Hilbert Series for the Coulomb branch of 3d $\mathcal{N} = 4$ quiver gauge theories

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Abstract: The study of the Coulomb branch moduli space of 3d gauge theories has been a long addressed problem due to the quantum corrections that affect this space. This project aims to illustrate a way to characterize this space with a method that does not involve the use of 3d mirror symmetry or perturbative techniques. In particular, the magnetic monopole formula will reveal itself to be the answer to our problem, because by counting the number of GIOs at a given degree that parameterizes the moduli space, via an Hilber Series we can completely characterize the Coulomb branch of such theories. An introductory discussion on the concepts of moduli space and Hilbert series will be presented, together with some basis on magnetic monopoles and supersymmetry. A collection of results for various simply laced quiver gauge theories is then reported.

Contents

1 Introduction

We focus on studying 3d supersymmetric theories which have been shown to have, given a sufficient amount of supersymmetry as $\mathcal{N} = 4$ theories, a particular kind of duality [27]. This duality is called 3*d mirror symmetry*, it states that given two dual supersymmetric conformal field theories that flow in the IR to the same theory, their Coulomb and Higgs branches are exchanged.

The aspect of these theories we want to characterize is their moduli space, which is the geometrical space parameterized by the vacuum expectation values of the scalar fields in the theory. In particular, for 3d theories, the moduli space splits into two Hyperkähler manifolds (usually quotients) called Coulomb and Higgs branches. The Higgs branch is the moduli space given by the scalars in the hypermultiplets of the theory while the Coulomb branch is the moduli space given by the scalars in the vector multiplets of the theory. Note that given a gauge group the vector field always transforms in the adjoint representation while the hypermultiplets can be in any representation. [1]

The study of the Higgs branch follows a standard procedure: by describing its Hyperkähler quotient nature, given by the zero loci of hypers F-terms over the complexified gauge group, the generating function counting chiral operators, known as Hilbert series, can then be evaluated using the Molien formula, see [24]. The Higgs branch is easier to study compared to its Coulomb counterpart due to the classical nature of the former space. So most of the analysis made on Coulomb branches used 3d mirror symmetry to study the Higgs branch of the dual theory, avoiding in this way to address the problem of quantum corrections. A perturbative approach computing correction to the metric is indeed possible but does not encompass all the aspects of the theory. Nevertheless, we will present a way, proposed by A.Hanany, S. Cremonesi, and A. Zaffaroni [12], to algebraically characterize, in a non perturbative way, the Coulomb branch without involving the dual theory.

The Hilbert series will be the mathematical tool that enables us to investigate the properties of moduli spaces. It was first introduced in the context of graded algebras and a summary of its properties can be found in the paper of R. Stanley [29]. We will use this tool to count gauge invariant operators, GIOs for short, that are known to parameterize the moduli space. Dimension of the moduli space, number of generators, and constraint relations can then be inferred via the Plethystic program introduced in [16, 4].

[1] In our discussion we will focus on quiver gauge theories where the hypermultiplets sit in the bifundamental representation of the gauge nodes they are within, see section 4.

In section 2, $4d$ SUSY will be recalled together with the multiplets splitting scheme. The tool of dimensional reduction used to obtain the $3d$ $\mathcal{N}=4$ theory we will be presented. The moduli space of $3d$ theories will be treated in terms of its geometrical structure and in terms of physics.
Focus on supersymmetric $3d$ $\mathcal{N}=4$ theories, elucidating their field content and the particular characteristic given by the space topology, will be put.

In section 3, a particular kind of operator will be introduced: the magnetic monopole. Starting from a QED example its characteristics and properties will be analyzed with the aim of extracting its conformal dimension [2, 3].

In section 4, will be described the formalism of quiver diagrams for $3d$ $\mathcal{N}=4$ theories along with its physical meaning.

In section 5, the Hilbert series will be presented in the context of graded algebras. A few examples of computations, based on its basic definition, via the Plethystic program [16, 4], will be carried out with the aim of elucidating how information is encoded in the series.
The highlight will then be shifted on ADE quiver gauge theories, a particular class of quiver gauge theories enjoying a connection with simply laced Dynkin diagrams.

In section 6, we will introduce the magnetic monopole formula for the Hilbert series of the Coulomb branch [12] of *good* or *ugly* theories [19].

In section 7, some computation involving Higgs and Coulomb branches will be reported.

2 Supersymmetry

2.1 $4d$ supersymmetric theories

The multiplets field content of $3d$ supersymmetric theories is closely related to the ones of $4d$ theories, for this reason, it is important to recall some basic concepts of $4d$ supersymmetry.

A supersymmetric algebra consists of a graded algebra of the form $V = V_0 \oplus V_1$, where V_0 is the Poincarè algebra, made of bosonic operators, and V_1 is the supercharges algebra, made of fermionic operators. A supersymmetric theory is characterized by the number of fermionic degrees of freedom $\mathcal{N}$; this number is given by $\mathcal{N} = \frac{N}{d}$ where N is the number of supercharges and d is the dimension of the smallest irreducible spinorial representation of the $SO(1, D-1)$ our theory lives in.
We first need to recall the supersymmetry algebra of a $4d$ $\mathcal{N}=1$ theory:

$$\begin{aligned}\{Q_a, \bar{Q}_b\} &= 2(\sigma^\mu P_\mu)_{ab} \\ \{Q_a, Q_b\} &= 0 \\ [Q_a, P_\mu] &= 0\end{aligned} \tag{2.1}$$

We will focus ourselves in the case of massless particles using the induced representation method, it means to first boost in an easy-to-handle reference frame and then classifying the representations according to the little group that leaves invariant P^μ, in this case, we boost to the frame where $P^\mu = (E, 0, 0, E)$ so that we have:

$$\{Q_a, \bar{Q}_b\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

Therefore we find a pair of creation annihilation operators $a = \frac{Q_1}{2\sqrt{E}}, a^\dagger = \frac{\bar{Q}_1}{2\sqrt{E}}$ with the following anticommutators:

$$\{a, a\} = \{a^\dagger, a^\dagger\} = 0 \ , \ \{a, a^\dagger\} = 1$$

Remembering that the Qs are Weyl spinors, under the little group $SO(2)$ that leaves P^μ invariant, the following commutator relation holds:

$$[Q_1, J_3] = \frac{1}{2}Q_1 \to [a, J_3] = \frac{1}{2}a$$

That allows us to label the reps with the helicity number λ, the operator a acts on a state with a certain λ transforming it into a state with helicity $\lambda - \frac{1}{2}$. It is important to remind that for each value of helicity in a multiplet, due to CPT invariance, there will also be the value $-\lambda$.

Now we are ready to define the $\mathcal{N} = 1$ multiplets, remember to add the CPT conjugate:

- Chiral $4d$ $\mathcal{N} = 1$ multiplet: $(0, \frac{1}{2})$. Field content: a scalar and a Weyl fermion: (φ, ψ).
- Vector $4d$ $\mathcal{N} = 1$ multiplet: $(\frac{1}{2}, 1)$. Field content: a Weyl fermion and a vector field: (χ, A_μ).

The generalization for $\mathcal{N} > 1$ is easy and we are interested in the resulting multiplets of $\mathcal{N} = 2$ and $\mathcal{N} = 4$, remember to add the CPT conjugate:

- $4d$ $\mathcal{N} = 2$ hypermultiplet: $(-\frac{1}{2}, 0, 0, \frac{1}{2})$. Field content: two scalars and two Weyl fermion: $(\psi_1, \varphi_1, \varphi_2, \psi_2)$. We can split it in two $\mathcal{N} = 1$ chiral multiplets.
- Vector $4d$ $\mathcal{N} = 2$ multiplet: $(0, \frac{1}{2}, \frac{1}{2}, 1)$. Field content: a scalar, two Weyl fermions and a vector field: $(\varphi, \chi_1, \chi_2, A_\mu)$. We can split it an $\mathcal{N} = 1$ chiral multiplet and an $\mathcal{N} = 1$ vector multiplet.
- Vector $4d$ $\mathcal{N} = 4$ multiplet: $(0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$. Field content: three scalars, four Weyl fermions and a vector field: $(\varphi_1, \varphi_2, \varphi_3, \chi_1, \chi_2, \chi_3, \chi_4, A_\mu)$. We can split it an $\mathcal{N} = 2$ hypermultiplet and an $\mathcal{N} = 2$ vector multiplet.

2.2 $3d$ $\mathcal{N}=4$ gauge theories

All the above discussions on $4d$ theories were made because we will now use a technique called *dimensional reduction*. This procedure is used to take a theory with a certain number of supercharges N in D dimensions to another theory with the same number of supercharges N but in a lower dimension D'.
In our case we want to take a $4d$ $\mathcal{N}=2$ theory (that has 8 supercharges) to a $3d$ $\mathcal{N}=4$ theory (here the smallest spinorial irreps has dimension 2 so it still has 8 supercharges).
We first embed $SO(1,2)$ in $SO(1,3)$, then we recall that $so(1,3)=su(2)\times su(2)$. Using the highest weight notation, for an element of $su(4)$ we have $[w_1,w_2]_{su(4)}$. The transformation table is then:

$$\begin{aligned}
[0,0]_{su(4)} &\to [0]_{su(2)} \\
[0,1]_{su(4)} &\to [1]_{su(2)} \\
[1,0]_{su(4)} &\to [1]_{su(2)} \\
[1,1]_{su(4)} &\to [2]_{su(2)} + [0]_{su(2)}
\end{aligned} \tag{2.3}$$

So after a dimensional reduction scalars remain scalars, fermions remain fermions but a vector field A_μ is reduced to a vector field $A_i(x)$ and a scalar A_0 both in the adjoint representation of the gauge group.

2.3 Moduli space

We gave a brief definition of moduli space in the introduction but it is far from being enough. This part will be taken from the lecture notes [9] of the Oxford course on Supersymmetry. Firstly, we need to recall that for a supersymmetric theory with n chiral multiplets, in presence of a gauge group G, the potential V_0 has the form:

$$V_0 = \sum_\phi |\frac{\partial \mathcal{W}(\phi)}{\partial \phi}|^2 + \frac{g^2}{2}\sum_{a=1}^{\dim(G)} (\bar{\phi} T_a^{\mathcal{R}} \phi)^2 \tag{2.4}$$

Where $\mathcal{W}$ is the superpotential and $T_a^{\mathcal{R}}$ are the generators of the group G in a certain representation $\mathcal{R}$. Define the "moment map operator" μ_a as:

$$\mu_a(\bar{\phi},\phi) = \bar{\phi} T_a^{\mathcal{R}} \phi$$

The moduli space of vacua $\mathcal{M}$ is defined by the configurations of fields that minimize V_0 modulo gauge equivalence:

$$\begin{aligned}
\mathcal{M} &= \left\{ \tfrac{\partial \mathcal{W}(\phi)}{\partial \phi} = 0 \ \ \forall \phi \ , \ \ \mu_a(\bar{\phi},\phi)\phi = 0 \ \ \forall a, \phi \right\} / G \\
\mathcal{M} &\cong \left\{ \tfrac{\partial \mathcal{W}(\phi)}{\partial \phi} = 0 \ \ \forall \phi \right\} / G^{\mathbb{C}}
\end{aligned} \tag{2.5}$$

The first line in equation (5) expresses $\mathcal{M}$ geometrically as a Kähler manifold (usually quotient) while the second line expresses $\mathcal{M}$ as an algebraic variety and this is the viewpoint we will adopt in our discussions.

The meaning of the gauge modulus in relation (5) is that, in supersymmetric field theories, there exists a set of inequivalent vacua. The inequivalence is due to the fact that two inequivalent vacuum states can't be connected by a gauge transformation, whilst for equivalent vacua there exists a gauge connection between them, so they represent the same physical state.

The physical description of moduli space is now clear, but it is worth recalling some mathematical definitions.

Definition 2.1. *Hermitian manifold: a Riemannian complex manifold* $(\mathcal{M}, g)$ *with a hermitian metric.*

This means that given a quasi complex structure J, which is a (1,1) tensor that satisfies $J^2 = -1$, then $g(Jx, Jy) = g(x, y)$.
Recalling that the metric can be written as:

$$g = \frac{1}{2} h_{ab}(dz^a \otimes d\bar{z}^b + d\bar{z}^b \otimes dz^a)$$

We can define an associated differential form ω as:

$$\omega = \frac{i}{2} h_{ab}\; dz^a \wedge d\bar{z}^b \tag{2.6}$$

Definition 2.2. *Kähler manifolds: an hermitian manifold which associated differential form* ω *is closed,* $d\omega = 0$.

Definition 2.3. *Hyperkähler manifold: a Riemannian manifold (M, g) equipped with three complex structures I, J, K which obey the quaternion algebra, and such that the metric tensor field g is hermitian with respect to all the three complex structures, and the three hermitian forms associated with I, J, K are closed.*

In the particular case of $3d$ $\mathcal{N} = 4$, the moduli space can be broken into two pieces:

- Coulomb Branch: which is the space parameterized by scalars in the vector multiplet that acquires a non zero vev.
- Higgs Branch: which is the space parameterized by scalars in the hypermultiplet that acquires a non zero vev.

Recalling the multiplets field content we can show that each of these subspaces is spanned by 4 scalars, see next subsection for an explanation, so they are Hyperkähler manifolds.
It is important to remind the reader that the Higgs branch is classical because is protected from quantum corrections while the Coulomb branch isn't. This difference will affect the computation of each of these branches, in particular, the Higgs branch can be computed directly from equation (5) putting the chiral multis that appear in the $\mathcal{N} = 2$ splitting of the $\mathcal{N} = 4$ vector multis equal to 0 in the F-terms.

2.4 $3d$ Theories

The features that emerge from considering a quantum field theory on a plane have yet to be discussed, to introduce them we will start by talking about the QED in 2+1 dimensions. The QED action is the usual:

$$S = \int d^3x \ -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} \tag{2.7}$$

The greek indices run from 0 to 2, from the definitions we have an electric field $E^i = F^{0i}$ with $i = 1, 2$ and a magnetic field $B = F^{12}$. The usual equation of motion leads to:

$$\partial_\mu F^{\mu\nu} = 0$$

And the Bianchi identity reads:

$$d(\star F) = \partial_\mu \left(\frac{1}{4\pi} \epsilon^{\mu\rho\sigma} F_{\rho\sigma} \right) = 0 \tag{2.8}$$

This means that there exists a conserved current, only due to topological reasons, which is not directly read from the Lagrangian. Therefore there could be operators which carry a non trivial charge under this symmetry, an example of them is the monopoles operators we will describe in the next subsection.
We can extend the description above to a general gauge group G, the action reads:

$$S = \int d^3x \ -\frac{1}{2e^2} \mathrm{tr}(F^{\mu\nu} F_{\mu\nu}) \tag{2.9}$$

The equation of motion can be obtained in the classical way, the difference with respect the previous case lies in the fact that, given the rank r of the group G, we will have a conserved current for each $U(1)$ in the Cartan subalgebra $U(1)^r$ of the gauge group G.

There is another characteristic of $3d$ theories we need to discuss, the so called *dualization of the photon* γ.[7] The 1-form $J = \star F$ is closed, so in virtue of the Poincarè lemma, in a contractible topological space is also an exact form. This means that we can think of J as the exterior derivative of a 0-form γ: $J = d\gamma$. Therefore we can regard the Bianchi identity for F as the equation of motion for J: $d(d\gamma) = 0$, and the equation of motion for F as the Bianchi identity for J.

The importance of the dual photon lies in the count of the scalars in the theory. We have said that the $3d$ $\mathcal{N} = 4$ vector multiplet splits in a $3d$ $\mathcal{N} = 2$ hypermultiplet, which contains 4 scalars, and a $3d$ $\mathcal{N} = 2$ vector multiplet. This last multiplet contains 3 scalars, one of which comes from the dimensional reduction procedure and a vector field which, as we have seen, is dual to a photon: in the end, even the vector multiplets contain 4 scalars.

3 Magnetic monopoles

In the last section, we started talking about QED in $3d$ theories without a matter field, so we need to discuss how the introduction of matter fields changes our discussion. The action for the theory is:

$$S = \int d^3x \; -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \psi_j^\dagger(\sigma \cdot iD_A)\psi^j \tag{3.1}$$

Where A is the vector field associated with the $U(1)$ symmetry, ψ is a complex spinor, and the index j runs from 1 to N_f. We said before that we want to construct operators which carry non trivial charge under the $U(1)_t$ topological symmetry with current:

$$j^\mu = \frac{1}{4\pi}\epsilon^{\mu\rho\sigma}F_{\rho\sigma} \tag{3.2}$$

The approach we will use follow closely the explanation given in [5]. A vortex-creating operator can be defined as an operator with a unit vortex charge, this means unit topological charge. The insertion of a vortex operator means that the OPE with J^μ is of the form:

$$J^\mu(x)O(0) \sim \frac{1}{4\pi}\frac{x^\mu}{|x|^3}O(0) + \text{ less singular terms} \tag{3.3}$$

So a vortex operator adds a point vortex charge, modifying equation (19) as:

$$\partial_\mu J^\mu = \delta(x) \tag{3.4}$$

This explicitly shows the change in the topology of the space near the insertion point x. So we can think about monopole operators as local operators introducing a Dirac singularity in the euclidean path integral. CFT teaches us that local operators are in one-on-one correspondence with states in the radially quantized theory. The reason for this is that there exists a conformal transformation that maps the insertion point of a local operator to infinity, so a local operator is exchanged for an incoming or outgoing state. In particular, a monopole operator with charge n is traded for a state on $S^2 \times \mathbb{R}$ with magnetic flux n.

We can realize a Dirac singularity in the insertion point x for a supersymmetric theory considering the following behavior of the $3d$ vector field and the adjoint scalar σ coming from the dimensional reduction:

$$A^{N/S} \sim \frac{m}{2}(\pm 1 - \cos\theta)d\phi \quad \sigma \sim \frac{m}{2r} \tag{3.5}$$

Where θ, ϕ are spherical coordinates around the insertion point, m is an element of the Lie algebra $\mathbf{g}$ of the gauge group G, and $A^{N/S}$ is the gauge field in the northern/southern hemisphere of the S^2 around the insertion point. In particular, we can choose a gauge where m sits in the Cartan subalgebra $\mathfrak{g}$ of the gauge group. Demanding the solution to be single valued for a transition between the two patches, the Dirac quantization condition arises [15], restricting m to be in the GNO dual of the gauge group G modulo Weyl action of this group [20].
Monopole operators can also be charged under the topological symmetry with a charge $J(m)$ equals the magnetic charge modulo elements of the coroot lattice of G.

As explained in [7, 12], in $\mathcal{N} = 4$ is allowed the presence of a background complex adjoint scalar ϕ so that the GNO magnetic charge m can break the gauge group G to a residual symmetry group H_m, which is the commutant in G of m. Thus we have two different types of BPS $\mathcal{N} = 4$ monopole operators in $3d$:

- Bare monopoles if the vev of $\phi = 0$:
- Dressed monopoles if the vev of $\phi \neq 0$

4 Quiver Diagrams

We will focus on a particular way to encode the Lagrangian of an SYM theory in a graph. We will read from the graph the gauge group of the theory and other related characteristics.

Definition 4.1. *Quiver diagram: a diagram characterized by:*

- *A set of vertices V usually represented with circles;*
- *A set of edges E;*
- *A function $s : V \to E$ and a function $t : E \to V$, which give the starting point and the ending point of an edge along with its direction.*

For $3d$ $\mathcal{N} = 4$ theories we can add another set of vertices F that encodes the flavor symmetry, examples of quiver graphs are in Figure 1. In these diagrams, the edges without

Figure 1. Example of quivers for $3d$ $\mathcal{N} = 4$ theories.

an arrow represent a double arrow between the two vertices.

As we can see in Figure 1, under each vertex it is put a group, and the vector multiplet associated transforms in the adjoint representation of that group. Hypermultiplets instead, transform under the bifundamental representation associated to the edge: in case of the left diagram in figure 1 the hypermultiplet transforms as $(m, \bar{n})$ for the edge going from $U(m)$ to $U(n)$ and as $(\bar{m}, n)$ for the edge going from $U(n)$ to $U(m)$. The gauge group of the theory can be read as the product of the groups associated with circle vertices, and in the case of box diagram quivers as in Figure 2, the number of flavors can be read by looking only at the box, so in this case, it is n.

The introduction of quiver diagrams allows us to define a particular class of theories: the ADE quiver gauge theories. These theories are characterized by the fact that their quiver diagrams are affine Dynkin diagrams. In particular thanks to the McKay correspondence we have the following identification (more words on this will be spent in Section 5.3.):

McKay graph for $SU(2)$ subgroup $\leftrightarrow$ ADE affine Dynkin diagram $\leftrightarrow$ ADE quiver gauge theory

It is useful to study the $3d$ $\mathcal{N}=4$ theories by looking at their $\mathcal{N}=2$ multiplets, so we need a prescription to convert an $\mathcal{N}=4$ quiver to an $\mathcal{N}=2$ one. The procedure is as follow:

- For each circle add $rank$(circle) arrows starting and ending in the circle.
- Substitute each edge between a circle/square to a circle with two arrows, one from the circle/square to the other circle and the other in the inverse way.

The physical meaning of the procedure for circles is that we can always split an $\mathcal{N}=4$ vector multiplet in an $\mathcal{N}=2$ vector multiplet (the circle in the diagram) and the associated $\mathcal{N}=2$ hypermultiplet (the loop arrow), the same reasoning applies to edges: we split an $\mathcal{N}=4$ hypermultiplet in two $\mathcal{N}=2$ hypermultiplets.
This subscription allows the possibility to write directly the superpotential $\mathcal{W}$ of the theory:

$$\mathcal{W} = tr\Bigg[\sum_{\text{edges}}\Bigg(\prod_{\substack{\text{arrows connected to}\\ \text{the left node}}} - \prod_{\substack{\text{arrows connected to}\\ \text{the right node}}}\Bigg)\Bigg] \tag{4.1}$$

Notice that to each arrow ending and starting from the same node we associate an adjoint scalar, while to arrows with different ending/starting points we associate a matrix $n_1 \times n_2$ where n_i is the dimension of the fundamental representation associated with that node.

From quiver diagrams, it is possible to infer the dimension of the Higgs branch of the $3d$ theory associated. In the Higgs branch, the Hypers can acquire non zero VEV, this means that the gauge group is completely broken, consequently, the dimension of this space is given by the number of $\mathcal{N}=4$ hypermultiplets, $\dim(R)$, minus the number of gauge fields that become massive due to complete Higgsing:

$$\dim(\mathcal{M}_H) = \sum_{hypers} \dim(R) - |G| \tag{4.2}$$

As an example, for the rightmost quiver in figure 1 the dimension is $kn - k^2$.

5 Hilbert Series

In physics, we are interested in counting the number of gauge invariant operators (GIOs) on the chiral ring [11] because they parameterize the moduli space, to perform such a task we make use of the Hilbert series.
An intuitive notion of Hilbert series (or HS) can be given in terms of monomials invariant: the HS counts the number of monomials invariant at a certain degree. More formally we can define it as in [29]:

Definition 5.1. *A Hilbert Series $HS(t, \mathcal{R})$ of a graded algebra $\mathcal{R} = \oplus_n \mathcal{R}_n$ is the formal power series:*

$$HS(t, \mathcal{R}) = \sum_{n=0}^{+\infty} H(\mathcal{R}, n) t^n$$

Where $H(\mathcal{R}, n)$ is the Hilbert function.
A powerful theorem that we will use is the following:

Theorem 1. *Hilbert-Serre, if $\mathcal{R}$ is finitely generated by s monomials of degree d_i, $i = 1, ..., s$ then:*

$$HS(t, \mathcal{R}) = \frac{P(t, \mathcal{R})}{\prod_{i=1}^{s} (1 - t^{d_i})}$$

To make explicit how all the information on the space $\mathcal{R}$ are included in its Hilbert series, we will use the plethystic program described in [17], from which we will recall the following definitions.

Definition 5.2. *The Plethystic Exponential for a multivariable function $f(t_1, ..., t_n)$ is:*

$$PE[f(t_1, ..., t_n)] = exp\left(\sum_{r=1}^{\infty} \frac{f(t_1^r, ..., t_n^r) - f(0, .., 0))}{r} \right)$$

Definition 5.3. *The Plethystic Logarithm for a multivariable function $g(t_1, ..., t_n)$ is:*

$$PL[g(t_1, ..., t_n)] = \sum_{r=1}^{\infty} \frac{\mu(r) \log g(t_1^r, ..., t_n^r)}{r}$$

Where $\mu(r)$ is the Möbius function:

$$\mu(r) = \begin{cases} 1, & \text{if } r \text{ is square-free with an even number of prime factors;} \\ -1, & \text{if } r \text{ is square-free with an odd number of prime factors;} \\ 0, & \text{if } r \text{ has a squared prime factor.} \end{cases} \tag{5.1}$$

5.1 Example $\mathbb{C}^n/\mathbb{Z}^n$

We want to count the invariant monomials of $\mathbb{C}^n/\mathbb{Z}^n$. The first thing that we need is the number and the structure of the invariant monomials. Without the quotient group, the Cartan algebra of $\mathbb{C}^n$ is $U(1)^n$ so we need n parameters t_i. The quotient group induces the equivalence relation $z_i \sim \xi_n z_i$ where ξ_n is chosen to be the one of n-th root of the unit. Namely, we are considering the action of the representation $\mathcal{R}$ of $\mathbb{Z}^n$ with $\dim \mathcal{R} = n$ on $\mathbb{C}^n$.

$$\begin{pmatrix} z_1' \\ \vdots \\ \vdots \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} \xi_m^{a_1} & 0 & 0 & \cdots & 0 \\ 0 & \xi_m^{a_2} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \xi_m^{a_n} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ \vdots \\ \vdots \\ z_n \end{pmatrix} \tag{5.2}$$

It is indeed easy to write the form of invariant monomials, in fact, we have:

$$z_1^{a_1} \cdots z_n^{a_n} \text{ with } a_1 + \cdots + a_n = 0 \mod n$$

Then the HS computation immediately follows:

$$HS(t_1, \cdots, t_n, \mathbb{C}^n/\mathbb{Z}^n) = \sum_{\substack{a_1 = 0, \cdots, a_n = 0 \\ a_1 + \cdots + a_n = 0 \mod n}}^{+\infty} t_1^{a_1} \cdots t_n^{a_n} = \frac{P(t_1, \cdots, t_m, \mathbb{C}^n/\mathbb{Z}^n)}{\prod_{i=1}^{m} (1 - t_i^n)^i} \tag{5.3}$$

Defining the fugacity map as:

$$t_1 = t x_1 \ , \quad t_2 = t \frac{x_2}{x_1} \ , \quad \cdots \ , \quad t_n = \frac{t}{x_{n-1}}$$

And using it as shown in [1] to link the HS coefficients to irreps of $SU(n)$ in the highest weight notation, we obtain:

$$HS(t^n, x_1, \cdots, x_n) = \sum_{k=0}^{+\infty} \chi([nk, \cdots, 0]) t^{nk} \tag{5.4}$$

5.1.1 The simple case of $\mathbb{C}^2/\mathbb{Z}^2$

In this case, we can explicitly compute the refined HS, obtaining [18]:

$$HS(t^2, x, \mathbb{C}^2/\mathbb{Z}^2) = \sum_{k=0}^{+\infty} \chi([2k]_{SU(2)}) t^{2k} = (1-t^4) PE(\chi([2]_{SU(2)}) t^2) = \frac{1 - (t^2)^2}{(1 - t^2)(1 - x^2 t^2)(1 - x^{-2} t^2)} \tag{5.5}$$

From which we can recover that there are 3 generators for $\mathbb{C}^2/\mathbb{Z}^2$, namely:

$$X = z_1 z_2 \quad Y = z_1^2 \quad Z = z_2^2$$

That is one on one with the denominator of equation (5.5), X with $1 - t^2$, Y with $1 - x^2 t^2$, and Z with $1 - x^{-2} t^2$; linked by the constraint relation $X^2 = YZ$.
We can make the substitution $Z = A + iB$,$Y = A - iB$,$X = iC$ to obtain the relation:

$$A^2 + B^2 + C^2 = 0$$

5.2 ADE classification and Invariant theory

A particular subset of quivers is of particular importance, ADE quivers are in correspondence with Dynkin diagrams of simply laced algebras. The number of Lie groups with a Dynkin diagram that satisfies the condition above is finite, listing all the possibilities we have:

- $A_n \leftrightarrow SU(n+1)$
- $D_n \leftrightarrow Spin(2n)$
- E_6, E_7, E_8 exceptional groups.

A pictorial representation is given in figure 2. What makes these graphs special is that they are the only graphs with the property that their vertices admit a labeling so that twice the value at a vertex is the sum of the values at the adjacent vertices, this property will be of fundamental importance when we will talk about *balance* in section 6.

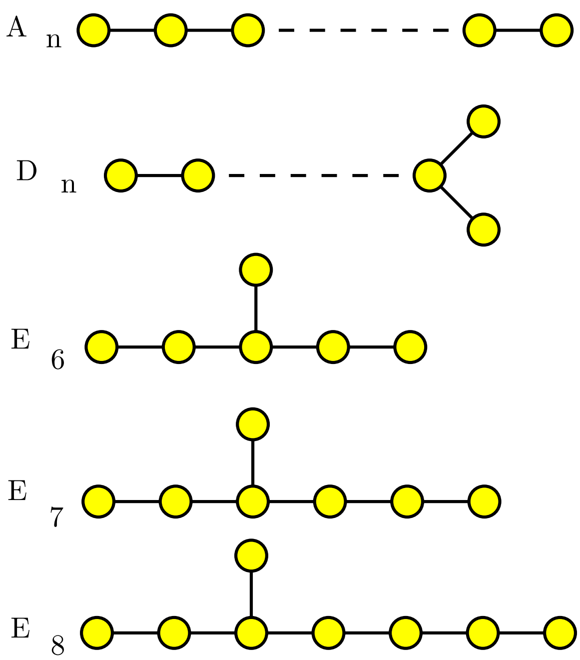

Figure 2. Simply laced Dynking diagrams [10]

Actually, we can take into account affine Lie algebras and their own Dynkin diagram to show that there are more simply laced Lie algebras, there is a thorough description of this in [8]. These affine algebras are important as, given a finite group G and its linear representation ρ_0, we can define the McKay graph of the pair (G, ρ_0) as[2]:

- For each irreducible representation ρ_i put a vertex;

[2]Actually as explained in [14] there are more rules but for subgroups of $SU(2)$ these ones suffice.

- For each vertex put a label to indicate $\dim \rho_i$;
- Two vertices i and j are connected if ρ_j appears in the direct sum of $\rho \otimes \rho_i$.

Then there is theorem thanks to McKay, which is also called the McKay correspondence, which states that:

Theorem 2. *For each G finite subgroup of SU(2), call ρ_0 its natural 2-dimensional representation defined by the inclusion $G \subset SU(2)$. Then the McKay graphs of each (G, ρ_0) are in one-on-one correspondence with the Dynkin diagrams of the affine Lie algebras of A_n, D_n, E_6, E_7, E_8*

For instance, the McKay graph for $\mathbb{Z}_n$ is the Dynkin diagram $\hat{A}_{n-1}$.

5.2.1 Invariant theory for subgroups of $SU(2)$

We are therefore interested in the natural action of finite subgroups of $SU(2)$ on $\mathbb{C}^2$. To classify these finite groups we use the homomorphism $\rho : SU(2) \to SO(3)$ and the fact that, given a subgroup S of $SO(3)$, which are well classified: $\rho^{-1}(S)$ is a finite subgroup if its order is even, otherwise, there exists a finite subgroup of $SU(2)$ of odd order isomorphic to S. The result is that the finite subgroups of $SU(2)$ are:

- The cyclic group of order $n+1$: $\mathbb{Z}_{n+1}$;
- The binary dihedral group of order 4n: Dic_n;
- The binary tetrahedral group of order 24: $\mathbb{BT}$,
- The binary octahedral group of order 48: $\mathbb{BO}$,
- The binary icosahedral group of order 120: $\mathbb{BI}$.

Our discussion will be based on the works of [18, 30, 28], we aim to calculate the Hilbert Series of $\mathbb{C}^2/\Gamma$ with Γ a finite subgroup of $SU(2)$.

Given a finite complex vector space V, we can define the ring of polynomial functions $\mathbb{C}[V]$ in the coordinates $\{z_i\}$, which acts on the basis $\{e_i\}$ of V as $z_i(e_j) = \delta_{ij}$. It is clear from the definitions given that $\{z_i\}$ is a basis of the dual space $V^\star$.
Recalling the definition of homogeneous function of degree n, we can split the polynomial ring in a graded decomposition:

$$\mathbb{C}[V] = \bigoplus_n \mathbb{C}[V]_n$$

With $\mathbb{C}[V]_1 = V^\star$ and more generally $\mathbb{C}[V]_n = S^n(V^\star)$ where S^n indicates the n-th symmetric algebra. In particular, the symmetric algebra of $V^\star$, $S(V^\star)$, inherits a graded decomposition. We define the invariant ring of polynomials under the action of a group G as:

$$\mathbb{C}[V]^G = \{f \in S(V^\star) \ \mid \ gf = f \quad \forall g \in G\} \tag{5.6}$$

Note that this space is finitely generated, thanks to a theorem you can find in [30].

Specializing for a subgroup Γ of $SU(2)$, the invariant ring is generated by 3 monomials X,Y,Z which obeys a constraint relation $c(X,Y,Z) = 0$. Via the isomorphism:

$$\chi : \mathbb{C}^2 \to \mathbb{C}^3 = \{(z_1, z_2) \to (X(z_1, z_2), Y(z_1, z_2), Z(z_1, z_2))\}$$

We can finally write the invariant ring for a subgroup of $SU(2)$ as:

$$\mathbb{C}[V]^\Gamma = \mathbb{C}[X, Y, Z]/\langle c \rangle \cong \mathbb{C}^2/\Gamma \tag{5.7}$$

The characteristic relations for each finite subgroup of $SU(2)$ are: We can now infer the

Γ	$c(X, Y, Z) = 0$
$\mathbb{Z}_{n+1}$	$X^2 + Y^2 + Z^{n+1} = 0$
Dic_{n-2}	$X^2 + Y^2 Z + Z^{n-1} = 0$
$\mathbb{BT}$	$X^2 + Y^3 + Z^4 = 0$
$\mathbb{BO}$	$X^2 + Y^3 + YZ^3 = 0$
$\mathbb{BI}$	$X^2 + Y^3 + Z^5 = 0$

Table 1. Constraint relation for each finite subgroup of $SU(2)$

Hilber series by looking at the degree of the constrain relation and of each generator in it.

5.2.2 Hilbert Series for $\mathbb{C}^2/\mathbb{Z}_{n+1}$

If we pick Z of degree 2 then X,Y have degree $n + 1$, the relation occurs at order $2n + 2$ hence:

$$HS(t) = \frac{1 - t^{2n+2}}{(1 - t^2)(1 - t^{n+1})(1 - t^{n+1})}$$

5.2.3 Hilbert Series for $\mathbb{C}^2/Dic_{n-2}$

If we pick Z of degree 2 then Y has degree $n - 2$ and X has degree $n - 1$, the relation occurs at order $2n - 2$ hence:

$$HS(t) = \frac{1 - t^{2n-2}}{(1 - t^2)(1 - t^{n-2})(1 - t^{n-1})}$$

5.2.4 Hilbert Series for $\mathbb{C}^2/\mathbb{BT}$

If we pick Z of degree 3 then Y has degree 4 and X has degree 6, the relation occurs at order 12 hence:

$$HS(t) = \frac{1 - t^{12}}{(1 - t^3)(1 - t^4)(1 - t^6)}$$

5.2.5 Hilbert Series for $\mathbb{C}^2/\mathbb{BO}$

If we pick Z of degree 4 then Y has degree 6 and X has degree 9, the relation occurs at order 18 hence:

$$HS(t) = \frac{1 - t^{18}}{(1 - t^4)(1 - t^6)(1 - t^9)}$$

5.2.6 Hilbert Series for $\mathbb{C}^2/\mathbb{BI}$

If we pick Z of degree 6 then Y has degree 10 and X has degree 15, the relation occurs at order 30 hence:

$$HS(t) = \frac{1-t^{30}}{(1-t^6)(1-t^{10})(1-t^{15})}$$

6 Coulomb Branch for $3d$ $\mathcal{N}=4$

We want to count the number of gauge invariant operators at a certain degree. We can define a conformal dimension for a BPS operator, this was done in [2, 3], and the result is:

$$\Delta(m) = -\sum_{\alpha\in\Delta_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in\mathcal{R}_i} |\rho_i(m)| \tag{6.1}$$

where the first sum running over the positive roots is the contribution of the $\mathcal{N}=4$ vector multiplets and the second sum over the weights of the matter field in the representation $\mathcal{R}_i$ of the gauge group G is the contribution of the $\mathcal{N}=4$ hypermultiplets H_i with $i=1,...,n$. The conformal dimension reveals itself to be a powerful tool to classify theories, in particular a theory is said to be [19]:

- *good*, if all the BPS monopoles in the theory have $\Delta > \frac{1}{2}$;
- *ugly*, if some BPS monopoles in the theory have $\Delta = \frac{1}{2}$ while other have $\Delta > \frac{1}{2}$;
- *bad*, if exist a BPS monopoles in the theory that has $\Delta < \frac{1}{2}$.

S.Cremonesi, A. Hanany, and A. Zaffaroni in [12] proposed a general formula for *good* and *ugly* theories to compute the Hilbert Series for the Coulomb branch:

$$H_G(t,z) = \sum_{m\in\Gamma^+_{\hat{G}}/\mathcal{W}_{\hat{G}}} z^{J(m)} t^{2\Delta(m)} P_G(t^2, m) \tag{6.2}$$

The fugacity z is for the topological symmetry and $J(m)$ is the topological charge of the monopole operator, $t^{2\Delta(m)}$ counts bare monopole operators with their conformal dimension. The last factor of equation (6.2) counts invariants for the residual gauge group which is unbroken under the presence of the monopole.

A notion of *balance* has been developed in order to test directly from the quiver diagram if its associated theory is good, ugly or bad. Focusing on ADE quivers (simply laced quivers) with gauge nodes $U(N)$, the balance for a node i is [22]:

$$Balance_{ADE}(i) = -2N_i + \sum_{j\in \substack{adjacent\\ nodes}} N_j \tag{6.3}$$

It is said that a quiver is balanced if all its nodes have balance. A positive balanced quiver is then defined as a quiver with all nodes of balance greater or equal to 0. A minimally unbalanced quiver instead has some nodes with balance -1 but the others are balanced. An unbalanced quiver has a least one node with a balance strictly less than -1. In this way, we have that the theory associated with the quiver is[3]:

- *good*: if the quiver is balanced or positively balanced;
- *ugly*: if the quiver is minimally unbalanced;
- *bad*: if the quiver is unbalanced.

In the following discussions, we will always work with *good* theories.

It is worth spending some lines on the ungauging scheme needed for unitary quivers without a flavor node for which a $U(1)$ center of mass factor needs to be ungauged. The ungauging procedure for a unitary node "i" is realized by inserting a delta function with argument one of the magnetic charges associated with that node. This arises the following question: does the choice of the gauge node, on which the ungauging procedure is carried out, affect the computation of the Hilber Series for the Coulomb Branch?
The answer to this has been given by A.Hanany and A.Zajac [26], they have shown that for simply laced quivers the Coulomb branch is unique in the sense that it is not affected by the choice of the ungauged node.
For our analysis of simply laced quivers, we will always try to perform the ungauging procedure on a $U(1)$ node when possible.

6.1 Highest Weight Generating function

The Hilbert Series is not the only generating function that can be used to encode information on the space, in particular in [21] a new method using Dynkin labels was proposed. The Highest Weight Generating function (HWG) is the base of this method we are going to present.

The aim of introducing a new generating function is to look at the thing with a more group theoretical perspective. In fact, consider a refined Hilbert Series:

$$HS(z_i, t_j) = \sum_{k_i} c(z_1, \cdots, z_N) t_1^{k_1} \cdots t_N^{k_N}$$

Where the c-s are coefficients of the term $t_1^{k_1} \cdots t_N^{k_N}$, this HS can be reduced to the form $HS(z_i, t)$ considering all the t_j fugacities equal to a fugacity t.
We can also use Dynkin labels of irreps of the group to put the series in the form:

$$HWG(m_i, t_j) = \sum_{n_i, k_j} b_{n_1, \cdots, n_r, k_1 \cdots, k_n} m_1^{n_1} \cdots m_r^{n_r} t_1^{k_1} \cdots t_N^{k_N} \tag{6.4}$$

This is exactly the Highest Weight Generating function we wanted to define. The mathematical properties of this series can be found in [21],but for what will be needed in our

[3]The identification can be seen by considering the shift to the conformal dimension given by the first non-zero magnetic charge associated to the node "i": $\delta_i(\Delta) = \frac{1}{2} Balance_{ADE}(i) + 1$

discussion suffices to think to the HWG as an information that can totally substitute the HS.

7 Results

In this section, we will present some results obtained with the methodology introduced for some quiver gauge theories.

7.1 $U(1)$ with n flavor gauge theory

7.1.1 Coulomb Branch

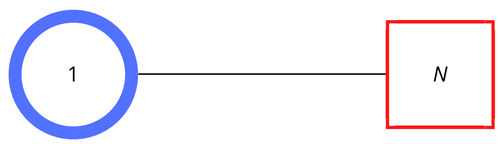

Figure 3. $\mathcal{N} = 4$ quiver for the $U(1)$ with N flavor theory.

In the simple case of $U(1)$ with n flavor gauge theory, the formula for the conformal dimension reads:

$$\Delta(m) = \frac{n}{2}|m| \tag{7.1}$$

The proof of this is straightforward, $U(1)$ is abelian so it has no positive root, the n hypermultiplets lie in the fundamental rep of the gauge group, which weight is 1 so $|\rho(m)| = |1 \cdot m| = |m|$.
The $P_G(t^2, m)$ factor is straightforward because the gauge group can't be broken further and has a Casimir of degree 1, hence:

$$HS(t) = \frac{1}{1-t^2} \sum_{m=-\infty}^{+\infty} t^{2\Delta(m)} = \frac{1-t^{2N}}{(1-t^2)(1-t^N)(1-t^N)} \tag{7.2}$$

Introducing a fugacity z for the topological charge we can refine the Hilbert Series obtaining:

$$HS(t,z) = \frac{1}{1-t^2} \sum_{m=-\infty}^{+\infty} z^m t^{2\Delta(m)} = \frac{1-t^{2N}}{(1-t^2)(1-zt^N)(1-z^{-1}t^N)} \tag{7.3}$$

This result reflects the fact that we have 3 generators for this kind of space, one with no topological charge called Φ, one with topological charge $+1$ called V_+ and one with topological charge -1 called V_-. These generators are constrained via a single relation at order N: $V_+V_- = \Phi^N$. So it can be recovered that in the case $n = 2$ this space is $\frac{\mathbb{C}^2}{\mathbb{Z}^2}$, more in general with n flavor this space is $\frac{\mathbb{C}^2}{\mathbb{Z}^n}$ (note that in section 5.2.2 we used a different normalization for the t: the t in section 5.2 here corresponds to t^2).

7.1.2 Higgs Branch

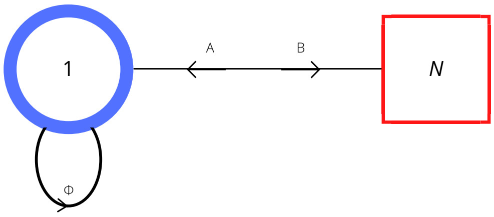

Figure 4. $\mathcal{N}=2$ quiver for the $U(1)$ with N flavor theory.

We are now interested to the Higgs Branch of the theory. First we proceed to compute the dimension of the Higgs branch, from the $\mathcal{N}=4$ quiver, we have $1*N$ quaternionic degrees of freedom (a chiral between a $U(r)$ and a $U(k)$ gives an $r*k$ addend) but, on a generic point of the Higgs branch the gauge group is completely broken so we have 1^2 broken generators (each U(r) gauge node has r^2 generators), hence the quaternionic dimension is

$$\dim \mathcal{M}_H = 1*N - 1^2 = N-1 \tag{7.4}$$

The superpotential, using the convention introduced in Section 4, reads $\mathcal{W}=A\Phi B$, hence the F-terms are:

$$\begin{cases} A\Phi = 0 \\ AB = 0 \\ B\Phi = 0 \end{cases}$$

On the Higgs branch the scalar fields coming from the vector multi do not contribute therefore we can put them equal to 0. We can now write the Hilbert Series for the F-flat space:

$$g(t,z,x_i) = (1-t^2)PE\big[[1,0,\cdots,0]_{SU(N)}\frac{t}{z} + [0,\cdots,0,1]_{SU(N)}\frac{z}{t}\big] \tag{7.5}$$

Where the first factor takes into account the constrain $AB=0$ coming from the non-trivial F-terms. Integrating over the Haar measure of $U(1)$ we get the Hilbert Series for the Higgs Branch:

$$HS(t,x_i) = \frac{1}{2\pi i}\oint \frac{dz}{z} g(t,z,x_i) = \sum_{p=0}[p,0,\cdots,0,p]_{SU(N)}t^{2p} \tag{7.6}$$

7.2 Affine A_2 quiver

7.2.1 Coulomb Branch

From the quiver diagram 5 we can read off the gauge group: $G=U(1)^2$.
Regarding the conformal dimension formula we need to introduce a magnetic charge for each $U(1)$ and, since the gauge group is abelian, there is no positive roots term contribution.

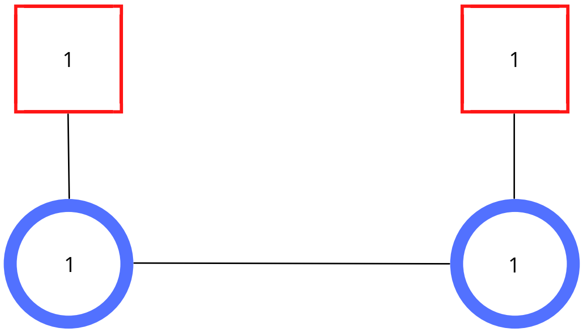

Figure 5. Affine A_2 quiver and its corresponding Dynkin diagram.

Calling the magnetic charges m_1 and m_2, the conformal dimension reads:

$$\Delta(m_1, m_2) = \frac{1}{2}\sum_{i=1}^{3}\sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m_1, m_2)| = \frac{1}{2}(|m_1 - m_2| + |m_1| + |m_2|) \tag{7.7}$$

Again, since the gauge group can't be broken further the unrefined Hilbert series reads:

$$HS(t) = \frac{1}{(1-t^2)^2}\sum_{m_1,m_2=-\infty}^{+\infty} t^{|m_1-m_2|+|m_1|+|m_2|} = \frac{1+4t^2+t^4}{(1-t^2)^4} \tag{7.8}$$

Introducing the fugacities z_1 and z_2 for the topological charge associated to each gauge node, the refined Hilbert series reads:

$$HS(t, z_1, z_2) = \frac{1}{(1-t^2)^2}\sum_{m_1,m_2=-\infty}^{+\infty} z_1^{m_1} z_2^{m_2} t^{(|m_1-m_2|+|m_1|+|m_2|)} \tag{7.9}$$

Via the fugacity map: $z_1 = \frac{y_2}{y_1^2}$ and $z_2 = y_1 y_2$, as suggested in [18], the HS is put in a form such that:

$$\begin{aligned} HS(t, y_1, y_2) &= \tfrac{1}{(1-t^2)^2}\sum_{m_1,m_2=-\infty}^{+\infty} y_1^{m_1} y_2^{m_2} t^{(|m_1-m_2|+|m_1|+|m_2|)} = \\ &= 1 + \left(y_1 y_2 + \tfrac{y_2}{y_1^2} + \tfrac{y_2^2}{y_1} + \tfrac{y_1^2}{y_2} + \tfrac{1}{y_1 y_2} + \tfrac{y_1}{y_2^2} + 2\right)t^2 + \cdots \end{aligned} \tag{7.10}$$

From which we can identify the character of $[1,1]_{SU(3)}$, thus it is possible to shoe that the HS can be expressed as:

$$HS(t, y_1, y_2) = \sum_{k=0}^{+\infty}[k,k]_{SU(3)} t^{2k} \tag{7.11}$$

This means that the classical $U(1)_t^2$ symmetry of the Coulomb branch is extended to $SU(3)$, in particular to $SU(3)/\mathbb{Z}_3$ [12].

7.2.2 Higgs Branch

Proceeding as in section 7.1.2, we start studying the Higgs Branch of this theory. First we want to calculate its dimension in quaternionic unit:

$$\dim \mathcal{M}_H = 1*1 + 1*1 + 1*1 - (1^2 + 1^2) = 1 \tag{7.12}$$

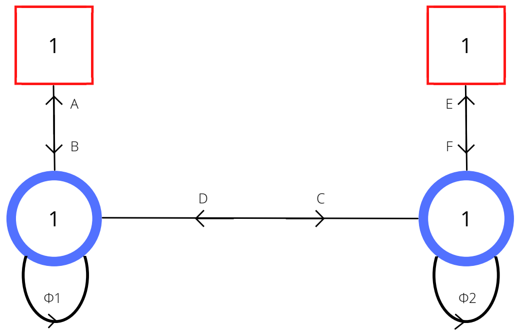

Figure 6. $\mathcal{N} = 2$ quiver for the A2 affine theory.

Then we can summarize in the following table the matter fields content:

Fields	$U(1)_f$	$U(1)$	$U(1)$	$U(1)_f$
Fugacities	x	y	z	q
A	1	-1	0	0
B	-1	1	0	0
C	0	1	-1	0
D	0	-1	1	0
E	0	0	1	-1
F	0	0	-1	1

Using the conventions introduced in section 4, the superpotential reads:

$$\mathcal{W} = \text{tr}\big(- B\Phi_1 A + C\Phi_1 D - D\Phi_2 C + E\Phi_2 F \big)$$

After imposing to be on the Higgs Branch i.e. $\Phi_i = 0$, the only non trivial F-terms are:

$$\begin{cases} -BA + CD = 0 \\ -DC + EF = 0 \end{cases}$$

Therefore the Hilbert Series for the F-flat space reads:

$$g(t, x, y, z, q) = \frac{PE\Big[\big(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{q} + \frac{q}{z}\big)t\Big]}{PE\big[2t^2\big]} \tag{7.13}$$

Integrating over the Haar measure of the Gauge Group $G = U(1) \times U(1)$ we get the Hilbert series for the Higgs Branch:

$$HS(t, x, q) = \frac{1}{(2\pi i)^2} \oint \frac{dydz}{yz} g(t, x, y, z, q) = -\frac{q\left(t^2 - t + 1\right)\left(t^2 + t + 1\right)x}{\left(qt^3 - x\right)\left(q - t^3 x\right)} \tag{7.14}$$

As expected the quaternionic dimension agrees with half the order of the pole for the unrefined HS in $t = 1$ (i.e. its complex dimension):

$$HS(t) = \frac{t^2 - t + 1}{(t-1)^2\left(t^2 + t + 1\right)}$$

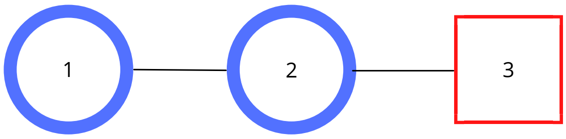

Figure 7. Quiver diagram corresponding to (1)-(2)-[3]

7.3 The $(1)-(2)-[3]$ quiver

7.3.1 Coulomb Branch

From the quiver diagram 7 we read that the gauge group is $G = U(1) \times U(2)$. We introduce a magnetic charge m_1 for the $U(1)$ node and two magnetic charges m_1, m_2 for the gauge node $U(2)$.
The formula for the conformal dimension now include a positive root term because the group $U(2)$ is not abelian. Therefore we need to find the positive root of $U(2)$, first recalling that $U(2) = U(1) \times SU(2)$ the root system splits in $\Delta = \Delta_{U(1)} \sqcup \Delta_{SU(2)}$. But $U(1)$ is abelian so it has no root, the roots of $SU(2)$ are very easy to find, recalling the $SU(2)$ algebra:

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

We can put it in a Cartan basis defining $J^{\pm} = (J_1 \pm iJ_2)$ and considering J_3 diagonal:

$$\begin{gathered}[J_3, J_+] = J_+ \\ [J_3, J_-] = -J_- \\ [J_+, J_-] = 2J_3\end{gathered} \tag{7.15}$$

Hence the root vector has components (1,-1) so it is also a positive root. So the conformal dimension is:

$$\begin{aligned}\Delta(m_1, m_2, m_3) = -\textstyle\sum_{\alpha\in\Delta_+} |\alpha(m_1, m_2, m_3)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in\mathcal{R}_i} |\rho_i(m_1, m_2, m_3)| = \\ = -|m_2 - m_3| + \tfrac{1}{2}(|m_1 - m_2| + |m_1 - m_3|) + \tfrac{3}{2}(|m_1| + |m_2| + |m_3|)\end{aligned} \tag{7.16}$$

Recalling the unrefined Hilbert Series formula:

$$HS(t) = \sum_{m_1,m_2,m_3=-\infty}^{+\infty} P_{U(1)}(t^2, m_1)P_{U(2)}(t^2, m_2, m_3)t^{2\Delta(m_1,m_2,m_3)} \tag{7.17}$$

We also need to calculate the classical factor $P_G(t^2, m_1, m_2, m_3) = P_{U(1)}(t^2, m_1)P_{U(2)}(t^2, m_2, m_3)$.
The $U(1)$ group has a one dimensional Casimir so $P_{U(1)}(t^2, m_1) = \frac{1}{1-t^2}$, while for the group $U(2)$ there are two cases:

- $U(2)$ is broken to is maximal torus $U(1)^2$ so $m_2 \neq m_3$ and $P_{U(2)}(t, m_1, m_2, m_3) = \frac{1}{(1-t^2)^2}$;
- $U(2)$ remains unbroken so $m_2 = m_3$ and $P_{U(2)}(t, m_1, m_2, m_3) = \frac{1}{(1-t^2)(1-t^4)}$.

We can now write the unrefined Hilbert series:

$$HS(t) == \frac{(1-t^4)(1-t^6)}{(1-t^2)^8} \tag{7.18}$$

The center of the group G is $U(1) \times U(1)$, so we introduce two fugacities z_1, z_{23} that allow us to write the refined Hilbert Series [7, 21]:

$$\begin{aligned} HS(t, z_1, z_{23}) &= \frac{1}{(1-t^2)(1-t^4)} \sum_{m_1, m_2 = -\infty}^{+\infty} z_1^{m_1} z_{23}^{2m_2} t^{-2|m_1 - m_2| + 3(|m_1| + 2|m_2|)} + \\ + \frac{1}{(1-t^2)^2} &\sum_{m_1, m_2 \neq m_3 = -\infty}^{+\infty} z_1^{m_1} z_{23}^{m_2+m_3} t^{-2|m_2-m_3| + (|m_1-m_2| + |m_1-m_3|) + 3(|m_1|+|m_2|+|m_3|)} \\ &= \frac{(1-t^4)(1-t^6)}{(1-t^2)^2(1-z_1^{-1}t^2)(1-t^2 z_1)(1-z_2^{-1}t^2)(1-t^2 z_2)(1-z_1^{-1}z_2^{-1}t^2)(1-t^2 z_1 z_2)} \end{aligned} \tag{7.19}$$

We can take an expansion in t:

$$HS(t, z_1, z_{23}) = 1 + \left(2 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t + O(t^2) \tag{7.20}$$

And by using the fugacity map [18] $z_1 = \frac{y_2}{y_1^2}$ and $z_{23} = y_1 y_2$ we have:

$$HS(t, y_1, y_2) = 1 + \left(2 + \frac{y_2}{y_1^2} + \frac{y_1^2}{y_2} + y_1 y_2 + \frac{1}{y_1 y_2} + \frac{y_2^2}{y_1} + \frac{y_1}{y_2^2}\right) t + O(t^2) = 1 + [1,1]_{SU(3)} t + O(t^2) \tag{7.21}$$

So we can infer that a symmetry enchantment has happened. Going further with the expansion we can get the following HWG function:

$$\begin{aligned} HWG(t, \mu_1, \mu_2) &= PE\left(\mu_1 \mu_2 (t + t^2) + (\mu_1^3 + \mu_2^3) t^3 - \mu_1^3 \mu_2^3 t^6\right) = \\ &= \frac{(1-\mu_1^3\mu_2^3 t^6)}{(1-\mu_1\mu_2 t)(1-\mu_1\mu_2 t^2)(1-\mu_1^3 t^3)(1-\mu_2^3 t^3)} \end{aligned} \tag{7.22}$$

Where μ_1, μ_2 are Dynkin labels for $SU(3)$.

7.3.2 Higgs Branch

Proceeding as in section 7.1.2, we start studying the Higgs Branch of this theory. First, we want to calculate its dimension in quaternionic unit:

$$\dim \mathcal{M}_H = 1 * 2 + 2 * 3 - (2^2 + 1^2) = 3 \tag{7.23}$$

Then we can summarize in the following table the matter fields content:

Fields	$U(1)$	$U(2)$	$U(3)_f$
Fugacities	q	a, b	x, y, z
A	1	[0,1]	[0,0,0]
B	-1	[1,0]	[0,0,0]
C	0	[1,0]	[0,0,1]
D	0	[0,1]	[1,0,0]

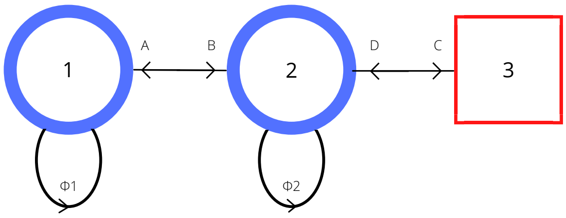

Figure 8. $\mathcal{N}=4$ $(1)-(2)-[3]$ quiver

Using the conventions introduced in section 4, the superpotential reads:

$$\mathcal{W} = \mathrm{tr}\big(A\Phi_1 B - B\Phi_2 A + C\Phi_2 D\big)$$

After imposing to be on the Higgs Branch i.e. $\Phi_i = 0$, the only non trivial F-terms are:

$$\begin{cases} AB = 0 \\ -BA + CD = 0 \end{cases}$$

Therefore the Hilbert Series for the F-flat space reads:

$$g(t,x,y,z,q,a,b) = \frac{PE\Big[\big(x[0,1] + \frac{1}{x}[1,0] + [1,0][0,0,1] + [0,1][1,0,0]\big)t\Big]}{PE\big[t^2 + [1,1]t^2\big]} \tag{7.24}$$

Integrating over the Haar measure of the Gauge Group $G = U(1)\times U(2)$ we get the Hilbert series for the Higgs Branch:

$$\begin{aligned} HS(t,x,y,z) &= \tfrac{1}{(2\pi i)^3 2!}\oint dq\ da\ db\ -\tfrac{a-b}{ab^2 q} g(t,x,y,z,q,a,b) = \\ &- \tfrac{(((1+t^2)(1-t+t^2)(1+t+t^2)x^2y^2z^2)}{((t^2x-y)(x-t^2y)(t^2x-z)(t^2y-z)(x-t^2z)(y-t^2z)))} \end{aligned} \tag{7.25}$$

The unrefined version is:

$$HS(t) = \frac{(1+t^2)(1-t+t^2)(1+t+t^2)}{(-1+t)^6(1+t)^6}$$

The pole in $t=1$ is of order 6, exactly twice the quaternionic dimension of the Higgs Branch.

7.4 The $(1)-(2)-(3)$ quiver with Adj on (3)

7.4.1 Coulomb Branch

From the quiver diagram 9 we read that the gauge group is $G = U(1)\times U(2)\times U(3)\ /\ U(1)$, where a $U(1)$ cent of mass has been ungauged. We need a total of 5 magnetic charges, two charges m_{21}, m_{22} for the $U(2)$ node and three charges m_{31}, m_{32}, m_{33} for the $U(3)$ node.

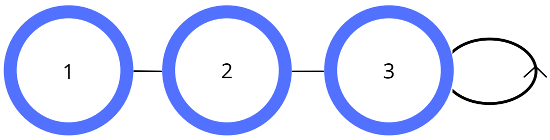

Figure 9. $\mathcal{N}=4$ quiver diagram for the $(1)-(2)-(3)$ quiver with Adj on (3)

Remembering that the loop hypermultiplet attached to the $U(3)$ node transforms in the adjoint representation of the associated gauge node, the conformal dimension formula reads:

$$\begin{gathered}\Delta(m)=-\sum_{\alpha\in\Delta_+}|\alpha(m)|+\tfrac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in\mathcal{R}_i}|\rho_i(m)|=\\ =-|m_{21}-m_{22}|-|m_{31}-m_{32}|-|m_{31}-m_{33}|+\\ +\tfrac{1}{2}\Big(|m_{21}|+|m_{22}|+\sum_{\substack{i\,=\,1,2\\ j\,=\,1,2,3}}|m_{2i}-m_{3j}|+\sum_{\substack{i\,=\,1,2,3\\ j\,=\,1,2,3}}|m_{3i}-m_{3j}|\Big)\end{gathered}\tag{7.26}$$

The refined HS for the Coulomb branch is, given that the center of G is $U(1)^2$:

$$HS_G(t,z)=\sum_{m\in\Gamma^+_{\hat{G}}/\mathcal{W}_{\hat{G}}}z_1^{m_{21}+m_{22}}z_2^{m_{31}+m_{32}+m_{33}}t^{2\Delta(m)}P_G(t,m)$$

Depending on how G is broken we have:

$$P_{U(3)}(t,m)=\begin{cases}\frac{1}{(1-t^2)^3} & m_{3i}=m_{3j}\\ \frac{1}{(1-t^2)^2(1-t^4)} & m_{31}=m_{32}\text{ or }m_{31}=m_{33}\\ \frac{1}{(1-t^2)(1-t^4)(1-t^6)} & m_{31}\neq m_{32}\text{ and }m_{31}\neq m_{33}\end{cases}\tag{7.27}$$

$$P_{U(2)}(t,m)=\begin{cases}\frac{1}{(1-t^2)^2} & m_{21}=m_{22}\\ \frac{1}{(1-t^2)^2(1-t^4)} & m_{21}\neq m_{22}\end{cases}$$

Summing the series leads to:

$$HS_G(t,z)=1+t^2\left(\left(z_1^2+z_1\right)z_2^3+\frac{z_1+1}{z_1^2z_2^3}+z_1z_2^2+\frac{1}{z_1z_2^2}+(z_1+1)z_2+\frac{\frac{1}{z_1}+1}{z_2}+z_1+\frac{1}{z_1}+2\right)+O(t^2)$$

With the help of the fugacity map: $z_1 \to x_2$ and $z_2 \to \frac{x_1}{x_2}$, we can recognize the character of [0,1] G_2. Going on with the expansion in powers of t and characters of G_2 we find:

$$\begin{aligned} HS_G(t,z) = 1 + [0,1]t^2 + ([2,0]+[0,2])t^4 + ([3,0]+[0,3]+[2,1])t^6 + \\ ([4,0]+[0,4]+[2,2]+[3,1]+[0,2])t^8 + \\ ([5,0]+[0,5]+[3,2]+[2,3]+[4,1]+[0,3]+[3,1])t^{10} + \cdots \end{aligned} \tag{7.28}$$

$$HWG(t,\mu_1,\mu_2) = 1 + \mu_1 t^2 + (\mu_2^2 + \mu_1^2)t^4 + (\mu_2^3 + \mu_1^3 + \mu_2^2\mu_1)t^6 + \cdots$$

The unrefined Hilbert Series reads:

$$HS(t) = \frac{t^{10} + 4t^8 + 9t^6 + 9t^4 + 4t^2 + 1}{(1-t^2)^{10}}$$

And associating the Dinkin labels $[i,j] \to \mu_2^i \mu_1^j$ the HWG reads:

$$HWG(t,\mu_1,\mu_2) = \frac{1 + \mu_1\mu_2^3 t^{10}}{(1-\mu_1 t^2)(1-\mu_2^2 t^4)(1-\mu_2^3 t^6)(1-\mu_1^2 t^8)} \tag{7.29}$$

7.4.2 Higgs Branch

We will now analyze the Higgs branch. First, we convert the quiver diagram in figure (6) to an $\mathcal{N}=2$ quiver. Using the conventions introduced in section 4, the superpotential

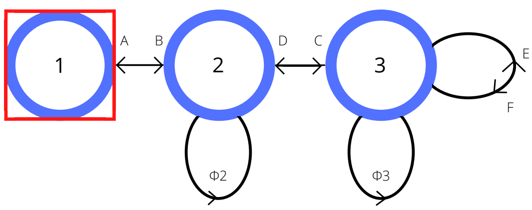

Figure 10. $\mathcal{N}=2$ quiver diagram for the $(1)-(2)-(3)$ quiver with Adj on (3)

reads:

$$\mathcal{W} = tr\big(-B\Phi_2 A + C\Phi_2 D - D\Phi_3 C + E\Phi_3 F - F\Phi_3 E\big) \tag{7.30}$$

To see the global symmetry it is better to rename the two fields coming from the Adj multiplet $E = \phi^\alpha$, $F = \phi^\beta$:

$$\begin{aligned} \mathcal{W} &= tr\big(-B\Phi_2 A + C\Phi_2 D - D\Phi_3 C + \phi^\alpha \Phi_3 \phi^\beta - \phi^\beta \Phi_3 \phi^\alpha\big) = \\ &= tr\big(-B\Phi_2 A + C\Phi_2 D - D\Phi_3 C + \epsilon_{\alpha\beta}\phi^\alpha \Phi_3 \phi^\beta\big) \end{aligned} \tag{7.31}$$

In this way an $SU(2)_g$ global symmetry is manifested. After imposing to be on the Higgs Branch i.e. $\Phi_i = 0$, the only non trivial F-terms are:

$$\begin{cases} -BA + CD = 0 \\ -DC + [E,F] = 0 \end{cases} \tag{7.32}$$

The space of solutions for the system is the F-flat space $\mathcal{F}$:

$$\begin{gathered}\mathcal{F} = \{\Phi_i = 0, \langle A\rangle \neq 0, \langle B\rangle \neq 0, \langle C\rangle \neq 0, \langle D\rangle \neq 0, \langle E\rangle \neq 0, \langle F\rangle \neq 0, \\ CD - BA = 0, [E,F] - DC = 0\}\end{gathered} \tag{7.33}$$

We can summarize the hypermultiplets field content in this table:

Fields	$U(1)_f$	$U(2)$	$U(3)$	$SU(2)_g$
Fugacities	x	y_1, y_2	z_1, z_2, z_3	c
A	1	[0,1]	0	[0]
B	-1	[1,0]	0	[0]
C	0	[1,0]	[0,0,1]	[0]
D	0	[0,1]	[1,0,0]	[0]
$E = \phi^\alpha$	0	[0,0]	[1,0,1]	[1]
$F = \phi^\beta$	0	[0,0]	[1,0,1]	[1]

We can write the Hilbert Series of the $\mathcal{F}$-flat space:

$$g(t,x,y,z) = \frac{PE\big(x[0,1]t + [1,0]\frac{t}{x} + ([1,0][0,0,1])t + ([0,1][1,0,0])t + [1,0,1][1]_{SU(2)_g}t\big)}{PE\big([1,1]t^2 + [1,0,1]t^2\big)} \tag{7.34}$$

Integrating over the Haar measure for the group G, we obtain the refined Hilbert Series where a fugacity c has been associated to $SU(2)_g$:

$$\begin{gathered}HS(t,c) = \frac{1}{(2\pi i)^5 3!2!}\oint \frac{(y_1-y_2)(z_1-z_2)(z_1-z_3)(z_2-z_3)}{y_1 y_2^2 z_1 z_2^2 z_3^3} g(t,x,y,z) = \\ \frac{c^7\left(t^2+1\right)\left(t^2-t+1\right)\left(t^2+t+1\right)\left(t^4-t^2+1\right)}{(c-t)^3(c+t)(ct-1)^3(ct+1)(c^2+ct+t^2)(c^2t^2+ct+1)(c-t^3)(ct^3-1)}\end{gathered} \tag{7.35}$$

Using the Plethystic program [16, 4] we can extract the PL of the Hilbert series to get:

$$PL[HS(t,c)] = ct + \frac{t}{c} + c^2t^2 + t^2 + \frac{t^2}{c^2} + c^3t^3 + ct^3 + \frac{t^3}{c} + \frac{t^3}{c^3} = [1]t + [2]t^2 + [3]t^3 - t^{12} \tag{7.36}$$

And infer that we have a moduli space that, removed the free part i.e. the $\mathbb{C}^2$ factor corresponding to the position of the instanton, is an hypersurface in $\mathbb{C}^7$ spanned by the 3 generators of order 2 and the 4 generators of order 3, constrained by a relation occurring at order 12.

Associating a Dynkin fugacity m to the character of A1 it is possible to write the HWG:

$$\begin{gathered}HWG(t,m) = \frac{1}{(1-t)^5(t+1)^5(t^2+1)^3(t^2-t+1)^2(t^2+t+1)^2(t^4+1)(mt-1)^3(mt+1)(m^2t^2+mt+1)(mt^3-1)} \cdot \\ \cdot\big(m^5t^{29} + 3m^5t^{21} + m^5t^{19} + 3m^5t^{17} + m^5t^9 + 2m^4t^{24} + 3m^4t^{22} - m^4t^{20} - 2m^4t^{18} + \\ -m^4t^{16} - m^4t^{14} + m^4t^{12} + m^4t^{10} - m^4t^8 - m^4t^6 + m^3t^{25} + m^3t^{23} - 2m^3t^{21} - 3m^3t^{19} + \\ +m^3t^{17} - 3m^3t^{15} - m^3t^{11} - 2m^3t^9 - 2m^3t^7 + m^3t^5 + m^2t^{24} - 2m^2t^{22} - 2m^2t^{20} - m^2t^{18} + \\ -3m^2t^{14} + m^2t^{12} - 3m^2t^{10} - 2m^2t^8 + m^2t^6 + m^2t^4 - mt^{23} - mt^{21} + mt^{19} + mt^{17} - mt^{15} + \\ -mt^{13} - 2mt^{11} - mt^9 + 3mt^7 + 2mt^5 + t^{20} + 3t^{12} + t^{10} + 3t^8 + 1\big)\end{gathered} \tag{7.37}$$

Now we want to compute the dimension of the Higgs branch: from the $\mathcal{N}=4$ quiver, we have $1*2+2*3+3^2$ quaternionic degrees of freedom (each of the loops around $U(r)$ gives an r^2 addend while an edge between a $U(r)$ and a $U(k)$ gives an $r*k$ addend), on a generic point of the Higgs branch the gauge group is completely broken so we have 2^2+3^2 broken generators (each $U(r)$ has r^2 generators and the $U(1)$ node is ungauged), hence the quaternionic dimension is:

$$\dim \mathcal{M}_H = 1*2+2*3+3^2-(2^2+3^2)=4 \tag{7.38}$$

Which agrees with the complex dimension given by the pole in $t=1$ of the unrefined HS:

$$HS(t)=\frac{\left(t^2+1\right)\left(t^2-t+1\right)\left(t^4-t^2+1\right)}{(t-1)^8(t+1)^2\left(t^2+t+1\right)^3}=\frac{t^8-t^7+t^6+t^2-t+1}{(t-1)^8(t+1)^2\left(t^2+t+1\right)^3} \tag{7.39}$$

8 Conclusions

In this project, we illustrated a simple formula that allowed us to characterize non perturbatively the Coulomb branch of $3d$ $\mathcal{N}=4$ gauge theories, our discussion was based on three main ingredients [12]:

- The correspondence between a magnetic monopole operator and a set of magnetic GNO charges m_j.
- The dimension Δ of a bare monopole operator and in particular, its contribution coming from vector and hyper multiplets.
- The classical factor $P_G(t^2,m)$ which counts the gauge invariants of the residual gauge group G_m, which is unbroken by the GNO magnetic flux m, according to their dimension.

Equipped with this machinery a huge class of theories can be considered, in particular, we can address the problem of quivers with gauge groups different from the unitary group.
Moreover, in our discussion, we introduced an important set of theories the ADE quiver gauge theories for which the instanton moduli space can now be directly computed from the Coulomb branch without involving the Higgs branch of the dual theory [13].
Further works were done in connecting these results to classical groups nilpotent orbits [23] allowing a study based on Hasse diagrams that groups different quivers in a family structure; for some quivers a connection with Type IIB string theory is possible [6] and a technique to mimic Kraft-Procesi transitions between quivers that belong to the same group (in the sense that their moduli space is the same as the closure of a nilpotent orbit or a nilpotent orbit of such a group) via Hanany-Witten transitions [25] has been developed.

9 Acknowledgement

I would like to thank Prof. Amihay Hanany for having introduced me to this quivers world and having me realize what does it means to study a topic of current interest, and in which

direction should I push myself in order to be able to contribute to the research frontier. I would like to thank also Julis Grimminger and Salvo Mancani with whom I had useful discussions that allowed me to understand better some concepts not so clear at first sight, without their aid this project would not be the way it is.

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