A new example 27th June 2024

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$ (It is controllable and observable)

$$G(s) = C_0 (sI - A)^{-1} B_0 = \begin{bmatrix} \dfrac{2s+3}{(s+1)(s+2)} & -\dfrac{1}{s+2} \\ \dfrac{2}{s+2} & -\dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\det G(s) = \frac{1}{(s+1)^2 (s+2)}$$

pole polynomial, $p_G(s) = (s+1)^2 (s+2)$

pole minimal polynomial, $p_G'(s) = (s+1)(s+2)$

$$= s^2 + 3s + 2 \equiv s^2 + d_1 s + d_2$$

Rewrite $G(s)$ as $G(s) = \dfrac{N(s)}{p_G'(s)}$ where $N(s)$ is a poly. matrix

$$G(s) = \frac{N_1 s + N_2}{s^2 + d_1 s + d_2} \; ; \; N_1 = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, \; N_2 = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$d_1 = 3, \; d_2 = 2$$

The block observer form is

$$A_1 = \begin{bmatrix} -d_1 I_2 & I_2 \\ -d_2 I_2 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} I_2 & 0 \end{bmatrix}$$

$$-\frac{1}{s+2} = -\frac{s+1}{(s+1)(s+2)} = \frac{-s-1}{=}$$

$$\frac{2}{s+2} = \frac{2(s+1)}{(s+1)(s+2)} = \frac{2s+2}{=}$$

$$G(s) = \frac{N(s)}{p'_G(s)} = \begin{bmatrix} \frac{2s+3}{(s+1)(s+2)} & -\frac{s+1}{(s+1)(s+2)} \\ \frac{2(s+1)}{(s+1)(s+2)} & -\frac{s}{(s+1)(s+2)} \end{bmatrix}$$

Now, construct a MFD.

$$G(s) = \left(p'_G(s)\, I_2\right)^{-1} N(s) = \begin{bmatrix} p'_G & 0 \\ 0 & p'_G \end{bmatrix}^{-1} N(s)$$

$$= \begin{bmatrix} \frac{1}{p'_G(s)} & 0 \\ 0 & \frac{1}{p'_G(s)} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} \frac{n_{11}}{p'_G} & \frac{n_{12}}{p'_G} \\ \frac{n_{21}}{p'_G} & \frac{n_{22}}{p'_G} \end{bmatrix}$$

ok!

$$D_L = \begin{bmatrix} p'_G & 0 \\ 0 & p'_G \end{bmatrix} \qquad N_L = N$$

$$G(s) = D_L(s)^{-1} N_L(s)$$

$$N_L(s) = p'_G(s)\, G(s)$$

The obtained left MFD of $G(s)$ is not coprime. We use the Hermite form to obtain a left coprime MFD.

We use another approach: First build a right MFD, then use the Hermite form construction to get a left-coprime MFD.

Construct a right MFD of $G(s)$:

$$G(s) = \frac{N(s)}{p'_G(s)} = \begin{bmatrix} \dfrac{2s+3}{(s+1)(s+2)} & -\dfrac{s+1}{(s+1)(s+2)} \\[2ex] \dfrac{2(s+1)}{(s+1)(s+2)} & -\dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$G(s) = N(s)\left(p'_G(s)\, I_2\right)^{-1} = N(s)\begin{bmatrix} p'_G & 0 \\ 0 & p'_G \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}\begin{bmatrix} \dfrac{1}{p'_G} & 0 \\ 0 & \dfrac{1}{p'_G} \end{bmatrix} = \begin{bmatrix} \dfrac{n_{11}}{p'_G} & \dfrac{n_{12}}{p'_G} \\[2ex] \dfrac{n_{21}}{p'_G} & \dfrac{n_{22}}{p'_G} \end{bmatrix}$$

$$\underset{p\times m}{G(s)} = \underset{p\times m}{N_R(s)}\ \underset{m\times m}{D_R^{-1}(s)}$$

$$N_R(s) = p'_G(s)\, G(s), \quad D_R(s) = p'_G(s)\, I_m$$

$$\begin{array}{c} \\ m \\ p \end{array}\begin{array}{c} m \qquad p \\ \begin{bmatrix} & & \\ & U(s) & \\ & & \end{bmatrix} \end{array}\begin{array}{c} m \\ \begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix} \end{array} = \begin{array}{c} m \\ \begin{bmatrix} R(s) \\ \hline 0 \end{bmatrix} \end{array} m.$$

$$R(s) = \begin{bmatrix} \circ & \cdots & \\ & \ddots & \\ 0 & & \circ \end{bmatrix}$$

$$D_L \Rightarrow \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

Recapitulation:

$$G(s) = \begin{bmatrix} \dfrac{2s+3}{(s+1)(s+2)} & -\dfrac{1}{s+2} \\ \dfrac{2}{s+2} & -\dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

pole polynomial, $P_G(s) = (s+1)^2(s+2)$

pole minimal polynomial, $P'_G(s) = (s+1)(s+2)$

Construct a right MFD of $G(s)$:

$$N_R(s) = P'_G(s)\, G(s) = \begin{bmatrix} 2s+3 & -s-1 \\ 2s+2 & -s \end{bmatrix}$$

$$D_R(s) = P'_G(s)\, I_m = \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+1)(s+2) \end{bmatrix}$$

$$G(s) = N_R(s)\, D_R^{-1}(s)$$

Apply the Hermite form procedure:

$$U(s) \begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix} =: H(s) \quad \text{Hermite form} = \begin{bmatrix} 1 & -1 \\ 0 & s+2 \\ \hline 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U(s) = \begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & -1 \\ 0 & 0 & -2s-2 & 2s+3 \\ \hline 1 & 0 & s^2+s & -s^2-2s-1 \\ 0 & 1 & 2s^2+4s+2 & -2s^2-5s-3 \end{array}\right]$$

$$G(s) = -U_{22}^{-1}(s)\, U_{21}(s) = \left[-U_{22}(s)\right]^{-1} U_{21}(s)$$

This is a left coprime MFD

$$-U_{22}(s) = \begin{bmatrix} -s^2-s & s^2+2s+1 \\ -2s^2-4s-2 & 2s^2+5s+3 \end{bmatrix}$$

$$U_{21}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$-U_{22}(s)$ is not row-reduced because the leading row coefficient matrix is singular:

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}.$$

To obtain a row-reduced form of $-U_{22}(s)$:

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -s^2-s & s^2+2s+1 \\ -2s^2-4s-2 & 2s^2+5s+3 \end{bmatrix} = \begin{bmatrix} -s^2-s & s^2+2s+1 \\ -2s-2 & s+1 \end{bmatrix}$$

the leading row coefficient matrix is $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$ (not singular)

$$P(s) := \begin{bmatrix} -s^2-s & s^2+2s+1 \\ -2s-2 & s+1 \end{bmatrix}$$

$$Q(s) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\underset{p \times m}{G(s)} = \underset{p \times p}{P^{-1}(s)} \, \underset{p \times m}{Q(s)}$$ row-reduced left-coprime MFD

Observer-form realization from $P^{-1}(s)\,Q(s)$ *

(it will be minimal because this MFD is row-reduced and (left-) coprime)

$l_i \triangleq$ degree of the i-th row of $P(s)$, $i=1,\dots,p$

$$A_0^\circ = \text{block diag}\left\{ \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}, l_i \times l_i,\ i=1,\dots,p \right\}$$

$$C_0^\circ = \text{block diag}\left\{ [1\,0 \cdots 0],\ 1\times l_i,\ i=1,\dots,p \right\}$$

$$B_0^\circ = I_n, \qquad n = \sum_{i=1}^{p} l_i \ (= \deg\det P(s))$$

The observer-form realization is

$$A_0 = A_0^\circ - P_{lr}\, P_{hr}^{-1}\, C_0^\circ, \qquad B_0 = Q_{lr}$$

$$C_0 = P_{hr}^{-1}\, C_0^\circ$$

Matrices P_{hr}, P_{lr} and Q_{lr} are determined as follows:

$$P(s) = S(s)\, P_{hr} + \Psi(s)\, P_{lr}$$

$$Q(s) = \Psi(s)\, Q_{lr}$$

$$S(s) \triangleq \text{diag}\{ s^{l_i},\ i=1,\dots,p\} = \begin{bmatrix} s^{l_1} & & 0 \\ & s^{l_2} & \\ & & \ddots \\ 0 & & & s^{l_p} \end{bmatrix}$$

$$\Psi(s) \triangleq \text{block diag}\left\{ [s^{l_i-1} \cdots 1],\ i=1,\dots,p \right\} =$$

* See Kailath 6.4.3

$$= \begin{bmatrix} s^{l_1-1}\ s^{l_1-2} \cdots 1 & & 0 \\ & \ddots & \\ 0 & & s^{l_p-1}\ s^{l_p-2} \cdots 1 \end{bmatrix}$$

Observer-form realization from $\begin{bmatrix} -s^2-s & s^2+2s+1 \\ -2s-2 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$\ell_1 = 2, \quad \ell_2 = 1 \quad (n = \ell_1 + \ell_2 = 3)$

$$A_o^o = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad C_o^o = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad B_o^o = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}}_{P_{hr} \triangleq} + \left[\begin{array}{cc|c} s & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \underbrace{\left[\begin{array}{c|c} -1 & 2 \\ 0 & 1 \\ \hline -2 & 1 \end{array}\right]}_{P_{\ell r} \triangleq}$$

$$Q(s) = \left[\begin{array}{cc|c} s & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \underbrace{\left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline -2 & 1 \end{array}\right]}_{Q_{\ell r} \triangleq}$$

$$A_o = A_o^o - P_{\ell r} P_{hr}^{-1} C_o^o = \begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B_o = Q_{\ell r} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$C_o = P_{hr}^{-1} C_o^o = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$