Discrete Math - Exam 1, Summer 2020

Question 1 (2 marks): Prove by induction that $3|(2^n + (-1)^{n+1})$ for all $n \in \mathbf{N}$.

Question 2 (4 marks): Let S be the set of bit strings defined as follows:

- $0 \in S$
- if $s \in S$, then $1s1 \in S$.

Write $1^n := 111...111$ (n copies of 1), and set

$$M := \{1^n 0 1^n : n \in \mathbf{N}\}$$

Show that S=M.

Question 3 (3+3+2 marks): On $\mathbf{Z} - \{0\}$, define relation $\sim$ by: $x \sim y$ if and only if $x \cdot y > 0$.

(a) Show that this is an equivalence relation.

(b) Extend the relation to $\mathbf{Z}$. How many of the required properties does it retain? Is it still an equivalence relation?

(c) Describe $(\mathbf{Z} - \{0\})/\sim$. What are the fibers of the partition?

Question 4 (1+1+2): Let M be a set and $\mathcal{P}(M)$ its power set. Consider the symmetric difference operation $\Delta : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathcal{P}(M)$ given by

$$A \Delta B := (A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)^C = (A \cap B^C) \cup (B \cap A^C)$$

(a) Is $(\mathcal{P}(M), \cap)$ a commutative group? Why or why not?

(b) Is $(\mathcal{P}(M), \Delta)$ a commutative group? Why or why not? (You may take as granted the fact that it is associative.)

(c) Consider $(\mathcal{P}(M), \cap, \Delta)$. You may take it as granted that the two operations distribute over one another. Is this then a commutative ring? An integral domain? A field? Why or why not?

Question 5 (5 marks): Calculate $11^{644} \bmod 645$. (Hint: if $x \equiv 11^{644} \bmod 645$, then since $645 = 3 \cdot 5 \cdot 43$, it is also true that $x \equiv 11^{644} \bmod 3$, $x \equiv 11^{644} \bmod 5$, and $x \equiv 11^{644} \bmod 43$.)