Entanglement Classification using Knots

PH3203 Term Project

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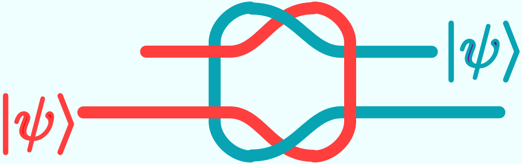

Basic Theoretical Background

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- A state is **entangled** if it is not separable.

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- If all eigenvalues positive, then, in 2×2 or 2×3 systems, this implies the state is separable.

Classifying Entanglement using Knots

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$$\mathrm{C}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

Here λ_i are the square root of the eigenvalues, in decreasing order, of the non-Hermitian matrix $\tilde{\rho}\rho$ where $\tilde{\rho}$ is defined by:

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According to the Concurrence test, a state represented by the density operator ρ is separable iff $C(\rho) = 0$.

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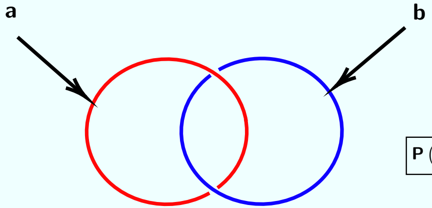

$$\boxed{\mathbf{P}\left(\mathbf{2^1}\right) = \mathbf{ab}}$$

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- An n-variable monomial M is irrelevant if all of its variables are already present as an n-ring link of lesser-order monomials, built only with the variables of M, eg ab + bc + abc is not valid as abc is irrelevant here.

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3. Full state $|\psi\rangle \longrightarrow$ sum of individual such states. Trace out d and obtain density matrix of reduced system.

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- **Full state becomes**:

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- Choose the term $t = ac$
- Two ring variables a and c, so 2 qubit GHZ state assigned to entangled part. $|\mathrm{E_q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv \left|2^1\right\rangle_{ac}$
- Separable qubit associated with b, keep it general $|\mathrm{S_q}\rangle = |q_1\rangle_b$
- Qudit state, take $|\mathrm{Q_d}\rangle = |1\rangle_d$
- **Full state becomes**: $|\psi_2\rangle = \left|2^1\right\rangle_{ac} \otimes |q_2\rangle_b \otimes |1\rangle_d$

Demonstration of Algorithm

- **Full state characterising the link**: $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$

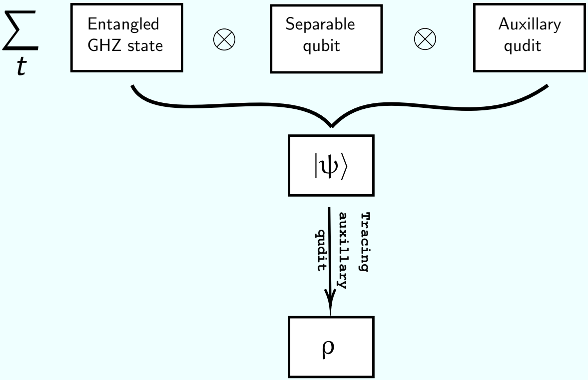

Figure: Schematic representation of the algorithm.

Demonstration of Algorithm

- **Full state characterising the link**: $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$
- Then construct the density matrix accordingly for the state.

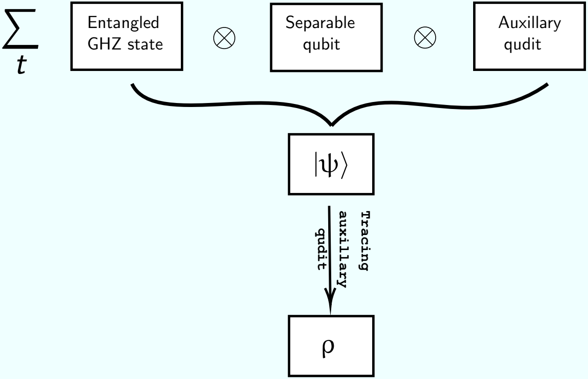

Figure: Schematic representation of the algorithm.

Some Examples...

Three Qubit System: 3^1 class

Pure State: $\left|3^1\right\rangle_{abc} = \frac{1}{\sqrt{2}}\left(|000\rangle_{abc} + |111\rangle_{abc}\right)$

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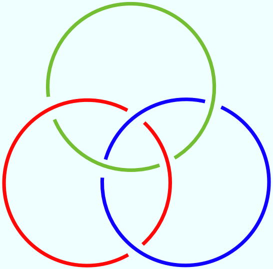

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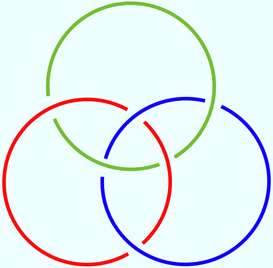

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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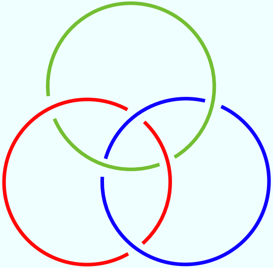

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$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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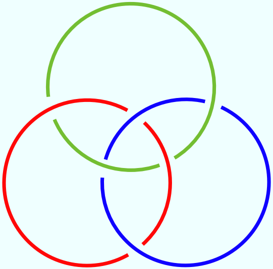

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- **Eigenvalues:** 0.0, 0.5, -0.5

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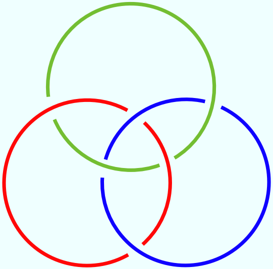

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- **Eigenvalues:**
  0.0, 0.5, -0.5
- One eigenvalue is negative $\rightarrow$ **Tripartite Entanglement**

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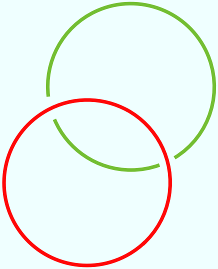

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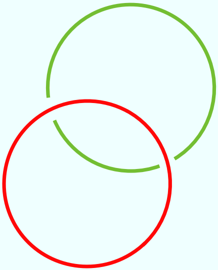

$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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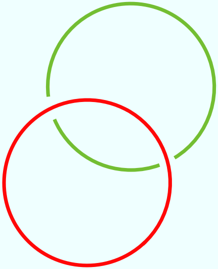

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$$\rho_{ab}^{Ta} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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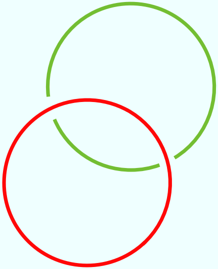

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- **Eigenvalues:**
  $0.0, \ 0.5 \geqslant 0$

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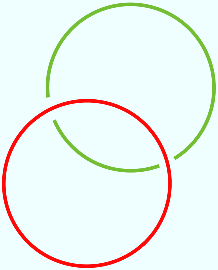

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- All eigenvalues are positive.

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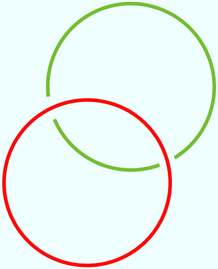

$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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- **Eigenvalues:** $0.0,\ 0.5 \geqslant 0$

- All eigenvalues are positive.
- System completely **separable** after cut.

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

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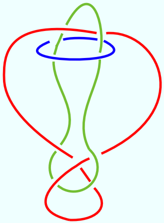

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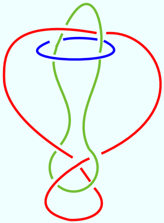

$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

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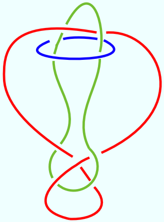

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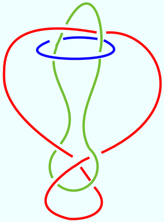

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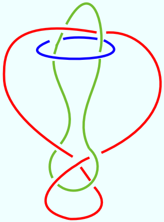
- **Eigenvalues:**

  $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

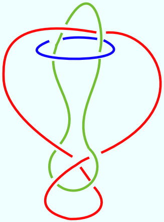
$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**

  $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



- **Eigenvalues:**

  $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$

  $-\mathbf{0.333}, 0.0, 0.127, 0.333, 0.872$

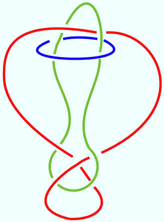
$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**

  $\mathbf{-0.471}, 0.0, 0.333, 0.471, 0.666$

  $\mathbf{-0.333}, 0.0, 0.127, 0.333, 0.872$

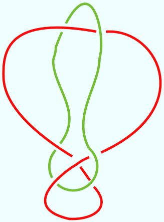
- One eigenvalue is negative → **Tripartite Entanglement**

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

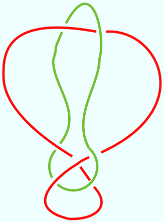
Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



Three Qubit System: 3^2 class: cut

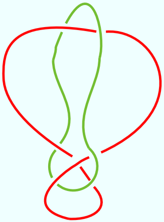
Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

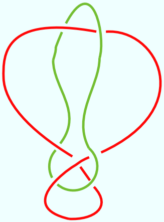


$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



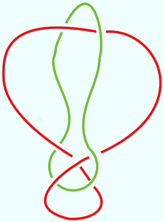
$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**
  $0.333, -\mathbf{0.333}, 0.666$

Three Qubit System: 3^2 class: cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

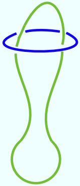
- **Eigenvalues:**
  $0.333, -\mathbf{0.333}, 0.666$
- One eigenvalue is negative $\rightarrow$ **Tripartite Entanglement**

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

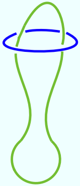
Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



Three Qubit System: 3^2 class: another cut

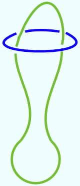
Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$

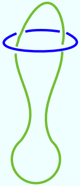


$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



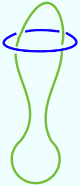
$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**
  0.0, 0.333, 0.666

Three Qubit System: 3^2 class: another cut

Pure State: $\left|3^2\right\rangle_{abc} = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$



$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

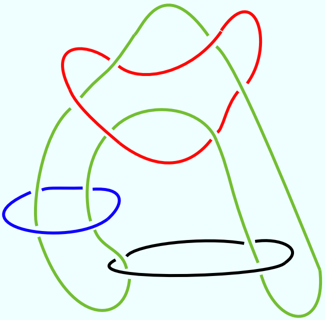
- **Eigenvalues:**
  0.0, 0.333, 0.666
- No eigenvalue is negative → **Separable**

Four Qubit System

Link Class: 4^{20}

Link polynomial: $abc + abd + ac$

$$|\psi_{20}\rangle = \left|3^1\right\rangle_{abc} |0\rangle_d |0\rangle_e + \left|3^1\right\rangle_{abd} |0\rangle_c |1\rangle_e + \left|2^1\right\rangle_{ac} |10\rangle_{bd} |2\rangle_e .$$



a: green b: black c: red d: blue

Four Qubit System

$$\hat{\rho}_{abcd} = \frac{\mathrm{Tr}_e(|\psi_{20}\rangle \langle\psi_{20}|)}{\sqrt{\langle\psi_{20}|\psi_{20}\rangle}}.$$

We need to check for Four-partite entanglement. So we have to compute the eigenvalues of the partial transpose with respect to each subsystem.

$\hat{\rho}_{abcd}^{T_a}$: Eigenvalues: $-0.270, -0.103, 0.000, 0.103, 0.270, 0.333$

$\hat{\rho}_{abcd}^{T_b}$: Eigenvalues: $-0.186, 0.000, 0.167, 0.209, 0.333, 0.477$

$\hat{\rho}_{abcd}^{T_c}$: Eigenvalues: $-0.236, 0.000, 0.064, 0.167, 0.236, 0.333, 0.436$

$\hat{\rho}_{abcd}^{T_d}$: Eigenvalues: $-0.167, 0.000, 0.033, 0.167, 0.259, 0.541$

Four Qubit System

$$\hat{\rho}_{abcd} = \frac{\mathrm{Tr}_e(|\psi_{20}\rangle\langle\psi_{20}|)}{\sqrt{\langle\psi_{20}|\psi_{20}\rangle}}.$$

We need to check for Four-partite entanglement. So we have to compute the eigenvalues of the partial transpose with respect to each subsystem.

$\hat{\rho}_{abcd}^{T_a}$: Eigenvalues: $-0.270, -0.103, 0.000, 0.103, 0.270, 0.333$

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$\hat{\rho}_{abcd}^{T_d}$: Eigenvalues: $-0.167, 0.000, 0.033, 0.167, 0.259, 0.541$

As the eigenvalues are negative, it has **FOUR PARTITE ENTANGLEMENT**.

Four Qubit System

All Possible Partial Traces:

Partial Trace with respect to system a :

$$\hat{\rho}_{bcd} = \begin{bmatrix} 0.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.166 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.166 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As this is a diagonal matrix, we can say directly that after measuring a, the rest of the system b,c,d becomes **separable**.

Four Qubit System

Partial Trace with respect to system b :

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

$$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

  $$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

- $\hat{\rho}_{acd}^{T_c}$:

  $$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

- $\hat{\rho}_{acd}^{T_c}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

- $\hat{\rho}_{acd}^{T_d}$:

  Eigenvalues: $0.000, 0.000, 0.000, 0.167, 0.230, 0.603.$

Four Qubit System

Partial Trace with respect to system b :

- $\hat{\rho}_{acd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

- $\hat{\rho}_{acd}^{T_c}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$

- $\hat{\rho}_{acd}^{T_d}$:

  Eigenvalues: $0.000, 0.000, 0.000, 0.167, 0.230, 0.603.$

Negative eigenvalues suggest, a **and** cd **are entangled**, and c **and** ad **are entangled.**

Four Qubit System

Partial Trace with respect to system b :

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  $$\text{Eigenvalues:} \; -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

- $\hat{\rho}_{acd}^{T_c}$:

  $$\text{Eigenvalues:} \; -0.167, 0.000, 0.167, 0.167, 0.333, 0.500.$$

- $\hat{\rho}_{acd}^{T_d}$:

  $$\text{Eigenvalues:} \; 0.000, 0.000, 0.000, 0.167, 0.230, 0.603.$$

Negative eigenvalues suggest, a **and** cd **are entangled**, and c **and** ad **are entangled.** We can not say anything about the subsystem d and ac.

Four Qubit System

Partial Trace with respect to system b :

$\mathbb{Q}$. How to conclude anything about the separability of the subsystem d and ac?

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Four Qubit System

Partial Trace with respect to system b :

$\mathbb{Q}$. How to conclude anything about the separability of the subsystem d and ac?

Computing eigenstates of ρ_{acd}:

$$\begin{bmatrix} 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.525 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.850 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.850 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.525 \\ 0 \end{bmatrix}$$

.

Four Qubit System

Partial Trace with respect to system b :

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If we can show that these vectors are separable as $|v_{ac}\rangle \otimes |v_d\rangle$, then that will show that the ac and d are separable.

Four Qubit System

Partial Trace with respect to system b :

Consider,

$$\begin{bmatrix} 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |0\rangle_a |0\rangle_c |0\rangle_d .$$

So this vector is separable in this form $|v_{ac}\rangle \otimes |v_d\rangle$.

Similarly, we can say all the vectors of this form are separable.

Four Qubit System

Partial Trace with respect to system b :

Consider,

$$\begin{bmatrix} 0.525 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.850 \\ 0 \end{bmatrix} = 0.525\,|0\rangle_a \otimes |0\rangle_c \otimes |0\rangle_d - 0.850\,|1\rangle_a \otimes |1\rangle_c \otimes |0\rangle_d$$

This can be written as,

$$(0.525\,|0\rangle_a \otimes |0\rangle_c - 0.850\,|1\rangle_a \otimes |1\rangle_c) \otimes |0\rangle_d\,.$$
$$= |v_{ac}\rangle \otimes |v\rangle_d$$

So, ac and d are separable.

Four Qubit System

Partial Trace with respect to system c :

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

$$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$$

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$

- $\hat{\rho}_{abd}^{T_b}$:

  Eigenvalues: $-0.103, 0.000, 0.167, 0.270, 0.333, 0.333.$

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$

- $\hat{\rho}_{abd}^{T_b}$:

  Eigenvalues: $-0.103, 0.000, 0.167, 0.270, 0.333, 0.333.$

- $\hat{\rho}_{abd}^{T_d}$:

  Eigenvalues: $-0.069, 0.000, 0.167, 0.167, 0.333, 0.402.$

.

Four Qubit System

Partial Trace with respect to system c :

- $\hat{\rho}_{abd}^{T_a}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.167, 0.167, 0.333.$

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  Eigenvalues: $-0.103, 0.000, 0.167, 0.270, 0.333, 0.333.$

- $\hat{\rho}_{abd}^{T_d}$:

  Eigenvalues: $-0.069, 0.000, 0.167, 0.167, 0.333, 0.402.$

Negative eigenvalues suggest, a **and** bd **are entangled**, b **and** ad **are entangled.** and d **and** ab **are entangled**.

Four Qubit System

Partial Trace with respect to system d :

Four Qubit System

Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

$$\text{Eigenvalues:} \quad -0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$$

Four Qubit System

Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

  $$\text{Eigenvalues:} \quad -0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$$

- $\hat{\rho}_{abc}^{T_b}$:

  $$\text{Eigenvalues:} \quad -0.122, 0.000, 0.167, 0.167, 0.333, 0.455.$$

Four Qubit System

Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

  Eigenvalues: $-0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$

- $\hat{\rho}_{abc}^{T_b}$:

  Eigenvalues: $-0.122, 0.000, 0.167, 0.167, 0.333, 0.455.$

- $\hat{\rho}_{abc}^{T_c}$:

  Eigenvalues: $-0.167, 0.000, 0.167, 0.333, 0.333, 0.333.$

.

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Partial Trace with respect to system d :

- $\hat{\rho}_{abc}^{T_a}$:

$$\text{Eigenvalues:} \quad -0.208, 0.000, 0.074, 0.167, 0.300, 0.333,$$

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$$\text{Eigenvalues:} \quad -0.122, 0.000, 0.167, 0.167, 0.333, 0.455.$$

- $\hat{\rho}_{abc}^{T_c}$:

$$\text{Eigenvalues:} \quad -0.167, 0.000, 0.167, 0.333, 0.333, 0.333.$$

Negative eigenvalues suggest, a **and** bc **are entangled**, b **and** ac **are entangled.** and c **and** ab **are entangled**.

Four Qubit System

Partial Trace with respect to system ab **:**

$$\hat{\rho}_{cd} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.166 & 0 & 0 \\ 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, c **and** d **are separated.**

Four Qubit System

Partial Trace with respect to system bc **:**

$$\hat{\rho}_{ad} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0.166 \end{bmatrix}.$$

So, a **and** d **are separated.**

Four Qubit System

Partial Trace with respect to system cd :

$$\hat{\rho}_{ab} = \begin{bmatrix} 0.333 & 0 & 0 & 0 \\ 0 & 0.166 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

So, a **and** b **are separated.**

Four Qubit System

Partial Trace with respect to system ad **:**

$$\hat{\rho}_{bc} = \begin{bmatrix} 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0.333 \end{bmatrix}.$$

So, b **and** c **are separated.**

Four Qubit System

Partial Trace with respect to system bd **:**

$$\hat{\rho}_{ac} = \begin{bmatrix} 0.5 & 0 & 0 & 0.166 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.166 & 0 \\ 0.166 & 0 & 0 & 0.333 \end{bmatrix}.$$

Four Qubit System

Partial Transpose with respect to a:

$$\hat{\rho}_{ac}^{T_a} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.166 & 0 \\ 0 & 0.166 & 0.166 & 0 \\ 0 & 0 & 0 & 0.333 \end{bmatrix}.$$

Eigenvalues are : -0.103, 0.270, 0.333, 0.500.
So, a **and** c **are entangled**.

Four Qubit System

Partial Trace with respect to system ac **:**

$$\hat{\rho}_{bd} = \begin{bmatrix} 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.166 \end{bmatrix}.$$

So, b **and** d **are separated.**

Four Qubit System

Result:

- After tracing out a, rest of the system b, c, d becomes separable.
- After tracing out b, a and cd remain entangled, c and ad remain entangled, and ac and d are separable.
- After tracing out c, a and bd, b and ad, and d and ab remain entangled.
- After tracing out d, a and bc, b and ac, and c and ab remain entangled.
- After tracing out ab, c and d are separable.
- After tracing out bc, a and d are separable.
- After tracing out cd, a and d are separable.
- After tracing out ad, b and c are separable.
- After tracing out bd, a and c are entangled.
- After tracing out ac, b and d are separable.

So, **the polynomial is** $abc + abd + ac$**.**

Application to Qubit Networks

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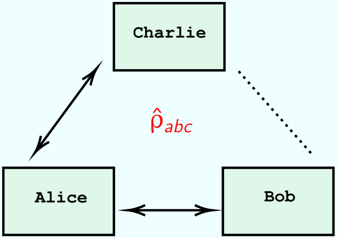

Application to Qubit Networks

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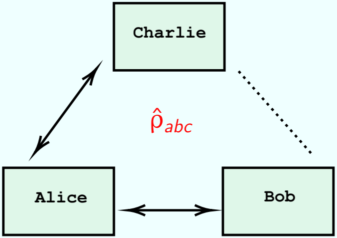
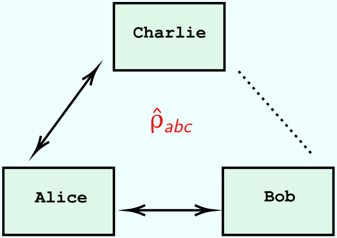
- If density operator of the subsystem consisting the parties wanting to perform the protocol is separable, then the protocol cannot be performed successfully.



Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.
- If density operator of the subsystem consisting the parties wanting to perform the protocol is separable, then the protocol cannot be performed successfully.
- If another party (not participating in the protocol) does not divulge information about their local operations, then results of the protocol cannot be correlated.



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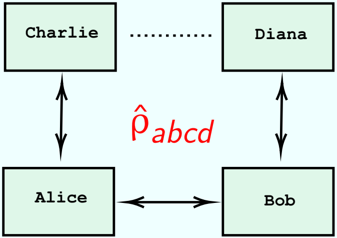

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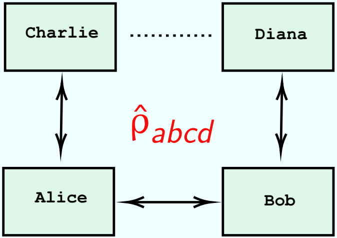

We can easily see that the polynomial describing this network is
$\mathrm{P}(\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}) = \mathrm{abc} + \mathrm{abd} + \mathrm{ac} \longrightarrow$

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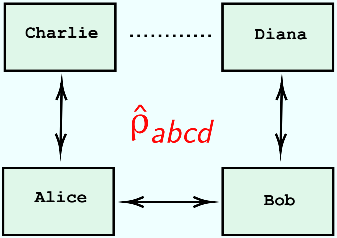

We can easily see that the polynomial describing this network is $\mathrm{P(a,b,c,d) = abc + abd + ac} \longrightarrow$ a state can immediately be constructed from the algorithm.

Conclusion

- We saw how preceding works have tried to use topological links to model entanglement and the problems with such analogies.

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Thank you!