Four Qubit System

Link Class: 4^{20}
Link polynomial: $abc + abd + ac$

$$|\psi_{20}\rangle = \left|3^1\right\rangle_{abc} |0\rangle_d |0\rangle_e + \left|3^1\right\rangle_{abd} |0\rangle_c |1\rangle_e + \left|2^1\right\rangle_{ac} |10\rangle_{bd} |2\rangle_e .$$

Four Qubit System

$$\hat{\rho}_{abcd} = \frac{\mathrm{Tr}_e(|\psi_{20}\rangle \langle\psi_{20}|)}{\sqrt{\langle\psi_{20}|\psi_{20}\rangle}}.$$

We need to check for Four-partite entanglement. So we have to compute the eigenvalues of the partial transpose with respect to each subsystem.

$\hat{\rho}_{abcd}^{T_a}$: Eigenvalues: $-0.270, -0.103, 0.000, 0.103, 0.270, 0.333$

$\hat{\rho}_{abcd}^{T_b}$: Eigenvalues: $-0.186, 0.000, 0.167, 0.209, 0.333, 0.477$

$\hat{\rho}_{abcd}^{T_c}$: Eigenvalues: $-0.236, 0.000, 0.064, 0.167, 0.236, 0.333, 0.436$

$\hat{\rho}_{abcd}^{T_d}$: Eigenvalues: $-0.167, 0.000, 0.033, 0.167, 0.259, 0.541$

Four Qubit System

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$$\hat{\rho}_{abcd}^{T_d}: \text{ Eigenvalues: } -0.167, 0.000, 0.033, 0.167, 0.259, 0.541$$

As the eigenvalues are negative, it has **FOUR PARTITE ENTANGLEMENT**.

Four Qubit System

All Possible Partial Traces:

Partial Trace with respect to system a **:**

$$\hat{\rho}_{bcd} = \begin{bmatrix} 0.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.166 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.166 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As this is a diagonal matrix, we can say directly that after measuring a, the rest of the system b,c,d becomes **separable**.

Four Qubit System

Partial Trace with respect to system b :

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Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

$$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

- $\hat{\rho}_{acd}^{T_c}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

- $\hat{\rho}_{acd}^{T_c}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

- $\hat{\rho}_{acd}^{T_d}$:

  Eigenvalues: $0.000, 0.000, 0.000, 0.167, 0.230, 0.603.$

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

$$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

- $\hat{\rho}_{acd}^{T_c}$:

$$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

- $\hat{\rho}_{acd}^{T_d}$:

$$\text{Eigenvalues:} \ 0.000, 0.000, 0.000, 0.167, 0.230, 0.603.$$

Negative eigenvalues suggest, a **and** cd **are entangled**, and c **and** ad **are entangled.**

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

  $$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

- $\hat{\rho}_{acd}^{T_c}$:

  $$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

- $\hat{\rho}_{acd}^{T_d}$:

  $$\text{Eigenvalues:} \; 0.000, 0.000, 0.000, 0.167, 0.230, 0.603.$$

Negative eigenvalues suggest, a **and** cd **are entangled**, and c **and** ad **are entangled.** We can not say anything about the subsystem d and ac.

Four Qubit System

Partial Trace with respect to system b :
$\mathcal{Q}$. How to conclude anything about the separability of the subsystem d and ac?

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Four Qubit System

Partial Trace with respect to system b :
$\mathcal{Q}$. How to conclude anything about the separability of the subsystem d and ac?
Computing eigenstates of ρ_{acd}:

$$\begin{bmatrix} 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.525 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.850 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.850 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.525 \\ 0 \end{bmatrix}$$

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Four Qubit System

Partial Trace with respect to system b :

$\mathcal{Q}$. How to conclude anything about the separability of the subsystem d and ac?

Computing eigenstates of ρ_{acd}:

$$\begin{bmatrix} 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.525 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.850 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.850 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.525 \\ 0 \end{bmatrix}$$

If we can show that these vectors are separable as $|v_{ac}\rangle \otimes |v_d\rangle$, then that will show that the ac and d are separable.

Four Qubit System

Partial Trace with respect to system b :

Consider,

$$\begin{bmatrix} 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |0\rangle_a |0\rangle_c |0\rangle_d .$$

So this vector is separable in this form $|v_{ac}\rangle \otimes |v_d\rangle$.

Similarly, we can say all the vectors of this form are separable.

Four Qubit System

Partial Trace with respect to system b :
Consider,

$$\begin{bmatrix} 0.525 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.850 \\ 0 \end{bmatrix} = 0.525 \left|0\right\rangle_a \otimes \left|0\right\rangle_c \otimes \left|0\right\rangle_d - 0.850 \left|1\right\rangle_a \otimes \left|1\right\rangle_c \otimes \left|0\right\rangle_d$$

This can be written as,

$$\left(0.525 \left|0\right\rangle_a \otimes \left|0\right\rangle_c - 0.850 \left|1\right\rangle_a \otimes \left|1\right\rangle_c\right) \otimes \left|0\right\rangle_d .$$
$$= \left|v_{ac}\right\rangle \otimes \left|v\right\rangle_d$$

So, ac and d are separable.

Four Qubit System

Partial Trace with respect to system c :

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

$$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$$

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$

- $\hat{\rho}_{abd}^{T_b}$:

  Eigenvalues: $-0.103, 0.000, 0.167, 0.270, 0.333, 0.333.$

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$

- $\hat{\rho}_{abd}^{T_b}$:

  Eigenvalues: $-0.103, 0.000, 0.167, 0.270, 0.333, 0.333.$

- $\hat{\rho}_{abd}^{T_d}$:

  Eigenvalues: $-0.069, 0.000, 0.167, 0.167, 0.333, 0.402.$

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$

- $\hat{\rho}_{abd}^{T_b}$:

  Eigenvalues: $-0.103, 0.000, 0.167, 0.270, 0.333, 0.333.$

- $\hat{\rho}_{abd}^{T_d}$:

  Eigenvalues: $-0.069, 0.000, 0.167, 0.167, 0.333, 0.402.$

Negative eigenvalues suggest, a **and** bd **are entangled**, b **and** ad **are entangled.** and d **and** ab **are entangled**.

Four Qubit System

Partial Trace with respect to system d :

Four Qubit System

Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

  Eigenvalues: $-0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$

Four Qubit System

Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

  Eigenvalues: $-0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$

- $\hat{\rho}_{abc}^{T_b}$:

  Eigenvalues: $-0.122, 0.000, 0.167, 0.167, 0.333, 0.455.$

Four Qubit System

Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

  $$\text{Eigenvalues:} \quad -0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$$

- $\hat{\rho}_{abc}^{T_b}$:

  $$\text{Eigenvalues:} \quad -0.122, 0.000, 0.167, 0.167, 0.333, 0.455.$$

- $\hat{\rho}_{abc}^{T_c}$:

  $$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.333, 0.333, 0.333.$$

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Four Qubit System

Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

  $$\text{Eigenvalues:} \quad -0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$$

- $\hat{\rho}_{abc}^{T_b}$:

  $$\text{Eigenvalues:} \quad -0.122, 0.000, 0.167, 0.167, 0.333, 0.455.$$

- $\hat{\rho}_{abc}^{T_c}$:

  $$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.333, 0.333, 0.333.$$

Negative eigenvalues suggest, *a* **and** *bc* **are entangled**, *b* **and** *ac* **are entangled.** and *c* **and** *ab* **are entangled**.

Four Qubit System

Partial Trace with respect to system ab **:**

$$\hat{\rho}_{cd} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.166 & 0 & 0 \\ 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, c **and** d **are separated.**

Four Qubit System

Partial Trace with respect to system bc **:**

$$\hat{\rho}_{ad} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0.166 \end{bmatrix}.$$

So, a **and** d **are separated.**

Four Qubit System

Partial Trace with respect to system cd :

$$\hat{\rho}_{ab} = \begin{bmatrix} 0.333 & 0 & 0 & 0 \\ 0 & 0.166 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

So, a **and** b **are separated.**

Four Qubit System

Partial Trace with respect to system ad **:**

$$\hat{\rho}_{bc} = \begin{bmatrix} 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0.333 \end{bmatrix}.$$

So, b **and** c **are separated.**

Four Qubit System

Partial Trace with respect to system bd **:**

$$\hat{\rho}_{ac} = \begin{bmatrix} 0.5 & 0 & 0 & 0.166 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.166 & 0 \\ 0.166 & 0 & 0 & 0.333 \end{bmatrix}.$$

Four Qubit System

Partial Transpose with respect to a:

$$\hat{\rho}_{ac}^{T_a} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.166 & 0 \\ 0 & 0.166 & 0.166 & 0 \\ 0 & 0 & 0 & 0.333 \end{bmatrix}.$$

Eigenvalues are : -0.103, 0.270, 0.333, 0.500.
So, a **and** c **are entangled**.

Four Qubit System

Partial Trace with respect to system ac **:**

$$\hat{\rho}_{bd} = \begin{bmatrix} 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.166 \end{bmatrix}.$$

So, b **and** d **are separated.**

Four Qubit System

Result:

- After tracing out a, rest of the system b, c, d becomes separable.
- After tracing out b, a and cd remain entangled, c and ad remain entangled, and ac and d are separable.
- After tracing out c, a and bd, b and ad, and d and ab remain entangled.
- After tracing out d, a and bc, b and ac, and c and ab remain entangled.
- After tracing out ab, c and d are separable.
- After tracing out bc, a and d are separable.
- After tracing out cd, a and d are separable.
- After tracing out ad, b and c are separable.
- After tracing out bd, a and c are entangled.
- After tracing out ac, b and d are separable.

So, **the polynomial is** $abc + abd + ac$**.**