JOINT INSTITUTE
交大密西根学院

Discrete Mathematics

Exercise Pool for the Final Exam

Relations

Exercise 1. Define

$$a_{m,n} := \begin{cases} 1 & m = n = 0, \\ a_{m-1,n} + 2 & m > 0, n = 0, \\ a_{m,n-1} + 2 & n > 0. \end{cases}$$

Use the lexicographic ordering and well-ordered indeuction to prove that

$$a_{m,n} = 2(m + n) + 1.$$

(4 Marks)

Last used: **Fall 2013**

Solution. The statement is true for $m = n = 0$, since

$$a_{0,0} = 1 = 2(0 + 0) + 1.$$

(1 Mark) Now fix (p, q) and suppose the stament is true for all (m, n) such that $(m, n) \prec (p, q)$, i.e., either $m < p$ or $m = p$ and $n < q$. **(1 Mark)**

- Suppose that $q = 0$. Then

  $$a_{p,q} = a_{p-1,0} = a_{p-1,0} + 2 = 2(p - 1) + 1 + 2 = 2p + 1,$$

  so the statement is true in this case. **(1 Mark)**

- Suppose that $q > 0$. Then

  $$a_{p,q} = a_{p,q-1} + 2 = 2(p + q - 1) + 1 + 2 = 2(p + q) + 1,$$

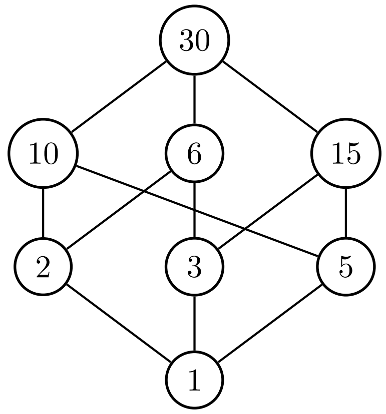
  and again the statement is true. **(1 Mark)**

This completes the proof by induction.

Exercise 2. Draw the Hasse diagram for the relation $a \sim b \quad \Leftrightarrow \quad a \mid b$ defined on the set $S = \{1, 2, 3, 5, 6, 10, 15, 30\}$.
(3 Marks)

Last used: **Fall 2012**

Solution.



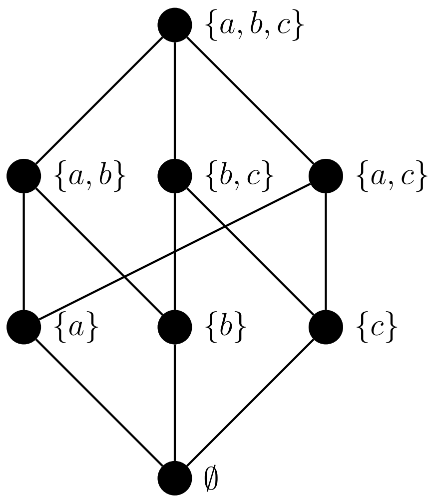
Exercise 3. Let $M = \{a, b, c\}$ and let $\subseteq$ be the usual subset relation. Draw the Hasse diagram for the partially ordered set $(\mathcal{P}(M), \subseteq)$, where $\mathcal{P}(M)$ is the power set of M.
(3 Marks)

Last used: **Fall 2014**

Solution. The power set has six elements:

$$\mathcal{P}(M) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

(1 Mark) The Hasse diagram is as follows:



(2 Marks)

Exercise 4. Let $A = \{1, 2, 3, 4\}$.

i) How many relations on A are there?

ii) Find a relation R on A that has exactly 3 ordered pair members and is both symmetric and antisymmetric.

iii) Prove that every relation R on A with 15 ordered pair members is not transitive.

iv) Prove that the intersection of two transitive relations on the set A is also transitive.

v) Prove that the union of two symmetric relations on the set A is also symmetric.

vi) Prove that the complement $\overline{R} = A^2 \setminus R$ of a symmetric relation R on the set A is symmetric.

vii) Give an example that shows that the union of two antisymmetric relations on the set A need not be antisymmetric.

(1 + 1 + 1 + 2 + 2 + 2 + 1 Marks)

Last used: **Fall 2012**

Solution.

i) Each relation can be represented by a 4×4 zero-one matrix. Hence there are 2^{16} possible relations.

ii) $R = \{(1,1),(2,2),(3,3)\}$.

iii) Suppose such a relation is transitive and (x,y) is the (only) ordered pair that does not belong to R. Take z different from x and y. Then xRz and zRy together with transitivity imply that xRy, a contradiction.

iv) Let R and S be transitive relations on the set A. To prove that $R \cap S$ is also transitive, suppose $x(R \cap S)y$ and $y(R \cap S)z$. Then xRz and xSz because each of the relations is transitive. Hence $x(R \cap S)z$. Thus $R \cap S$ is transitive.

v) Let R and S be symmetric relations on the set A. To prove that $R \cup S$ is also symmetric, suppose $x(R \cup S)y$. Then xRy or xSy by the meaning of union. Since R and S are both symmetric, it follows that either yRx or ySx. But this is just what is needed to prove that $y(R \cup S)x$. Thus $R \cup S$ is symmetric.

vi) Suppose R is symmetric. To prove R is symmetric, we use the definition of symmetry. Suppose xRy. Then (x,y) does not belong to R. If (y,x) belongs to R, then by symmetry of R, the ordered pair (x,y) would have to belong as well. Therefore (y,x) does not belong to R. But this means that (y,x) does belong to R. Thus R is symmetric, by the definition of symmetry.

vii) Take $R = \{(1,1),(1,2)\}$ and $S = \{(1,1),(2,1)\}$.

Graph Theory

Exercise 1. Prove or disprove: the hypercube graphs Q_n, $n \in \mathbb{N} \setminus \{0\}$, are planar.
(5 Marks)

Solution. Suppose that Q_n has v_n vertices and e_n edges. Since Q_{n+1} is constructed by making two copies of Q_n and joining the pairwise "copies" of the vertices, we have

$$v_{n+1} = 2v_n, \qquad\qquad e_{n+1} = 2e_n + v_n.$$

(1 Mark) Since $v_1 = 2$, we have $v_n = 2^n$, so

$$e_{n+1} = 2e_n + 2^n.$$

In particular, $v_3 = 8$ and $e_3 = 12$ but $v_4 = 16$ and $e_4 = 30$. **(1 Mark)** Since a hypercube does not have simple ciruits of length 3 (because two adjacent vertices are associated with binary numbers differing in exactly one digit), **(1 Mark)** it follows that Q_n is not planar if $e_n > 2v_n - 4$. **(1 Mark)** This is the case for $n = 4$ (and all greater values of n), so it is not true that all hypercube graophs are planar. **(1 Mark)**

Last used: **Fall 2013**

Exercise 2. If $G = (V, E)$ is a simple graph, then we define the *complement* of G as the graph $\overline{G} := (V, E')$, where $E' = \{\{x, y\} : x \neq y,\ \{x, y\} \notin E\}$. In other words, $\overline{G}$ is the simple graph having the same vertices as G and containing precisely those edges that are not E.

Suppose that G has n vertices. Prove that if both G and $\overline{G}$ are planar, then $n \leq 10$.
(4 Marks)

Last used: **Fall 2014**

Solution. Let $|E| = e$ and $|E'| = e'$. If G and $\overline{G}$ are both planar, then $e \leq 3n - 6$ and $e' \leq 3n - 6$. **(1 Mark)** Since a complete graph has $n(n-1)/2$ edges, we know that

$$e' = \frac{n(n-1)}{2} - e$$

(1 Mark) which gives

$$\frac{n(n-1)}{2} - e + e \leq 3n - 6 + 3n - 6 = 6n - 12$$

or

$$n^2 - 13n + 24 \leq 0.$$

(1 Mark) This quadratic equation has solutions $n = 13/2 \pm \sqrt{169/4 - 96/4} = 6.5 \pm \sqrt{73/4} < 6.5 + 4.5 = 11$. Since $n \in \mathbb{N}$, it follows that $n \leq 10$. **(1 Mark)**

Exercise 3. Let G be a simple graph with n vertices. Show that if G has more than

$$\frac{(n-1)(n-2)}{2}$$

edges, then G must be connected.
(4 Marks)

Last used: **Fall 2012**

Solution. Suppose that G is not connected. Then there are two components, with $m < n$ and $n - m$ vertcies, respectively, that are not joined by any edges. Each component can have at most $m(m-1)/2$ and $(n-m)(n-m-1)/2$ edges, respectively. Since there are no edges joining the two components, the total number of edges of G is at most

$$\frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2} = \frac{n(n-1)}{2} - m(n-m).$$

This number is maximal when $m = 1$ or $m = n - 1$, yielding

$$\frac{(n-1)(n-2)}{2}$$

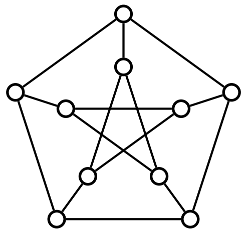
for the maximal number of edges of G if G is not connected.

Exercise 4. Show that a simple graph with n vertices and e edges has at least $n - e$ connected components.
(2 Marks)

Last used: **Fall 2012**

Solution. We can prove this by induction in e: Any graph with n edges and $e = 1$ edges will have at least $n-1$ connected components, because the one edge joins at most two vertices. Now suppose that any graph with n vertices and e edges has at least $n - e$ components. Consider a graph with n vertices and $e+1$ edges that has k connected components. Remove an edge. Then the resulting graph has at most $k+1$ components. We then have $k+1 \le n-e$, i.e., $k \le n-e-1$. This proves the assertion for any graph with $e+1$ edges. **(2 Marks)**

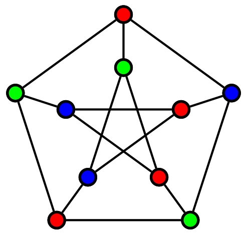
Exercise 5. Find the chromatic number of the *Petersen graph*, shown below:



Prove your answer by exhibiting a k-coloring and showing that fewer colors will not be sufficient.
(3 Marks)

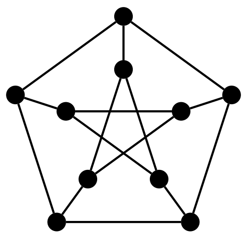
Last used: **Fall 2014**

Solution. Since the Petersen graph contains a cycle of odd length (the outer pentagon), at least three colors are necessary. **(2 Marks)** A 3-coloring is shown below:



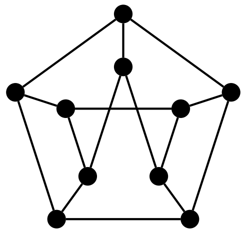
(~~2~~ 4 Marks)

Exercise 6. The *crossing number* $k \in \mathbb{N}$ of a graph is the minimum number of edges in the graph such that the graph with these edges deleted has a planar representation. The crossing number of a planar graph is zero. Find the crossing number of the *Petersen graph*, shown below:



Prove your answer by showing which edges need to be deleted and giving the corresponding planar representation. Show also that deleting fewer edges will not be sufficient. (You do not need to formally prove that the planar representation is isomorphic to the original representation with deleted edges.)
(3 Marks)

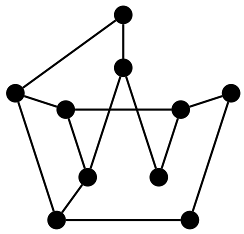
Exercise 7. Mark two edges in the following graph such that it remains nonplanar after both are deleted. Prove your answer.



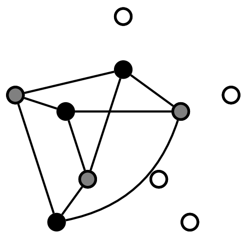
(3 Marks)

Last used: **Fall 2012**

Solution. The following two edges can be deleted:

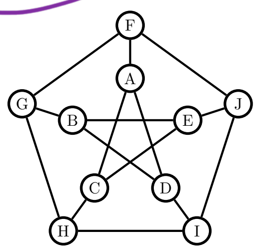


(1 Mark) The resulting graph is homeomorphic to

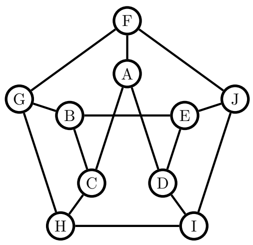


This graph is just $K_{3,3}$, so the graph with two edges deleted has a subgraph homeomorphic to $K_{3,3}$. Hence it is non-planar. **(2 Marks)**

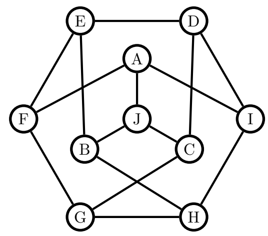
Exercise 8. Consider the following three graphs:



Graph I



Graph II



Graph III

Determine which of these graphs are isomorphic (I and II?, II and III?, I and III?). If they are, give the graph isomorphism and prove that it is an isomorphism. If they are not, prove this.
(6 Marks)

Last used: **Fall 2014**

Solution. The graphs I and III are isomorphic. **(1/2 Mark)** An isomorphism is given by $\varphi\colon V_{\mathrm{III}} \to V_{\mathrm{I}}$,

$$\begin{aligned} \varphi\colon A &\mapsto H, & B &\mapsto A, & C &\mapsto E, & D &\mapsto J, & E &\mapsto F,\\ F &\mapsto G, & G &\mapsto B, & H &\mapsto D, & I &\mapsto I, & J &\mapsto C, \end{aligned}$$

(1/2 Mark). To prove that φ is an isomorphism, consider the adjacency matrix for Graph III:

$$A_{\mathrm{III}} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \end{array} \begin{array}{c} \begin{array}{cccccccccc} A & B & C & D & E & F & G & H & I & J \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The adjacency matrix for $\varphi(III)$ is

$$A_{\varphi(\mathrm{III})} = \begin{array}{c} \\ H \\ A \\ E \\ J \\ F \\ G \\ B \\ D \\ I \\ C \end{array} \begin{array}{c} \begin{array}{cccccccccc} H & A & E & J & F & G & B & D & I & C \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Since the two matrices coincide, φ is a graph isomorphism and the graphs I and III are isomorphic. **(2 Marks)**

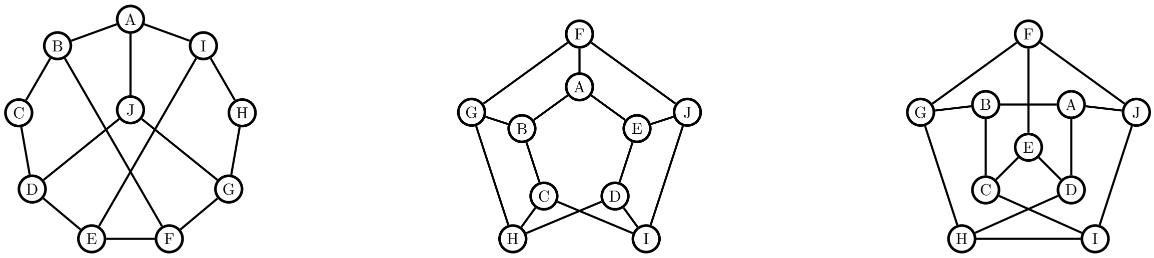
Graph II contains a simple circuit of length 4 (B,C,H,G,B). Graph I does not contain such a circuit, as follows: let A,B,C,D,E be the “inner vertices” and F,G,H,I,J the “outer vertices”. Since the subgraphs consisting of only the inner vertices or only the outer vertices are both 5-cycles C_5, there can not be a simple circuit of length 4 passing only through the inner or only through the outer vertices.

Every inner vertex is adjacent to precisely one outer vertex and vice-versa. Furthermore, if two outer vertices are adjacent then the two respective adjacent inner vertices are not adjacent. Therefore, a simple circuit of length 4 starting at an outer vertex would have to pass first to an inner vertex, then to another inner vertex but would then not be able to return to the original outer vertex. Similarly, there can not be a simple circuit of length 4 starting at an inner vertex.

Since the existence of a simple circuit of length n is a graph invariant, it follows that graphs II and I are not isomorphic. **(2 Marks)**

Since graph I is isomorphic to graph III, it follows that graphs III and II are also not isomorphic. **(1 Mark)**

Exercise 9. Consider the following three graphs:



Graph I Graph II Graph III

Determine which of these graphs are isomorphic (I and II?, II and III?, I and III?). If they are, give the graph isomorphism and prove that it is an isomorphism. If they are not, prove this.
(6 Marks)

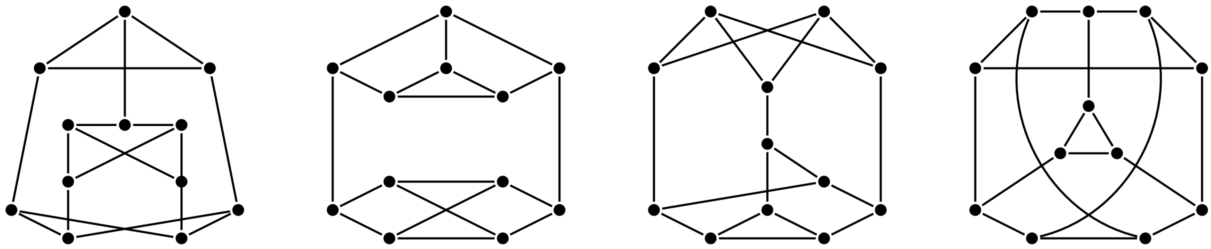
Last used: **Fall 2013**

Solution. We first note that Graph II is bipartite; a bipartition of the vertex set $V = V_1 \cup V_2$ is given by

$$V_1 = \{B, E, F, H, I\}, \qquad V_2 = \{A, C, D, G, J\}.$$

Graph III contains a simple circuit of length 5, given by (F, G, B, A, J, F). Therefore, Graph III can not be bipartite (let $F \in V_1$; then $G \in V_2$, $B \in V_1$, $A \in V_2$, $J \in V_1$, $F \in V_2$ - contradiction). Graph II and Graph III are not isomorphic.
Graph I contains a vertex of degree 2 (H), while all vertices of Graphs II and III have degree 3, so Graph I is not isomorphic to either of them. In conclusion, none of the three graphs are isomorphic.
(2 Marks) for each of the three non-isomorphisms, shown by whatever means.

Exercise 10. If two graphs are isomorphic, they either both have a hamiltonian circuit or they both do not have such a circuit. Use this to show that the graphs below are non-isomorphic as follows:



A B C D

Each of the graphs above has a single simple circuit of length 3; in each of the graphs, delete the vertices in the circuit and all edges incident to them, obtaining subgraphs A', B', C', D'.

i) Which of the A', B', C', D' graphs have a hamiltonian circuit? Draw the circuit in those that have one.

ii) Show that the graphs among A', B', C', D' with a hamiltonian circuit are not isomorphic to each other.

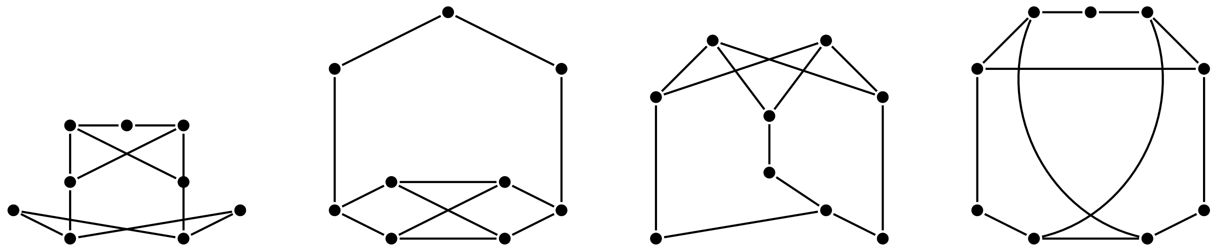
iii) Show that the remaining graphs among A', B', C', D' are not isomorphic to each other.

iv) Conclude that the four graphs A, B, C, D are mutually non-isomnorphic.

(2+3+3+2 Marks)

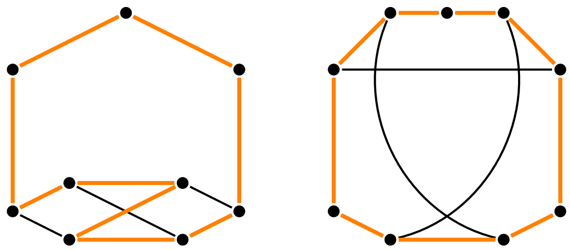
Last used: **Fall 2012**

Solution. We first plot the four graphs with the simple circuit of length 3 removed:



A' B' C' D'

i) The graphs B' and D' have a hamiltonian circuit, the other two do not.



B' D'

ii) For example: B' has a simple circuit of length 4 (needs to be drawn), while D' does not (proof by adjacency matrix needed).

Alternatively: The subgraph of vertices of degree 2 in B' is connected, while the corresponding subgraph in D' is not connected.

Therefore, B' and D' are not isomorphic.

iii) The subgraph of A' consisting of vertices of order 3 and their connecting edges is connected. The corresponding subgraph of C' is not connected, so these subgraphs can not be isomorphic. It follows that A' and C' are not isomorphic.

iv) From the above discussion, we see that A', B', C, D' are miutually non-isomorphic. If any two of A, B, C, D were isomorphic, then there would exist a graph isomorphism which maps the simple circuits of length 3 onto each other. This isomorphism would then also map the remaining vertices into each other, so the "primed" graphs with the circuits removed would be isomorphic. Since we have shown that they are mutually-non-isomorphic, such an isomorphism can not exist, and therefore A, B, C, D are mutually non-isomorphic.

Trees

Exercise 1. Show that a connected graph is a tree if and only if the removal of any edge causes it to become disconnected.
(5 Marks)

Last used: **Fall 2012**

Solution. Suppose a tree is given. Designate any vertex to be the root and remove an edge. The edge is incident to two vertices, a and b. There is a unique path from the root of the tree to a and to b. Suppose that a path to b starts from the root, passes through a and ends at b. Then removing the edge will eliminate this path from the tree. Since the path was unique, there is no longer a path joining the root to b and so the tree has become disconnected.

Conversely, suppose a graph has the property that it becomes disconnected after removing any edge. Then the graph can not have any simple circuits, so it must be a tree.

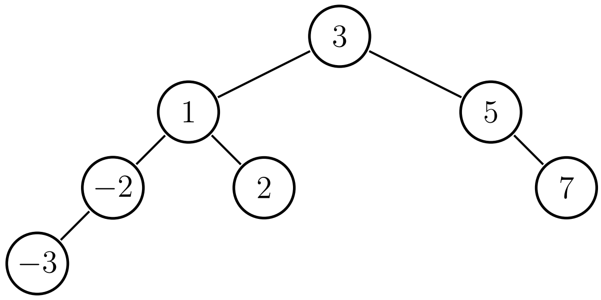
Exercise 2. Enter the following numbers, in the order given, into a binary search tree based on the natural ordering in $\mathbb{Z}$:

$$3,\ 1,\ 5,\ 7,\ -2,\ 2,\ -3$$

(2 Marks)

Last used: **Fall 2014**

Solution. The tree should look as follows:



Exercise 3. Consider a list of four distinct natural numbers $a_1, a_2, a_3, a_4 \in \mathbb{N}$.

i) In order to sort this list, how many comparisons are at most necessary? How many possible list orderings are there?

ii) What is a lower bound for the number of comparisons needed to sort the list? Is it possible to find an algorithm that uses exactly this lower bound? Draw a decision tree to prove your assertion.

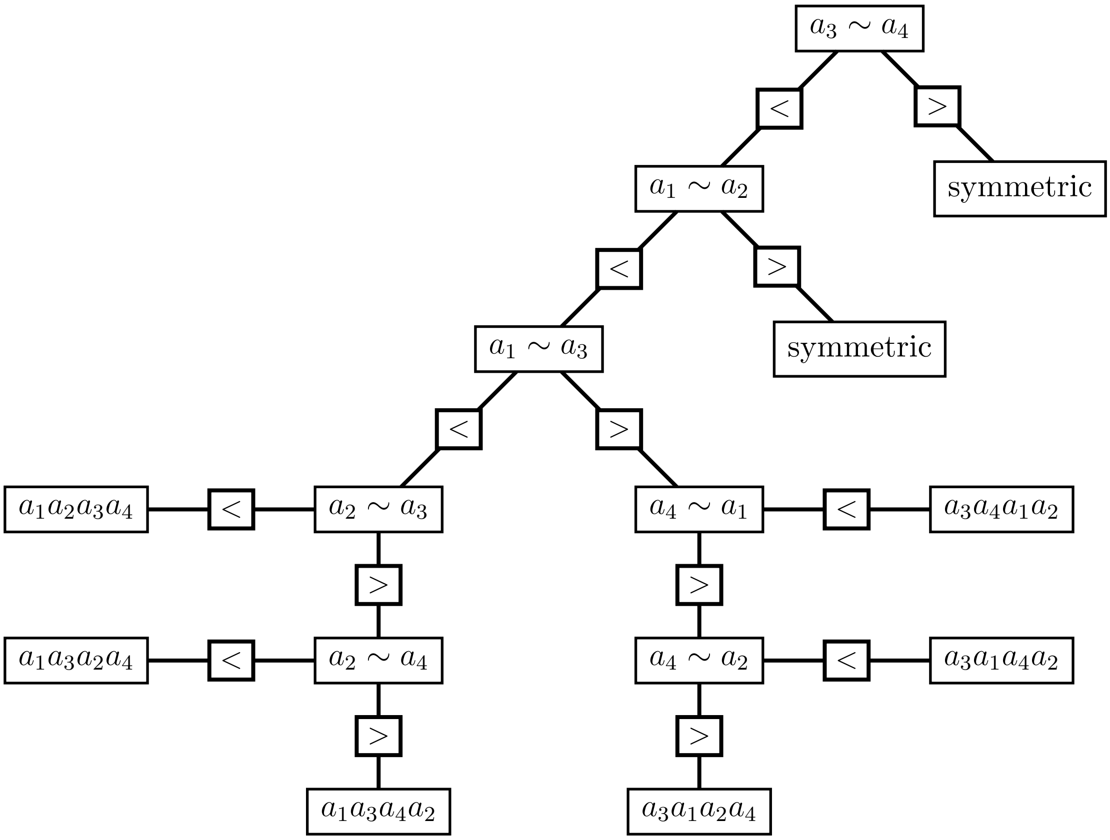
(1+4 Marks)

Last used: **Fall 2012**

Solution.

i) At most $\binom{4}{2} = 6$ comparisons are necessary. There are $4!$ possible list orderings.

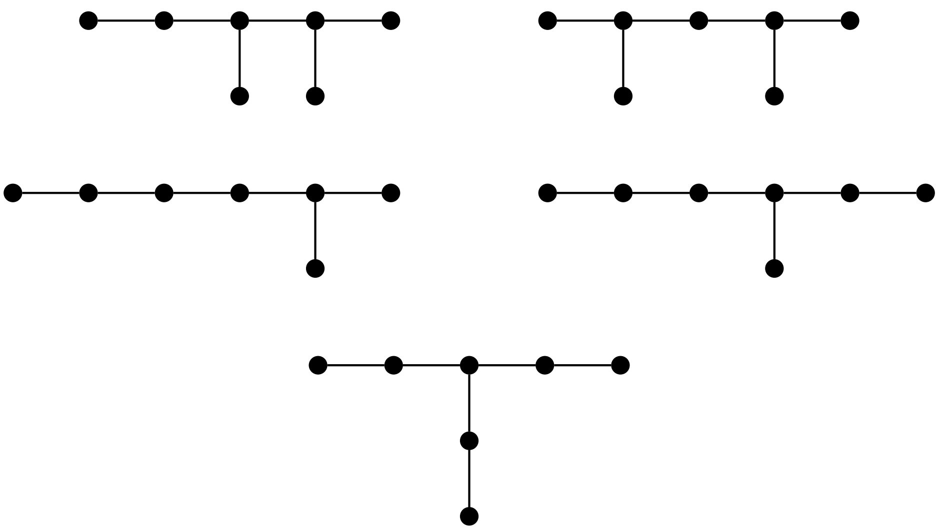
ii) A lower bound is given by $\lceil \log_2(4!) \rceil = \lceil \log_2(24) \rceil = 5$ comparisons.



Exercise 4. Draw all possibe trees with 7 vertices such that the largest degree of the vertices is equal to 3.
(4 Marks)

Last used: **Fall 2012**

Solution. A tree with 7 vertices has 6 edges. The total sum of the degrees of the 7 vertices is then 12. There are an even number of vertices of odd degree, so we can have degree sequences 3,2,2,2,1,1,1 or 3,3,2,1,1,1,1. No other degree sequences are possible. This leaves the following possible trees:



Hence, there are 5 possible trees.

Generating Functions

Exercise 1. Consider the recurrence equation

$$3a_n - 3a_{n-1} = n^2, \qquad n \geq 1, \qquad\qquad a_0 = 1.$$

i) Use the recurrence equation to find the generating function given by $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

ii) Expand $a(x)$ into a series at $x = 0$ to find an explicit representation for a_n, $n \in \mathbb{N}$.

(3+4 Marks)

Last used: **Fall 2014**

Solution.

i) The ansatz yields

$$\begin{aligned} a(x) &= \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} a_{n-1} x^n + \frac{1}{3} \sum_{n=1}^{\infty} n^2 x^n \\ &= 1 + x \sum_{n=0}^{\infty} a_n x^n + \frac{1}{3} \sum_{n=2}^{\infty} n(n-1) x^n + \frac{1}{3} \sum_{n=1}^{\infty} n x^n \\ &= 1 + x a(x) + \frac{x^2}{3} \sum_{n=2}^{\infty} n(n-1) x^{n-2} + \frac{x}{3} \sum_{n=1}^{\infty} n x^{n-1} \\ &= 1 + x a(x) + \frac{x^2}{3} \left(\frac{1}{1-x} \right)'' + \frac{x}{3} \left(\frac{1}{1-x} \right)' \\ &= 1 + x a(x) + \frac{2x^2}{3(1-x)^3} + \frac{x}{3(1-x)^2} \end{aligned}$$

Hence,

$$a(x) = \frac{1}{1-x} + \frac{2x^2}{3(1-x)^4} + \frac{x}{3(1-x)^3}$$

ii) From the above calculation,

$$\begin{aligned} a(x) &= \frac{1}{1-x} + \frac{1}{3(1-x)} \sum_{n=1}^{\infty} n^2 x^n \\ &= \sum_{n=0}^{\infty} x^n + \frac{1}{3} \Big(\sum_{n=0}^{\infty} x^n \Big) \Big(\sum_{n=0}^{\infty} n^2 x^n \Big) \\ &= \sum_{n=0}^{\infty} x^n + \frac{1}{3} \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} 1 \cdot k^2 \Big) x^n. \end{aligned}$$

Since

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

we obtain

$$a(x) = \sum_{n=0}^{\infty} \left(1 + \frac{n(n+1)(2n+1)}{18} \right) x^n$$

and conclude

$$a_n = \frac{n(n+1)(2n+1)}{18}.$$

Exercise 2. The *Schröder numbers* S_n are related to the Catalan numbers as follows: The alphabet consists of $\{\uparrow, \downarrow, \rightarrow\rightarrow\}$. A Schröder word is the empty string or a word w characterized by

P1 $\#(\uparrow) = \#(\downarrow)$

P2 In any word consisting of the first k letters of w, $\#(\uparrow) \geq \#(\downarrow)$ $(k = 1, \ldots, l(w))$

The length of a Schröder word w is given by $l(w) = \#(\uparrow) + \#(\downarrow) + 2\#(\rightarrow\rightarrow)$. For example, the possible Schröder words of length 4 are illustrated below:

The *Schröder number* S_n is the number of Schröder words of length $2n$. One defines $S_0 := 1$.

i) Give a recursive definition of a Schröder word and from it derive the recurrence relation

$$S_n = S_{n-1} + \sum_{i+j=n-1} S_i S_j, \qquad n \geq 1.$$

(You do not have to prove that the recursive definition coincides with that of P! and P2 above).

ii) Find the generationg function $S(x) = \sum_{n=0}^{\infty} S_n x^n$.

(4+3 Marks)

Solution.

i) We use the following recursive definition:

- The empty string is a Schröder word.
- If w is a Schröder word, then $\rightarrow\rightarrow w$ is a Schröder word.
- If w_1 and w_2 are Schröder words, then $\uparrow w_1 \downarrow w_2$ is a Schröder word.

The set of Schröder words of length $2n$ is then

$$\begin{aligned}\{w \colon l(w) = 2n\} = \{&w = \rightarrow\rightarrow w_1 \colon l(w_1) = 2n-2\} \\ &\cup \{w = \uparrow w_2 \downarrow w_3 \colon l(w_2) + l(w_3) = 2n-2\}.\end{aligned}$$

These sets are disjoint, so it follows that

$$\begin{aligned}S_n &= |\{w \colon l(w) = 2n\}| = \left|\{w = \rightarrow\rightarrow w_1 \colon l(w_1) = 2n-2\}\right| \\ &\quad + \left|\{w = \uparrow w_2 \downarrow w_3 \colon l(w_2) + l(w_3) = 2n-2\}\right| \\ &= S_{n-1} + \sum_{i+j=n-1} S_i S_j.\end{aligned}$$

ii) We have

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} S_n x^n = 1 + \sum_{n=1}^{\infty} S_n x^n \\ &= 1 + \sum_{n=1}^{\infty} S_{n-1} x^n + \sum_{n=1}^{\infty} \Big(\sum_{i+j=n-1} S_i S_j \Big) x^n \\ &= 1 + xS(x) + xS(x)^2. \end{aligned}$$

This implies

$$xS(x)^2 + (x-1)S(x) + 1 = 0$$

or

$$S(x) = \frac{1 - x \pm \sqrt{1 - 6x + x^2}}{2x}$$

Exercise 3. Let a_n be the number of words of length n that comprise the letters $0, 1, 2$ but which never include two successive zeroes. For example, $a_1 = 3$, $a_2 = 8$, $a_3 = 22$ etc.

i) Derive a recurrence relation for a_n.

ii) Find the generating function for the sequence (a_n).

iii) Use the generating function to obtain an explicit formula for a_n.

(2+3+3 Marks)

Last used: **Fall 2012**

Solution.

i) Consider the first letter. If it is a 1 or 2, the next $n-1$ digits can be chosen in a_{n-1} ways. If it is 0, the next digit can be either a 1 or 2 and the succeding digits can be chosen in a_{n-2} ways. We hence have

$$a_n = 2a_{n-1} + 2a_{n-2}.$$

for $n \geq 3$ (or even for $n \geq 2$ if we set $n_0 = 1$).

ii) Let

$$f(x) := \sum_{n=0}^{\infty} a_n x^n,$$

where we set $a_0 = 1$ for completeness. Then

$$\begin{aligned} f(x) &= 1 + 3x + \sum_{n=2}^{\infty} a_n x^n = 3x + \sum_{n=2}^{\infty} (2a_{n-1} + 2a_{n-2}) x^n \\ &= 1 + 3x + 2\sum_{n=2}^{\infty} a_{n-1} x^n + 2\sum_{n=2}^{\infty} a_{n-2} x^n \\ &= 1 + 3x + 2x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= 1 + 3x + 2x(f(x) - 1) + 2x^2 f(x). \end{aligned}$$

This gives

$$f(x) = \frac{1+x}{1 - 2x - 2x^2}.$$

iii) The roots of $1-2x-2x^2$ are given by

$$-\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{1}{2}}=-\frac{1\mp\sqrt{3}}{2}=\frac{1}{1\pm\sqrt{3}}$$

so

$$\begin{aligned}
f(x)&=\frac{1+x}{\left(x-\frac{1}{1+\sqrt{3}}\right)\left(x-\frac{1}{1-\sqrt{3}}\right)}\\
&=-\frac{1+x}{2\sqrt{3}}\left(\frac{1}{x-\frac{1}{1+\sqrt{3}}}-\frac{1}{x-\frac{1}{1-\sqrt{3}}}\right)\\
&=-\frac{1+x}{2\sqrt{3}}\left(\frac{-(1+\sqrt{3})}{1-(1+\sqrt{3})x}+\frac{(1-\sqrt{3})}{1-(1-\sqrt{3})x}\right)\\
&=-\frac{1+x}{2\sqrt{3}}\sum_{n=0}^{\infty}\left[(1-\sqrt{3})^{n+1}-(1+\sqrt{3})^{n+1}\right]x^n\\
&=1-\frac{1}{2\sqrt{3}}\sum_{n=1}^{\infty}\left[(1-\sqrt{3})^{n}-(1+\sqrt{3})^{n}+(1-\sqrt{3})^{n+1}-(1+\sqrt{3})^{n+1}\right]x^n\\
&=1-\sum_{n=1}^{\infty}\left[\frac{2-\sqrt{3}}{2\sqrt{3}}(1-\sqrt{3})^n-\frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^n\right]x^n\\
&=1+\sum_{n=1}^{\infty}\left[\frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n+\frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n\right]x^n
\end{aligned}$$

It follows that

$$a_n=\frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n+\frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n$$

for $n\in\mathbb{N}$.

Exercise 4. Let a_n be the number of words of length n that comprise the letters $0,1,2$ but which never include two successive zeroes. For example, $a_1=3$, $a_2=8$, $a_3=22$ etc.

i) Derive a recurrence relation for a_n.

ii) Find the generating function for the sequence (a_n).

iii) Use the generating function to obtain an explicit formula for a_n.

(2+3+3 Marks)

Last used: **Fall 2012**

Solution.

i) Consider the first letter. If it is a 1 or 2, the next $n-1$ digits can be chosen in a_{n-1} ways. If it is 0, the next digit can be either a 1 or 2 and the succeding digits can be chosen in a_{n-2} ways. We hence have

$$a_n=2a_{n-1}+2a_{n-2}.$$

for $n\geq 3$ (or even for $n\geq 2$ if we set $n_0=1$).

ii) Let

$$f(x) := \sum_{n=0}^{\infty} a_n x^n,$$

where we set $a_0 = 1$ for completeness. Then

$$\begin{aligned} f(x) &= 1 + 3x + \sum_{n=2}^{\infty} a_n x^n = 3x + \sum_{n=2}^{\infty} (2a_{n-1} + 2a_{n-2}) x^n \\ &= 1 + 3x + 2\sum_{n=2}^{\infty} a_{n-1} x^n + 2\sum_{n=2}^{\infty} a_{n-2} x^n \\ &= 1 + 3x + 2x\sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= 1 + 3x + 2x(f(x) - 1) + 2x^2 f(x). \end{aligned}$$

This gives

$$f(x) = \frac{1+x}{1-2x-2x^2}.$$

iii) The roots of $1 - 2x - 2x^2$ are given by

$$-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{2}} = -\frac{1 \mp \sqrt{3}}{2} = \frac{1}{1 \pm \sqrt{3}}$$

so

$$\begin{aligned} f(x) &= \frac{1+x}{\left(x - \frac{1}{1+\sqrt{3}}\right)\left(x - \frac{1}{1-\sqrt{3}}\right)} \\ &= -\frac{1+x}{2\sqrt{3}} \left(\frac{1}{x - \frac{1}{1+\sqrt{3}}} - \frac{1}{x - \frac{1}{1-\sqrt{3}}} \right) \\ &= -\frac{1+x}{2\sqrt{3}} \left(\frac{-(1+\sqrt{3})}{1-(1+\sqrt{3})x} + \frac{(1-\sqrt{3})}{1-(1-\sqrt{3})x} \right) \\ &= -\frac{1+x}{2\sqrt{3}} \sum_{n=0}^{\infty} \left[(1-\sqrt{3})^{n+1} - (1+\sqrt{3})^{n+1} \right] x^n \\ &= 1 - \frac{1}{2\sqrt{3}} \sum_{n=1}^{\infty} \left[(1-\sqrt{3})^n - (1+\sqrt{3})^n + (1-\sqrt{3})^{n+1} - (1+\sqrt{3})^{n+1} \right] x^n \\ &= 1 - \sum_{n=1}^{\infty} \left[\frac{2-\sqrt{3}}{2\sqrt{3}} (1-\sqrt{3})^n - \frac{2+\sqrt{3}}{2\sqrt{3}} (1+\sqrt{3})^n \right] x^n \\ &= 1 + \sum_{n=1}^{\infty} \left[\frac{3+2\sqrt{3}}{6} (1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6} (1-\sqrt{3})^n \right] x^n \end{aligned}$$

It follows that

$$a_n = \frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n$$

for $n \in \mathbb{N}$.

Exercise 5. The *Jacobsthal sequence* (J_n) is defined by the recurrence relation

$$J_0 = 0, \qquad J_1 = 1, \qquad J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \in \mathbb{N} \setminus \{0,1\}.$$

i) Solve the recurrence relation to obtain an explicit formula for J_n, $n \in \mathbb{N}$.

ii) Find the generating function for the sequence (J_n).

(4+4 Marks)

Last used: **Summer 2012**

Solution.

i) We make the ansatz $J_n = r^n$. Then

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

leads to the quadratic equation $r^2 - r - 2 = 0$, **(1 Mark)** which has solutions $r = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} = \frac{1}{2} \pm \frac{3}{2}$. **(1 Mark)** We hence see that

$$J_n = c_1 \cdot (-1)^n + c_2 \cdot 2^n, \qquad c_1, c_2 \in \mathbb{R}.$$

(1 Mark) Since $J_0 = 0$ and $J_1 = 1$ we have $c_1 + c_2 = 0$ and $2c_2 - c_1 = 1$. This gives $c_1 = -\frac{1}{3}$ and $c_2 = \frac{1}{3}$, so

$$J_n = \frac{2^n - (-1)^n}{3}.$$

(1 Mark)

ii) Define

$$f(x) = \sum_{n=0}^{\infty} J_n x^n$$

(1 Mark) Then

$$\begin{aligned} f(x) &= x + \sum_{n=2}^{\infty} J_n x^n \\ &= x + \sum_{n=2}^{\infty} (J_{n-1} + 2J_{n-2}) x^n \\ &= x + x \sum_{n=2}^{\infty} J_{n-1} x^{n-1} + 2x^2 \sum_{n=2}^{\infty} J_{n-2} x^{n-2} \\ &= x + x f(x) + 2x^2 f(x) \end{aligned}$$

(1 Mark) We hence have

$$f(x)(1 - x - 2x^2) = x$$

or

$$f(x) = \frac{x}{1 - x - 2x^2} = \frac{x}{(1+x)(1-2x)}$$

(2 Marks) for either expression