Basic Concepts in Logic

Propositional Logic, Statements

A ***statement*** (also called a ***proposition***) is anything we can regard as being either ***true*** or ***false***. We do not define here what the words "statement", "true" or "false" mean. This is beyond the purview of mathematics and falls into the realm of philosophy. Instead, we apply the principle that "we know it when we see it."

Contrary to the textbook, we will generally not use examples from the "real world" as statements. The reason is that in general objects in the real world are much to loosely define for the application of strict logic to make any sense. For example, the statement "It is raining." may be considered true by some people ("Yes, raindrops are falling out of the sky.") while at the same time false by others ("No, it is merely drizzling.") Furthermore, important information is missing (Where is it raining? When is it raining?). Some people may consider this information to be implicit in the statement (It is raining ***here*** and ***now***.) but others may not, and this causes all sorts of problems. Generally, applying strict logic to colloquial expressions is pointless.

The Natural Numbers

Instead, our examples will be based on numbers. For now, we assume that the set of natural numbers

$$\mathbb{N} := \{0, 1, 2, 3, \ldots\}$$

has been constructed. In particular, we assume that we know what a ***set*** is! If n is a natural number, we write $n \in \mathbb{N}$. (We will later discuss naive set theory and give a formal construction of the natural numbers.) We also assume that on $\mathbb{N}$ we have defined the operations of addition $+ \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and multiplication $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and that their various properties (commutativity, associativity, distributivity) hold.

The Natural Numbers

1.1. Definition. Let $m, n \in \mathbb{N}$ be natural numbers.

(i) We say that n is ***greater than or equal to*** m, writing $n \geq m$, if there exists some $k \in \mathbb{N}$ such that $n = m + k$. If we can choose $k \neq 0$, we say n is ***greater than*** m and write $n > m$.

(ii) We say that m ***divides*** n, writing $m \mid n$, if there exists some $k \in \mathbb{N}$ such that $n = m \cdot k$.

(iii) If $2 \mid n$, we say that n is even.

(iv) If there exists some $k \in \mathbb{N}$ such that $n = 2k + 1$, we say that n is odd.

(v) Suppose that $n > 1$. If there does not exist any $k \in \mathbb{N}$ with $1 < k < n$ such that $k \mid n$, we say that n is ***prime***.

It can be proven that every number is either even or odd and not both. We also assume this for the purposes of our examples.

Statements

1.2. Examples.

- "$3 > 2$" is a ***true statement***.
- "$x^3 > 10$" is not a statement, because we can not decide whether it is true or not.
- "the cube of any natural number is greater than 10" is a ***false statement***.

The last example can be written using a ***statement variable*** n:

- "For any natural number n, $n^3 > 10$"

The first part of the statement is a ***quantifier*** ("for any natural number n"), while the second part is called a ***statement form*** or ***predicate*** ("$n^3 > 10$").

A statement form becomes a statement (which can then be either true or false) when the variable takes on a specific value; for example, $3^3 > 10$ is a true statement and $1^3 > 10$ is a false statement.

Working with Statements

We will denote statements by capital letters such as $A, B, C, \ldots$ and statement forms by symbols such as $A(x)$ or $B(x, y, z)$ etc.

1.3. Examples.

- A: 4 is an even number.
- $B: 2 > 3$.
- $A(n)$: $1 + 2 + 3 + \ldots + n = n(n+1)/2$.

We will now introduce logical operations on statements. The simplest possible type of operation is a ***unary operation***, i.e., it takes a statement A and returns a statement B.

1.4. Definition. Let A be a statement. Then we define the ***negation of*** A, written as $\neg A$, to be the statement that is true if A is false and false if A is true.

Negation

1.5. Example. If A is the statement $A\colon 2 > 3$, then the negation of A is $\neg A\colon 2 \not> 3$.

We can describe the action of the unary operation $\neg$ through the following table:

A	$\neg A$
T	F
F	T

If A is true (T), then $\neg A$ is false (F) and vice-versa. Such a table is called a ***truth table***.

We will use truth tables to define all our operations on statements.

Conjunction

The next simplest type of operations on statements are ***binary operations***. The have two statements as arguments and return a single statement, called a ***compound statement***, whose truth or falsehood depends on the truth or falsehood of the original two statements.

1.6. Definition. Let A and B be two statements. Then we define the ***conjunction*** of A and B, written $A \wedge B$, by the following truth table:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

The conjunction $A \wedge B$ is spoken "A and B." It is true only if both A and B are true, false otherwise.

Disjunction

1.7. Definition. Let A and B be two statements. Then we define the ***disjunction*** of A and B, written $A \vee B$, by the following truth table:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

The conjunction $A \vee B$ is spoken "A or B." It is true only if either A or B is true, false otherwise.

1.8. Example.

- Let A: $2 > 0$ and B: $1 + 1 = 1$. Then $A \wedge B$ is false and $A \vee B$ is true.
- Let A be a statement. Then the compound statement "$A \vee (\neg A)$" is always true, and "$A \wedge (\neg A)$" is always false.

Proofs using Truth Tables

How do we prove that "$A \vee (\neg A)$" is an always true statement? We are claiming that $A \vee (\neg A)$ will be a true statement, regardless of whether the statement A is true or not. To prove this, we go through all possibilities using a truth table:

A	$\neg A$	$A \vee (\neg A)$
T	F	T
F	T	T

Since the column for $A \vee (\neg A)$ only lists T for "true," we see that $A \vee (\neg A)$ is always true. A compound statement that is always true is called a ***tautology***.

Correspondingly, the truth table for $A \wedge (\neg A)$ is

A	$\neg A$	$A \wedge (\neg A)$
T	F	F
F	T	F

so $A \wedge (\neg A)$ is always false. A compound statement that is always false is called a ***contradiction***.

Implication

1.9. Definition. Let A and B be two statements. Then we define the ***implication*** of B and A, written $A \Rightarrow B$, by the following truth table:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

We read "$A \Rightarrow B$" as "A implies B," "if A, then B" or "A only if B". (The last formulation refers to the fact that A can not be true unless B is true.)

To illustrate why the implication is defined the way it is, it is useful to look at a specific implication of predicates: we expect the predicate

$$A(n)\colon n \text{ is prime} \Rightarrow n \text{ is odd}, \qquad\qquad n \in \mathbb{N}, \tag{1.1}$$

to be false if and only if we can find a prime number n that is not odd.

Implication

By selecting different values of n we obtain the following types of statements

- $n = 3$. Then n is prime and n is odd, so we have $\mathrm{T} \Rightarrow \mathrm{T}$.
- $n = 4$. Then n is not prime and n is not odd, so we have $\mathrm{F} \Rightarrow \mathrm{F}$.
- $n = 9$. Then n is not prime, but n is odd. We have $\mathrm{F} \Rightarrow \mathrm{T}$.

None of these values of n would cause us to designate (1.1) as generating false statements. Therefore, we should assign the truth value "T" to each of these three cases.

However, let us take

- $n = 2$. Then n is prime, but n is not odd. We have $\mathrm{T} \Rightarrow \mathrm{F}$.

This is clearly a value of n for which (1.1) should be false. Hence, we we should assign the truth value "F" to the implication $\mathrm{T} \Rightarrow \mathrm{F}$.

Equivalence

1.10. Definition. Let A and B be two statements. Then we define the ***equivalence*** of A and B, written $A \Leftrightarrow B$, by the following truth table:

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

We read "$A \Leftrightarrow B$" as "A is equivalent to B" or "A if and only if B". Some textbooks abbreviate "if and only if" by "iff."

If A and B are both true or both false, then they are equivalent. Otherwise, they are not equivalent. In propositional logic, "equivalence" is the closest thing to the "equality" of arithmetic.

Equivalence

On the one hand, logical equivalence is strange; two statements A and B do not need to have anything to do with each other to be equivalent. For example, the statements "$2 > 0$" and "$100 = 99 + 1$" are both true, so they are equivalent.

On the other hand, we use equivalence to manipulate compound statements.

1.11. Definition. Two compound statements A and B are called ***logically equivalent*** if $A \Leftrightarrow B$ is a tautology. We then write $A \equiv B$.

1.12. Example. The two ***de Morgan rules*** are the tautologies

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B), \qquad \neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B).$$

In other words, they state that $\neg(A \vee B)$ is logically equivalent to $(\neg A) \wedge (\neg B)$ and $\neg(A \wedge B)$ is logically equivalent to $(\neg A) \vee (\neg B)$.

Contraposition

An important tautology is the ***contrapositive*** of the compound statement $A \Rightarrow B$.

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A).$$

For example, for any natural number n, the statement "$n > 0 \Rightarrow n^3 > 0$" is equivalent to "$n^3 \not> 0 \Rightarrow n \not> 0$." This principle is used in proofs by contradiction.

We prove the contrapositive using a truth table:

A	B	$\neg A$	$\neg B$	$\neg B \Rightarrow \neg A$	$A \Rightarrow B$	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Rye Whiskey

The following song is an old Western-style song, called “Rye Whiskey” and performed by Tex Ritter in the 1930’s and 1940’s.

If the ocean was whiskey and I was a duck,
I’d swim to the bottom and never come up.

But the ocean ain’t whiskey, and I ain’t no duck,
So I’ll play jack-of-diamonds and trust to my luck.

For it’s whiskey, rye whiskey, rye whiskey I cry.
If I don’t get rye whiskey I surely will die.

The lyrics make sense (at least as much as song lyrics generally do).

Rye Whiskey

One can use de Morgan's rules and the contrapositive to re-write the song lyrics as follows

> *If I never reach bottom or sometimes come up,*
> *Then the ocean's not whiskey, or I'm not a duck.*
>
> *But my luck can't be trusted, or the cards I'll not buck,*
> *So the ocean is whiskey or I am a duck.*
>
> *For it's whiskey, rye whiskey, rye whiskey I cry.*
> *If my death is uncertain, then I get whiskey (rye).*

These lyrics seem to be logically equivalent to the original song, but are just humorous nonsense. This again illustrates clearly why it is futile to apply mathematical logic to everyday language.

This example is due to (clickable link) W. P. Cooke, The American Mathematical Monthly, Vol. 76, No. 9 (Nov., 1969), p. 1051.

Some Logical Equivalencies

The following logical equivalencies can be established using truth tables or by using previously proven equivalencies. Here T is the compound statement that is always true, $T: A \vee (\neg A)$ and F is the compound statement that is always false, $F: A \wedge (\neg A)$

Equivalence	Name
$A \wedge T \equiv A$	Identity for $\wedge$
$A \vee F \equiv A$	Identity for $\vee$
$A \wedge F \equiv F$	Dominator for $\wedge$
$A \vee T \equiv T$	Dominator for $\vee$
$A \wedge A \equiv A$	Idempotency of $\wedge$
$A \vee A \equiv A$	Idempotency of $\vee$
$\neg(\neg A) \equiv A$	Double negation

Some Logical Equivalencies

Equivalence	Name
$A \wedge B \equiv B \wedge A$	Commutativity of $\wedge$
$A \vee B \equiv B \vee A$	Commutativity of $\vee$
$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$	Associativity of $\wedge$
$(A \vee B) \vee C \equiv A \vee (B \vee C)$	Associativity of $\vee$
$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	Distributivity
$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$	Distributivity
$A \vee (A \wedge B) \equiv A$	Absorption
$A \wedge (A \vee B) \equiv A$	Absorption

These laws include all that are necessary for a ***boolean algebra generated by*** $\wedge$ ***and*** $\vee$ (identity element, commutativity, associativity, distributivity). Hence the name ***boolean logic*** for this calculus of logical statements.

Some Logical Equivalencies

We omitted de Morgan's laws from the previous table. We now list some equivalences involving conditional statements.

Equivalence
$A \Rightarrow B \equiv \neg A \vee B \equiv \neg B \Rightarrow \neg A$
$(A \Rightarrow B) \wedge (A \Rightarrow C) \equiv A \Rightarrow (B \wedge C)$ $(A \Rightarrow B) \vee (A \Rightarrow C) \equiv A \Rightarrow (B \vee C)$
$(A \Rightarrow C) \wedge (B \Rightarrow C) \equiv (A \vee B) \Rightarrow C$ $(A \Rightarrow C) \vee (B \Rightarrow C) \equiv (A \wedge B) \Rightarrow C$
$(A \Leftrightarrow B) \equiv ((\neg A) \Leftrightarrow (\neg B))$ $(A \Leftrightarrow B) \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ $(A \Leftrightarrow B) \equiv (A \wedge B) \vee ((\neg A) \wedge (\neg B))$ $\neg(A \Leftrightarrow B) \equiv A \Leftrightarrow (\neg B)$

Logical Quantifiers

In the previous examples we have used predicates $A(x)$ with the words "for all x." This is an instance of a ***logical quantifier*** that indicates for which x a predicate $A(x)$ is to be evaluated to a statement.

In order to use quantifiers properly, we clearly need a universe of objects x which we can insert into $A(x)$ (a ***domain*** for $A(x)$). This leads us immediately to the definition of a ***set***. We will discuss set theory in detail later. For the moment it is sufficient for us to view a set as a "collection of objects" and assume that the following sets are known:

- the set of natural numbers $\mathbb{N}$ (which includes the number 0),
- the set of integers $\mathbb{Z}$,
- the set of real numbers $\mathbb{R}$,
- the empty set $\emptyset$ (also written $\varnothing$ or $\{\}$) that does not contain any objects.

If M is a set containing x, we write $x \in M$ and call x an ***element*** of M.

Logical Quantifiers

There are two types of quantifiers:

- the ***universal quantifier***, denoted by the symbol $\forall$, read as "for all" and
- the ***existential quantifier***, denoted by $\exists$, read as "there exists."

1.13. Definition. Let M be a set and $A(x)$ be a predicate. Then we define the quantifier $\forall$ by

$$\underset{x\in M}{\forall} A(x) \quad\Leftrightarrow\quad A(x) \text{ is true for all } x \in M$$

We define the quantifier $\exists$ by

$$\underset{x\in M}{\exists} A(x) \quad\Leftrightarrow\quad A(x) \text{ is true for at least one } x \in M$$

We may also write $\forall x \in M\colon A(x)$ instead of $\underset{x\in M}{\forall} A(x)$ and similarly for $\exists$.

Logical Quantifiers

We may also state the domain before making the statements, as in the following example.

1.14. Examples. Let x be a real number. Then

- $\forall x \colon x > 0 \Rightarrow x^3 > 0$ is a true statement;
- $\forall x \colon x > 0 \Leftrightarrow x^2 > 0$ is a false statement;
- $\exists x \colon x > 0 \Leftrightarrow x^2 > 0$ is a true statement.

Sometimes mathematicians put a quantifier at the end of a statement form; this is known as a ***hanging quantifier***. Such a hanging quantifier will be interpreted as being located just before the statement form:

$$\exists y \colon y + x^2 > 0 \qquad\qquad \forall x$$

is equivalent to $\exists y \forall x \colon y + x^2 > 0$.

Contraposition and Negation of Quantifiers

We do not actually need the quantifier $\exists$ since

$$\begin{aligned} \underset{x\in M}{\exists} A(x) \quad &\Leftrightarrow \quad A(x) \text{ is true for at least one } x \in M \\ &\Leftrightarrow \quad A(x) \text{ is not false for all } x \in M \\ &\Leftrightarrow \quad \neg \underset{x\in M}{\forall} (\neg A(x)) \end{aligned} \tag{1.2}$$

The equivalence (1.2) is called ***contraposition of quantifiers***. It implies that the negation of $\exists x \in M : A(x)$ is equivalent to $\forall x \in M : \neg A(x)$. For example,

$$\neg \left(\exists x \in \mathbb{R} : x^2 < 0\right) \qquad \Leftrightarrow \qquad \forall x \in \mathbb{R} : x^2 \not< 0.$$

Conversely,

$$\neg \left(\forall x \in M : A(x)\right) \qquad \Leftrightarrow \qquad \exists x \in M : \neg A(x).$$

Vacuous Truth

If the domain of the universal quantifier $\forall$ is the empty set $M = \varnothing$, then the statement $\forall x \in M \colon A(x)$ is defined to be true regardless of the predicate $A(x)$. It is then said that $A(x)$ is ***vacuously true***.

1.15. Example. Let M be the set of real numbers x such that $x = x + 1$. Then the statement

$$\underset{x \in M}{\forall} x > x$$

is true.

This convention reflects the philosophy that a universal statement is true unless there is a counterexample to prove it false. While this may seem a strange point of view, it proves useful in practice.

This is similar to saying that “All pink elephants can fly.” is a true statement, because it is impossible to find a pink elephant that can't fly.

Nesting Quantifiers

We can also treat predicates with more than one variable as shown in the following example.

1.16. Examples. Here x, y are taken from the real numbers.

- $\forall x \forall y \colon x^2 + y^2 - 2xy \geq 0$ is equivalent to $\forall y \forall x \colon x^2 + y^2 - 2xy \geq 0$. Therefore, one often writes $\forall x, y \colon x^2 + y^2 - 2xy \geq 0$.
- $\exists x \exists y \colon x + y > 0$ is equivalent to $\exists y \exists x \colon x + y > 0$, often abbreviated to $\exists x, y \colon x + y > 0$.
- $\forall x \exists y \colon x + y > 0$ is a true statement.
- $\exists x \forall y \colon x + y > 0$ is a false statement.

As is clear from these examples, the order of the quantifiers is important if they are different.

Examples from Calculus

Let I be an interval in $\mathbb{R}$. Then a function $f\colon I \to \mathbb{R}$ is said to be ***continuous*** on I if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{x\in I}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall} \quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon.$$

The function f is ***uniformly continuous*** on I if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in I}{\forall}\ \underset{y\in I}{\forall} \quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon.$$

It is easy to see that a function that is uniformly continuous on I must also be continuous on I.

If I is a closed interval, $I=[a,b]$, it can also be shown that a continuous function is also uniformly continuous. However, that requires techniques from calculus and is not obvious just by looking at the logical structure of the definitions.

Examples from Calculus

Negating complicated expressions can be done step-by-step. For example, the statement that f is not continuous on I is equivalent to

$$\begin{aligned}
&\neg\Big(\underset{\varepsilon>0}{\forall}\ \underset{x\in I}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall}\quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\Big)\\
\Leftrightarrow &\Big(\underset{\varepsilon>0}{\exists}\ \neg\ \underset{x\in I}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall}\quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\Big)\\
\Leftrightarrow &\Big(\underset{\varepsilon>0}{\exists}\ \underset{x\in I}{\exists}\ \neg\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall}\quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\Big)\\
\Leftrightarrow &\Big(\underset{\varepsilon>0}{\exists}\ \underset{x\in I}{\exists}\ \underset{\delta>0}{\forall}\ \neg\ \underset{y\in I}{\forall}\quad |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\Big)\\
\Leftrightarrow &\Big(\underset{\varepsilon>0}{\exists}\ \underset{x\in I}{\exists}\ \underset{\delta>0}{\forall}\ \underset{y\in I}{\exists}\quad (|x-y|<\delta)\wedge\neg(|f(x)-f(y)|<\varepsilon)\Big)\\
\Leftrightarrow &\Big(\underset{\varepsilon>0}{\exists}\ \underset{x\in I}{\exists}\ \underset{\delta>0}{\forall}\ \underset{y\in I}{\exists}\quad (|x-y|<\delta)\wedge(|f(x)-f(y)|\geq\varepsilon)\Big)
\end{aligned}$$

Examples from Calculus

1.17. Example. The ***Heaviside function*** $H\colon \mathbb{R} \to \mathbb{R}$,

$$H(x) := \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is not continuous on $I = \mathbb{R}$. To see this, we need to show that there exists an $\varepsilon > 0$ (take $\varepsilon = 1/2$) and an $x \in \mathbb{R}$ (take $x = 0$) such that for any $\delta > 0$ there exists a $y \in \mathbb{R}$ such that

$$|x - y| = |y| < \delta \qquad \text{and} \qquad |H(x) - H(y)| = |1 - H(y)| \geq \varepsilon = \frac{1}{2}.$$

Given any $\delta > 0$ we can choose $y = -\delta/2$. Then $|y| = \delta/2 < \delta$ and $|1 - H(y)| = 1 > 1/2$. This proves that H is not continuous on $\mathbb{R}$.

Arguments in Mathematics

The previous example contains a ***mathematical argument*** to show that the Heaviside function is not continuous on its domain. The argument boils down to the following:

(i) We know that

$$A\colon \underset{\varepsilon>0}{\exists}\ \underset{x\in\mathbb{R}}{\exists}\ \underset{\delta>0}{\forall}\ \underset{y\in\mathbb{R}}{\exists} \quad (|x-y|<\delta)\wedge(|H(x)-H(y)|\geq\varepsilon)$$

implies

$$B\colon\ H \text{ is not continuous on its domain.}$$

(ii) We show that A is true.

(iii) Therefore, B is true.

Logically, we can express this argument as

$$\big(A\wedge(A\Rightarrow B)\big)\Rightarrow B.$$

Arguments and Argument Forms

1.18. Definition.

(i) An ***argument*** is a finite sequence of statements. All statements except for the final statement are called ***premises*** while the final statement is called the ***conclusion***. We say that an argument is ***valid*** if the truth of all premises implies the truth of the conclusion.

(ii) An ***argument form*** is a finite sequence of predicates (statement forms). An argument form is ***valid*** if it yields a valid argument whenever statements are substituted for the predicates.

From the definition of an argument it is clear that an argument consisting of a sequence of premises $P_1, \dots, P_n$ and a conclusion C is valid of and only if

$$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Rightarrow C \tag{1.3}$$

is a tautology, i.e., a true statement for any values of the premises an the conclusion.

Arguments and Argument Forms

An argument is a finite list of premises $P_1, \ldots, P_n$ followed by a conclusion C. We usually write this list as

$$\begin{array}{rc} & P_1 \\ & P_2 \\ & \vdots \\ & P_n \\ \hline \therefore & C \end{array}$$

where the symbol $\therefore$ is pronounced "therefore". You may only use this symbol when constructing a logical argument in the notation above. Do not use it as a general-purpose abbreviation of "therefore".

Certain basic valid arguments in mathematics are given latin names and called ***rules of inference***.

Modus Ponendo Ponens

1.19. Example. The rule of inference

$$\begin{array}{rl} & A \\ & A \Rightarrow B \\ \hline \therefore & B \end{array}$$

is called ***modus ponendo ponens*** (latin for "mode that affirms by affriming"); it is often abbreviated simply "modus ponens". The associated tautology is

$$\big(A \wedge (A \Rightarrow B)\big) \Rightarrow B \tag{1.4}$$

We verify that (1.4) actually is a tautology using the truth table:

A	B	$A \Rightarrow B$	$A \wedge (A \Rightarrow B)$	$\big(A \wedge (A \Rightarrow B)\big) \Rightarrow B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

Hypothetical Syllogisms

A ***syllogism*** is an argument that has exactly two premises. We first give three ***hypothetical syllogisms***, i.e., syllogisms involving the implication "$\Rightarrow$".

Rule of Inference	Name
$\begin{array}{rl} & A \\ & A \Rightarrow B \\ \hline \therefore & B \end{array}$	Modus (Ponendo) Ponens ***Mode that affirms (by affirming)***
$\begin{array}{rl} & \neg B \\ & A \Rightarrow B \\ \hline \therefore & \neg A \end{array}$	Modus (Tollendo) Tollens ***Mode that denies (by denying)***
$\begin{array}{rl} & A \Rightarrow B \\ & B \Rightarrow C \\ \hline \therefore & A \Rightarrow C \end{array}$	Transitive Hypothetical Syllogism

Hypothetical Syllogisms

1.20. Examples.

(i) Modus ponendo ponens:

3 is both prime and greater than 2
If 3 is both prime and greater than 2, then 3 is odd

∴ 3 is odd.

(ii) Modus tollendo tollens:

4 is not odd
If 4 is both prime and greater than 2, then 4 is odd

∴ 4 is not both prime and greater than 2.

(iii) Transitive hypothetical syllogism:

If 5 is greater than 4, then 5 is greater than 3
If 5 is greater than 3, then 5 is greater than 2

∴ If 5 is greater than 4, then 5 is greater than 2.

Disjunctive and Conjunctive Syllogisms

There are two important syllogisms involving the disjunction "$\vee$" and the conjunction "$\wedge$":

Rule of Inference	Name
$\begin{array}{rl} & A \vee B \\ & \neg A \\ \hline \therefore & B \end{array}$	Modus Tollendo Ponens ***Mode that affirms by denying***
$\begin{array}{rl} & \neg(A \wedge B) \\ & A \\ \hline \therefore & \neg B \end{array}$	Modus Ponendo Tollens ***Mode that denies by affirming***
$\begin{array}{rl} & A \vee B \\ & \neg A \vee C \\ \hline \therefore & B \vee C \end{array}$	Resolution

Disjunctive and Conjunctive Syllogisms

1.21. Examples.

(i) Modus tollendo ponens:

$$\begin{array}{rl} & \text{4 is odd or even} \\ & \text{4 is not odd} \\ \hline \therefore & \text{4 is even.} \end{array}$$

(ii) Modus ponendo tollens:

$$\begin{array}{rl} & \text{4 is not both even and odd} \\ & \text{4 is even} \\ \hline \therefore & \text{4 is not odd.} \end{array}$$

(iii) Resolution:

$$\begin{array}{rl} & \text{4 is even or 4 is greater than 2} \\ & \text{4 is odd or 4 is prime} \\ \hline \therefore & \text{4 is greater than 2 or 4 is prime.} \end{array}$$

Some Simple Arguments

Finally, we give some seemingly obvious, but nevertheless useful, arguments:

Rule of Inference	Name
$\begin{array}{ll} & A \\ & B \\ \hline \therefore & A \wedge B \end{array}$	Conjunction
$\begin{array}{ll} & A \wedge B \\ \hline \therefore & A \end{array}$	Simplification
$\begin{array}{ll} & A \\ \hline \therefore & A \vee B \end{array}$	Addition

Examples for these are left to the reader!

Validity and Soundness

The previous rules of inference are all ***valid arguments***. In the examples we gave, the arguments always led to a correct conclusion. This was, however, only because all the premises were true statements. It is possible for a valid argument to lead to a wrong conclusion if one or more of its premises are false.

If, in addition to being valid, an argument has only true premises, we say that the argument is ***sound***. In that case, its conclusion is true.

1.22. Example. The following argument is valid (it is based on the rule of resolution), but not sound:

$$\begin{array}{rl} & \text{4 is even or 4 is prime} \\ & \text{4 is odd or 4 is prime} \\ \hline \therefore & \text{4 is prime.} \end{array}$$

(The second premise is false, so the conclusion doesn't have to be true.)

Non Sequitur

The term ***non sequitur*** (latin for "it does not follow") is often used to describe logical fallacies, i.e., inferences that invalid because they are not based on tautologies. Some common fallacies are listed below:

Rule of Inference	Name
$\begin{array}{rl} & B \\ & A \Rightarrow B \\ \hline \therefore & A \end{array}$	Affirming the Consequent
$\begin{array}{rl} & \neg A \\ & A \Rightarrow B \\ \hline \therefore & \neg B \end{array}$	Denying the Antecedent
$\begin{array}{rl} & A \vee B \\ & A \\ \hline \therefore & \neg B \end{array}$	Affirming a Disjunct

Non Sequitur

1.23. Examples.

(i) Affirming the consequent:

$$\begin{array}{rl} & \text{If 9 is prime, then it is odd} \\ & \text{9 is odd} \\ \hline \therefore & \text{9 is prime.} \end{array}$$

(ii) Denying the antecedent

$$\begin{array}{rl} & \text{If 9 is prime, then it is odd} \\ & \text{9 is not prime} \\ \hline \therefore & \text{9 is not odd.} \end{array}$$

(iii) Affirming a disjunct:

$$\begin{array}{rl} & \text{2 is even or 2 is prime} \\ & \text{2 is even} \\ \hline \therefore & \text{2 is not prime.} \end{array}$$

Rules of Inference for Quantified Statements

Without proof or justification, we give the following rules of inference for quantified statements. They are often assumed as axioms in abstract logic systems.

Rule of Inference	Name
$\dfrac{\forall_{x\in M} P(x)}{\therefore \quad P(x_0) \text{ for any } x_0 \in M}$	Universal Instantiation
$\dfrac{P(x) \text{ for any arbitrarily chosen } x \in M}{\therefore \quad \forall_{x\in M} P(x)}$	Universal Generalization
$\dfrac{\exists_{x\in M} P(x)}{\therefore \quad P(x_0) \text{ for a certain (unknown) } x_0 \in M}$	Existential Instantiation
$\dfrac{P(x_0) \text{ for some (known) } x_0 \in M}{\therefore \quad \exists_{x\in M} P(x)}$	Existential Generalization

Constructing Arguments

Often, complex arguments can be broken down into syllogisms. As an example, we give a logical proof of the following theorem:

1.24. Theorem. Let $n \in \mathbb{N}$ be a natural number and suppose that n^2 is even. Then n is even.

Proof.
We use the following premises:

$$P_1\colon \underset{n\in\mathbb{N}}{\forall} \neg(n \text{ even} \wedge n \text{ odd}),$$

$$P_2\colon n \text{ odd} \Rightarrow n^2 \text{ odd},$$

$$P_3\colon n^2 \text{ even} \wedge (n \text{ even} \vee n \text{ odd})$$

and we wish to arrive at the conclusion

$$C\colon n \text{ even}.$$

Constructing Arguments

Proof (continued).

Premise P_2 can be easily checked: if n is odd, there exists some k such that $n = 2k + 1$, so

$$n^2 = (2k+1)^2 = 2(2k^2 + 2k) + 1 = 2k' + 1$$

where $k' = 2k^2 + 2k$. Hence n^2 is also odd. We have

$$\begin{array}{rl} & P_3\colon\ n^2 \text{ even} \wedge (n \text{ even} \vee n \text{ odd}) \\ \hline \therefore & P_4\colon\ n^2 \text{ even.} \end{array}$$

by the Rule of Simplification. By Universal Instantiation, we obtain

$$\begin{array}{rl} & P_1\colon\ \underset{n\in\mathbb{N}}{\forall} \neg(n \text{ even} \wedge n \text{ odd}) \\ \hline \therefore & P_5\colon\ \neg(n^2 \text{ even} \wedge n^2 \text{ odd}). \end{array}$$

Constructing Arguments

Proof (continued).

Furthermore, by Modus Ponendo Tollens,

$$\begin{array}{rl} & P_4 : n^2 \text{ even} \\ & P_5 : \neg(n^2 \text{ even} \wedge n^2 \text{ odd}) \\ \hline \therefore & P_6 : \neg(n^2 \text{ odd}). \end{array}$$

Using Modus Tollendo Tollens,

$$\begin{array}{rl} & P_6 : \neg(n^2 \text{ odd}) \\ & P_2 : n \text{ odd} \Rightarrow n^2 \text{ odd} \\ \hline \therefore & P_7 : \neg(n \text{ odd}). \end{array}$$

Simplification yields

$$\begin{array}{rl} & P_3 : n^2 \text{ even} \wedge (n \text{ even} \vee n \text{ odd}) \\ \hline \therefore & P_8 : n \text{ even} \vee n \text{ odd}. \end{array}$$

Constructing Arguments

Proof (continued).

Finally, Modus Tollendo Ponens gives

$$\begin{array}{rl} & P_7 : \neg(n \text{ odd}) \\ & P_8 : n \text{ even} \vee n \text{ odd} \\ \hline \therefore & C : n \text{ even.} \end{array}$$

This completes the proof. □

1.25. Remark. Of course, this proof could have been shortened and simplified if we had replaced “odd” with “not even” throughout, and we might have formulated premise P_3 slightly differently (as two separate premises) to avoid using the rule of simplification. However, our goal was to illustrate the usage of a wide variety of rules of inference and that writing down a logically valid proof is in most cases extremely tedious; in most mathematics, many of the mentioned rules of inference are used implicitly without being stated.

Basic Concepts in Set Theory

Naive Set Theory: Sets via Predicates

We want to be able to talk about "collections of objects"; however, we will be unable to strictly define what an "object" or a "collection" is (except that we also want any collection to qualify as an "object"). The problem with "naive" set theory is that any attempt to make a formal definition will lead to a contradiction - we will see an example of this later. However, for our practical purposes we can live with this, as we won't generally encounter these contradictions.

We indicate that an object (called an ***element***) x is part of a collection (called a ***set***) X by writing $x \in X$. We characterize the elements of a set X by some predicate P:

$$x \in X \qquad \Leftrightarrow \qquad P(x). \tag{2.1}$$

We write $X = \{x \colon P(x)\}$.

Notation for Sets

We define the empty set $\varnothing := \{x : x \neq x\}$. The empty set has no elements, because the predicate $x \neq x$ is never true.

We may also use the notation $X = \{x_1, x_2, \dots, x_n\}$ to denote a set. In this case, X is understood to be the set

$$X = \{x : (x = x_1) \vee (x = x_2) \vee \cdots \vee (x = x_n)\}.$$

We will frequently use the convention

$$\{x \in A : P(x)\} = \{x : x \in A \wedge P(x)\}$$

2.1. Example. The set of even positive integers is

$$\{n \in \mathbb{N} : \underset{k \in \mathbb{N}}{\exists} \, n = 2k\}$$

Subsets and Equality of Sets

If every object $x \in X$ is also an element of a set Y, we say that X is a subset of Y, writing $X \subset Y$; in other words,

$$X \subset Y \qquad \Leftrightarrow \qquad \forall x \in X : x \in Y.$$

We say that $X = Y$ if and only if $X \subset Y$ and $Y \subset X$.

We say that X is a ***proper subset*** of Y if $X \subset Y$ but $X \neq Y$. In that case we write $X \subsetneq Y$.

Some authors write $\subseteq$ for $\subset$ and $\subset$ for $\subsetneq$. Pay attention to the convention used when referring to literature.

Examples of Sets and Subsets

2.2. Examples.

1. For any set X, $\varnothing \subset X$. Since $\varnothing$ does not contain any elements, the domain of the statement $\forall x \in X \colon x \in Y$ is empty. Therefore, it is vacuously true and hence $\varnothing \subset X$.
2. Consider the set $A = \{a, b, c\}$ where a, b, c are arbitrary objects, for example, numbers. The set

$$B = \{a, b, a, b, c, c\}$$

is equal to A, because it satisfies $A \subset B$ and $B \subset A$ as follows:

$$x \in A \quad \Leftrightarrow \quad (x = a) \vee (x = b) \vee (x = c) \quad \Leftrightarrow \quad x \in B.$$

Therefore, neither order nor repetition of the elements affects the contents of a set.

If $C = \{a, b\}$, then $C \subset A$ and in fact $C \subsetneqq A$. Setting $D = \{b, c\}$ we have $D \subsetneqq A$ but $C \not\subset D$ and $D \not\subset C$.

Power Set and Cardinality

If a set X has a finite number of elements, we define the ***cardinality*** of X to be this number, denoted by $\#X$, $|X|$ or $\operatorname{card} X$.

We define the ***power set***

$$\mathscr{P}(X) := \{A\colon A \subset X\}.$$

Here the elements of the set $\mathscr{P}(X)$ are themselves sets; $\mathscr{P}(X)$ is the "set of all subsets of X." Therefore, the statements

$$A \subset X \qquad \text{and} \qquad A \in \mathscr{P}(X)$$

are equivalent.

2.3. Example. The power set of $\{a, b, c\}$ is

$$\mathscr{P}(\{a, b, c\}) = \big\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\big\}.$$

The cardinality of $\{a, b, c\}$ is 3, the cardinality of the power set is $|\mathscr{P}(\{a, b, c\})| = 8$.

Operations on Sets

If $A = \{x\colon P_1(x)\}$, $B = \{x\colon P_2(x)\}$ we define the ***union***, ***intersection*** and ***difference*** of A and B by

$$A \cup B := \{x\colon P_1(x) \vee P_2(x)\}, \qquad A \cap B := \{x\colon P_1(x) \wedge P_2(x)\},$$
$$A \setminus B := \{x\colon P_1(x) \wedge (\neg P_2(x))\}.$$

Let $A \subset M$. We then define the ***complement*** of A by

$$A^{\mathrm{c}} := M \setminus A.$$

If $A \cap B = \emptyset$, we say that the sets A and B are ***disjoint***.

Occasionally, the notation $A - B$ is used for $A \setminus B$ and A^{c} is sometimes denoted by $\overline{A}$.

2.4. Example. Let $A = \{a, b, c\}$ and $B = \{c, d\}$. Then

$$A \cup B = \{a, b, c, d\}, \qquad A \cap B = \{c\}, \qquad A \setminus B = \{a, b\}.$$

Operations on Sets

The laws for logical equivalencies immediately lead to several rules for set operations. For example, the distributive laws for $\wedge$ and $\vee$ imply

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Other such rules are, for example,

- $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
- $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
- $A \setminus B = B^{\mathrm{c}} \cap A$
- $(A \setminus B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B$

Some of these will be proved in the recitation class and the exercises.

Operations on Sets

Occasionally we will need the following notation for the union and intersection of a finite number $n \in \mathbb{N}$ of sets:

$$\bigcup_{k=0}^{n} A_k := A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n,$$

$$\bigcap_{k=0}^{n} A_k := A_0 \cap A_1 \cap A_2 \cap \cdots \cap A_n.$$

This notation even extends to $n = \infty$, but needs to be properly defined:

$$x \in \bigcup_{k=0}^{\infty} A_k \quad :\Leftrightarrow \quad \underset{k \in \mathbb{N}}{\exists} \; x \in A_k,$$

$$x \in \bigcap_{k=0}^{\infty} A_k \quad :\Leftrightarrow \quad \underset{k \in \mathbb{N}}{\forall} \; x \in A_k.$$

Operations on Sets

In particular,

$$\bigcap_{k=0}^{\infty} A_k \subset \bigcup_{k=0}^{\infty} A_k.$$

2.5. Example. Let $A_k = \{0, 1, 2, \dots, k\}$ for $k \in \mathbb{N}$. Then

$$\bigcup_{k=0}^{\infty} A_k = \mathbb{N}, \qquad \bigcap_{k=0}^{\infty} A_k = \{0\}.$$

To see the first statement, note that $\mathbb{N} \subset \bigcup_{k=0}^{\infty} A_k$ since $x \in \mathbb{N}$ implies $x \in A_x$ implies $x \in \bigcup_{k=0}^{\infty} A_k$. Furthermore, $\bigcup_{k=0}^{\infty} A_k \subset \mathbb{N}$ since $x \in \bigcup_{k=0}^{\infty} A_k$ implies $x \in A_k$ for some $k \in \mathbb{N}$ implies $x \in \mathbb{N}$.

For the second statement, note that $\bigcap_{k=0}^{\infty} A_k \subset \mathbb{N}$. Now $0 \in A_k$ for all $k \in \mathbb{N}$. Thus $\{0\} \subset \bigcap_{k=0}^{\infty} A_k$. On the other hand, for any $x \in \mathbb{N} \setminus \{0\}$ we have $x \notin A_{x-1}$ whence $x \notin \bigcap_{k=0}^{\infty} A_k$.

Ordered Pairs

A set does not contain any information about the order of its elements, e.g.,

$$\{a, b\} = \{b, a\}.$$

Thus, there is no such a thing as the "first element of a set". However, sometimes it is convenient or necessary to have such an ordering. This is achieved by defining an ***ordered pair***, denoted by

$$(a, b)$$

and having the property that

$$(a, b) = (c, d) \qquad \Leftrightarrow \qquad (a = c) \wedge (b = d). \tag{2.2}$$

We define

$$(a, b) := \{\{a\}, \{a, b\}\}.$$

It is not difficult to see that this definition guarantees that (2.2) holds.

Cartesian Product of Sets

If A, B are sets and $a \in A$, $b \in B$, then we denote the set of all ordered pairs by

$$A \times B := \{(a, b) \colon a \in A, \ b \in B\}.$$

$A \times B$ is called the ***cartesian product*** of A and B.

In this manner we can define an ***ordered triple*** (a, b, c) or, more generally, an ordered ***n-tuple*** $(a_1, \dots, a_n)$ and the n-fold cartesian product $A_1 \times \dots \times A_n$ of sets A_k, $k = 1, \dots, n$.

If we take the cartesian product of a set with itself, we may abbreviate it using exponents, e.g.,

$$\mathbb{N}^2 := \mathbb{N} \times \mathbb{N}.$$

Problems in Naive Set Theory

If one simply views sets as arbitrary "collections" and allows a set to contain arbitrary objects, including other sets, then fundamental problems arise. We first illustrate this by an analogy:

Suppose a library contains not only books but also catalogs of books, i.e., books listing other books. For example, there might be a catalog listing all mathematics books in the library, a catalog listing all history books, etc. Suppose that there are so many catalogs, that you are asked to create catalogs of catalogs, i.e., catalogs listing other catalogs. In particular, you are asked to create the following:

(i) A catalog of all catalogs in the library. This catalog lists all catalogs contained in the library, so it must of course also list itself.

(ii) A catalog of all catalogs that list themselves. Does this catalog also list itself?

(iii) A catalog of all catalogs that do not list themselves. Does this catalog also list itself?

The Russel Antinomy

In the previous analogy, we can view "catalogs" as "sets" and being "listed in a catalog" as "being an element of a set". Then we have

(i) The set of all sets must have itself as an element.

(ii) The set of all sets that have themselves as elements may or may not contain itself. (This may be decidable by adding some rule to set theory.)

(iii) It is not decidable whether the "set of all sets that do not have themselves as elements" has itself as an element.

The Russel Antinomy

Formally, this paradox is known as the ***Russel antinomy:***

2.6. Russel Antinomy. The predicate $P(x)\colon x \notin x$ does not define a set $A = \{x \colon P(x)\}$.

Proof.
If $A = \{x \colon x \notin x\}$ were a set, then we should be able to decide for any set y whether $y \in A$ or $y \notin A$. We show that for $y = A$ this is not possible because either assumption leads to a contradiction:

(i) Assume $A \in A$. Then $P(A)$ by (2.1), i.e., $A \notin A$. $\lightning$

(ii) Assume $A \notin A$. Then $\neg P(A)$ by (2.1), therefore $A \in A$. $\lightning$

Since we cannot decide whether $A \in A$ or $A \notin A$, A can not be a set. $\square$

The Russel Antinomy

There are several examples in classical literature and philosophy of the Russel antimony:

(i) Epimenides the Cretan says, “All Cretans are liars.”

(ii) In a mountain village, there is a barber. Some villagers shave themselves (always) while the others never shave themselves. The barber shaves those and only those villagers that never shave themselves. Who shaves the barber?

Russel Antinomy

We will simply ignore the existence of such contradictions and build on naive set theory. There are further paradoxes (antinomies) in naive set theory, such as ***Cantor's paradox*** and the ***Burali-Forti paradox***. All of these are resolved if naive set theory is replaced by a ***modern axiomatic set theory*** such as ***Zermelo-Fraenkel set theory.***

Further Information:

- ***Set Theory***, Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/set-theory/
- P.R. Halmos, ***Naive Set Theory***,Aavilable here: http://link.springer.com/book/10.1007/978-1-4757-1645-0
- T. Jech, ***Set Theory: The Third Millennium Edition, Revised and Expanded***, Available here: http://link.springer.com/book/10.1007/3-540-44761-X

Special Topic: Construction of the Natural Numbers

The Natural Numbers

The "counting numbers" $0, 1, 2, 3, \ldots$ are the basis of discrete mathematics. We refer to their totality as the set of ***natural numbers*** and denote it by $\mathbb{N}$. We have up to now used them to supply examples for our introduction to logic and to enumerate sets. It is time we briefly discuss how they can be formally defined.

We will represent the natural numbers as set of objects (denoted by $\mathbb{N}$) together with a relation called "succession": If n is a natural numbers, the "successor" of n, $\operatorname{succ}(n)$, is defined and also in $\mathbb{N}$. In elementary terms, 1 is the successor to 0, 2 is the successor to 1, etc.

There is no unique set of natural numbers in the sense that we can exhibit "the" set $\mathbb{N}$. Rather, any pair $(\mathbb{N}, \operatorname{succ})$, can qualify as a ***realization of the natural numbers*** if it satisfies certain axioms.

The Peano Axioms and the Induction Axiom

3.1. Definition. Let $\mathbb{N}$ be any set and suppose that the successor of any element of $\mathbb{N}$ has been defined. The ***Peano axioms*** are

(i) $\mathbb{N}$ contains at least one object, called zero.

(ii) $\mathbb{N}$ is closed under the successor function, i.e., if n is in $\mathbb{N}$, the successor of n is in $\mathbb{N}$.

(iii) Zero is not the successor of a number.

(iv) Two numbers of which the successors are equal are themselves equal.

(v) Induction axiom. If a set $S \subset \mathbb{N}$ contains zero and also the successor of every number in S, then $S = \mathbb{N}$.

Any set with a successor relation satisfying these axioms is called a realization of the natural numbers.

The von Neumann Realization

The Hungarian mathematician John von Neumann gave a very elegant realization of the natural numbers that is based on set theory:

Assume the empty set $\emptyset$ exists. Our idea is to define

$$\begin{aligned} 0 &:= \emptyset, \\ 1 &:= \{0\} = \{\emptyset\}, \\ 2 &:= \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \\ 3 &:= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

and so on. It is clear that we want 1 to be the successor of 0, 2 the successor of 1 and so on.

We formally define

$$\operatorname{succ}(n) := n \cup \{n\}, \tag{3.1a}$$

$$\mathbb{N} := \{\emptyset\} \cup \Big\{ n \colon \underset{m \in \mathbb{N}}{\exists}\, n = \operatorname{succ}(m) \Big\}. \tag{3.1b}$$

The von Neumann Realization

Several questions arise:

- Is it possible to define a set $\mathbb{N}$ in this ***recursive*** way?
- Does the von Neumann realization actually satisfy the Peano axioms?

The answer to both questions is affirmative:

- The possibility of defining $\mathbb{N}$ in this way is taken as an ***axiom*** of set theory.
- The proof that Peano axiom (iv) is satisfied is much harder than it looks - try it!

For a discussion of both questions (and more) we refer to P.R. Halmos, ***Naive Set Theory*** (see Slide 63). We will briefly sketch the proof of the Peano axiom (iv) here.

The von Neumann Realization

One interesting feature of the realization (3.1) is that

$$n \in \operatorname{succ}(n) \qquad \text{and} \qquad n \subsetneq \operatorname{succ}(n).$$

Furthermore, we have a natural definition of the ordering relation "$<$" that does not require any concept of addition:

$$m < n \qquad :\Leftrightarrow \qquad m \subsetneq n.$$

Of course, we still need to check that the Peano axioms are satisfied:

(i) $0 = \emptyset \in \mathbb{N}$ by definition.

(ii) For any $n \in \mathbb{N}$, $\operatorname{succ}(n) \in \mathbb{N}$ by definition.

(iii) $0 = \emptyset$ and for any n, $\operatorname{succ}(n) = n \cup \{n\} \neq \emptyset$, so 0 is not the successor of any number.

(v) Any set S with the properties given is equal to $\mathbb{N}$ by the definition of $\mathbb{N}$.

The von Neumann Realization

We have skipped over the fourth Peano axiom: We need to show $\operatorname{succ}(m) = \operatorname{succ}(n)$ implies $m = n$. This is a little complicated and requires two further lemmas, which we discuss below.

3.2. Lemma. Let n be a natural number in the von Neumann realization (3.1). Then

$$\underset{m\in\mathbb{N}}{\forall} \quad m \in n \quad \Rightarrow \quad n \not\subset m, \tag{3.2}$$

or, equivalently,

$$\underset{m\in\mathbb{N}}{\forall} \quad n \subset m \quad \Rightarrow \quad m \notin n \tag{3.3}$$

Proof.
Let S be the set of natural numbers such that (3.2) holds,

$$S := \Big\{ n \in \mathbb{N} : \underset{m\in\mathbb{N}}{\forall} \; m \in n \;\Rightarrow\; n \not\subset m \Big\}.$$

The von Neumann Realization

Proof (continued).

Our goal is to show that $S = \mathbb{N}$, so that (3.2) is established for all $n \in \mathbb{N}$. Of course, at first it is possible that $S = \emptyset$.

We first show that $0 \in S$. Since $0 = \emptyset$, $m \in 0$ is false for all $m \in \mathbb{N}$ and so the implication

$$m \in 0 \quad \Rightarrow \quad 0 \not\subset m$$

is true for all $m \in \mathbb{N}$. Hence, $0 \in S$.

We next aim to show that if $n \in S$, then $\operatorname{succ}(n) \in S$. This then yields $S = \mathbb{N}$ by the induction axiom.

Let $n \in S$. We first derive a basic fact: since $n = n$, we have, in particular, that $n \subset n$ so $n \notin n$ by (3.3) for $m = n$. Then

$$\operatorname{succ}(n) = n \cup \{n\} \not\subset n.$$

The von Neumann Realization

Proof (continued).

Now suppose that $\operatorname{succ}(n) \subset m$ for some m. Then $n \subset \operatorname{succ}(n) \subset m$. Since $n \in S$, it follows that $m \notin n$. By contraposition, if $m \in n$, then $\operatorname{succ}(n) \not\subset m$.

Hence, $\operatorname{succ}(n) \not\subset n$ and $\operatorname{succ}(n) \not\subset m$ for all $m \in n$. Since the elements of $\operatorname{succ}(n)$ consist of n and the elements of n, it follows that

$$m \in \operatorname{succ}(n) \quad \Rightarrow \quad \operatorname{succ}(n) \not\subset m$$

Therefore, if $n \in S$, then $\operatorname{succ}(n) \in S$.

We have shown that the set S contains 0 and the successor of every element of S. By the induction axiom (which we have already established for the von Neumann construction) it follows that $S = \mathbb{N}$. $\square$

The von Neumann Realization

3.3. Definition. A set A is called transitive if

$$y \in x \wedge x \in A \quad \Rightarrow \quad y \in A.$$

In particular, a set A is transitive if and only if $x \subset A$ for all $x \in A$.

3.4. Lemma. Let n be a natural number in the von Neumann realization (3.1). Then n is transitive.

Proof.
We again proceed by induction. Let S be the set of transitive natural numbers. Then $0 \in S$ is vacuously true. Now let $n \in S$. If $x \in \operatorname{succ}(n)$, then either $x \in n$ or $x = n$. If $x \in n$, then $x \subset n \subset \operatorname{succ}(n)$, because $n \in S$. If $x = n$, then $x \subset n \cup \{n\} = \operatorname{succ}(n)$. Hence, $\operatorname{succ}(n) \in S$ and we again deduce $S = \mathbb{N}$ by the induction axiom. $\square$

The von Neumann Realization

Finally, we can prove that the von Neumann numbers satisfy the fourth Peano axiom:

3.5. Lemma. Let m and n be natural numbers in the von Neumann realization (3.1). Then n

$$\operatorname{succ}(m) = \operatorname{succ}(n) \quad \Rightarrow \quad m = n$$

Proof.
Let $n \cup \{n\} = m \cup \{m\}$. Then $n \in m$ or $n = m$. Similarly, $m = n$ or $m \in n$. Suppose that $m \neq n$. Then $n \in m$ and $m \in n$. By Lemma 3.4, $n \in n$. However, Lemma 3.2 then implies $n \not\subset n$, which is a contradiction. $\square$