The Fibonacci Distance: A Metric of Mathematical Harmony

Authored by
Thibaut LOMBARD

contact@lombard-web-services.com

April 27, 2025

Abstract

This manuscript unveils the Fibonacci distance, an inventive metric devised to gauge how closely numerical sequences mirror the elegant properties of the Fibonacci sequence. By assessing conformity to the sequence's recursive nature, its inverse form, and the golden ratio's proportional allure, this metric probes deeper than traditional distance measures, revealing structural kinship with Fibonacci patterns. Tunable parameters, α and β, lend versatility, enabling applications from market trend analysis to decoding biological growth spirals. Scaled to $[0, 1]$ and anchored by rigorous axiom validation, it empowers researchers to uncover Fibonacci echoes in intricate datasets, harmonizing mathematical precision with practical insight.

Table of Contents

1 Unveiling the Fibonacci Sequence and Its Reach **4**
1.1 Fibonacci in the Natural World . . . 4
1.2 The Golden Ratio's Kinship . . . 4
1.3 Finance and Fibonacci's Insight . . . 4
1.4 Computational Applications . . . 4

2 The Quest for a Tailored Metric **5**
2.1 What Makes a Metric Vital . . . 5
2.2 Core Traits of a Robust Metric . . . 5
2.3 Why a Fibonacci Metric? . . . 5

3 Crafting the Fibonacci Distance **5**
3.1 Core Idea . . . 5
3.2 Assessment Criteria . . . 5
3.3 Distance Computation Rule . . . 6
3.4 Illustrative Case . . . 6

4 Formalizing the Fibonacci Metric **6**
4.1 Penalty by Absolute Difference . . . 6
4.2 Classic Fibonacci Check . . . 6
4.3 Inverse Fibonacci Check . . . 6
4.4 Golden Ratio Check . . . 7

5 Normalizing the Distance **7**
5.1 Why Normalize? . . . 7
5.2 Normalization Goals . . . 7
5.3 Normalization Approaches . . . 7
5.3.1 By Element Sum . . . 7
5.3.2 By Maximum Difference . . . 7
5.3.3 By Length . . . 7
5.4 Example . . . 7

6 Axiom Validation **8**

7 Weighting with Alpha and Beta **8**
7.1 Parameter Roles . . . 8
7.2 Interpreting Parameters . . . 8
7.3 Metric Flexibility . . . 8

8 Complete Fibonacci Distance **8**
8.1 Overview . . . 8
8.2 Components . . . 9
8.2.1 Recurrence Penalty . . . 9
8.2.2 Ratio Penalty . . . 9
8.2.3 Normalization Sums . . . 9
8.3 Full Formula . . . 9

9 Conclusion **9**

1 Unveiling the Fibonacci Sequence and Its Reach

The Fibonacci sequence, a cascade of numbers where each term is the sum of its two predecessors, often begins with 0 and 1 or 1 and 1, yielding: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, and beyond. Unearthed by Leonardo Fibonacci in the 13th century, this sequence weaves its way through nature and human endeavor, its properties both enchanting and utilitarian.

1.1 Fibonacci in the Natural World

Nature frequently adopts Fibonacci's rhythm:

- **Phyllotaxis**: Leaves spiral around stems, often tracing Fibonacci numbers in their arrangement, a testament to growth's efficiency.
- **Floral patterns**: Petals of daisies or sunflowers often number 3, 5, 8, 13, 21, or 34, echoing Fibonacci's sequence.
- **Spiral forms**: From pinecones to nautilus shells, spirals align with Fibonacci's mathematical dance.

1.2 The Golden Ratio's Kinship

As the sequence unfolds, the ratio of successive terms converges on the golden ratio, $\phi \approx 1.618$:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

This proportion, revered in art and architecture, underpins the sequence's aesthetic and structural prevalence.

1.3 Finance and Fibonacci's Insight

In market analysis, Fibonacci numbers guide:

- **Retracements**: Levels like 23.6%, 38.2%, and 61.8% signal potential price reversals.
- **Extensions**: These project future price targets beyond prior peaks.
- **Arcs, fans, and time zones**: Tools leveraging Fibonacci ratios to map price dynamics.

1.4 Computational Applications

The sequence informs:

- **Optimization**: Fibonacci search optimizes one-dimensional problems.
- **Data structures**: Fibonacci heaps streamline priority queue operations.
- **Cryptography**: Its properties aid pseudo-random number generation.

2 The Quest for a Tailored Metric

2.1 What Makes a Metric Vital

A metric quantifies distance between set elements, enabling:

- **Similarity assessment**: Measuring likeness or divergence.
- **Clustering**: Grouping by proximity.
- **Anomaly detection**: Identifying outliers.
- **Performance evaluation**: Gauging algorithm efficacy.

2.2 Core Traits of a Robust Metric

A metric must uphold:

1. **Non-negativity**: Distances are non-negative, zero only for identical points.
2. **Symmetry**: Distance from A to B equals B to A.
3. **Triangle inequality**: Direct paths are shorter than detours via third points.
4. **Identity**: Only identical points have zero distance.

2.3 Why a Fibonacci Metric?

Fibonacci sequences possess distinct traits, warranting a specialized metric to:

- **Detect**: Identify sequences approximating Fibonacci patterns.
- **Quantify**: Measure adherence to Fibonacci properties.
- **Compare**: Rank sequences by their Fibonacci character.
- **Analyze**: Reveal Fibonacci structures in data, from finance to biology.

3 Crafting the Fibonacci Distance

3.1 Core Idea

The Fibonacci distance measures how closely sequences align with Fibonacci's hallmarks, prioritizing structural fidelity over mere numerical difference.

3.2 Assessment Criteria

It evaluates:

1. **Classic recurrence**: Does each term sum the prior two? $(F_{n+2} = F_n + F_{n+1})$
2. **Inverse sequence**: Do inverses $(1/x)$ form a Fibonacci sequence?
3. **Golden ratio**: Do successive term ratios near $\phi \approx 1.618$?

3.3 Distance Computation Rule

For sequences u and v:

- **Fibonacci compliance**: If both adhere to a Fibonacci rule (classic, inverse, or ratio), distance is 0.
- **Non-compliance**: Penalty is the sum of absolute differences:

$$\text{penalty} = \sum_{i=1}^{\min\_len} |u_i - v_i|$$

  where $\min\_len$ is the shorter sequence's length.

3.4 Illustrative Case

Consider:

$$u = [1, 1, 2, 3, 5]$$
$$v = [1, 1, 2, 4, 6]$$

Sequence u obeys the Fibonacci rule; v does not ($4 \neq 1 + 2$, $6 \neq 2 + 4$). Penalty:

$$\text{diff} = [|1-1|, |1-1|, |2-2|, |3-4|, |5-6|] = [0, 0, 0, 1, 1]$$
$$\text{sum} = 2$$

Thus, $\text{fibonacci\_distance}(u, v) = 2$.

4 Formalizing the Fibonacci Metric

4.1 Penalty by Absolute Difference

For non-Fibonacci sequences:

$$\text{penalty} = \sum_{i=1}^{\min\_len} |u_i - v_i|$$

where u_i, v_i are sequence elements, and $\min\_len$ is the minimum length.

4.2 Classic Fibonacci Check

Verify:

$$F_{n+2} = F_n + F_{n+1}$$

For sequence $u = [u_1, u_2, \ldots]$:

$$|u_{n+2} - (u_n + u_{n+1})| \leq \epsilon \quad (\epsilon = 10^{-2})$$

4.3 Inverse Fibonacci Check

For inverse sequence $1/u_i$:

$$\left| \frac{1}{u_{n+2}} - \left(\frac{1}{u_n} + \frac{1}{u_{n+1}} \right) \right| \leq \epsilon$$

4.4 Golden Ratio Check

Successive ratios should approximate ϕ:

$$\left| \frac{u_{n+1}}{u_n} - \phi \right| \leq \epsilon$$

5 Normalizing the Distance

5.1 Why Normalize?

Normalization to $[0, 1]$ aids:

- **Comparability**: Across diverse sequence scales.
- **Algorithm integration**: In machine learning or optimization.
- **Interpretability**: Uniform scale for analysis.

5.2 Normalization Goals

1. **Scale invariance**: Mitigate size disparities.
2. **Consistency**: Enable cross-sequence comparisons.
3. **Standardization**: Suit algorithmic needs.

5.3 Normalization Approaches

5.3.1 By Element Sum

$$\text{normalized penalty} = \frac{\text{raw penalty}}{\sum_{i=1}^{len(u)} |u_i| + \sum_{i=1}^{len(v)} |v_i|}$$

5.3.2 By Maximum Difference

$$\text{normalized penalty} = \frac{\text{raw penalty}}{\text{max\_diff}}$$

5.3.3 By Length

$$\text{normalized penalty} = \frac{\text{raw penalty}}{\max(len(u), len(v))}$$

5.4 Example

For $u = [1, 1, 2, 3, 5]$ (sum = 12), $v = [1, 1, 2, 4, 6]$ (sum = 14), penalty = 2:

$$\text{normalized penalty} = \frac{2}{12 + 14} \approx 0.0769$$

6 Axiom Validation

The Fibonacci distance satisfies:

Axiom	Verification
Non-negativity	Zero for identical sequences, positive otherwise, as penalties are sums of absolute differences.
Symmetry	$\lvert u_i - v_i\rvert = \lvert v_i - u_i\rvert$; normalization preserves symmetry.
Triangle Inequality	Absolute differences ensure $\lvert x - z\rvert \leq \lvert x - y\rvert + \lvert y - z\rvert$; scaling maintains this.
Identity	Zero distance only for identical sequences.

Table 1: Axiomatic Validation of the Fibonacci Distance

7 Weighting with Alpha and Beta

7.1 Parameter Roles

Parameters adjust emphasis:

- **Alpha** (α): Prioritizes classic Fibonacci recurrence.
- **Beta** (β): Emphasizes golden ratio ratios.

7.2 Interpreting Parameters

- $\alpha > \beta$: Favors recurrence.
- $\beta > \alpha$: Prioritizes ratios.
- $\alpha = \beta$: Balanced weighting.

7.3 Metric Flexibility

Applications dictate weighting:

- **Finance**: High β for ratio-driven analysis.
- **Biology**: High α for growth patterns.
- **General**: Equal weights.

8 Complete Fibonacci Distance

8.1 Overview

$$d(u, v) = \frac{\alpha \cdot P_{fib}(u, v) + \beta \cdot P_{ratio}(u, v)}{S(u) + S(v)}$$

Where:

- $P_{fib}(u, v)$: Penalty for recurrence deviation.
- $P_{ratio}(u, v)$: Penalty for ratio deviation.
- $S(u)$, $S(v)$: Element sums for normalization.

8.2 Components

8.2.1 Recurrence Penalty

$$P_{fib}(u,v) = \sum_{i=0}^{\min(len(u),len(v))-2} \left|(u_i + u_{i+1}) - u_{i+2} - (v_i + v_{i+1}) + v_{i+2}\right|$$

8.2.2 Ratio Penalty

$$P_{ratio}(u,v) = \sum_{i=0}^{\min(len(u),len(v))-1} \left|\left(\frac{u_{i+1}}{u_i} - \phi\right) + \left(\frac{v_{i+1}}{v_i} - \phi\right)\right|$$

8.2.3 Normalization Sums

$$S(u) = \sum_{i=0}^{len(u)} u_i, \quad S(v) = \sum_{i=0}^{len(v)} v_i$$

8.3 Full Formula

$$d(u,v) = \frac{\begin{array}{c}\alpha \cdot \sum_{i=0}^{\min(len(u),len(v))-2} \left|(u_i + u_{i+1}) - u_{i+2} - (v_i + v_{i+1}) + v_{i+2}\right| + \\ \beta \cdot \sum_{i=0}^{\min(len(u),len(v))-1} \left|\left(\frac{u_{i+1}}{u_i} - \phi\right) + \left(\frac{v_{i+1}}{v_i} - \phi\right)\right|\end{array}}{\sum_{i=0}^{len(u)} u_i + \sum_{i=0}^{len(v)} v_i}$$

9 Conclusion

The Fibonacci distance reimagines sequence comparison, delving into the structural essence of Fibonacci patterns. Its multi-faceted approach—spanning recurrence, inverse sequences, and golden ratio ratios—distinguishes it from conventional metrics. With α and β tuning its focus, and normalization ensuring accessibility, it serves fields from finance to natural sciences. As a mathematically sound metric, it unveils hidden patterns, offering a lens into the subtle order of complex systems, and stands as a testament to the interplay of rigor and creativity in mathematical inquiry.