

# Entanglement Classification using Knots

## PH3203 Term Project

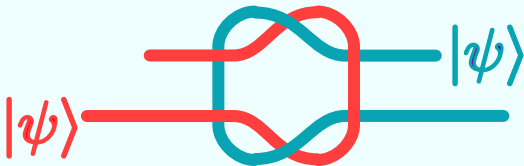
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Instructor: Prof. Sourin Das

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# Basic Theoretical Background

# Introduction to Knots

# Introduction to Quantum Information

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where  $p_i \geq 0$ ,  $\sum_i p_i = 1$ , and  $|\psi_i\rangle$  are pure states.

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- A state is **entangled** if it is **not separable**.

# Peres-Horodecki Criterion

# Classifying Entanglement using Knots

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$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

Here  $\lambda_i$  are the square root of the eigenvalues, in decreasing order, of the non-Hermitian matrix  $\tilde{\rho}\rho$  where  $\tilde{\rho}$  is defined by:

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According to the Concurrence test, a state represented by the density operator  $\rho$  is separable iff  $C(\rho) = 0$ .

# Polynomial Approach to Entanglement

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- 3 Full state  $|\psi\rangle \rightarrow$  sum of individual such states. Trace out  $d$  and obtain density matrix of reduced system.

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- **Full state becomes:**

$$|\psi_1\rangle = |2^1\rangle_{ab} \otimes |q_1\rangle_c \otimes |0\rangle_d$$



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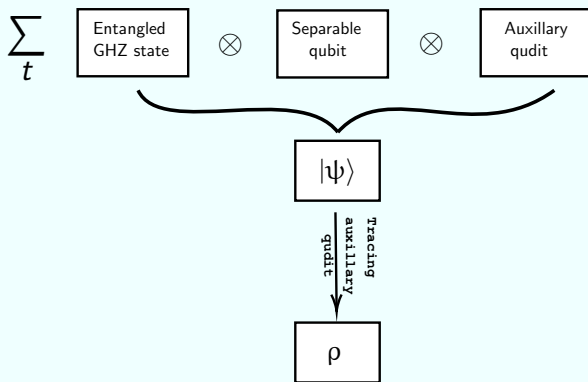
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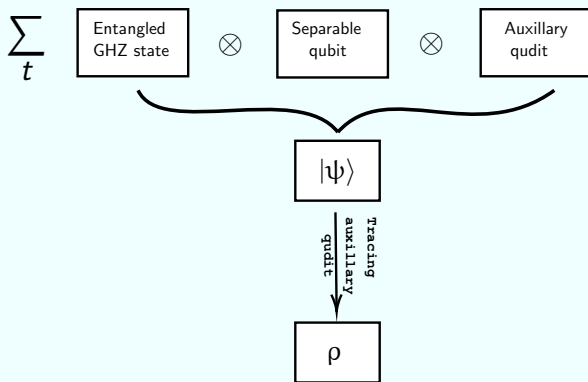
- **Mixed state characterising the link:**  $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$



**Figure:** Schematic representation of the algorithm.

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- **Mixed state characterising the link:**  $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$
- Then construct the density matrix accordingly for the state.



**Figure:** Schematic representation of the algorithm.

# Some Examples...

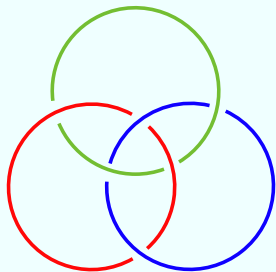


# Three Qubit System: $3^1$ class

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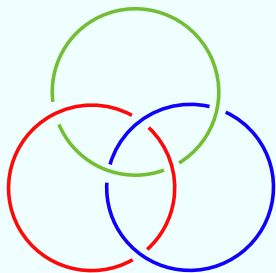
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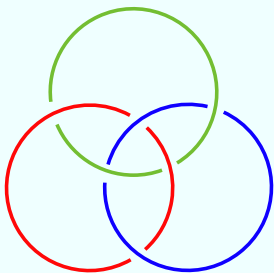
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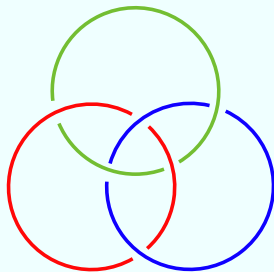


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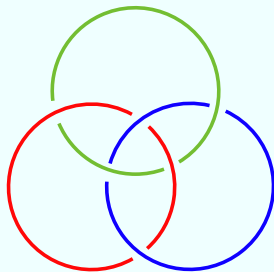
• **Eigenvalues:**  
0.0, 0.5,  $-0.5$

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

# Three Qubit System: $3^1$ class

**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



- **Eigenvalues:**

0.0, 0.5,  $-0.5$

- One eigenvalue is negative  $\rightarrow$

**Tripartite  
Entanglement**

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

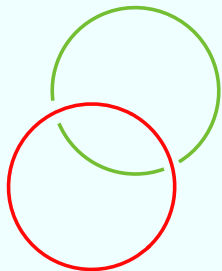
$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

## Three Qubit System: $3^1$ class

**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$

# Three Qubit System: $3^1$ class

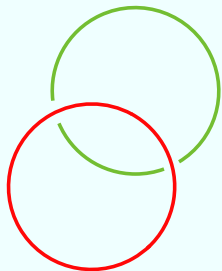
**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$





# Three Qubit System: $3^1$ class

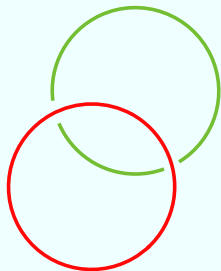
**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



$$\rho_{ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

# Three Qubit System: $3^1$ class

**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$

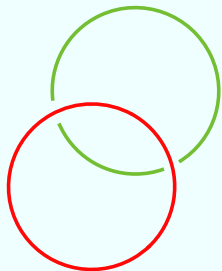


$$\rho_{ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{ab}^{Ta} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

# Three Qubit System: $3^1$ class

**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



$$\rho_{ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

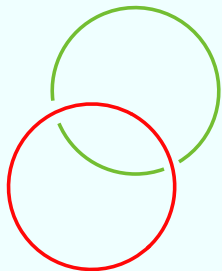
$$\rho_{ab}^{Ta} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

● **Eigenvalues:**

$$0.0, 0.5 \geq 0$$

# Three Qubit System: $3^1$ class

**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



$$\rho_{ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

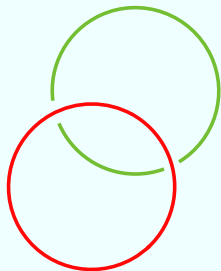
$$\rho_{ab}^{Ta} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

● **Eigenvalues:**  
 $0.0, 0.5 \geq 0$

● All eigenvalues are positive.

# Three Qubit System: $3^1$ class

**Pure State:**  $|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} (|000\rangle_{abc} + |111\rangle_{abc})$



$$\rho_{ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\rho_{ab}^{Ta} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

• **Eigenvalues:**  
 $0.0, 0.5 \geq 0$

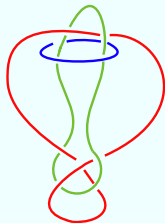
- All eigenvalues are positive.
- System completely **separable** after cut.

## Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

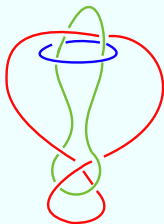
## Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



# Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

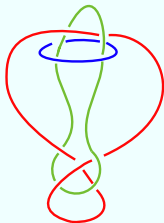


$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$



# Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

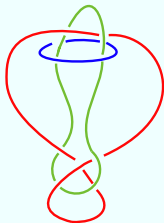


$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

# Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



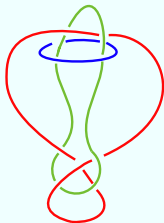
## ● Eigenvalues:

$-0.471, 0.0, 0.333, 0.471, 0.666$

$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$
$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

# Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



## ● Eigenvalues:

$-0.471, 0.0, 0.333, 0.471, 0.666$

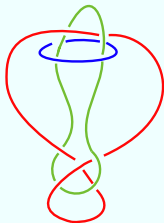
$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

# Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



## ● Eigenvalues:

$-0.471, 0.0, 0.333, 0.471, 0.666$

$-0.333, 0.0, 0.127, 0.333, 0.872$

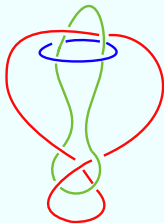
$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

# Three Qubit System: $3^2$ class

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$



$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

## ● Eigenvalues:

−0.471, 0.0, 0.333, 0.471, 0.666

−0.333, 0.0, 0.127, 0.333, 0.872

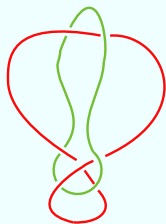
- One eigenvalue is negative → **Tripartite Entanglement**

## Three Qubit System: $3^2$ class: cut

**Pure State:**  $|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} (|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc})$

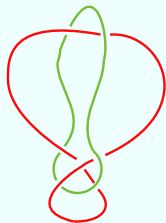
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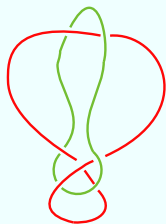


$$\rho_{ab} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$



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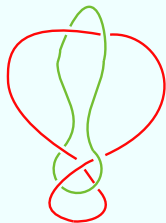


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$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

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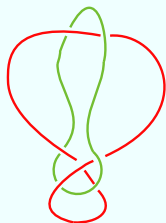
● **Eigenvalues:**  
 $0.333, -0.333, 0.666$

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$$\rho_{bc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

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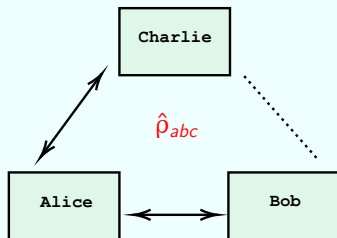
# Four Qubit System

# Application to Qubit Networks

- Different parties possessing entangled qubits, want to perform protocols with certain restrictions.

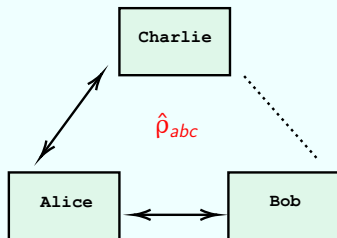
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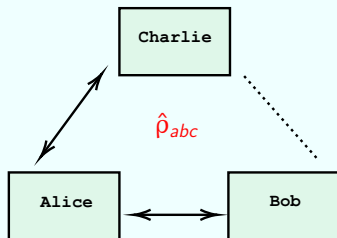
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- If density operator of the subsystem consisting the parties wanting to perform the protocol is separable, then the protocol cannot be performed successfully.
- If another party (not participating in the protocol) does not divulge information about their local operations, then results of the protocol cannot be correlated.



## Application to Qubit Networks

Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

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## Application to Qubit Networks

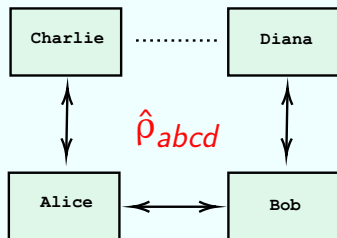
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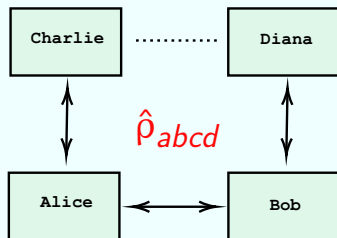
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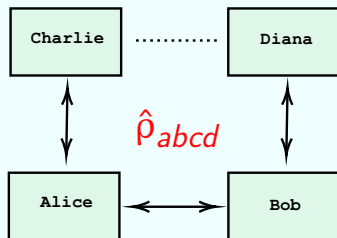


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We can easily see that the polynomial describing this network is  $P(a, b, c, d) = abc + abd + ac \rightarrow$  a state can immediately be constructed from the algorithm.

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*Thank you!*