I. INTRODUCTION

II. NOTATIONS AND PRELIMINARIES

Notations: The real number set is denote as $\mathbb{R}$. $\|z\|$ denotes the ℓ_2-norm of z. $[z_i]_{vec}$ where $i \in \{1, 2, ..., N\}$ is defined as a column vector whose dimension is $N \times 1$ and the ith element is z_i. $\text{diag}\{k_i\}$ for $i \in \{1, 2, ..., N\}$ is a diagonal matrix whose dimension is $N \times N$ and the ith diagonal element is k_i. $\text{diag}\{a_{ij}\}$ where $i, j \in \{1, 2, ..., N\}$ gives a diagonal matrix whose dimension is $N^2 \times N^2$ and diagonal elements are $a_{11}, a_{12}, ..., a_{1N}, a_{21}, ..., a_{NN}$, successively. $\mathcal{A} = [a_{ij}]$ is a matrix whose (i, j)th entry is a_{ij}. Given that matrix Q is symmetric and real, $\lambda_{min}(Q)(\lambda_{max}(Q))$ stands for the smallest(largest) eigenvalue of Q. $max_{i\in\{1,2,...,N\}}\{l_i\}$ denotes the largest value of l_i for $i \in \{1, 2, ..., N\}$. $\mathsf{I}_{N\times N}$ is an identity matrix with its dimension being $N \times N$ and $\mathbf{1}(\mathbf{0})$ is a column vector with its entries being $1(0)$. Moreover, $\otimes$ is the Kronecker product.

Algebraic Graph Theory: A graph $\mathcal{G}$ is given by $\mathcal{G} = (\mathcal{V}, \mathcal{E}_g)$, in which $\mathcal{V} = \{1, 2, ..., N\}$, $\mathcal{E}_g \subseteq \mathcal{V} \times \mathcal{V}$ respectively are the node set and edge set. The edge $(i, j) \in \mathcal{E}_g$ indicates are the node j can receive information from node i, but not necessarily vice versa. The

III. PROBLEM STATEMENT

假设条件如下：

1. f 为二阶连续可微函数，且 ∇f 为全局 Lipschitz 的；
2. 拓扑结构为有向强连通图；
3. ∇f 为强凸函数；

4. $\nabla_{ij}^2 f_i(\mathbf{x}) = \frac{\partial^2 f_i(\mathbf{x})}{\partial x_i \partial x_j}$ 是有界的;

5. 误差 $d(t)$ 是有界的。

令 $\beta_i \geq |d_i(t)|, \dot{\varrho}_i(t) = -\alpha_i \varrho_i(t), \varrho_i > 0, \alpha_i > 0$，且

$$\xi_i(t) = \frac{s_i(t)}{|s_i(t)| + \varrho_i(t)}$$

算法设计如下:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\tau \mathbf{k}\mathbf{s} - [\beta_i \xi_i(t)]_{vec} + \mathbf{d}(t) \\ \mathbf{s} = \mathbf{v} + [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ \dot{\mathbf{y}} = -\theta\bar{\theta}(\mathcal{L} \otimes \mathbf{I}_{N\times N} + \mathcal{A}_q)\tilde{\mathbf{y}} \end{cases}$$

选取 Lyapunov 函数为

$$V(t) = \frac{1}{2}\tilde{\mathbf{x}}^T\tilde{\mathbf{x}} + \frac{1}{2}\mathbf{s}^T\mathbf{s} + \tilde{\mathbf{y}}^T P\tilde{\mathbf{y}} + \sum_{i=1}^{N} \frac{\beta_i}{\alpha_i}\varrho_i(t)$$

其中，$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$，$\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{1}_N \otimes \mathbf{x}$。

证明如下:

首先，令 $V_1(t) = \frac{1}{2}\tilde{\mathbf{x}}^T\tilde{\mathbf{x}}$，

其次，令 $V_2(t) = \frac{1}{2}\tilde{\mathbf{y}}^T P\tilde{\mathbf{y}}$，

再令 $V_3(t) = \frac{1}{2}\mathbf{s}^T\mathbf{s} + \sum_{i=1}^{N} \frac{\beta_i}{\alpha_i}\varrho_i(t)$

最后，我们得到

$$\begin{aligned}\dot{V} \leq & -m\|\tilde{\mathbf{x}}\|^2-\tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\ & -(\theta\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{y}}\|^2 \\ & +(\max_{i\in\mathcal{V}}\{l_i\}+2N\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{y}}\| \\ & +(2\sqrt{N}\|P\|+\theta L_1)\|\tilde{\mathbf{y}}\|\|\mathbf{s}\|+\|\tilde{\mathbf{x}}\|\|\mathbf{s}\| \\ & -\sum_{i=1}^{N}\frac{(\beta_i-|d_i(t)|)|s_i(t)|(|s_i(t)|+\varrho_i(t))+\beta_i\varrho_i^2(t)}{|s_i(t)|+\varrho_i(t)} \\ \leq & -\lambda_{\min}(F)\|\mathbf{E}_1\|^2-\tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\ & +(1+2\sqrt{N}\|P\|+\theta L_1)\|\mathbf{E}_1\|\|\mathbf{s}\|\end{aligned}$$

其中 $F=\begin{bmatrix} m & a \\ a & \theta\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\}\end{bmatrix}$, $\mathbf{E}=[\tilde{\mathbf{x}}^T,\tilde{\mathbf{y}}^T]$, $a=-\frac{\max_{i\in\mathcal{V}}\{l_i\}+2N\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{2}$，当 $\theta>\frac{(\max_{i\in\mathcal{V}}\{l_i\}+2N\|P\|\max_{i\in\mathcal{V}}(l_i\})^2}{4m\lambda_{\min}(Q)}+\frac{2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{\lambda_{\min}(Q)}$，则 $\lambda_{min}(F)>0$。令 $\bar{H}(\mathbf{y})=[h_{ij}]$，在 $i=j$ 时，我们有 $h_{ii}=[\nabla_{i1}^2 f_i(\mathbf{y}_i),\nabla_{i2}^2 f_i(\mathbf{y}_i),\ldots,\nabla_{iN}^2 f_i(\mathbf{y}_i)]$，$\nabla_{ij}^2 f_i(\mathbf{y}_i)=\frac{\partial^2 f_i(\mathbf{x})}{\partial x_i\partial x_i}\mid_{\mathbf{x}=\mathbf{y}_i}$，否则 $h_{ij}=\mathbf{0}_N^T$，L_1 满足 $\|\bar{H}(\mathbf{y})\|\|\bar{\boldsymbol{\theta}}(\mathcal{L}\otimes\mathbf{I}_{N\times N}+\mathcal{A}_q)\|\leq L_1$。

因此

$$\dot{V}(t)\leq-\lambda_{\min}(G)\|\mathbf{E}\|^2$$

其中 $G=\begin{bmatrix}\lambda_{\min}(F) & -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} \\ -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} & \tau\lambda_{\min}(\mathbf{k})\end{bmatrix}$，$\mathbf{E}=[\tilde{\mathbf{x}}^T,\tilde{\mathbf{y}}^T,\tilde{\mathbf{s}}^T]$，当 $\tau>\frac{(1+2\sqrt{N}\|P\|+\theta L_1)^2}{4\lambda_{\min}(F)\lambda_{\min}(\mathbf{k})}$，则 $\lambda_{min}(G)>0$。