Chapitre 1

Introduction

1.1 Definitions

In this section we shall introduce the basics of *linear programming* and develop them in the further chapters.

Definition 1.1.1 (Linear program)**.** A linear program is an optimisation problème composed of a *cost vector* $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ and a *cost function* over all $n-$dimensional vectors of the form

$$\mathbf{c} \cdot \mathbf{x} = \sum_{i=1}^{n} c_i x_i$$

which we seek to minimise. This cost function is subject to a set of *linear constraints* and *linear equalities*. The problem is then of the form

$$\begin{array}{rll} \text{minimise} & \mathbf{c} \cdot \mathbf{x} & \\ \text{subject to} & \mathbf{a}_i' \cdot \mathbf{x} \geq b_i & i \in M_1 \\ & \mathbf{a}_i' \cdot \mathbf{x} \leq b_i & i \in M_2 \\ & \mathbf{a}_i' \cdot \mathbf{x} = b_i & i \in M_3 \\ & x_j \geq 0 & j \in N_1 \\ & x_j \leq 0 & j \in N_2 \end{array}$$

The variables $x_1, \ldots x_n$ are called *decision variables* and a vector satisfying all the constraints is called a *feasible solution*, the set of such vectors is hence called *feasible set* or *feasible region.* A solution to the minimisation problem amongst the feasible solutions is called an *optimal feasible solution.*

Remark. We could have defined a linear program as a maximisation problem, this is equivalent as maximising the cost function $\mathbf{c} \cdot \mathbf{x}$ is equivalent to minimising $-\mathbf{c} \cdot \mathbf{x}$.

Remark. Some constraints are furthermore equivalent. Indeed $\mathbf{a}_i' \cdot \mathbf{x} = b_i$ is equivalent to two constraints $\mathbf{a}_i' \cdot \mathbf{x} \leq b_i$ and $\mathbf{a}_i' \cdot \mathbf{x} \geq b_i$, and the primer is equivalent to $-(\mathbf{a}_i') \cdot \mathbf{x} \geq -b_i$. Hence a linear problem can be written in a simpler form as

$$\begin{aligned} \text{minimise} \quad & \mathbf{c} \cdot \mathbf{x} && (1.1) \\ \text{subject to} \quad & A\mathbf{x} \geq \mathbf{b} && (1.2) \end{aligned}$$

where A is the $n \times m$ matrix with rows $\mathbf{a}_i'$ and $\mathbf{b} = (b_1, \ldots, b_m)$.

Definition 1.1.2 (Equality Standard Form)**.** A linear program is in the *equality standard form* if is it of the form

$$\begin{aligned} \text{minimise} \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned}$$

We shall work with programs of this form for the simplex method in further chapters.

Given a linear program one can always rewrite it in equality standard by following a simple algorithm.

1. *Elimination of free variables* : given an unrestricted variable x_j in the general form one can rewrite $x_j := x_j^+ - x_j^-$ and impose the constraints $x_j^+, x_j^- \geq 0$.
2. *Elimination of inequality constraints* : given an inequality constraint of the form

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

we introduce *slack variables* s_i and write the problem in equality standard form as

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j + s_i &= b_i \\ s_i &\geq 0 \end{aligned}$$

If instead we had the sum greater than or usual to b_i we would swap the $+$ for a $-$ in the last sum.

1.2 Graphical Representation

We shall consider a few simple examples to provide geometric insight on linear programs.