The Fibonacci Distance

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Abstract

This paper introduces the Fibonacci distance, a novel metric crafted to quantify the structural alignment of numerical sequences with the intrinsic properties of the Fibonacci sequence. By evaluating adherence to the classic Fibonacci recurrence, its inverse, and the golden ratio, this metric transcends conventional distance measures, capturing the mathematical essence of Fibonacci patterns. With adjustable parameters α and β, the distance adapts to diverse applications, from financial market analysis to biological pattern recognition. Normalized to the interval [0,1] and rigorously validated against metric axioms, it offers a robust tool for detecting Fibonacci-like behaviors in complex datasets. This work bridges mathematical elegance with practical utility, illuminating hidden structures in nature and science.

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1 Introduction to the Fibonacci Sequence and Its Applications

The Fibonacci sequence is a mathematical sequence where each number is the sum of the two preceding ones, typically starting with 0 and 1 or 1 and 1. The first terms of the sequence are thus: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

This mathematical sequence, discovered by Leonardo Fibonacci in the 13th century, has remarkable properties and appears in many natural phenomena and application domains.

1.1 Applications in Nature

The Fibonacci sequence is found in various biological structures:

- **Leaf arrangement** (phyllotaxis): Leaves on a stem are often arranged in spirals following Fibonacci numbers.
- **Flowers and fruits**: The number of petals in many flowers corresponds to a Fibonacci number (3, 5, 8, 13, 21, 34, ...).
- **Spiral structures**: Sunflower seeds, pinecones, and snail shells exhibit spiral patterns that follow the Fibonacci sequence.

1.2 The Golden Ratio and the Fibonacci Sequence

The ratio between two consecutive Fibonacci numbers approaches the golden ratio ($\phi \approx 1.618$) as the sequence progresses:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

This relationship with the golden ratio, considered the ideal proportion ratio in art and architecture, explains why the Fibonacci sequence appears in many artistic and architectural creations.

1.3 Applications in Finance and Trading

In financial technical analysis, Fibonacci numbers are used for:

- **Fibonacci retracement levels**: These levels (23.6%, 38.2%, 61.8%, etc., derived from Fibonacci ratios) predict potential support and resistance levels after a price movement.
- **Fibonacci extensions**: Used to project potential price targets beyond previous levels.
- **Fibonacci arcs, fans, and time zones**: Tools based on Fibonacci ratios to analyze price movements in space and time.

1.4 Applications in Computer Science and Optimization

The Fibonacci sequence and its properties are also used in:

- **Optimization algorithms**: Fibonacci search is a one-dimensional optimization method.
- **Data structures**: Fibonacci heaps are used to implement priority queues.
- **Pseudo-random number generation** and **cryptography**.

2 Need for Using a Metric

2.1 Definition and Importance of Metrics

A metric is a mathematical function that defines a distance between two elements of a set. Metrics are essential in many fields because they allow:

- **Quantifying similarity** or difference between two objects
- **Classifying and grouping** objects based on their proximity
- **Detecting anomalies** or outliers
- **Evaluating performance** of algorithms or models

2.2 Essential Properties of a Good Metric

For a distance function to be considered a valid mathematical metric, it must satisfy four fundamental axioms:

1. **Non-negativity**: The distance between two objects is always positive or zero.
2. **Symmetry**: The distance from A to B is the same as from B to A.
3. **Triangle inequality**: The direct distance between two points is never greater than the sum of distances via a third point.
4. **Identity of indiscernibles**: The distance from an object to itself is always zero.

These properties ensure the metric behaves intuitively and consistently, enabling its use in various algorithms and applications.

2.3 Why a Fibonacci-Specific Metric?

Sequences following a Fibonacci pattern have unique mathematical properties distinguishing them from random sequences. A dedicated Fibonacci metric allows:

- **Identifying** sequences that approximately follow a Fibonacci model
- **Measuring** the degree of conformity of a sequence to Fibonacci properties
- **Comparing** different sequences based on their "Fibonacci-ness"
- **Detecting** Fibonacci structures in experimental data

This is particularly useful in fields like financial analysis, computational biology, or studing natural phenomena with growth patterns.

3 The Fibonacci Distance: Definition and Criteria

3.1 Fundamental Concept

The Fibonacci distance is a metric designed to measure how closely two numerical sequences or vectors adhere to the characteristic properties of Fibonacci sequences. This distance evaluates similarity not just by absolute values but by conformity to specific Fibonacci-related mathematical models.

3.2 Evaluation Criteria

The Fibonacci metric checks several properties between two sequences or vectors u and v:

1. **Conformity to the classic Fibonacci sequence**: Is each term the sum of the two preceding ones? ($F_{n+2} = F_n + F_{n+1}$)
2. **Conformity to an inverse Fibonacci sequence**: If the inverse ($1/x$) of each term is taken, does the resulting sequence follow Fibonacci properties?
3. **Conformity to the golden ratio**: Are the ratios between successive terms close to the golden ratio ($\phi \approx 1.618$)?

3.3 Fundamental Rule for Distance Calculation

The Fibonacci distance between two sequences or vectors u and v is calculated as follows:

- **If both sequences follow a Fibonacci rule** (classic, inverse, or ratio) → the distance is 0 (no penalty)
- **Otherwise** → a penalty is calculated as the sum of absolute differences between corresponding elements:

  $$\text{penalty} = \sum_{i=1}^{\min\_len} |u_i - v_i|$$

  where $\min\_len$ is the length of the shorter sequence.

3.4 Simple Example

Consider two sequences:

$$u = [1, 1, 2, 3, 5]$$
$$v = [1, 1, 2, 4, 6]$$

The sequence u perfectly follows the classic Fibonacci rule, while v deviates (since $4 \neq 1+2$ and $6 \neq 1+4$).

Since both sequences do not follow a Fibonacci rule, the penalty is calculated:

$$\begin{aligned}\text{diff} &= [|1-1|, |1-1|, |2-2|, |3-4|, |5-6|] \\ &= [0, 0, 0, 1, 1] \\ \text{sum} &= 0 + 0 + 0 + 1 + 1 = 2\end{aligned}$$

Thus, $\text{fibonacci\_distance}(u, v) = 2$.

4 Formalization of the Fibonacci Metric

4.1 Penalty Formula (Absolute Difference)

When two sequences u and v do not follow a Fibonacci model, the penalty is calculated as the sum of absolute differences between corresponding elements:

$$\text{penalty} = \sum_{i=1}^{\min\_len} |u_i - v_i|$$

where:

- u_i and v_i are elements of sequences u and v, respectively
- $\min\_len$ is the minimum length between u and v

4.2 Verification of the Classic Fibonacci Sequence

The classic Fibonacci sequence satisfies:

$$F_{n+2} = F_n + F_{n+1}$$

The function checks if, for a sequence $u = [u_1, u_2, u_3, ...]$, this rule holds for each consecutive triplet:

$$|u_{n+2} - (u_n + u_{n+1})| \leq \epsilon$$

where ϵ is a small tolerance threshold (e.g., $\epsilon = 10^{-2}$).

4.3 Verification of the Inverse Fibonacci Sequence

If sequence u follows an inverse Fibonacci sequence, the rule is checked:

$$F(n) \to \frac{1}{F(n)}$$

The function calculates the inverses of the sequence elements and checks if these inverses follow a classic Fibonacci sequence:

$$\left| \frac{1}{u_{n+2}} - \left(\frac{1}{u_n} + \frac{1}{u_{n+1}} \right) \right| \leq \epsilon$$

4.4 Verification of the Fibonacci Ratio (Golden Ratio)

The Fibonacci ratio approaches the golden ratio $\phi \approx 1.618$. The function checks if the ratio between successive elements approximates this value:

$$\frac{u_{n+1}}{u_n} \approx 1.618$$

Specifically, for each pair of successive elements u_n and u_{n+1}:

$$\left| \frac{u_{n+1}}{u_n} - \phi \right| \leq \epsilon$$

5 Normalization of the Fibonacci Distance

5.1 Importance of Normalization

Normalizing a distance to the range [0, 1] is often used to make results comparable, especially when working with:

- Metrics integrated into optimization systems
- Machine learning algorithms
- Comparisons of objects in varied contexts

For the Fibonacci distance, normalizing to [0, 1] allows for more universal and consistent interpretation.

5.2 Normalization Objectives

1. **Avoid large value discrepancies** for sequences of different sizes or magnitudes
2. **Facilitate comparison** of distances between sequences regardless of length or absolute value
3. **Standardize the metric** for use in algorithms like machine learning or optimization

5.3 Normalization Methods

5.3.1 Normalization by Sum of Elements

The simplest method divides the penalty by the total sum of elements in both sequences u and v:

$$\text{normalized penalty} = \frac{\text{raw penalty}}{\sum_{i=1}^{len(u)} |u_i| + \sum_{i=1}^{len(v)} |v_i|}$$

5.3.2 Normalization by Maximum Possible Difference

Another approach normalizes by the maximum possible difference in sequence elements:

$$\text{normalized penalty} = \frac{\text{raw penalty}}{\text{max\_diff}}$$

where max_diff is the largest possible difference between elements of the two sequences.

5.3.3 Normalization by Sequence Length

A third approach divides the penalty by the total length of the sequences:

$$\text{normalized penalty} = \frac{\text{raw penalty}}{\max(len(u), len(v))}$$

5.4 Normalization Example

Revisiting the previous example:

$$u = [1, 1, 2, 3, 5] \quad (\text{sum} = 12)$$
$$v = [1, 1, 2, 4, 6] \quad (\text{sum} = 14)$$

With a raw penalty of 2, normalization by the sum of elements yields:

$$\text{normalized penalty} = \frac{2}{12 + 14} = \frac{2}{26} \approx 0.0769$$

This value lies between 0 and 1, facilitating interpretation and comparison with other normalized measures.

6 Verification of Metric Axioms

For a distance function to be a valid metric, it must satisfy four fundamental axioms. Let's verify that the Fibonacci distance complies:

Axiom	**Verification for Fibonacci Distance**
Non-negativity	The distance is zero when sequences are identical and always positive otherwise. The penalty is a sum of absolute differences, which are always non-negative.
Symmetry	The distance is independent of sequence order since absolute differences are symmetric: $\lvert u_i - v_i \rvert = \lvert v_i - u_i \rvert$. Normalization by the sum of elements is also symmetric.
Triangle Inequality	The raw distance satisfies the triangle inequality because absolute differences do: $\lvert x - z \rvert \leq \lvert x - y \rvert + \lvert y - z \rvert$. Normalization preserves this property as it is a scaling of the raw penalty.
Identity of Indiscernibles	The distance between a sequence and itself is 0, as absolute differences between an element and itself are zero. Distances are positive for different sequences.

Table 1: Verification of Metric Axioms for Fibonacci Distance

The Fibonacci distance satisfies all four axioms, making it a valid mathematical metric.

7 Introduction of Alpha and Beta Parameters

7.1 Role of Weighting Parameters

In calculating the Fibonacci distance, weighting parameters can be introduced to modulate the relative importance of different Fibonacci properties:

- **Alpha** (α): Weights the importance of verifying classic Fibonacci properties (relation $F_{n+2} = F_n + F_{n+1}$)
- **Beta** (β): Weights the importance of the ratio between sequence elements (conformity with the golden ratio)

7.2 Parameter Interpretation

- **If** $\alpha > \beta$: The distance is more influenced by conformity to the classic Fibonacci rule than the golden ratio.
- **If** $\beta > \alpha$: The distance is more influenced by ratios close to the golden ratio than the classic rule.
- **If** $\alpha = \beta$: Both properties are equally important.

7.3 Impact on the Metric

These parameters allow the metric's sensitivity to be adjusted based on application needs:

- In financial analysis, ratios may be prioritized (β high).
- In growth phenomena studies, the recurrence relation may be favored (α high).
- For balanced analysis, $\alpha = \beta$ may be chosen.

8 Complete Formulation of the Fibonacci Distance

8.1 Summary of the Distance Function

The complete Fibonacci distance, incorporating α and β weightings, is expressed as:

$$d(u,v) = \frac{\alpha \cdot P_{fib}(u,v) + \beta \cdot P_{ratio}(u,v)}{S(u) + S(v)}$$

Where:

- $P_{fib}(u,v)$ is the raw penalty for differences in conformity to the classic Fibonacci rule
- $P_{ratio}(u,v)$ is the raw penalty for differences in conformity of element ratios to the golden ratio
- α and β are weighting factors
- $S(u)$ and $S(v)$ are the sums of the sequence elements for normalization

8.2 Component Details

8.2.1 Classic Fibonacci Penalty $P_{fib}(u,v)$

$$P_{fib}(u,v) = \sum_{i=0}^{\min(len(u),len(v))-2} |(u_i + u_{i+1}) - u_{i+2} - (v_i + v_{i+1}) + v_{i+2}|$$

This penalty measures the deviation between sequences relative to the classic Fibonacci rule.

8.2.2 Fibonacci Ratio Penalty $P_{ratio}(u,v)$

$$P_{ratio}(u,v) = \sum_{i=0}^{\min(len(u),len(v))-1} \left| \left(\frac{u_{i+1}}{u_i} - \phi \right) + \left(\frac{v_{i+1}}{v_i} - \phi \right) \right|$$

This penalty measures the difference in ratios relative to the golden ratio ϕ.

8.2.3 Sum of Elements $S(u)$ and $S(v)$

$$S(u) = \sum_{i=0}^{len(u)} u_i, \quad S(v) = \sum_{i=0}^{len(v)} v_i$$

8.3 Complete Expanded Formula (Formatted for One Row)

The complete expanded formula, reformatted to fit a single row for display, is:

$$d(u,v) = \frac{\begin{array}{c} \alpha \cdot \sum_{i=0}^{\min(len(u),len(v))-2} |(u_i + u_{i+1}) - u_{i+2} - (v_i + v_{i+1}) + v_{i+2}| + \\ \beta \cdot \sum_{i=0}^{\min(len(u),len(v))-1} \left| \left(\frac{u_{i+1}}{u_i} - \phi \right) + \left(\frac{v_{i+1}}{v_i} - \phi \right) \right| \end{array}}{\sum_{i=0}^{len(u)} u_i + \sum_{i=0}^{len(v)} v_i}$$

This compact formulation adapts the Fibonacci metric to various application contexts by adjusting α and β based on the desired emphasis on different Fibonacci properties.

9 Conclusion

The Fibonacci distance is an innovative approach to measuring similarity between numerical sequences based on the mathematical properties of Fibonacci sequences. Beyond a simple element-wise difference, this metric evaluates conformity to specific Fibonacci-related mathematical models.

Key features of this metric include:

- **Multi-criteria**: It considers multiple aspects of Fibonacci sequences (classic recurrence, inverse, and golden ratio ratios).
- **Adaptable**: Parameters α and β allow adjustment based on application needs.
- **Normalizable**: The distance can be scaled to [0,1], aiding interpretation and use in algorithms.
- **Mathematically valid**: It satisfies the four fundamental metric axioms (non-negativity, symmetry, triangle inequality, and identity of indiscernibles).

This metric provides a powerful tool for identifying and measuring Fibonacci-like structures in fields such as finance, natural sciences, data analysis, and any domain where Fibonacci-related growth properties are relevant.

The Fibonacci distance fills a gap in existing metrics by offering an approach that goes beyond simple value comparisons to capture fundamental structural and relational properties. Its flexibility and robust mathematical foundation make it a valuable tool for sequence analysis in numerous scientific and practical contexts.