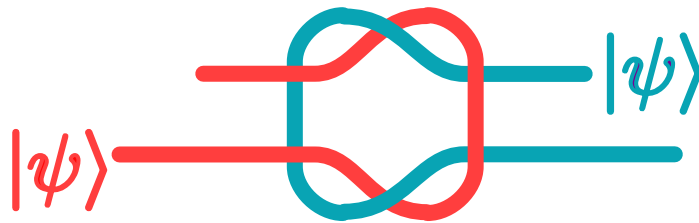# Classification Of Entanglement Using Knots

A Topological Approach to Quantum States

PH3203 Term Project Report
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1 Introduction

Classifying entanglement is essential because not all quantum states are equally useful for quantum information tasks. Different types of entanglement serve as distinct resources, each suited to specific applications such as quantum algorithms or secure communication protocols like quantum key distribution. Understanding and identifying these entanglement types helps determine how quantum states can be used and manipulated effectively. hi hi hi this is testing

SLOCC (Stochastic Local Operations and Classical Communication) is a method for classifying quantum entanglement. It defines equivalence classes of quantum states based on whether they can be converted into each other using local operations (on individual qubits) and classical communication.

This idea has been used successfully to study three-qubit states, as shown in [1,2], classifying four-qubit states [3–6] Methods have been developed for handling systems with even more qubits [7,8].[0.3cm] In this paper, the authors have proposed an alternative classification scheme for quantum entanglement based on topological links.

One of the first images that comes to mind when we think of entanglement is that of entangled threads. Naturally, one wonders if we could study quantum entanglement using entangled 'knots'. Aravind [1] was the first to point out the connections between entangled quantum states and classical knot configurations, focussing on similarity between 3-particle GHZ state and Borromean rings. He associated each particle with a ring, entanglement of any set of particles as inability to separate their corresponding rings, and measurement of particle state as cutting its ring. But he noted that performing the measurement in different basis would not lead to the same conclusions. This limit in analogy was dealt with by Sugita [2]. He proposed that cutting the ring is equivalent to tracing out the corresponding particle from the density operator, which is a basis-independent operation. This represents viewing the system as though that particle is no longer present. Moreover, the trace operation helps to generalise the idea to quantum systems with more than 2 levels.

2 Classification of Links: A Polynomial Approach

2.1 Formalism of the Link Polynomial

2.2 Obtaining a Link from an Entangled Quantum State

2.3 Obtaining an Entangled Quantum State from a Link

In this section, we will see how we can obtain a link from a quantum state. This is in general a difficult task to obtain an entangled quantum states from the polynomial as the number of qubits increases. The process in the paper mentions an algorithm which provides an 'incomplete' map betwee a given link and a quantum state. Using the procedure, the general structure of the quantum state can be obtained, however, some free coefficients remain which needs to be fixed computationally. Moreover, presently only mixed states satisfying the link can be obtained using this procedure.

Note that although incomplete, the map is still useful since we can ascertain the structure of each state contained in the mixed state. That is, from a possibility of 2^N (for N qubits, there are 2^N basis states, namely $|0\rangle, |1\rangle, \ldots |2^N - 1\rangle$ where each ket is to be assumed in the binary representation) states, we are reducing it to a much smaller number.

For this, we will use the GHZ type of state as a building block which are of the form:

$$|\text{N}^1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes\text{N}} + |1\rangle^{\otimes\text{N}}\right)$$

Here $|0\rangle^{\otimes\text{N}}$ is the tensor product of N number of $|0\rangle$ states, that is, $|0\rangle^{\otimes\text{N}} = |\underbrace{0000\ldots0}_{\text{N times}}\rangle$. The state $|\text{N}^1\rangle$ is a maximally entangled state of N qubits. The general algorithm to obtain a state from the link is as follows:

1. Let a polynomial P be given. Select a term of the given 'link' polynomial, say t.

2. The term t is then mapped to a state of the form $|\text{E}_\text{q}\rangle \otimes |\text{S}_\text{q}\rangle \otimes |\text{Q}_\text{d}\rangle$ where:

   - $|\text{E}_\text{q}\rangle$ is the entangled qubit of the GHZ type as specified above, associated to ring variables contained in t.

   - $|\text{S}_\text{q}\rangle$ is a separable qubit associated with ring variables not contained in t. There are a number of possibilities for this separable qubit and we have to find it computationally.

   - $|\text{Q}_\text{d}\rangle$ is a qudit state which is associated with an artifically introduced ring variable (which is alphabetically the next letter of the largest ring variable present). The states always starts from 0 for the first term and is increased by 1 for each successive term of the polynomial. This will later be traced out, hence is of less significance.

3. The full state $|\psi\rangle$ is constructed by summing these individual states obtained for each term of the polynomial.

4. The full mixed state characterised by this polynomial is then obtained by tracing out the qudit state $|\text{Q}_\text{d}\rangle$.

$$\boxed{\hat{\rho}(\text{P}) = \frac{\text{Tr}_\text{d}\,|\psi\rangle\,\langle\psi|}{\sqrt{\langle\psi|\psi\rangle}}}$$

Example demonstrating the algorithm:

We will see a simple example of the algorithm to obtain a state from a link. Consider the polynomial $\text{P}(a, b, c) = ab + ac$. This is a three-ring link.

- Let us choose the term t = ab. This term has two ring variables thus we will associate a two qubit GHZ type of state to $|\text{E}_\text{q}\rangle$. Thus, we have $|\text{E}_\text{q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv |2^1\rangle_\text{ab}$.

  Since the separable qubit has large possibility, we will denote it generally by $|\text{q}_1\rangle$ and this will be associated with the remaining ring variable which is c. Thus, $|\text{S}_\text{q}\rangle = |\text{q}_1\rangle_\text{c}$. The remaining term is the qudit state which will be associated to d (since d is alphabetical successor of the largest ring variable c). Then we will have the full state:

  $$|\psi_1\rangle = |2^1\rangle_\text{ab} \otimes |\text{q}_1\rangle_\text{c} \otimes |0\rangle_\text{d}$$

- Now, let us choose the next term in the polynomial which is t = ac. Similar to above, to the entangled qubit we will associate the two qubit GHZ state, thus, $|\text{E}_\text{q}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv |2^1\rangle_\text{ac}$.

  The separable qubit will be associated with the remaining ring variable b and we will denote it by $|\text{q}_2\rangle_\text{b}$. The qudit state will be associated with d which is the alphabetical successor of c but this time we will use $|1\rangle_\text{d}$ as for each successive term, the qudit state increases to the next level. Thus, we have:

  $$|\psi_2\rangle = |2^1\rangle_\text{ac} \otimes |\text{q}_2\rangle_\text{b} \otimes |1\rangle_\text{d}$$

- The full state $|\psi\rangle$ is then obtained by summing the two states obtained above with some coefficients:

$$\begin{aligned}
|\psi\rangle &= c_1 |\psi_1\rangle + c_2 |\psi_2\rangle \\
&= c_1(|2^1\rangle_{\mathrm{ab}} \otimes |q_1\rangle_{\mathrm{c}} \otimes |0\rangle_{\mathrm{d}}) + c_2(|2^1\rangle_{\mathrm{ac}} \otimes |q_2\rangle_{\mathrm{b}} \otimes |1\rangle_{\mathrm{d}})
\end{aligned}$$

Then we can trace out the qudit state $|d\rangle$ to obtain the density matrix of the ring variables:

$$\boxed{\hat{\rho}_{\mathrm{abc}} = \frac{\mathrm{Tr_d} |\psi\rangle \langle \psi|}{\sqrt{\langle \psi | \psi \rangle}}}$$

2.4 Applying to Three Qubit Systems

As a demonstration, we will apply our algorithm to three qubit systems. Note that from the rules of the 'link' polynomial, the possible basis terms for three qubit system are: $\{ab, ac, bc, abc\}$. Using this, four distinct classes of polynomials are possible:

$$\begin{aligned}
P_1(a, b, c) &= abc \\
P_2(a, b, c) &= abc + ab \\
P_3(a, b, c) &= ab + ac \\
P_4(a, b, c) &= ab + ac + bc
\end{aligned}$$

3^1 Link Class

Let us start with the 3^1 link class, which corresponds to the Borromean Link and is given by $P_1(a, b, c) = abc$. Cutting any of a, b or c will lead to complete separabability and loss of entanglement.
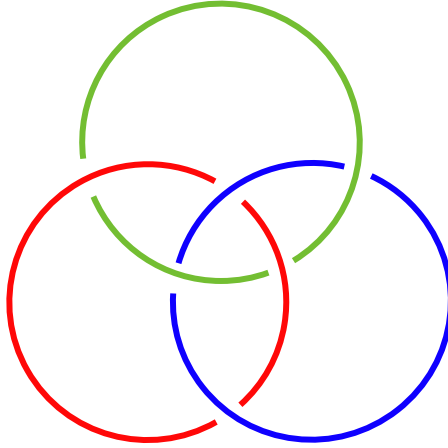

Figure 1: The Borromean link, characterising the 3^1 link class.

We already know that on its own the GHZ state characterises the 3^1 link, as discussed in the preceding works. Thus, we have the pure state:

$$|3^1\rangle_{abc} = \frac{1}{\sqrt{2}} \left(|000\rangle_{abc} + |111\rangle_{abc}\right)$$

The density matrix corresponding to the state is found to be:

$$\rho_{abc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The partial transpose with respect to any of the subsystem (since the polynomial is symmetric) is same and is given by:

$$\rho_{abc}^{T_a/T_b/T_c} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The eigenvalues corresponding to this matrix are $0.0, 0.5$ and -0.5. Since there are negative eigenvalues, we can conclude that the system exhibits **tripartite entanglement** as a whole.

Now, let us reduce the system by tracing out one of the variable. Since the polynomial is symmetric, we can choose any of the variable to trace out, say c. The reduced density matrix is given by:

$$\rho_{ab} \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The partial transpose with respect to a or b results in the same above matrix which have eigenvalues 0.5 and 0.0 which are all positive, thus concluding the absence of any entanglement in the system. This is consistent with the fact that for the Borromean link, cutting any link will result in complete separability of the links.

3^2 Link Class

Let us now consider the 3^2 link class given by $P_2(a, b, c) = abc + ab$. Cutting any of a, b will lead to complete separabality but if we cut c, then the other rings will remain entangled. The link can be represented as:
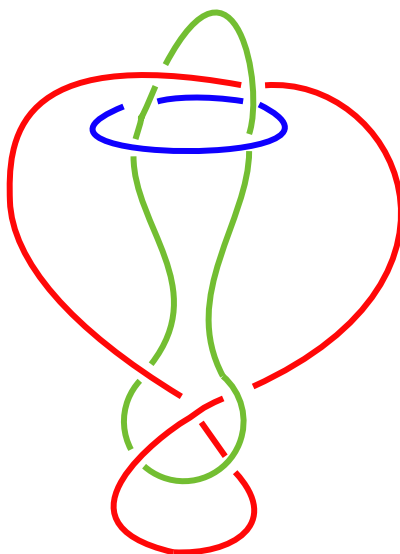
Figure 2: The knot diagram, characterising the 3^2 link class.

Here the blue knot corresponds to c while the other two correspond to a and b (a,b are symmetric in the polynomial). An example of a pure state is found, as mentioned in the paper:

$$|3^2\rangle_{abc} = \frac{1}{\sqrt{3}} \left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc} \right)$$

To check that this state indeed statistfies the link, let us calculate the density operator $\hat{\rho}_{abc} = |3^2\rangle_{abc} \langle 3^2|_{abc}$ and then check for the PPT test for each cuts. The density matrix obtained is:

$$\rho_{abc} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

Since the variables a and b are symmetrix, we can choose to analyse only one of them and c.

We then see the partial transpose with respect to a and c. The matrices are given by:

$$\rho_{\mathrm{abc}}^{\mathrm{T_a}} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{\mathrm{abc}}^{\mathrm{T_c}} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

The above matrices have eigenvalues $\lambda_{\mathrm{a}} = \mathbf{-0.471}, 0.0, 0.333, 0.471, 0.666$ and $\lambda_{\mathrm{c}} = -0.333, 0.0, 0.127, 0.333, 0.872$ respectively. Since there are negative eigenvalues, we can conclude that the system exhibits **tripartite entanglement** as a whole.

The reduced density matrices are given by:

$$\rho_{\mathrm{bc}} = \begin{bmatrix} 0.333 & 0.333 & 0.0 & 0.0 \\ 0.333 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

$$\rho_{\mathrm{ab}} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

We now obtain the partial tranpose for the PPT test. We note that $\rho_{\mathrm{bc}}^{\mathrm{T_b/T_c}}$ is the same as that of the above matrix ρ_{bc} whose eigenvalues are $0.0, 0.333, 0.666$ which are all positive. Thus, we can conclude that the system is separable. On the other hand, we obtain:

$$\rho_{ab}^{T_a} = \begin{bmatrix} 0.666 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.333 \end{bmatrix}$$

The eigenvalues of this matrix are $0.333, -0.333, 0.666$, one of which is negative, thus denoting the presence of entanglement. This successfully verifies the behaviour of the link abc + ab.

Using the above algorithm, we can also construct the mixed state corresponding to the link by considering the state:

$$|\psi_2\rangle = |3^1\rangle_{abc} |0\rangle_d + |2^1\rangle_{ab} |0\rangle_c |1\rangle_d$$

It is to be noted that this class has not been described in the previous works [1, 2]. The density operator is then given by:

$$\hat{\rho}(a, b, c) = \frac{\mathrm{Tr}_d |\psi_2\rangle \langle \psi_2|}{\sqrt{\langle \psi_2 | \psi_2 \rangle}}$$

Using numerical calculation and tracing out subsystem d, we found the density matrix to be:

$$
\rho_{abc} =
\begin{bmatrix}
0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.25 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\
0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25
\end{bmatrix}
$$

Let us now calculate the partial transposes for the PPT test:

$$
\rho_{abc}^{T_a} =
\begin{bmatrix}
0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.25 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25
\end{bmatrix}
$$

$$
\rho_{abc}^{T_b} =
\begin{bmatrix}
0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.25 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25
\end{bmatrix}
$$

$$
\rho_{abc}^{T_c} =
\begin{bmatrix}
0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.25 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25
\end{bmatrix}
$$

The set of eigenvalues for the above matrices are:

- $\lambda_a = \mathbf{\color{red}{-0.354}}, 0.0, 0.0, 0.0, 0.25, 0.25, 0.354, 0.5$

- $\lambda_b = \mathbf{\color{red}{-0.354}}, 0.0, 0.0, 0.0, 0.25, 0.25, 0.354, 0.5$

- $\lambda_c = \mathbf{\color{red}{-0.183}}, 0.0, 0.0, 0.0, 0.0, 0.25, 0.25, 0.683$

Each of the above matrices have atleast one negative eigenvalues, thus denoting the presence of entanglement in the system as a whole. Let us now analyse the reduced density matrices and

their PPT matrices which corresponds to the act of 'cutting a link':

$$\rho_{\rm bc} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{\rm bc}^{\rm T_b} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\rm ac} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{\rm ac}^{\rm T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\rm ab} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.5 \end{bmatrix} \quad \rho_{\rm ab}^{\rm T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The set of eigenvalues for the above matrices are:

- $0.0, 0.25, 0.25, 0.5 \longrightarrow$ no eigenvalue is negative, thus cutting a separates the systems b and c.

- $0.0, 0.25, 0.25, 0.5 \longrightarrow$ no eigenvalue is negative, thus cutting b separates the systems a and c.

- $\mathbf{-0.25}, 0.25, 0.5, 0.5 \longrightarrow$ one eigenvalue is negative, thus on cutting c, the systems a and b remain entangled.

This faithfully reproduces the behaviour of the link $abc + ab$ and thus, the state characterises the link.

3^3 Link Class

We now analyse the link class given by $P_3(a, b, c) = ab + ac$. The link diagram is given by:
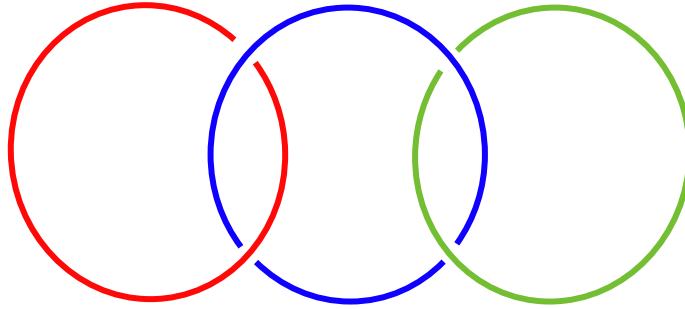

Figure 3: The knot diagram, characterising the 3^3 link class.

We can see that the knot polynomial is symmetric in b and c. Thus, if we cut either b or c, the other will remain entangled with a but complete separation results from cutting a. This case has already been discussed in section 2.2 and a pure state has been obtained. The mixed state can be obtained using the state:

$$|\psi_3\rangle = |2^1\rangle_{ab} |0\rangle_c |0\rangle_d + |2^1\rangle_{ac} |1\rangle_b |1\rangle_d$$

The density matrix obtained for the system is:

$$\rho_{abc} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

Let us calculate the partial transposes for the PPT test:

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{abc}^{T_b} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.25 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

The set of eigenvalues obtained for the above matrices are:

- $\lambda_a = \textcolor{red}{\mathbf{-0.155}}, \textcolor{red}{\mathbf{-0.155}}, 0.0, 0.0, 0.25, 0.25, 0.405, 0.405$

- $\lambda_b = \textcolor{red}{\mathbf{-0.2}}, 0.0, 0.0, 0.0, 0.139, 0.25, 0.25, 0.562$

- $\lambda_c = \textcolor{red}{\mathbf{-0.2}}, 0.0, 0.0, 0.0, 0.139, 0.25, 0.25, 0.562$

Since atleast one of the eigenvalues is negative for each matrix, we can conclude that the system exhibits **tripartite entanglement** as a whole. Let us now consider the reduced systems.

We obtain the density matrices and their PPT matrices for the reduced system as:

$$\rho_{bc} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{bc}^{T_b} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{ac} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.25 \end{bmatrix} \quad \rho_{ac}^{T_a} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{ab} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.5 \end{bmatrix} \quad \rho_{ab}^{T_a} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.25 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

The set of eigenvalues obtained for the above matrices are:

- $0.0, 0.25, 0.25, 0.5 \longrightarrow$ no negative eigenvalues, so cutting a makes the system separable.

- $-\mathbf{0.155}, 0.25, 0.405, 0.5 \longrightarrow$ negative eigenvalue, so on cutting b, system a and c remain entangled.

- $-\mathbf{0.155}, 0.25, 0.405, 0.5 \longrightarrow$ negative eigenvalue, so on cutting c system a and b remain entangled.

This is consistent with the link properties and hence the mixed state obtained is indeed characterises the link.

3^4 Link Class

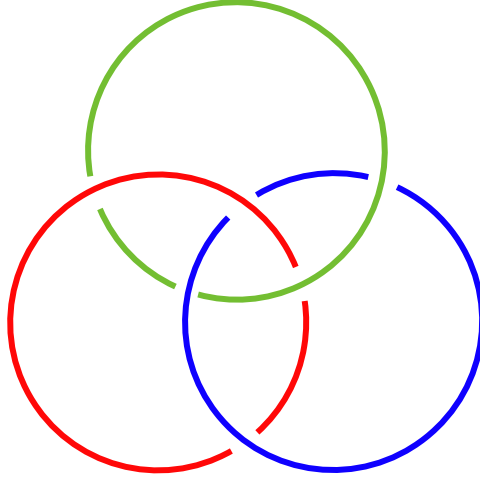Let us now analyse the class $P_4(a, b, c) = ab + ac + bc$. The link diagram is given by:

Figure 4: The knot diagram, characterising the 3^4 link class.

This link demostrates a behaviour opposite to that of the Borromean ring, that is, setting any one variable to zero with not result in complete separability. The other two links remain entangled. The pure state having the characteristic of the link is the **W state** which has been documented in previous works:

$$|3^4\rangle = \frac{1}{\sqrt{3}} \left(|001\rangle_{abc} + |010\rangle_{abc} + |100\rangle_{abc} \right)$$

Let us check whether this state satisfies the link property. The density matrix is given by:

$$\rho abc = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The corresponding partial transpose with respect to the three variables are given by:

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\rho_{abc}^{T_b} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.333 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 & 0.333 & 0.0 & 0.0 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

All the three matrices have the same set of eigenvalues $-\mathbf{0.471}, 0.0, 0.333, 0.471, 0.666$, one of which is negative, thus denoting the presence of entanglement. The reduced density matrix in each case is the same and is given by:

$$\rho_{reduced} = \begin{bmatrix} 0.333 & 0.0 & 0.0 & 0.333 \\ 0.0 & 0.333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.333 & 0.0 \\ 0.333 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The partial transpose of this is the same as that of the matrix. The eigenvalues of the PPT matrix are $-0.206, 0.333, 0.333, 0.539$, one of which is negative, thus denoting the present of entanglement. This verifies the behaviour of the link $ab + ac + bc$.

From the algorithm, the mixed state can be obtained using the state:

$$|\psi_4\rangle = |2^1\rangle_{ab} |0\rangle_c |0\rangle_d + |2^1\rangle_{ac} |1\rangle_b |1\rangle_d + |2^1\rangle_{bc} |0\rangle_a |2\rangle_d$$

Using the above state after tracing out subsystem d, we obtained the density matrix as:

$$\rho_{abc} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \\ 0.125 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

Let us now check the partial transpose for the entire system. The PPT matrices are found to be:

$$\rho_{abc}^{T_a} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{abc}^{T_b} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.125 & 0.25 & 0.0 & 0.125 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{abc}^{T_c} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.125 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.125 & 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.25 & 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

The set of eigenvalues of $\rho_{abc}^{T_a}$ and $\rho_{abc}^{T_c}$ are the same, namely **−0.146**, **−0.052**, 0.0, 0.0, 0.222, 0.25, 0.302, 0.424 while the eigenvalues of $\rho_{abc}^{T_b}$ are **−0.138**, 0.0, 0.0, 0.107, 0.125, 0.125, 0.25, 0.531. We see that there exists atleast one negative eigenvalue in all the three matrices, thus denoting the presence of tripartite entanglement. Let us now check the reduced density matrices.

$$\rho_{\mathrm{bc}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.375 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.375 \end{bmatrix} \qquad \rho_{\mathrm{bc}}^{\mathrm{T_b}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 \\ 0.0 & 0.125 & 0.375 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.375 \end{bmatrix}$$

$$\rho_{\mathrm{ac}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.25 \\ 0.0 & 0.125 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.125 & 0.0 \\ 0.25 & 0.0 & 0.0 & 0.25 \end{bmatrix} \qquad \rho_{\mathrm{ac}}^{\mathrm{Ta}} = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.125 & 0.25 & 0.0 \\ 0.0 & 0.25 & 0.125 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 \end{bmatrix}$$

$$\rho_{\mathrm{ab}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.125 \\ 0.0 & 0.375 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.125 & 0.0 & 0.0 & 0.375 \end{bmatrix} \qquad \rho_{\mathrm{ab}}^{\mathrm{T_a}} = \begin{bmatrix} 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.375 & 0.125 & 0.0 \\ 0.0 & 0.125 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.375 \end{bmatrix}$$

The set of eigenvalues of PPT matrices of ρ_{bc} and ρ_{ab} are the same given by **−0.038**, $0.25, 0.375, 0.413$ while the eigenvalues of PPT matrix of ρ_{ac} are **−0.125**, $0.25, 0.375, 0.5$. Since in each set, a negative eigenvalue exists, we can conclude that the system exhibits entanglement, thus verifying the behaviour of the link ab + ac + bc, that is, even after cutting one variable, the other two remain entangled.

2.5 Applying to Four Qubit Systems

3 Physical Significance: Use in Qubit Networks

The knot theoretic approach to entangled quantum states can be of great use in qubit networks, where different parties posses different qubits, entangled together and they want to perform some protocols (Here protocol means some series of operations/measurements to obtain some result, like the **BB84 protocol** for secure key distribution).
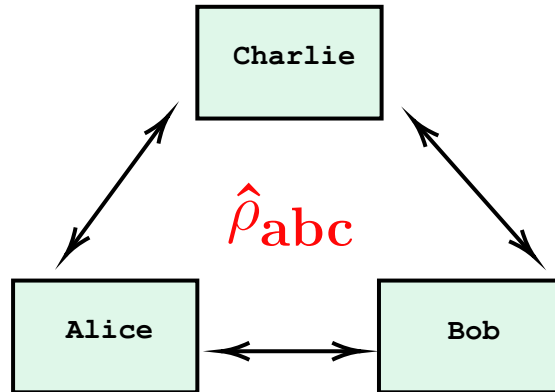

Figure 5: A qubit network, showing three parties, Alice, Bob and Charlie sharing an entangled state, represented by density operator $\hat{\rho}_{\mathrm{abc}}$.

For successfully performing a protocol between two parties, entanglement must exist between the qubits possessed by the two parties. The above diagram shows three parties Alice, Bob and Charlie, each possessing a qubit and their combined state is described by the density operator $\hat{\rho}_{\mathrm{abc}}$. Then, suppose Alice and Bob want to perform a protocol while Charlie does not want to participate. In this case, a general operator for the system will be of the form:

$$\hat{O} = \sum_i (\hat{M}_{\mathrm{i,ab}} \otimes \hat{\mathbb{1}}_{\mathrm{c}}) \hat{\rho}_{abc} (\hat{M}_{\mathrm{i,ab}}^{\dagger} \otimes \hat{\mathbb{1}}_{\mathrm{c}})$$

where $\hat{M}_{i,ab}$ is some operator acting on the combined Hilbert space of Alice and Bob and the index i runs over all the possible outcomes. Note that the operator does not interfere with Charlie's qubit. The probability of the ith outcome is given by:

$$p_i = \text{Tr}_{abc}[(\hat{M}_{i,ab}^\dagger \otimes \hat{1}_c)(\hat{M}_{i,ab} \otimes \hat{1}_c)\hat{\rho}_{abc}] \quad \xrightarrow[\text{system } c]{\text{Tracing out}} \text{Tr}_{ab}[\hat{M}_{i,ab}^\dagger \hat{M}_{i,ab}\hat{\rho}_{ab}]$$

In this case, knowledge of Charlie's system is not neccessary for successfully applying the protocol by Alice and Bob. In the case Charlie decides to act on his qubit by some operator $\hat{N}_{i,c}$, the total operator for the system will now be modified by:

$$\hat{M}_{i,ab} \otimes \hat{1}_c \quad \longrightarrow \hat{M}_{i,ab} \otimes \hat{N}_{i,c}$$

Now, for successfully performing the protocol, Alice and Bob must have a kmowledge of Charlie's operation since it could potentially affect the outcome of the protocol as the states are entangled. Thus, we have the following observations from the above discussion:

- If the operator $\hat{\rho}_{ab}$ is separable, then the protocol cannot be performed successfully between Alice and Bob, as the two parties are not entangled.

- If Charlie decides not to divulge any information about his system, the protocol cannot be applied as no correlation can be determined between the two parties Alice and Bob.

In this case, the above discussion on the knot theoretic approach to entangled quantum states can be of great use. As a demonstration, we consider building a three-qubit network such that:

- Bob and Charlie may never successfully perform any protocols without external help from Alice.

- Alice may have a chance to communicate with either Bob or Charlie if she wishes to.

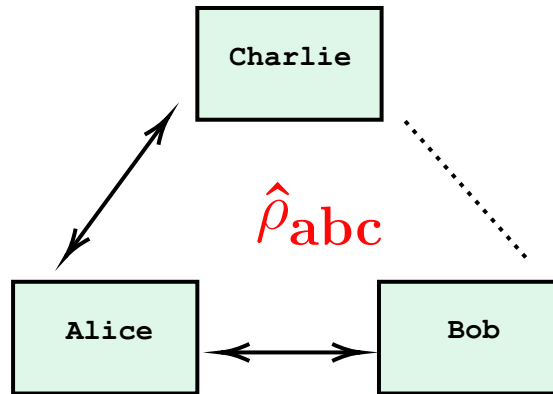The network is shown in the figure below:

Figure 6: A qubit network, showing three parties, Alice, Bob and Charlie sharing an entangled state, represented by density operator $\hat{\rho}_{abc}$. The dotted line represent no connection between the two parties.

The network can be easily described using the link polynomial language as: $P(a, b, c) = ac + ab$ and thus we can immediately find a state that satisfies the above condition. The situation becomes more helpful when we consider higher number of qubits. Let us consider a four qubit network, where Alice, Bob, Charlie and Diana are the parties such that only the following parties can communicate:

- Alice, Bob, and Charlie
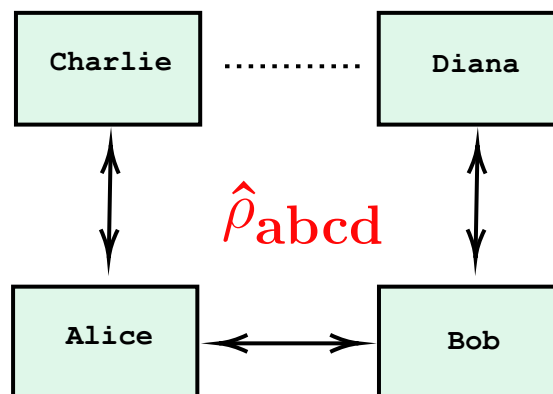
- Alice, Bob, and Diana

- Alice and Charlie

Figure 7: The four qubit network, showing four parties, Alice, Bob, Charlie and Diana sharing an entangled state, represented by density operator $\hat{\rho}_{abcd}$. The dotted line represent no connection between the two parties.

We can easily see that the polynomial describing this network is $P(a, b, c, d) = abc + abd + ac$. Thus a state can immediately be obtained using the algorithm described previously.

4   Discussion and Conclusion

5   Acknowledgements

Appendix A: Quantum Information Basics

5.1 Density Matrix

5.2 Peres-Horodecki Criterion

Appendix B: **Knot Theory Basics**

Appendix C: Code for Numeric Calculations

We used the `QuantumInformation.jl` package in `Julia` to perform the numerical calculations. The code provided below shows some basic calculations that we had used in this report.

```julia
using QuantumInformation, LinearAlgebra, Latexify
#example of a 3 qubit system: 3^2 link class mixed class

function ghz_n(n::Int64)
    up_f=zeros(2^n)
    a=up
    b=down
    for i =2:n
        a=kron(a,up)
        b=kron(b,down)
    end
    return real.((a+b)/sqrt(2))
end
up = ket(1,2)
down = ket(2,2)

#defining the mixed state
psi_3_2 = kron(ghz_n(3), ket(1,2))+kron(ghz_n(2), ket(1,2), ket(2,2));

#calculating the density matrix after tracing out the last qubit
rho_3_2 = round.(real.((ptrace(psi_3_2*psi_3_2', [2,2,2,2], 4))), digits=3)
rho_3_2 = rho_3_2 ./tr(rho_3_2);

#calculating the partial transpose of the entire system
ppt_rho_3_2 = []
for i=1:3
    push!(ppt_rho_3_2, ptranspose(rho_3_2, [2,2,2], i));
end

#calculating the reduced traces
red_rho_3_2 = []
for i= 1:3
    push!(red_rho_3_2, ptrace(rho_3_2, [2,2,2], i))
end

#calculating the eigenvalues of the PPT matrices
for i = 1:3
    println(round.(eigvals((ppt_rho_3_2[i])), digits=3))
end

#calculating the PPT matrices and their eigenvalues of the reduced system
for i = 1:3
    for j=1:2
    latexify(ptranspose(red_rho_3_2[i], [2,2], j))|>print
    end
end
for i = 1:3
    for j=1:1
    (round.(eigvals(ptranspose(red_rho_3_2[i], [2,2], j)), digits=3))|>println
    end
end
```

References

[1] P. K. Aravind. *Borromean Entanglement of the GHZ State*, pages 53–59. Springer Netherlands, Dordrecht, 1997.

[2] Ayumu Sugita. Borromean entanglement revisited. 2007.