

## I. INTRODUCTION

## II. NOTATIONS AND PRELIMINARIES

*Notations:* The real number set is denote as  $\mathbb{R}$ .  $\|z\|$  denotes the  $\ell_2$ -norm of  $z$ .  $[z_i]_{vec}$  where  $i \in \{1, 2, \dots, N\}$  is defined as a column vector whose dimension is  $N \times 1$  and the  $i$ th element is  $z_i$ .  $\text{diag}\{k_i\}$  for  $i \in \{1, 2, \dots, N\}$  is a diagonal matrix whose dimension is  $N \times N$  and the  $i$ th diagonal element is  $k_i$ .  $\text{diag}\{a_{ij}\}$  where  $i, j \in \{1, 2, \dots, N\}$  gives a diagonal matrix whose dimension is  $N^2 \times N^2$  and diagonal elements are  $a_{11}, a_{12}, \dots, a_{1N}, a_{21}, \dots, a_{NN}$ , successively.  $\mathcal{A} = [a_{ij}]$  is a matrix whose  $(i, j)$ th entry is  $a_{ij}$ . Given that matrix  $Q$  is symmetric and real,  $\lambda_{min}(Q)(\lambda_{max}(Q))$  stands for the smallest(largest) eigenvalue of  $Q$ .  $\max_{i \in \{1, 2, \dots, N\}}\{l_i\}$  denotes the largest value of  $l_i$  for  $i \in \{1, 2, \dots, N\}$ .  $\mathbf{I}_{N \times N}$  is an identity matrix with its dimension being  $N \times N$  and  $\mathbf{1}(0)$  is a column vector with its entries being 1(0). Moreover,  $\otimes$  is the Kronecker product.

*Algebraic Graph Theory:* A graph  $\mathcal{G}$  is given by  $\mathcal{G} = (\mathcal{V}, \mathcal{E}_g)$ , in which  $\mathcal{V} = \{1, 2, \dots, N\}$ ,  $\mathcal{E}_g \subseteq \mathcal{V} \times \mathcal{V}$  respectively are the node set and edge set. The edge  $(i, j) \in \mathcal{E}_g$  indicates are the node  $j$  can receive information from node  $i$ , but not necessarily vice versa. The in-neighbor set of node  $i$  is given as  $\mathcal{N}_i^{in} = \{j | (j, i) \in \mathcal{E}_g\}$ . A directed path is a sequence of edges of the form  $(i_1, i_2), (i_2, i_3), \dots$ . A directed graph is strongly connected if for every pair of two distinct nodes, there is a path. Let  $\mathcal{A} = [a_{ij}]$  be the adjacency matrix in which  $a_{ij} > 0$  if  $(i, j) \in \mathcal{E}_g$  and  $a_{ij} = 0$  otherwise. The Laplacian matrix  $\mathcal{L}$  is defined as  $\mathcal{L} = \mathcal{D} - \mathcal{A}$ , where  $\mathcal{D} = \text{diag}\{d_i\}$  and  $d_i = \sum_{j=1}^N a_{ij}$ .

### III. PROBLEM STATEMENT

In the concerned game,  $N$  players with labels from 1 to  $N$  are engaged and each player  $i$  has a local objective function  $f_i(\mathbf{x}) : \mathbb{R}^N \rightarrow \mathbb{R}$ , in which  $\mathbf{x} = [x_1, x_2, \dots, x_N]^T$  and  $x_i \in \mathbb{R}$  is the action of player  $i$ . Moreover, for second-order players, player  $i$ 's action is governed by

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i + d_i(t) \end{cases} \quad (1)$$

where  $v_i(t)$  is the velocity-like state of player  $i$ ,  $u_i$  is the control input and  $d_i(t)$  is the disturbance for  $i \in \mathcal{V}$ . Each player  $i$  aims to minimize its own objective function  $f_i(\mathbf{x})$  by adjusting its action  $x_i(t)$ , that is

$$\begin{aligned} \min_{x_i} \quad & f_i(\mathbf{x}) \\ \text{s.t.} \quad & (1) \text{ for second-order players} \end{aligned} \quad (2)$$

*Definition 1:* An action profile  $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$  is a Nash equilibrium if for all  $i \in \mathcal{V}$ , we have

$$f_i(x_i^*, \mathbf{x}_{-i}^*) \leq f_i(x_i, \mathbf{x}_{-i}^*), \forall x_i \in \mathbb{R} \quad (3)$$

where  $\mathbf{x}_{-i} = [x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_N]^T$ .

To facilitate later analysis, we made the following assumptions.

*Assumptions 1:* For  $i \in \mathcal{V}$ ,  $f_i(\mathbf{x})$  is  $\mathcal{C}^2$  and  $\nabla f_i(\mathbf{x})$  is globally Lipschitz with  $l_i$ .

*Assumptions 2:* The digraph  $\mathcal{G}$  is strongly connected.

*Lemma 1:* Let  $D$  be a nonnegative diagonal matrix and  $\mathcal{H} = \mathcal{L} + D$ . Under Assumptions 2, there are symmetric positive definite matrices  $P$  and  $Q$  such that

$$\mathcal{H}^T P + P \mathcal{H} = Q \quad (4)$$

*Assumptions 3:* For  $\mathbf{x}, \mathbf{z} \in \mathbb{R}^N$ ,

$$(\mathbf{x} - \mathbf{z})^T ([\nabla_i f_i(\mathbf{x})]_{vec} - [\nabla_i f_i(\mathbf{z})]_{vec}) \geq m \|\mathbf{x} - \mathbf{z}\|^2 \quad (5)$$

where  $\nabla_i f_i(\mathbf{x}) = \partial f_i(\mathbf{x}) / \partial x_i$  and  $m$  is a positive constant.

*Assumptions 4:* For  $i \in \mathcal{V}$ ,  $\nabla_{ij}^2 f_i(\mathbf{x}) = \partial^2 f_i(\mathbf{x}) / \partial x_i \partial x_j$  is bounded.

*Assumptions 5:* The disturbance  $d(t)$  is bounded, i.e.,  $\forall i \in \mathcal{V}, |d_i(t)| \leq \bar{d}_i(t)$  for a positive constant  $\bar{d}_i$ .

To realize asymptotic Nash equilibrium seeking for games with second-order integrator-type players distributively, the control input is designed as

$$\begin{cases} u_i = -\tau k_i s_i - \beta_i \xi_i(t) \\ s_i = v_i + \nabla_i f_i(\mathbf{y}_i) \\ \dot{y}_{ij} = -\theta \bar{\theta}_{ij} \left( \sum_{k=1}^N a_{ik}(y_{ij} - y_{kj}) + a_{ij}(y_{ij} - x_j) \right) \\ \dot{\varrho}_i(t) = -\alpha_i \varrho_i(t) \text{ with } \varrho_i(t) > 0 \end{cases} \quad (6)$$

where  $i, j \in \mathcal{V}$ ,  $\theta, \tau$  are adjustable positive parameters,  $\bar{\theta}_{ij}, k_i, \alpha_i$  are fixed positive parameters,  $\beta_i \geq \bar{d}_i$  is a positive constant,  $s_i$  and  $\varrho_i(t)$  represents an auxiliary variable for player  $i$  and  $\xi_i(t) = s_i(t) / (|s_i(t)| + \varrho_i(t))$ .

By (1) and (6), the closed-loop system is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\tau \mathbf{k} \mathbf{s} - [\beta_i \xi_i(t)]_{vec} + \mathbf{d}(t) \\ \mathbf{s} = \mathbf{v} + [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ \dot{\mathbf{y}} = -\theta \bar{\theta} (\mathcal{L} \otimes \mathbf{I}_{N \times N} + \mathcal{A}_q) \tilde{\mathbf{y}} \end{cases} \quad (7)$$

where  $\mathbf{k} = \text{diag}\{k_i\}$ ,  $\bar{\theta} = \text{diag}\{\bar{\theta}_{ij}\}$ ,  $\mathcal{A}_q = \text{diag}\{a_{ij}\}$ ,  $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \dots, \mathbf{y}_N^T]^T$  and  $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{1}_N \otimes \mathbf{x}$ .

Let  $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$ .

*Theorem 1:* Under Assumptions 1-5, there exist a  $\theta^* > 0$  such that for each  $\theta > \theta^*$ , there exists a  $\tau^* > 0$  such that for each  $\tau > \tau^*$ , players' actions globally exponentially converge to the Nash equilibrium by (7).

*Proof:* Define  $V = V_1 + V_2 + V_3$ , where

$$\begin{aligned} V_1(t) &= \frac{1}{2} \tilde{\mathbf{x}}^T \tilde{\mathbf{x}} \\ V_2(t) &= \frac{1}{2} \tilde{\mathbf{y}}^T P \tilde{\mathbf{y}} \\ V_3(t) &= \frac{1}{2} \mathbf{s}^T \mathbf{s} + \sum_{i=1}^N \frac{\beta_i}{\alpha_i} \varrho_i(t) \end{aligned} \tag{8}$$

Then, taking the time derivative of  $V_1$  yields

$$\begin{aligned} \dot{V}_1 &= \tilde{\mathbf{x}}^T (\mathbf{s} - [\nabla_i f_i(\mathbf{y}_i)]_{vec}) \\ &\leq -m \|\tilde{\mathbf{x}}\|^2 + \max_{i \in \mathcal{V}} \{l_i\} \|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{y}}\| + \|\tilde{\mathbf{x}}\| \|\mathbf{s}\| \end{aligned} \tag{9}$$

where in the inequality, we have used Assumptions 3 and  $\|[\nabla_i f_i(\mathbf{y}_i)]_{vec} - [\nabla_i f_i(\mathbf{x})]_{vec}\| \leq \max_{i \in \mathcal{V}} \{l_i\} \|\mathbf{y}\|$ .

Then, taking the time derivative of  $V_2$  yields

$$\begin{aligned} \dot{V}_2 &= \dot{\tilde{\mathbf{y}}}^T P \tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T P \dot{\tilde{\mathbf{y}}} \\ &= -\theta \tilde{\mathbf{y}}^T Q \tilde{\mathbf{y}} - 2\tilde{\mathbf{y}}^T P \mathbf{1}_N \otimes \mathbf{s} \\ &\quad + 2\tilde{\mathbf{y}}^T P \mathbf{1}_N \otimes [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ &\leq -(\theta \lambda_{\min}(Q) - 2\sqrt{N} \|P\| \max_{i \in \mathcal{V}} \{l_i\}) \|\tilde{\mathbf{y}}\|^2 \\ &\quad + 2\sqrt{N} \|P\| \|\tilde{\mathbf{y}}\| \|\mathbf{s}\| + 2N \|P\| \max_{i \in \mathcal{V}} \{l_i\} \|\tilde{\mathbf{y}}\| \|\tilde{\mathbf{x}}\|. \end{aligned} \tag{10}$$

where in the inequality, by Assumption 2 and Lemma 1, we let  $P, Q \in \mathbb{R}^{N^2 \times N^2}$  and  $Q = P\bar{\theta}(\mathcal{L} \otimes \mathbf{I}_{N \times N} + \mathcal{A}_q) + (\mathcal{L} \otimes \mathbf{I}_{N \times N} + \mathcal{A}_q)^T \bar{\theta} P = Q$ .

Moreover, taking the time derivative of  $V_3$  yields

$$\begin{aligned}
\dot{V}_3 &= \mathbf{s}^T(\dot{\mathbf{v}} + \bar{H}(\mathbf{y})\dot{\mathbf{y}}) - \sum_{i=1}^N \beta_i \varrho_i(t) \\
&= \mathbf{s}^T(-\tau \mathbf{k} \mathbf{s} - [\beta_i \xi_i(t)]_{vec} + \mathbf{d}(t) + \bar{H}(\mathbf{y})\dot{\mathbf{y}}) - \sum_{i=1}^N \beta_i \varrho_i(t) \\
&= \mathbf{s}^T(-\tau \mathbf{k} \mathbf{s} + \bar{H}(\mathbf{y})\dot{\mathbf{y}}) - \sum_{i=1}^N \frac{(\beta_i - |d_i(t)|)|s_i(t)|(|s_i(t)| + \varrho_i(t)) + \beta_i \varrho_i^2(t)}{|s_i(t)| + \varrho_i(t)}
\end{aligned} \tag{11}$$

where  $\bar{H}(\mathbf{y}) = [h_{ij}]$  in which for  $i \neq j$ ,  $h_{ij} = \mathbf{0}_N^T$  and for  $i = j$ ,  $h_{ii} = [\nabla_{i1}^2 f_i(\mathbf{y}_i), \nabla_{i2}^2 f_i(\mathbf{y}_i), \dots, \nabla_{iN}^2 f_i(\mathbf{y}_i)]$ ,  $\nabla_{ij}^2 f_i(\mathbf{y}_i) = \frac{\partial^2 f_i(\mathbf{x})}{\partial x_i \partial x_j} |_{\mathbf{x}=\mathbf{y}_i}$ . By Assumption 4,  $\|\bar{H}\mathbf{y}\|$  is bounded. Hence, There are some positive constant  $L_1$  meet  $\|\bar{H}(\mathbf{y})\| \|\bar{\boldsymbol{\theta}}(\mathcal{L} \otimes \mathbf{I}_{N \times N} + \mathcal{A}_q)\| \leq L_1$ . Therefore,

$$\dot{V}_3 \leq -\tau \lambda_{\min}(\mathbf{k}) \|\mathbf{s}\|^2 + \theta L_1 \|\mathbf{s}\| \|\tilde{\mathbf{y}}\| \tag{12}$$

Hence,

$$\begin{aligned}
\dot{V} &\leq -m \|\tilde{\mathbf{x}}\|^2 - \tau \lambda_{\min}(\mathbf{k}) \|\mathbf{s}\|^2 \\
&\quad - (\theta \lambda_{\min}(Q) - 2\sqrt{N} \|P\| \max_{i \in \mathcal{V}} \{l_i\}) \|\tilde{\mathbf{y}}\|^2 \\
&\quad + (\max_{i \in \mathcal{V}} \{l_i\} + 2N \|P\| \max_{i \in \mathcal{V}} \{l_i\}) \|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{y}}\| \\
&\quad + (2\sqrt{N} \|P\| + \theta L_1) \|\tilde{\mathbf{y}}\| \|\mathbf{s}\| + \|\tilde{\mathbf{x}}\| \|\mathbf{s}\| \\
&\quad - \sum_{i=1}^N \frac{(\beta_i - |d_i(t)|)|s_i(t)|(|s_i(t)| + \varrho_i(t)) + \beta_i \varrho_i^2(t)}{|s_i(t)| + \varrho_i(t)} \\
&\leq -\lambda_{\min}(F) \|\mathbf{E}_1\|^2 - \tau \lambda_{\min}(\mathbf{k}) \|\mathbf{s}\|^2 \\
&\quad + (1 + 2\sqrt{N} \|P\| + \theta L_1) \|\mathbf{E}_1\| \|\mathbf{s}\|
\end{aligned} \tag{13}$$

where  $F = \begin{bmatrix} m & a \\ a & \theta \lambda_{\min}(Q) - 2\sqrt{N} \|P\| \max_{i \in \mathcal{V}} \{l_i\} \end{bmatrix}$ ,  $\mathbf{E}_1 = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T]^T$ ,  $a = -\frac{\max_{i \in \mathcal{V}} \{l_i\} + 2N \|P\| \max_{i \in \mathcal{V}} \{l_i\}}{2}$ , choose  $\theta > \frac{(\max_{i \in \mathcal{V}} \{l_i\} + 2N \|P\| \max_{i \in \mathcal{V}} \{l_i\})^2}{4m \lambda_{\min}(Q)} + \frac{2\sqrt{N} \|P\| \max_{i \in \mathcal{V}} \{l_i\}}{\lambda_{\min}(Q)}$ .

Then,  $\lambda_{\min}(F)$  is a real positive number and  $\lambda_{\min}(F) > 0$ . Thus,

$$\dot{V}(t) \leq -\lambda_{\min}(G)\|\mathbf{E}\|^2 \quad (14)$$

where  $G = \begin{bmatrix} \lambda_{\min}(F) & -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} \\ -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} & \tau\lambda_{\min}(\mathbf{k}) \end{bmatrix}$ ,  $\mathbf{E} = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T, \tilde{\mathbf{s}}^T]^T$ . By choosing  $\tau > \frac{(1+2\sqrt{N}\|P\|+\theta L_1)^2}{4\lambda_{\min}(F)\lambda_{\min}(\mathbf{k})}$ ,  $\lambda_{\min}(G)$  is a real positive number and  $\lambda_{\min}(G) > 0$ .

The conclusion can be drawn.