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Hilbert Series for the Coulomb branch of $3d$ $\mathcal{N} = 4$ gauge theories

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Introduction

Our discussion will be held around the concept of **Moduli Space**:

- Can be thought as the set of zero energy configurations of the field theory.
- Bridge between supersymmetric gauge theories and geometry.
- Parametrized by gauge invariant chiral operators.
- In $3d$ the gauge invariant chiral operators are 't Hooft monopole operators.

Cremonesi 1701.00641 [4]

Outline

- Moduli Space
  - Supersymmetric gauge theories
  - Definition
  - Example
  - $3d\ \mathcal{N} = 4$

- Monopole operators

- Hilbert Series
  - HS for the Coulomb Branch
  - Results for $U(1)$ with N flavor

Moduli Space: Supersymmetric gauge theories

A $3d$ $\mathcal{N}=4$ supersymmetric gauge theory is specified by:

- A **gauge group** G with associated $rk(G)$ vector multiplets V^a. Each V^a decomposes in an $\mathcal{N}=2$ V-multi $\mathcal{V}$ and an $\mathcal{N}=2$ chiral multi Φ.
- A **matter content**, given a representation $\mathcal{R}$ of G we associate $\dim(\mathcal{R})$ hypermultiples Y^i. Each Y decomposes in chiral $\mathcal{N}=2$ multis X and $\bar{X}$ which sits respectively in $\mathcal{R}$ and $\bar{\mathcal{R}}$.
- A holomorphic gauge invariant **superpotential**[1] $\mathcal{W}$.

Cremonesi 1701.00641 [4]

[1] Actually in $\mathcal{N}=4$ we restrict the form of the superpotential to be $\mathcal{W}=\mathrm{tr}\bar{X}\Phi X$, with matter fields transforming in the bifundamental rep.

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Moduli Space: Definition

Given the ingredients above we can define a potential V_0: [2]

$$V_0 = \sum_{\phi} |\frac{\partial \mathcal{W}(\phi)}{\partial \phi}|^2 + \frac{g^2}{2} \sum_{a=1}^{\dim(G)} (\bar{\phi} T_a^{\mathcal{R}} \phi)^2$$

The **moduli space of vacua** $\mathcal{M}$ is defined by the configurations of fields that minimize V_0 modulo gauge equivalence:

$$\mathcal{M} = \left\{ \frac{\partial \mathcal{W}(\phi)}{\partial \phi} = 0 \ \ \forall \phi \ , \ \ \bar{\phi} T_a^{\mathcal{R}} \phi = 0 \ \ \forall a, \phi \right\} / G$$

$$\mathcal{M} \cong \left\{ \frac{\partial \mathcal{W}(\phi)}{\partial \phi} = 0 \ \ \forall \phi \right\} / G^{\mathbb{C}}$$

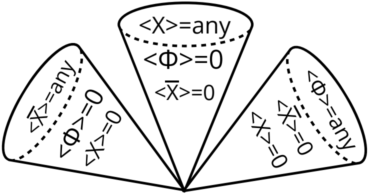
Cremonesi 1701.00641 [4]

[2] Here ϕ stands for each chiral multi.

Moduli Space: Example

In the simple example of $\mathcal{W} = \bar{X}\Phi X$ the F-terms are:

$$F\text{-terms} = \begin{cases} \Phi X = 0 \\ \bar{X}\Phi = 0 \\ \bar{X}X = 0 \end{cases}$$



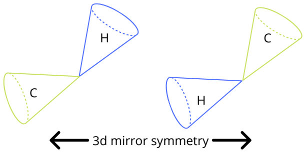
From which we deduce the chiral ring:

$$\mathbb{C}[X, \bar{X}, \Phi]/\langle \Phi X, \bar{X}\Phi, \bar{X}X \rangle$$

Cremonesi 1701.00641 [4]

Moduli Space: $3d\ \mathcal{N} = 4$

- $3d$ supersymmetric theories enjoy a powerful duality called $3d$ **mirror symmetry**.
- Under this duality the **Coulomb Branch** and the **Higgs Branch** of dual theories are exchanged.



- **Coulomb Branch** is the moduli space parametrized by scalars in the V-multi.
- **Higgs Branch** is the moduli space parametrized by scalars in the H-multi.

K. Intriligator and N. Seiberg 9607207 [8]

Monopole Operators

- Obtained by inserting an operator with a Dirac singularity at the insertion point x:

$$A_\pm \sim \frac{m}{2}(\pm 1 - \cos\theta)d\phi \qquad \sigma \sim \frac{m}{2r}$$

- Specified by the **magnetic charge** $m \in \mathbf{h}/\mathcal{W}$.
  Well-definedness of the gauge bundle implies $e^{2\pi i m} = \mathbf{id}_G$ so $m \in \Lambda_{G^\vee}/\mathcal{W}$.
- Subject to **quantum relations**.
- Conformal dimension given by:

$$\Delta(m) = -\sum_{\alpha\in\Delta_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in\mathcal{R}_i} |\rho_i(m)|$$

A.Kapustin 0501015 [9],P. Goddard Nucl.Phys.B125(1977) [7]

Monopole Operators

- In $3d$ there is a **topological symmetry** associated to the current:

$$j^{\mu} = \frac{1}{4\pi}\epsilon^{\mu\rho\sigma}F_{\rho\sigma}$$

- Magnetic monopoles can be charged under this symmetry with a charge $J(m)$ equal to the magnetic charge modulo elements of the coroot lattice of G.

A.Kapustin 0501015 [9],

P. Goddard Nucl.Phys.B125(1977) [7]

Further discussions in: D.Bashkirov and A.Kapustin [1],Benna Klebanov Klose[2],V.Borokhov A.Kapustin X.Wu[3],F. Englert and P. Windey[6]

Hilbert Series: HS for the Coulomb Branch

- Hilbert Series counts invartiant gauge operators at a given degree.
- It encodes information on the number of generators and the relations among them.
- For the Coulomb branch of $3d$ $\mathcal{N} = 4$ gauge theories the HS is given by the **magnetic monopole formula**:

$$H_G(t, z) = \sum_{m \in \Gamma^+_{\hat{G}} / \mathcal{W}_{\hat{G}}} z^{J(m)} t^{2\Delta(m)} P_G(t, m)$$

where $P_G(t, m) = \prod_{i=1}^{r} \frac{1}{1 - t^{2d_i(G_m)}}$ counts polynomials in the Casimir invariant of the residual gauge group G_m.

A.Hanany S.Cremonesi A.Zaffaroni 1309.2657 [5]

Hilbert Series: Results for $U(1)$ with N flavor

- $U(1)$ Casimir degree is 1;
- The conformal dimension reads: $\Delta(m) = \frac{N}{2}|m|$

Therefore the Hilbert Series is:

$$HS(t^2, z) = \frac{1}{1-t^2} \sum_{m=-\infty}^{+\infty} z^m t^{2\Delta(m)} = \frac{1-t^{2N}}{(1-t^2)(1-zt^N)(1-z^{-1}t^N)}$$

The generators satisfy $V_+ V_- = Z^N$, and the space is $\mathbb{C}^2/\mathbb{Z}^N$. If $N = 2$ then $HS(t, w) = \frac{1}{1-t^4} PE([2]_w t^2)$, with $z = w^2$, symmetry enchantment to $SU(2)$.

A.Hanany S.Cremonesi A.Zaffaroni 1309.2657 [5]

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Q & A

Thanks for your attention!

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