

# Ve203 Discrete Mathematics (Fall 2016)

## Assignment 5: Sunzi, Fermat, Legendre

Date Due: 4:00 PM, Thursday, the 3<sup>rd</sup> of November 2016



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This assignment has a total of (40 Marks).

### \* Exercise 5.1 Binary Insertion Sort

The *binary insertion sort* is a variation of the insertion sort that uses a binary search technique rather than a linear search technique to insert the  $i$  element in the correct place among the previously sorted elements.

- i) Express the binary insertion sort in pseudocode.  
(2 Marks)
- ii) Compare the number of comparisons of elements used by the insertion sort and the binary insertion sort when sorting the list 7, 4, 3, 8, 1, 5, 4, 2.  
(2 Marks)
- iii) Show that the insertion sort uses  $O(n^2)$  comparisons of elements.  
(2 Marks)
- iv) Find the complexity of the binary insertion sort. Is it significantly faster?  
(2 Marks)

### \* Exercise 5.2

Order the letters M, I, C, H, I, G, A, N alphabetically using

- i) merge sort,  
(2 Marks)
- ii) insertion sort,  
(2 Marks)
- iii) bubble sort  
(2 Marks)

algorithms. (Note that it does not matter that the letter “I” is repeated.) For each algorithm, show what the arrangement is after each pass/merge. How many comparisons of letters are made using each algorithm?

### \* Exercise 5.3 Application to Feng Shui

The *iterated integer sum* of  $n \in \mathbb{N} \setminus \{0\}$  is calculated as follows: The decimal digits of  $n$  are added to yield a sum  $n_1$ . If  $n_1$  is greater than 9, the integers of  $n_1$  are added. This process is repeated until a number between 0 and 9 is obtained. For example, the iterated integer sum of 54469 is calculated as follows:  $5 + 4 + 4 + 6 + 9 = 28$ ,  $2 + 8 = 10$ ,  $1 + 0 = 1$ .

- i) Give the worst-case number of additions that need to be performed to calculate the iterated integer sum of  $n \in \mathbb{N} \setminus \{0\}$  in this way.  
(2 Marks)
- ii) How is the iterated integer sum of a number  $n$  related to  $n \bmod 9$ ? Prove your assertion!  
(3 Marks)
- iii) Generalize (ii) to integers represented in arbitrary base  $b$ .  
(2 Marks)

(The iterated integer sum plays a role in Feng Shui; see, for example, [http://fengshui.about.com/od/fengshuicures/qt/kua\\_number.htm](http://fengshui.about.com/od/fengshuicures/qt/kua_number.htm).)

(2+3+2 Marks)

**Exercise 5.4 Modular Exponentiation**

Find  $4102^{1042} \bmod 2014$  using the algorithm for modular exponentiation given in the lecture. Show all the steps in the algorithm.

(2 Marks)

**Exercise 5.5 Stein's Algorithm for the GCD in base 2**

Let  $a > b$  be two natural numbers.

- i) How can multiplication or division by 2 be efficiently performed in base 2?

(1 Mark)

- ii) If  $a$  and  $b$  are both even, express  $\gcd(a, b)$  in terms of  $\gcd(\frac{a}{2}, \frac{b}{2})$ .

(1 Mark)

- iii) If  $a$  and  $b$  are both odd, express  $\gcd(a - b, b)$  in terms of  $\gcd(a, b)$ .

(1 Mark)

- iv) Work out an algorithm to calculate the gcd of two natural numbers in base 2.

(2 Marks)

(According to D. Knuth, this algorithm, called *Stein's algorithm*, was probably known in 1st-century China.)

**\* Exercise 5.6**

Solve the following recurrence relations:

$$\begin{array}{lll} a_n = a_{n-1} + 6a_{n-2}, & n \geq 2, & a_0 = 3, a_1 = 6, \\ a_{n+2} = -4a_{n+1} + 5a_n, & n \geq 0, & a_0 = 2, a_1 = 8. \end{array}$$

(4 Marks)

**Exercise 5.7**

Prove Theorem 2.3.8 of the lecture, which states that all solution to a linear homogeneous recurrence relation of degree two are of the form

$$a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot nr_0^n, \quad \alpha_1, \alpha_2 \in \mathbb{R}, \quad n \in \mathbb{N}.$$

if there is only a single characteristic root  $r_0$ .

(2 Marks)

**\* Exercise 5.8**

Find all solutions of the following recurrence relations:

$$\begin{array}{l} a_n = 5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n, \\ a_n = -5a_{n-1} - 6a_{n-2} + 2^n + 3n, \\ a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n. \end{array}$$

(6 Marks)