Classification of Entanglement using Knots

PH3203 Term Project

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1 Introduction

Classifying entanglement is essential because not all quantum states are equally useful for quantum information tasks. Different types of entanglement serve as distinct resources, each suited to specific applications such as quantum algorithms or secure communication protocols like quantum key distribution. Understanding and identifying these entanglement types helps determine how quantum states can be used and manipulated effectively. hi hi hi this is testing

SLOCC (Stochastic Local Operations and Classical Communication) is a method for classifying quantum entanglement. It defines equivalence classes of quantum states based on whether they can be converted into each other using local operations (on individual qubits) and classical communication.

This idea has been used successfully to study three-qubit states, as shown in [1,2], classifying four-qubit states [3–6] Methods have been developed for handling systems with even more qubits [7,8].[0.3cm] In this paper, the authors have proposed an alternative classification scheme for quantum entanglement based on topological links.

One of the first images that comes to mind when we think of entanglement is that of entangled threads. Naturally, one wonders if we could study quantum entanglement using entangled 'knots'. Aravind [1] was the first to point out the connections between entangled quantum states and classical knot configurations, focussing on similarity between 3-particle GHZ state and Borromean rings. He associated each particle with a ring, entanglement of any set of particles as inability to separate their corresponding rings, and measurement of particle state as cutting its ring. But he noted that performing the measurement in different basis would not lead to the same conclusions. This limit in analogy was dealt with by Sugita [2]. He proposed that cutting the ring is equivalent to tracing out the corresponding particle from the density operator, which is a basis-independent operation. This represents viewing the system as though that particle is no longer present. Moreover, the trace operation helps to generalise the idea to quantum systems with more than 2 levels.

2 Classification of Links: A Polynomial Approach

2.1 Formalism of the Link Polynomial

2.2 Obtaining a Link from an Entangled Quantum State

2.3 Obtaining an Entangled Quantum State from a Link

In this section, we will see how we can obtain a link from a quantum state. This is in general a difficult task to obtain an entangled quantum states from the polynomial as the number of qubits increases. The process in the paper mentions an algorithm which provides an 'incomplete' map betwee a given link and a quantum state. Using the procedure, the general structure of the quantum state can be obtained, however, some free coefficients remain which needs to be fixed computationally. Moreover, presently only mixed states satisfying the link can be obtained using this procedure.

Note that although incomplete, the map is still useful since we can ascertain the structure of each state contained in the mixed state. That is, from a possibility of 2^N (for N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots |2^N - 1\rangle$ where each ket is to be assumed in the binary representation) states, we are reducing it to a much smaller number.

For this, we will use the GHZ type of state as a building block which are of the form:

$$|N^1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$$

Here $|0\rangle^{\otimes N}$ is the tensor product of N number of $|0\rangle$ states, that is, $|0\rangle^{\otimes N} = |\underbrace{0000\ldots0}_{\text{N times}}\rangle$. The state $|N^1\rangle$ is a maximally entangled state of N qubits. The general algorithm to obtain a state from the link is as follows:

1. Let a polynomial P be given. Select a term of the given 'link' polynomial, say t.
2. The term t is then mapped to a state of the form $|\mathrm{E_q}\rangle \otimes |\mathrm{S_q}\rangle \otimes |\mathrm{Q_d}\rangle$ where:
   - $|\mathrm{E_q}\rangle$ is the entangled qubit of the GHZ type as specified above, associated to ring variables contained in t.
   - $|\mathrm{S_q}\rangle$ is a separable qubit associated with ring variables not contained in t. There are a number of possibilities for this separable qubit and we have to find it computationally.
   - $|\mathrm{Q_d}\rangle$ is a qudit state which is associated with an artifically introduced ring variable (which is alphabetically the next letter of the largest ring variable present). The states always starts from 0 for the first term and is increased by 1 for each successive term of the polynomial. This will later be traced out, hence is of less significance.
3. The full state $|\psi\rangle$ is constructed by summing these individual states obtained for each term of the polynomial.
4. The full mixed state characterised by this polynomial is then obtained by tracing out the qudit state $|\mathrm{Q_d}\rangle$.

$$\boxed{\hat{\rho}(P) = \frac{\mathrm{Tr}_d \, |\psi\rangle \langle\psi|}{\sqrt{\langle\psi|\psi\rangle}}}$$

Example demonstrating the algorithm:

We will see a simple example of the algorithm to obtain a state from a link. Consider the polynomial $P(a,b,c) = ab + ac$. This is a three-ring link.

- Let us choose the term $t = ab$. This term has two ring variables thus we will associate a two qubit GHZ type of state to $|\mathrm{E_q}\rangle$. Thus, we have $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \equiv |2^1\rangle_{ab}$.

  Since the separable qubit has large possibility, we will denote it generally by $|q_1\rangle$ and this will be associated with the remaining ring variable which is c. Thus, $|\mathrm{S_q}\rangle = |q_1\rangle_c$. The remaining term is the qudit state which will be associated to d (since d is alphabetical successor of the largest ring variable c). Then we will have the full state:

$$|\psi_1\rangle = |2^1\rangle_{ab} \otimes |q_1\rangle_c \otimes |0\rangle_d$$

- Now, let us choose the next term in the polynomial which is $t = ac$. Similar to above, to the entangled qubit we will associate the two qubit GHZ state, thus, $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \equiv |2^1\rangle_{ac}$.

  The separable qubit will be associated with the remaining ring variable b and we will denote it by $|q_2\rangle_b$. The qudit state will be associated with d which is the alphabetical successor of c but this time we will use $|1\rangle_d$ as for each successive term, the qudit state increases to the next level. Thus, we have:

$$|\psi_2\rangle = |2^1\rangle_{ac} \otimes |q_2\rangle_b \otimes |1\rangle_d$$

- The full state $|\psi\rangle$ is then obtained by summing the two states obtained above with some coefficients:

$$\begin{aligned} |\psi\rangle &= c_1 |\psi_1\rangle + c_2 |\psi_2\rangle \\ &= c_1(|2^1\rangle_{ab} \otimes |q_1\rangle_c \otimes |0\rangle_d) + c_2(|2^1\rangle_{ac} \otimes |q_2\rangle_b \otimes |1\rangle_d) \end{aligned}$$

  Then we can trace out the qudit state $|d\rangle$ to obtain the density matrix of the ring variables:

$$\boxed{\hat{\rho}_{abc} = \frac{\mathrm{Tr}_d \, |\psi\rangle \langle\psi|}{\sqrt{\langle\psi|\psi\rangle}}}$$

2.4 Applying to Three Qubit Systems

As a demonstraction, we will apply our algorithm to three qubit systems. Note that from the rules of the 'link' polynomial, the possible basis terms for three qubit system are: $\{ab, ac, bc, abc\}$. Using this, four distinct classes of polynomials are possible:

$$\begin{aligned}
P_1(a,b,c) &= abc \\
P_2(a,b,c) &= abc + ab \\
P_3(a,b,c) &= ab + ac \\
P_4(a,b,c) &= ab + ac + bc
\end{aligned}$$

Let us start with the 3^1 link class, which correspond to the Borromean Link. Cutting any of a, b or c will lead to complete separabability and loss of entanglement.

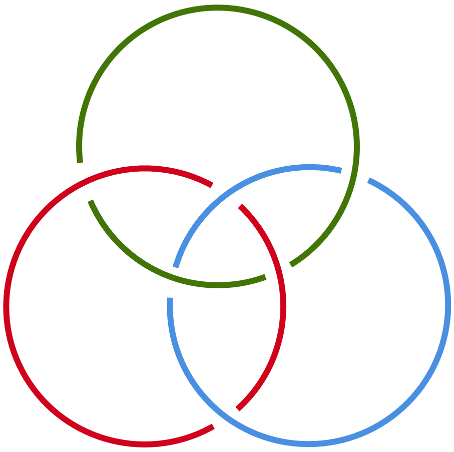

Figure 1: The Borromean link, characterising the 3^1 link class.

We already know that on its own the GHZ state characterises the 3^1 link, as discussed in the preceding works. Thus, we have the pure state:

$$|3^1\rangle_{abc} = \frac{1}{\sqrt{2}}(|000\rangle_{abc} + |111\rangle_{abc})$$

The density matrix corresponding to the state is found to be:

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix} \tag{1}$$

The partial transpose with respect to any of the subsystem (since the polynomial is symmetric) is same and is given by:

$$\rho^{T_a/T_b/T_c} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix} \tag{2}$$

The eigenvalues corresponding to this matrix are 0.0, 0.5 and -0.5. Since there are negative eigenvalues, we can conclude that the system exhibits **tripartite entanglement** as a whole.

Now, let us reduce the system by tracing out one of the variable. Since the polynomial is symmetric, we can choose any of the variable, say c. The reduced density matrix is given by:

$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix} \tag{3}$$

The partial transpose with respect to a or b results in the same above matrix which have eigenvalues 0.5 and 0.0 which are all positive, thus concluding the absence of any entanglement in the system. This is consistent with the fact that for the Borromean link, cutting any link with result in complete separability of the links.

Let us now consider the 3^2 link class given by $P_2(a,b,c) = abc + ab$. Cutting any of a, b will lead to complete separability but if we cut c, then the other rings will remain entangled. The link can be represented as:

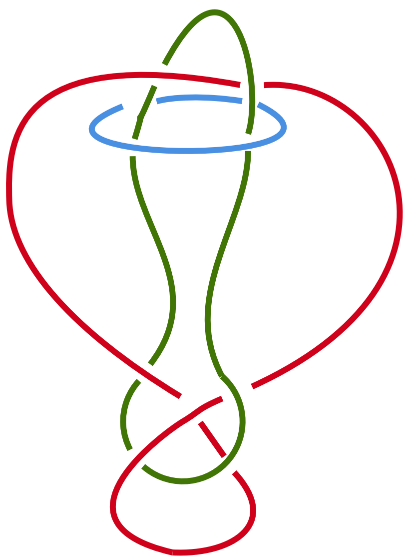

Figure 2: The knot diagram, characterising the 3^2 link class.

Here the blue knot corresponds to c while the other two correspond to a and b (a,b are symmetric in the polynomial). An example of a pure state is found, as mentioned in the paper:

$$|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} \left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$$

To check that this state indeed statistfies the link, let us calculate the density operator $\hat{\rho}_{abc} = |3^2\rangle_{abc} \langle 3^2|_{abc}$

and then check for the PPT test for each cuts. The density matrix obtained is:

$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix} \tag{4}$$

Since the variables a and b are symmetrix, we can choose to analyse only one of them and c. We then see the partial transpose with respect to a and c. The matrices are given by:

$$\rho^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix} \tag{5}$$

$$\rho^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix} \tag{6}$$

The above matrices have eigenvalues $\lambda_a = -0.471, 0.0, 0.333, 0.471, 0.666$ and $\lambda_c = -0.333, 0.0, 0.127, 0.333, 0.872$ respectively. Since there are negative eigenvalues, we can conclude that the system exhibits **tripartite entanglement** as a whole.

The reduced density matrices are given by:

$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix} \tag{7}$$

$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix} \tag{8}$$

We now obtain the partial tranpose for the PPT test. We note that $\rho_{bc}^{T_b/T_c}$ is the same as that of the above matrix whose eigenvalues are $0.0, 0.333, 0.666$ which are all positive. Thus, we can conclude that the system is separable. On the other hand, we obtain:

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix} \tag{9}$$

The eigenvalues of this matrix are $0.333, -0.333, 0.666$, one of which is negative, thus denoting the presence of entanglement. This successfully verifies the behaviour of the link $abc + ac$.

Using the above algorithm, we can construct the mixed state corresponding to the link by considering the state:

$$|\psi_2\rangle = |3^1\rangle_{abc} |0\rangle_d + |2^1\rangle_{ab} |0\rangle_c |1\rangle_d$$

It is to be noted that this class has not been described in the previous works [1, 2]. The density operator is then given by:

$$\hat{\rho}(a, b, c) = \frac{\mathrm{Tr}_d |\psi_2\rangle \langle\psi_2|}{\sqrt{\langle\psi_2|\psi_2\rangle}}$$

We now analyse the class $P_3(a, b, c) = ab + ac$. The link diagram is given by:

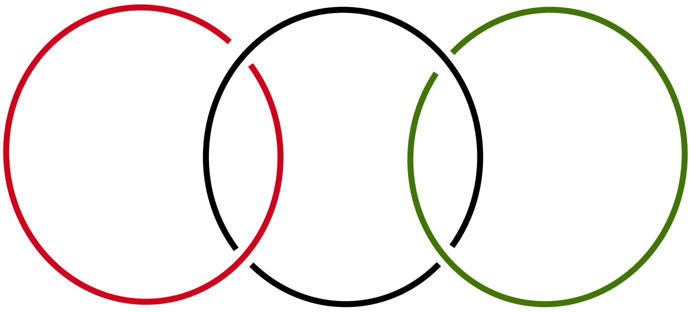

Figure 3: The knot diagram, characterising the 3^3 link class.

We can see that the knot polynomial is symmetric in b and c. Thus, if we cut either b or c, the other will remain entangled with a but complete separation results from cutting a. This case has already been discussed in section 2.2 and a pure state has been obtained. The mixed state can be obtained using the state:

$$|\psi_3\rangle = |2^1\rangle_{ab} |0\rangle_c |0\rangle_d + |2^1\rangle_{ac} |1\rangle_b |1\rangle_d$$

Let us now analyse the class $P_4(a, b, c) = ab + ac + bc$. The link diagram is given by:

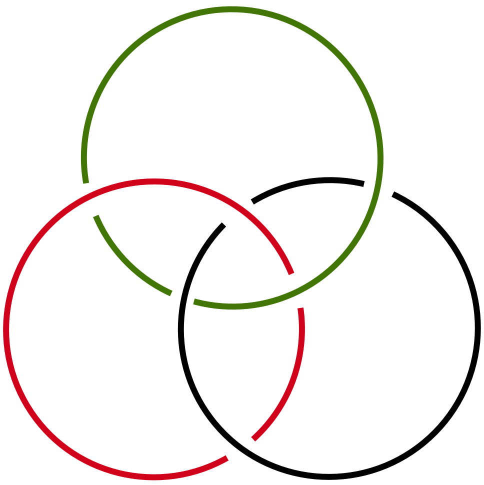

Figure 4: The knot diagram, characterising the 3^4 link class.

This link demostrates a behaviour opposite to that of the Borromean ring, that is, setting any one variable to zero with not result in complete separability. The other two links remain entangled. The pure state having the characteristic of the link is the **W state** which has been documented in previous works:

$$|3^4\rangle = \frac{1}{\sqrt{3}} \left(|001\rangle_{abc} + |010\rangle_{abc} + |100\rangle_{abc}\right)$$

From the algorithm, the mixed state can be obtained using the state:

$$|\psi_4\rangle = |2^1\rangle_{ab} |0\rangle_c |0\rangle_d + |2^1\rangle_{ac} |1\rangle_b |1\rangle_d + |2^1\rangle_{bc} |0\rangle_a |2\rangle_d$$

2.5 Applying to Four Qubit Systems

3 Physical Significance: Use in Quantum Networks

4 Discussion and Conclusion

Appendix A: Quantum Information Basics

4.1 Density Matrix

4.2 Peres-Horodecki Criterion

Appendix B: Knot Theory Basics

Appendix C: Code for Numeric Calculations

We used the `QuantumInformation.jl` package in `Julia` to perform the numerical calculations. The code provided below shows some basic calculations that we had used in this report.

```
using QuantumInformation, LinearAlgebra, Latexify
```

References

[1] P. K. Aravind. *Borromean Entanglement of the GHZ State*, pages 53–59. Springer Netherlands, Dordrecht, 1997.

[2] Ayumu Sugita. Borromean entanglement revisited. 2007.