INTRODUCTION

This experiment is concerned with the measurement and interpretation of the Faraday effect - magnetically induced circular birefringence. The experiment reveals a fundamental connection between optics and electromagnetism, and leads to an evaluation of the effective ratio of charge to mass of the particles in flint glass with which light most strongly interacts, i.e., electrons.

Any transparent substance whose molecules are not mirror symmetric and which is not a racemic mixture of its enantiomorphic stereoisomers is 'optically active' to some degree, i.e. it exhibits the phenomenon of circular birefringence whereby the plane of polarization of a beam of linearly polarized light passing through the substance is rotated by an amount proportional to the distance traveled. Dextrose (otherwise known as *d*-glucose, one of the 16 stereoisomers of pentahydroxyaldehyde) and *d*-tartaric acid (used in soft drinks) are nutritious examples of optically active substances. Pure or in solution, they both rotate the plane of polarization counterclockwise relative to the direction of propagation; their mirror images do the opposite and, incidentally, are not nutritious. Some crystals, e.g. natural quartz, exhibit both optical activity and birefringence.

In 1845 Faraday (1846) discovered that a magnetic field in the direction of propagation of a light beam in a transparent medium produces the effects of circular birefringence. Thus he established for the first time a direct connection between optics and electromagnetism - a connection he had long suspected to exist and for which he had searched for many years. However, a half century passed before an explanation of the Faraday effect was formulated in terms of Maxwell's electromagnetic theory of light (developed in the 1860's) and the concept of the atomicity of charge which had evolved during the 19^{th} century under the influence of electrochemistry and been validated by Thomson's discovery of the electron in 1897.

CLASSICAL THEORY OF THE FARADAY EFFECT

Representation of Linearly Polarized Light as a Superposition of Circularly Polarized Components

The electric vector $\boldsymbol{E}$ of a plane wave of linearly polarized light of frequency ω traveling in the +X direction with its polarization in the Y direction can be represented by the expression

$$\boldsymbol{E}(x,t) = \boldsymbol{E}_r + \boldsymbol{E}_l, \tag{1a}$$

where

$$\boldsymbol{E}_r(x,t) = \{0, (A/2)\cos[\omega(t - nx/c)], (A/2)\sin[\omega(t - nx/c)]\} \tag{1b}$$

and

$$\boldsymbol{E}_l(x,t) = \{0, (A/2)\cos[\omega(t - nx/c)], -(A/2)\sin[\omega(t - nx/c)]\} \tag{1c}$$

represent the electric vectors in plane waves of right and left circularly polarized light, respectively. (Note that the sum of the z components is zero.) The amplitude of $\boldsymbol{E}$ is A, and the velocity of propagation is c/n where n is the index of refraction. Thus a beam of linearly polarized light can be considered as a linear superposition of right and left circularly polarized components.

Under certain circumstances the velocities of right and left circularly polarized light may be different, as in *d*-glucose or in a substance in which there is a magnetic field. In such a circumstance one can calculate separately the propagation of the two components and then recombine them to obtain the electric vector of the resultant linearly polarized wave. The net effect, as we now show, is a rotation of the plane of polarization.

To indicate a difference in velocities for right and left circularly polarized waves we replace the index n in equations (1b) and (1c) by n_r and n_l, respectively. As illustrated in Figure 1, the electric vector on the y-z plane at $x=D$ is then

$$\boldsymbol{E}(D,t) = \boldsymbol{E}_r(D,t) + \boldsymbol{E}_l(D,t). \tag{2a}$$

Using the identities

$$\cos(\alpha) + \cos(\beta) = 2\cos(\alpha/2 + \beta/2)\cos(\alpha/2 - \beta/2),$$

and

$$\sin(\alpha) - \sin(\beta) = 2\cos(\alpha/2 + \beta/2)\sin(\alpha/2 - \beta/2),$$

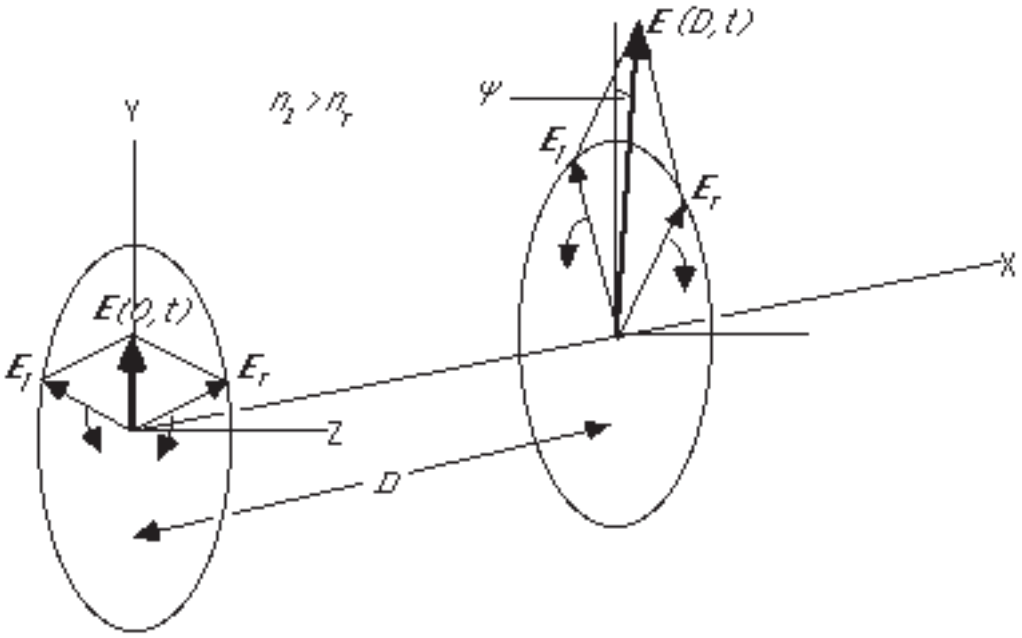

Figure 1. Addition of right and left circularly polarized components on two planes defined by x=0 and x=D, respectively. Because $n_l > n_r$, E_l is more retarded in phase at x=D relative to x=0 than E_r, with the result that their sum at x=D is rotated clockwise by the angle $\psi = \pi(n_l - n_r)D/\lambda$..

we find for the electric field at $x = D$ the expression

$$\boldsymbol{E}(D,t) = \{0,\ A\cos[\omega(t-(n_r+n_l)D/2c)]\cos[\omega(n_l-n_r)D/2c],$$
$$A\cos[\omega(t-(n_r+n_l)D/2c)]\sin[\omega(n_l-n_r)D/2c]\}. \qquad (2b)$$

This represents an oscillating electric field of amplitude A inclined with respect to the y-axis by an angle

$$\psi = \tan^{-1}\left(\frac{E_z}{E_y}\right) = \frac{\omega(n_l-n_r)D}{2c} = \frac{\pi(n_l-n_r)D}{\lambda}, \qquad (3)$$

where λ is the wavelength of the light in vacuum. Thus circular birefringence can be explained as an effect of a difference in the propagation velocities of right and left circularly polarized light. The difference is generally very small, amounting to only a few parts per million in optically active materials as well as in the Faraday effect at moderate magnetic field strengths. However, in a

typical setup $D/\lambda >>1$ so that even minute differences between n_l and n_r yield values of ψ which are large enough to be measured accurately.

MAGNETIC CIRCULAR BIREFRINGENCE – BECQUEREL'S THEORY OF THE FARADAY EFFECT

Experiments established early on that the Faraday rotation angle ψ is proportional to the product of D and the magnetic field B. Thus

$$\psi = VDB , \tag{4}$$

where V is a proportionality factor called the Verdet constant. The question remained as to how one might calculate the value of V from the fundamental physics of light, magnetism, and matter.

Up to this point in our development we have only used the two ideas; 1)light is a linearly superposable transverse wave with velocity c/n, and 2) $n_r \neq n_l$. The first step in understanding the Faraday effect in terms of electromagnetic theory and the atomic structure of matter was taken by H. Becquerel, in 1897. His theory was based on the concept of the newly discovered electron, but it did not include the idea of quantized energy states introduced by Planck in 1900, or the concept of light quanta, developed by Einstein in 1905. Even though the Becquerel theory does not make use of quantum principles, it does provide a simple conceptual framework for a preliminary understanding of the phenomenon, and yields quantitative predictions for the Verdet constant that are remarkable close to the measured values.

The next substantial steps beyond this theory were not taken until the early 1930's, but then only in gases (Serber, 1932; Van Vleck, 1932). A large amount of theoretical and experimental work has been done on the Faraday effect in solids since 1960, because of its application to semiconductors and the construction of microwave switching devices. Recent theoretical treatments are very complex, as a perusal of reviews (e.g. Mavroides 1972) will show.

Incidentally, the theory of the Faraday effect in ionized gas, where the active agents are free electrons, is not so complicated. The effect is important in radio astronomy because it provides a measure of the average value of the product $B_r n_e$ along the line of sight to a source of linearly polarized radio emission, where B_r is the component of the magnetic field along the line of sight and n_e is the free electron density (typically $B_r \approx 10^{-6}$ gauss, $n_e \approx 0.1$ cm^{-3}).

Following Becquerel and subsequent presentations of his theory (see e.g. Rossi 1957), we assume that a transparent material contains particles of mass m and charge q embedded in a continuum of opposite charge and restrained by elastic force to vibrate about fixed sites. We consider the situation in which there is a steady magnetic field $\boldsymbol{B}$ of magnitude B in the +X direction and a plane right circularly polarized electromagnetic wave which, at a given point,

produces a rapidly varying electric field $\boldsymbol{E}$ of constant magnitude E rotating clockwise with respect to $\boldsymbol{B}$ in a plane perpendicular to $\boldsymbol{B}$.In the steady state the charged particles move in a circle of radius r governed by the equation (cgs units)

$$-m\omega^2 r = -kr + Eq + Bq\omega r/c \ , \qquad (5)$$

where k is the "spring constant" of the restraining force. For left circularly polarized light the sign of the magnetic term is reversed. Equation (5) is readily solved to obtain for r the formula

$$r = (Eq/m)/(\omega_0{}^2 - \omega^2 - Bq\omega/mc) \ , \qquad (6)$$

where

$$\omega_0 = (k/m)^{1/2} \ .$$

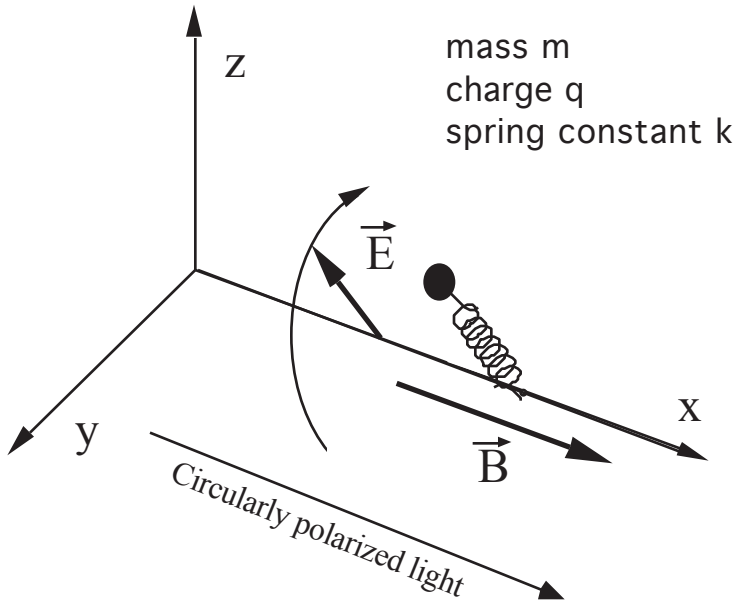

Figure 2. The particle, with mass m and charge q, is fixed to the end of a spring, with spring constant k, attached rigidly to the x axis. The directions of light propagation and the **B** field are parallel to the x axis; the **E** field is perpendicular to the x axis and rotates in the yz plane.

Displacement of a charge q from its oppositely charged equilibrium site by the distance r creates an electric dipole of magnitude qr. If there are N such dipoles per unit volume, then the polarization $\boldsymbol{P}$ has the magnitude Nqr, and the permittivity is $\varepsilon = (1+4\pi \boldsymbol{P}/\boldsymbol{E})$. Therefore, in a dielectric where the permeability $\mu=1$ and the propagation velocity $v=c/\varepsilon^{1/2}$ and in the presence of a magnetic field, we obtain for the index of refraction for right circularly polarized light the expression

$$n_r(\omega) = c/V = [1 + (4\pi Nq^2/m)/(\omega_0^2 - \omega^2 - B\omega q/mc)]^{1/2}, \quad (7a)$$

and for left circularly polarized light

$$n_l(\omega) = c/V = [1 + (4\pi Nq^2/m)/(\omega_0^2 - \omega^2 + B\omega q/mc)]^{1/2}. \quad (7b)$$

Equations (7) show that the indices "blow up" as the frequency of the light approaches a value such that $\omega_0^2 - \omega^2 \pm B\omega q/mc = 0$. (One can avoid this unphysical consequence of our simple assumptions by adding to the equation of motion a damping term representing a drag force proportional to the velocity of the charged particle. Such a term can give a classical account of the phenomenon of absorption. Both the indices of refraction and the absorption coefficient have maxima near ω_0. Fortunately, omission of a damping term has very little effect on the accuracy of the theory at frequencies far from ω_0, as in the present case.) The sign of the effect of the magnetic field on the indices of refraction for circularly polarized light depends on the sign of the product $B\omega q$. In particular, for right circular polarization with $\boldsymbol{B}$ in the direction of propagation ($B_x>0$) and negatively charged particles, $B\omega q<0$, which implies a decrease in n_r and a corresponding increase in n_l . Therefore, under these assumptions $(n_l - n_r)>0$ and, according to equation (3), $\psi > 0$.

The change of n_r caused by turning on a magnetic field B is equal to the change of n_r when the frequency of the light is changed from ω to $\omega + \Delta\omega_r$, where $\Delta\omega_r$ is defined by the quadratic equation

$$\omega^2 + B\omega q/mc = (\omega + \Delta\omega_r)^2 .$$

The solution of this equation is

$$\Delta\omega_r = Bq/2mc. \quad (8a)$$

Similarly,

$$\omega^2 - B\omega q/mc = (\omega + \Delta\omega_l)^2$$

and

$$\Delta\omega_l = -Bq/2mc. \quad (8b)$$

These $\Delta\omega$'s are very small compared to ω. Thus the difference in the indices of refraction can be written with high accuracy as

$$n_l - n_r = (dn/d\omega)(\Delta\omega_l - \Delta\omega_r) = (dn/d\lambda)(\lambda^2/2\pi c)(Bq/mc), \qquad (9)$$

where $\lambda=2\pi c/\omega$ is the vacuum wavelength of the light. Combining equations (3) and (9) we obtain for the angle of Faraday rotation

$$\psi = \frac{dn}{d\lambda}\ \frac{\lambda}{2c^2}\ \frac{q}{m}\ DB\ . \qquad (10)$$

Thus the Verdet constant is related to the constants of the interacting particles by the formula

$$V = \frac{dn}{d\lambda}\ \frac{\lambda}{2c^2}\ \frac{q}{m}\ . \qquad (11)$$

Equation (11) is the formula derived by H. Becquerel. It was characterized by Van Vleck (1932) as "rather too simple". Indeed, the quantum treatment of the problem in even the simplest of materials is complicated, and the results depend critically on the nature of the medium, i.e. whether it is dielectric, semiconducting, diamagnetic, paramagnetic, ferromagnetic, etc. Of course this means that measurements of the Faraday effect can be of value in the study of the electronic structure of matter provided an adequate theoretical framework for its interpretation is available.

We will bury the complications by defining a constant C such that equation (11) is written as

$$V = -\frac{dn}{d\lambda}\ \frac{\lambda}{2c^2}\ C\frac{e}{m_e}\ , \qquad (12)$$

where $-e$ and m_e are the charge and mass of the free electron. It is reassuring that for visible light in ordinary molecular hydrogen gas, H_2, the value of C is 0.99. For other substances and other spectral bands the values of C can be substantially different from unity and, in some cases, even negative. In the case of the interstellar medium where the effect is due to the presence of free electrons one has $\omega_0=0$ and $C=1$.

To get a rough idea of what to expect in this experiment we estimate the angle of rotation under the following set of assumptions:

a) Suppose $n=2.0$ at $\lambda=5000$ Å, and the peak absorption is at $\lambda_0=2000$Å.

b) $B=2000$ gauss and $D=10$ cm.

c) $C=1.0$.

For convenience we rewrite equation (7) to express the dependence of the index of refraction on wavelength ($B=0$) and obtain the formula

$$n = \left\{ 1 + \frac{K}{1 - (\lambda_0/\lambda)^2} \right\}^{1/2} \qquad (13)$$

where $\lambda_0 = 2\pi c/\omega_0$ and $K = 4\pi N q^2/m\omega_0^2$. Assumption a) implies $K = 2.52$. Differentiating equation (13) we find $\lambda(dn/d\lambda) = -0.286$. Substituting this in equation (10) with $q = -4.8 \times 10^{-10}$esu and $m = 9.1 \times 10^{-28}$g, we find $\psi = +1.67$ radians or +96°.