I. INTRODUCTION

II. NOTATIONS AND PRELIMINARIES

Notations: The real number set is denote as $\mathcal{R}$

III. PROBLEM STATEMENT

假设条件如下：

1. f 为二阶连续可微函数，且 ∇f 为全局 Lipschitz 的；
2. 拓扑结构为有向强连通图；
3. ∇f 为强凸函数；
4. $\nabla_{ij}^2 f_i(\mathbf{x}) = \frac{\partial^2 f_i(\mathbf{x})}{\partial x_i \partial x_j}$ 是有界的；
5. 误差 $d(t)$ 是有界的。

令 $\beta_i \geq |d_i(t)|, \dot{\varrho}_i(t) = -\alpha_i \varrho_i(t), \varrho_i > 0, \alpha_i > 0$，且

$$\xi_i(t) = \frac{s_i(t)}{|s_i(t)| + \varrho_i(t)}$$

算法设计如下：

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\tau \mathbf{k}\mathbf{s} - [\beta_i \xi_i(t)]_{vec} + \mathbf{d}(t) \\ \mathbf{s} = \mathbf{v} + [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ \dot{\mathbf{y}} = -\theta\bar{\theta}(\mathcal{L} \otimes \mathbf{I}_{N\times N} + \mathcal{A}_q)\tilde{\mathbf{y}} \end{cases}$$

选取 Lyapunov 函数为

$$V(t) = \frac{1}{2}\tilde{\mathbf{x}}^T\tilde{\mathbf{x}} + \frac{1}{2}\mathbf{s}^T\mathbf{s} + \tilde{\mathbf{y}}^T P \tilde{\mathbf{y}} + \sum_{i=1}^{N} \frac{\beta_i}{\alpha_i}\varrho_i(t)$$

其中，$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$，$\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{1}_N \otimes \mathbf{x}$。

证明如下：

首先，令 $V_1(t) = \frac{1}{2}\tilde{\mathbf{x}}^T\tilde{\mathbf{x}}$，

其次，令 $V_2(t) = \frac{1}{2}\tilde{\mathbf{y}}^T P\tilde{\mathbf{y}}$，

再令 $V_3(t) = \frac{1}{2}\mathbf{s}^T\mathbf{s} + \sum_{i=1}^N \frac{\beta_i}{\alpha_i}\varrho_i(t)$

最后，我们得到

$$\begin{aligned}
\dot{V} \leq & -m\|\tilde{\mathbf{x}}\|^2 - \tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\
& -(\theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{y}}\|^2 \\
& +(\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{y}}\| \\
& +(2\sqrt{N}\|P\| + \theta L_1)\|\tilde{\mathbf{y}}\|\|\mathbf{s}\| + \|\tilde{\mathbf{x}}\|\|\mathbf{s}\| \\
& -\sum_{i=1}^N \frac{(\beta_i - |d_i(t)|)|s_i(t)|(|s_i(t)| + \varrho_i(t)) + \beta_i\varrho_i^2(t)}{|s_i(t)| + \varrho_i(t)} \\
\leq & -\lambda_{\min}(F)\|\mathbf{E}_1\|^2 - \tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\
& +(1 + 2\sqrt{N}\|P\| + \theta L_1)\|\mathbf{E}_1\|\|\mathbf{s}\|
\end{aligned}$$

其中 $F = \begin{bmatrix} m & a \\ a & \theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\} \end{bmatrix}$，$\mathbf{E} = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T]$，$a = -\frac{\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{2}$，当 $\theta > \frac{(\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}(l_i\})^2}{4m\lambda_{\min}(Q)} + \frac{2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{\lambda_{\min}(Q)}$，则 $\lambda_{min}(F) > 0$。令 $\bar{H}(\mathbf{y}) = [h_{ij}]$，在 $i = j$ 时，我们有 $h_{ii} = [\nabla_{i1}^2 f_i(\mathbf{y}_i), \nabla_{i2}^2 f_i(\mathbf{y}_i), \ldots, \nabla_{iN}^2 f_i(\mathbf{y}_i)]$，$\nabla_{ij}^2 f_i(\mathbf{y}_i) = \frac{\partial^2 f_i(\mathbf{x})}{\partial x_i \partial x_i}|_{\mathbf{x}=\mathbf{y}_i}$，否则 $h_{ij} = \mathbf{0}_N^T$，L_1 满足 $\|\bar{H}(\mathbf{y})\|\|\bar{\boldsymbol{\theta}}(\mathcal{L}\otimes\mathbf{I}_{N\times N} + \mathcal{A}_q)\| \leq L_1$。

因此

$$\dot{V}(t) \leq -\lambda_{\min}(G)\|\mathbf{E}\|^2$$

其中 $G = \begin{bmatrix} \lambda_{\min}(F) & -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} \\ -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} & \tau\lambda_{\min}(\mathbf{k}) \end{bmatrix}$，$\mathbf{E} = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T, \tilde{\mathbf{s}}^T]$，当 $\tau > \frac{(1+2\sqrt{N}\|P\|+\theta L_1)^2}{4\lambda_{\min}(F)\lambda_{\min}(\mathbf{k})}$，则 $\lambda_{min}(G) > 0$。