

Entanglement Classification using Knots

PH3203 Term Project

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Basic Theoretical Background

Introduction to Knots

Introduction to Quantum Information

Classifying Entanglement using Knots

Polynomial Approach to Entanglement

Obtaining a Link from given State

Obtaining State from a given Link

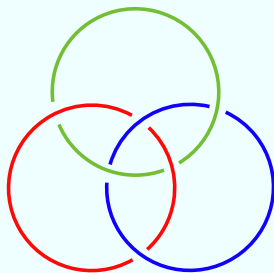
Some Examples...

Three Qubit System: 3^1 class

Pure State: $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$

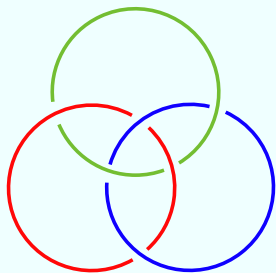
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Three Qubit System: 3^1 class

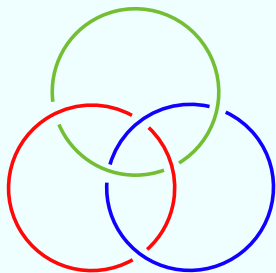
Pure State: $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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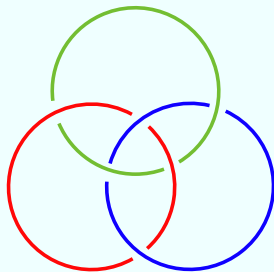


$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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Pure State: $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



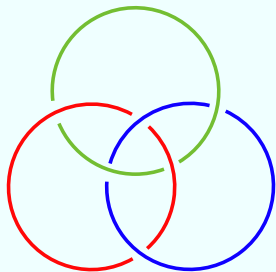
• **Eigenvalues:**
0.0, 0.5, -0.5

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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Three Qubit System: 3^1 class

Pure State: $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



- **Eigenvalues:**

0.0, 0.5, -0.5

- One eigenvalue is negative \rightarrow

**Tripartite
Entanglement**

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

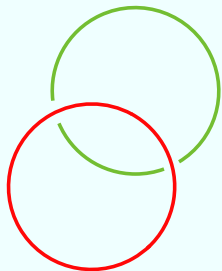
$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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Pure State: $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$

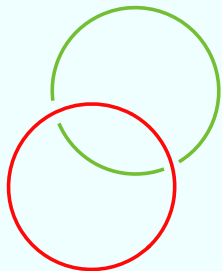
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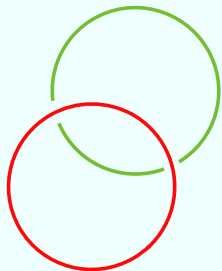
Pure State: $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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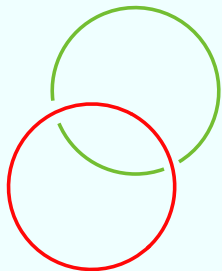


$$\rho_{ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{ab}^{Ta} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

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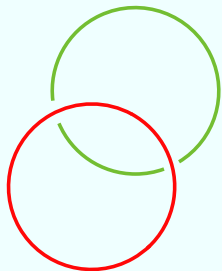
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• **Eigenvalues:**

$$0.0, 0.5 \geq 0$$

Three Qubit System: 3^1 class

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$$\rho_{ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

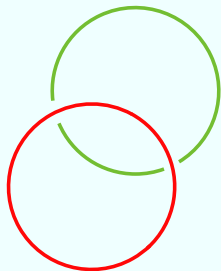
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● **Eigenvalues:**
 $0.0, 0.5 \geq 0$

● All eigenvalues are positive.

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• **Eigenvalues:**
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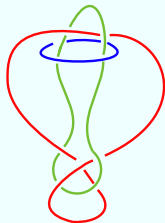
- All eigenvalues are positive.
- System completely **separable** after cut.

Three Qubit System: 3^2 class

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

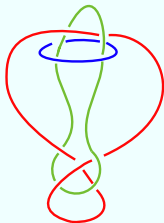
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Three Qubit System: 3^2 class

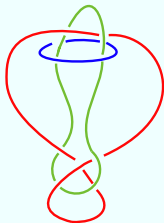
Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

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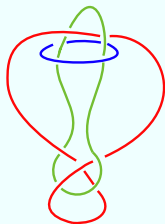


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$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

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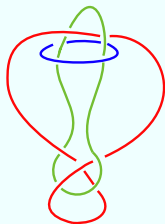
● Eigenvalues:

$-0.471, 0.0, 0.333, 0.471, 0.666$

$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$
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Three Qubit System: 3^2 class

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



● Eigenvalues:

$-0.471, 0.0, 0.333, 0.471, 0.666$

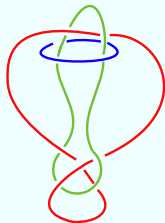
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$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

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Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



● Eigenvalues:

$-0.471, 0.0, 0.333, 0.471, 0.666$

$-0.333, 0.0, 0.127, 0.333, 0.872$

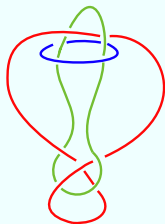
$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

● Eigenvalues:

−0.471, 0.0, 0.333, 0.471, 0.666

−0.333, 0.0, 0.127, 0.333, 0.872

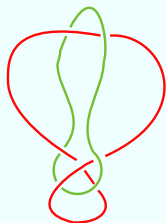
- One eigenvalue is negative → **Tripartite Entanglement**

Three Qubit System: 3^2 class: cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

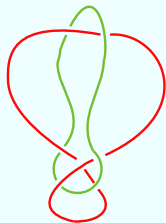
Three Qubit System: 3^2 class: cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



Three Qubit System: 3^2 class: cut

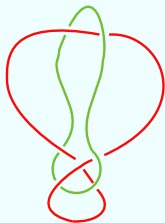
Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

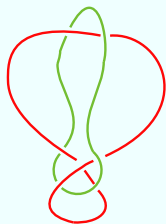


$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



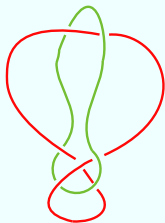
● **Eigenvalues:**
 $0.333, -0.333, 0.666$

$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



- **Eigenvalues:**

$0.333, -0.333, 0.666$

- One eigenvalue is negative \rightarrow **Tripartite Entanglement**

$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$
$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

Three Qubit System: 3^2 class: another cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



Three Qubit System: 3^2 class: another cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



● **Eigenvalues:**
 $0.0, 0.333, 0.666$

$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Three Qubit System: 3^2 class: another cut

Pure State: $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{bc}^{T_b} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

- **Eigenvalues:**

0.0, 0.333, 0.666

- No eigenvalue is negative \rightarrow **Separable**

Three Qubit System: 3^3 class

Three Qubit System: 3^4 class

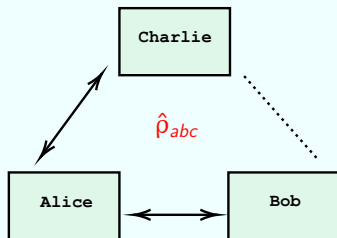
Four Qubit System

Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.

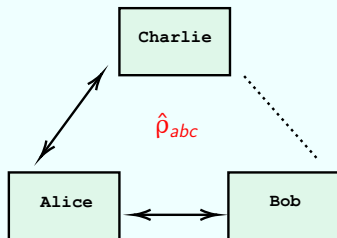
Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.



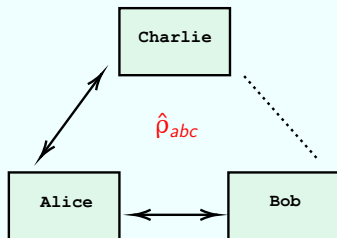
Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.
- If density operator of the subsystem consisting the parties wanting to perform the protocol is separable, then the protocol cannot be performed successfully.



Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.
- If density operator of the subsystem consisting the parties wanting to perform the protocol is separable, then the protocol cannot be performed successfully.
- If another party (not participating in the protocol) does not divulge information about their local operations, then results of the protocol cannot be correlated.



Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

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Application to Qubit Networks

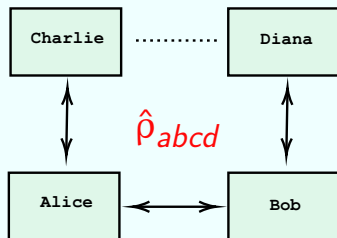
Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie
- Alice, Bob, and Diana
- Alice and Charlie

Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

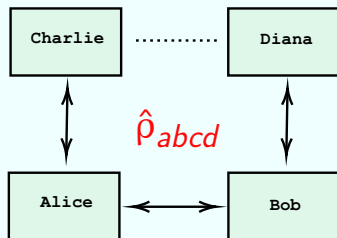
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Application to Qubit Networks

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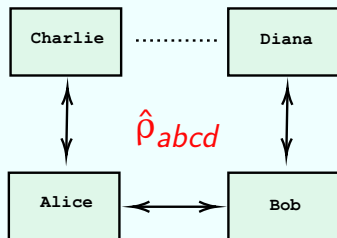


We can easily see that the polynomial describing this network is
 $P(a, b, c, d) = abc + abd + ac \longrightarrow$

Application to Qubit Networks

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We can easily see that the polynomial describing this network is $P(a, b, c, d) = abc + abd + ac \rightarrow$ a state can immediately be constructed from the algorithm.

Conclusion