2.2 Calculating Exact Default Time τ_i from simulated u_i

Exact default time $\tau = t_{m-1} + \delta t$ is estimated by first, identifying the year by an iterative procedure (comparing $\log(1-u)$ to the running sum of hazard rate) and second, calculating the year fraction δt.

Extending derivation for $\tau \stackrel{D}{=} F^{-1}(u)$ as the inverse of Exponential Distribution **CDF**

$$\tau \stackrel{D}{=} -\frac{\log(1-u)}{\lambda_\tau} \quad \Rightarrow \quad \log(1-u) = -\lambda_\tau \tau = -\int_0^\tau \lambda_s ds = \log P(0,\tau)$$

If $1-u = P(0,\tau)$ is the exact probability of survival, and default occurs as $t_{m-1} \leq \tau \leq t_m$

$$P(0,t_m) \;\leq\; P(0,\tau) \;\leq\; P(0,t_{m-1})$$

because survival probability is cumulative and decreasing over time (see CDS Bootstrapping). Re-specified in terms of intensities $\log P(0,t_m) = -\sum_{j=1}^{m} \lambda_j \Delta t_j$, the inequality becomes:

$$-\sum_{j=1}^{m} \lambda_j \Delta t_j \;\leq\; \log(1-u) \;\leq\; -\sum_{j=1}^{m-1} \lambda_j \Delta t_j \tag{2.1}$$

First, the inequality is used in the iterative procedure to determine the year of default.

We can continue to re-arrange around u

$$\begin{aligned} P(0,t_m) \;\leq\; 1-u \;\leq\; P(0,t_{m-1}) \\ -P(0,t_{m-1}) \;\leq\; u-1 \;\leq\; -P(0,t_m) \\ 1-P(0,t_{m-1}) \;\leq\; u \;\leq\; 1-P(0,t_m) \\ \mathrm{PD}_{m-1} \;\leq\; u \;\leq\; \mathrm{PD}_m \end{aligned}$$

Since we compare the threshold u to the cumulative probability of default, then if u is large (more than cumulative PD for year 5) it has a meaning of default occurring after year 5.

Second, calculating the year fraction also relies on $1-u = P(0,\tau)$, where $\tau = t_{m-1} + \delta t$

$$\delta t = -\frac{1}{\lambda_m} \log\left(\frac{1-u}{P(0,t_{m-1})}\right) = -\frac{1}{\lambda_m} \log\left(\frac{P(0,t_{m-1}+\delta t)}{P(0,t_{m-1})}\right) \tag{2.2}$$

This essentially refers to calculation of hazard rate as a log-ratio of survival probabilities.