Topological Insulators and the SSH Model

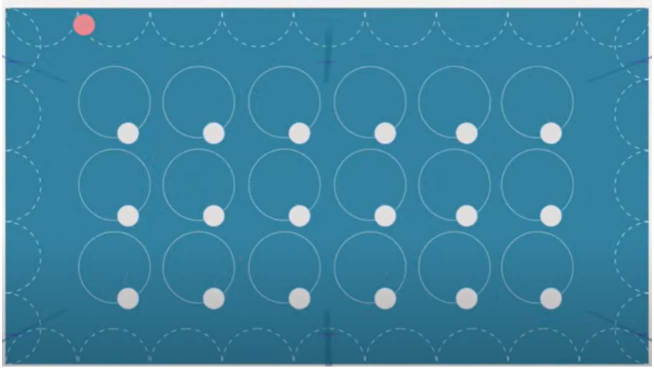
Topological Matter

- It was believed that in lower dimensions, phase transitions are not possible.
- However, it was seen that in low dimensional materials, unexpected collective phenomena can occur when the atoms interact with each other.
- Topological concepts are necessary to explain these phenomena like phase transitions.
- Eg: Topological insulators, Topological superconductors, topological half-metals.

Thouless, Haldane and Kosterlitz were awarded the 2016 Nobel Prize for discovery of topological phases of matter.

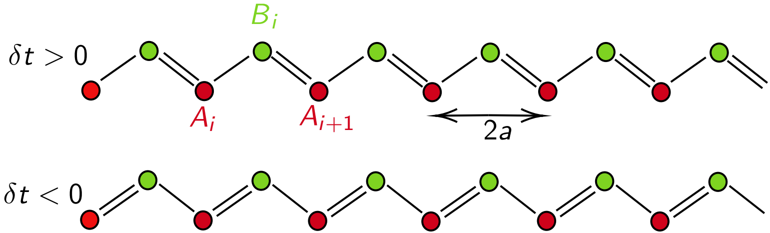
Introduction: Topological Insulators

- Material with insulating bulk but conducting edges.
- Presence of Bulk energy gap separating the highest occupied electronic band from the lowest empty band.
- Edge/Surface possesses gapless states, protected by time-reversal symmetry.



Su-Schrieffer-Heeger (SSH) Model

- 1D tight-binding model
- Each unit cell $\rightarrow$ two sites A and B
- For N unit cells $\longrightarrow$ 2N sites.
- Different hopping parameters $\rightarrow$ t (intercellular), t' (intracellular)
- $\delta t = t - t'$



$$\boxed{\hat{H}_{\text{SSH}} = t' \sum_{i=1}^{N} c^\dagger_{A,i} c_{B,i} + c^\dagger_{B,i} c_{A,i} + t \sum_{i=1}^{N} c^\dagger_{A,i+1} c_{B,i} + c^\dagger_{B,i} c_{A,i+1}}$$

Solving the system: BULK STATES

Bulk States:

1. Comprises of the central part of the chain.
2. Bulk does not depend on how edge is defined.
3. We apply Periodic Boundary Condition where $(N+1)^{th}$ site is equivalent to the 1^{st} site.

Let $\overrightarrow{c_x} = \begin{pmatrix} \hat{c}_{x,A} \\ \hat{c}_{x,B} \end{pmatrix}$ for x = 1,2,3,4.....

Consider the Fourier Transform over k-space:

$$\overrightarrow{c_x} = \frac{1}{\sqrt{N}} \sum_k e^{ikx} \overrightarrow{c_k}$$

This can be done since PBC makes the system translationally invariant, hence Bloch theorem applies.

Solving the System

Substituting this in the Hamiltonian, we get after simplification:

$$\hat{H}_{\mathrm{SSH}} = \sum_k \overrightarrow{c_x}^\dagger H_k \overrightarrow{c_x}$$

where H_k is the matrix of Hamiltonian in momentum-space.

$$H_k = \begin{pmatrix} 0 & t' + te^{-ik} \\ t' + te^{ik} & 0 \end{pmatrix} = (t' + t \ \cos k) \ \hat{\sigma}_x + t \sin k \ \hat{\sigma}_y$$

where $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are the Pauli-Matrices. Thus, in a more compact way, H_k can be written as:

$$H_k = \overrightarrow{d}(k).\overrightarrow{\sigma}$$

$$\vec{d}(k) = \begin{pmatrix} t' + t \cos k \\ t \sin k \\ 0 \end{pmatrix} \text{ and } \vec{\sigma} = \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}$$

Eigenvalues of $\mathbf{H_k}$ are: $E_k = \pm\sqrt{t^2 + t'^2 + 2tt'\cos k}$

Corresponding eigenstates are:

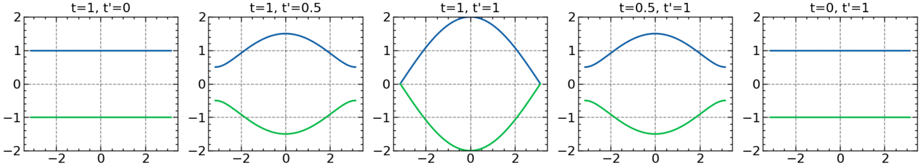
$$|\pm k\rangle = \begin{pmatrix} \pm e^{-i\phi(k)} \\ 1 \end{pmatrix}$$

where $\phi(k) = \tan^- 1\left(\frac{t\sin k}{t' + t\cos k}\right)$

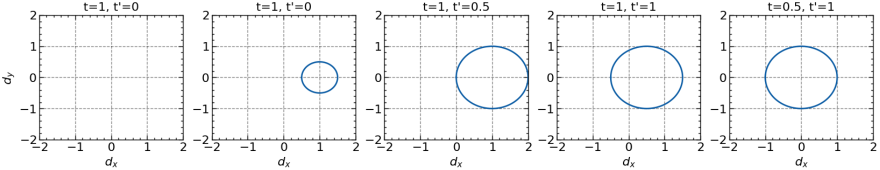
- d(k) tells us about the eigenstates.
- We plot d(k) vector over d_x and d_y plane
- We see the even though dispersion relation is same, the trajectory of d(k) is different.
- Topologically different phases $\longrightarrow$ Insulators in one phase have to cross through Metallic phase to reach another insulating phase .

Energy Eigenvalues

Eigenvalues of $\mathbf{H_k}$ are: $E_k = \pm\sqrt{t^2 + t'^2 + 2tt' \cos k}$



Variation of E_k with k for varying t, t'



Trajectory of endpoints of d(k)

Winding Number

We see in previous plot that the circles 'wind' about the origin (where d(k)=0) after t=t' case.
This leads us to the concept of Winding Number

- Total number of times that curve travels counterclockwise around the point
- Topological Invariant: Invariant under homeomorphisms

Winding Number ν is given by the formula:

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\vec{d}(k) \times \frac{\partial}{\partial k} \vec{d}(k) \right) \, dk$$

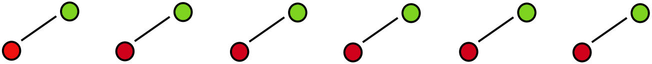
Winding Number is either 0 or 1 in this case but can be increased if we increase the degree of hopping.

Solving the System: EDGE STATES

Edge States:

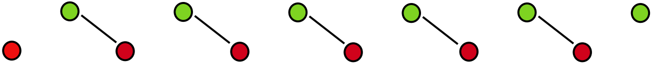
1. Can have two possible configurations in the fully dimerised limit.
2. **If intercell hopping vanishes: Trivial Phase**

   $t = 0$

3. **If intracell hopping vanishes: Topological Phase**

   $t' = 0$

   1 isolated site per edge

   Must contain zero-energy eigenstates: $H\,|A,1\rangle = H\,|B,N\rangle = 0$.

As seen earlier, Bulk has flat bands in the dimerised limit.

Zero Energy Eigenstates:

In the case when intracellular hopping vanishes, the 2 edge states become isolated.

Thus each end of the chain has a single eigenstate at zero-energy (E=0 since there is no onsite potential term in the model): $\hat{H}\left|\psi_0\right\rangle = 0$

We take an ansatz of the form:

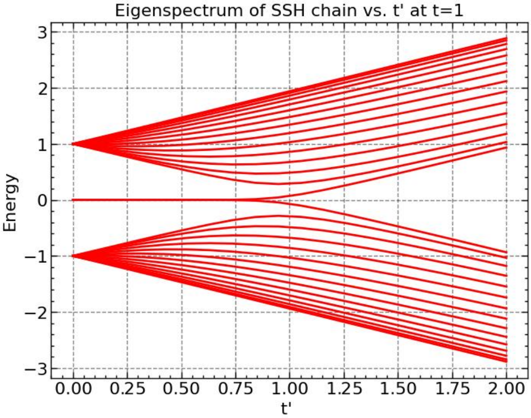
$$\left|\psi_0\right\rangle = \sum_{y=1}^{N} \left(u_{A,y} c^{\dagger}_{A,y} + u_{B,y} c^{\dagger}_{B,y} \right) \left|\text{vac}\right\rangle$$

After simplification and calculation, we get: $u_{A,x} = \left(\frac{-t'}{t}\right)^{x-1} u_{A,1}$ If $t' < t$, $u_{A,x}$ decays as we move from boundary.

Hence $|\left|\psi_0\right\rangle = 0|^2 \approx |u_{A,x}|^2$ also goes down.

The localisation length is given by: $\xi \approx |\psi_0(x)^2| \approx \left(\frac{x}{x-1}\right) \frac{1/2}{\ln\left(\frac{t}{t'}\right)}$

Energy Eigenspectrum

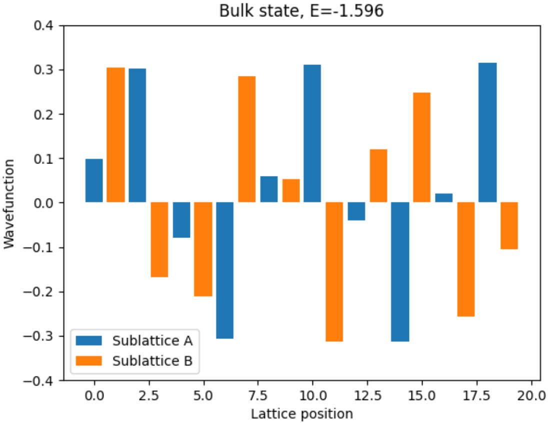


Zero energy states exists till $t' < t$ but after that, no zero energy edge states.

Bulk-Energy Correspondence

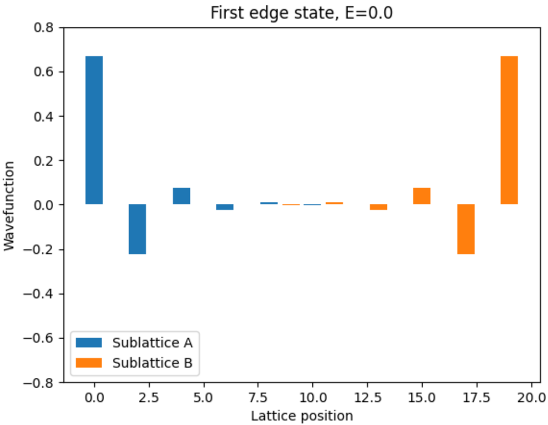
- The twofold degenerate edge states have a correspondence to the topology of the system, namely the bulk-edge correspondence.
- Whenever the system bulk has certain non-zero topological invariants, it allows for edge states, robust to adiabatic transformations of the Hamiltonian, to exist.
- There exists generally a one-to-one correspondence between the number of the protected edge/surface states and the value of the topological invariant (Winding Number) defined in the bulk.

Bulk State Wavefunction



Wavefunction of the Bulk State of SSH Model

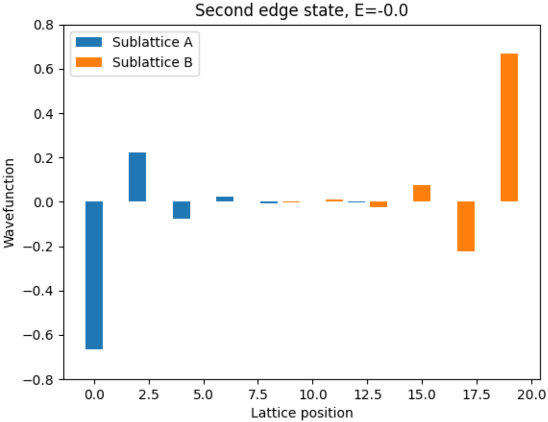
Edge States Wavefunction



Wavefunction of the First Edge State of SSH Model

We see that the wavefunction is Symmetric with respect to the lattice position.

Edge States Wavefunction



Wavefunction of the Second Edge State of SSH Model

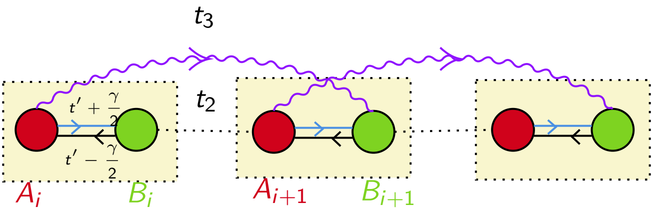
We see that the wavefunction is Anti-Symmetric with respect to the lattice position.

Non-Hermitian SSH Model

We now generalise the SSH model by adding non-Hermiticity to it.

$$\hat{H}_{\text{SSH}} = \sum_{i=1}^{N} \left(t' - \frac{\gamma}{2}\right) c^\dagger_{A,i} c_{B,i} + \left(t' + \frac{\gamma}{2}\right) c^\dagger_{B,i} c_{A,i} +$$

$$\sum_{i=1}^{N} t_2 (c^\dagger_{A,i+1} c_{B,i} + c^\dagger_{B,i} c_{A,i+1}) + \sum_{i=1}^{N} t_3 (c^\dagger_{B,i+1} c_{A,i} + c^\dagger_{A,i} c_{B,i+1})$$



Solving NHSSH Model

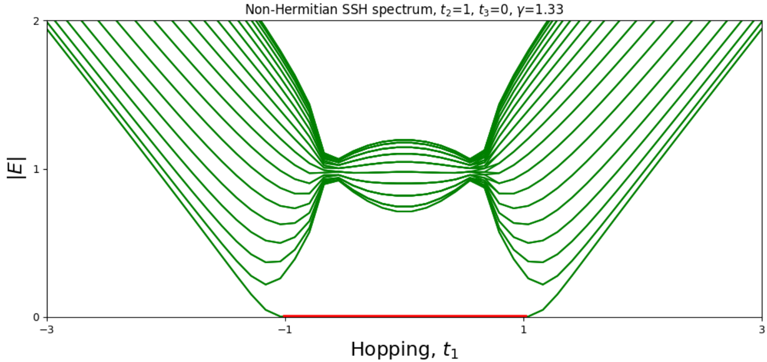
Similar to the previous transformation, we obtain:

$$H(k) = d_x\sigma_x + \left(d_y + i\frac{\gamma}{2}\right)\sigma_y$$

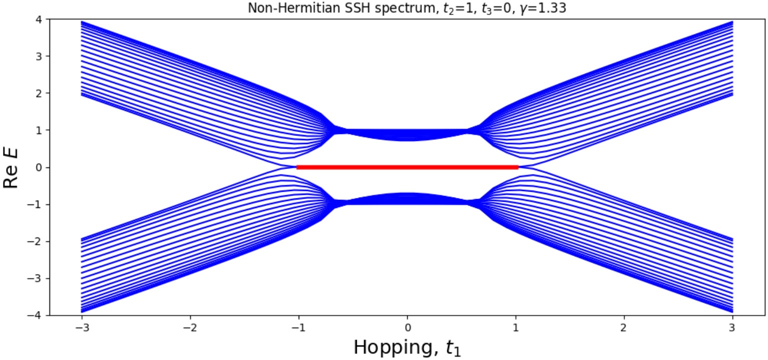
where $d_x = t_1 + (t_2 + t_3)\cos k$, $d_y = (t_2 - t_3)\sin k$

The energy eigenvalues are calculated to be : $E_\pm = \sqrt{d_x^2 + \left(d_y + i\frac{\gamma}{2}\right)^2}$

Due to the addition of non-Hermiticity, the energy eigenvalues can become imaginary and hence, Exceptional Points will arise in the plot of the energy spectrum.



Red denotes the zero mode line.



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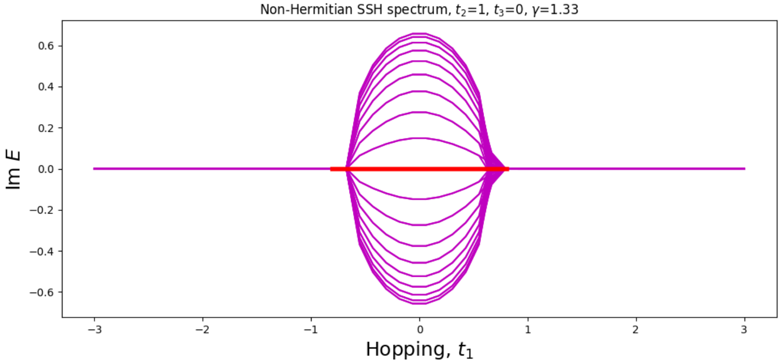
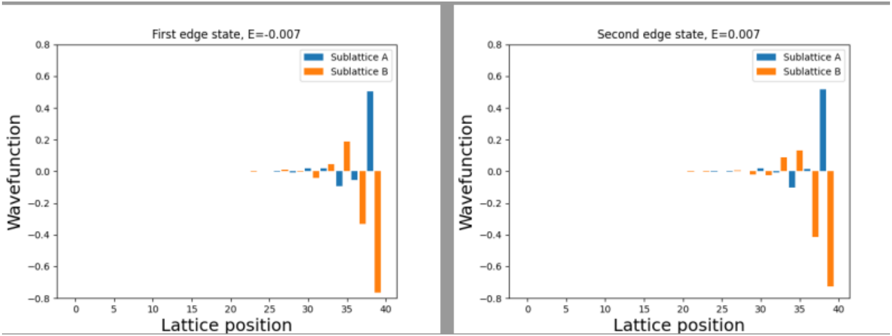


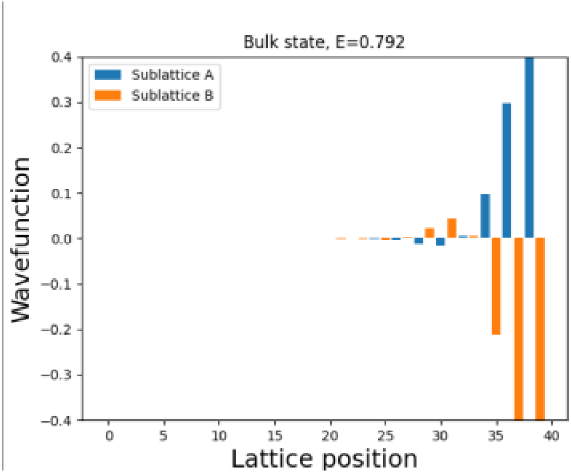
Red denotes the zero mode line.

Wavefunctions



Edge States

Wavefunctions



Bulk State

This exhibits the phenonmenon of Non-Hermitian Skin Effect which states that irrespective of bulk or edge, all eigenstates are spatially localized at the boundary of a system.

NHSE is responsible for several unidirectional physical effects; the massive accumulation of eigenstates at the boundaries hints an extreme sensitivity to weak boundary couplings. This could have many potential applications in many different fields and hence is a very demanding topic for research.

Thank You