Classification Of Entanglement Using Knots

A Topological Approach to Quantum States

PH3203 Term Project Report

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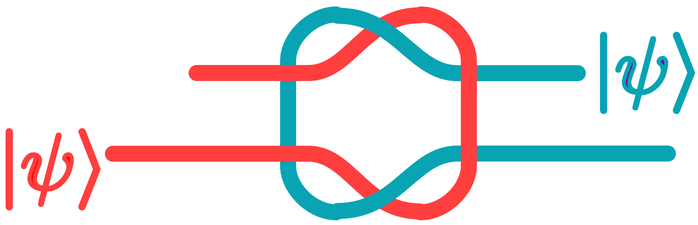

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1 Introduction

Classifying entanglement is essential because not all quantum states are equally useful for quantum information tasks. Different types of entanglement serve as distinct resources, each suited to specific applications such as quantum algorithms or secure communication protocols like quantum key distribution. Understanding and identifying these entanglement types helps determine how quantum states can be used and manipulated effectively. hi hi hi this is testing

SLOCC (Stochastic Local Operations and Classical Communication) is a method for classifying quantum entanglement. It defines equivalence classes of quantum states based on whether they can be converted into each other using local operations (on individual qubits) and classical communication.

This idea has been used successfully to study three-qubit states, as shown in [1,2], classifying four-qubit states [3–6] Methods have been developed for handling systems with even more qubits [7,8].[0.3cm] In this paper, the authors have proposed an alternative classification scheme for quantum entanglement based on topological links.

One of the first images that comes to mind when we think of entanglement is that of entangled threads. Naturally, one wonders if we could study quantum entanglement using entangled 'knots'. Aravind [1] was the first to point out the connections between entangled quantum states and classical knot configurations, focussing on similarity between 3-particle GHZ state and Borromean rings. He associated each particle with a ring, entanglement of any set of particles as inability to separate their corresponding rings, and measurement of particle state as cutting its ring. But he noted that performing the measurement in different basis would not lead to the same conclusions. This limit in analogy was dealt with by Sugita [2]. He proposed that cutting the ring is equivalent to tracing out the corresponding particle from the density operator, which is a basis-independent operation. This represents viewing the system as though that particle is no longer present. Moreover, the trace operation helps to generalise the idea to quantum systems with more than 2 levels.

2 How it all began . . .

In his 1997 paper, Aravind points out the connection between three-particle GHZ (Greenberger–Horne–Zeilinger) state and Borromean Rings. The latter is a set of three interlinked rings that can not be pulled apart. However, if one of the rings is cut, the other two are separated. The GHZ state of 3 spin-$\frac{1}{2}$ particles has the form:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 |0\rangle_2 |0\rangle_3 - |1\rangle_1 |1\rangle_2 |1\rangle_3)$$

where $|0\rangle$ and $|1\rangle$ are eigenvectors of the operator $\hat{\mathrm{S}}_{\mathrm{z}}$

$$\hat{S}_z |0\rangle = \frac{\hbar}{2} |0\rangle \qquad \hat{S}_z |1\rangle = -\frac{\hbar}{2} |1\rangle$$

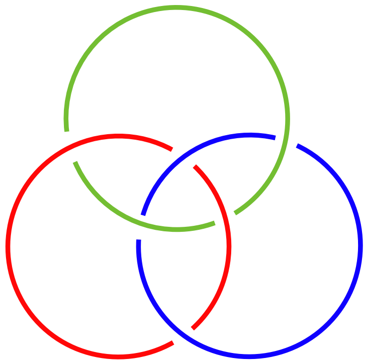

Figure 1: The Borromean Link

He made the following associations between the GHZ state and the Borromean rings:

- Each particle is associated with a ring
- Measuring the spin of a particle along the z-direction is equivalent to cutting the corresponding ring
- The entanglement of any set of particles along is modelled by the inability to separate the corresponding rings.

Aravind claims, that as long as no measurement is made on particle 1, particles 2 and 3 are in an entangled state, as their reduced density operator $\hat{\rho}_{23}$ can not be written as a product of density operators of particles 2 and 3. However, as pointed out in Sugita's paper, $\hat{\rho}_{23}$ is a separable state. We compute $\hat{\rho}_{23}$ and show that it is a mixed separable state. The density matrix for the entire system is given as:

$$\hat{\rho}_{123} = |\Psi\rangle\langle\Psi| = \frac{1}{2}(|000\rangle\langle 000| - |000\rangle\langle 111| - |111\rangle\langle 000| + |111\rangle\langle 111|)$$

From this, we can find the reduced density operator as $\hat{\rho}_{23} = \mathrm{Tr}_1(\hat{\rho}_{123}) = \sum_{\mathrm{i}} \langle \mathrm{i}| \hat{\rho}_{23} |\mathrm{i}\rangle$, where $\{|\mathrm{i}\rangle\}$ are the basis of particle 3. Thus, we obtain:

$$\hat{\rho}_{23} = \frac{1}{2}\left[(\mathrm{Tr}_1 |0\rangle\langle 0|) \cdot |00\rangle\langle 00| - \mathrm{Tr}_1(|0\rangle\langle 1|) \cdot |00\rangle\langle 11| - \mathrm{Tr}_1(|1\rangle\langle 0|) \cdot |11\rangle\langle 00| + \mathrm{Tr}_1(|1\rangle\langle 1|) \cdot |11\rangle\langle 11|\right]$$

We calculate partial trace as follows:

$$\mathrm{Tr}_1(|0\rangle\langle 0|) = \mathrm{Tr}\begin{bmatrix} \langle 0|0\rangle\langle 0|0\rangle & \langle 0|0\rangle\langle 0|1\rangle \\ \langle 1|0\rangle\langle 0|0\rangle & \langle 1|0\rangle\langle 0|1\rangle \end{bmatrix} = \mathrm{Tr}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1$$

$$\mathrm{Tr}_1(|0\rangle\langle 1|) = \mathrm{Tr}\begin{bmatrix} \langle 0|0\rangle\langle 1|0\rangle & \langle 0|0\rangle\langle 1|1\rangle \\ \langle 1|0\rangle\langle 1|0\rangle & \langle 1|0\rangle\langle 1|1\rangle \end{bmatrix} = \mathrm{Tr}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

We have used the fact that $|0\rangle$ and $|1\rangle$ are orthonormal state vectors in the Hilbert space. Similarly, we can show that $\mathrm{Tr}_1(|1\rangle\langle 0|)=0$ and $\mathrm{Tr}_1(|1\rangle\langle 1|)=1$

Therefore, their reduced density operator is $\hat{\rho}_{23} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, proving $\hat{\rho}_{23}$ is a mixed state as it cannot be written as an outer product of pure state. Note that $\hat{\rho}_{23}$ can be written as:

$$\hat{\rho}_{23} = \frac{1}{2}(|0\rangle\langle 0|) \otimes (|0\rangle\langle 0|) + \frac{1}{2}(|1\rangle\langle 1|) \otimes (|1\rangle\langle 1|)$$

Thus, our claim that $\hat{\rho}_{23}$ is a separable state is proved.

Now, we will see the effect of measurement on the GHZ state. If we measure the spin of the first particle in GHZ state along z-direction, the result is either $\frac{\hbar}{2}$ (corresponding to the first particle being spin-up) or $-\frac{\hbar}{2}$ (corresponding to first particle being spin-down). In both cases, the second and third particle collapses in a definite separable state. Thus, particles 2 and 3 become disentangled on measuring the first particle. Since GHZ state is symmetric in all the three particles, the above argument suffices for measurement along z-direction for all the three particles. Thus, the GHZ state can be 'modelled' by the Borromean ring.

Now, suppose that instead of measuring the spin along the z-direction, we measure them along the x-direction. It is then natural to associate the cutting of a knot with a spin measurement on the corresponding particle along the x-direction. However, with this 'altered' meaning of 'cutting a knot', we show that GHZ state is no longer modelled by the Borromean rings. For this, let us recall that the eigenvectors and eigenvalues of $\hat{\mathrm{S}}_{\mathrm{x}}$ are given by:

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) \\
&= \frac{1}{2\sqrt{2}}\left[(|000\rangle - |011\rangle + |100\rangle - |111\rangle) + (|000\rangle + |011\rangle - |100\rangle - |111\rangle)\right] \\
&= \frac{1}{2\sqrt{2}}\left[(|0\rangle + |1\rangle)(|00\rangle - |11\rangle) + (|0\rangle - |1\rangle)(|00\rangle + |11\rangle)\right] \\
&= \frac{1}{\sqrt{2}}\left[\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) + \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\right] \\
&= \frac{|+\rangle}{\sqrt{2}}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) + \frac{|-\rangle}{\sqrt{2}}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)
\end{aligned}$$

Now, if the spin of the first particle is measured along the x-axis, the outcome is either $\frac{\hbar}{2}$ (corresponding to the first particle being spin-up along x) or $-\frac{\hbar}{2}$ (corresponding to the first particle being spin-down along x). In both the cases, second and third particles become entangled, as the state $\frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ is not separable.

Therefore, the GHZ state can no longer be modelled by the Borromean rings, rather, it can be modelled by 3-Hopf rings, in which each pair of rings is linked and can not be separated even if the third ring is cut.

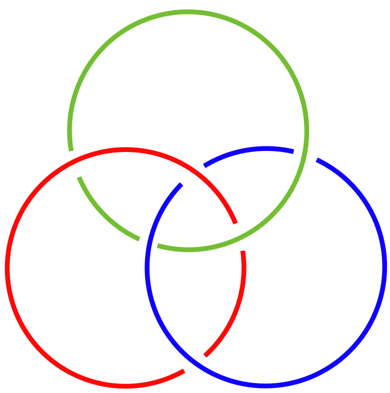

Figure 2: 3-Hopf Rings

The above discussion demonstrates that an entangled quantum state can correspond to more than one knot configuration if we define cutting of knot to be equivalent to measurement in a particular basis. As observables to be measured can be chosen in many different ways(for example spin can be measured along different directions in space), there seems to be no unique quantum process that corresponds to the mathematical act of cutting a knot.

Now we investigate another three particle state, known as the $|W\rangle$ state, given by:

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

We measure spin of the first particle in z-direction. The outcome is $\frac{\hbar}{2}$ with probaility $\frac{2}{3}$, and $-\frac{\hbar}{2}$ with probability $\frac{1}{3}$.

In the former case, after measurement, particles 2 and 3 are in state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This state is one of the Bell States and hence, maximally entangled. These can be modelled by 3-Hopf rings.

In the latter case, they are separable after the measurement and therefore modelled by Borromean rings.

Thus we see that the $|W\rangle$ state can be modeled by Borromean rings with probability $\frac{1}{3}$ and by 3-Hopf rings with probability $\frac{2}{3}$.

Aravind writes about one more linked 3-knots, where the middle ring in linked to the outer two rings, which are not connected to each other (we will call it the *3-chain*). He shows that there exists a state corresponding to this link configuration.

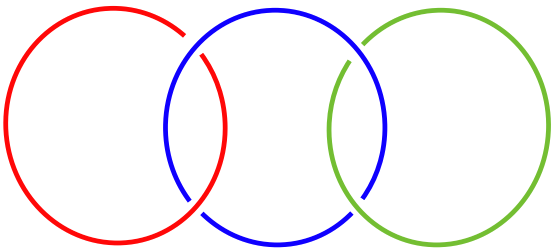

Figure 3: 3-chain

Notice that in previous sections, we started with a multi-particle quantum state, and checked how it behaves under various measurements and partial traces. Then, we tried to find a suitable link configuration but here we do just the reverse. Consider the state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0++\rangle + |1--\rangle)$$

We will show that this state can be modelled appropriately by the 3-chain.

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{2}}\left[\left(|0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \left(|1\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|00+\rangle + |01+\rangle) + \frac{1}{\sqrt{2}}(|10-\rangle - |11-\rangle)\right] \\
&= \frac{|0\rangle_2}{\sqrt{2}}\left(\frac{|0\rangle_1 |+\rangle_3 + |1\rangle_1 |-\rangle_3}{\sqrt{2}}\right) + \frac{|1\rangle_2}{\sqrt{2}}\left(\frac{|0\rangle_1 |+\rangle_3 - |1\rangle_1 |-\rangle_3}{\sqrt{2}}\right) \\
&= \frac{|0\rangle_3}{\sqrt{2}}\left(\frac{|0\rangle_1 |+\rangle_2 + |1\rangle_1 |-\rangle_2}{\sqrt{2}}\right) + \frac{|1\rangle_3}{\sqrt{2}}\left(\frac{|0\rangle_1 |+\rangle_2 - |1\rangle_1 |-\rangle_2}{\sqrt{2}}\right) \quad \text{[Changing } 2 \leftrightarrow 3 \text{ since symmetric]}
\end{aligned}$$

Another alternate way of writing the same state is:

$$|\Psi\rangle = \frac{|+\rangle_1}{\sqrt{2}}\left(\frac{|+\rangle_2|+\rangle_3 + |-\rangle_2|-\rangle_3}{\sqrt{2}}\right) + \frac{|-\rangle_1}{\sqrt{2}}\left(\frac{|+\rangle_2|+\rangle_3 - |-\rangle_1|-\rangle_3}{\sqrt{2}}\right)$$

Now, we measure spin of the three particles in z-direction:

- Measure spin of 1: After the measuement, particles 2 and 3 are in state $|+\rangle_2|+\rangle_3$ or $|-\rangle_2|-\rangle_3$. Thus, irrespective of the outcome, particles 2 and 3 are disentangled.
- Measure spin of 2: After the measurement, particles 1 and 3 are in state $|0\rangle_1|+\rangle_3 + |1\rangle_1|-\rangle_3$ or $|0\rangle_1|+\rangle_3 - |1\rangle_1|-\rangle_3$. Thus, irrespective of the outcome, particles 1 and 3 are maximally entangled (Using equation $(*_1)$).
- Measure spin of 3: After the measurement, particles 1 and 2 are in state $|0\rangle_1|+\rangle_2 + |1\rangle_1|-\rangle_2$ or $|0\rangle_1|+\rangle_2 - |1\rangle_1|-\rangle_2$. Thus, irrespective of the outcome, particles 1 and 2 are maximally entangled (Using equation $(*_2)$).

Aravind concludes his paper by noting the low possiblility to develop the analogy between entangled quantum states and knot configurations in any systematic fashion. A part of the difficulty arises because there is no single quantum process that corresponds to the mathematical act of cutting a knot. It appears very unlikely that the classification of knot configurations could have any systematic application or utility in the study of entangled quantum states.

However he listed some possible use cases for this classification scheme:

- If some complicated knot becomes equivalent to the unknot, then there is a possibility that the corresponding quantum state is separable in some basis.
- Two equivalent knots might imply that two entangled states (represented by those knots) are equivalent, only written in different bases.
- More entangled knots might represent a higher degree of entanglement, leading to some topological entanglement measure.

We now consider another paper by Sugita (2007) extrapolating Aravind's work. Sugita introduced a basis-independent correspondence between quantum states and links. In the following, we consider composite systems consisting of qubits, and associate a ring with a qubit. Two entangled qubits are represented by two entangled rings, and two separable qubits are represented by two unentangled rings.

If we associate the measurement with cutting of the corresponding ring (as was done by Aravind), then the correspondence depends on the choice of measurement basis. Instead, in this paper, partial trace was used instead of measurement as a counterpart of 'cutting of a link'. Physically speaking, it means that the corresponding qubit is ignored and we focus only on the remaining qubits.

We start with the 3-particle GHZ state and trace out qubit A from the original density matrix for the entire system, leading to the reduced density operator:

$$\hat{\rho}_{bc} = \mathrm{Tr}_a\,\hat{\rho} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

which is a separable mixed state, as shown before. Therefore this state corresponds to the Borromean ring. Thus we can establish a connection between qubits and rings in a basis-independent way, since partial trace requires us to find trace, which is basis-independent.

Now we will see how the correspondence between tracing out a particle (finding the reduced density operator) and cutting of the corresponding ring gives a natural basis-independent way to associate topological rings to quantum entangled states. We have the $|\mathrm{W}\rangle$ state as follows:

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

The density operator for the W state is then

$$\rho = |\mathrm{W}\rangle\langle\mathrm{W}|$$

Using the partial trace with respect to particle A, we find the reduced density operator for particles B and C:

$$\rho_{\mathrm{BC}} = {}_{\mathrm{A}}\langle 0|\,\rho\,|0\rangle_{\mathrm{A}} + {}_{\mathrm{A}}\langle 1|\,\rho\,|1\rangle_{\mathrm{A}}$$

From this calculation we can find the reduced density operator of particles 2 and 3 to be:

$$\rho_{BC} = \frac{1}{3}\left(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 10| + |01\rangle\langle 01| + |10\rangle\langle 01|\right)$$

Define

$$|\Psi\rangle = |01\rangle + |10\rangle \Rightarrow \rho_{BC} = \frac{1}{3}|00\rangle\langle 00| + \frac{1}{3}|\Psi\rangle\langle\Psi|$$

Note that $|\Psi\rangle$ is not normalized. Let $|\Psi\rangle_{\mathrm{N}}$ denote normalized $|\Psi\rangle$

$$|\Psi\rangle_N = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$\Rightarrow |\Psi\rangle_N\langle\Psi|_N = \frac{1}{2}|\Psi\rangle\langle\Psi| \Rightarrow |\Psi\rangle\langle\Psi| = 2\,|\Psi\rangle_N\langle\Psi|_N$$

Thus

$$\rho^{BC} = \frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}|\Psi\rangle_N\langle\Psi|_N$$

Here, $|00\rangle$ and $|\Psi\rangle_{\mathrm{N}}$ are normalized. Hence ρ^{BC} is a mixed state. Sugita uses the Concurrence Test[3] for separability. The concurrence of a density operator is defined as:

$$\mathrm{C}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

Here λ_{i} are the sqaure root of the eigenvalues, in decreasing order, of the non-Hermitian matrix $\tilde{\rho}\rho$ where $\tilde{\rho}$ is defined by:

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

According to the Concurrence test, a state represented by the density operator ρ is separable iff $\mathrm{C}(\rho) = 0$.

For the above reduced density matrix, we find:

$$\tilde{\rho} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix} \quad \text{and} \quad \tilde{\rho}\rho = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.148 & 0.148 & 0.0 \\ 0.0 & 0.148 & 0.148 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The eigenvalues of $\tilde{\rho}\rho$ are $0.444, 0.0, 0.0, 0.0$ and hence $\mathrm{C}(\rho) = 0.444$ which is positive and hence the state is entangled.

Now we consider the 3-chain state:

$$|\Psi\rangle = a\,|000\rangle + b\,|{+}1{+}\rangle\,, \text{ with } \ a, b \in \mathbb{R},\ |a|^2 + |b|^2 = 1$$

Then the density matrix is obtained as:

$$\rho^{ABC} = |\Psi\rangle\langle\Psi| = |a|^2(|000\rangle\langle000|) + |b|^2(|{+}1{+}\rangle\langle{+}1{+}|)$$

Now we trace out B and find ρ^{AC}:

$$\rho^{AC} = \mathrm{tr}_B(\rho^{ABC}) = |a|^2\mathrm{tr}_B(|0\rangle\langle0|).(|00\rangle\langle00|) + |b|^2\mathrm{tr}_B(|1\rangle\langle1|).(|{+}{+}\rangle\langle{+}{+}|)$$

Recall that $\mathrm{tr_B}(|0\rangle\langle0|) = \mathrm{tr_B}(|1\rangle\langle1|) = 1$. Therefore ρ^{AC} represents a mixed state. Note that it can be written as:

$$\rho^{AC} = |a|^2|00\rangle\langle00| + |b|^2|{+}{+}\rangle\langle{+}{+}| = |a|^2|0\rangle\langle0| \otimes |0\rangle\langle0| + |b|^2|{+}\rangle\langle{+}| \otimes |{+}\rangle\langle{+}|$$

Thus, we see that ρ^{AC} is also separable. Similarly, we have ρ^{BC} and ρ^{AB}:

$$\rho^{AB} = |a|^2|00\rangle\langle00| + |b|^2|{+}1\rangle\langle{+}1|$$
$$\rho^{BC} = |a|^2|00\rangle\langle00| + |b|^2|1{+}\rangle\langle1{+}|$$

Let us now calculate the concurrence of ρ^{AB} and ρ^{BC}. From the definition, we calculated:

$$\rho_{ab} = \begin{bmatrix} \mathrm{a}^2 & \frac{1}{2}\mathrm{ab} & 0 & \frac{1}{2}\mathrm{ab} \\ \frac{1}{2}\mathrm{ab} & \frac{1}{2}\mathrm{b}^2 & 0 & \frac{1}{2}\mathrm{b}^2 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}\mathrm{ab} & \frac{1}{2}\mathrm{b}^2 & 0 & \frac{1}{2}\mathrm{b}^2 \end{bmatrix} \qquad \tilde{\rho}_{ab}\rho_{ab} = \frac{ab}{4}\begin{bmatrix} 3\mathrm{ab} & 3\mathrm{b}^2 & 0 & 3\mathrm{b}^2 \\ 0 & 0 & 0 & 0 \\ -3\mathrm{ab} & -3\mathrm{b}^2 & 0 & -3\mathrm{b}^2 \\ 4\mathrm{a}^2 & 3\mathrm{ab} & 0 & 3\mathrm{ab} \end{bmatrix}$$

$$\rho_{bc} = \begin{bmatrix} \mathrm{a}^2 & 0 & \frac{1}{2}\mathrm{ab} & \frac{1}{2}\mathrm{ab} \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}\mathrm{ab} & 0 & \frac{1}{2}\mathrm{b}^2 & \frac{1}{2}\mathrm{b}^2 \\ \frac{1}{2}\mathrm{ab} & 0 & \frac{1}{2}\mathrm{b}^2 & \frac{1}{2}\mathrm{b}^2 \end{bmatrix} \qquad \tilde{\rho}_{bc}\rho_{bc} = \frac{ab}{4}\begin{bmatrix} 3\mathrm{ab} & 0 & 2\mathrm{b}^2 & 2\mathrm{b}^2 \\ 3\mathrm{ab} & 0 & -2\mathrm{b}^2 & -2\mathrm{b}^2 \\ 0 & 0 & 0 & 0 \\ 4\mathrm{a}^2 & 0 & 3\mathrm{ab} & 3\mathrm{ab} \end{bmatrix}$$

These matrices have the same eigenvalues (since the system is symmetric with respect to A and C). The eigenvalues are: $|\mathrm{ab}|\frac{(1+\sqrt{2})}{2}, |\mathrm{ab}|\frac{(\sqrt{2}-1)}{2}, 0, 0$. Thus the concurrence is:

$$\mathrm{C}(\rho) = |ab| > 0$$

Calculating the concurrence reveals that ρ_{ab} and ρ_{bc} are not separable. Thus, BC and AB are still entangled. This is clearly mirrored in the 3-chain link configuration, in which A and C are the outer rings and B is the middle one. So, if B is cut (or in the language of Quantum mechanics, traced out) A and C are separable, they are not linked. However if either A or C are cut, then, other 2 are still linked.

3 Classification of Links: A Polynomial Approach

3.1 Formalism of the Link Polynomial

3.2 Obtaining a Link from an Entangled Quantum State

3.3 Obtaining an Entangled Quantum State from a Link

In this section, we will see how we can obtain a link from a quantum state. This is in general a difficult task to obtain an entangled quantum states from the polynomial as the number of qubits increases. The process in the paper mentions an algorithm which provides an 'incomplete' map betwee a given link and a quantum state. Using the procedure, the general structure of the quantum state can be obtained, however, some free coefficients remain which needs to be fixed computationally. Moreover, presently only mixed states satisfying the link can be obtained using this procedure.

Note that although incomplete, the map is still useful since we can ascertain the structure of each state contained in the mixed state. That is, from a possibility of 2^{N} (for N qubits, there are 2^{N} basis states, namely $|0\rangle, |1\rangle, \dots |2^{\mathrm{N}}-1\rangle$ where each ket is to be assumed in the binary representation) states, we are reducing it to a much smaller number.

For this, we will use the GHZ type of state as a building block which are of the form:

$$|\mathrm{N}^1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes \mathrm{N}} + |1\rangle^{\otimes \mathrm{N}}\right)$$

Here $|0\rangle^{\otimes \mathrm{N}}$ is the tensor product of N number of $|0\rangle$ states, that is, $|0\rangle^{\otimes \mathrm{N}} = |\underbrace{0000\dots0}_{\mathrm{N\ times}}\rangle$. The state $|\mathrm{N}^1\rangle$ is a maximally entangled state of N qubits. The general algorithm to obtain a state from the link is as follows:

1. Let a polynomial P be given. Select a term of the given 'link' polynomial, say t.
2. The term t is then mapped to a state of the form $|\mathrm{E_q}\rangle \otimes |\mathrm{S_q}\rangle \otimes |\mathrm{Q_d}\rangle$ where:
   - $|\mathrm{E_q}\rangle$ is the entangled qubit of the GHZ type as specified above, associated to ring variables contained in t.
   - $|\mathrm{S_q}\rangle$ is a separable qubit associated with ring variables not contained in t. There are a number of possibilities for this separable qubit and we have to find it computationally.
   - $|\mathrm{Q_d}\rangle$ is a qudit state which is associated with an artifically introduced ring variable (which is alphabetically the next letter of the largest ring variable present). The states always starts from 0 for the first term and is increased by 1 for each successive term of the polynomial. This will later be traced out, hence is of less significance.
3. The full state $|\psi\rangle$ is constructed by summing these individual states obtained for each term of the polynomial.
4. The full mixed state characterised by this polynomial is then obtained by tracing out the qudit state $|\mathrm{Q_d}\rangle$.

$$\boxed{\hat{\rho}(\mathrm{P}) = \frac{\mathrm{Tr_d}\,|\psi\rangle\langle\psi|}{\mathrm{Tr}\{\mathrm{Tr_d}\,|\psi\rangle\langle\psi|\}}}$$

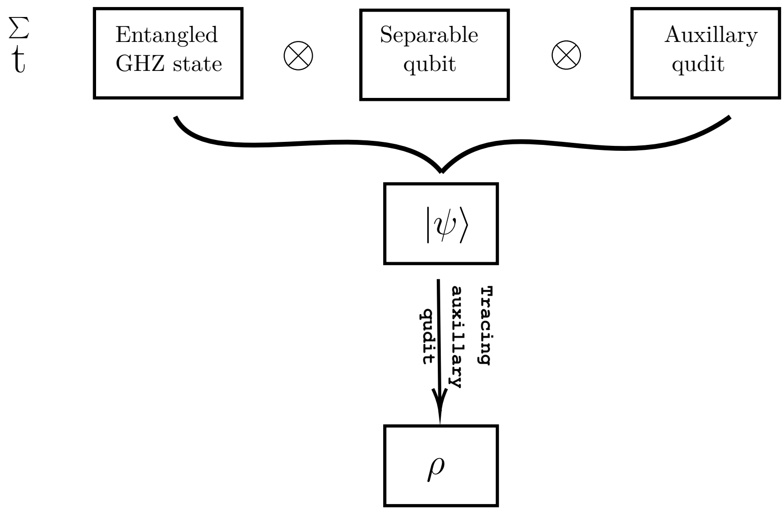

Figure 4: A schematic diagram of the algorithm

Example demonstrating the algorithm:

We will see a simple example of the algorithm to obtain a state from a link. Consider the polynomial $\mathrm{P(a, b, c) = ab + ac}$. This is a three-ring link.

- Let us choose the term $\mathrm{t = ab}$. This term has two ring variables thus we will associate a two qubit GHZ type of state to $|\mathrm{E_q}\rangle$. Thus, we have $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \equiv |2^1\rangle_{\mathrm{ab}}$.

  Since the separable qubit has large possibility, we will denote it generally by $|\mathrm{q_1}\rangle$ and this will be associated with the remaining ring variable which is c. Thus, $|\mathrm{S_q}\rangle = |\mathrm{q_1}\rangle_{\mathrm{c}}$. The remaining term is the qudit state which will be associated to d (since d is alphabetical successor of the largest ring variable c). Then we will have the full state:

$$|\psi_1\rangle = |2^1\rangle_{\mathrm{ab}} \otimes |\mathrm{q_1}\rangle_{\mathrm{c}} \otimes |0\rangle_{\mathrm{d}}$$

- Now, let us choose the next term in the polynomial which is $\mathrm{t = ac}$. Similar to above, to the entangled qubit we will associate the two qubit GHZ state, thus, $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \equiv |2^1\rangle_{\mathrm{ac}}$.

  The separable qubit will be associated with the remaining ring variable b and we will denote it by $|\mathrm{q_2}\rangle_{\mathrm{b}}$. The qudit state will be associated with d which is the alphabetical successor of c but this time we will use $|1\rangle_{\mathrm{d}}$ as for each successive term, the qudit state increases to the next level. Thus, we have:

$$|\psi_2\rangle = |2^1\rangle_{\mathrm{ac}} \otimes |\mathrm{q_2}\rangle_{\mathrm{b}} \otimes |1\rangle_{\mathrm{d}}$$

- The full state $|\psi\rangle$ is then obtained by summing the two states obtained above with some coefficients:

$$\begin{aligned} |\psi\rangle &= \mathrm{c_1}\,|\psi_1\rangle + \mathrm{c_2}\,|\psi_2\rangle \\ &= \mathrm{c_1}(|2^1\rangle_{\mathrm{ab}} \otimes |\mathrm{q_1}\rangle_{\mathrm{c}} \otimes |0\rangle_{\mathrm{d}}) + \mathrm{c_2}(|2^1\rangle_{\mathrm{ac}} \otimes |\mathrm{q_2}\rangle_{\mathrm{b}} \otimes |1\rangle_{\mathrm{d}}) \end{aligned}$$

  Then we can trace out the qudit state $|\mathrm{d}\rangle$ to obtain the density matrix of the ring variables:

$$\boxed{\hat{\rho}_{\mathrm{abc}} = \frac{\mathrm{Tr_d}\,|\psi\rangle\langle\psi|}{\mathrm{Tr}\{\mathrm{Tr_d}\,|\psi\rangle\langle\psi|\}}}$$

3.4 Applying to Three Qubit Systems

As a demonstration, we will apply our algorithm to three qubit systems. Note that from the rules of the 'link' polynomial, the possible basis terms for three qubit system are: $\{\mathrm{ab}, \mathrm{ac}, \mathrm{bc}, \mathrm{abc}\}$. Using this, four distinct classes of polynomials are possible:

$$\begin{aligned}
\mathrm{P}_1(\mathrm{a},\mathrm{b},\mathrm{c}) &= \mathrm{abc}\\
\mathrm{P}_2(\mathrm{a},\mathrm{b},\mathrm{c}) &= \mathrm{abc}+\mathrm{ab}\\
\mathrm{P}_3(\mathrm{a},\mathrm{b},\mathrm{c}) &= \mathrm{ab}+\mathrm{ac}\\
\mathrm{P}_4(\mathrm{a},\mathrm{b},\mathrm{c}) &= \mathrm{ab}+\mathrm{ac}+\mathrm{bc}
\end{aligned}$$

3^1 Link Class

Let us start with the 3^1 link class, which corresponds to the Borromean Link and is given by $\mathrm{P}_1(\mathrm{a},\mathrm{b},\mathrm{c}) = \mathrm{abc}$. Cutting any of a, b or c will lead to complete separabability and loss of entanglement.

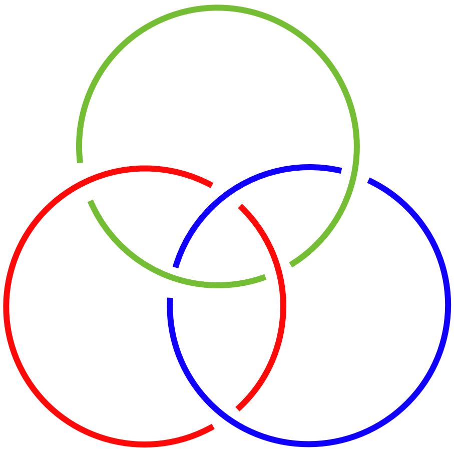

Figure 5: The Borromean link, characterising the 3^1 link class.

We already know that on its own the GHZ state characterises the 3^1 link, as discussed in the preceding works. Thus, we have the pure state:

$$|3^1\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$$

The density matrix corresponding to the state is found to be:

$$\rho_{abc} = \begin{bmatrix}
0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5\\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\
0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5
\end{bmatrix}$$

The partial transpose with respect to any of the subsystem (since the polynomial is symmetric) is same and is given by:

$$\rho_{abc}^{T_a/T_b/T_c} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The eigenvalues corresponding to this matrix are $0.0, 0.5$ and $\mathbf{-0.5}$. Since there are negative eigenvalues, we can conclude that the system exhibits **tripartite entanglement** as a whole.

Now, let us reduce the system by tracing out one of the variable. Since the polynomial is symmetric, we can choose any of the variable to trace out, say c. The reduced density matrix is given by:

$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The partial transpose with respect to a or b results in the same above matrix which have eigenvalues 0.5 and 0.0 which are all positive, thus concluding the absence of any entanglement in the system. This is consistent with the fact that for the Borromean link, cutting any link will result in complete separability of the links.

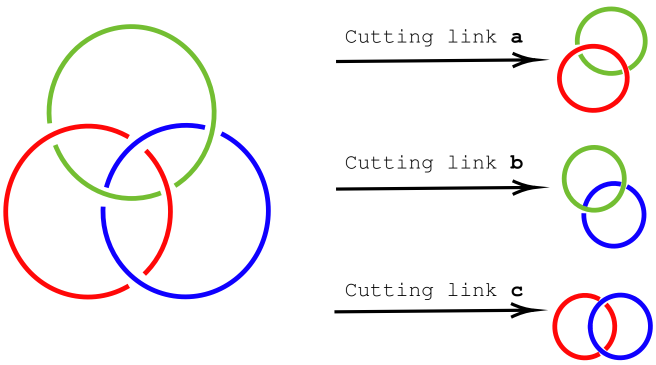

Figure 6: Possible link cuts for the Borromean link. All the cuts lead to the same configuration.

3^2 Link Class

Let us now consider the 3^2 link class given by $\mathrm{P}_2(\mathrm{a}, \mathrm{b}, \mathrm{c}) = \mathrm{abc} + \mathrm{ab}$. Cutting any of a, b will lead to complete separabality but if we cut c, then the other rings will remain entangled. The link can be represented as:

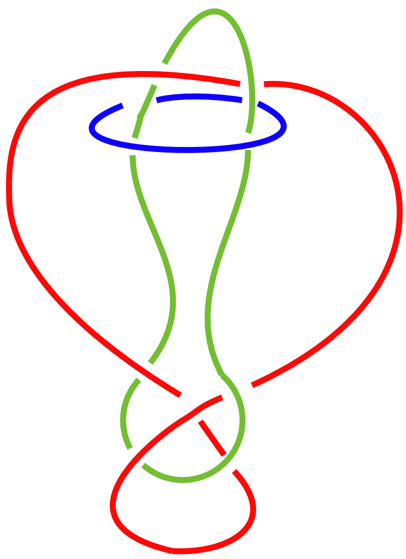

Figure 7: The knot diagram, characterising the 3^2 link class.

Here the blue knot corresponds to c while the other two correspond to a and b (a,b are symmetric in the polynomial). An example of a pure state is found, as mentioned in the paper:

$$|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} \left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc} \right)$$

To check that this state indeed statistfies the link, let us calculate the density operator $\hat{\rho}_{\text{abc}} = |3^2\rangle_{\text{abc}} \langle 3^2|_{\text{abc}}$ and then check for the PPT test for each cuts. The density matrix obtained is:

$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Since the variables a and b are symmetrix, we can choose to analyse only one of them and c.

We then see the partial transpose with respect to a and c. The matrices are given by:

$$\rho_{\text{abc}}^{\text{T}_\text{a}} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{\text{abc}}^{\text{T}_\text{c}} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

The above matrices have eigenvalues $\lambda_\text{a} = \mathbf{-0.471}, 0.0, 0.333, 0.471, 0.666$ and $\lambda_\text{c} = -0.333, 0.0, 0.127, 0.333, 0.872$ respectively. Since there are negative eigenvalues, we can conclude that the system exhibits **tripartite entanglement** as a whole.

The reduced density matrices are given by:

$$\rho_{\text{bc}} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{\text{ab}} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

We now obtain the partial tranpose for the PPT test. We note that $\rho_{\text{bc}}^{\text{T}_\text{b}/\text{T}_\text{c}}$ is the same as that of the above matrix ρ_{bc} whose eigenvalues are $0.0, 0.333, 0.666$ which are all positive. Thus, we can conclude that the system is separable. On the other hand, we obtain:

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

The eigenvalues of this matrix are $0.333, -0.333, 0.666$, one of which is negative, thus denoting the presence of entanglement. This successfully verifies the behaviour of the link abc + ab.

Using the above algorithm, we can also construct the mixed state corresponding to the link by considering the state:

$$|\psi_2\rangle = |3^1\rangle_{abc} |0\rangle_d + |2^1\rangle_{ab} |0\rangle_c |1\rangle_d$$

It is to be noted that this class has not been described in the previous works [1, 2]. The density operator is then given by:

$$\hat{\rho}(a, b, c) = \frac{\text{Tr}_d |\psi_2\rangle \langle\psi_2|}{\sqrt{\langle\psi_2|\psi_2\rangle}}$$

Using numerical calculation and tracing out subsystem d, we found the density matrix to be:

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

Let us now calculate the partial transposes for the PPT test:

$$\rho_{\mathrm{abc}}^{\mathrm{T_a}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{abc}}^{\mathrm{T_b}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{abc}}^{\mathrm{T_c}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

The set of eigenvalues for the above matrices are:

- $\lambda_{\mathrm{a}} = \mathbf{-0.354}, 0.0, 0.0, 0.0, 0.25, 0.25, 0.354, 0.5$
- $\lambda_{\mathrm{b}} = \mathbf{-0.354}, 0.0, 0.0, 0.0, 0.25, 0.25, 0.354, 0.5$
- $\lambda_{\mathrm{c}} = \mathbf{-0.183}, 0.0, 0.0, 0.0, 0.0, 0.25, 0.25, 0.683$

Each of the above matrices have atleast one negative eigenvalues, thus denoting the presence of entanglement in the system as a whole. Let us now analyse the reduced density matrices and their PPT matrices which corresponds to the act of 'cutting a link':

$$\rho_{\mathrm{bc}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{\mathrm{bc}}^{\mathrm{T_b}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{ac}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{\mathrm{ac}}^{\mathrm{T_a}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{ab}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.5 \end{bmatrix} \quad \rho_{\mathrm{ab}}^{\mathrm{T_a}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The set of eigenvalues for the above matrices are:

- $0.0, 0.25, 0.25, 0.5 \longrightarrow$ no eigenvalue is negative, thus cutting a separates the systems b and c.
- $0.0, 0.25, 0.25, 0.5 \longrightarrow$ no eigenvalue is negative, thus cutting b separates the systems a and c.
- $\mathbf{-0.25}$, $0.25, 0.5, 0.5 \longrightarrow$ one eigenvalue is negative, thus on cutting c, the systems a and b remain entangled.

This faithfully reproduces the behaviour of the link abc + ab and thus, the state characterises the link.

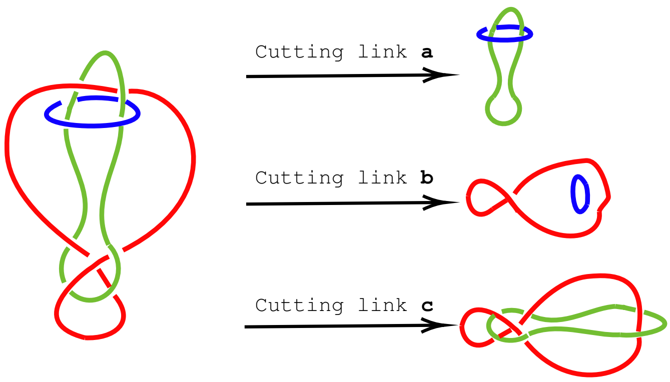

Figure 8: Possible link cuts for the 3^2 link class. Two cuts lead to complete separability while other cut lead to different entangled configuration.

3^3 Link Class

We now analyse the link class given by $P_3(a, b, c) = ab + ac$. The link diagram is given by:

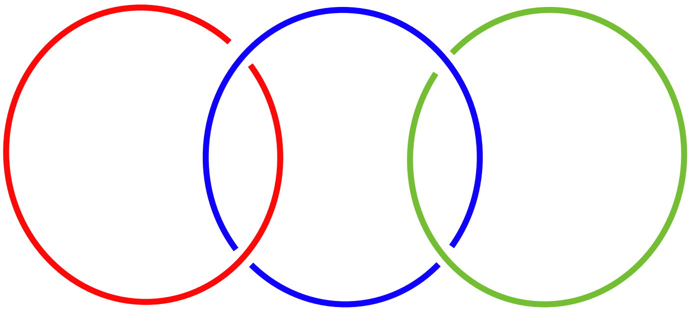

Figure 9: The knot diagram, characterising the 3^3 link class.

We can see that the knot polynomial is symmetric in b and c. Thus, if we cut either b or c, the other will remain entangled with a but complete separation results from cutting a. This case has already been discussed in section 3.2 and a pure state has been obtained. The mixed state can be obtained using the state:

$$|\psi_3\rangle = |2^1\rangle_{ab} |0\rangle_c |0\rangle_d + |2^1\rangle_{ac} |1\rangle_b |1\rangle_d$$

The density matrix obtained for the system is:

$$\rho_{abc} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

Let us calculate the partial transposes for the PPT test:

$$\rho_{\mathrm{abc}}^{\mathrm{T_a}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{abc}}^{\mathrm{T_b}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.25 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{abc}}^{\mathrm{T_c}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

The set of eigenvalues obtained for the above matrices are:

- $\lambda_{\mathrm{a}} = \mathbf{-0.155}, \mathbf{-0.155}, 0.0, 0.0, 0.25, 0.25, 0.405, 0.405$
- $\lambda_{\mathrm{b}} = \mathbf{-0.2}, 0.0, 0.0, 0.0, 0.139, 0.25, 0.25, 0.562$
- $\lambda_{\mathrm{c}} = \mathbf{-0.2}, 0.0, 0.0, 0.0, 0.139, 0.25, 0.25, 0.562$

Since atleast one of the eigenvalues is negative for each matrix, we can conclude that the system exhibits **tripartite entanglement** as a whole. Let us now consider the reduced systems. We obtain the density matrices and their PPT matrices for the reduced system as:

$$\rho_{\mathrm{bc}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{\mathrm{bc}}^{\mathrm{T_b}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{ac}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{\mathrm{ac}}^{\mathrm{T_a}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{ab}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.5 \end{bmatrix} \quad \rho_{\mathrm{ab}}^{\mathrm{T_a}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.25 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The set of eigenvalues obtained for the above matrices are:

- $0.0, 0.25, 0.25, 0.5 \longrightarrow$ no negative eigenvalues, so cutting a makes the system separable.
- $\mathbf{-0.155}, 0.25, 0.405, 0.5 \longrightarrow$ negative eigenvalue, so on cutting b, system a and c remain entangled.
- $\mathbf{-0.155}, 0.25, 0.405, 0.5 \longrightarrow$ negative eigenvalue, so on cutting c system a and b remain entangled.

This is consistent with the link properties and hence the mixed state obtained is indeed characterises the link.

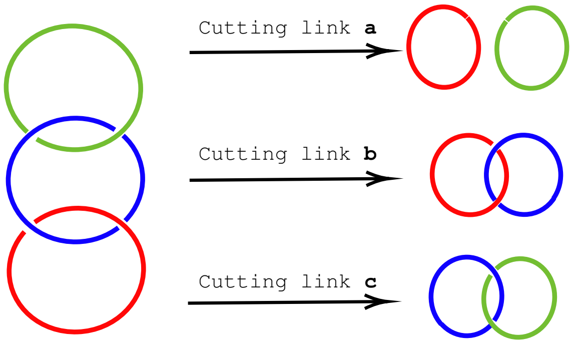

Figure 10: Possible link cuts for the 3^3 link class. One cut lead to complete separability while two of the cuts lead to entangled configurations.

3^4 Link Class

Let us now analyse the class $\mathrm{P}_4(\mathrm{a}, \mathrm{b}, \mathrm{c}) = \mathrm{ab} + \mathrm{ac} + \mathrm{bc}$. The link diagram is given by:

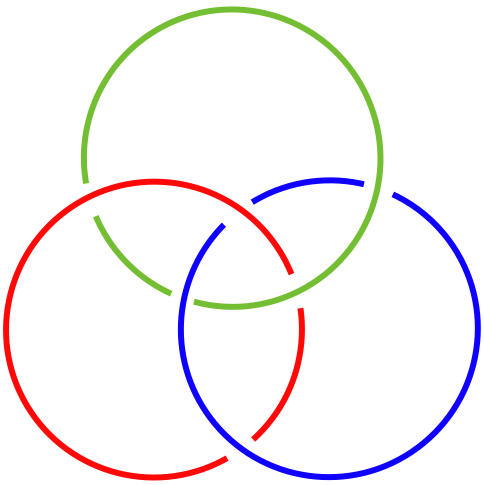

Figure 11: The knot diagram, characterising the 3^4 link class.

This link demostrates a behaviour opposite to that of the Borromean ring, that is, setting any one variable to zero with not result in complete separability. The other two links remain entangled. The pure state having the characteristic of the link is the **W state** which has been documented in previous works:

$$|3^4\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle_{abc} + |010\rangle_{abc} + |100\rangle_{abc}\right)$$

Let us check whether this state satisfies the link property. The density matrix is given by:

$$\rho abc = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The corresponding partial transpose with respect to the three variables are given by:

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\rho_{abc}^{T_b} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

All the three matrices have the same set of eigenvalues $\mathbf{-0.471}, 0.0, 0.333, 0.471, 0.666$, one of which is negative, thus denoting the presence of entanglement. The reduced density matrix in each case is the same and is given by:

$$\rho_{reduced} = \begin{bmatrix} 0.333 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The partial transpose of this is the same as that of the matrix. The eigenvalues of the PPT matrix are $-0.206, 0.333, 0.333, 0.539$, one of which is negative, thus denoting the present of entanglement. This verifies the behaviour of the link ab + ac + bc.

From the algorithm, the mixed state can be obtained using the state:

$$|\psi_4\rangle = |2^1\rangle_{ab}|0\rangle_c|0\rangle_d + |2^1\rangle_{ac}|1\rangle_b|1\rangle_d + |2^1\rangle_{bc}|0\rangle_a|2\rangle_d$$

Using the above state after tracing out subsystem d, we obtained the density matrix as:

$$\rho_{abc} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \\ 0.125 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

Let us now check the partial transpose for the entire system. The PPT matrices are found to be:

$$\rho_{\mathrm{abc}}^{\mathrm{T_a}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{abc}}^{\mathrm{T_b}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.125 & 0.25 & 0.0 & 0.125 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{abc}}^{\mathrm{T_c}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.125 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

The set of eigenvalues of $\rho_{\mathrm{abc}}^{\mathrm{T_a}}$ and $\rho_{\mathrm{abc}}^{\mathrm{T_c}}$ are the same, namely $\mathbf{-0.146}, \mathbf{-0.052}, 0.0, 0.0, 0.222, 0.25, 0.302, 0.424$ while the eigenvalues of $\rho_{\mathrm{abc}}^{\mathrm{T_b}}$ are $\mathbf{-0.138}, 0.0, 0.0, 0.107, 0.125, 0.125, 0.25, 0.531$. We see that there exists atleast one negative eigenvalue in all the three matrices, thus denoting the presence of tripartite entanglement. Let us now check the reduced density matrices.

$$\rho_{\rm bc} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.375 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.375 \end{bmatrix} \quad \rho_{\rm bc}^{\rm T_b} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.125 & 0.375 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.375 \end{bmatrix}$$

$$\rho_{\rm ac} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.125 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{\rm ac}^{\rm Ta} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.125 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\rm ab} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 \\ 0.0 & 0.375 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.375 \end{bmatrix} \quad \rho_{\rm ab}^{\rm T_a} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.375 & 0.125 & 0.0 \\ 0.0 & 0.125 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.375 \end{bmatrix}$$

The set of eigenvalues of PPT matrices of $\rho_{\rm bc}$ and $\rho_{\rm ab}$ are the same given by $\mathbf{-0.038}, 0.25, 0.375, 0.413$ while the eigenvalues of PPT matrix of $\rho_{\rm ac}$ are $\mathbf{-0.125}, 0.25, 0.375, 0.5$. Since in each set, a negative eigenvalue exists, we can conclude that the system exhibits entanglement, thus verifying the behaviour of the link $\rm ab + ac + bc$, that is, even after cutting one variable, the other two remain entangled.

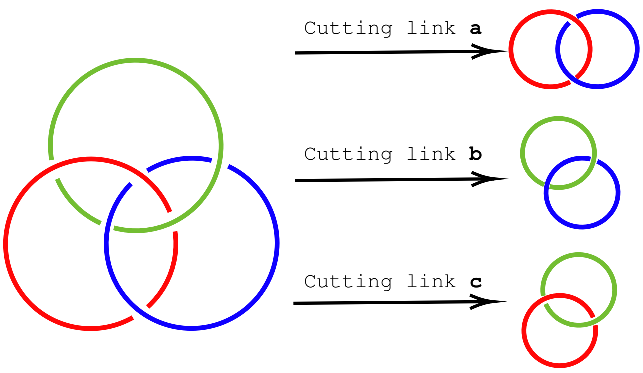

Figure 12: Possible link cuts for the 3^4 link class. All the cuts lead to entangled configuration.

3.5 Applying to Four Qubit Systems

4 Physical Significance: Use in Qubit Networks

The knot theoretic approach to entangled quantum states can be of great use in qubit networks, where different parties posses different qubits, entangled together and they want to perform some protocols (Here protocol means some series of operations/measurements to obtain some result, like the **BB84 protocol** for secure key distribution).

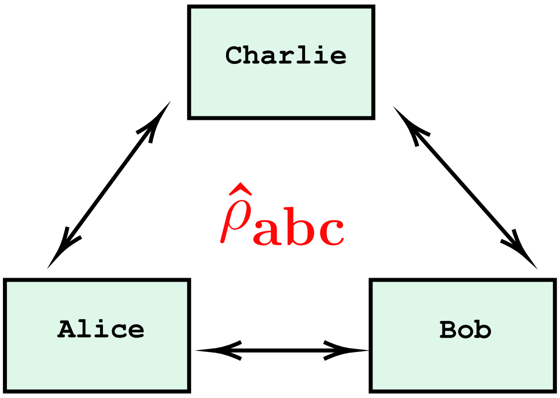

Figure 13: A qubit network, showing three parties, Alice, Bob and Charlie sharing an entangled state, represented by density operator $\hat{\rho}_{\mathrm{abc}}$.

For successfully performing a protocol between two parties, entanglement must exist between the qubits possessed by the two parties. The above diagram shows three parties Alice, Bob and Charlie, each possessing a qubit and their combined state is described by the density operator $\hat{\rho}_{\mathrm{abc}}$. Then, suppose Alice and Bob want to perform a protocol while Charlie does not want to participate. In this case, a general operator for the system will be of the form:

$$\hat{\mathrm{O}} = \sum_i (\hat{\mathrm{M}}_{\mathrm{i,ab}} \otimes \hat{\mathbb{1}}_{\mathrm{c}})\hat{\rho}_{abc}(\hat{\mathrm{M}}^{\dagger}_{\mathrm{i,ab}} \otimes \hat{\mathbb{1}}_{\mathrm{c}})$$

where $\hat{\mathrm{M}}_{\mathrm{i,ab}}$ is some operator acting on the combined Hilbert space of Alice and Bob and the index i runs over all the possible outcomes. Note that the operator does not interfere with Charlie's qubit. The probability of the i^{th} outcome is given by:

$$p_i = \mathrm{Tr}_{abc}[(\hat{\mathrm{M}}^{\dagger}_{\mathrm{i,ab}} \otimes \hat{\mathbb{1}}_{\mathrm{c}})(\hat{\mathrm{M}}_{\mathrm{i,ab}} \otimes \hat{\mathbb{1}}_{\mathrm{c}})\hat{\rho}_{abc}] \xrightarrow[\text{system } c]{\text{Tracing out}} \mathrm{Tr}_{ab}[\hat{\mathrm{M}}^{\dagger}_{\mathrm{i,ab}}\hat{\mathrm{M}}_{\mathrm{i,ab}}\hat{\rho}_{ab}]$$

In this case, knowledge of Charlie's system is not neccessary for successfully applying the protocol by Alice and Bob. In the case Charlie decides to act on his qubit by some operator $\hat{\mathrm{N}}_{\mathrm{i,c}}$, the total operator for the system will now be modified by:

$$\hat{\mathrm{M}}_{\mathrm{i,ab}} \otimes \hat{\mathbb{1}}_{\mathrm{c}} \quad \longrightarrow \hat{\mathrm{M}}_{\mathrm{i,ab}} \otimes \hat{\mathrm{N}}_{\mathrm{i,c}}$$

Now, for successfully performing the protocol, Alice and Bob must have a kmowledge of Charlie's operation since it could potentially affect the outcome of the protocol as the states are entangled. Thus, we have the following observations from the above discussion:

- If the operator $\hat{\rho}_{\mathrm{ab}}$ is separable, then the protocol cannot be performed successfully between Alice and Bob, as the two parties are not entangled.
- If Charlie decides not to divulge any information about his system, the protocol cannot be applied as no correlation can be determined between the two parties Alice and Bob.

In this case, the above discussion on the knot theoretic approach to entangled quantum states can be of great use. As a demonstration, we consider building a three-qubit network such that:

- Bob and Charlie may never successfully perform any protocols without external help from Alice.
- Alice may have a chance to communicate with either Bob or Charlie if she wishes to.

The network is shown in the figure below:

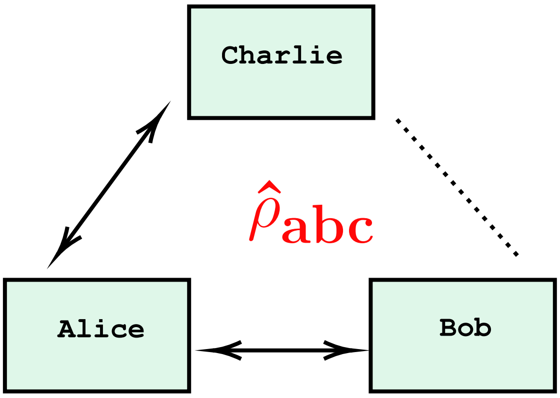

Figure 14: A qubit network, showing three parties, Alice, Bob and Charlie sharing an entangled state, represented by density operator $\hat{\rho}_{\mathrm{abc}}$. The dotted line represent no connection between the two parties.

The network can be easily described using the link polynomial language as: $\mathrm{P(a,b,c) = ac + ab}$ and thus we can immediately find a state that satisfies the above condition. The situation becomes more helpful when we consider higher number of qubits. Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie
- Alice, Bob, and Diana
- Alice and Charlie

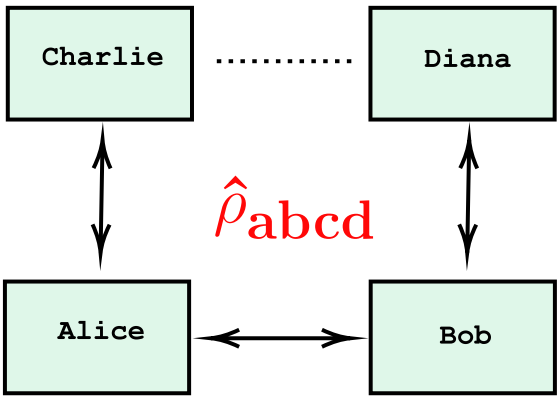

Figure 15: The four qubit network, showing four parties, Alice, Bob, Charlie and Diana sharing an entangled state, represented by density operator $\hat{\rho}_{\mathrm{abcd}}$. The dotted line represent no connection between the two parties.

We can easily see that the polynomial describing this network is $\mathrm{P(a,b,c,d) = abc + abd + ac}$. Thus a state can immediately be obtained using the algorithm described previously.

5 Discussion and Conclusion

6 Acknowledgements

Appendix A: Quantum Information Basics

6.1 Density Matrix

6.2 Peres-Horodecki Criterion

Appendix B: Knot Theory Basics

Appendix C: Code for Numeric Calculations

We used the `QuantumInformation.jl` package in `Julia` to perform the numerical calculations. The code provided below shows some basic calculations that we had used in this report.

```
using QuantumInformation, LinearAlgebra, Latexify
#example of a 3 qubit system: 3^2 link class mixed class

function ghz_n(n::Int64)
    up_f=zeros(2^n)
    a=up
    b=down
    for i =2:n
        a=kron(a,up)
        b=kron(b,down)
    end
    return real.((a+b)/sqrt(2))
end
up = ket(1,2)
down = ket(2,2)

#defining the mixed state
psi_3_2 = kron(ghz_n(3), ket(1,2))+kron(ghz_n(2), ket(1,2), ket(2,2));

#calculating the density matrix after tracing out the last qubit
rho_3_2 = round.(real.((ptrace(psi_3_2*psi_3_2', [2,2,2,2], 4))), digits=3)
rho_3_2 = rho_3_2 ./tr(rho_3_2);

#calculating the partial transpose of the entire system
ppt_rho_3_2 = []
for i=1:3
    push!(ppt_rho_3_2, ptranspose(rho_3_2, [2,2,2], i));
end

#calculating the reduced traces
red_rho_3_2 = []
for i= 1:3
    push!(red_rho_3_2, ptrace(rho_3_2, [2,2,2], i))
end

#calculating the eigenvalues of the PPT matrices
for i = 1:3
    println(round.(eigvals((ppt_rho_3_2[i])), digits=3))
end

#calculating the PPT matrices and their eigenvalues of the reduced system
for i = 1:3
    for j=1:2
    latexify(ptranspose(red_rho_3_2[i], [2,2], j))|>print
    end
end
for i = 1:3
    for j=1:1
    (round.(eigvals(ptranspose(red_rho_3_2[i], [2,2], j)), digits=3))|>println
    end
end
```

References

[1] P. K. Aravind. *Borromean Entanglement of the GHZ State*, pages 53–59. Springer Netherlands, Dordrecht, 1997.

[2] Ayumu Sugita. Borromean entanglement revisited. 2007.

[3] William K. Wootters. Entanglement of formation of an arbitrary state of two qubits. *Physical Review Letters*, 80(10):2245–2248, March 1998.