STATE TO LINKS

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1. Introduction

We have till now only shown existence of a polynomial invariant of a link characterized by the way it behaves after cutting. In this section, we will show that this polynomial can be connected to any quantum state, and using this we can study the entanglement property of the state.

2. Obtaining A Link From Quantum System

The polynomial gives us the behavior of the topological link after cutting any particular knot. In the case, we the operation of cutting a particular is equivalent to taking a partial trace with respect to that system. That means if a measurement is done for some system then the other states are entangled or not.

As cutting the link is equivalent to tracing out the state. First step to write down the polynomial expression for the state is to perform all possible partial traces, and then we have to check if the resulting state is entangled or not. This information gives us the polynomial which is essentially the same as finding out the topological link.

2.1. **Example.** Consider the three qubit system, given by the wavefunction,

$$|\psi\rangle = \frac{1}{2}(|100\rangle_{abc} + |010\rangle_{abc} + |110\rangle_{abc} + |011\rangle_{abc}).$$

The density matrix here is given by the matrix $\rho = |\psi\rangle\langle\psi|$. Also we chose the convention that the state a is identified as the ring variable a, state b for the ring variable b, and state c for the ring variable c.

$$\rho = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Firstly we have to find out if all the wavefunctions are in entangled initially or not. This will tell us if the polynomial has any three variable term or not. To do this we use the PPT test (positive partial transpose) with respect to each subsystem. We denote the partial transpose with respect to subsystem a as ρ^{T_a}. Here we present the partial transpose with respect to each subsystem.

$$\rho^{T_a} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0 & 0.25 & 0 & 0.25 & 0 \\ 0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The set of eigenvalues of this matrix is $-0.433, 0.0, 0.0, 0.0, 0.0, 0.25, 0.433, 0.75$, one of which is negative.

$$\rho^{T_b} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The set of eigenvalues of this matrix is $-0.354, 0.0, 0.0, 0.0, 0.0, 0.146, 0.354, 0.854$, one of which is negative.

$$\rho^{T_c} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0 & 0.25 & 0 & 0.25 & 0 \\ 0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is the same as the matrix ρ^{T_a}. As partial transpose of all the subsystem has negative eigenvalues we can say that all the subsystems are entangled with each other. So there exists a **tripartite entanglement.**

Now we want to see if the system remains entangled after tracing out with respect to each subsystem or not. Here the reduced density matrix is denoted as ρ_{ab} when the system c is traced out, similarly we have reduced density matrix ρ_{bc}, and ρ_{ac}. To see if this reduced density matrices are separable or entangled, we again use PPT test. Here as the system size is 2×2, presence of atleast one negative eigenvalue will imply entanglement between the subsystems, but also if none of the eigenvalues are negative then the systems are separable. The last statement is only true for 2×2 and 2×3 systems.

The reduced density matrices, $\rho_{bc}, \rho_{ab}, \rho_{ac}$ are the following:

$$\rho_{bc} = \begin{bmatrix} 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.25 \end{bmatrix}.$$

$$\rho_{ac} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\rho_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0.25 & 0.25 & 0.25 \\ 0 & 0.25 & 0.25 & 0.25 \end{bmatrix}.$$

Partial transpose of ρ_{bc} with respect to subsystem b is $\rho_{bc}^{T_b}$.

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.25 \end{bmatrix}.$$

This has all positive eigenvalues: $0.0, 0.0, 0.25, 0.75$. So, the subsystem b and c are separated after a is traced out. Now, partial transpose of ρ_{ac} with respect to the subsystem a is $\rho_{ac}^{T_a}$ and it is

$$\rho_{ac}^{T_a} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0 & 0 \\ 0.25 & 0 & 0.5 & 0 \\ 0.25 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are $-0.233, 0.121, 0.379, 0.733$, one of which is negative. So after tracing out subsystem b, the subsystem a and c remains entangled. Partial transpose of ρ_{ab} with respect to the subsystem a is the following,

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0 & 0 & 0 & 0.25 \\ 0 & 0.5 & 0 & 0.25 \\ 0 & 0 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}.$$

The eigenvalues of this matrix are $-0.233, 0.121, 0.379, 0.733$, one of which is negative, so after tracing out c, the subsystem a and b remains entangled. So to summarize, we want to a 3 variable polynomial with the following property:

- If we put $a = 0$, the polynomial is zero.
- If we put $b = 0$, the polynomial is just ac, corresponding to the entanglement of a and c after tracing out b.
- If we put $c = 0$, the polynomial is bc, as tracing out c, gives an entangled state of b and c.

From these information we can say that the polynomial corresponding to this state is $ac + bc$. This corresponds to the link class 3^3.

3. Density Matrix Formalism

If a Hilbert space is given, then any wavefunction is given by an element of the Hilbert space. For isolated system, any quantum state is given by these elements of the Hilbert space. This kind of states are also called pure state. As the whole quantum state is given by one state vector from the Hilbert space.

But in general, it can be possible that the system is not isolated and connected to an environment. In that case even if we are interested in the wavefunction corresponds to the Hilbert space of the system, the effect of environment makes a difference. In this case, the actual Hilbert space that we should consider should be the tensor product of the Hilbert space of the system and the environment. So any state should corresponds to both the Hilbert space of envirnoment and the system. But we are only interested in the state of the system. So it might be possible that we do not get any pure state but an incoherent superposition of pure states. This is called a mixed state. Mixed states are ensembles of pure states. And each of the states have a probabilty corresponding to it. These states can not be represented as a state vector from the Hilbert space.

To tackle this problem of representing mixed state, we introduce the formalism of density matrix. First we define density matrix for a pure state. Let $|\psi\rangle$ is an state vector from the Hilbert space $\mathcal{H}$. Then we define the density matrix corresponding to this state as $\rho = |\psi\rangle\langle\psi|$. Note that, for pure state $\rho^2 = \rho$, this is an alternate definition of pure state. We can also see that this formalism is gauge invariant, as in quantum mechanics any vector if multiplied with a $U(1)$ phase represents the same phase, but in density matrix formalism density matrix remains same. Trace of density matrix is always one. it corresponds to probability conservation. Assume an ensemble of pure state $\{\psi_1, \psi_2, \dots\}$ are given where ψ_i has probability λ_i, then the density corresponding to this mixed state is given by

$$\rho = \lambda_1 |\psi_1\rangle\langle\psi_1| + \lambda_2 |\psi_2\rangle\langle\psi_2| + \dots.$$

Here also, we see that trace of the density matrix is 1. Mathematically, we can define density matrix as a trace class operator on the Hilbert space $\mathcal{H}$ with trace 1.

Let B is observable corresponds to the Hilbert space $\mathcal{H}$. The expectation of this observable is given by $\mathrm{Tr}\{B\rho\}$. The proof follows as following,

$$\begin{aligned}
\langle B\rangle &= \sum_j \lambda_j \langle\psi_j| B |\psi_j\rangle \\
&= \sum_{j,k} \lambda_j \langle\psi_j| B |\psi_k\rangle \langle\psi_k|\psi_j\rangle \\
&= \sum_{j,k} \langle\psi_j| B\lambda_j |\psi_k\rangle \langle\psi_k|\psi_j\rangle \\
&= \sum_{j,k} \langle\psi_j| B\lambda_k |\psi_k\rangle \langle\psi_k|\psi_j\rangle \\
&= \sum_j \langle\psi_j| B\rho |\psi_j\rangle \\
&= \mathrm{Tr}\{B\rho\}
\end{aligned}$$

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