

Решения задач

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10 апреля 2025 г.

Задача 22

$$\begin{aligned}\int \frac{dx}{\cos^4 x} &= \int \frac{\cos^2 x + \sin^2 x}{\cos^4 x} dx = \int \frac{dx}{\cos^2 x} + \int \frac{\operatorname{tg}^2 x}{\cos^2 x} dx = \operatorname{tg} x + \int \frac{\operatorname{tg}^2 x}{\cos^2 x} dx = \left[\begin{array}{l} t = \operatorname{tg} x \\ dt = \frac{dx}{\cos^2 x} \end{array} \right] = \\ &= \operatorname{tg} x + \int t^2 dt = \operatorname{tg} x + \frac{t^3}{3} + C = \operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + C\end{aligned}$$

Ответ: $\operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + C$

Задача 27

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^3 x dx - \int \sin^5 x dx \\ \int \sin^5 x dx &= \left[\begin{array}{l} u = \sin^4 x, \quad du = 4 \sin^3 x \cos x dx \\ dv = \sin x dx, \quad v = -\cos x \end{array} \right] = uv - \int v du = -\sin^4 x \cos x + 4 \int \sin^3 x \cos^2 x dx \\ \int \sin^3 x \cos^2 x dx &= \int \sin^3 x dx + \sin^4 x \cos x - 4 \int \sin^3 x \cos^2 x dx \\ 5 \int \sin^3 x \cos^2 x dx &= \int \sin^3 x dx + \sin^4 x \cos x; \quad \int \sin^3 x \cos^2 x dx = \frac{\int \sin^3 x dx + \sin^4 x \cos x}{5} \\ \int \sin^3 x dx &= \left[\begin{array}{l} u = \sin^2 x, \quad du = 2 \sin x \cos x dx \\ dv = \sin x dx, \quad v = -\cos x \end{array} \right] = uv - \int v du = -\sin^2 x \cos x + 2 \int \sin x \cos^2 x dx = \\ &= -\sin^2 x \cos x + 2 \int \sin x dx - 2 \int \sin^3 x dx = -\sin^2 x \cos x - 2 \cos x - 2 \int \sin^3 x dx \\ 3 \int \sin^3 x dx &= -\sin^2 x \cos x - 2 \cos x; \quad \int \sin^3 x dx = -\frac{\sin^2 x \cos x + 2 \cos x}{3} \\ \int \sin^3 x \cos^2 x dx &= \frac{-\frac{\sin^2 x \cos x + 2 \cos x}{3} + \sin^4 x \cos x}{5} = \cos x \cdot \frac{3 \sin^4 x - \sin^2 x - 2}{15} + C\end{aligned}$$

Ответ: $\cos x \cdot \frac{3 \sin^4 x - \sin^2 x - 2}{15} + C$

Задача 28

$$\begin{aligned}\int \frac{dx}{\operatorname{tg}^8 x} &= \int \frac{\cos^8 x}{\sin^8 x} dx = \int \frac{(1 - \sin^2 x)^4}{\sin^8 x} dx = \int \frac{\sin^8 x - 4 \sin^6 x + 6 \sin^4 x - 4 \sin^2 x + 1}{\sin^8 x} dx = \int dx - 4 \int \frac{dx}{\sin^2 x} + \\ &\quad + 6 \int \frac{dx}{\sin^4 x} - 4 \int \frac{dx}{\sin^6 x} + \int \frac{dx}{\sin^8 x} \\ \int \frac{dx}{\sin^{2n} x} &= \left[\begin{array}{l} u = \frac{1}{\sin^{2n-2} x}, \quad du = \frac{-(2n-3) \sin^{2n-3} x \cos x}{\sin^{4n-4} x} dx = -(2n-3) \frac{\cos x}{\sin^{2n-1} x} \\ dv = \frac{dx}{\sin^2 x}, \quad v = \int dv = \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x = \frac{\cos x}{\sin x} \end{array} \right] = \int u dv = uv - \int v du = \\ &= -\frac{\cos x}{\sin^{2n-1} x} - (2n-3) \int \frac{\cos^2 x}{\sin^{2n} x} dx = -\frac{\cos x}{\sin^{2n-1} x} - (2n-3) \int \frac{dx}{\sin^{2n} x} + (2n-3) \int \frac{dx}{\sin^{2n-2} x} \\ (2n-2) \int \frac{dx}{\sin^{2n} x} &= -\frac{\cos x}{\sin^{2n-1} x} + (2n-3) \int \frac{dx}{\sin^{2n-2} x} \\ \int \frac{dx}{\sin^{2n} x} &= \frac{-\frac{\cos x}{\sin^{2n-1} x} + (2n-3) \int \frac{dx}{\sin^{2n-2} x}}{(2n-2)} = \frac{2n-3}{2n-2} \int \frac{dx}{\sin^{2n-2} x} - \frac{\cos x}{(2n-2) \sin^{2n-1} x}\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{\operatorname{tg}^8 x} &= \int dx - 4 \int \frac{dx}{\sin^2 x} + 6 \int \frac{dx}{\sin^4 x} - 4 \int \frac{dx}{\sin^6 x} + \int \frac{dx}{\sin^8 x} = \int dx - 4 \int \frac{dx}{\sin^2 x} + 6 \int \frac{dx}{\sin^4 x} - \\
&- 4 \int \frac{dx}{\sin^6 x} + \frac{5}{6} \int \frac{dx}{\sin^6 x} - \frac{\cos x}{6 \sin^7 x} = \int dx - 4 \int \frac{dx}{\sin^2 x} + 6 \int \frac{dx}{\sin^4 x} - \frac{19}{6} \int \frac{dx}{\sin^6 x} - \frac{\cos x}{6 \sin^7 x} = \\
&= \int dx - 4 \int \frac{dx}{\sin^2 x} + 6 \int \frac{dx}{\sin^4 x} - \frac{19}{6} \left(\frac{3}{4} \int \frac{dx}{\sin^4 x} - \frac{\cos x}{4 \sin^5 x} \right) - \frac{\cos x}{6 \sin^7 x} = \int dx - 4 \int \frac{dx}{\sin^2 x} + \\
&+ 6 \int \frac{dx}{\sin^4 x} - \frac{19}{8} \int \frac{dx}{\sin^4 x} + \frac{19 \cos x}{24 \sin^5 x} - \frac{\cos x}{6 \sin^7 x} = \int dx - 4 \int \frac{dx}{\sin^2 x} + \frac{29}{8} \int \frac{dx}{\sin^4 x} + \frac{19 \cos x}{24 \sin^5 x} - \\
&- \frac{\cos x}{6 \sin^7 x} = \int dx - 4 \int \frac{dx}{\sin^2 x} + \frac{29}{8} \left(\frac{1}{2} \int \frac{dx}{\sin^2 x} - \frac{\cos x}{2 \sin^3 x} \right) + \frac{19 \cos x}{24 \sin^5 x} - \frac{\cos x}{6 \sin^7 x} = \int dx - \\
&- 4 \int \frac{dx}{\sin^2 x} + \frac{29}{16} \int \frac{dx}{\sin^2 x} - \frac{29 \cos x}{16 \sin^3 x} + \frac{19 \cos x}{24 \sin^5 x} - \frac{\cos x}{6 \sin^7 x} = \int dx - \frac{35}{16} \int \frac{dx}{\sin^2 x} - \frac{29 \cos x}{16 \sin^3 x} + \\
&+ \frac{19 \cos x}{24 \sin^5 x} - \frac{\cos x}{6 \sin^7 x} = x + \frac{35}{16} \operatorname{ctg} x - \frac{29 \cos x}{16 \sin^3 x} + \frac{19 \cos x}{24 \sin^5 x} - \frac{\cos x}{6 \sin^7 x} + C
\end{aligned}$$

Ответ: $x + \frac{35}{16} \operatorname{ctg} x - \frac{29 \cos x}{16 \sin^3 x} + \frac{19 \cos x}{24 \sin^5 x} - \frac{\cos x}{6 \sin^7 x} + C$

Задача 29

$$\begin{aligned}
\int \frac{dx}{\sin^3 x} &= \left[u = \frac{1}{\sin x}, \quad du = \frac{-\cos x}{\sin^2 x} dx \right. \\
&\left. dv = \frac{dx}{\sin^2 x}, \quad v = \int dv = \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x = \frac{\cos x}{\sin x} \right] = \int u dv = uv - \int v du = -\frac{\cos x}{\sin^2 x} - \\
&- \int \frac{\cos^2 x}{\sin^3 x} dx = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \\
&+ \int \frac{1}{2} \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \frac{1}{\cos^2 \frac{x}{2}} dx = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \int \frac{1}{2} \frac{1}{\operatorname{tg} \frac{x}{2}} \frac{1}{\cos^2 \frac{x}{2}} dx = \left[t = \operatorname{tg} \frac{x}{2} \right. \\
&\left. dt = \frac{1}{2} \frac{1}{\cos^2 \frac{x}{2}} dx \right] = \\
&= -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} - \int \frac{dt}{t} = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \ln |t| = -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right| \\
2 \int \frac{dx}{\sin^3 x} &= -\frac{\cos x}{\sin^2 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right| + 2C \\
\int \frac{dx}{\sin^3 x} &= \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right| - \frac{1}{2} \frac{\cos x}{\sin^2 x} + C
\end{aligned}$$

Ответ: $\frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right| - \frac{1}{2} \frac{\cos x}{\sin^2 x} + C$

Задача 30

$$\begin{aligned}
\int \frac{dx}{1 + \sin^2 x} &= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}} dx = \int \frac{1}{1 + 2 \operatorname{tg}^2 x} \frac{1}{\cos^2 x} dx = \left[t = \sqrt{2} \operatorname{tg} x \right. \\
&\left. dt = \sqrt{2} \frac{1}{\cos^2 x} dx \right] = \frac{1}{\sqrt{2}} \int \frac{dt}{1 + t^2} = \\
&= \frac{\operatorname{arctg} t}{\sqrt{2}} + C = \frac{\operatorname{arctg}(\sqrt{2} \operatorname{tg} x)}{\sqrt{2}} + C
\end{aligned}$$

Ответ: $\frac{\operatorname{arctg}(\sqrt{2} \operatorname{tg} x)}{\sqrt{2}} + C$

Задача 31

$$\begin{aligned}
\int \frac{dx}{5 - 4 \sin x + 3 \cos x} &= \frac{1}{5} \int \frac{dx}{1 - \frac{4}{5} \sin x + \frac{3}{5} \cos x} = \frac{1}{5} \int \frac{dx}{1 + \cos(\operatorname{arccos}(\frac{3}{5}) + x)} = \left[t = \operatorname{arccos}(\frac{3}{5}) + x \right. \\
&\left. dt = dx \right] = \\
&= \frac{1}{5} \int \frac{dt}{1 + \cos t} = \frac{1}{10} \int \frac{dt}{\cos^2 \frac{t}{2}} = \left[y = \frac{t}{2} \right. \\
&\left. dy = \frac{1}{2} dt \right] = \frac{1}{5} \int \frac{dy}{\cos^2 y} = \frac{1}{5} \operatorname{tg} y + C = \frac{1}{5} \operatorname{tg} \frac{t}{2} + C = \\
&= \frac{1}{5} \operatorname{tg} \frac{\operatorname{arccos}(\frac{3}{5}) + x}{2} + C
\end{aligned}$$

Ответ: $\frac{1}{5} \operatorname{tg} \frac{\arccos(\frac{3}{5}) + x}{2} + C$

Задача 32

$$\begin{aligned} \int (x^2 + 3x + 5) \cos 2x \, dx &= \left[\begin{array}{l} u = x^2 + 3x + 5, \, du = (2x + 3) \, dx \\ dv = \cos 2x \, dx \quad v = \int dv = \int \cos 2x \, dx = \frac{1}{2} \sin 2x \end{array} \right] = \int u \, dv = uv - \int v \, du = \\ &= \frac{1}{2} (x^2 + 3x + 5) \sin 2x - \frac{1}{2} \int (2x + 3) \sin 2x \, dx = \left[\begin{array}{l} u = 2x + 3, \quad du = 2 \, dx \\ dv = \sin 2x \, dx \quad v = \int dv = \int \sin 2x \, dx = -\frac{1}{2} \cos 2x \end{array} \right] = \\ &= \frac{1}{2} (x^2 + 3x + 5) \sin 2x + \frac{1}{4} (2x + 3) \cos 2x - \frac{1}{2} \int \cos 2x \, dx = \\ &= \frac{1}{2} (x^2 + 3x + 5) \sin 2x + \frac{1}{4} (2x + 3) \cos 2x - \frac{1}{4} \sin 2x + C \end{aligned}$$

Ответ: $\frac{1}{2} (x^2 + 3x + 5) \sin 2x + \frac{1}{4} (2x + 3) \cos 2x - \frac{1}{4} \sin 2x + C$

Задача 33

$$\begin{aligned} \int x e^{x^{1/3}} \, dx &= \left[\begin{array}{l} t = x^{1/3} \\ x = t^3 \\ dx = 3t^2 \, dt \end{array} \right] = 3 \int t^5 e^t \, dt = \left[\begin{array}{l} u = t^5, \quad du = 5t^4 \, dt \\ dv = e^t \, dt \quad v = e^t \end{array} \right] = \int v \, du = uv - \int v \, du = \\ &= 3t^5 e^t - 15 \int t^4 e^t \, dt = 3t^5 e^t - 15t^4 e^t + 60 \int t^3 e^t \, dt = 3t^5 e^t - 15t^4 e^t + 60t^3 e^t - 180 \int t^2 e^t \, dt = \\ &= 3t^5 e^t - 15t^4 e^t + 60t^3 e^t - 180t^2 e^t + 360 \int t e^t \, dt = 3t^5 e^t - 15t^4 e^t + 60t^3 e^t - 180t^2 e^t + 360t e^t - \\ &\quad - 360 \int e^t \, dt = 3t^5 e^t - 15t^4 e^t + 60t^3 e^t - 180t^2 e^t + 360t e^t - 360e^t + C = \\ &= e^t (3t^5 - 15t^4 + 60t^3 - 180t^2 + 360t - 360) + C = \\ &= e^{x^{1/3}} (3x^{5/3} - 15x^{4/3} + 60x^{3/3} - 180x^{2/3} + 360x^{1/3} - 360) + C \end{aligned}$$

Ответ: $e^{x^{1/3}} (3x^{5/3} - 15x^{4/3} + 60x^{3/3} - 180x^{2/3} + 360x^{1/3} - 360) + C$

Задача 34

$$\begin{aligned} \int \frac{dx}{x\sqrt{2+x-x^2}} &= \int \frac{dx}{x\sqrt{(2-x)(1+x)}} = \left[\begin{array}{l} t = \sqrt{\frac{x+1}{2-x}}, \quad x = \frac{2t^2-1}{t^2+1} \\ dx = \frac{4t(t^2+1) - 2t(2t^2-1)}{(t^2+1)^2} \, dt = \frac{6t}{(t^2+1)^2} \, dt \end{array} \right] = \\ &= \int \frac{\frac{6t}{(t^2+1)^2}}{\frac{2t^2-1}{t^2+1} t (2 - \frac{2t^2-1}{t^2+1})} \, dt = 2 \int \frac{dt}{(2t^2-1)} = 2 \int \frac{dt}{(\sqrt{2}t-1)(\sqrt{2}t+1)} = \\ &= 2 \int \frac{dt}{\sqrt{2}t-1} - 2 \int \frac{dt}{\sqrt{2}t+1} = \left[\begin{array}{l} y = \sqrt{2}t-1 \quad z = \sqrt{2}t+1 \\ dy = \sqrt{2} \, dt \quad dz = \sqrt{2} \, dt \end{array} \right] = \\ &= \frac{1}{\sqrt{2}} \int \frac{dy}{y} - \frac{1}{\sqrt{2}} \int \frac{dz}{z} = \frac{1}{\sqrt{2}} (\ln|y| - \ln|z|) + C = \frac{1}{\sqrt{2}} (\ln|\sqrt{2}t-1| - \ln|\sqrt{2}t+1|) + C = \\ &= \frac{1}{\sqrt{2}} (\ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}} - 1| - \ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}} + 1|) + C \end{aligned}$$

Ответ: $\frac{1}{\sqrt{2}} (\ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}} - 1| - \ln|\sqrt{2}\sqrt{\frac{x+1}{2-x}} + 1|) + C$

Задача 35

$$\begin{aligned} \int \sin^8 x \, dx &= \int \sin^2 x \sin^6 x \, dx = \int \sin^2 x (\sin^2 x)^3 \, dx = \int \sin^2 x (1 - \cos^2 x)^3 \, dx = \int \sin^2 x (1 - 3\cos^2 x + 3\cos^4 x - \cos^6 x) \, dx = \\ &= \int \sin^2 x \, dx - 3 \int \sin^2 x \cos^2 x \, dx + 3 \int \sin^2 x \cos^4 x \, dx - \int \sin^2 x \cos^6 x \, dx = \int \sin^2 x \, dx - 3 \int \sin x \cos x \, dx + \\ &\quad + (2n-1) \int \sin^{2n-2} x \cos^2 x \, dx = -\cos x \sin^{2n-1} x - (2n-1) \int \sin^{2n-2} x (1 - \sin^2 x) \, dx = -\cos x \sin^{2n-1} x + \\ &\quad + (2n-1) \int \sin^{2n-2} x \, dx - (2n-1) \int \sin^{2n} x \, dx \end{aligned}$$

$$\begin{aligned}
2n \int \sin^{2n} x \, dx &= -\cos x \sin^{2n-1} x + (2n-1) \int \sin^{2n-2} x \, dx \\
\int \sin^{2n} x \, dx &= -\frac{1}{2n} \cos x \sin^{2n-1} x + \frac{(2n-1)}{2n} \int \sin^{2n-2} x \, dx \\
\int \sin^8 x \, dx &= -\frac{1}{8} \cos x \sin^7 x + \frac{7}{8} \int \sin^6 x \, dx = -\frac{1}{8} \cos x \sin^7 x - \frac{7}{48} \cos x \sin^5 x + \frac{35}{48} \int \sin^4 x \, dx = -\frac{1}{8} \cos x \sin^7 x - \\
&\quad -\frac{7}{48} \cos x \sin^5 x - \frac{35}{192} \cos x \sin^3 x + \frac{105}{192} \int \sin^2 x \, dx = -\frac{1}{8} \cos x \sin^7 x - \frac{7}{48} \cos x \sin^5 x - \frac{35}{192} \cos x \sin^3 x - \\
&\quad -\frac{105}{384} \cos x \sin x + \frac{105}{384} \int dx = -\frac{1}{8} \cos x \sin^7 x - \frac{7}{48} \cos x \sin^5 x - \frac{35}{192} \cos x \sin^3 x - \frac{105}{384} \cos x \sin x + \frac{105}{384} x + \frac{105}{384} x + C \\
\text{Ответ: } &-\frac{1}{8} \cos x \sin^7 x - \frac{7}{48} \cos x \sin^5 x - \frac{35}{192} \cos x \sin^3 x - \frac{105}{384} \cos x \sin x + \frac{105}{384} x + C
\end{aligned}$$

Задача 36

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}(x-1)} &= \left[\begin{array}{l} t = \sqrt{x} \\ x = t^2 \\ dx = 2t \, dt \end{array} \right] = \int \frac{2t}{t(t^2-1)} \, dt = 2 \int \frac{dt}{(t-1)(t+1)} = \int \frac{dt}{t-1} - \int \frac{dt}{t+1} = \left[\begin{array}{l} y = t-1 \\ dy = dt \end{array} \right] \left[\begin{array}{l} z = t+1 \\ dz = dt \end{array} \right] = \\
&= \int \frac{dy}{y} - \int \frac{dz}{z} = \ln |y| - \ln |z| + C = \ln \left| \frac{y}{z} \right| + C = \ln \left| \frac{t-1}{t+1} \right| + C = \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C
\end{aligned}$$

Ответ: $\ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C$