Chapter 6

More on polarized light

In Chapter 4 it was noted that the experiments carried out by Fresnel and Young led to the discovery of transverse character of light that could satisfactorily describe the phenomenon of interference using polarized light. The basic definition of state of polarization of light via the polarization ellipse of the transverse EM field was briefly introduced in Chapter 4. When such polarized light passes through any anisotropic medium, its polarization state is transformed depending upon medium characteristics. The variations in the state of polarization of a wave thus enable us to characterize the system under consideration. A number of mathematical formalisms have been developed over the years to deal with the propagation of polarized light and its interaction with optical systems. Among these, the Jones calculus and the Stokes-Mueller calculus have been the most widely used. The former is a field-based model that assumes coherent addition of the phase and amplitude of EM waves, and the latter is an intensity-based model that instead utilizes the incoherent addition of wave intensities. In this chapter we define the various states of polarization of light waves and discuss the interaction of such polarized light with material media using the Jones and Stokes-Mueller calculus. We also briefly introduce the concepts of polarimetric measurements and touch upon representative applications of experimental polarimetry.

6.1 State of polarization of light waves

As introduced in Chapter 4, the classical concept of polarized light represents the state of polarization of a transverse electromagnetic (EM) wave by the evolution of transverse electric field vector **E** as a function of time at a given point of the space. If the vector extremity describes a stationary curve in their temporal evolution, the wave is *polarized*. Accordingly, the shape of the curve traced out by the **E** vector defines the polarization state of the wave in question. On the other hand, if the vector extremity follows random paths during the observation or measurement time, it is *unpolarized*. In the corresponding quantum mechanical description, it is assumed that each individual photon (energy quanta) is polarized, and its associated state vector corresponds to one of the classical polarization states. When a large number of photons are considered, their collective behavior is consistent with the classical limit (the wave solution to Maxwell's equations). In the case when all of the photons exhibit the same polarization, the light is said to be completely polarized. On the other hand, when there are photons of different polarizations but with a distribution favoring one particular state, the light is *partially polarized*, and when the photons are uniformly distributed over all possible polarization states, the light is said to be *unpolarized*. Nevertheless, the quantum polarization state vector for the photon is analogous to the Jones vector (described subsequently) in its classical description. Thus the quantum mechanical view of polarization and the corresponding classical formalisms are mutually consistent.

6.1.1 Jones vector representation of pure polarization states

In the classical description, the electric field vector of any transverse EM plane monochromatic wave of frequency ω, propagating along the z direction, can be expressed in terms of the two orthogonal components (x and y; note that other orthonormal coordinates are possible) in the right-handed Cartesian coordinate system as [28, 29, 30, 31, 32]

$$\mathbf{E}(z,t) = \begin{bmatrix} E_{0x}\cos(kz-\omega t-\delta_x) \\ E_{0y}\cos(kz-\omega t-\delta_y) \end{bmatrix}, \tag{6.1}$$

with

$$k = (n' + in'')\frac{\omega}{c}. \tag{6.2}$$

which is the modulus of the propagation vector $\mathbf{k}$, c is the speed of light in vacuum, and n' and n'' are the real and imaginary parts of the refractive index, which determine the speed of light and the absorption in the medium, respectively.

The *polarization* of the wave is defined by the shape of the trajectory described by E in the xy plane. This shape depends on the ratio of the amplitudes $\tan\nu$ and the phase difference δ, defined as

$$\tan\nu = \frac{E_{0y}}{E_{0x}}; \delta = \delta_y - \delta_x. \tag{6.3}$$

This trajectory is in general elliptical and is represented in Fig. 6.1. Besides the parameters defined above, the ellipse can also be described by the orientation (azimuth) α of its major axis and its ellipticity ϵ, which is positive (negative) for left- (right-) handedness. The ellipticity ϵ varies between the two limits of zero (linearly polarized light) and $\pm 45°$ (circularly polarized light), representing the two limits of generally elliptical polarization. R. Clark

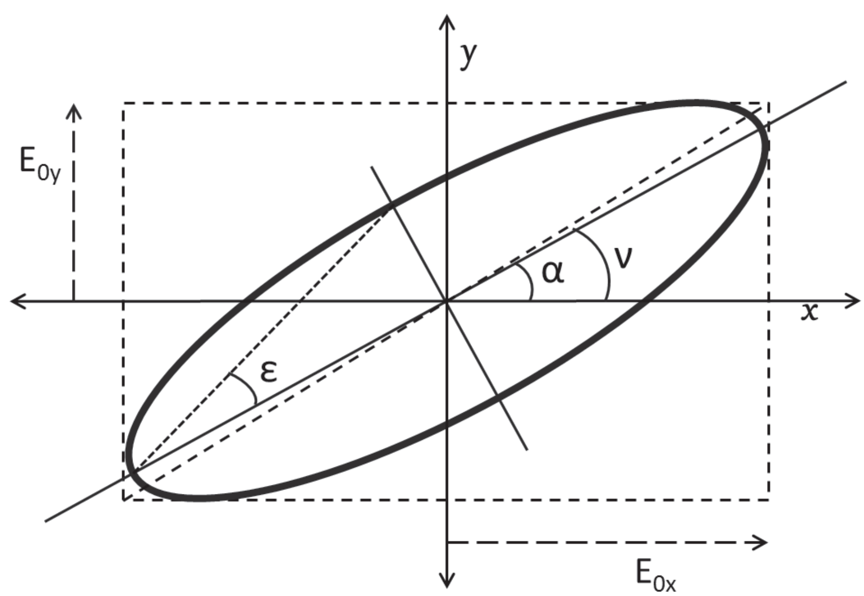

FIGURE 6.1: The polarization ellipse of a wave propagating in the z direction. Here E_{0x} and E_{0y} are the amplitudes of the x and y field oscillations; their ratio is given by $\tan\nu$. The parameter α is the azimuth of the major axis of the ellipse and ϵ is its ellipticity. ϵ is positive or negative for left- or right-handed polarization states, respectively.

Jones represented (between 1941 and 1947) the polarization state of a quasi-monochromatic transverse plane wave by a two-dimensional column matrix or a vector whose elements are complex amplitudes of the field vector along the two orthogonal directions, known as the Jones vector [33]. Accordingly, the Jones vector is defined as

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x} \exp(-i\delta_x) \\ E_{0y} \exp(-i\delta_y) \end{pmatrix}. \tag{6.4}$$

Depending on the relative amplitudes and phases of the two orthogonal components of the electric field in Eq. (6.4), the Jones vectors corresponding to the different pure polarization states are listed in Table 6.1 (with H, V, P and M, for linear polarizations along the horizontal, vertical, $+45°$and $-45°$ directions, respectively and L and R for left and right circular polarizations).

The intensity of a fully polarized wave characterized by the Jones vector is given by

$$I = I_x + I_y = \frac{1}{2}(E_{0x}^2 + E_{0y}^2) = \frac{1}{2}(E \cdot E^*), \tag{6.5}$$

where E^* is the conjugate of E.

Experimentally, one can determine the azimuth α of a linearly polarized light beam propagating along the z direction by observing its *extinction* through a linear analyzer set perpendicular to α. This type of characterization may also be extended to elliptically polarized beams as illustrated in Fig. 6.2.

To determine the ellipticity ϵ, a quarter-wave plate (QWP) is inserted in the beam path with its slow axis oriented at the azimuth α. Due to the $90°$ phase shift introduced by the QWP, the initial elliptical polarization state is transformed into a linear one, oriented at $\alpha + \epsilon$ from the x reference axis. Then, a linear analyzer with its pass axis oriented at ϵ from the fast axis of the QWP will lead to complete extinction of the beam. In practice, the extinction is achieved by a trial-and-error procedure, and the azimuth α and the ellipticity ϵ are eventually determined from the angular settings of the quarter-wave plate and the analyzer when maximum extinction is obtained.

Note that this vectorial description of polarization state enables the matrix treatment for describing the polarizing transfer of light in its interaction with any medium. An optical element, like a retardation plate or a partial polarizer, is therefore represented by a 2×2 matrix, whose four elements are generally complex. This can be represented as [33]

$$\begin{gathered} \mathbf{E}' = J\mathbf{E}, \\ \begin{pmatrix} E'_x \\ E'_y \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \end{gathered} \tag{6.6}$$

where J is a 2×2 complex matrix, known as the Jones matrix of the interacting medium, and $\mathbf{E}$ and $\mathbf{E}'$ are the input and the output Jones vectors of light, respectively. Applying the associative properties of matrices, the matrix operator equivalent to a combination of several optical elements can then be

TABLE 6.1: Usual polarization states: Jones vectors, azimuths, ellipticities and shapes of the ellipses.

State	**H**	**V**	**P**	**M**	**L**	**R**	**Elliptical**
E	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$	$\begin{pmatrix} \cos\alpha\cos\epsilon - i\sin\alpha\sin\epsilon \\ \sin\alpha\cos\epsilon + i\cos\alpha\sin\epsilon \end{pmatrix}$
α	0	90°	+45°	−45°	Undefined	Undefined	α
ϵ	0	0	0	0	+45°	−45°	ϵ
Shape of the ellipse	$\longrightarrow$	$\uparrow$	$\nearrow$	$\nwarrow$	$\circlearrowleft$	$\circlearrowright$	$\downarrow$

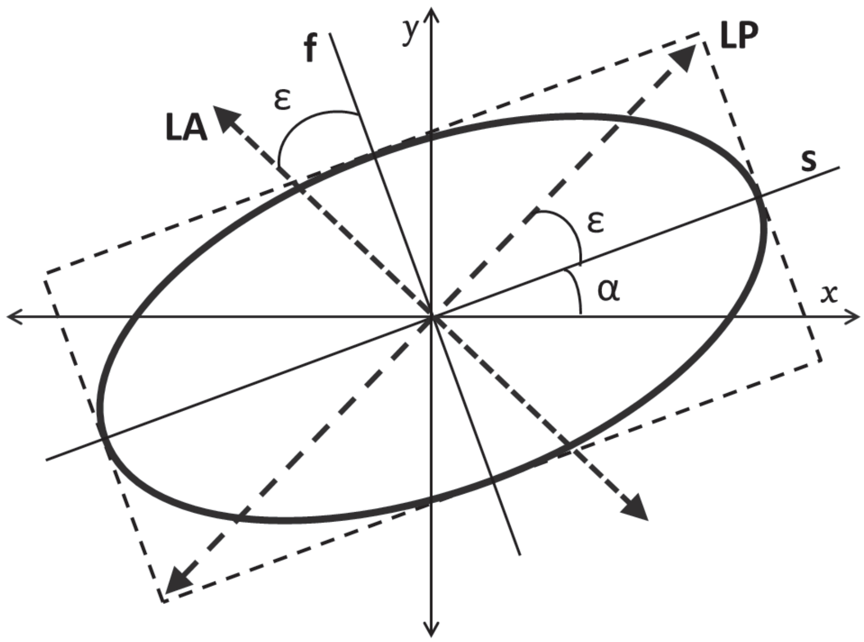

FIGURE 6.2: Extinction method for the analysis of arbitrary elliptical polarizations. The input elliptical polarization is transformed into a linear polarization state (**LP**) by inserting a quarter-wave plate with its slow axis **s** oriented at azimuth α. Complete extinction is then observed by setting a linear analyzer **LA** at perpendicular orientation to **LP**. The ellipticity ϵ is then measured as the angle between the analyzer axis for extinction and the fast axis **f** of the quarter-wave plate.

easily determined. It is the result of the multiplication of the matrices of each optical element, in the same order as that of light passing through. Hence, the Jones vector of an optical wave that emerges from a system of n optical systems can be written as

$$J = J_n J_{n-1} \dots J_2 J_1. \tag{6.7}$$

We shall address the Jones matrices corresponding to various polarization transforming interactions of medium in subsequent sections.

While theoretically interesting, the Jones formalism is limited in that it can only describe pure polarization states (completely polarized waves), and it is thus ill-suited for applications in which it is necessary to consider partial polarization or depolarizing interactions (polarization loss). Yet, quasi-monochromatic radiation is not necessarily completely polarized and many of the naturally occurring optical materials tend to be depolarizing. Such general cases can be better addressed by the coherency matrix and the Stokes-Mueller formalisms, as we describe next.

6.1.2 Partially polarized states

The previous subsection dealt with completely polarized waves. In such an idealized situation, the transverse components of the optical field (E_x and E_y) describe a perfect polarization ellipse (or some special form of an ellipse, such as a circle or a straight line, depending upon the relative amplitudes and phases) in the xy (transverse) plane. Note that the time scale at which the light vector traces out an instantaneous ellipse is of the order of 10^{-15} seconds. This period of time is clearly too short to allow us to follow the tracing of the ellipse. This fact, therefore, immediately prevents us from ever observing the polarization ellipse. More important, such a description is only applicable for light that is completely polarized, waves for which transverse components of the field amplitudes E_{ox}, E_{oy} and the associated phases δ_x and δ_y can be considered as constant during the measurement time. Yet, in nature, light is very often unpolarized or partially polarized. Thus, the polarization ellipse is an idealization of the true behavior of light; it is only correct at any given instant of time. These limitations force us to consider an alternative description of polarized light in which only observed average values or measured quantities ('intensities' rather than instantaneous field) enter.

Before we invoke mathematical formulation of partial polarization states via the measurable intensities (time average of the square of the amplitude), it might be useful to gain some qualitative idea on the phenomenon of partial polarization (or depolarization) from practical extinction measurements. For example, if we try the extinction method to characterize natural light directly coming from a source, such as the sun or a light bulb, the detected intensity can be independent of the settings of the quarter-wave plate and the analyzer. We can thus conclude that the light coming from the sun or the light bulb is *totally depolarized*. In other cases—for example, the light coming from a bulb but reflected from a floor en route to the observer—the intensity detected through the quarter-wave plate and the analyzer may vary between I_{min} and I_{max}. This provides an experimental definition of the *degree of polarization* (DOP) of the light beam, :

$$DOP = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}. \tag{6.8}$$

For totally polarized states, I_{min} vanishes leading to $DOP = 1$. At the other extreme, for totally unpolarized light, $I_{min} = I_{max}$ and $DOP = 0$. For partially polarized states, on the other hand, the DOP may take any intermediate values between zero and one. For such partially polarized states, the motion of the electric field in the xy plane is no longer a perfect ellipse, but rather a somewhat *disordered* one. In case of a totally random motion of the electric vector $\mathbf{E}$, in the extinction procedure the analyzer would detect the same

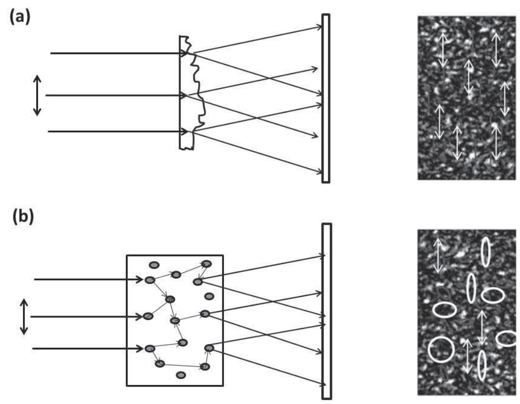

FIGURE 6.3: Scattering of a linearly polarized coherent light beam by static samples. Top: single scattering from an optically thin sample. The state of polarization of the speckle spots remains the same as that of the incident beam. Bottom: multiple scattering by an optically thick sample. The state of polarization varies considerably from speckle to speckle.

constant intensity. What is implicitly assumed in this description is that the light polarization may be defined at any instant, but may vary over time scales much shorter than the integration time of the detector. As a result, this detector takes the *temporal averages of the intensities*, which is sequentially generated by *different totally polarized states*. We note here that that the averaging of intensities (i.e., the *incoherent* sum) of polarized contributions is not necessarily temporal (it may be *spatial* as well, as illustrated below).

Consider the scattering experiments shown in Fig. 6.3. In one case (top panel) the object is optically thin and the laser undergoes single scattering by the rough surface. In the other case, the object is optically thick leading to strong multiple scattering effect. In both cases, the incident laser beam is spatially coherent, and the scattering objects are static (we ignore for the moment any possible thermal/Brownian motions). It is well known that in these conditions we can observe a speckle pattern in the screen due to the interferences (at each point of the screen) of many scattered waves having random (but static) amplitudes and relative phases.

The major difference between single and multiple scattering regimes is that for the former, the polarization of all scattered waves is the same as that of the incident wave, while the polarization states become random in the case of multiple scattering. Consequently, as outlined in Fig. 6.3, all the speckles feature the same polarization as the incident laser for single scattering, while in the other case, each speckle is still fully polarized, but this polarization varies randomly from one speckle to the next.

To summarize, *true* depolarization requires that the detected signal is the sum of intensities due to various polarized contributions with different state of polarization. The summing may take place temporally, spatially or even spectrally, and it depends not only on the sample itself but also on the characteristics of the illumination beam and of the detection system.

In 1852 Sir George Gabriel Stokes discovered that the polarization behavior could be represented in terms of observables [34]. He found that any state of polarized light could be completely described by four measurable quantities now known as the Stokes polarization parameters. As we saw earlier, the amplitude of the optical field cannot be observed; rather, the quantity that can be observed is the intensity, which is derived by taking a time average of the square of the amplitude. This suggests that if we take a time average of the unobserved polarization ellipse, we will be led to the observables of the polarization ellipse. As we shall show shortly, these observables of the polarization ellipse (measured as four sets of intensity values) are exactly the Stokes polarization parameters. Importantly, these Stokes parameters can encompass any polarization state of light, whether it is natural, totally or partially polarized (and can thus deal with both polarizing and depolarizing optical interactions). Before we address that, we shall introduce the concept of the *coherency matrix*, which deals with the time-averaged description of the transverse field components (amplitudes and phases). The definition of degree of polarization (DOP) will be introduced via this so-called coherency matrix formalism and it will be shown that the four measurable Stokes polarization parameters actually follow from combinations of the various elements of the coherency matrix.

6.1.3 Concept of 2×2 coherency matrix

The coherency matrix (also called the matrix of polarization in the literature) includes partial polarization effects by taking the *temporal average* of the direct product of the Jones vector by its Hermitian conjugate. In this way, the 2×2 coherency matrix ϕ is defined as [30, 35, 36, 37]

$$\phi = \left\langle \mathbf{E} \otimes \mathbf{E}^{\dagger} \right\rangle = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix}, \tag{6.9}$$

where $\langle \cdots \rangle$ denotes temporal (ensemble) average, $\otimes$ denotes the tensorial or Kronecker product, $\mathbf{E}^*$ is the conjugate of $\mathbf{E}$ and $\mathbf{E}^{\dagger}$ is the transpose conjugate of the Jones vector $\mathbf{E}$. The two defining properties of the coherency matrix are its Hermiticity ($\phi = \phi^{\dagger}$, by its definition) and non-negativity ($\phi \geq 0$): every 2×2 matrix obeying these two conditions is a valid coherency matrix and represents some physically realizable polarization state. The non-negativity condition for this 2×2 matrix can also be written as

$$tr\phi > 0 \text{ and } det\phi \geq 0. \tag{6.10}$$

It is pertinent to note that the trace of the coherency matrix ($tr\phi$) represents an experimentally measurable quantity, the total intensity of light that corresponds to the addition of the two orthogonal component intensities. As it will be discussed shortly (in context with the Stokes parameters), this corresponds to the sum of intensities measured using two orthogonal orientations of a polarizer. Thus, the first non-negativity condition of the coherency matrix ($tr\phi > 0$) directly follows from the non-negativity of total intensity. The second condition ($det\phi \geq 0$), on the other hand, follows from the limiting condition of degree of polarization ($0 \leq DOP \leq 1$). This can be understood by noting that the off-diagonal elements of ϕ are defined by taking the time average over the product of a field component with the conjugate of its transverse component. The quantity $det\phi$ therefore represents the fluctuations in the phases of the field components. As we can easily see, for a perfectly coherent source (where the phases of the transverse field components and their difference can be considered as a constant over a finite measurement time), the determinant of the coherency matrix should vanish ($det\phi = 0$). This corresponds to the fully polarized light (the ideal situation that we discussed in context with the Jones formalism and polarization ellipse). For partially coherent (or incoherent) sources, on the other hand, $det\phi > 0$, representing partially (mixed) polarized states or even completely unpolarized states. In fact, the quantity $det\phi$ (its square root, rather, as we shall show later) is a quantitative measure of the unpolarized intensity component of any partially polarized light (the natural intensity component that is independent of the polarizer/wave plate orientation in the experiment described in the beginning to define the partial polarization states). This definition of the state of polarization via the coherency matrix thus enables us to relate the DOP of light to the coherence characteristics of the source. It follows that the coherence property of the source itself limits the maximum achievable polarization ($DOP = 1$ can only be produced by an idealized perfectly coherent source). The definition of DOP can in fact be invoked from the coherency matrix using the ratio of determinant of ϕ (representing the completely unpolarized component of intensity) and the trace of ϕ (representing the total intensity) as

$$DOP = \frac{I_{pol}}{I_{tot}} = \sqrt{1 - \frac{4det(\phi)}{[Tr(\phi)]^2}}. \tag{6.11}$$

Here, I_{pol} is the polarized fraction of the intensity and I_{tot} is the total intensity. As we can observe now, the empirical definition of DOP (Eq. 6.8), which was introduced rather naively in context to the experiment discussed in the previous section, is consistent with the definition of DOP from the coherency matrix. It is clear that fully polarized light corresponds to $det\phi = 0$ ($DOP = 1$) and partially polarized or mixed polarization states correspond to $det\phi > 0$ ($DOP < 1$). The physical significance of these and other relevant issues dealing with DOP will become more apparent when we discuss (in the following section) the relationship between the coherency matrix elements and the experimentally measurable Stokes polarization parameters. In passing, we note that for polarization preserving (nondepolarizing) interactions, the transformation of $\phi(\phi \to \phi')$ (the changes in polarization state of light represented by coherency matrix transformation) can be represented by the action of the Jones matrix J as $\phi' = J\phi J^\dagger$, implying Jones systems map pure states ($det\phi = 0$) into pure states. Depolarizing transformation involving mixed (partial) polarization states ($det\phi > 0$), on the other hand, is handled by Stokes-Mueller formalism, as discussed subsequently in Section 6.2.3.

6.1.4 Stokes parameters: Intensity-based representation of polarization states

Following the presentation above, polarized states are not characterized in terms of well-determined field amplitudes, but rather by intensities (the time average of the square of the field amplitudes). These measurable intensities are grouped in a 4×1 vector (four row, single-column array) known as the Stokes vector **S**, which is sufficient to characterize any polarization state of light (pure, partial or unpolarized). These are defined as [30, 34, 35, 36]

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle \\ \langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle \\ \langle E_x E_y^* \rangle + \langle E_y E_x^* \rangle \\ i\left(\langle E_y E_x^* \rangle - \langle E_x E_y^* \rangle\right) \end{bmatrix} = \begin{bmatrix} \langle E_{0x}^2 + E_{0y}^2 \rangle \\ \langle E_{0x}^2 - E_{0y}^2 \rangle \\ \langle 2E_{0x}E_{0y}cos\delta \rangle \\ \langle 2E_{0x}E_{0y}sin\delta \rangle \end{bmatrix}, \quad (6.12)$$

where once again, $\langle \cdots \rangle$ denotes temporal average and the electric field components (E_{0x} and E_{0y}) and the corresponding phase difference $\delta(= \delta_y - \delta_x)$ are also temporally averaged over the measurement time. As apparent, these four Stokes parameters are real experimental quantities (intensities) typically measured with conventional square-law photo-detectors, usually in energy-like dimensions. I is the total detected light intensity that corresponds to the addition of the two orthogonal component intensities; Q is the difference in intensity between horizontal and vertical polarization states; U is the difference between the intensities of linear $+45°$ and $+45°(135°)$ polarization states; and V is the difference between intensities of right circular and left circular polarization states (note that if a difference was replaced by a sum in any of these pairs, total intensity I would result). Thus these parameters can be directly determined by the following six intensity measurements (I) performed with ideal polarizers: I_H, horizontal linear polarizer (0°); I_V, vertical linear polarizer (90°); I_P, 45° linear polarizer; I_M, 135° (−45°) linear polarizer; I_R, right circular polarizer, and I_L, left circular polarizer.

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} I_H + I_V \\ I_H - I_V \\ I_P + I_M \\ I_R - I_L \end{bmatrix}. \quad (6.13)$$

TABLE 6.2: Normalized Stokes vectors for usual totally polarized states (of Table 6.1)

State	**H**	**V**	**P**	**M**	**R**	**L**	**Elliptical**
S	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ \cos 2\alpha \cos 2\epsilon \\ \sin 2\alpha \cos 2\epsilon \\ -\sin 2\epsilon \end{bmatrix}$

Also note that **S** is not a vector in the geometric space; rather, this array of intensity values represent a directional vector in the polarization state space (the Poincaré sphere, described subsequently). For totally polarized states defined by Jones vectors of the form given by Eq. (6.4), the corresponding Stokes vectors are

$$\mathbf{S} = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}. \tag{6.14}$$

Usually, Stokes vectors are represented in intensity normalized form (normalized by the first element I). The normalized Stokes vectors for fully polarized states are listed in Table 6.2. At the other extreme, for *totally unpolarized states*, $Q = U = V = 0$, which corresponds to the fact that no matter how the analyzer is oriented, for such states the transmitted intensity is always the same, equal to one-half of the total intensity.

Using this formalism, the following polarization parameters of any light beam are defined:

- net degree of polarization

$$DOP = \frac{\sqrt{(Q^2 + U^2 + V^2)}}{I}, \tag{6.15}$$

- degree of linear polarization

$$DOP = \frac{\sqrt{(Q^2 + U^2)}}{I}, \tag{6.16}$$

- degree of circular polarization

$$DOP = \frac{V}{I}. \tag{6.17}$$

Note that the degree of polarization of light should not exceed unity. This therefore imposes the following restriction on the Stokes parameters,

$$I \geq \sqrt{Q^2 + U^2 + V^2}, \tag{6.18}$$

where the equality and the inequality signs correspond to completely and partially polarized states, respectively. It is worth noting that this restriction of DOP actually originates from the non-negativity condition of the coherency matrix (Eq. (6.10)), which implies the physical realizability of any polarization state. Moreover, the definition of DOP (Eq. (6.15)) is also commensurate with the definition based on the coherency matrix elements (Eq. (6.11)). In order to understand this, we relate the Stokes vector elements with the elements of the coherency matrix (Eq. (6.9)) as

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle \\ \langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle \\ \langle E_x E_y^* \rangle + \langle E_y E_x^* \rangle \\ i\left(\langle E_y E_x^* \rangle - \langle E_x E_y^* \rangle\right) \end{bmatrix} = \begin{bmatrix} \phi_{xx} + \phi_{yy} \\ \phi_{xx} - \phi_{yy} \\ \phi_{xy} + \phi_{yx} \\ i(\phi_{yx} - \phi_{xy}) \end{bmatrix}. \tag{6.19}$$

In the literature, the coherency matrix is also sometimes written as a 4×1 vector (four row, single-column array, analogous to the Stokes vector) and is denoted as the coherency vector $\mathbf{L}$. Thus, $\mathbf{S}$ and $\mathbf{L}$ are related by the 4×4 matrix A as

$$\mathbf{S} = A \begin{bmatrix} \phi_{xx} \\ \phi_{xy} \\ \phi_{yx} \\ \phi_{yy} \end{bmatrix} = A\mathbf{L}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{pmatrix}. \tag{6.20}$$

We are now in a position to inspect the non-negativity condition of 2×2 coherency matrix and the corresponding definition of DOP via the coherency matrix elements. By performing simple algebraic manipulations using Eq. (6.20), we can see that the determinant of the coherency matrix can be written in terms of the Stokes parameters as

$$det\phi = \frac{1}{4}\left[I^2 - (Q^2 + U^2 + V^2)\right]. \tag{6.21}$$

Thus, the non-negativity condition of coherency matrix is equivalent to the condition that the DOP should not exceed unity (Eq. (6.18)). Moreover, the quantity $(Q^2 + U^2 + V^2)^{1/2}$ signifies polarized component of the detected intensity (as each of the quantities, Q, U and V are differences in intensities between orthogonal polarizations). Thus, either in Eq. (6.11) or in Eq. (6.15), the degree of polarization is defined as the ratio of the polarized component of the detected intensity (I_{pol}) to the total detected intensity (I_{tot}). It thus follows, as we noted earlier, the determinant of the coherency matrix (the quantity $det\phi$) is an absolute measure of the unpolarized intensity of any partially polarized light:

$$det\phi = \frac{1}{4}\left[I_{tot}^2 - I_{pol}^2\right]. \tag{6.22}$$

6.1.5 The Poincaré sphere representation of Stokes polarization parameters

The Poincaré sphere is a very convenient geometrical representation of all possible polarization states. The intensity-normalized Stokes parameters (q, u and v) are used as coordinate axes to form the Poincaré sphere. The intensity-normalized form of Stokes vector is [30]

$$\mathbf{S}^T = I\left(1, \frac{Q}{I}, \frac{U}{I}, \frac{V}{I}\right) = I(1, q, u, v) = I(1, \mathbf{s}^T). \tag{6.23}$$

This is illustrated in Fig. 6.4. In this space, the *DOP* is nothing else but the distance of the representative point from origin. Thus the physical realizability condition given by Eq. (6.18) implies that all acceptable Stokes vectors are represented by points located within the unit radius sphere, the *Poincaré sphere*. Totally polarized states are found at the surface of the sphere (point A) while partially polarized states are inside (point B). The other spherical coordinates, the points 'latitude' and 'longitude,' are nothing else but twice the azimuth α and ellipticity ϵ, as shown by the last column of Table 6.2 for totally polarized states. It is clear that in this geometric representation, the equatorial circle of the sphere represents the set of linear polarization states (with zero ellipticity); the poles are the points of ellipticity ∓ 1 representing right (north pole) and left (south pole) circular polarization states, respectively; the north hemisphere and the south hemisphere correspond to right-handed and left-handed elliptical polarizations, respectively. This geometrical representation provides simple and intuitive descriptions of many aspects of the interactions between polarized light and samples and/or instruments. As we shall discuss subsequently, any type of polarization transformation introduced by interaction with a medium can be conveniently described by a characteristic trajectory in the Poincaré sphere.

6.1.6 Decomposition of mixed polarization states

The Stokes formalism enables us to express the incoherent superposition of two light waves. The Stokes vector $\mathbf{S}$ of a partially polarized wave can be decomposed into a completely polarized part and an unpolarized part; this type of decomposition is unique [30, 37].

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{bmatrix} \tag{6.24}$$

(Partially polarized wave = Unpolarized wave + Completely polarized wave.) While the above decomposition of the Stokes vector in Eq. (6.24) is relatively easy to implement, analogous decomposition of the coherency matrix is much more important in conceptual and practical grounds, as we discuss here. The

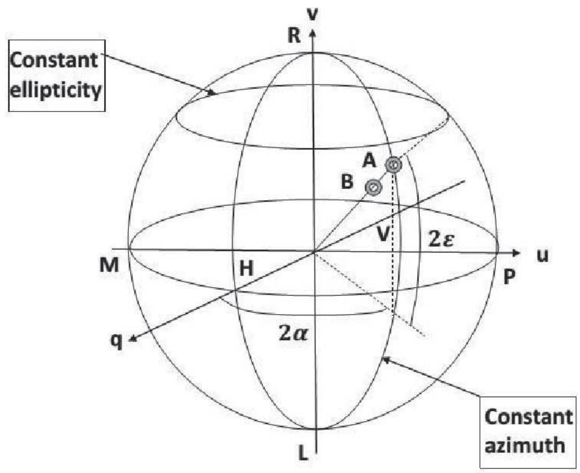

FIGURE 6.4: Geometrical representation of Stokes vectors within the Poincaré sphere. Any given polarization state is represented by a point whose Cartesian coordinates are the intensity-normalized coordinates (q, u, v). The radial coordinate is the DOP, and the 'longitude' and 'latitude' are, respectively, 2α and 2ϵ. Totally polarized states are found at the surface of the unit radius sphere, while partially polarized states are inside (e.g., points A and B, respectively). Linearly polarized states, among which are the H, V, P and M states, are on the 'equator' while the L and R circular states are found at the 'poles.'

coherency matrix of a partially polarized wave can indeed be decomposed into incoherent superposition of two independent completely polarized waves. Here, we briefly outline the steps of such a decomposition, which leads to the useful concept of polarization entropy related to the degree of polarization of a wave. Since the coherency matrix is Hermitian by construction, a unitary matrix can always be found permitting its diagonalization. The diagonalized coherency matrix can be written in the form

$$\phi = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{6.25}$$

The two eigenvalues, λ_1 and λ_2, of the diagonalized coherency can be obtained from the solution of the characteristic equation as

$$\lambda_1 = \frac{1}{2}\mathrm{tr}\phi\left[1 + \left[1 - \frac{4\det(\phi)}{[Tr\phi]^2}\right]^{1/2}\right] \quad \lambda_2 = \frac{1}{2}\mathrm{tr}\phi\left[1 - \left[1 - \frac{4\det(\phi)}{[Tr\phi]^2}\right]^{1/2}\right]. \tag{6.26}$$

It is clear that the two constituent completely polarized waves of the coherency matrix are polarized in the direction corresponding to the eigenvectors associated to λ_1 and λ_2, respectively, and the resulting decomposition can be written as

$$\frac{1}{\lambda_1+\lambda_2}\phi = \frac{1}{\lambda_1+\lambda_2}\begin{bmatrix}\lambda_1 & 0\\ 0 & 0\end{bmatrix} + \frac{1}{\lambda_1+\lambda_2}\begin{bmatrix}0 & 0\\ 0 & \lambda_2\end{bmatrix}. \tag{6.27}$$

Eq. (6.27) leads to the probabilistic interpretation of λ_1 and λ_2, which consequently leads to the concept of entropy in the study of partial polarization. Note that entropy ζ of a given system describes the degree of disorder of the system and is usually defined as

$$\zeta = \sum_{i=1}^{N} p_i \log p_i = \frac{\lambda_i}{\sum_{i=1}^{N}\lambda_j}. \tag{6.28}$$

Considering the condition $(\lambda_1+\lambda_2) = constant = C$, we can obtain conditions for minimum and maximum values for entropy ζ with Eq. (6.29):

$$\begin{aligned}\zeta_{min} =& 0,\ \text{when}\ \lambda_j = C,\ \lambda_i = 0\ (i \neq j;\ i,j = 1,2),\\ \zeta_{max} =& 1,\ \text{when}\ \lambda_j = C/2,\ \text{for any}\ i = 1,\ 2.\end{aligned} \tag{6.29}$$

The minimum and maximum values for entropy $\zeta_{min} = 0$ and $\zeta_{max} = 1$ correspond to completely polarized waves (having a single eigenvalue of the diagonalized coherency matrix) and completely unpolarized waves (having two equal eigenvalues), respectively. The concept of entropy is thus directly related to the degree of polarization of the wave. For any completely polarized input wave, any depolarizing interactions with the medium thus leads to entropy production, and accordingly, the depolarization property of the medium can be quantified by the entropy of the output polarization state (which can be obtained from the decomposition of the coherency matrix above).

Having used the Jones vector, coherency matrix and Stokes vector formalism to describe the state of polarization of light waves, we now turn to the more interesting problem of transformation (or even loss) of polarization of light waves in their interaction with any material medium. In the following section, we define the various medium polarimetry characteristics and their mathematical representation through Jones matrix and Mueller matrix formalisms.

6.2 Interaction of polarized light with material media

As we discussed in Section 6.1.1 (in the context of Jones vector representations of pure polarization states of light), a vectorial description of the polarization state enables the matrix treatment to describe the polarizing transfer of light in its interaction with any medium. This is true for either the Jones vector (representing completely polarized states) or the Stokes vector (representing both complete and partial polarization states). As it is clear now, the interactions that only transform a pure polarization state into another pure polarization state (polarization-preserving interactions keeping the degree of polarization unity) can be tackled using the Jones formalism [33]. In contrast, the Stokes-Mueller formalism can deal with both the polarization-preserving and depolarizing interactions (which lead to loss of polarization, leading to reduction in DOP) [34, 38]. In this section, we shall address both these formalisms. First, we shall introduce the fundamental medium polarimetric characteristics, and then these medium polarization properties will be represented via their characteristic transformation matrices, namely, the Jones matrix (2×2 field transformation matrix) and the Mueller matrix (4×4 intensity-based Stokes vector transformation matrix). On the way, we shall establish useful relationships between the Jones and Mueller matrices (for polarization-preserving interactions).

6.2.1 Basic medium polarimetry characteristics

The three basic medium polarization properties are *retardance*, *diattenuation* and *depolarization*. The first two effects represent polarization-preserving interaction and can accordingly be modeled using both Jones and Stokes-Mueller formalisms. The third one, on the other hand, leads to loss of polarization and thus cannot be handled using Jones formalism.

The two polarization effects *retardance* and *diattenuation* arise from differences in the refractive indices for different polarization states, and they are often described in terms of ordinary and extraordinary axes and indices. Differences in the real parts of refractive indices result in linear and circular birefringence (retardance), whereas differences in the imaginary parts can cause linear and circular dichroism (which manifests itself as diattenuation, described below) [30, 31, 32]. Mathematically, retardance and birefringence are related simply via $R = k.L.\Delta n$, where R is the retardance, k is the wave vector of the light, L is the pathlength in the medium and Δn is the difference in the real parts of the refractive index known as birefringence. *Linear retardance*, denoted δ, is therefore the relative phase shift between orthogonal linear polarization components (between vertical and horizontal, or between $+45°$ and $-45°$) upon propagation through any medium. The different types of wave plates (half-wave plate, quarter-wave plate, etc.) made of anisotropic materials are examples of perfect linear retarders. Usually, a linear retarder converts input linearly polarized light into elliptically polarized light by introducing phase difference between orthogonal linear polarization components. The output state of polarization depends upon the magnitude of linear retardance and orientation angle (θ) of the principal axis of the retarder with respect to the input linear polarization direction. Analogously, *circular retardance* (δ_C) arises from phase differences between right circularly polarized

(RCP) and left circularly polarized (LCP) states. Such effects are usually introduced by asymmetric chiral structures, and they are manifested as rotation of input linear polarization ($\delta_C = 2 \times$ optical rotation ψ).

The *diattenuation (D)* of an optical element is a measure of the differential attenuation of orthogonal polarization states for both linear and circular polarization. This is analogous to dichroism, which is the differential absorption of two orthogonal polarization states (linear or circular); however, diattenuation is more general, since the differential attenuation need not be caused by absorption alone, rather, it can be the result of various other effects (e.g., scattering, reflection, refraction, etc.). *Linear diattenuation* is defined as differential attenuation of two orthogonal linear polarization states and *circular diattenuation* is defined as differential attenuation of RCP and LCP states. Like linear retardance, polarization transformation by a linear diattenuator also depends upon the magnitude of diattenuation and the orientation angle (θ) of the principal axis of the diattenuator. The simplest form of a diattenuator is the ideal polarizer that transforms incident unpolarized light to completely polarized light ($D = 1$ for ideal polarizer), although often with a significant reduction in the overall intensity.

If an incident state is completely polarized and the exiting state after interaction with the sample has a degree of polarization less than unity, then the sample possesses the *depolarization* property. Depolarization is usually encountered due to multiple scattering of photons (although randomly oriented uniaxial birefringent domains can also depolarize light); incoherent addition of amplitudes and phases of the scattered field results in scrambling of the output polarization state.

6.2.2 Relationship between Jones and Mueller matrices

In Section 1.1, we defined the 2×2 Jones (Eq. (6.6)) to represent the polarization transfer function of a medium in its interaction with completely polarized light. The Jones matrix J is generally complex and contains eight independent parameters (real and imaginary parts of each of the four matrix elements), or seven parameters if the absolute phase is excluded. The polarizing interactions of any medium are contained in the elements of this matrix J; the medium polarization characteristics associated with alterations of relative amplitudes and phases (of orthogonal polarization states) are encoded in the real and imaginary parts of the elements, respectively. As we noted, matrix algebra enables us to compute the Jones matrix of an optical system formed by a series of elements through sequential multiplication of the individual matrices of these elements. Moreover, rotation (by an angle α) of any optical element can also be conveniently modeled by the rotational transformation of Jones matrices ($J\rangle J'$) via the usual coordinate rotation matrix $R(\alpha)$:

$$R(\alpha) = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}, \qquad J' = R^{-1}(\alpha) J R(\alpha). \tag{6.30}$$

Analogous to the Jones matrix, a 4×4 matrix M known as the Mueller matrix (developed by Hans Mueller in the 1940s) describes the transformation of the Stokes vector (polarization state) in its interaction with a medium [36, 37, 39]:

$$\mathbf{S_0} = M\mathbf{S_i}, \tag{6.31}$$

$$\begin{aligned}
\begin{bmatrix} I_o \\ Q_o \\ U_o \\ V_o \end{bmatrix} &= \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} I_i \\ Q_i \\ U_i \\ V_i \end{bmatrix} \\
&= \begin{bmatrix} m_{11}I_i & m_{12}Q_i & m_{13}U_i & m_{14}V_i \\ m_{21}I_i & m_{22}Q_i & m_{23}U_i & m_{24}V_i \\ m_{31}I_i & m_{32}Q_i & m_{33}U_i & m_{34}V_i \\ m_{41}I_i & m_{42}Q_i & m_{43}U_i & m_{44}V_i \end{bmatrix},
\end{aligned}$$

with $\mathbf{S_i}$ and $\mathbf{S_o}$ being the Stokes vectors of the input and the output light, respectively. The 4×4 real Mueller matrix M has at most sixteen independent parameters (or fifteen if the absolute intensity is excluded), including depolarization information. All the medium polarization properties are encoded in the various elements of the Mueller matrix, which can thus be thought of as the complete optical polarization fingerprint of a sample. Similar to the Jones formalism, matrix properties allow us to determine the resultant Mueller matrix equivalent to a system formed by a series of optical elements through sequential multiplication of the individual Mueller matrices of the elements.

The fundamental requirement real Mueller matrices must meet is that they map physical incident Stokes vectors into physically realizable resultant Stokes vectors (satisfying Eq. (6.18)). Similarly, a Mueller matrix cannot output a state with negative flux. In fact, conditions for physical realizability of Mueller matrices have been studied extensively in the literature, and many necessary conditions have been derived [39, 40, 41, 42, 43]; this is outside the scope of this book. We note below the other important necessary condition for physical realizability of a Mueller matrix (Eq. (6.32)), and we refer the reader to references [30, 39, 40, 41, 42, 43] for a more detailed account of the necessary and sufficient conditions that any 4×4 real matrix should satisfy to qualify as a Mueller matrix of any physical system.

$$tr(MM^T) = \sum_{i,j=1}^{4} m_{ij} \leq 4m_{11}^2, \tag{6.32}$$

where M^T is the transpose of matrix M and the indices $i, j = 1, 2, 3, 4$ denote its rows and columns, respectively. Here the equality and the inequality signs correspond to nondepolarizing and depolarizing systems, respectively.

Relationships between the Jones formalism and the Stokes-Mueller formalism are worth a brief mention here. For the special case of a nondepolarizing linear optical system (a deterministic system, satisfying the equality in Eqs. (6.10) and (6.32)), a one-to-one correspondence between the real 4×4 Mueller matrix M and the complex 2×2 Jones matrix J can be derived via the coherency matrix formalism. Such a relationship can be obtained by using the following set of equations describing the transformation of the input Jones vector ($\mathbf{E_i}$), coherency vector ($\mathbf{L_i}$, defined in Eq. (6.20)) and Stokes vector ($\mathbf{S_i}$).

$$\mathbf{E_0} = J\mathbf{E_i}, \qquad \mathbf{L_0} = W\mathbf{L_i}, \qquad \mathbf{S_0} = M\mathbf{S_i}. \tag{6.33}$$

Here, $\mathbf{E_o}$, $\mathbf{L_o}$ and $\mathbf{S_o}$ are the output Jones, coherency and Stokes vectors, respectively, after medium interaction. The Jones and Mueller matrices J and M have been defined earlier. The matrix W is a 4×4 matrix that describes the transformation of the coherency vector in its interaction with the medium and is known as the Wolf matrix. Using Eqs. (6.9) and (6.20) and by performing simple algebraic manipulations, we can show that the Wolf matrix W and the Mueller matrix M are related to the Jones matrix J as

$$W = J \otimes J^*, \quad M = A \cdot (J \otimes J^*) \cdot A^{-1}. \tag{6.34}$$

Here, A is the 4×4 matrix defined in Eq. (6.20), relating the Stokes vector and the coherency vector.

Thus every Jones matrix (that can only describe a special case of a nondepolarizing optical system) can be transformed into an equivalent Mueller matrix (and a Wolf matrix); however, the converse is not necessarily true. The resulting nondepolarizing Mueller matrix contains seven independent parameters and is accordingly termed a Mueller-Jones matrix. The examples of such Mueller-Jones matrices are the matrices for retardance (both linear and circular) and diattenuation (linear and circular) effects. We show below an interesting example of transforming the Jones matrix to the Mueller-Jones matrix. Consider the rotational transformation of Jones matrices ($J\rangle J'$) via the usual coordinate rotation matrix $R(\alpha)$ (Eq. (6.30)). Apparently, $R(\alpha)$ represents coordinate rotation of the electric field vector. This warrants that analogous rotational transformation should also exist for Mueller-Jones matrices. Employing Eq. (6.34), we can determine the analogous rotational transformation of the Mueller-Jones matrix ($M \to M'$) as

$$M' = T^{-1}(\alpha) M T(\alpha), \qquad T(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.35}$$

where the rotation matrix $T(\alpha)$ implies rotation of the Stokes vector in the polarization state space (i.e., in the Poincaré sphere; see Fig. 6.4) rather than in the coordinate space. This also implies that a rotation of the field vector by an angle α leads to a rotation of 2α of the Stokes vector (around the v-axis describing circular polarization) in the Poincaré sphere.

We conclude this section by once again noting, while both the Jones and the Stokes-Mueller formalisms describe polarization change using matrix/vector equations, the latter provides a framework with which partial polarization states can be handled and depolarizing materials can be described. Since in nature, light is often partially polarized (or unpolarized) and in most practical situations, loss of polarization is unavoidable, the Stokes-Mueller formalism has been used in most practical polarimetry applications. In contrast, the use of the Jones formalism has been limited as a complementary theoretical approach to the Mueller matrix calculus, or to studies in clear media, specular reflections and thin films where polarization loss is not an issue.

6.2.3 Jones matrices for nondepolarizing interactions: Examples and parametric representation

We now provide explicit expressions for the Jones matrices corresponding to the two polarization preserving effects, *retardance (linear and circular)* and *diattenuation (linear and circular)*, and we briefly discuss the resulting effect on the state of polarization introduced by these transformations.

Retardance (birefringence): Linear retardance originates from the difference in the real part of the refractive index between two orthogonal linear polarization states and accordingly leads to a difference in phase between these states while propagating through an 'anisotropic' medium exhibiting this effect. The Jones matrix for this effect can be written as [28, 30, 33];

$$J_{LR} = \begin{bmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{bmatrix}. \tag{6.36}$$

Here, ϕ_x and ϕ_y are the respective phases of the two orthogonal linear polarization states (x- and y-polarized, respectively; corresponding Jones vectors are noted as H and V states in Table 6.1). The resulting magnitude of linear retardance is

$$\delta = \frac{2\pi}{\lambda}(n_y - n_x)L,$$

where n_x and n_y are the real part of the refractive indices for x- and y-polarized light, respectively; L is the pathlength. Note that this diagonal form of the Jones matrix (J_{LR} in Eq. (6.36)) is obtained for an anisotropic medium whose principal axis is oriented along the laboratory x/y direction. In general, J_{LR} may have off-diagonal elements based on the orientation angle (θ) of the principal axis with respect to the laboratory x-/y-axes. As noted in Eq. (6.30), the general form of the Jones matrix for the arbitrary orientation angle θ can be obtained using rotational transformation as

$$\begin{aligned} J_{LR}(\delta,\theta) &= R^{-1}(\theta) J_{LR} R(\theta) \qquad (6.37) \\ &= \begin{pmatrix} e^{i\phi_x}\cos^2\theta + e^{i\phi_y}\sin^2\theta & (e^{i\phi_x} - e^{i\phi_y})\cos\theta\sin\theta \\ (e^{i\phi_x} - e^{i\phi_y})\cos\theta\sin\theta & e^{i\phi_x}\sin^2\theta + e^{i\phi_y}\cos^2\theta \end{pmatrix}, \end{aligned}$$

where $R(\theta)$ is the rotation matrix of Eq. (6.30).

For example, the Jones matrix of a quarter-waveplate ($\delta = \pi/2$) with its principal axis aligned along the laboratory x-axis ($\theta = 0°$) is

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The state of polarization of light emerging from such a medium exhibiting linear birefringence obviously depends upon the input polarization state, the magnitude of retardance δ and orientation angle of the principal axis θ. For example, if input $+45°$ linearly polarized light characterized by Jones vector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}^T$) is incident on the above quarter-waveplate, the output polarization state will be left circularly polarized (LCP) with Jones vector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \end{bmatrix}^T$).

Analogously, circular retardance (δ_C) originates from the difference in the real part of the refractive index ($n_L - n_R$) between two orthogonal circular polarization states (LCP/RCP) and is manifested as the rotation of the plane of polarization (optical rotation $\psi; \delta_C = 2\psi$):

$$\delta_C = \frac{2\pi}{\lambda}(n_L - n_R)L.$$

The Jones matrix corresponding to this effect is a pure rotation matrix:

$$J_{CR}(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin\psi & \cos(\psi) \end{bmatrix}. \tag{6.38}$$

Diattenuation (dichroism): As previously mentioned diattenuation arises due to differential attenuation of orthogonal polarization states (for both linear and circular) and originates from the differences in the imaginary part of the refractive index for orthogonal polarization states. The Jones matrix for linear diattenuation effect can be written as [28, 30, 33]

$$J_{LD} = \frac{1}{\sqrt{a^2 + b^2}}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}. \tag{6.39}$$

Here, a and b are real numbers because they are related to the amplitudes of the two orthogonal linear polarization states (x- and y-polarized, respectively). The magnitude of linear diattenuation ($-1 \le D \le +1$) can be written as

$$D = \frac{a^2 - b^2}{a^2 + b^2}.$$

Note that like the linear retarder, the Jones matrix of a linear diattenuator also depends upon the magnitude of diattenuation D and the orientation angle (θ) of the principal axis of the diattenuator, and the general form of the diattenuator oriented at an angle θ can be obtained as

$$J_{LD}(d, \theta) = R^{-1}(\theta) J_{LD} R(\theta). \tag{6.40}$$

An example of a perfect diattenuator matrix is that of a linear polarizer (magnitude of diattenuation $D = \pm 1$):

$$J_{LD}(D = \pm 1, \theta) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.41}$$

Apparently, light emerging from a perfect linear diattenuator is always linearly polarized along the direction of the principal axis of the diattenuator, irrespective of the polarization state of the incident light.

6.2.4 Standard Mueller matrices for basic interactions (diattenuation, retardance, depolarization): Examples and parametric representation

Having defined the Jones matrices for the various polarization-preserving interactions, we now turn to the corresponding representation using Mueller matrices. We must note that although both the Jones and the Stokes-Mueller approaches rely on linear algebra and matrix formalisms, they differ in many aspects. Specifically, the Stokes-Mueller formalism has certain advantages. First of all, it can encompass any polarization state of light, whether it is natural, totally or partially polarized (can thus deal with both polarizing and depolarizing optical systems). Second, the Stokes vectors and Mueller matrices can be measured with relative ease using intensity-measuring conventional (square-law detector) instruments, including most polarimeters, radiometers and spectrometers.

We also note that in the conventional Mueller matrix representation of the retardance and diattenuation effects, the optical elements exhibiting these two effects are often referred to as the *homogeneous retarder* and the *homogenous diattenuator*. In this convention, polarimetric elements are called homogeneous if they exhibit two fully polarized orthogonal eigenstates, i.e., two polarization states that are transmitted without alteration and that do not interfere with each other. In practice, such light states are linearly polarized along two perpendicular directions, or circularly polarized and rotating in opposite senses. The normalized Stokes vectors $\mathbf{S_1}$ and $\mathbf{S_2}$ of such orthogonal states are of the form

$$\mathbf{S_1}^T = (1, \mathbf{s}^T), \mathbf{S_2}^T = (1, -\mathbf{s}^T), \tag{6.42}$$

with $\parallel \mathbf{s} \parallel = 1$ as these states are fully polarized. Orthogonal states are thus found on the surface of the Poincaré sphere at diametrically opposed positions. For any homogeneous polarimetric element, there are thus two (and only two) such states that are left invariant on the Poincaré sphere.

Homogeneous retarders: The elements exhibiting the retardance effects are characterized by two orthogonal eigenpolarization states, each of which is transmitted without modification. The corresponding orthogonal Stokes eigenvectors are of the form given by Eq. (6.42). Homogenous retarders transmit both eigenstates with the same intensity coefficients, but different phases. This phase difference is the scalar retardation δ, as we defined earlier in context with Jones matrix representation. A pure retarder can be described geometrically as rotation in the space of Stokes vectors. Mathematically, the Mueller matrix M_R of the retarder can be written as [44, 45, 46]

$$M_R = \begin{pmatrix} 1 & 0^T \\ 0 & m_R \end{pmatrix}, \tag{6.43}$$

where 0 represents the null vector and the 3×3 submatrix, m_R, is a rotation matrix in the Poincaré (q,u,v) space. The action of a retarder on an arbitrary incident Stokes vector $\mathbf{S}$ is a rotation of its representative point on the Poincaré sphere, described by m_R. Moreover, the axis of this rotation is defined by the two diametrically opposed points representing the two eigenpolarizations, and the rotation angle is the retardation δ.

For linear retarders with eigenstates linearly polarized along θ and $\theta + 90°$ azimuths, the form of the Mueller matrix $(M_{LR}(\tau, \delta, \theta))$ can be obtained by applying the transformation of Eq. (6.34) on the Jones matrix of a retarder (Eq. 6.37) [44, 45, 46]

$$\tau \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \sin^2 2\theta \cos\delta & \cos 2\theta \sin 2\theta (1 - \cos\delta) & -\sin 2\theta \sin\delta \\ 0 & \cos 2\theta \sin 2\theta (1 - \cos\delta) & \sin^2 2\theta + \cos^2 2\theta \cos\delta & \cos 2\theta \sin\delta \\ 0 & \sin 2\theta \sin\delta & -\cos 2\theta \sin\delta & \cos\delta \end{pmatrix}, \tag{6.44}$$

where τ is the intensity transmission for incident unpolarized light, and can be taken to be unity if the optical material is nonabsorbing (lossless).

A straightforward calculation indeed shows that two fully polarized orthogonal eigenstates (linearly polarized states with azimuths θ and $\theta + 90°$) are transmitted unchanged:

$$M_{LR}(\tau, \delta, \theta) \begin{bmatrix} 1 \\ \pm\cos 2\theta \\ \pm\sin 2\theta \\ 0 \end{bmatrix} = \tau \begin{bmatrix} 1 \\ \pm\cos 2\theta \\ \pm\sin 2\theta \\ 0 \end{bmatrix}. \tag{6.45}$$

An example of a Mueller matrix of a homogeneous linear retarder is that of a quarter-wave plate (with $\theta = 0°$)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

As noted before, the Mueller matrix of the quarter wave plate above for its principal axis oriented at any arbitrary angle θ can easily be determined using the rotational transformation of Eq. (6.35) (yielding Eq. (6.44) with $\delta = \pi/2$).

We now consider *circular retarders*, i.e., elements for which the eigenpolarizations are the opposite circular polarization states. The Mueller matrices of such elements are of the form [44, 45, 46]

$$M_{CR}(\psi) = \tau \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & \cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.46}$$

When a linearly polarized wave interacts with a circular retarder, its polarization remains linear, but it is rotated by an angle ψ (known as *optical rotation*). The effect may also be interpreted as a rotation of the incident linearly polarized Stokes vector $\mathbf{S}$ in the Poincaré sphere by an amount equal to the circular retardance ($\delta_C = 2\psi$).

Finally, we point out that the scalar retardation of any homogeneous retarder (linear or circular) can be determined from the general Mueller matrix M_R of a retarder as

$$\delta, \psi = \cos^{-1}\left(\frac{Tr(M_R)}{2} - 1\right). \tag{6.47}$$

The formula above is valid for media exhibiting both linear retardance δ and circular retardance (optical rotation ψ). We can readily verify this from the Mueller matrix of the combined effect by multiplying individual matrices for the linear and circular retarder (in either order). The total retardance in such case can be obtained employing Eq. (6.47) on the Mueller matrix representing the combined effects.

Homogeneous diattenuators: For a diattenuating system, the output intensity depends on the input polarization state. If we consider an intensity normalized input Stokes vector $\mathbf{S}$ such that

$$\mathbf{S_{in}}^T = (1, \mathbf{s}^T), \qquad \text{with} \qquad \| \mathbf{s} \| = DOP \leq 1, \tag{6.48}$$

which corresponds to arbitrary polarizations at constant intensity (normalized to unity), then the output intensity (i.e., the first component of S_{out}) is simply given by

$$I_{out} = m_{11}(1 + \mathbf{D} \cdot \mathbf{s}). \tag{6.49}$$

This output intensity reaches its maximum (minimum) value I_{max} (I_{min}) when the scalar product $\mathbf{D} \cdot \mathbf{s}$ is maximum (minimum) under the constraint $\| \mathbf{s} \| = DOP \leq 1$, i.e., when $s = \pm \frac{D}{\|D\|}$. We thus obtain:

$$\mathbf{S_{max}}^T = \left(1, \frac{\mathbf{D}^T}{\| \mathbf{D} \|}\right) \text{ and } I_{max} = m_{11}(1 + \| \mathbf{D} \|),$$

$$\mathbf{S_{min}}^T = \left(1, -\frac{\mathbf{D}^T}{\| \mathbf{D} \|}\right) \text{ and } I_{min} = m_{11}(1 - \| \mathbf{D} \|), \tag{6.50}$$

from which we immediately obtain the *scalar diattenuation* D:

$$D = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \| \mathbf{D} \| \,. \tag{6.51}$$

The diattenuator vector $\mathbf{D}$ thus defines both the scalar diattenuation D and the polarization states transmitted with the highest (or the lowest) intensity. Note that the two polarization states giving these extremal intensity transmission values are totally polarized, and they are located at diametrically opposite positions on the Poincaré sphere.

The elements of the Mueller matrix of a diattenuator are uniquely determined by their diattenuation vector $\mathbf{D}$. Their (totally polarized) eigenpolarization states corresponding, respectively, to maximum and minimum transmissions are given by Eqs. (6.50). The corresponding Mueller matrix is then given by

$$M_D = \tau \begin{pmatrix} 1 & \mathbf{D}^T \\ \mathbf{D} & m_d \end{pmatrix}, \quad \text{where} \quad m_D = \sqrt{1 - D^2} I_3 + (1 - \sqrt{1 - D^2}) \mathbf{D}\mathbf{D}^T. \tag{6.52}$$

Once again, τ is the intensity transmission for incident unpolarized light. Diattenuation may occur due to reflection and/or refraction at an interface, or to propagation in anisotropic or chiral materials. Anisotropy may introduce *linear dichroism*. For any propagation direction, the imaginary part of the wave vector (corresponding to the imaginary part of the refractive index n'') may take two different values, n''_L and n''_H; the former, corresponding to the lowest absorption, is valid for a wave linearly polarized at azimuth θ and the latter for the orthogonal polarization, at $\theta + 90°$. The linear (scalar) dichroism is then defined as

$$\Delta n'' = n''_H - n''_L > 0. \tag{6.53}$$

For a parallel slab of thickness L, the intensity transmissions for the two eigenpolarizations are, respectively,

$$T_{max} = \exp(-2n''_L L), \quad T_{min} = \exp(-2n''_H L), \tag{6.54}$$

yielding a scalar diattenuation

$$D = \frac{T_{max} - T_{min}}{T_{max} + T_{min}} = \sinh(\Delta n'' L), \tag{6.55}$$

where $\Delta n'' L$ is the dichroism integrated over the slab thickness L.

Chiral media (e.g., a biological fluid such as glucose) can also exhibit dichroism, but usually it is *circular dichroism*. The formulas above are still valid, but in this case the eigenpolarizations for which the absorption coefficients are well defined are left and right circular ones.

The Mueller matrix for a linear diattenuator ($M_{LD}(\tau, D, \theta)$) can be written in the following symmetric form [44, 45, 46];

$$\frac{\tau}{2}\begin{bmatrix} 1 & D\cos 2\theta & D\sin 2\theta & 0 \\ D\cos 2\theta & \cos^2 2\theta + \sqrt{1-D^2}\sin^2 2\theta & \left(1-\sqrt{1-D^2}\right)\cos 2\theta \sin 2\theta & 0 \\ D\sin 2\theta & \left(1-\sqrt{1-D^2}\right)\cos 2\theta \sin 2\theta & \sin^2 2\theta + \sqrt{1-D^2}\cos^2 2\theta & 0 \\ 0 & 0 & 0 & \sqrt{1-D^2} \end{bmatrix}, \tag{6.56}$$

implying that the maximum and minimum intensity transmittances are obtained for linearly polarized states with azimuths θ and $\theta + 90°$. It can be easily checked that these eigenpolarization states are unchanged by $M_{LD}(\tau, D, \theta)$ but are transmitted with intensity factors $\frac{\tau}{2}(1 \pm D)$.

Similarly, for circular diattenuators, the general form of the Mueller matrix is

$$M_{CD}(\tau, D) = \frac{\tau}{2}\begin{pmatrix} 1 & 0 & 0 & D \\ 0 & \sqrt{1-D^2} & 0 & 0 \\ 0 & 0 & \sqrt{1-D^2} & 0 \\ D & 0 & 0 & 1 \end{pmatrix}, \tag{6.57}$$

and of course in this case there is no need to define any partial azimuth θ.
Depolarizers: A depolarizer is an object that reduces the degree of polarization of the incoming light. The simplest depolarizers are those for which the Mueller matrix M_Δ is diagonal [44, 45, 46];

$$M_\Delta = \tau\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \tag{6.58}$$

with absolute values of a, b and c smaller than unity. If so, any incident Stokes vector S_i of the form

$$\mathbf{S_i}^T = I(1, q, u, v)$$

is transformed into

$$\mathbf{S_{out}}^T = \tau I(1, aq, bu, cv),$$

which gives the output degree of polarization

$$DOP_{out} = \sqrt{a^2q^2 + b^2u^2 + c^2v^2} \leq \sqrt{q^2 + u^2 + v^2} = DOP_{in}. \tag{6.59}$$

In the geometrical representation, the action of a depolarizer defined in Eq. (6.58) is to pull the representative point of the incoming Stokes vector toward the origin. As a result, the Poincaré sphere is transformed into an ellipsoid limited by the segments $[-a, a], [-b, b]$ and $[-c, c]$ along the q, u, v-axes.

As discussed previously, depolarization occurs due to incoherent addition of intensities of polarized states with different polarizations. Depolarization may occur due to multiple scattering in the first place, together with spatially varying, randomly oriented birefringent domains. However, these effects alone are not sufficient to cause real depolarization but would give rise to a speckle pattern with $DOP = 1$ everywhere but with different polarizations from one point to another. True depolarization occurs if this speckle pattern is blurred by the motion of the scattering sample, the lack of spatial coherence of the illumination beam, the sample motion, and the like.

In the most general case, the Mueller matrix of a depolarizer, M_Δ, is given in compact notation as

$$M_\Delta = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & m_\Delta \end{pmatrix}, \tag{6.60}$$

where m_Δ is a 3×3 real symmetric matrix. This matrix can be diagonalized to recover the form given by Eq. (6.58) where the eigenvalues a, b, c are real numbers varying between -1 and 1. Thus the Mueller matrix of the most general depolarizer depends on six parameters (as can be seen from the very definition of the m_Δ matrix as a 3×3 symmetric matrix, or by the fact that the diagonalization process involves not only the three eigenvalues but also the basis formed by the eigenvectors of m_Δ). General depolarizers are thus rather complex mathematical objects, this complexity being related to situations like multiple scattering in anisotropic media. Here we will not discuss the properties of general depolarizers any further, and in the following section we will only consider depolarizers of the form given by Eq. (6.58). We thus define the depolarizing power of such samples as

$$\Delta = 1 - \frac{1}{3}(\mid a \mid + \mid b \mid + \mid c \mid), \tag{6.61}$$

which can be further separated in depolarizing powers for linear and circular polarizations as

$$\Delta_L = 1 - \frac{1}{2}(\mid a \mid + \mid b \mid) \text{ and } \Delta_C = 1 - \mid c \mid . \tag{6.62}$$

Another widely used approach of defining depolarization property of a sample using Mueller matrix M uses the following definition of the depolarization index P_d:

$$P_d = \sqrt{\frac{\sum_{i=1}^{4} \sum_{j=1}^{4} M_{ij}^2 - M_{11}^2}{3M_{11}^2}} = \sqrt{\frac{Tr(M^T M) - M_{11}^2}{3M_{11}^2}}. \tag{6.63}$$

As a final remark, the depolarization powers Δ defined in Eq. (6.61) vary from zero, for nondepolarizing samples, to one for totally depolarizing ones, while the opposite holds for the general depolarization index P_d defined in Eq. (6.63).

To summarize, thus far in this chapter we have discussed mathematical formalisms to deal with the state of polarization of light and interaction of polarized light with material media. Specifically, the two most widely used formalisms, namely, the Jones calculus and the Stokes-Mueller calculus, have been discussed. As previously noted, the former is a field-based model and is limited to describing pure polarization states (a completely polarized wave) and polarization-preserving (nondepolarizing) interactions only. The latter, on the other hand, is an intensity-based model and is more encompassing in the sense that it provides a framework with which partial polarization states can be handled and depolarizing interactions can also be modeled. Caution must, however, be exercised while implementing these formalisms either for practical purposes of polarimetric measurements or for conceptual reasons for polarimetric modeling. It may be worthwhile to spend a few words on the validity regime of such algebra. Note that both the Jones vector (in the field-based representation) and the Stokes vector (in the intensity-based representation) deal with a two-dimensional electromagnetic field, and are applicable when the light wave is completely transverse in nature (plane electromagnetic waves or more generally to uniformly polarized elementary beams). Note that even for paraxial beam-like fields, the spatial mode (distribution of field) and polarization are not always separable (unlike plane waves or elementary beams) and accordingly, we need different algebra to describe such inhomogeneous polarization. This so-called *classical entanglement* between polarization and spatial mode is handled by defining the beam coherency polarization matrix (a variant of the 2×2 coherency matrix incorporating simultaneously both the field polarization and its spatial distribution) [47]. In other general cases involving three-dimensional fields (as encountered in tight focusing, scattering and the near field), the two-dimensional polarimetry formalisms have been extended via the definition of the 3×3 coherency matrix and the generalized nine-element Stokes vector [48]. Moreover, there is other emerging 'un-conventional' polarization algebra involving vector beams, geometric phases (Pancharatnam-Berry phase) arising from spin-orbit interactions of light, radial and azimuthal polarization of light beams and so forth [49, 50]. Some of these issues related to advanced topics in polarization optics are discussed in Chapter 11. For now, we restrict our discussion to the conventional polarization algebra. In the following, we briefly introduce the concepts of polarimetric measurements (based on conventional polarization algebra), and we touch upon representative applications of experimental polarimetry.

6.3 Experimental polarimetry and representative applications

Polarimeters can be regarded as optical instruments used for the determination of polarization characteristics of light and the sample. Based on this