int dx = -\frac{1}{8}\cos x\sin^{7}x - \frac{7}{48}\cos x\sin^{5}x - \frac{35}{192}\cos x\sin^{3}x - \frac{105}{384}\cos x\sin x + \frac{105}{384}x + \frac{105}{384}x + C$$

Ответ: $-\frac{1}{8}\cos x\sin^{7}x - \frac{7}{48}\cos x\sin^{5}x - \frac{35}{192}\cos x\sin^{3}x - \frac{105}{384}\cos x\sin x + \frac{105}{384}x + C$

Задача 36

$$\int \frac{dx}{\sqrt{x}(x-1)} = \begin{bmatrix} t=\sqrt{x} \\ x=t^2 \\ dx = 2t\,dt \end{bmatrix} = \int \frac{2t}{t(t^2-1)}\,dt = 2\int \frac{dt}{(t-1)(t+1)} = \int \frac{dt}{t-1} - \int \frac{dt}{t+1} = \begin{bmatrix} y=t-1 & z=t+1 \\ dy=dt & dz=dt \end{bmatrix} =$$
$$= \int \frac{dy}{y} - \int \frac{dz}{z} = \ln|y| - \ln|z| + C = \ln|\frac{y}{z}| + C = \ln|\frac{t+1}{t-1}| + C = \ln|\frac{\sqrt{x}-1}{\sqrt{x}+1}| + C$$

Ответ: $\ln|\frac{\sqrt{x}-1}{\sqrt{x}+1}| + C$