I. INTRODUCTION

II. NOTATIONS AND PRELIMINARIES

Notations: The real number set is denote as $\mathbb{R}$. $\|z\|$ denotes the ℓ_2-norm of z. $[z_i]_{vec}$ where $i \in \{1, 2, ..., N\}$ is defined as a column vector whose dimension is $N \times 1$ and the ith element is z_i. $\mathrm{diag}\{k_i\}$ for $i \in \{1, 2, ..., N\}$ is a diagonal matrix whose dimension is $N \times N$ and the ith diagonal element is k_i. $\mathrm{diag}\{a_{ij}\}$ where $i, j \in \{1, 2, ..., N\}$ gives a diagonal matrix whose dimension is $N^2 \times N^2$ and diagonal elements are $a_{11}, a_{12}, ..., a_{1N}, a_{21}, ..., a_{NN}$, successively. $\mathcal{A} = [a_{ij}]$ is a matrix whose (i, j)th entry is a_{ij}. Given that matrix Q is symmetric and real, $\lambda_{min}(Q)(\lambda_{max}(Q))$ stands for the smallest(largest) eigenvalue of Q. $max_{i\in\{1,2,...,N\}}\{l_i\}$ denotes the largest value of l_i for $i \in \{1, 2, ..., N\}$. $\mathbf{I}_{N\times N}$ is an identity matrix with its dimension being $N \times N$ and $\mathbf{1}(\mathbf{0})$ is a column vector with its entries being $1(0)$. Moreover, $\otimes$ is the Kronecker product.

Algebraic Graph Theory: A graph $\mathcal{G}$ is given by $\mathcal{G} = (\mathcal{V}, \mathcal{E}_g)$, in which $\mathcal{V} = \{1, 2, ..., N\}$, $\mathcal{E}_g \subseteq \mathcal{V} \times \mathcal{V}$ respectively are the node set and edge set. The edge $(i, j) \in \mathcal{E}_g$ indicates are the node j can receive information from node i, but not necessarily vice versa. The in-neighbor set of node i is given as $\mathcal{N}_i^{in} = \{j|(j, i) \in \mathcal{E}_g\}$. A directed path is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), ...$ A directed graph is strongly connected if for every pair of two distinct nodes, there is a path. Let $\mathcal{A} = [a_{ij}]$ be the adjacency matrix in which $a_{ij} > 0$ if $(i, j) \in \mathcal{E}_g$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}\{d_i\}$ and $d_i = \sum_{j=1}^{N} a_{ij}$.

III. PROBLEM STATEMENT

In the concerned game, N players with labels from 1 to N are engaged and each player i has a local objective function $f_i(\mathbf{x}) : \mathbb{R}^N \to \mathbb{R}$, in which $\mathbf{x} = [x_1, x_2, ..., x_N]^T$ and $x_i \in \mathbb{R}$ is the action of player i. Moreover, for second-order players, player i's action is governed by

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i + d_i(t) \end{cases} \tag{1}$$

where $v_i(t)$ is the velocity-like state of player i, u_i is the control input and $d_i(t)$ is the disturbance for $i \in \mathcal{V}$. Each player i aims to minimize its own objective function $f_i(\mathbf{x})$ by adjusting its action $x_i(t)$, that is

$$\begin{aligned} &\min_{x_i} \quad f_i(\mathbf{x}) \\ &\text{s.t.} \qquad (1) \quad \text{for second-order players} \end{aligned} \tag{2}$$

Definition 1: An action profile $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ is a Nash equilibrium if for all $i \in \mathcal{V}$, we have

$$f_i(x_i^*, \mathbf{x}_{-i}^*) \leq f_i(x_i, \mathbf{x}_{-i}^*), \forall x_i \in \mathbb{R} \tag{3}$$

where $\mathbf{x}_{-i} = [x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N]^T$.

To facilitate later analysis, we made the following assumptions.

Assumptions 1: For $i \in \mathcal{V}$, $f_i(\mathbf{x})$ is $\mathcal{C}^2$ and $\nabla f_i(\mathbf{x})$ is globally Lipschitz with l_i.

Assumptions 2: The digraph $\mathcal{G}$ is strongly connected.

Lemma 1: Let D be a nonnegative diagonal matrix and $\mathcal{H} = \mathcal{L} + D$. Under Assumptions 2, there are symmetric positive definite matrices P and Q such that

$$\mathcal{H}^T P + P\mathcal{H} = Q \tag{4}$$

Assumptions 3: For $\mathbf{x}, \mathbf{z} \in \mathbb{R}^N$,

$$(\mathbf{x}-\mathbf{z})^T([\nabla_i f_i(\mathbf{x})]_{vec} - [\nabla_i f_i(\mathbf{z})]_{vec}) \geq m\|\mathbf{x}-\mathbf{z}\|^2 \tag{5}$$

where $\nabla_i f_i(\mathbf{x}) = \partial f_i(\mathbf{x})/\partial x_i$ and m is a positive constant.

Assumptions 4: For $i \in \mathcal{V}$, $\nabla^2_{ij} f_i(\mathbf{x}) = \partial^2 f_i(\mathbf{x})/\partial x_i \partial x_j$ is bounded.

Assumptions 5: The disturbance $d(t)$ is bounded, i.e., $\forall i \in \mathcal{V}, |d_i(t)| \leq \bar{d}_i(t)$ for a positive constant $\bar{d}_i$.

To realize asymptotic Nash equilibrium seeking for games with second-order integrator-type players distributively, the control input is designed as

$$\begin{cases} u_i = -\tau k_i s_i - \beta_i \xi_i(t) \\ s_i = v_i + \nabla_i f_i(\mathbf{y}_i) \\ \dot{y}_{ij} = -\theta \bar{\theta}_{ij}\left(\sum_{k=1}^N a_{ik}(y_{ij} - y_{kj}) + a_{ij}(y_{ij} - x_j)\right) \\ \dot{\varrho}_i(t) = -\alpha_i \varrho_i(t) \text{ with } \varrho_i(t) > 0 \end{cases} \tag{6}$$

where $i, j \in \mathcal{V}, \theta, \tau$ are adjustable positive parameters, $\bar{\theta}_{ij}, k_i, \alpha_i$ are fixed positive parameters, $\beta_i \geq \bar{d}_i$ is a positive constant, s_i and $\varrho_i(t)$ represents an auxiliary variable for player i and $\xi_i(t) = s_i(t)/(|s_i(t)| + \varrho_i(t))$.

By (1) and (6), the closed-loop system is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\tau \mathbf{k}\mathbf{s} - [\beta_i \xi_i(t)]_{vec} + \mathbf{d}(t) \\ \mathbf{s} = \mathbf{v} + [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ \dot{\mathbf{y}} = -\theta \bar{\theta}(\mathcal{L} \otimes \mathbf{I}_{N\times N} + \mathcal{A}_q)\tilde{\mathbf{y}} \end{cases} \tag{7}$$

where $\mathbf{k} = \mathrm{diag}\{k_i\}, \bar{\theta} = \mathrm{diag}\{\bar{\theta}_{ij}\}, \mathcal{A}_q = \mathrm{diag}\{a_{ij}\}, \mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, ..., \mathbf{y}_N^T]^T$ and $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{1}_N \otimes \mathbf{x}$.

Let $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$.

Theorem 1: Under Assumptions 1-5, there exist a $\theta^* > 0$ such that for each $\theta > \theta^*$, there exists a $\tau^* > 0$ such that for each $\tau > \tau^*$, players' actions globally exponentially converge to the Nash equilibrium by (7).

Proof: Define $V = V_1 + V_2 + V_3$, where

$$\begin{aligned} V_1(t) &= \frac{1}{2}\tilde{\mathbf{x}}^T\tilde{\mathbf{x}} \\ V_2(t) &= \frac{1}{2}\tilde{\mathbf{y}}^T P\tilde{\mathbf{y}} \\ V_3(t) &= \frac{1}{2}\mathbf{s}^T\mathbf{s} + \sum_{i=1}^{N}\frac{\beta_i}{\alpha_i}\varrho_i(t) \end{aligned} \tag{8}$$

Then, taking the time derivative of V_1 yields

$$\begin{aligned} \dot{V}_1 &= \tilde{\mathbf{x}}^T(\mathbf{s} - [\nabla_i f_i(\mathbf{y}_i)]_{vec}) \\ &\leq -m\|\tilde{\mathbf{x}}\|^2 + \max_{i\in\mathcal{V}}\{l_i\}\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{y}}\| + \|\tilde{\mathbf{x}}\|\|\mathbf{s}\| \end{aligned} \tag{9}$$

where in the inequality, we have used Assumptions 3 and $\|[\nabla_i f_i(\mathbf{y}_i)]_{vec} - [\nabla_i f_i(\mathbf{x})]_{vec}\| \leq \max_{i\in\mathcal{V}}\{l_i\}\|\tilde{\mathbf{y}}\|$.

Then, taking the time derivative of V_2 yields

$$\begin{aligned} \dot{V}_2 =& \dot{\tilde{\mathbf{y}}}^T P\tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T P\dot{\tilde{\mathbf{y}}} \\ =& -\theta\tilde{\mathbf{y}}^T Q\tilde{\mathbf{y}} - 2\tilde{\mathbf{y}}^T P\mathbf{1}_N \otimes \mathbf{s} \\ & + 2\tilde{\mathbf{y}}^T P\mathbf{1}_N \otimes [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ \leq& -(\theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{y}}\|^2 \\ & + 2\sqrt{N}\|P\|\|\tilde{\mathbf{y}}\|\|\mathbf{s}\| + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\}\|\tilde{\mathbf{y}}\|\|\tilde{\mathbf{x}}\|. \end{aligned} \tag{10}$$

where in the inequality, by Assumption 2 and Lemma 1, we let $P, Q \in \mathbb{R}^{N^2\times N^2}$ and $Q = P\bar{\theta}(\mathcal{L}\otimes\mathbf{I}_{N\times N} + \mathcal{A}_q) + (\mathcal{L}\otimes\mathbf{I}_{N\times N} + \mathcal{A}_q)^T\bar{\theta}P = Q$.

Moreover, taking the time derivative of V_3 yields

$$\begin{aligned} \dot{V}_3 &= \mathbf{s}^T(\dot{\mathbf{v}} + \bar{H}(\mathbf{y})\dot{\mathbf{y}}) - \sum_{i=1}^{N}\beta_i\varrho_i(t) \\ &= \mathbf{s}^T(-\tau\mathbf{k}\mathbf{s} - [\beta_i\xi_i(t)]_{vec} + \mathbf{d}(t) + \bar{H}(\mathbf{y})\dot{\mathbf{y}}) - \sum_{i=1}^{N}\beta_i\varrho_i(t) \end{aligned} \tag{11}$$

where $\bar{H}(\mathbf{y}) = [h_{ij}]$ in which for $i \neq j$, $h_{ij} = \mathbf{0}_N^T$ and for $i = j$, $h_{ii} = [\nabla_{i1}^2 f_i(\mathbf{y}_i), \nabla_{i2}^2 f_i(\mathbf{y}_i), \ldots, \nabla_{iN}^2 f_i(\mathbf{y}_i)]$, $\nabla_{ij}^2 f_i(\mathbf{y}_i) = \frac{\partial^2 f_i(\mathbf{x})}{\partial x_i \partial x_i} |_{\mathbf{x}=\mathbf{y}_i}$. By Assumption 4, $\|\bar{H}\mathbf{y}\|$ is bounded. Hence, There are some positive constant L_1 meet $\|\bar{H}(\mathbf{y})\|\|\bar{\boldsymbol{\theta}}(\mathcal{L} \otimes \mathbf{I}_{N\times N} + \mathcal{A}_q)\| \leq L_1$. Therefore,

$$\dot{V}_3 \leq -\tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 + \theta L_1\|\mathbf{s}\|\|\tilde{\mathbf{y}}\| - \sum_{i=1}^{N} \frac{(\beta_i - |d_i(t)|)|s_i(t)|(|s_i(t)| + \varrho_i(t)) + \beta_i\varrho_i^2(t)}{|s_i(t)| + \varrho_i(t)} \tag{12}$$

Hence,

$$\begin{aligned}
\dot{V} \leq & -m\|\tilde{\mathbf{x}}\|^2 - \tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\
& - (\theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{y}}\|^2 \\
& + (\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{y}}\| \\
& + (2\sqrt{N}\|P\| + \theta L_1)\|\tilde{\mathbf{y}}\|\|\mathbf{s}\| + \|\tilde{\mathbf{x}}\|\|\mathbf{s}\| \\
& - \sum_{i=1}^{N} \frac{(\beta_i - |d_i(t)|)|s_i(t)|(|s_i(t)| + \varrho_i(t)) + \beta_i\varrho_i^2(t)}{|s_i(t)| + \varrho_i(t)} \\
\leq & -\lambda_{\min}(F)\|\mathbf{E}_1\|^2 - \tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\
& + (1 + 2\sqrt{N}\|P\| + \theta L_1)\|\mathbf{E}_1\|\|\mathbf{s}\|
\end{aligned} \tag{13}$$

where $F = \begin{bmatrix} m & a \\ a & \theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\} \end{bmatrix}$, $\mathbf{E}_1 = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T]^T$, $a = -\frac{\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{2}$, choose $\theta > \frac{(\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}(l_i\})^2}{4m\lambda_{\min}(Q)} + \frac{2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{\lambda_{\min}(Q)}$. Then, $\lambda_{min}(F)$ is a real positive number and $\lambda_{min}(F) > 0$. Thus,

$$\dot{V}(t) \leq -\lambda_{\min}(G)\|\mathbf{E}\|^2 \tag{14}$$

where $G = \begin{bmatrix} \lambda_{\min}(F) & -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} \\ -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} & \tau\lambda_{\min}(\mathbf{k}) \end{bmatrix}$, $\mathbf{E} = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T, \tilde{\mathbf{s}}^T]^T$. By choosing $\tau > \frac{(1+2\sqrt{N}\|P\|+\theta L_1)^2}{4\lambda_{\min}(F)\lambda_{\min}(\mathbf{k})}$, $\lambda_{min}(G)$ is a real positive number and $\lambda_{min}(G) > 0$.

The conclusion can be drawn.