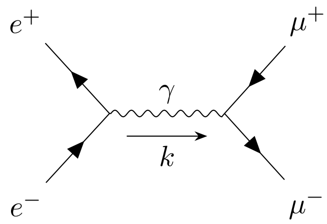
Quantum Field Theory

Lecture Notes

Instructor: Prof. Ritesh Kumar Singh



Sagnik Seth

Dept of Physical Sciences
IISER Kolkata

Contents

Lecture 01: Quantum SHO

People do 'weird' stuffs for earning,
You can do the same for learning!

Since the foundational aspect of QFT (and many other topics in Physics) is the Harmonic Oscillator, let us discuss the quantum Harmonic oscillator for introduction. For that, note that the classical Hamiltonian for the harmonic oscillator is given by:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

We do not like the things like m and ω which prevent us from seeing things clearly 😅. Hence, we invoke the holy action of making things dimensionless. For that, we define:

$$\begin{aligned} X &= \sqrt{\frac{m\omega}{\hbar}}x \qquad \longrightarrow \qquad x^2 = \frac{\hbar}{m\omega}X^2 \\ P &= \frac{1}{\sqrt{m\omega\hbar}}p \qquad \longrightarrow \qquad p^2 = m\omega\hbar P^2 \end{aligned}$$

Substituting these in the Hamiltonian, we have:

$$H = \frac{\hbar\omega}{2}(X^2 + P^2)$$

Note that, we are still within the classical domain. Now, let us elevate x and p to operators and we define:

$$[x, p] = i\hbar\,\mathbb{1} \implies [X, P] = \sqrt{\frac{\cancel{m\omega}}{\hbar}}\frac{i\hbar\,\mathbb{1}}{\sqrt{\cancel{m\omega\hbar}}} = i$$

Introducing the commutator bracket brings us to the quantum world. Now, we invoke our very own ladder operators:

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) \qquad &&: \text{annihilation operator} \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \qquad &&: \text{creation operator} \end{aligned}$$

Then we will have [1]:

$$\left[a, a^\dagger\right] = \frac{1}{2}[X + iP, X - iP] = \frac{-i}{2}([X, P] - [P, X]) = -i \times i = 1$$

Also, note that:

$$a^\dagger a = \frac{1}{2}(X^2 + P^2 + i\underbrace{(XP - PX)}_{i}) = \frac{1}{2}(X^2 + P^2) - \frac{1}{2} \implies H = \frac{\hbar\omega}{2}(a^\dagger a + \frac{1}{2})$$

Let us consider the *complete set of commuting observables* (CSCO) for this problem. Evidently, the set $\{H\}$ itself satisfies the condition since the eigenvalues are all non-degenerate (hence, we can label each state with only one index). To understand why, let us consider the action of the the annihilation operator on a (normalised) state $|\psi\rangle$. For that we note the following:

$$\begin{aligned} \left[a^\dagger a, a\right] &= -a \implies [H, a] = -\hbar\omega\ a \\ \left[a^\dagger a, a\right] &= a^\dagger \implies \left[H, a^\dagger\right] = \hbar\omega\ a^\dagger \end{aligned}$$

[1] henceforth, forsaking the hat symbol and identity operator $\mathbb{1}$, since they cause nothing but trouble, when the context is clear

Now, we have:

$$Ha\left|\psi\right\rangle - aH\left|\psi\right\rangle = [H,a]\left|\psi\right\rangle = -\hbar\omega a\left|\psi\right\rangle \implies H(a\left|\psi\right\rangle) = (E-\hbar\omega)(a\left|\psi\right\rangle)$$

Thus, if $|\psi\rangle$ has an energy eigenvalue E, then $a|\psi\rangle$ will have an energy eigenvalue $E-\hbar\omega$. Thus, starting from any eenrgy state, we can change to another state with energy reduced by one unit of $\hbar\omega$, using the annihilation operator. Similarly, we will have:

$$H(a^\dagger\left|\psi\right\rangle) = (E+\hbar\omega)(a^\dagger\left|\psi\right\rangle)$$

Let us denote the states $a^\dagger|\psi\rangle$ and $a|\psi\rangle$ by $|\psi_+\rangle$ and $|\psi_-\rangle$ respectively. Then, we will have

$$\langle\psi_-|\psi_-\rangle = \langle\psi|\, a^\dagger a\left|\psi\right\rangle = \langle\psi|\left(\frac{H}{\hbar\omega}-\frac{1}{2}\right)|\psi\rangle$$

Now, since $|\psi_-\rangle$ is a valid vector of the Hilbert space, its norm must be non-negative and finite. Hence, we obtain the condition:

$$0 \le \frac{E}{\hbar\omega} - \frac{1}{2} < \infty \implies \frac{\hbar\omega}{2} \le E$$

Hence, we get a lower bound on the energy eigenvalue, that is, there exists a state $|\psi_{\min}\rangle$ such that $H|\psi_{\min}\rangle = E_{min}|\psi_{\min}\rangle$ where $E_{min} = \frac{\hbar\omega}{2}$. Then, starting from this state, if we apply the creation operator, we will get successively increasing energies (and hence the system is non-degenerate). We thus obtain:

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \qquad n = 0,1,\ldots$$

Lecture 02: Dimensional Deep Dive

For the quantum harmonic oscillator, we had seen that how making things dimensionless eases calculations a bit. We consider three fundamental constants, each occupying their own special place in their field of action.

$$\begin{aligned} c &\equiv [LT^{-1}] && \text{: relativity} \\ \hbar &\equiv [L^2MT^{-1}] && \text{: quantum} \\ G &\equiv [L^3M^{-1}T^{-2}] && \text{: gravity} \end{aligned}$$

Lengths are tractable for us, since we can see the 'length', same goes for mass, atleast we can 'feel' it. However, time is an *enigma*. *We can't hold two ends of time at the same time* unlike holding two ends of a rod to measure its length. The absolute truth is: *Time passes!*

Note that, in physics, we are mostly concerned with equations like $E = mc^2$ and $E = \hbar\omega$. To that extend, we define the natural units:

$$\begin{aligned} c &= 1 \\ \hbar &= 1 \end{aligned}$$

We want these quantitites to be numerically equal to 1 and dimensionless. Note that, we had also done this kind of things before. When writing Newton's law, we had said that

$$F \propto m, F \propto a \implies F \propto ma \implies F = kma \qquad \text{for some } k$$

Now, we chose unit and dimension of force in a way such that $k = 1$ (dimensionless and unit value) which gave us the celebrated law.

Note that:

- Making c dimensionless:
$$[LT^{-1}] \equiv 0 \implies [L] = [T]$$
- Making $\hbar$ dimensionless:
$$[L^2MT^{-1}] = [L^2ML^{-1}] = [LM] \equiv 0 \implies [L] = [M^{-1}]$$

Hence, we see that length and time are equivalent while mass and length have inverse relation. In this natural units, we have that energy is equivalent to mass and any other unit can be represented in terms of mass. Thus, by our convention, we choose mass or energy as the only important dimension.

Note that
$$\hbar c \equiv Jm = 1 \qquad \text{(in natural units)}$$

From this, we can say heuristically that increasing length scale is decreases the energy (mass) scale. We mainly use the length scale in context of *de Broglie wavelength*.

This is perhaps not the only way to define natural units. In cosmology, G plays a more important role and hence it is better to set $G = 1$, leaving aside $\hbar$. Using this *natural units*, we find:
$$[L] = [M]$$
$$[L] = [T]$$

Here, we see that length and mass scale are directly related. Well, in this regard, we treat the length scale to be that of the Schwarzschild radius of a blackhole [1] which intuitively grows with mass. Since the above two natural units are widely distinct, there is as such no problem, however, in the unfortunate case where we have to consider both de Broglie wavelength and the Schwarzschild radius (that is, in the infamous domain of quantum gravity 😨), one needs to be very careful.

Theorem 1 ($\pi-$Theorem):
Let $q_1, \dots, q_n$ be n variables which are physically relevant to a problem and which are related by an expression, that is,
$$F(q_1, \dots, q_n) = 0 \iff q_i = \widetilde{F}(q_1, \dots \hat{q}_i, \dots q_n)$$
If k is the number of fundamental dimensions required to describe the n variables, then we can group these in $(n-k)$ groups of dimensionless variables $\Pi_1, \dots, \Pi_{n-k}$ such that for some f, we have:
$$f(\Pi_1, \dots, \Pi_{n-k}) = 0 \iff \Pi_i = \widetilde{f}(\Pi_1, \dots \hat{\Pi}_i, \dots \Pi_{n-k})$$
.

The theorem seems a bit vague (and pointless too). Let us take a physical example. Consider a spherical ball in a viscous fluid. The variable in the problem are:

$$\begin{aligned}
\text{Drag force:} \quad & q_1 \to \quad F \quad [MLT^{-2}] \\
\text{Sphere diameter:} \quad & q_2 \to \quad d \quad [L] \\
\text{Fluid density:} \quad & q_3 \to \quad \rho \quad [ML^{-3}] \\
\text{Fluid velocity:} \quad & q_4 \to \quad v \quad [LT^{-1}] \\
\text{Fluid viscosity:} \quad & q_5 \to \quad \eta \quad [ML^{-1}T^{-1}]
\end{aligned}$$

[1]This crap is the radius of an object such that if the body is squeezed to a radius lesser than the Schwarzschild radius, the gravitational attraction between the constituents of the body causes its irreversible collapse, turning it to a black hole ⚫

So, there are 5 such parameters and only three units viz. M, L, T are needed to describe them. Hence we will have two Π groups. It is a good thing to choose the repeating variables (variables which will be in both groups) that relate to mass, geometry and the kinematics of the problem. Also, note that since the Π groups are dimensionless, we can take the non-repeating variable's power to be 1. Hence, in this problem we choose them to be ρ, d, v. Thus, we will have:

$$\Pi_1 = \rho^{a_1} d^{a_2} v^{a_3} F \equiv [ML^{-3}]^{a_1}[L]^{a_2}[LT^{-1}]^{a_3}[MLT^{-2}] = [M^{a_1+1}L^{-3a_1+a_2+a_3+1}T^{-a_3-2}]$$
$$\Pi_2 = \rho^{b_1} d^{b_2} v^{b_3} \eta \equiv [ML^{-3}]^{b_1}[L]^{b_2}[LT^{-1}]^{b_3}[ML^{-1}T^{-1}] = [M^{b_1+1}L^{-3b_1+b_2+b_3-1}T^{-b_3-1}]$$

Hence we obtain two sets of equations:

$$\begin{aligned}
a_1 + 1 = 0 &\implies a_1 = -1 \\
-a_3 - 2 = 0 &\implies a_3 = -2 \\
-3a_1 + a_2 + a_3 + 1 = 0 &\implies 3 + a_2 - 2 + 1 = 0 \implies a_2 = -2 \\
b_1 &= -1 \\
b_3 &= -1 \\
-3b_1 + b_2 + b_3 - 1 = 0 &\implies 3 + b_2 - 2 = 0 \implies b_2 = -1
\end{aligned}$$

Then we obtain the Π groups as:

$$\Pi_1 = \frac{F}{\rho d^2 v^2} \qquad \Pi_2 = \frac{\eta}{\rho d v} = \frac{1}{\frac{\rho d v}{\eta}}$$

We identity $\frac{\rho d v}{\eta}$ to be the *Reynold's numer* $\mathcal{R}$. Then we can say, for some ϕ,

$$\frac{F}{\rho d^2 v^2} = \phi(\mathcal{R})$$

To some extraterrestrial being, the physical laws will (hopefully) be valid, however, they might not understand the human-made units (like metres and seconds) in which we measure these quantitites.

We need something natural, based on Nature and hence we used the natural units. For this purpose, we will use things like $c, \hbar, G, k_b \dots$ and we set all of them to 1. Doing this will lead to change in all scales. For that, we define the Planck units, made of fundamental constants of Nature:

- **Planck mass**: $E_p = \sqrt{\frac{\hbar c}{G}}$
- **Planck length**: $l_p = \sqrt{\frac{\hbar G}{c^3}}$
- **Planck time**: $l_p = \sqrt{\frac{\hbar c}{G^5}}$

Let us now analyse the physical regimes in which the fundamental constants become important. For that, we will use a tuple $(G, \frac{1}{c}, \hbar)$ (note that all the elements of the tuple are written in terms of very small quantitites of the SI scale).

$$\begin{aligned}
(0,0,0) &\rightarrow \text{Classical mechanics} \\
(G,0,0) &\rightarrow \text{Newtonian gravity} \\
\left(0,\tfrac{1}{c},0\right) &\rightarrow \text{Special relativity} \\
(0,0,\hbar) &\rightarrow \text{Basic quantum mechanics} \\
\left(G,\tfrac{1}{c},0\right) &\rightarrow \text{General relativity} \\
\left(0,\tfrac{1}{c},\hbar\right) &\rightarrow \text{QFT and relativistic QM} \\
(G,0,\hbar) &\rightarrow \text{Non-relativistic gravity} \\
\left(G,\tfrac{1}{c},\hbar\right) &\rightarrow \text{Quantum gravity}
\end{aligned}$$

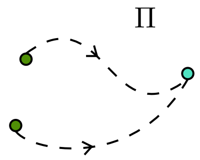
Lecture 03: From Classical to Quantum

Inspecting the transition from a classical to quantum description is necessary for understanding QFT, which is just a *sophisticated mechanics*.

We know that the position q and momentum p uniquely define the state of a *classical* particle.

$$(q, p) \in \Pi \text{ (phase space)} \qquad q, p \in \mathbb{R}$$

The Hamiltonian is just a way of doing time evolution. Using the Hamiltonian, we can 'fibrate' the phase space, that is, starting from one point, we can go to some other point. Also, note that changing the parameters in the Hamiltonian (eg. ω in SHO Hamiltonian), we change the fibration patterns.



In the quantum world, the phase space changes to the Hilbert space $\mathcal{H}$ while a point in the phase space (q, p) changes to a vector $|\psi\rangle$ in the Hilbert space.

Also, these real variables become Hermitian (self-adjoint) operators [1]and the classical Poisson bracket now transforms to the commutator.

$$\text{CM:} \qquad \{q, p\} = 1$$
$$\text{QM:} \qquad [\hat{q}, \hat{p}] = i$$

1.1 Time Evolution

From classical Hamilton's equations of motion, we have:

$$\dot{q} = \{q, H\} = \frac{\partial H}{\partial p}$$
$$\dot{p} = \{p, H\} = -\frac{\partial H}{\partial q}$$

If $z = \begin{pmatrix} p \\ q \end{pmatrix}$ is a $2n$ dimensional vector, then $\dot{z} = \{z, H\}$. Now, in quantum we know:

$$i\hbar \frac{\partial |\psi\rangle}{\partial dt} = H |\psi\rangle$$

In both classical and quantum case, the time derivative of a quantity is equal to some action of the Hamiltonian on that quantity (classical $\to$ Poisson bracket, quantum $\to$ Multiplication with Hamiltonian)[2].

The Hamiltonian in position basis becomes:

$$H = \frac{-\hbar^2}{2m} \boldsymbol{\nabla}^2 + V$$

And the energy and momentum operators become $E \to i\hbar \frac{\partial}{\partial t}$ and $p \to -i\hbar \boldsymbol{\nabla}$.

[1] A self-adjoint operator $\mathcal{O}$ is such that $\mathcal{O} = \mathcal{O}^\dagger$ in all respect, that is, $\mathcal{O}$ and $\mathcal{O}^\dagger$ have the same domain and action. In general, $\mathbb{D}(\mathcal{O}) \subseteq \mathbb{D}(\mathcal{O}^\dagger)$ where $\mathbb{D}(\cdot)$ represents the domain of some operator. If the domains are not equal but action is same on a restricted domain (mainly occurs in infinite-dimensional spaces), then those operators are not self-adjoint but called *symmetric/Hermitian* (in some places Hermiticity also requires a symmetric operator to be bounded).

[2] A better analogy would be to use density matrices which gives the von Neumann equation which uses commutator bracket

1.1.1 Free Particle

The dispersion relation for a non-relativistic free particle is:

$$E = \frac{p^2}{2m}$$

The typical solution can be written as $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)} \equiv \exp\left(-\frac{i}{\hbar}p^\mu x_\mu\right)$ where we have used the index repeated summation notation and taken the $(+,-,-,-)$ convention.

For a relativistic free particle, the dispersion relation becomes:

$$E^2 = mc^4 + p^2c^2$$

Substituting the energy and momentum operators here, we get:

$$\begin{aligned}
(i\hbar)^2\frac{\partial^2\psi}{\partial t^2} &= (-i\hbar)^2c^2\boldsymbol{\nabla}^2\psi + m^2c^4\psi \\
\implies \hbar^2\frac{\partial^2\psi}{\partial t^2} - \hbar^2c^2\boldsymbol{\nabla}^2\psi &= -m^2c^4\psi \\
\implies \quad \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \boldsymbol{\nabla}^2\psi \quad &= \frac{-m^2c^2}{\hbar^2}\psi
\end{aligned}$$

The above equation is called the *Klein-Gordon equation*. In covariant notation, we can compactly write it as:

$$\left(\partial_\mu\partial^\mu + \left(\frac{mc}{\hbar}\right)^2\right)\psi = 0$$

1.1.2 Continuity Equation

Taking the complex cojungate of the Schrödinger's equation and then after some algebraic manipulation, we obtain:

$$\frac{\partial(\psi^*\psi)}{\partial t} = \frac{i\hbar}{2m}\boldsymbol{\nabla}\cdot[\psi^*(\boldsymbol{\nabla}\psi) - \psi(\boldsymbol{\nabla}\psi^*)]$$

Here, we identify:

$$\rho := \psi^*\psi \qquad\qquad \mathbf{J} = \frac{-i\hbar}{2m}[\psi^*(\boldsymbol{\nabla}\psi) - \psi(\boldsymbol{\nabla}\psi^*)]$$

which yields the well-known form of the *continuity equation*:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$$

Note that in this case, ρ is a positive-definite quantity and can indeed have the interpretation of probability. Also, note that if ψ somehow becomes real-valued, then $\mathbf{J} = 0$ which implies that ρ is constant in time (though it can change in space).

Doing the same thing to Klein-Gordon equation yields:

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \boldsymbol{\nabla}\cdot(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*) = 0$$

From this equation, we can identify:

$$\rho := \left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \qquad\qquad \mathbf{J} := -c^2(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)$$

Note the apparent problems with this identification:

- It is not apriori obvious that ρ is positive-definite and hence has problem with probability interpretation.
- The equation is second-order in time and hence ρ seems to have a term evolving forward and one term evolving backward in time.
- The dispersion relation does not have a single solution; the solutions are $\pm E$

The thing is, KG equation treats time and space on equal footing (unlike Schrödinger equation where time was in first order and space was in second order). Hence we have to consider all possibilities of moving back and forth in both space and time.