Distributed Nash Equilibrium Seeking For Second-Order Systems With Unknown Dynamics

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Abstract—1

Index Terms—distributed Nash equilibrium seeking; second-order systems; unknown dynamics; boundary layer technique.

I. INTRODUCTION

OVER the past two decades, game theory has spread across diverse research fields, including biology, economics, and computer science. As game theory evolves, the quest for Nash equilibrium in noncooperative games gains significance in theory and practice, as evidenced by studies [1], [2], [3], [4], [5], [6]. Substantial progress has been made in distributed control and optimization of networked systems based on Nash equilibrium seeking and its applications [7], [8], [9], [10], [11]. For example, building upon the framework introduced in [12], numerous consensus-based distributed Nash equilibrium seeking strategies have been devised, such as distributed Nash equilibrium seeking in multiagent game under switching communication topologies digraph [13] and fully distributed Nash equilibrium seeking [14]. Nevertheless, the majority of existing findings don't account for the influence of system disturbances, which is unrealistic considering that many practical engineering systems frequently encounter disturbances.

The rest of this paper is organized as follows. Section II gives notations and preliminaries. Problem statement is introduced in Section III and main results are given in Section IV where we utilize boundary layer technique to design a continuous function method for distributed Nash equilibrium seeking for second-order players. Simulation studies are presented in Section V and conclusions are drawn in Section VI.

II. NOTATIONS AND PRELIMINARIES

Notations: The real number set is denote as $\mathbb{R}$. $\|z\|$ denotes the ℓ_2-norm of z. $[z_i]_{vec}$ where $i \in \{1,2,...,N\}$ is defined as a column vector whose dimension is $N \times 1$ and the ith element is z_i. $\text{diag}\{k_i\}$ for $i \in \{1,2,...,N\}$ is a diagonal matrix whose dimension is $N \times N$ and the ith diagonal element is k_i. $\text{diag}\{a_{ij}\}$ where $i,j \in \{1,2,...,N\}$ gives a diagonal matrix whose dimension is $N^2 \times N^2$ and diagonal elements are $a_{11}, a_{12}, ..., a_{1N}, a_{21}, ..., a_{NN}$, successively. $\mathcal{A} = [a_{ij}]$ is a matrix whose (i,j)th entry is a_{ij}. Given that matrix Q is symmetric and real, $\lambda_{min}(Q)(\lambda_{max}(Q))$ stands for the smallest(largest) eigenvalue of Q. $max_{i\in\{1,2,...,N\}}\{l_i\}$ denotes the largest value of l_i for $i \in \{1,2,...,N\}$. $\mathbf{I}_{N\times N}$ is an identity matrix with its dimension being $N \times N$ and $\mathbf{1}(\mathbf{0})$ is a column vector with its entries being $1(0)$. Moreover, $\otimes$ is the Kronecker product.

Algebraic Graph Theory: A graph $\mathcal{G}$ is given by $\mathcal{G} = (\mathcal{V}, \mathcal{E}_g)$, in which $\mathcal{V} = \{1,2,...,N\}$, $\mathcal{E}_g \subseteq \mathcal{V}\times\mathcal{V}$ respectively are the node set and edge set. The edge $(i,j) \in \mathcal{E}_g$ indicates are the node j can receive information from node i, but not necessarily vice versa. The in-neighbor set of node i is given as $\mathcal{N}_i^{in} = \{j|(j,i) \in \mathcal{E}_g\}$. A directed path is a sequence of edges of the form $(i_1,i_2),(i_2,i_3),....$ A directed graph is strongly connected if for every pair of two distinct nodes, there is a path. Let $\mathcal{A} = [a_{ij}]$ be the adjacency matrix in which $a_{ij} > 0$ if $(i,j) \in \mathcal{E}_g$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}\{d_i\}$ and $d_i = \sum_{j=1}^{N} a_{ij}$.

III. PROBLEM STATEMENT

In the concerned game, N players with labels from 1 to N are engaged and each player i has a local objective function $f_i(\mathbf{x}) : \mathbb{R}^N \to \mathbb{R}$, in which $\mathbf{x} = [x_1, x_2, ..., x_N]^T$ and $x_i \in \mathbb{R}$ is the action of player i. Moreover, for second-order players, player i's action is governed by

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i + d_i(t) \end{cases} \tag{1}$$

where $v_i(t)$ is the velocity-like state of player i, u_i is the control input and $d_i(t)$ is the disturbance for $i \in \mathcal{V}$. Each player i aims to minimize its own objective function $f_i(\mathbf{x})$ by adjusting its action $x_i(t)$, that is

$$\begin{aligned} &\min_{x_i} \quad f_i(\mathbf{x}) \\ &\text{s.t.} \quad\ \ (1) \quad \text{for second-order players} \end{aligned} \tag{2}$$

Definition 1: An action profile $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ is a Nash equilibrium if for all $i \in \mathcal{V}$, we have

$$f_i(x_i^*, \mathbf{x}_{-i}^*) \leq f_i(x_i, \mathbf{x}_{-i}^*), \forall x_i \in \mathbb{R} \tag{3}$$

where $\mathbf{x}_{-i} = [x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N]^T$.

To facilitate later analysis, we made the following assumptions.

Assumptions 1: For $i \in \mathcal{V}$, $f_i(\mathbf{x})$ is $\mathcal{C}^2$ and $\nabla f_i(\mathbf{x})$ is globally Lipschitz with l_i.

Assumptions 2: The digraph $\mathcal{G}$ is strongly connected.

Lemma 1: Let D be a nonnegative diagonal matrix and $\mathcal{H} = \mathcal{L} + D$. Under Assumptions 2, there are symmetric positive definite matrices P and Q such that

$$\mathcal{H}^T P + P\mathcal{H} = Q \tag{4}$$

Assumptions 3: For $\mathbf{x}, \mathbf{z} \in \mathbb{R}^N$,

$$(\mathbf{x}-\mathbf{z})^T([\nabla_i f_i(\mathbf{x})]_{vec} - [\nabla_i f_i(\mathbf{z})]_{vec}) \geq m\|\mathbf{x}-\mathbf{z}\|^2 \quad (5)$$

where $\nabla_i f_i(\mathbf{x}) = \partial f_i(\mathbf{x})/\partial x_i$ and m is a positive constant.

IV. MAIN RESULTS

Assumptions 4: For $i \in \mathcal{V}$, $\nabla^2_{ij} f_i(\mathbf{x}) = \partial^2 f_i(\mathbf{x})/\partial x_i \partial x_j$ is bounded.

Assumptions 5: The disturbance $d(t)$ is bounded, i.e., $\forall i \in \mathcal{V}, |d_i(t)| \leq \bar{d}_i(t)$ for a positive constant $\bar{d}_i$.

To realize asymptotic Nash equilibrium seeking for games with second-order integrator-type players distributively, the control input is designed as

$$\begin{cases} u_i = -\tau k_i s_i - \beta_i \xi_i(t) \\ s_i = v_i + \nabla_i f_i(\mathbf{y}_i) \\ \dot{y}_{ij} = -\theta \bar{\theta}_{ij} \left(\sum_{k=1}^N a_{ik}(y_{ij} - y_{kj}) + a_{ij}(y_{ij} - x_j) \right) \\ \dot{\varrho}_i(t) = -\alpha_i \varrho_i(t) \text{ with } \varrho_i(t) > 0 \end{cases} \quad (6)$$

where $i, j \in \mathcal{V}, \theta, \tau$ are adjustable positive parameters, $\bar{\theta}_{ij}, k_i, \alpha_i$ are fixed positive parameters, $\beta_i \geq \bar{d}_i$ is a positive constant, s_i and $\varrho_i(t)$ represents an auxiliary variable for player i and $\xi_i(t) = s_i(t)/(|s_i(t)| + \varrho_i(t))$.

By (1) and (6), the closed-loop system is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\tau \mathbf{k}\mathbf{s} - [\beta_i \xi_i(t)]_{vec} + \mathbf{d}(t) \\ \mathbf{s} = \mathbf{v} + [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ \dot{\mathbf{y}} = -\theta \bar{\theta}(\mathcal{L} \otimes \mathbf{I}_{N\times N} + \mathcal{A}_q)\tilde{\mathbf{y}} \end{cases} \quad (7)$$

where $\mathbf{k} = \text{diag}\{k_i\}, \bar{\theta} = \text{diag}\{\bar{\theta}_{ij}\}, \mathcal{A}_q = \text{diag}\{a_{ij}\}, \mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, ..., \mathbf{y}_N^T]^T$ and $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{1}_N \otimes \mathbf{x}$.

Let $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$.

Theorem 1: Under Assumptions 1-5, there exist a $\theta^* > 0$ such that for each $\theta > \theta^*$, there exists a $\tau^* > 0$ such that for each $\tau > \tau^*$, players' actions globally exponentially converge to the Nash equilibrium by (7).

Proof: Define $V = V_1 + V_2 + V_3$, where

$$\begin{aligned} V_1(t) &= \frac{1}{2}\tilde{\mathbf{x}}^T\tilde{\mathbf{x}} \\ V_2(t) &= \frac{1}{2}\tilde{\mathbf{y}}^T P \tilde{\mathbf{y}} \\ V_3(t) &= \frac{1}{2}\mathbf{s}^T\mathbf{s} + \sum_{i=1}^N \frac{\beta_i}{\alpha_i}\varrho_i(t) \end{aligned} \quad (8)$$

Then, taking the time derivative of V_1 yields

$$\begin{aligned} \dot{V}_1 &= \tilde{\mathbf{x}}^T(\mathbf{s} - [\nabla_i f_i(\mathbf{y}_i)]_{vec}) \\ &\leq -m\|\tilde{\mathbf{x}}\|^2 + \max_{i\in\mathcal{V}}\{l_i\}\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{y}}\| + \|\tilde{\mathbf{x}}\|\|\mathbf{s}\| \end{aligned} \quad (9)$$

where in the inequality, we have used Assumptions 3 and $\|[\nabla_i f_i(\mathbf{y}_i)]_{vec} - [\nabla_i f_i(\mathbf{x})]_{vec}\| \leq \max_{i\in\mathcal{V}}\{l_i\}\|\tilde{\mathbf{y}}\|$.

Then, taking the time derivative of V_2 yields

$$\begin{aligned} \dot{V}_2 =& \dot{\tilde{\mathbf{y}}}^T P \tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T P \dot{\tilde{\mathbf{y}}} \\ =& -\theta \tilde{\mathbf{y}}^T Q \tilde{\mathbf{y}} - 2\tilde{\mathbf{y}}^T P \mathbf{1}_N \otimes \mathbf{s} \\ &+ 2\tilde{\mathbf{y}}^T P \mathbf{1}_N \otimes [\nabla_i f_i(\mathbf{y}_i)]_{vec} \\ \leq& -(\theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{y}}\|^2 \\ &+ 2\sqrt{N}\|P\|\|\tilde{\mathbf{y}}\|\|\mathbf{s}\| + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\}\|\tilde{\mathbf{y}}\|\|\tilde{\mathbf{x}}\|. \end{aligned} \quad (10)$$

where in the inequality, by Assumption 2 and Lemma 1, we let $P, Q \in \mathbb{R}^{N^2\times N^2}$ and $Q = P\bar{\theta}(\mathcal{L}\otimes \mathbf{I}_{N\times N} + \mathcal{A}_q) + (\mathcal{L}\otimes \mathbf{I}_{N\times N} + \mathcal{A}_q)^T \bar{\theta} P = Q$.

Moreover, taking the time derivative of V_3 yields

$$\begin{aligned} \dot{V}_3 &= \mathbf{s}^T(\dot{\mathbf{v}} + \bar{H}(\mathbf{y})\dot{\mathbf{y}}) - \sum_{i=1}^N \beta_i \varrho_i(t) \\ &= \mathbf{s}^T(-\tau\mathbf{k}\mathbf{s} - [\beta_i\xi_i(t)]_{vec} + \mathbf{d}(t) + \bar{H}(\mathbf{y})\dot{\mathbf{y}}) - \sum_{i=1}^N \beta_i\varrho_i(t) \end{aligned} \quad (11)$$

where $\bar{H}(\mathbf{y}) = [h_{ij}]$ in which for $i \neq j$, $h_{ij} = \mathbf{0}_N^T$ and for $i = j$, $h_{ii} = [\nabla^2_{i1} f_i(\mathbf{y}_i), \nabla^2_{i2} f_i(\mathbf{y}_i), \ldots, \nabla^2_{iN} f_i(\mathbf{y}_i)]$ $\nabla^2_{ij} f_i(\mathbf{y}_i) = \frac{\partial^2 f_i(\mathbf{x})}{\partial x_i \partial x_i}|_{\mathbf{x}=\mathbf{y}_i}$. By Assumption 4, $\|\bar{H}\mathbf{y}\|$ is bounded. Hence, There are some positive constant L_1 meet $\|\bar{H}(\mathbf{y})\|\|\bar{\boldsymbol{\theta}}(\mathcal{L}\otimes \mathbf{I}_{N\times N} + \mathcal{A}_q)\| \leq L_1$. Therefore,

$$\begin{aligned} \dot{V}_3 \leq& -\tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 + \theta L_1\|\mathbf{s}\|\|\tilde{\mathbf{y}}\| \\ &- \sum_{i=1}^N \frac{(\beta_i - |d_i(t)|)|s_i(t)|(|s_i(t)| + \varrho_i(t)) + \beta_i\varrho_i^2(t)}{|s_i(t)| + \varrho_i(t)} \end{aligned} \quad (12)$$

Hence,

$$\begin{aligned} \dot{V} \leq& -m\|\tilde{\mathbf{x}}\|^2 - \tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\ &- (\theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{y}}\|^2 \\ &+ (\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\})\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{y}}\| \\ &+ (2\sqrt{N}\|P\| + \theta L_1)\|\tilde{\mathbf{y}}\|\|\mathbf{s}\| + \|\tilde{\mathbf{x}}\|\|\mathbf{s}\| \\ &- \sum_{i=1}^N \frac{(\beta_i - |d_i(t)|)|s_i(t)|(|s_i(t)| + \varrho_i(t)) + \beta_i\varrho_i^2(t)}{|s_i(t)| + \varrho_i(t)} \\ \leq& -\lambda_{\min}(F)\|\mathbf{E}_1\|^2 - \tau\lambda_{\min}(\mathbf{k})\|\mathbf{s}\|^2 \\ &+ (1 + 2\sqrt{N}\|P\| + \theta L_1)\|\mathbf{E}_1\|\|\mathbf{s}\| \end{aligned} \quad (13)$$

where $F = \begin{bmatrix} m & a \\ a & \theta\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\} \end{bmatrix}$ $\mathbf{E}_1 = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T]^T$ $a = -\frac{\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{2}$, choose $\theta > \frac{(\max_{i\in\mathcal{V}}\{l_i\} + 2N\|P\|\max_{i\in\mathcal{V}}(l_i\})^2}{4m\lambda_{\min}(Q)} + \frac{2\sqrt{N}\|P\|\max_{i\in\mathcal{V}}\{l_i\}}{\lambda_{\min}(Q)}$. Then, $\lambda_{min}(F)$ is a real positive number and $\lambda_{min}(F) > 0$. Thus,

$$\dot{V}(t) \leq -\lambda_{\min}(G)\|\mathbf{E}\|^2 \quad (14)$$

where $G = \begin{bmatrix} \lambda_{\min}(F) & -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} \\ -\frac{1+2\sqrt{N}\|P\|+\theta L_1}{2} & \tau\lambda_{\min}(\mathbf{k}) \end{bmatrix}, \mathbf{E} = [\tilde{\mathbf{x}}^T, \tilde{\mathbf{y}}^T, \tilde{\mathbf{s}}^T]^T$. By choosing $\tau > \frac{(1+2\sqrt{N}\|P\|+\theta L_1)^2}{4\lambda_{\min}(F)\lambda_{\min}(\mathbf{k})}$, $\lambda_{min}(G)$ is a real positive number and $\lambda_{min}(G) > 0$.

The conclusion can be drawn.

V. SIMULATION STUDIES

VI. CONCLUSIONS

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