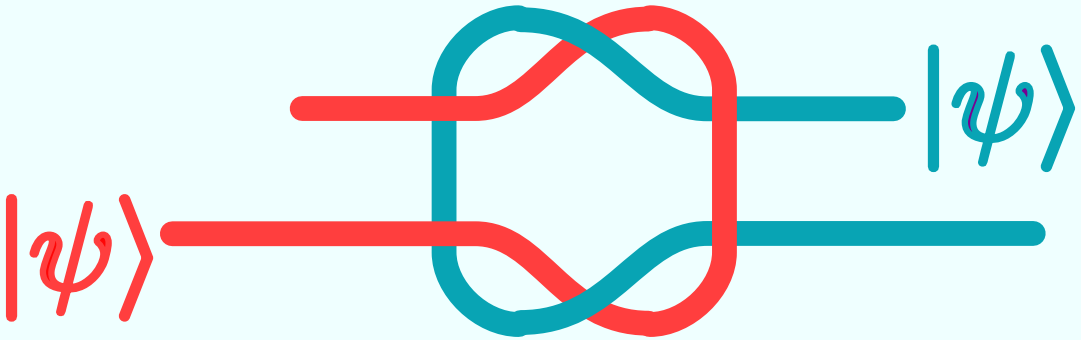
Entanglement Classification using Knots

PH3203 Term Project

Sagnik Seth 22MS026 Jessica Das 22MS157 Sayan Karmakar 22MS163

Instructor: Prof. Sourin Das

Department of Physics, IISER Kolkata



IISER KOLKATA

Basic Theoretical Background

Introduction to Knots

Introduction to Quantum Information

Peres-Horodecki Criterion

Classifying Entanglement using Knots

How it all began...

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.

.

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

Borromean Rings model:

Observations

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

Borromean Rings model:

- $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ in computational basis

Observations

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

Borromean Rings model:

- $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ in computational basis

Observations

- Measuring in Z basis: first particle measured, then remaining collapses to separable state $\rightarrow$ separable$\rightarrow$

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

Borromean Rings model:

- $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ in computational basis

Observations

- Measuring in Z basis: first particle measured, then remaining collapses to separable state $\rightarrow$ separable$\rightarrow$ modelled by Borromean ring

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

Borromean Rings model:

- $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ in computational basis
- $|\Psi\rangle = \frac{|+\rangle}{\sqrt{2}}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) + \frac{|-\rangle}{\sqrt{2}}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$ in X basis

Observations

- Measuring in Z basis: first particle measured, then remaining collapses to separable state $\rightarrow$ separable$\rightarrow$ modelled by Borromean ring

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

Borromean Rings model:

- $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ in computational basis
- $|\Psi\rangle = \frac{|+\rangle}{\sqrt{2}}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) + \frac{|-\rangle}{\sqrt{2}}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$ in X basis

Observations

- Measuring in Z basis: first particle measured, then remaining collapses to separable state $\rightarrow$ separable$\rightarrow$ modelled by Borromean ring
- Measuring in X basis: first particle measured, then remaining collapses to an entangled state $\rightarrow$ remains entangled

How it all began...

- Aravind, 1997 $\rightarrow$ modelled entanglement using knots.
- Used basis-dependent measurement as an analogy to 'cut the knots'.

Borromean Rings model:

- $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ in computational basis
- $|\Psi\rangle = \frac{|+\rangle}{\sqrt{2}}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) + \frac{|-\rangle}{\sqrt{2}}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$ in X basis

Observations

- Measuring in Z basis: first particle measured, then remaining collapses to separable state $\rightarrow$ separable$\rightarrow$ modelled by Borromean ring
- Measuring in X basis: first particle measured, then remaining collapses to an entangled state $\rightarrow$ remains entangled$\rightarrow$ modelled by 3-Hopf rings

Polynomial Approach to Entanglement

Obtaining a Link from given State

Obtaining State from a given Link

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots \left|2^N - 1\right\rangle$

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots \left|2^N - 1\right\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots \left|2^N - 1\right\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots \left|2^N - 1\right\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Algorithm:

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots \left|2^N - 1\right\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Algorithm:

1. Take a term t of the link polynomial $P(\{t\})$

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots |2^N - 1\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Algorithm:

1. Take a term t of the link polynomial $P(\{t\})$
2. t mapped to a state: $|\mathrm{E_q}\rangle \otimes |\mathrm{S_q}\rangle \otimes |\mathrm{Q_d}\rangle$

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots \left|2^N - 1\right\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Algorithm:

1. Take a term t of the link polynomial $P(\{t\})$
2. t mapped to a state: $|\mathrm{E_q}\rangle \otimes |\mathrm{S_q}\rangle \otimes |\mathrm{Q_d}\rangle$
   - $|\mathrm{E_q}\rangle \longrightarrow$ entangled qubit of GHZ type, associated to ring variables in t.

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots \left|2^N - 1\right\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Algorithm:

1. Take a term t of the link polynomial $P(\{t\})$
2. t mapped to a state: $|\mathrm{E_q}\rangle \otimes |\mathrm{S_q}\rangle \otimes |\mathrm{Q_d}\rangle$
   - $|\mathrm{E_q}\rangle \longrightarrow$ entangled qubit of GHZ type, associated to ring variables in t.
   - $|\mathrm{S_q}\rangle \longrightarrow$ separable qubit associated with ring variables not in t. Generally, large number of possibilities for this separable qubit.

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots |2^N - 1\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Algorithm:

1. Take a term t of the link polynomial $P(\{t\})$
2. t mapped to a state: $|\mathrm{E_q}\rangle \otimes |\mathrm{S_q}\rangle \otimes |\mathrm{Q_d}\rangle$
   - $|\mathrm{E_q}\rangle \longrightarrow$ entangled qubit of GHZ type, associated to ring variables in t.
   - $|\mathrm{S_q}\rangle \longrightarrow$ separable qubit associated with ring variables not in t. Generally, large number of possibilities for this separable qubit.
   - $|\mathrm{Q_d}\rangle \longrightarrow$ qudit state associated with *artificially* introduced ring variable (alphabetical successor of the largest ring variable present). To be traced out later, hence is of less significance. Can be a random state, hence it becomes **mixed**.

Obtaining State from a given Link

- In general, difficult to obtain state, given a link polynomial.
- Given N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots |2^N - 1\rangle$
- An algorithm has been devised to obtain a general structure of the state from the link.
- But it only produces a mixed state only ☹ !

Algorithm:

1. Take a term t of the link polynomial $P(\{t\})$
2. t mapped to a state: $|\mathrm{E_q}\rangle \otimes |\mathrm{S_q}\rangle \otimes |\mathrm{Q_d}\rangle$
   - $|\mathrm{E_q}\rangle \longrightarrow$ entangled qubit of GHZ type, associated to ring variables in t.
   - $|\mathrm{S_q}\rangle \longrightarrow$ separable qubit associated with ring variables not in t. Generally, large number of possibilities for this separable qubit.
   - $|\mathrm{Q_d}\rangle \longrightarrow$ qudit state associated with *artificially* introduced ring variable (alphabetical successor of the largest ring variable present). To be traced out later, hence is of less significance. Can be a random state, hence it becomes **mixed**.
3. Full state $|\psi\rangle \longrightarrow$ sum of individual such states. Trace out d and obtain density matrix of reduced system.

Demonstration of Algorithm

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ab$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ab$
- Two ring variables a and b, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ab}$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ab$
- Two ring variables a and b, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ab}$
- Separable qubit associated with c, keep it general $|\mathrm{S_q}\rangle = |q_1\rangle_c$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ab$
- Two ring variables a and b, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ab}$
- Separable qubit associated with c, keep it general $|\mathrm{S_q}\rangle = |q_1\rangle_c$
- Qudit state, take $|\mathrm{Q_d}\rangle = |0\rangle_d$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ab$
- Two ring variables a and b, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ab}$
- Separable qubit associated with c, keep it general $|\mathrm{S_q}\rangle = |q_1\rangle_c$
- Qudit state, take $|\mathrm{Q_d}\rangle = |0\rangle_d$
- **Full state becomes**:

$$|\psi_1\rangle = \left|2^1\right\rangle_{ab} \otimes |q_1\rangle_c \otimes |0\rangle_d$$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ac$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ac$
- Two ring variables a and c, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ac}$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ac$
- Two ring variables a and c, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ac}$
- Separable qubit associated with b, keep it general $|\mathrm{S_q}\rangle = |q_1\rangle_b$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ac$
- Two ring variables a and c, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ac}$
- Separable qubit associated with b, keep it general $|\mathrm{S_q}\rangle = |q_1\rangle_b$
- Qudit state, take $|\mathrm{Q_d}\rangle = |1\rangle_d$

Demonstration of Algorithm

Let us take a look at a demonstration of the algorithm. Take:

$$\mathrm{P(a, b, c) = ab + ac}$$

- Choose the term $t = ac$
- Two ring variables a and c, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ac}$
- Separable qubit associated with b, keep it general $|\mathrm{S_q}\rangle = |q_1\rangle_b$
- Qudit state, take $|\mathrm{Q_d}\rangle = |1\rangle_d$
- **Full state becomes**: $|\psi_2\rangle = \left|2^1\right\rangle_{ac} \otimes |q_2\rangle_b \otimes |1\rangle_d$

Demonstration of Algorithm

- **Mixed state characterising the link**: $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$

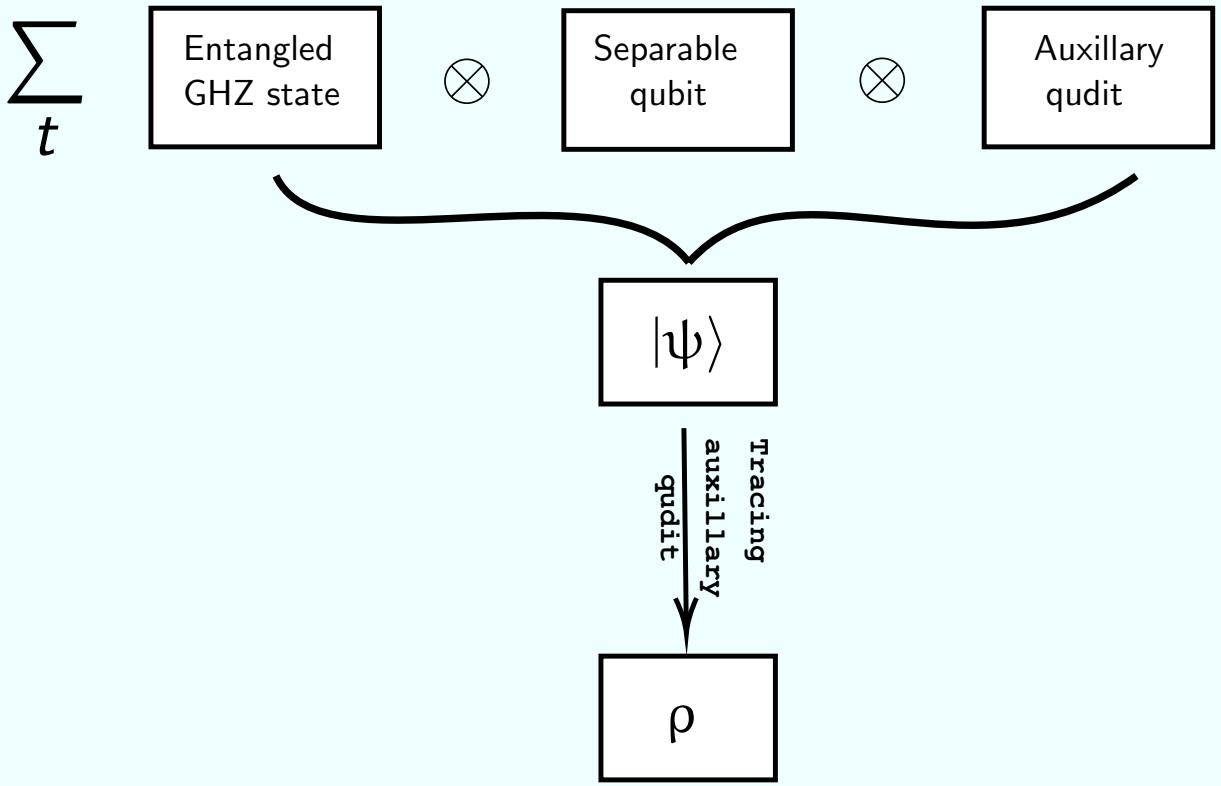


Figure: Schematic representation of the algorithm.

Demonstration of Algorithm

- **Mixed state characterising the link**: $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$
- Then construct the density matrix accordingly for the state.

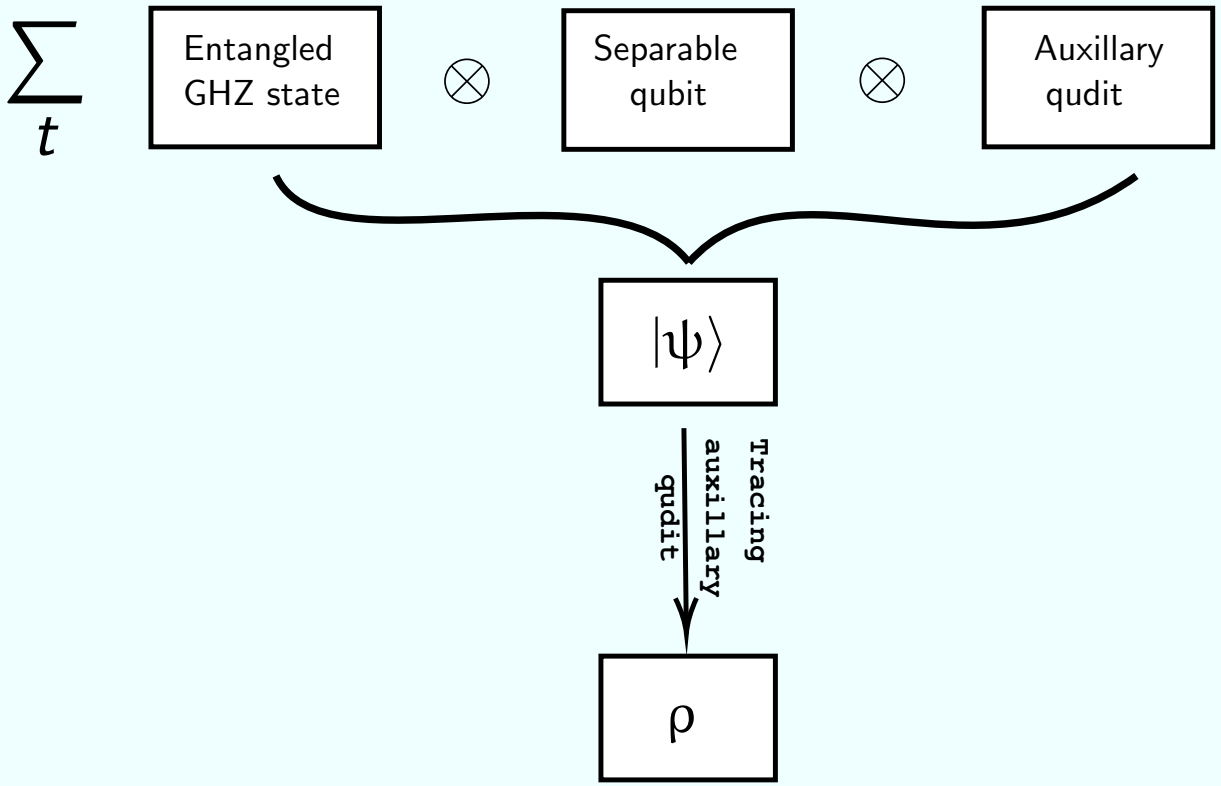


Figure: Schematic representation of the algorithm.

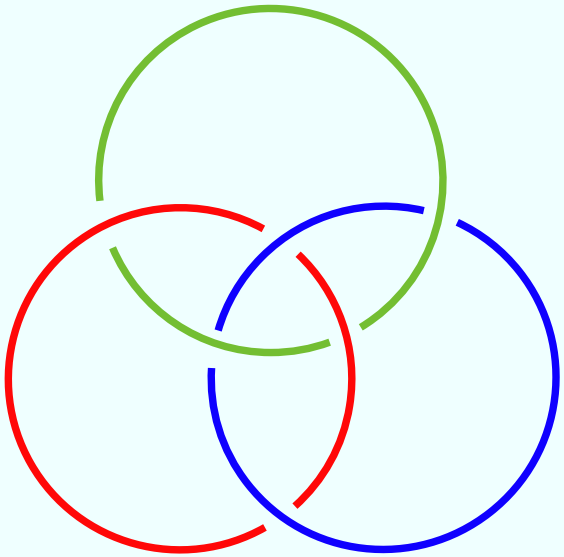
Some Examples...

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$

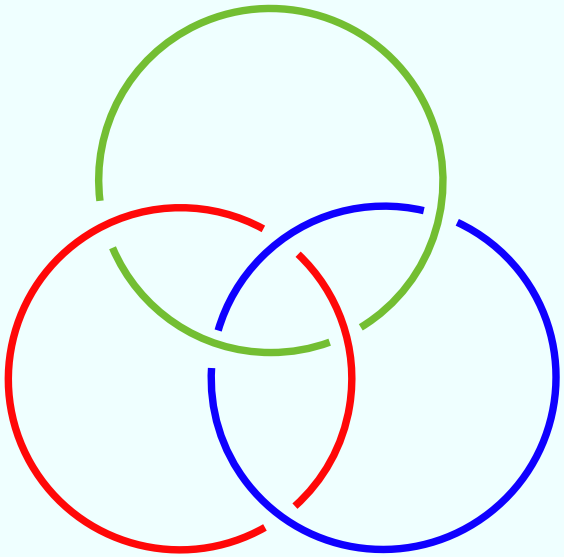
Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



Three Qubit System: 3^1 class

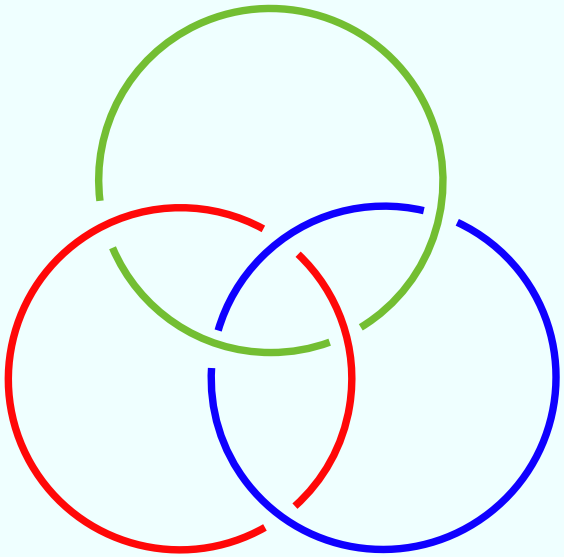
Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$

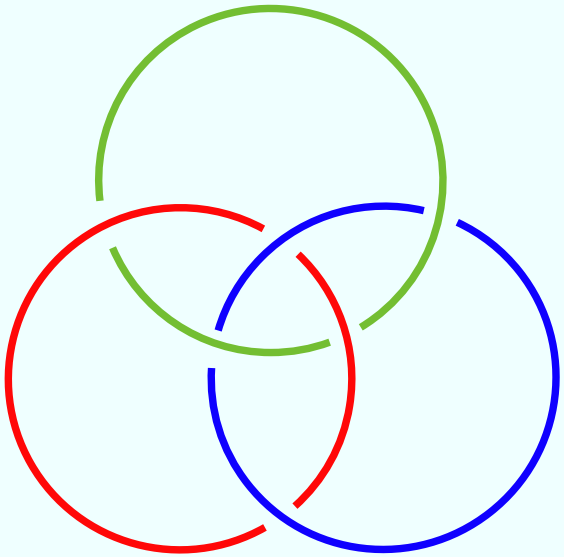


$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



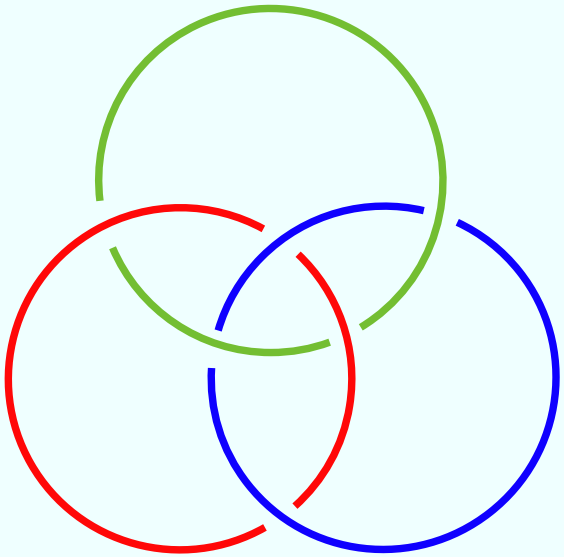
$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- **Eigenvalues:**
  0.0, 0.5, −0.5

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- **Eigenvalues:** 0.0, 0.5, -0.5
- One eigenvalue is negative $\rightarrow$ **Tripartite Entanglement**

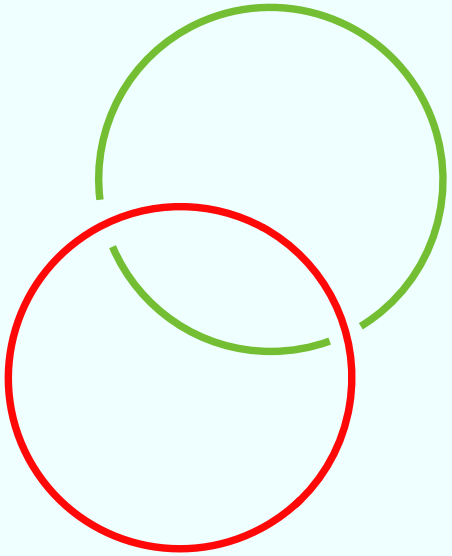
$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$

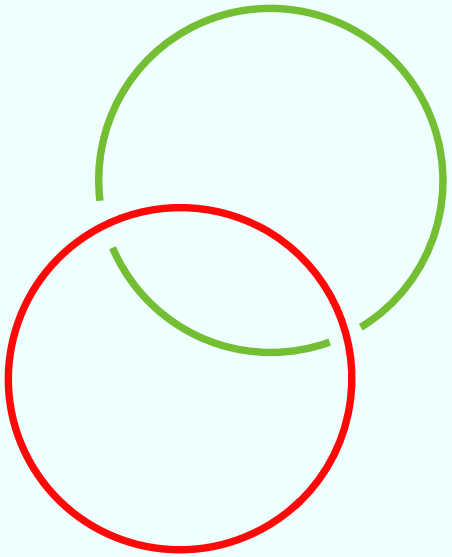
Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



Three Qubit System: 3^1 class

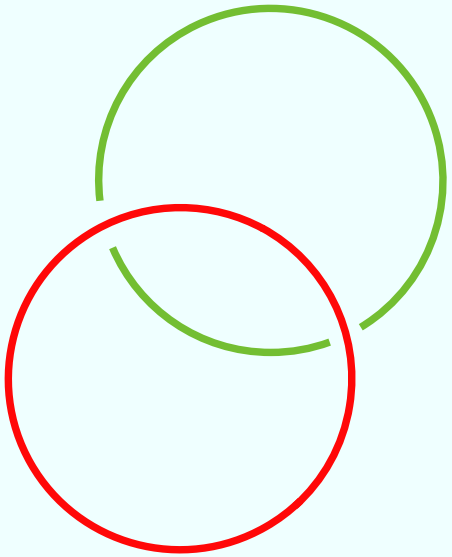
Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



$$\rho_{ab}\begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$

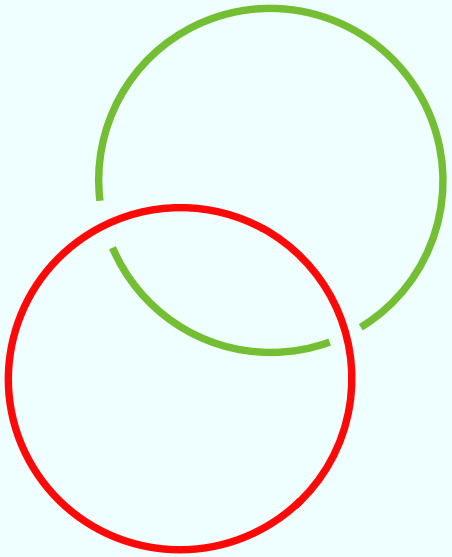


$$\rho_{ab}\begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{ab}^{Ta}\begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



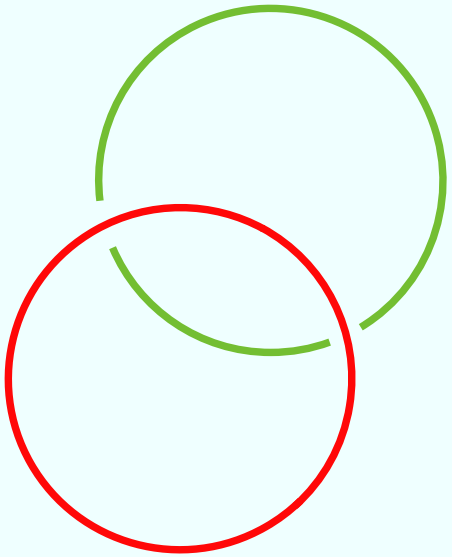
$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{ab}^{Ta} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- **Eigenvalues:**
  $0.0,\ 0.5 \geqslant 0$

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



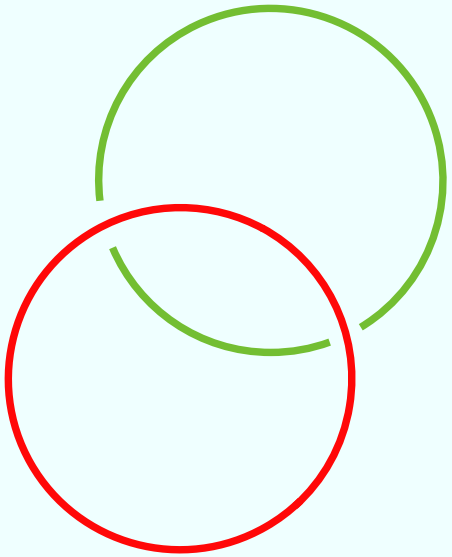
$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{ab}^{Ta} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- **Eigenvalues:**
  $0.0,\ 0.5 \geqslant 0$
- All eigenvalues are positive.

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$



- **Eigenvalues:**
  $0.0,\ 0.5 \geqslant 0$

$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{ab}^{Ta} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

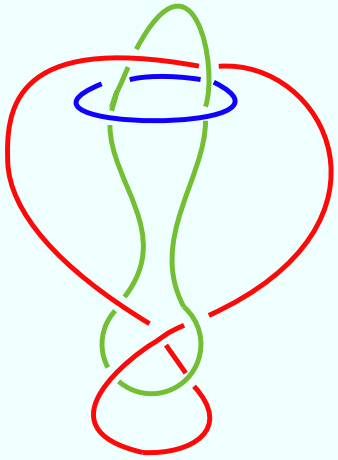
- All eigenvalues are positive.
- System completely **separable** after cut.

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

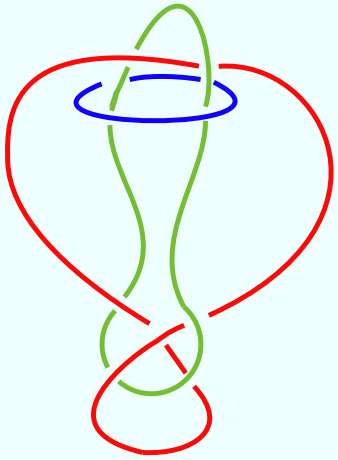
Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



Three Qubit System: 3^2 class

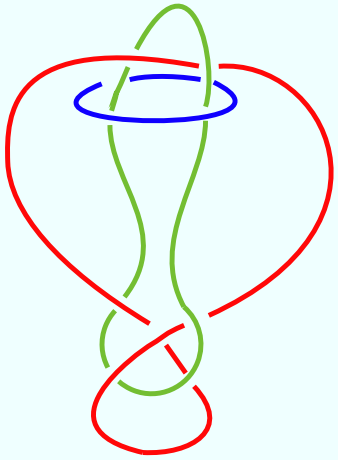
Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

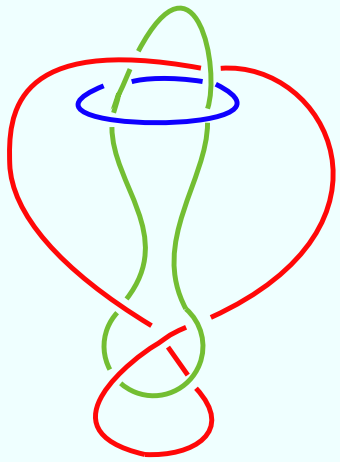


$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

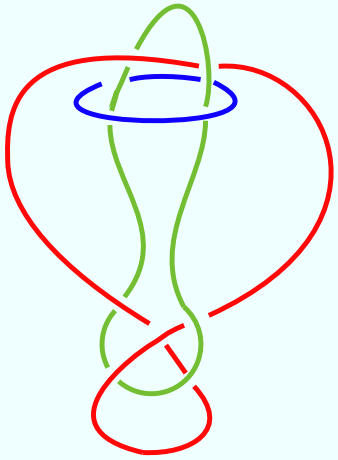
$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**

  $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

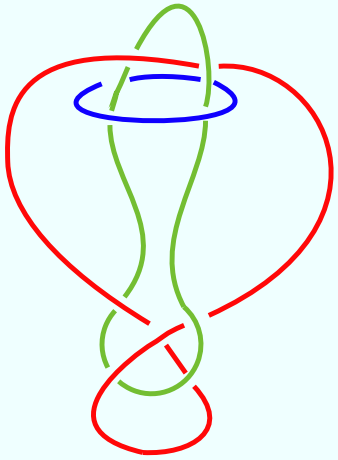
$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**

  $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

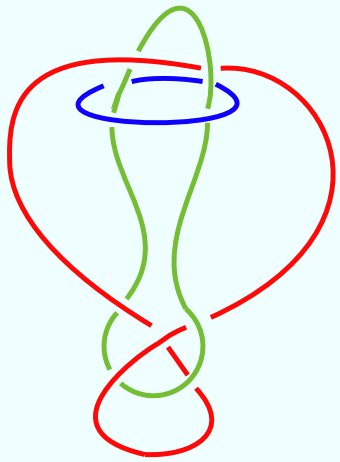
- **Eigenvalues:**

  $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$

  $-\mathbf{0.333}, 0.0, 0.127, 0.333, 0.872$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**

  $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$

  $-\mathbf{0.333}, 0.0, 0.127, 0.333, 0.872$

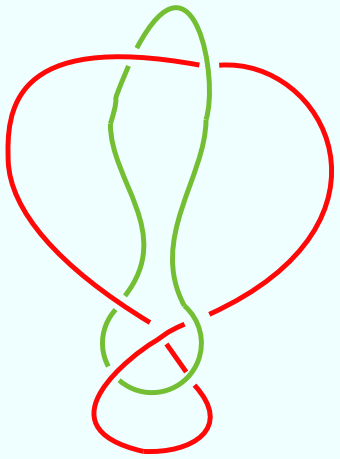
- One eigenvalue is negative $\rightarrow$ **Tripartite Entanglement**

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

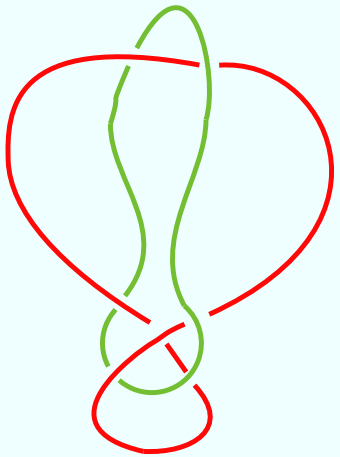
Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



Three Qubit System: 3^2 class: cut

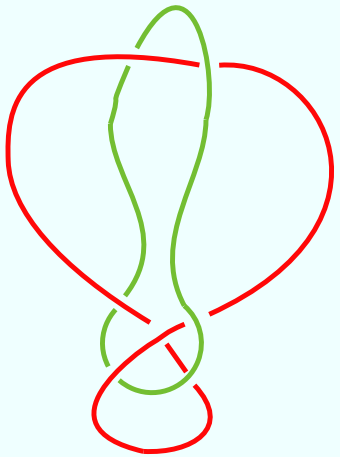
Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

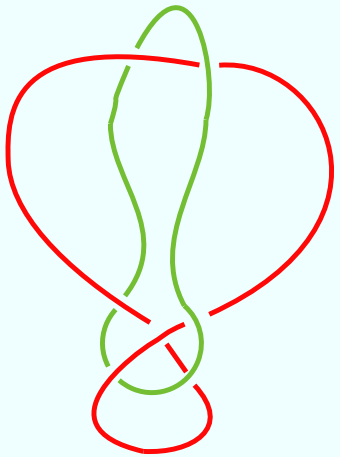


$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

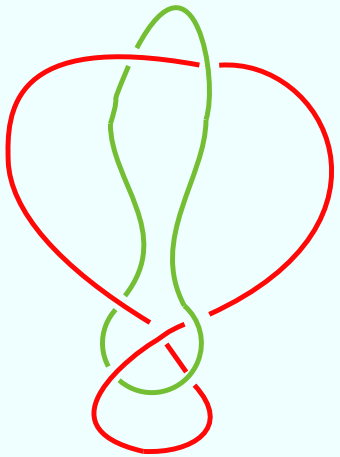
$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**

  $0.333, -\mathbf{0.333}, 0.666$

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

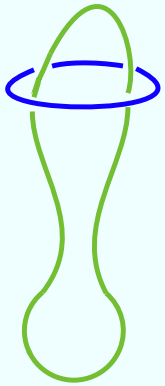
- **Eigenvalues:**
  $0.333, -\mathbf{0.333}, 0.666$
- One eigenvalue is negative $\rightarrow$ **Tripartite Entanglement**

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

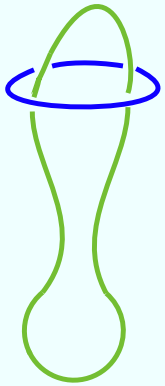
Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



Three Qubit System: 3^2 class: another cut

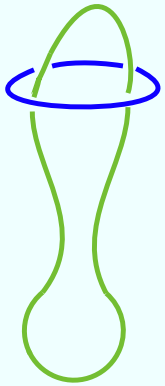
Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

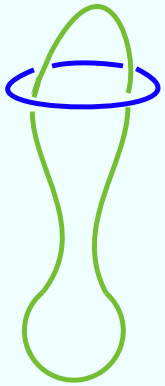


$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



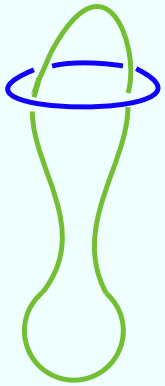
- **Eigenvalues:**
  0.0, 0.333, 0.666

$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**
  $0.0, 0.333, 0.666$
- No eigenvalue is negative $\rightarrow$ **Separable**

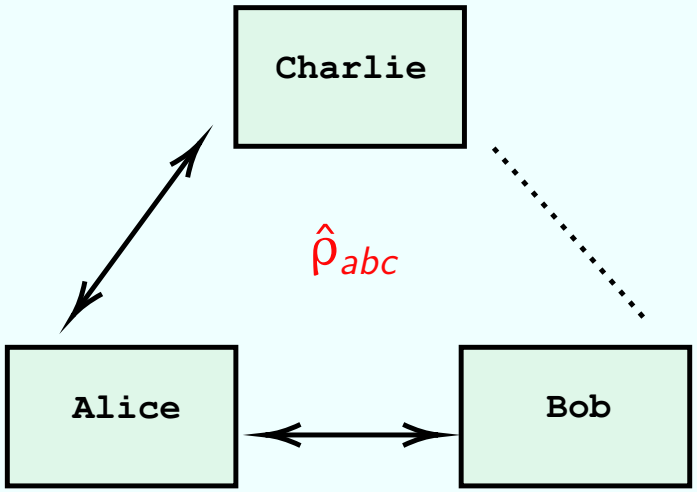
Four Qubit System

Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.

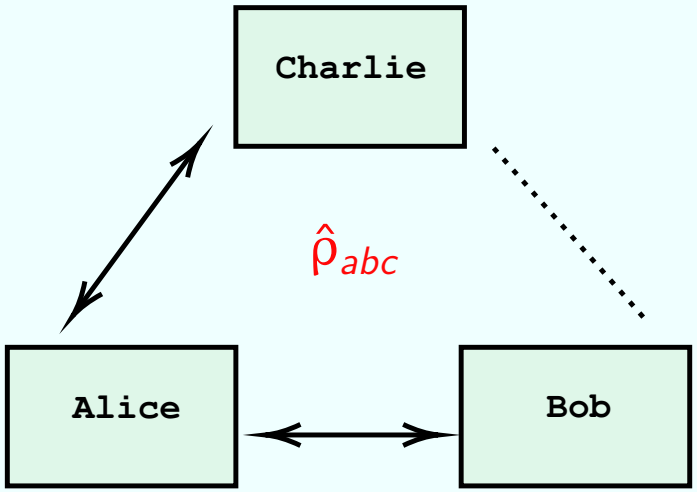
Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.



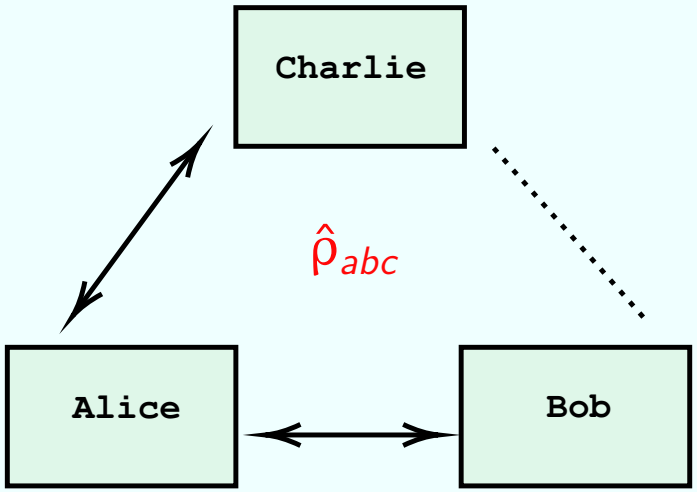
Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.
- If density operator of the subsystem consisting the parties wanting to perform the protocol is separable, then the protocol cannot be performed successfully.



Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.
- If density operator of the subsystem consisting the parties wanting to perform the protocol is separable, then the protocol cannot be performed successfully.
- If another party (not participating in the protocol) does not divulge information about their local operations, then results of the protocol cannot be correlated.



Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie

Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie
- Alice, Bob, and Diana

Application to Qubit Networks

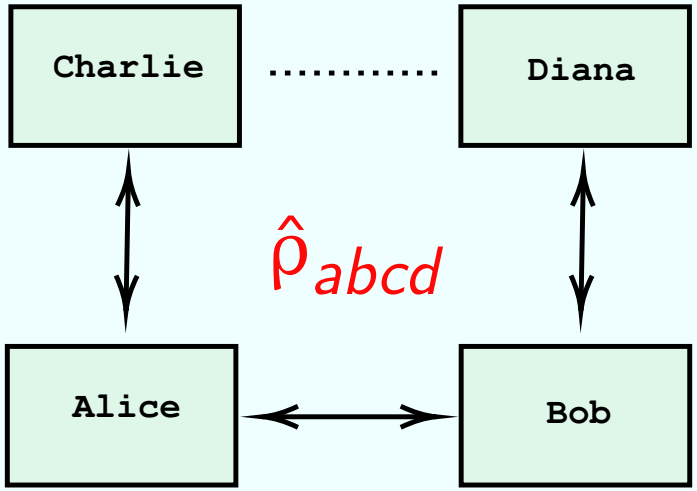
Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie
- Alice, Bob, and Diana
- Alice and Charlie

Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

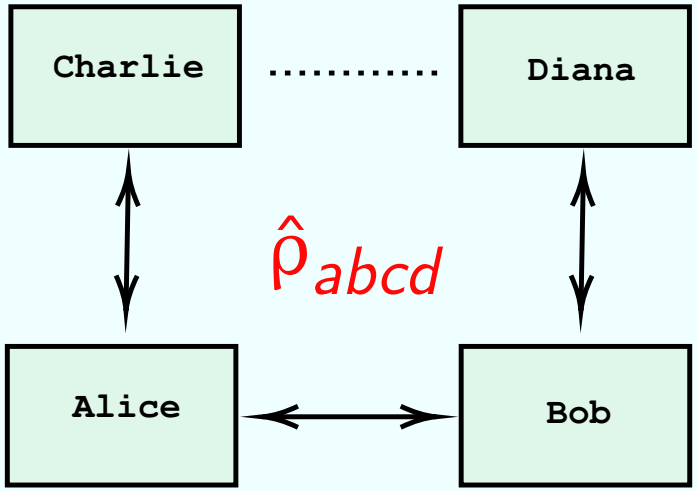
- Alice, Bob, and Charlie
- Alice, Bob, and Diana
- Alice and Charlie



Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie
- Alice, Bob, and Diana
- Alice and Charlie

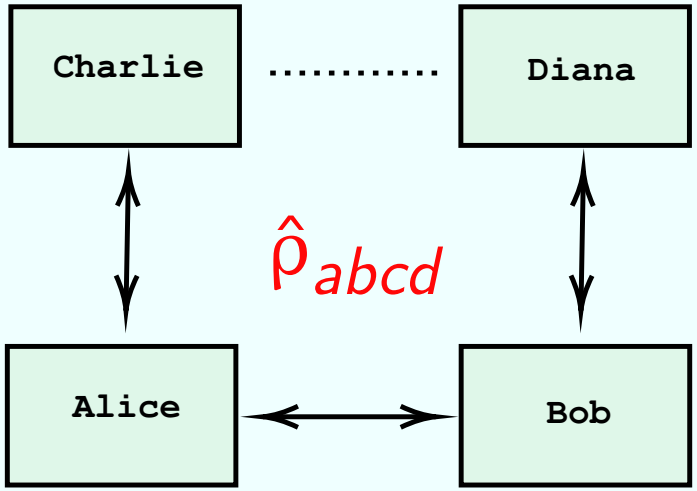


We can easily see that the polynomial describing this network is $\mathrm{P(a,b,c,d) = abc + abd + ac} \longrightarrow$

Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie
- Alice, Bob, and Diana
- Alice and Charlie



We can easily see that the polynomial describing this network is $\mathrm{P(a,b,c,d) = abc + abd + ac} \longrightarrow$ a state can immediately be constructed from the algorithm.

Conclusion