Ve215 Electric Circuits

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Chapter 9

Sinusoids and Phasors

9.1 Introduction

- Circuits driven by sinusoidal current or voltage sources are called alternating current (ac) circuits.
- We now begin the analysis of ac circuits.
- We are interested in sinusoidal steady-state response of ac circuits.

9.2 Sinusoids

A sinusoid is a signal that has the form of the sine or cosine function. For example,

$$v(t) = V_m \sin(\omega t + \phi)$$

where

V_m : amplitude

ω: angular frequency

ϕ: initial phase

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t$$

$$v_2(t) = V_m \sin(\omega t + \phi)$$

shown in Fig. 9.2.

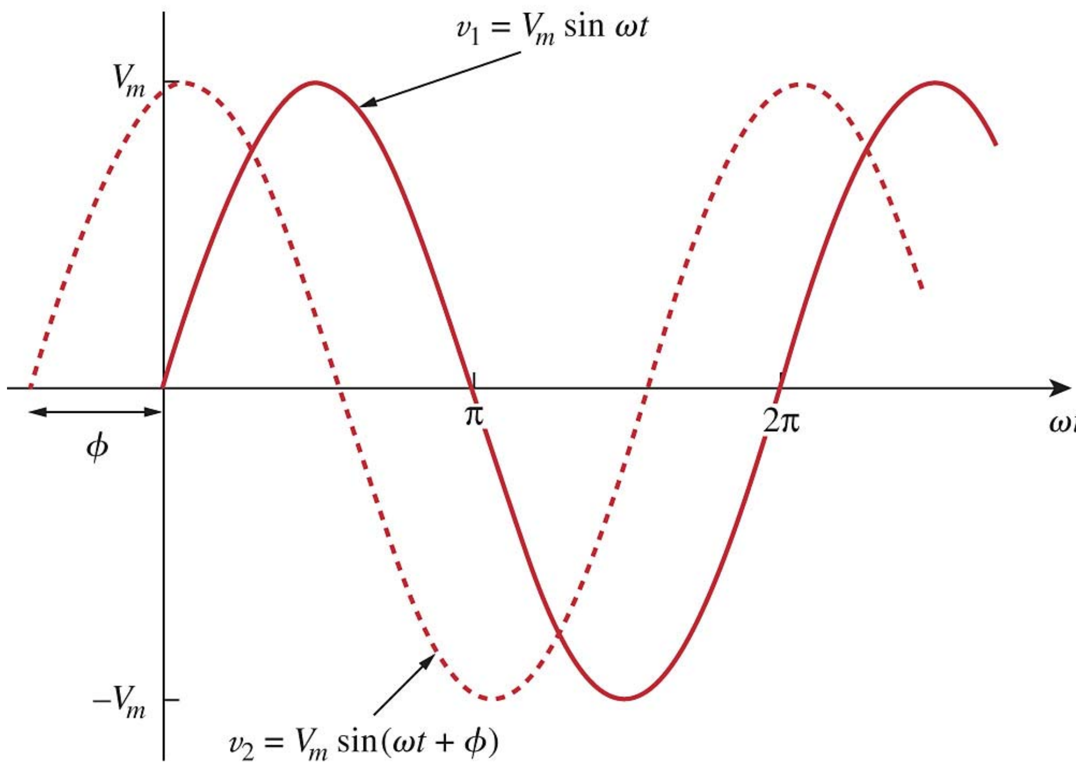


Figure 9.2 Two sinusoids with different phases.

The starting point of v_2 in Fig. 9.2 occurs first in time. Therefore, we say that v_2 leads v_1 by ϕ or that v_1 lags v_2 by ϕ. If $\phi \neq 0$, we say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*.

9.3 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions. A phasor is a complex number that represents the amplitude and phase of a sinusoid.

Given a sinusoid

$$v(t) = V_m \cos(\omega t + \phi)$$

$$= \mathrm{Re}\left(V_m e^{j(\omega t + \phi)}\right)$$

$$= \mathrm{Re}\left(V_m e^{j\phi} e^{j\omega t}\right)$$

$$= \mathrm{Re}(\widetilde{V} e^{j\omega t})$$

where $\widetilde{V} = V_m e^{j\phi} = V_m \angle \phi$ is the phasor representation of the sinusoid $v(t)$.

Figure 9.7 shows the *sinor* $\tilde{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction. We may regard $v(t)$ as the projection of the sinor on the real axis.

$$\underbrace{\tilde{V}e^{j\omega t}}_{\text{Sinor}} = V_m e^{j(\omega t+\phi)}$$

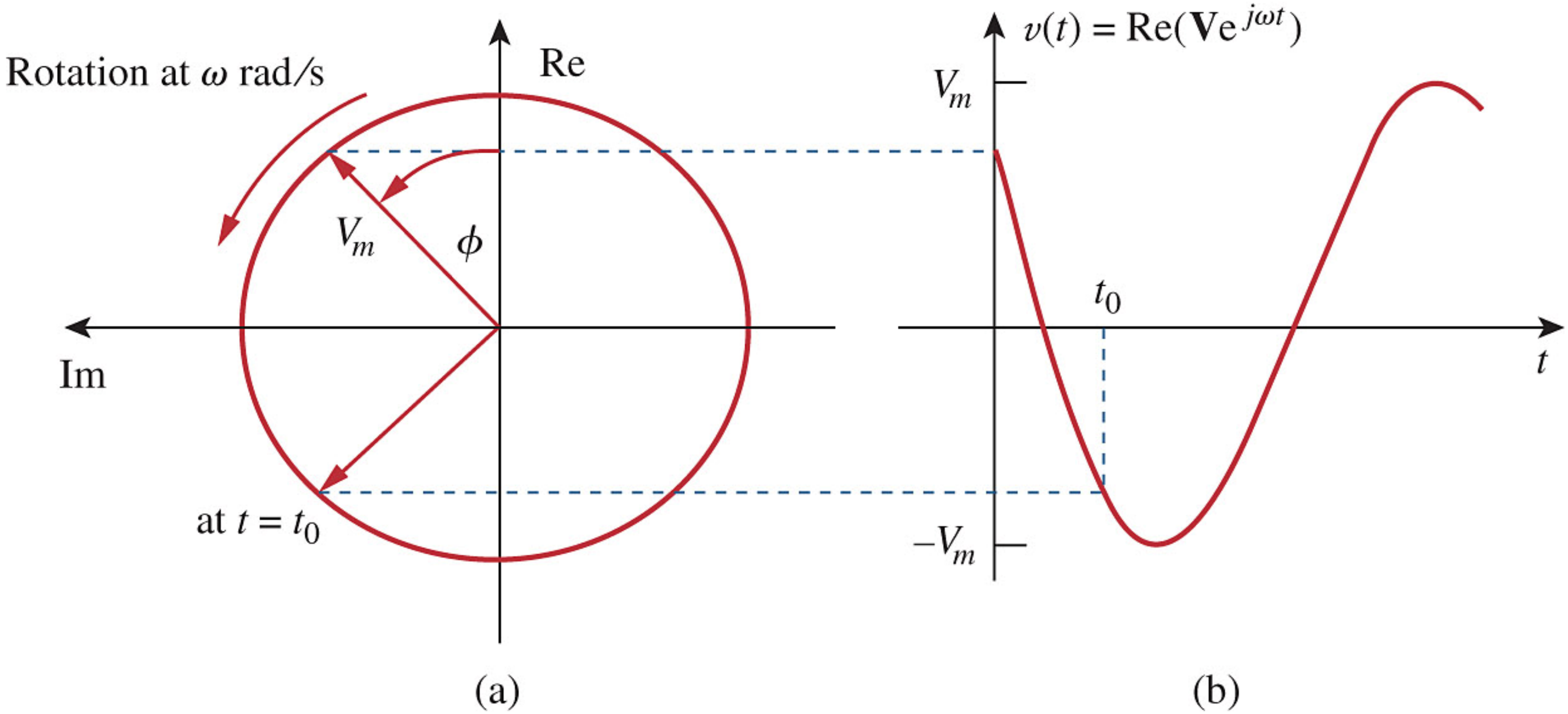


Figure 9.7 Representation of sinor: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

$$\tilde{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$$

Sinor

The value of the sinor at time $t = 0$ is the phasor $\tilde{V}$. The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present.

Since a phasor has magnitude and phase ("direction"), it behaves as a vector. Figure 9.8 shows two phasors: $\tilde{V} = V_m \angle \phi$ and $\tilde{I} = I_m \angle -\theta$. Such a graphical representation of phasors is known as a *phasor diagram*.

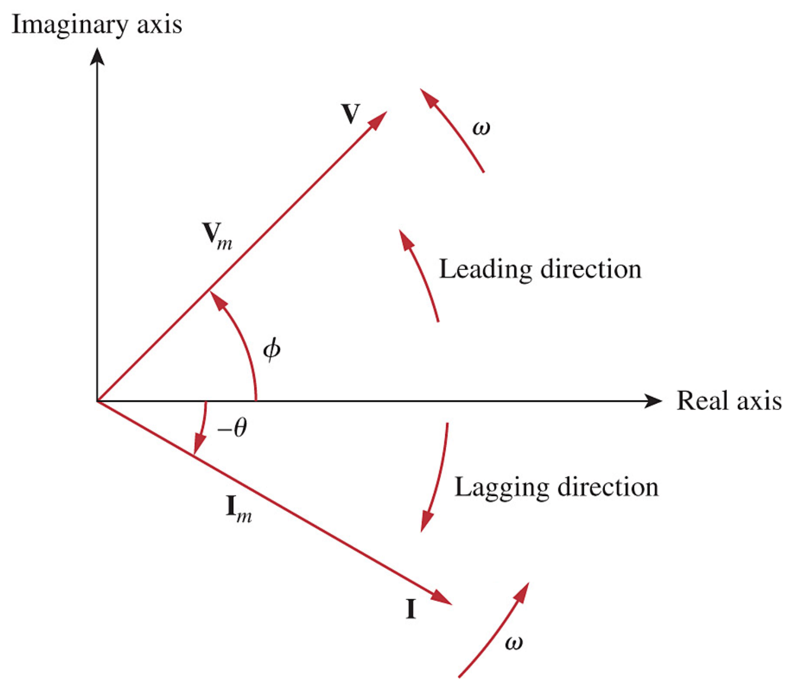


Figure 9.8 A phasor diagram.

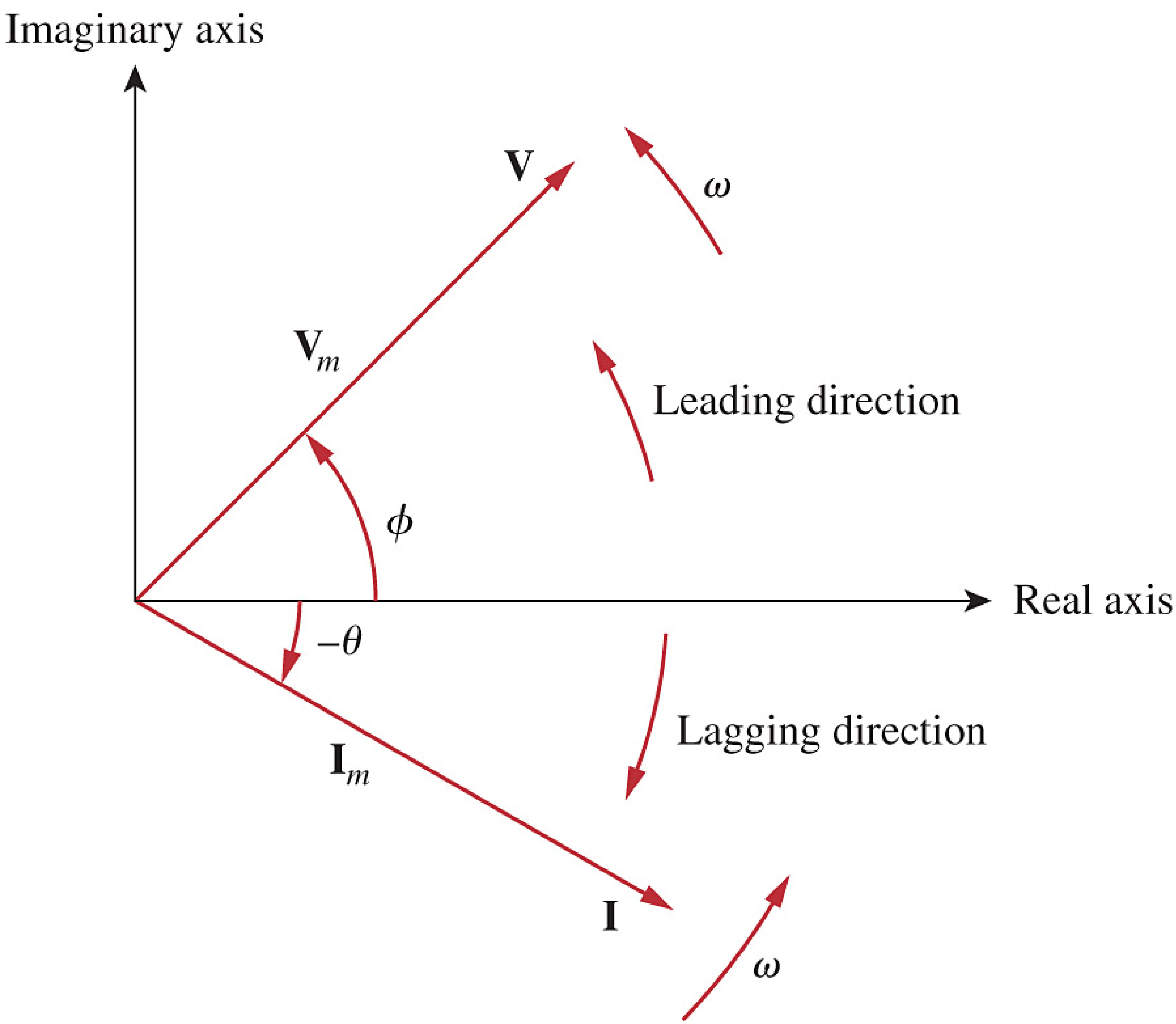


Figure 9.8 A phasor diagram.

In conclusion, a sinusoid has a time-domain representation $v(t) = V_m \cos(\omega t + \phi)$ and a phasor-domain representation $\widetilde{V} = V_m \angle \phi$.
The phasor domain is also known as the frequency domain.

$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \widetilde{V} = V_m \angle \phi$$

9.4 Phasor Relationships for Circuit Elements

We begin with the resistor. If the current through a resistor R is $i = \underline{I_m} \cos(\omega t + \underline{\phi})$, the voltage across it is given by Ohm's law as

$$v = iR = \underline{RI_m} \cos(\omega t + \underline{\phi})$$

The phasor form of this voltage is

$$\tilde{V} = RI_m \angle \phi$$

But the phasor representation of the current is $\tilde{I} = I_m \angle \phi$. Hence,

$$\tilde{V} = R\tilde{I}$$

showing the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain.

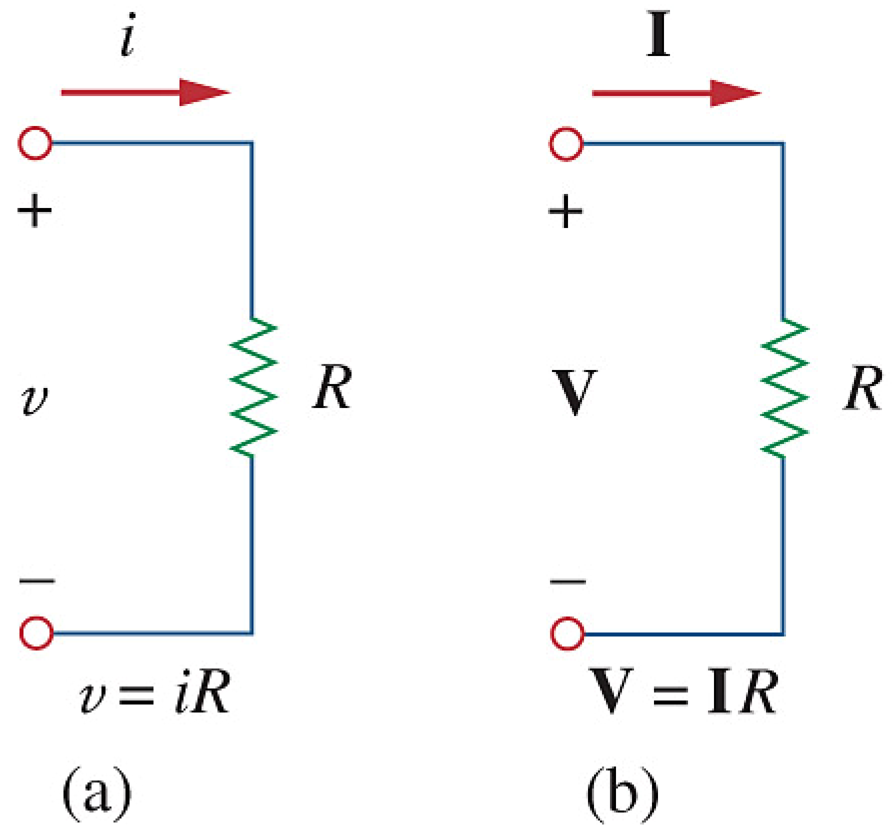


Figure 9.9 Voltage-current relations for a resistor in the (a) time domian, (b) frequency domain.

$$\tilde{V} = R\tilde{I}$$

$$\tilde{V} = RI_m \angle \phi \qquad \tilde{I} = I_m \angle \phi$$

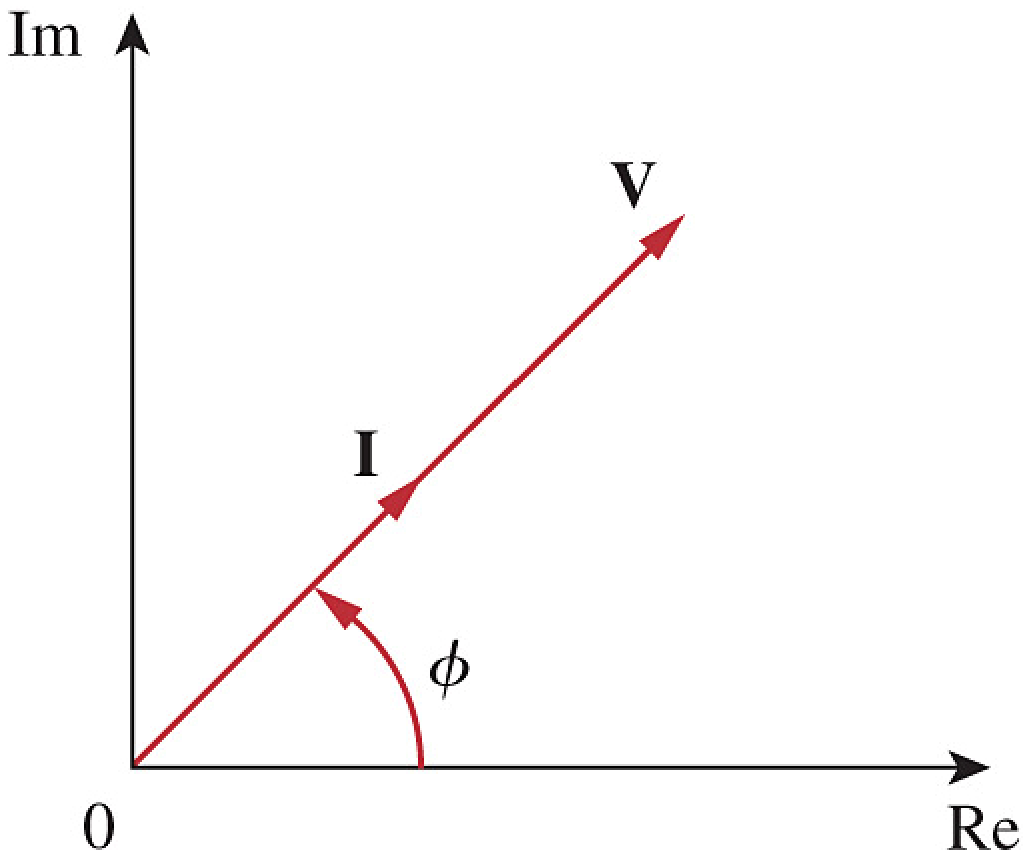


Figure 9.10 Phasor diagram for the resistor.

For the inductor L, if $i = I_m \cos(\omega t + \phi)$

$\Leftrightarrow \tilde{I} = I_m \angle\phi$, then

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

$$= \omega L I_m \cos(\omega t + \phi + 90°) \Leftrightarrow$$

$$\tilde{V} = \omega L I_m \angle(\phi + 90°) = j\omega L I_m \angle\phi$$

$$= j\omega L \tilde{I}$$

$\angle 90° = e^{j90°} = j$

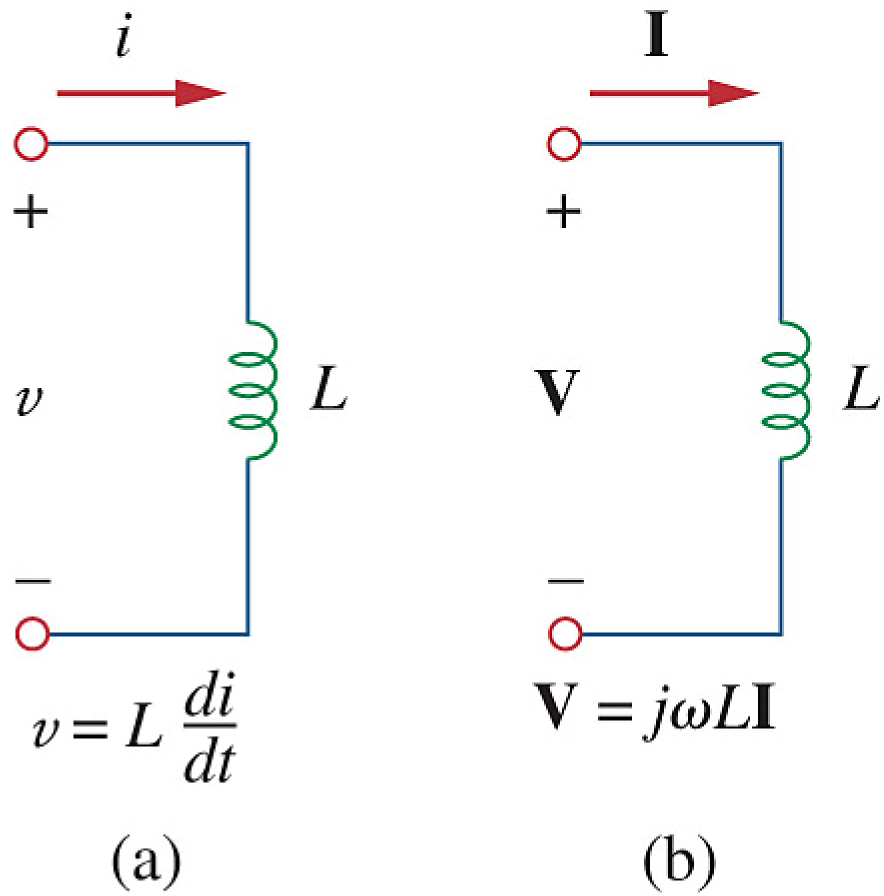


Figure 9.11 Voltage-current relations for an inductor in the (a) time domain, (b) frequency domain.

$$\tilde{I} = I_m \angle \underline{\phi} \qquad \tilde{V} = \omega L I_m \angle \underline{(\phi + 90°)} = j\omega L \tilde{I}$$

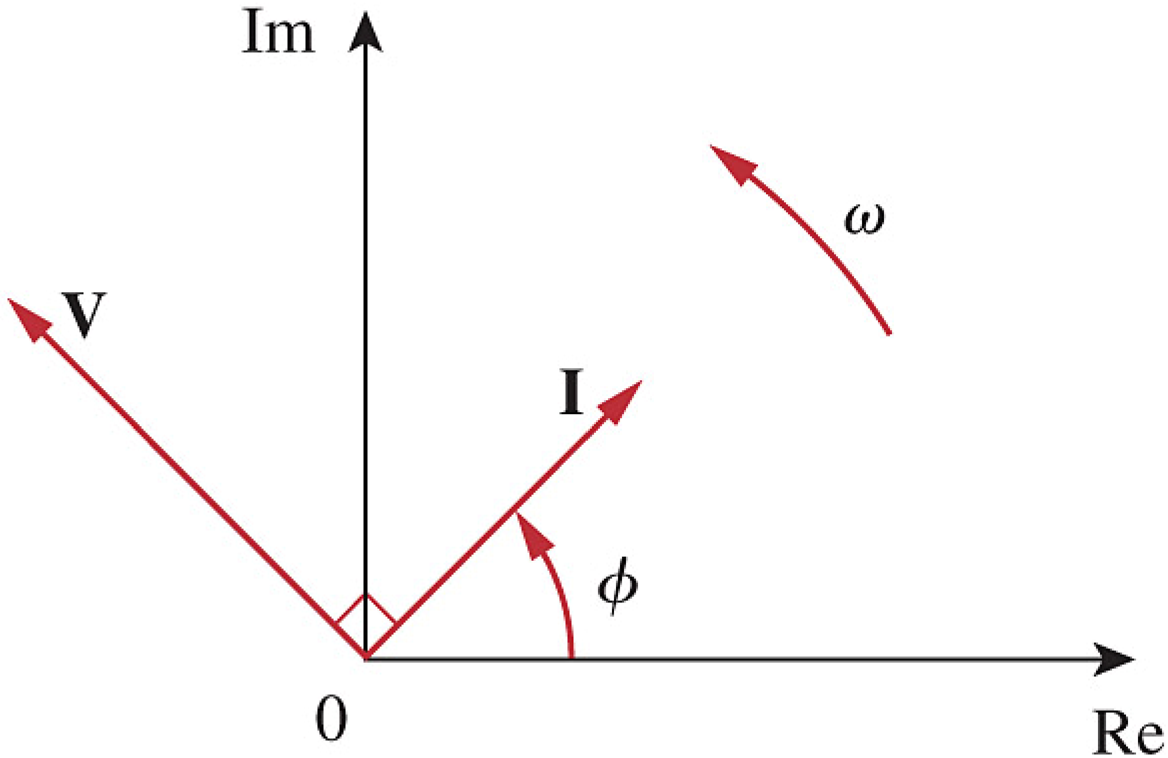


Figure 9.12 Phasor diagram for the inductor.

For the capacitor C, if $v = V_m \cos(\omega t + \phi)$

$\Leftrightarrow \tilde{V} = V_m \angle \phi$, then

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$

$$= \omega C V_m \cos(\omega t + \phi + 90^\circ) \Leftrightarrow$$

$$\tilde{I} = \omega C V_m \angle(\phi + 90^\circ) = j\omega C V_m \angle \phi$$

$$= j\omega C \tilde{V}$$

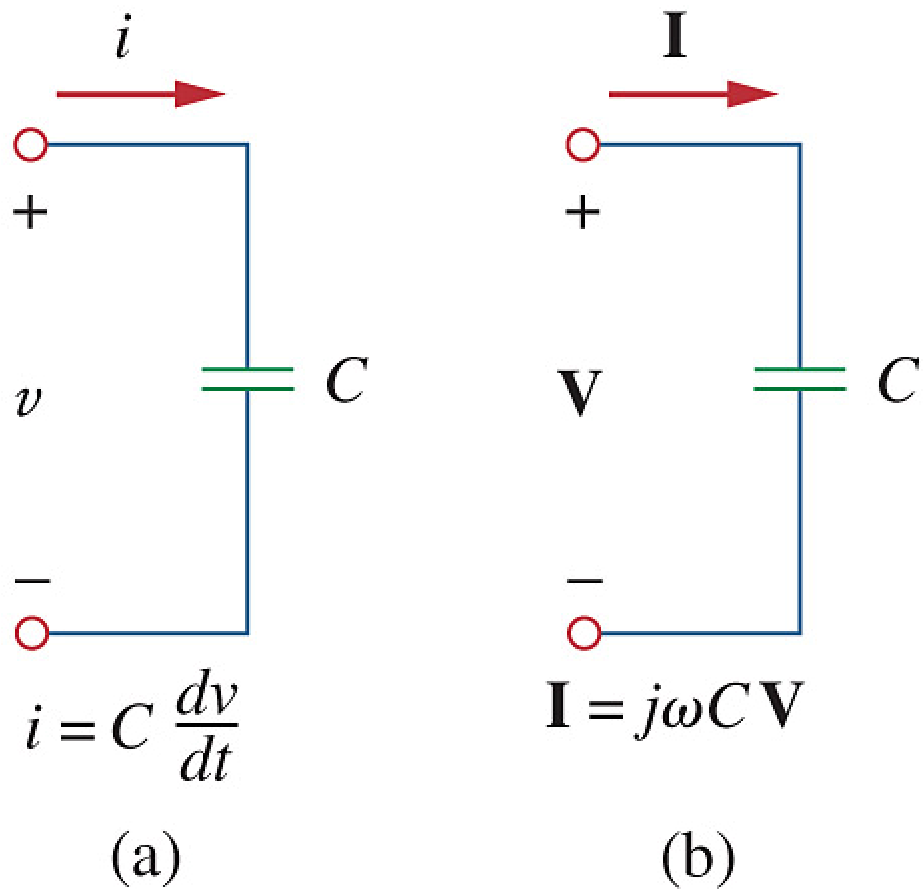


Figure 9.13 Voltage-current relations for a capacitor in the (a) time domain, (b) frequency domain.

$$\tilde{V} = V_m \underline{\angle \phi} \qquad \tilde{I} = \omega C V_m \underline{\angle(\phi + 90°)} = j\omega C \tilde{V}$$

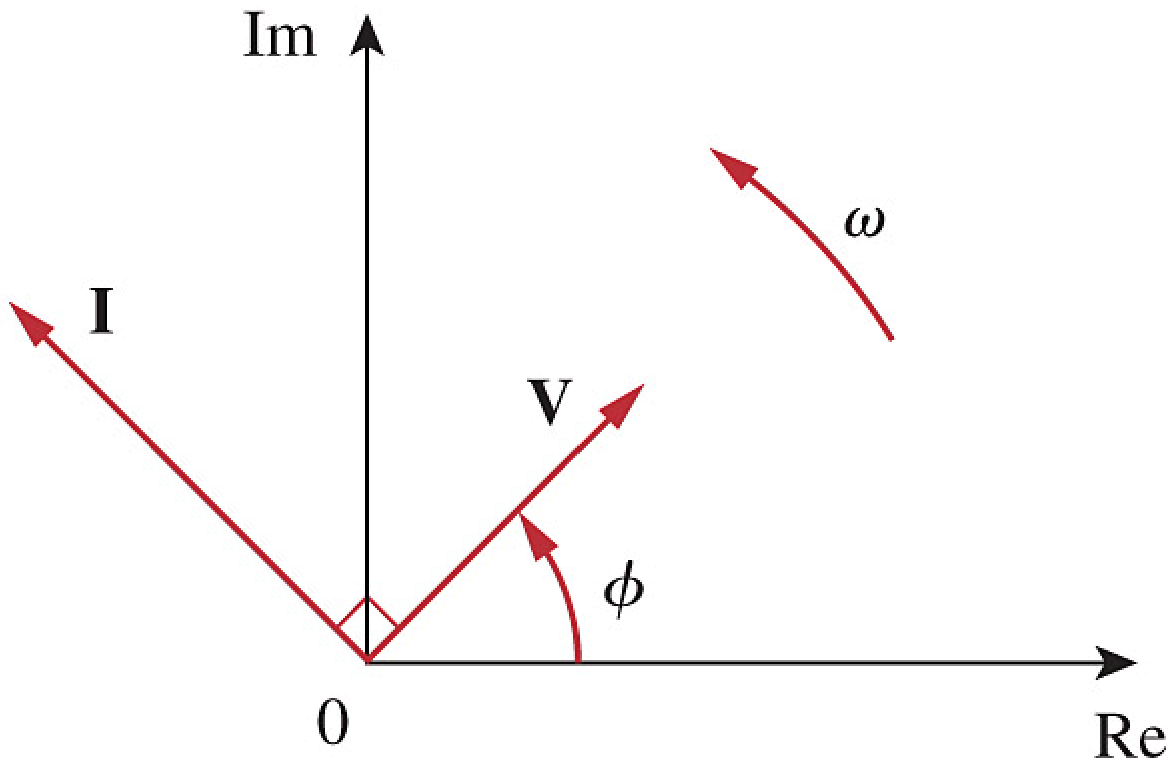


Figure 9.14 Phasor diagram for the capacitor.

TABLE 9.2

Summary of voltage - current relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\tilde{V} = R\tilde{I}$
L	$v = L\dfrac{di}{dt}$	$\tilde{V} = j\omega L\tilde{I}$
C	$i = C\dfrac{dv}{dt}$	$\tilde{V} = \dfrac{1}{j\omega C}\tilde{I}$

9.5 Impedance and Admittance

The impedance Z of a circuit is the ratio of the phasor voltage $\tilde{V}$ to the phasor current $\tilde{I}$, measured in ohms (Ω).

$$Z = \frac{\tilde{V}}{\tilde{I}}$$

$$\boxed{\begin{aligned} \tilde{V} &= R\tilde{I} \\ \tilde{V} &= j\omega L\tilde{I} \\ \tilde{V} &= \frac{1}{j\omega C}\tilde{I} \end{aligned}}$$

For the three passive elements, we have $Z = R$, $Z = j\omega L$, and $Z = 1/\, j\omega C$, respectively.

The admittance Y of a circuit is the ratio of the phasor current $\tilde{I}$ to the phasor voltage $\tilde{V}$, measured in siemens (S).

$$Y = \frac{\tilde{I}}{\tilde{V}} = \frac{1}{Z}$$

$$\boxed{\begin{gathered}\tilde{V} = R\tilde{I} \\ \tilde{V} = j\omega L\tilde{I} \\ \tilde{V} = \frac{1}{j\omega C}\tilde{I}\end{gathered}}$$

For the three passive elements, we have $Y = 1/R$, $Y = 1/j\omega L$, and $Y = j\omega C$, respectively.

General definition of R

R: Resistance
G: Conductance
$R=1/G$

(Real number)

$\Rightarrow$

Z: Impedance
Y: Admittance
$Z=1/Y$

(complex number)

The Ohm's law in phasor form is

$$\tilde{V} = Z\tilde{I} \text{ or } \tilde{I} = Y\tilde{V}$$

As a complex quantity, the impedence may be expressed in rectangular form or polar form

$$Z = R + jX = |Z| \angle \theta$$

where

R : resistance

X : reactance

If $X > 0$, we say that the impedance is inductive or lagging since current lags voltage; If $X < 0$, we say that the impedance is capacitive or leading because current leads voltage.

The impedance, resistance, and reactance are all measured in ohms.

$$\tilde{I} = I_m \angle \phi \qquad \tilde{V} = \omega L I_m \angle(\phi + 90^\circ) = j\omega L \tilde{I}$$

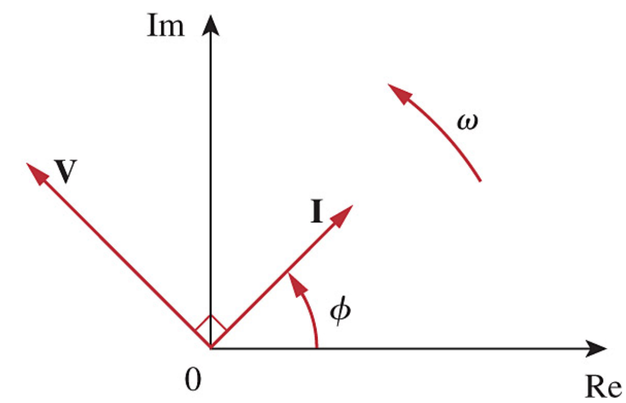


Figure 9.12 Phasor diagram for the inductor.

$$Z = R + jX$$

For an inductor, Z=jωL
→X=ωL>0
→Impedance is inductive
or lagging (**I** lags **V**)

$$\tilde{V} = V_m \angle \phi \qquad \tilde{I} = \omega C V_m \angle(\phi + 90^\circ) = j\omega C \tilde{V}$$

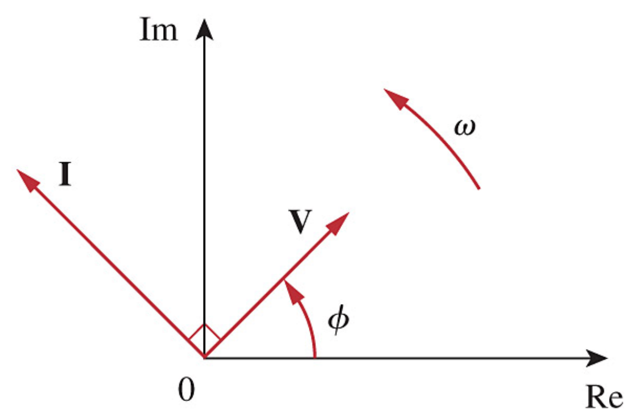


Figure 9.14 Phasor diagram for the capacitor.

$$Z = R + jX$$

For an inductor, Z=1/(jωC)
→X=–1/(ωC)<0
→Impedance is capacitive
or leading (**I** leads **V**)

The admittance can be written as

$$Y = G + jB$$

where

G : conductance

B : susceptance

The admittance, conductance, and susceptance are all measured in siemens.

$$\left.\begin{array}{l} R: \ \text{resistance} \\ X: \ \text{reactance} \end{array}\right\} \quad \text{Impedance Z}$$

$$\left.\begin{array}{l} G: \ \text{conductance} \\ B: \ \text{susceptance} \end{array}\right\} \quad \text{Admittance Y}$$

9.6 Kirchhoff's Laws in the Frequency Domain

For KVL, Let $v_1, v_2, \cdots, v_n$ be the voltages around a closed loop. Then

$$\sum_{i=1}^{n} v_i = 0 \qquad (9.51)$$

In sinusoidal steady state,

$$v_i = V_{mi}\cos(\omega t + \phi_i) = \mathrm{Re}\left(\tilde{V}_i e^{j\omega t}\right)$$

$$\sum_{i=1}^{n} \mathrm{Re}\left(\tilde{V}_i e^{j\omega t}\right) = 0$$

$$\mathrm{Re}\left(\left(\sum_{i=1}^{n} \widetilde{V}_i\right) e^{j\omega t}\right) = 0$$

but $e^{j\omega t} \neq 0$,

$$\sum_{i=1}^{n} \widetilde{V}_i = 0$$

indicating that KVL holds for phasors.

For KCL, if $i_1, i_2, \cdots, i_n$ are the currents leaving or entering a closed surface in a circuit at time t, and $\tilde{I}_1, \tilde{I}_2, \ldots \tilde{I}_n$ are the phasor forms of $i_1, i_2, \cdots, i_n$, then

$$\sum_{i=1}^{n} i_i = 0 \Rightarrow \sum_{i=1}^{n} \tilde{I}_i = 0$$

Since basic circuit laws, Kirchoff's and Ohm's, hold in phasor domain, it is not difficult to analyze ac circuits.

9.7 Impedance Combinations

For the N series-connected impedances shown in Fig. 9.18, the equivalent impedance at the input terminals is

KVL

$$Z_{eq} = \frac{\widetilde{V}}{\widetilde{I}} = \frac{\sum_{i=1}^{N} \widetilde{V}_i}{\widetilde{I}} = \sum_{i=1}^{N} \frac{\widetilde{V}_i}{\widetilde{I}} = \sum_{i=1}^{N} Z_i$$

Same I

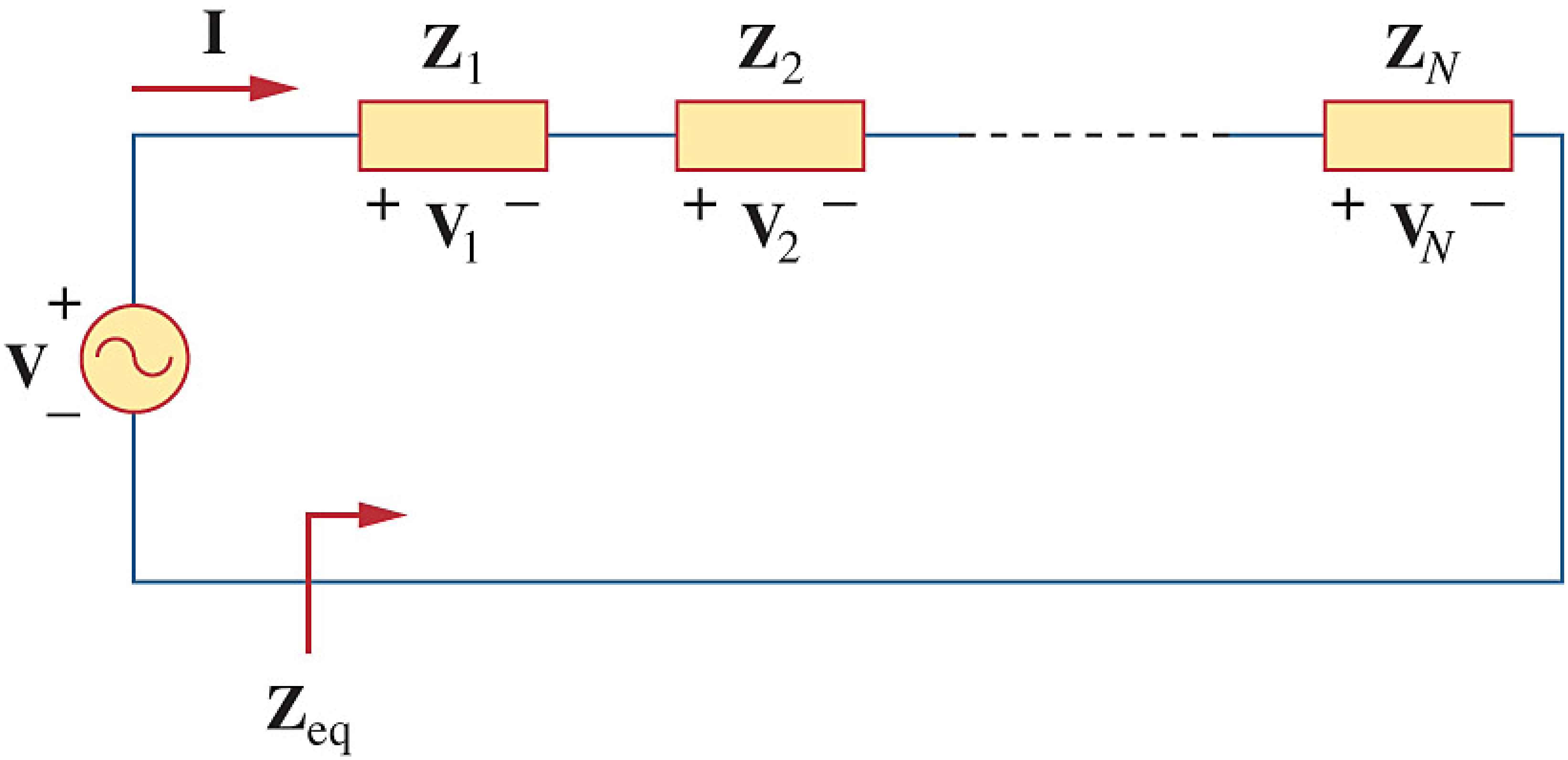


Figure 9.18 N impedances in series.

For the N parallel-connected impedances shown in Fig. 9.20, the equivalent admittance at the input terminals is

$$Y_{eq} = \frac{\tilde{I}}{\tilde{V}} = \frac{\overbrace{\sum_{i=1}^{N} \tilde{I}_i}^{\text{KCL}}}{\underbrace{\tilde{V}}_{\text{Same V}}} = \sum_{i=1}^{N} \frac{\tilde{I}_i}{\tilde{V}} = \sum_{i=1}^{N} Y_i$$

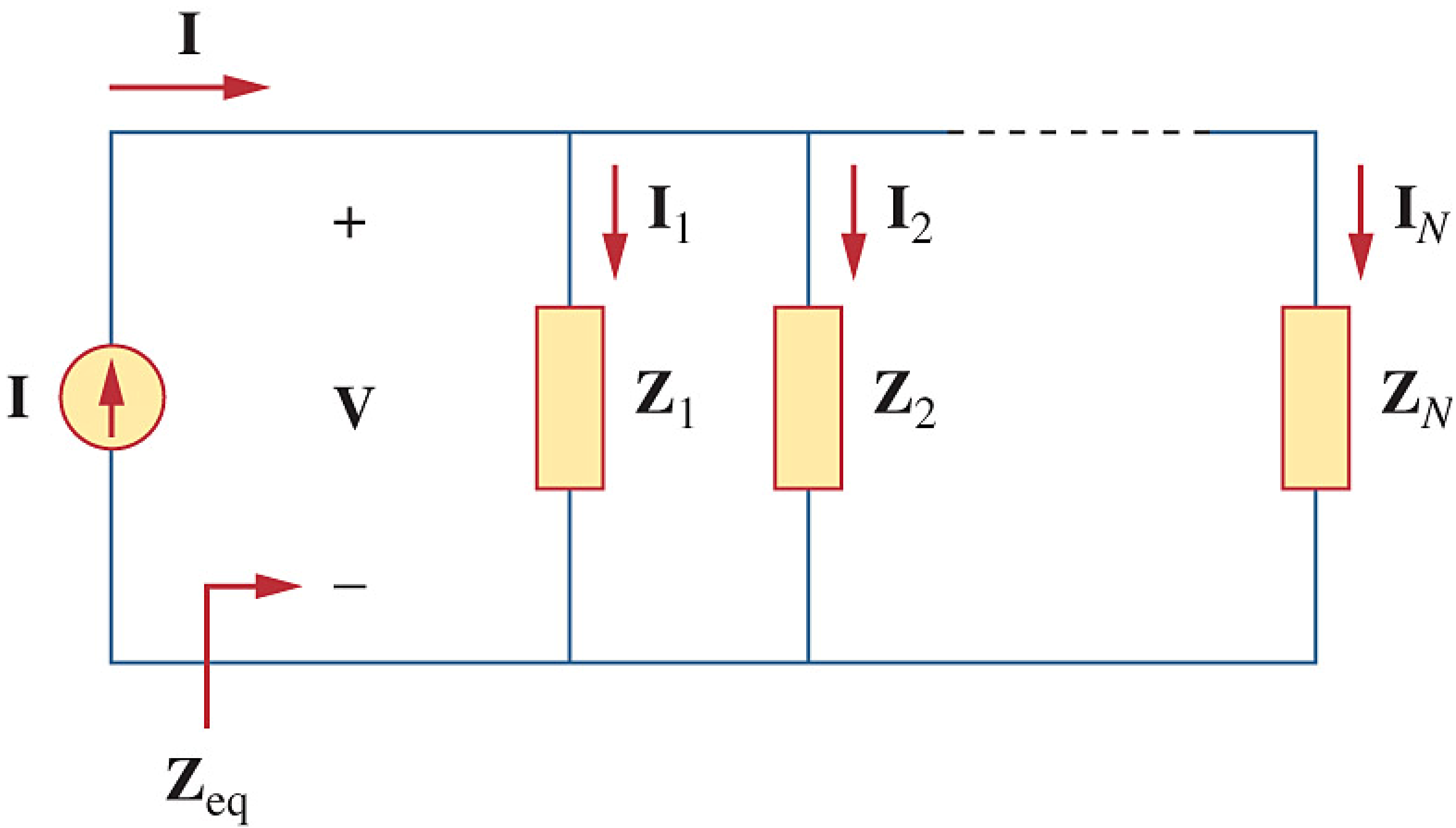


Figure 9.20 N impedances in parallel.

The delta-to-wye and wye-to-delta transformations that we applied to resistive circuits are also valid for impedances. With reference to Fig. 9.22, the conversion formulas are as follows:

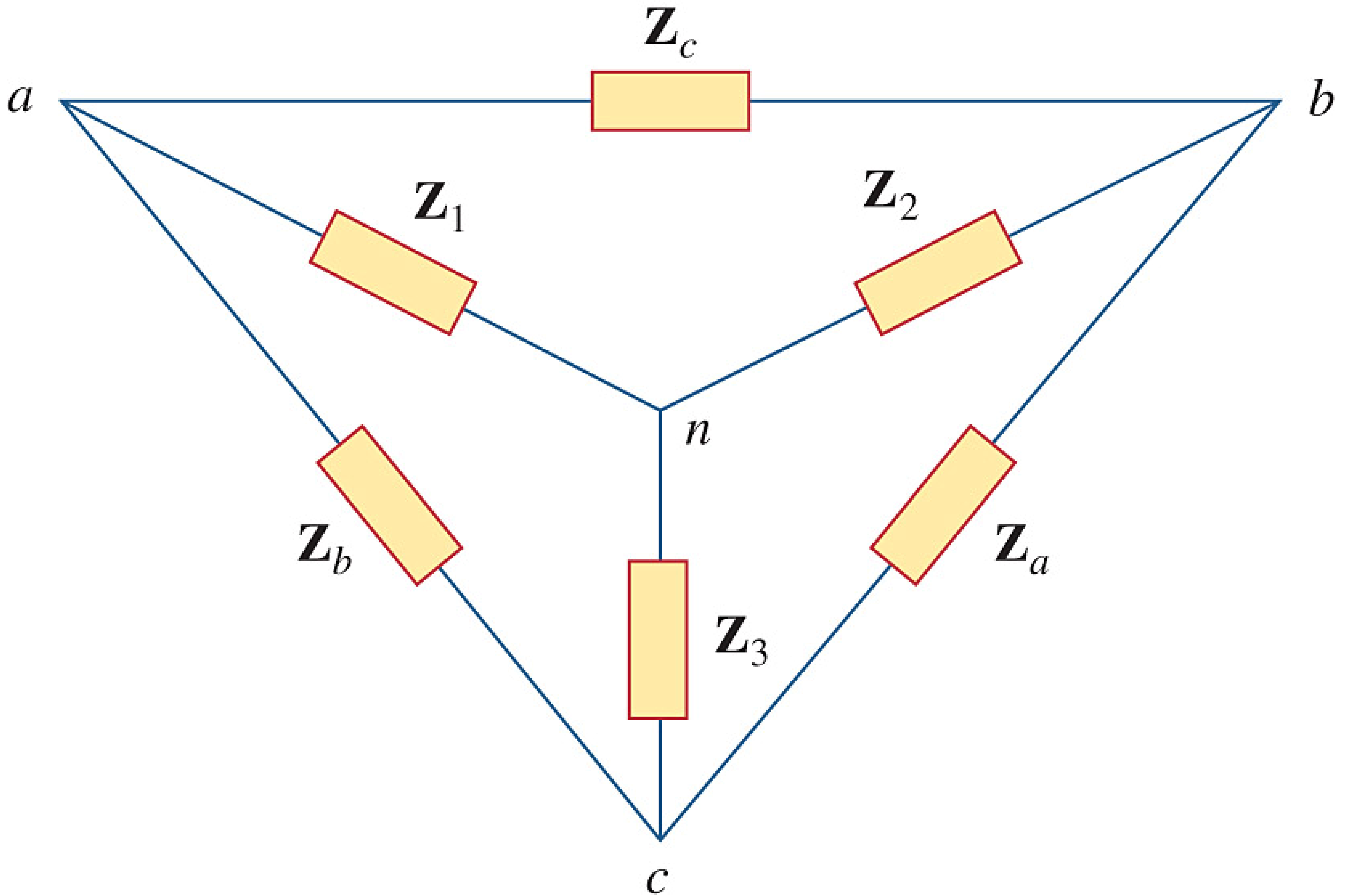


Figure 9.22 Superimposed wye and delta networks.

Y-Δ conversion:

$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

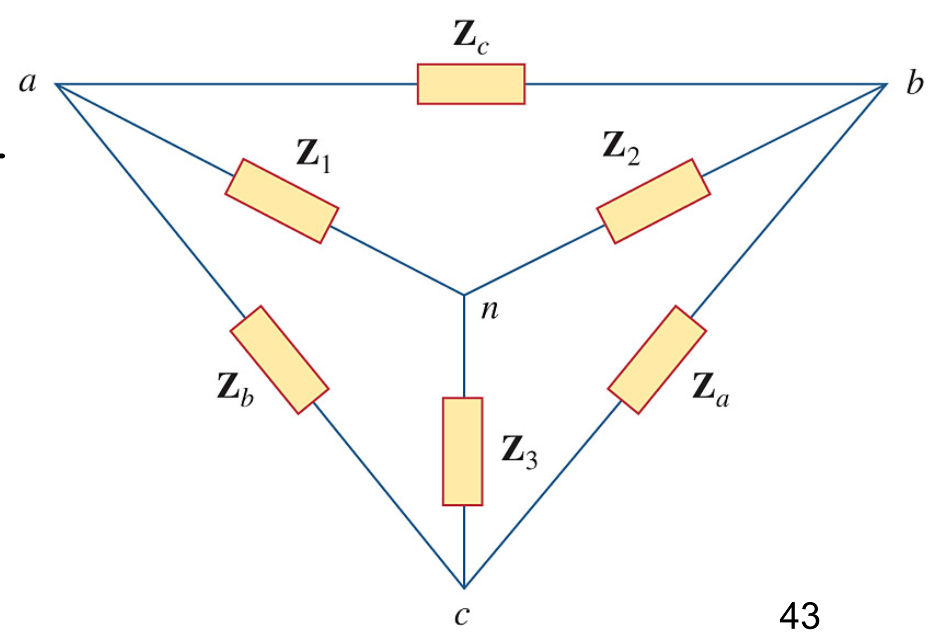


Figure 9.22 Superimposed wye and delta networks.

Δ-Y conversion:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

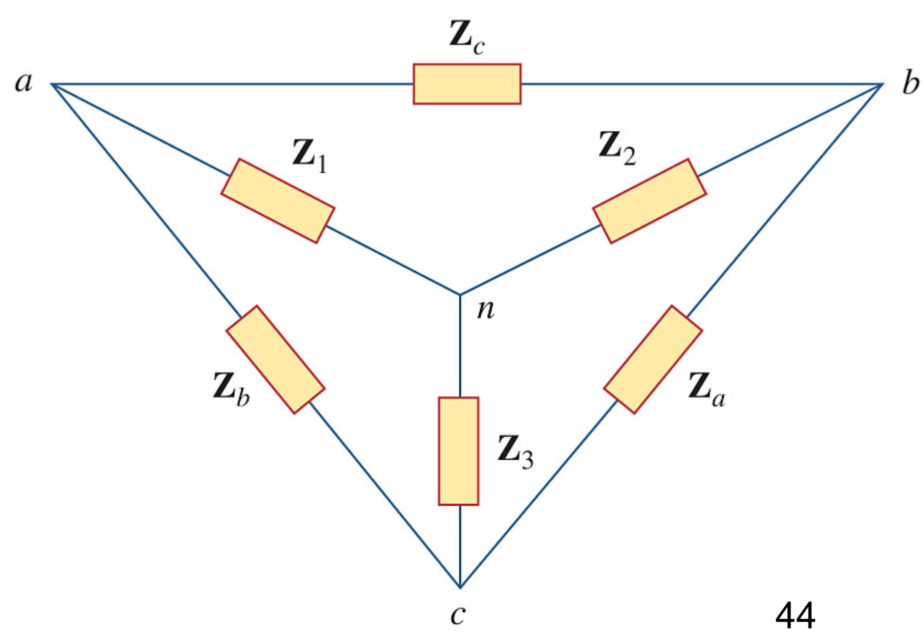


Figure 9.22 Superimposed wye and delta networks.

Practice Problem 9.10 Find the input impedance of the circuit in Fig. 9.24 at $\omega = 10$ rad/s.

Solution :

8-H inductor: $Z_1 = j10 \times 8 = j80\ (\Omega)$

0.5-mF capacitor: $Z_2 = \dfrac{1}{j10 \times (0.5 \times 10^{-3})}$

$= -j200\ (\Omega)$

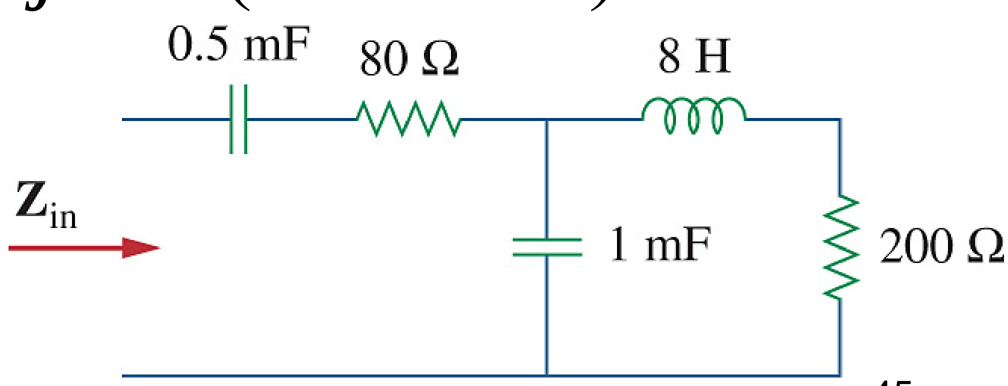


Figure 9.24

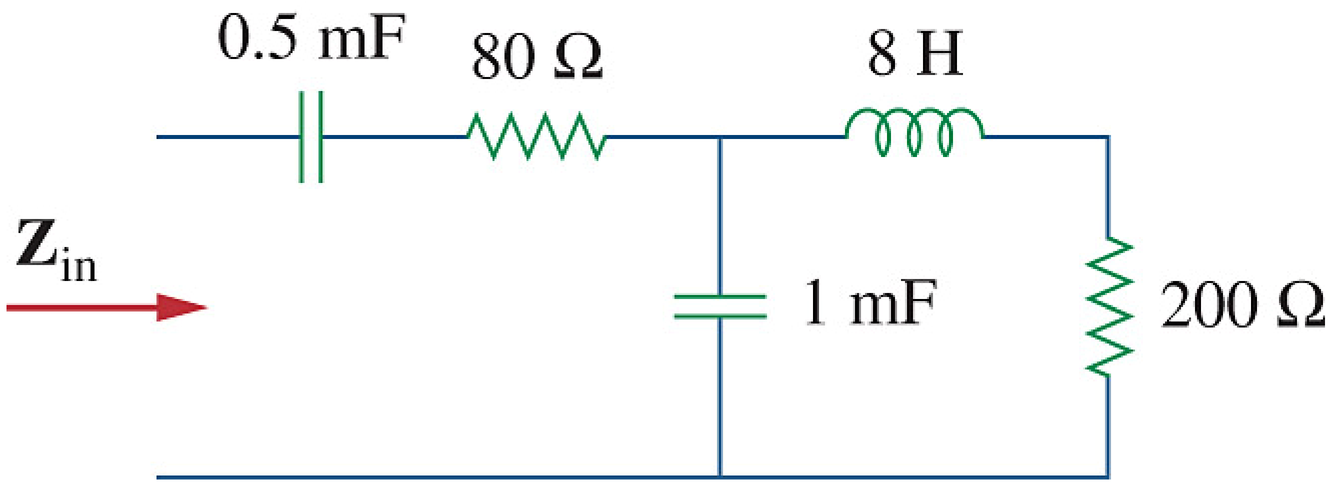


Figure 9.24

$$\text{1-mF capacitor: } Z_3 = \frac{1}{j10 \times (1 \times 10^{-3})}$$

$$= -j100\ (\Omega)$$

$$Z_{in} = Z_2 + 80 + Z_3 \parallel (Z_1 + 200)$$

where

$$Z_3 \parallel (Z_1 + 200) = (-j100) \parallel (j80 + 200)$$

$$= \frac{(-j100) \times (j80 + 200)}{(-j100) + (j80 + 200)}$$

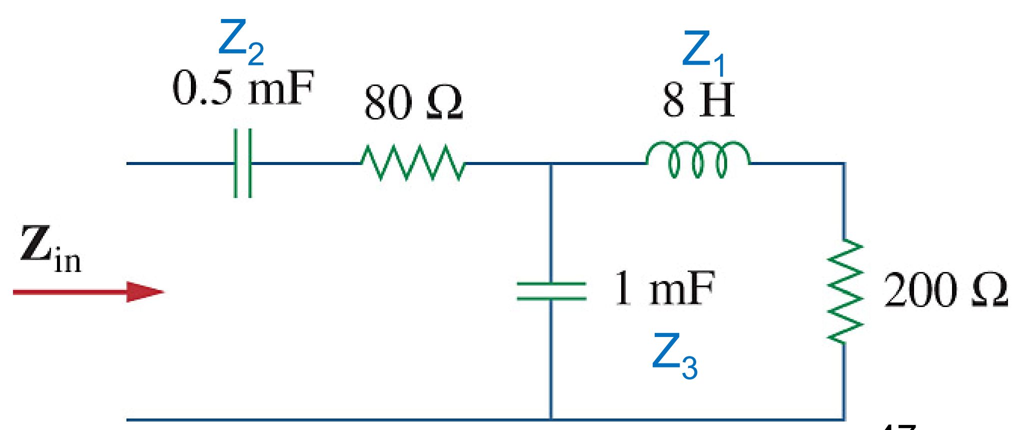


Figure 9.24

$$= \frac{(-j100)\times(200+j80)}{200-j20}$$

Rectangular form

$$\approx \frac{(100\angle -90^\circ)\times 215.4066\angle 21.80^\circ}{200.9975\angle -5.71^\circ}$$

Polar form

$$\approx 107.1688\angle -62.49^\circ$$

$$\approx 49.5016 - j95.0512\ (\Omega)$$

$$Z_{in} = -j200 + 80 + 49.5016 - j95.0512$$

$$\approx 129.50 - j295.05\ (\Omega)$$

Practice Problem 9.11 Calculate v_o in the circuit of Fig. 9.27.

Solution :

0.5-H inductor: $Z_1 = j10 \times 0.5 = j5\ (\Omega)$

$\frac{1}{20}$-F capacitor: $Z_2 = \dfrac{1}{j10 \times (1/20)}$

$= -j2\ (\Omega)$

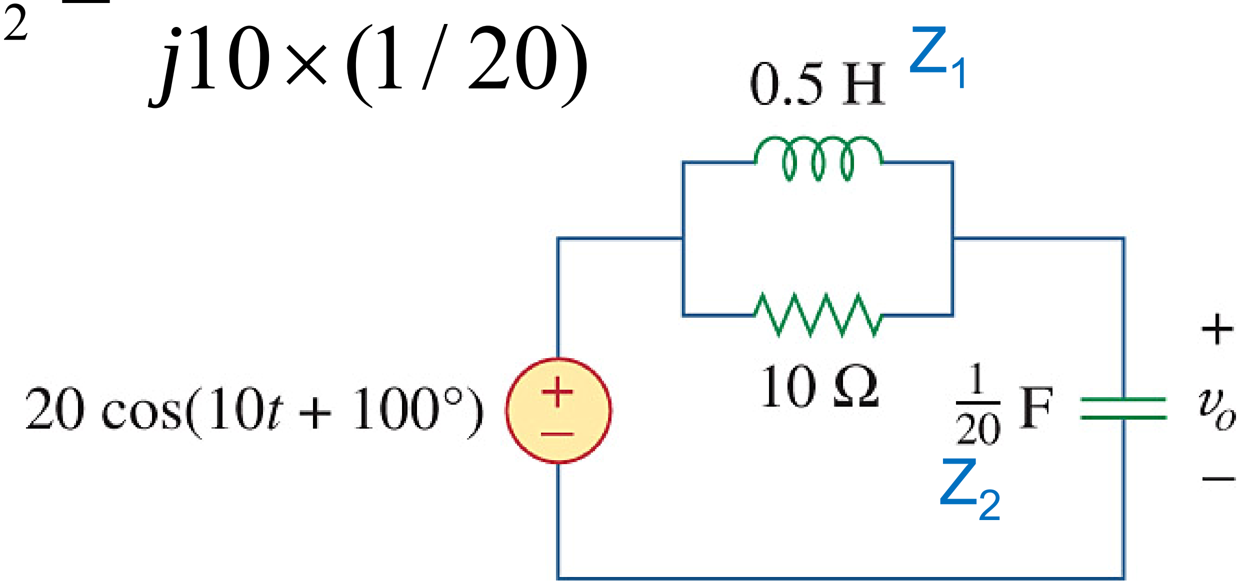


Figure 9.27

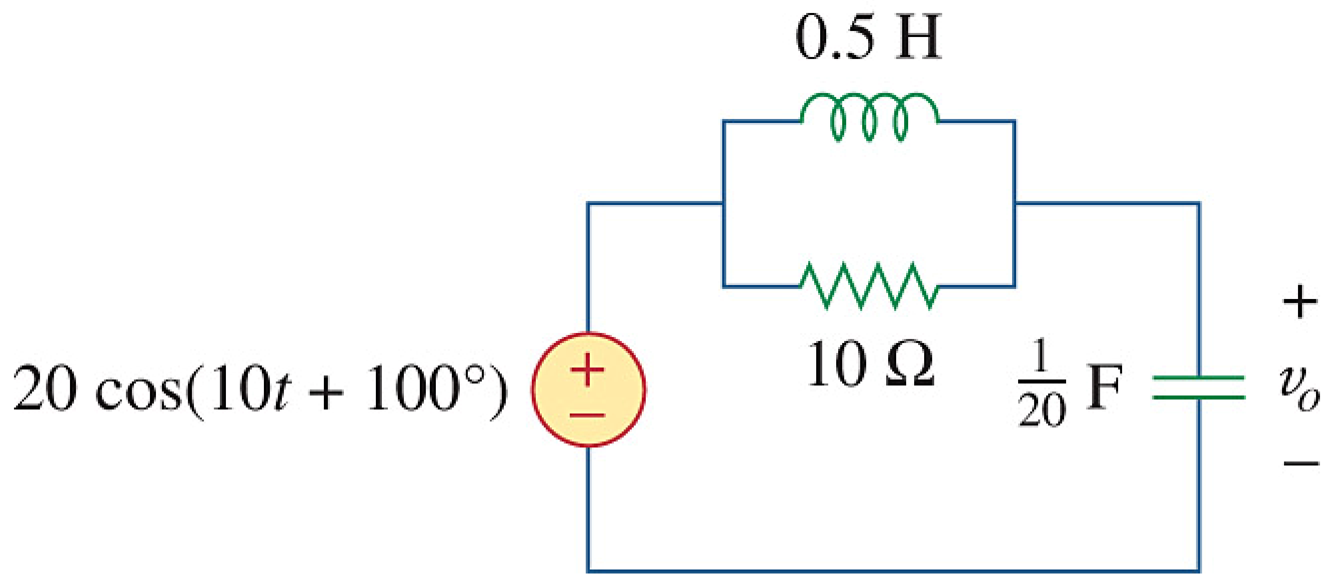


Figure 9.27

$$20\cos(10t+100^\circ)\ \text{V}: \ 20\angle 100^\circ\ \text{V}$$

$$\tilde{V}_o = 20\angle 100^\circ \times \frac{-j2}{-j2+10 \parallel j5}$$

Voltage division

$$10 \parallel j5 = \frac{10\times j5}{10+j5} = \frac{j10}{2+j} = 2+j4$$

$$\tilde{V}_o = 20\angle 100^\circ \times \frac{-j2}{2+j2} = 10\sqrt{2}\angle -35^\circ\ (\text{V})$$

$$v_o(t) = 10\sqrt{2}\cos(10t-35^\circ)\ (\text{V})$$

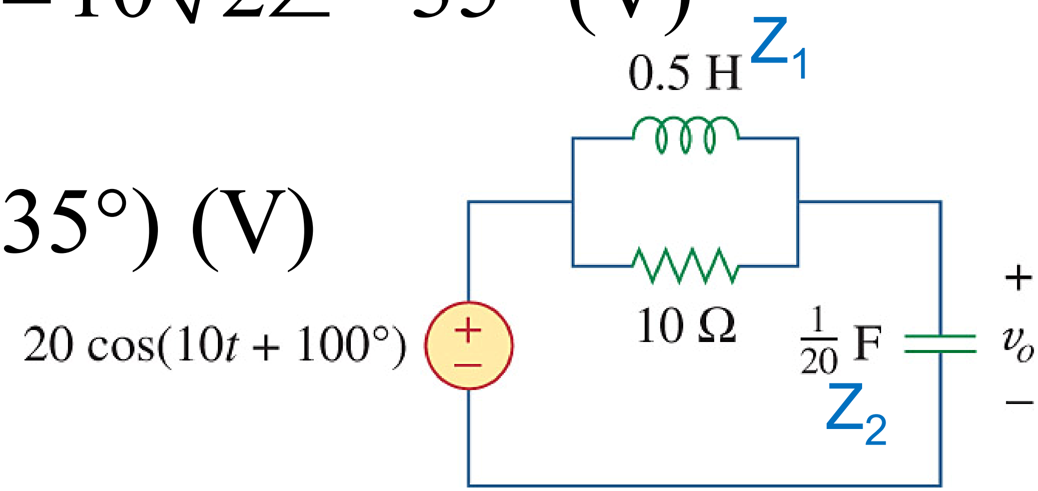


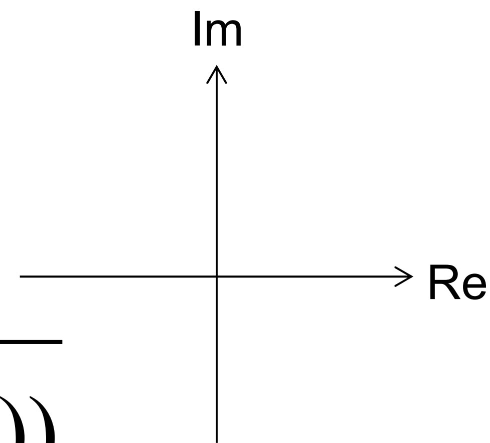
Figure 9.27

9.8 Applications

Phase - Shifters In Fig. 9.31(a),

$$\tilde{V}_o = \tilde{V}_i \frac{R}{R + 1/(j\omega C)} = \tilde{V}_i \frac{R}{R - j(1/\omega C)}$$

$$= \tilde{V}_i \frac{R}{\sqrt{R^2 + (1/\omega C)^2} \angle -\tan^{-1}(1/(\omega RC))}$$



Z=R+jX
$\angle Z = \tan^{-1}(X/R)$

$\tilde{V}_o$ leads $\tilde{V}_i$ by $\theta = \tan^{-1}(1/(\omega RC))$,

$0° < \theta < 90°$, as shown in Fig. 9.32(a).

$\angle \mathbf{V_i} + \theta = \angle \mathbf{V_o}$
$\angle \mathbf{V_o} > \angle \mathbf{V_i}$

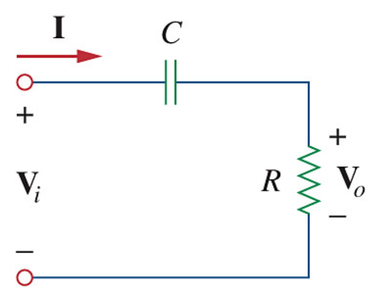


Figure 9.31(a) Series RC shift circuit: leading output.

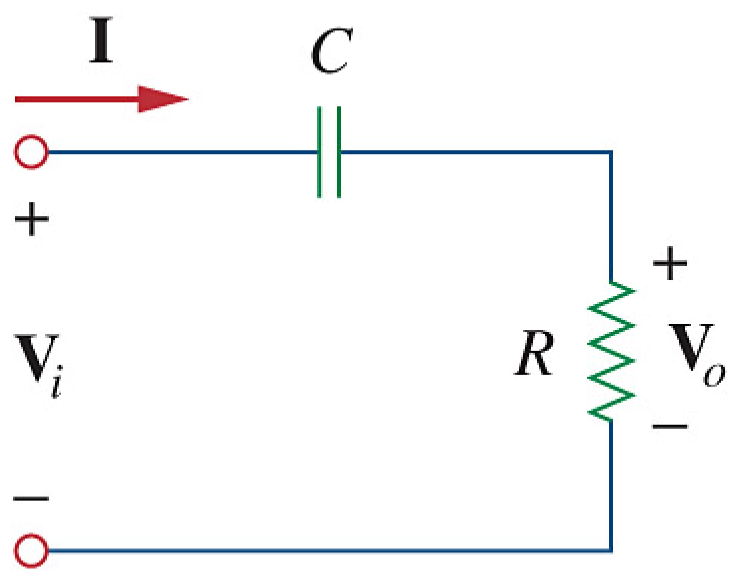


Figure 9.31(a) Series RC shift circuit: leading output.

$$\angle \mathbf{V_i} + \theta = \angle \mathbf{V_o}$$
$$\angle \mathbf{V_o} > \angle \mathbf{V_i}$$
$$0^\circ < \theta < 90^\circ$$

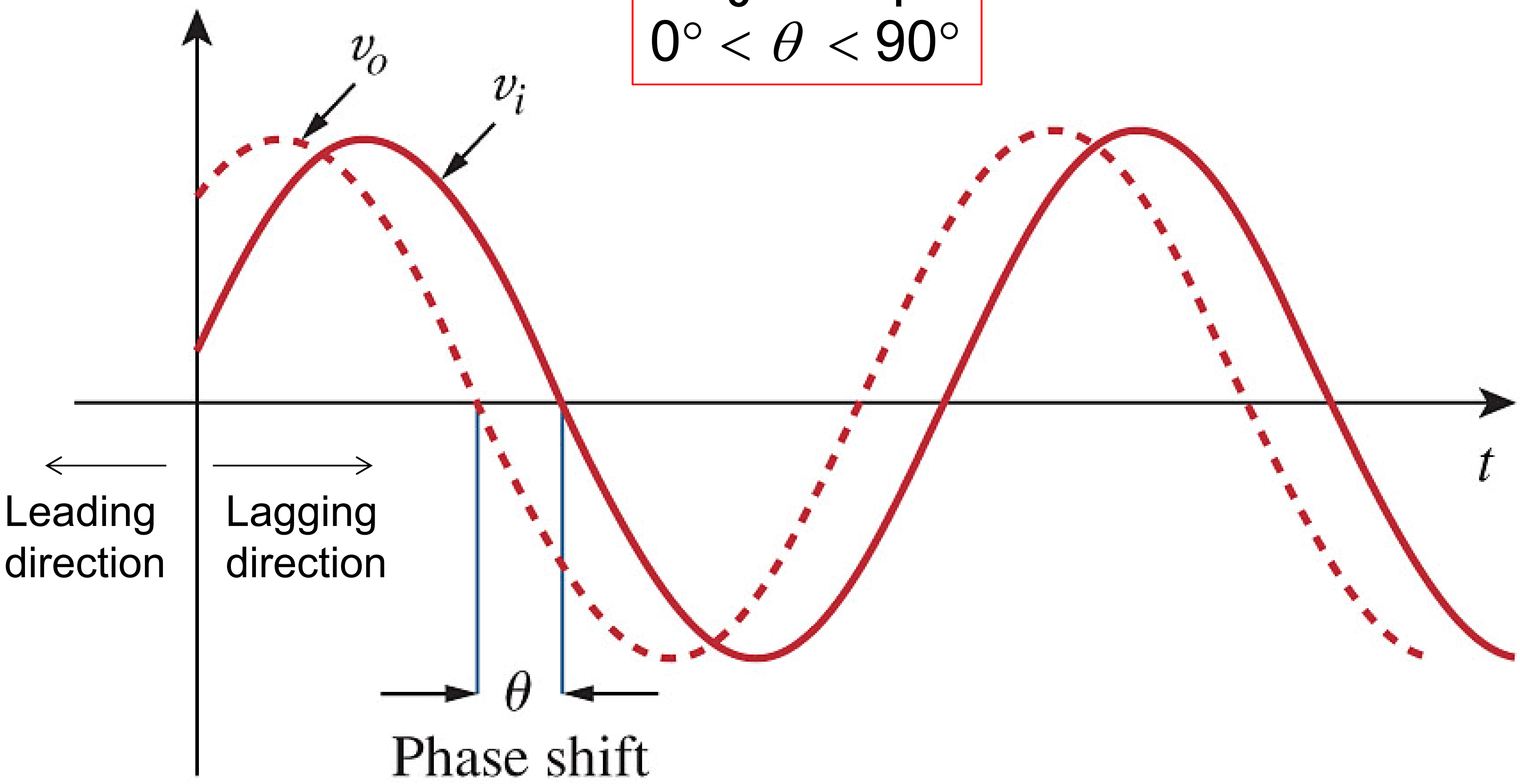


Figure 9.32(a) Phase shift in RC circuits: leading output.

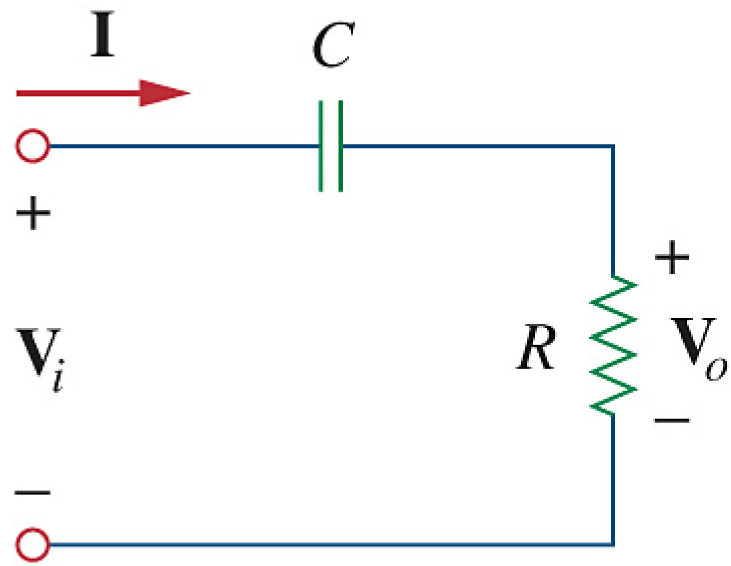


Figure 9.31(a) Series RC shift circuit: leading output.

Another way to check leading/lagging relation:

(1) $\mathbf{V_o}$=$\mathbf{I}$R → $\angle\mathbf{I}=\angle\mathbf{V_o}$
(2) $\mathbf{V_i}=\mathbf{IZ}=\mathbf{I}(R+1/j\omega C)$ → $-90° < \angle\mathbf{Z} < 0$ → $\angle\mathbf{I} > \angle\mathbf{V_i}$
(3) Thus, $\angle\mathbf{V_o} > \angle\mathbf{V_i}$, leading output

Issue of 90° shift:

To produce 90° shift, $\omega RC \rightarrow 0$

$\tilde{V}_o$ leads $\tilde{V}_i$ by $\theta = \tan^{-1}(1/(\omega RC))$

$|\mathbf{V_o}| = 1/(1+1/(\omega RC)^2)^{1/2}|\mathbf{V_i}| \rightarrow$
$1/(1+\infty)^{1/2}|\mathbf{V_i}| \rightarrow 0$

$$\tilde{V}_o = \tilde{V}_i \frac{R}{R+1/(j\omega C)} = \tilde{V}_i \frac{R}{R-j(1/\omega C)}$$

$$= \tilde{V}_i \frac{R}{\sqrt{R^2+(1/\omega C)^2}\angle -\tan^{-1}(1/(\omega RC))}$$

No output voltage!

In Fig. 9.31(b),

$$\tilde{V}_o = \tilde{V}_i \frac{1/(j\omega C)}{R + 1/(j\omega C)} = \tilde{V}_i \frac{1}{1 + j\omega RC}$$

$$= \tilde{V}_i \frac{1}{\sqrt{1+(\omega RC)^2} \angle \tan^{-1}(\omega RC)}$$

$\tilde{V}_o$ lags $\tilde{V}_i$ by $\theta = \tan^{-1}(\omega RC)$, $0° < \theta < 90°$,

as shown in Fig. 9.32(b).

$$\boxed{\begin{array}{l} \angle \mathbf{V_i} - \theta = \angle \mathbf{V_o} \\ \angle \mathbf{V_o} < \angle \mathbf{V_i} \end{array}}$$

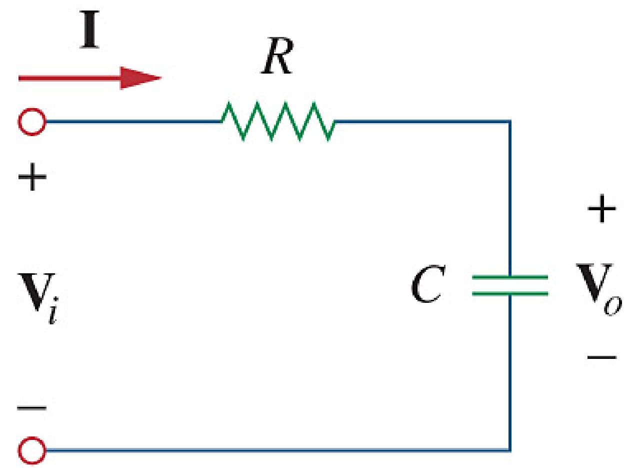


Figure 9.31(b) Series RC shift circuit: lagging output.

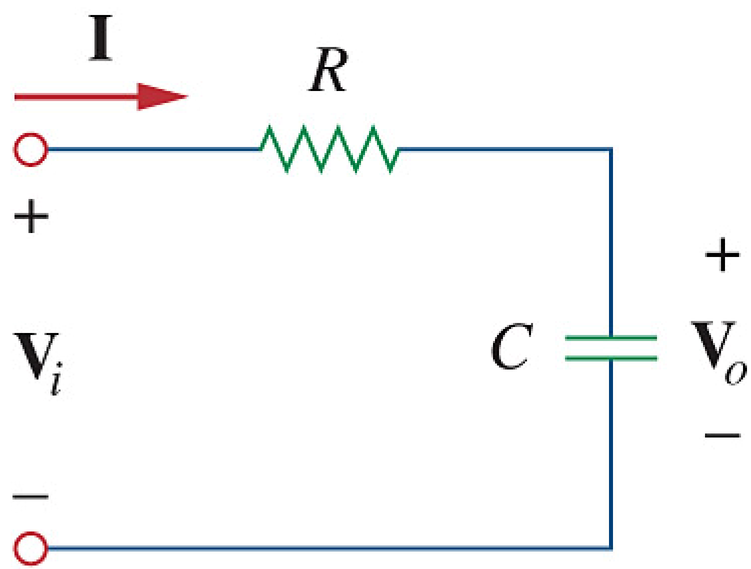


Figure 9.31(b) Series RC shift circuit: lagging output.

$\angle \mathbf{V_i} - \theta = \angle \mathbf{V_o}$
$\angle \mathbf{V_o} < \angle \mathbf{V_i}$
$0° < \theta < 90°$

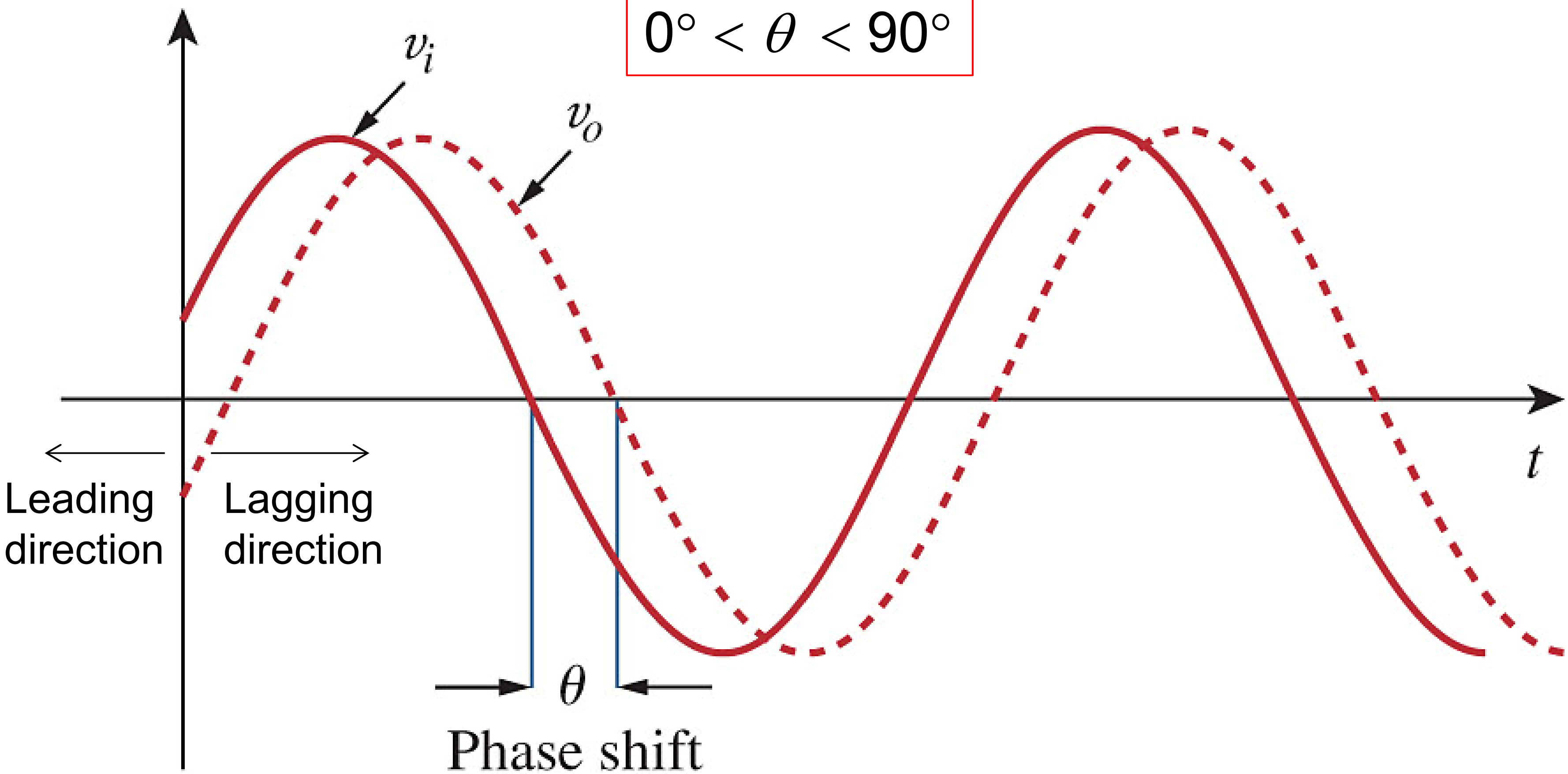


Figure 9.32(b) Phase shift in RC circuits: lagging output.

Practice Problem 9.13 Design an *RC* circuit to provide a phase shift of 90° leading.

Solution :

We need two stages, with each stage providing a phase shift of 45°.

Select $R_1 = R_2 = 20\ \Omega$,

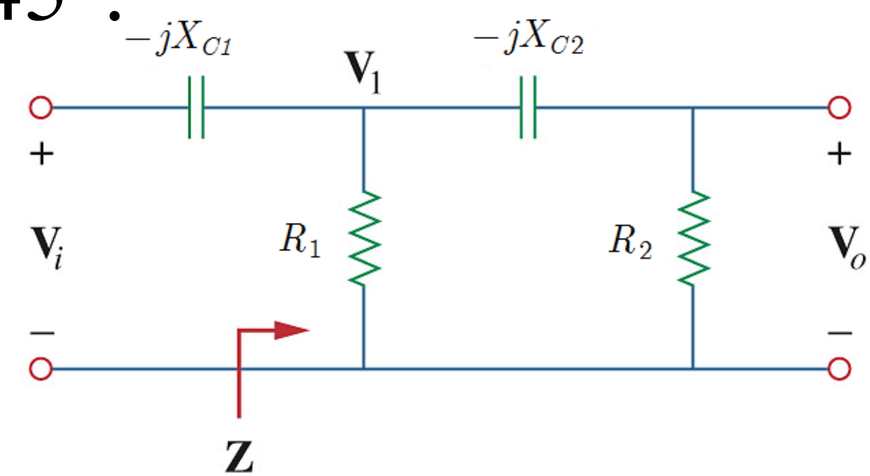


Figure 9.33 An *RC* phase shift circuit with 90° leading phase shift; for Example 9.13.

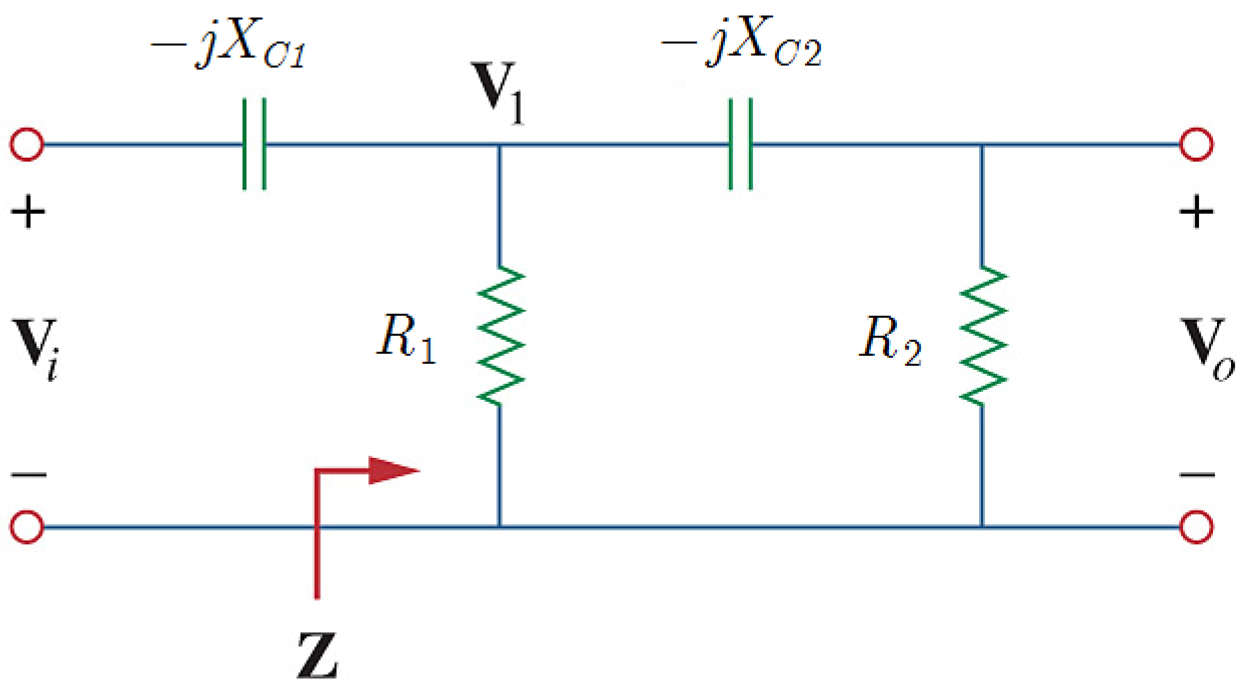


Figure 9.33 An RC phase shift circuit with 90° leading phase shift; for Example 9.13.

$$\tilde{V}_o = \tilde{V}_1 \frac{20}{20 - jX_{C2}} = \tilde{V}_1 \frac{20(20 + jX_{C2})}{20^2 + X_{C2}^2}$$

If $X_{C2} = 20\ \Omega$, then the second stage produces a 45° phase shift.

$$Z = 20 \parallel (20 - j20) = \frac{20 \times (20 - j20)}{20 + (20 - j20)}$$

$$= 12 - j4\ (\Omega)$$

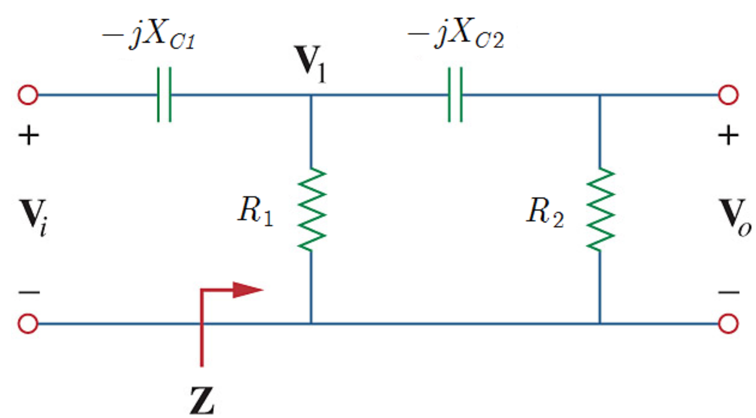


Figure 9.33 An RC phase shift circuit with 90° leading phase shift; for Example 9.13.

$$\tilde{V}_1 = \tilde{V}_i \frac{Z}{-jX_{C1} + Z} = \tilde{V}_i \frac{12 - j4}{12 - j(4 + X_{C1})}$$

$$= \tilde{V}_i \frac{(12 - j4)(12 + j(4 + X_{C1}))}{12^2 + (4 + X_{C1})^2}$$

$$= \tilde{V}_i \frac{(160 + 4X_{C1}) + j(12X_{C1})}{12^2 + (4 + X_{C1})^2}$$

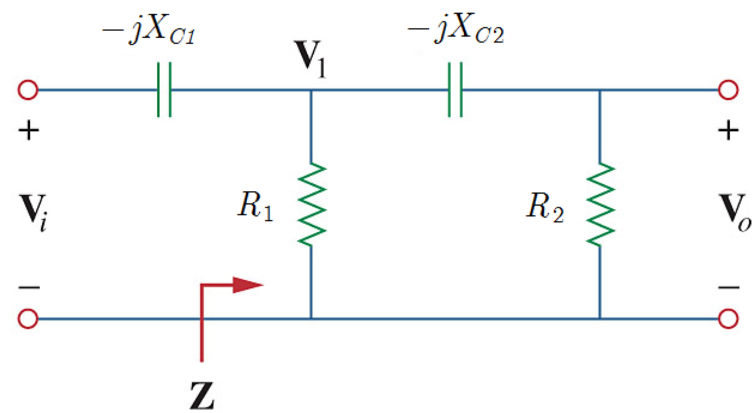


Figure 9.33 An RC phase shift circuit with 90° leading phase shift; for Example 9.13.

For the first stage to produce another 45°, we require $160 + 4X_{C1} = 12X_{C1}$, i.e.,

$X_{C1} = 20\ \Omega$.

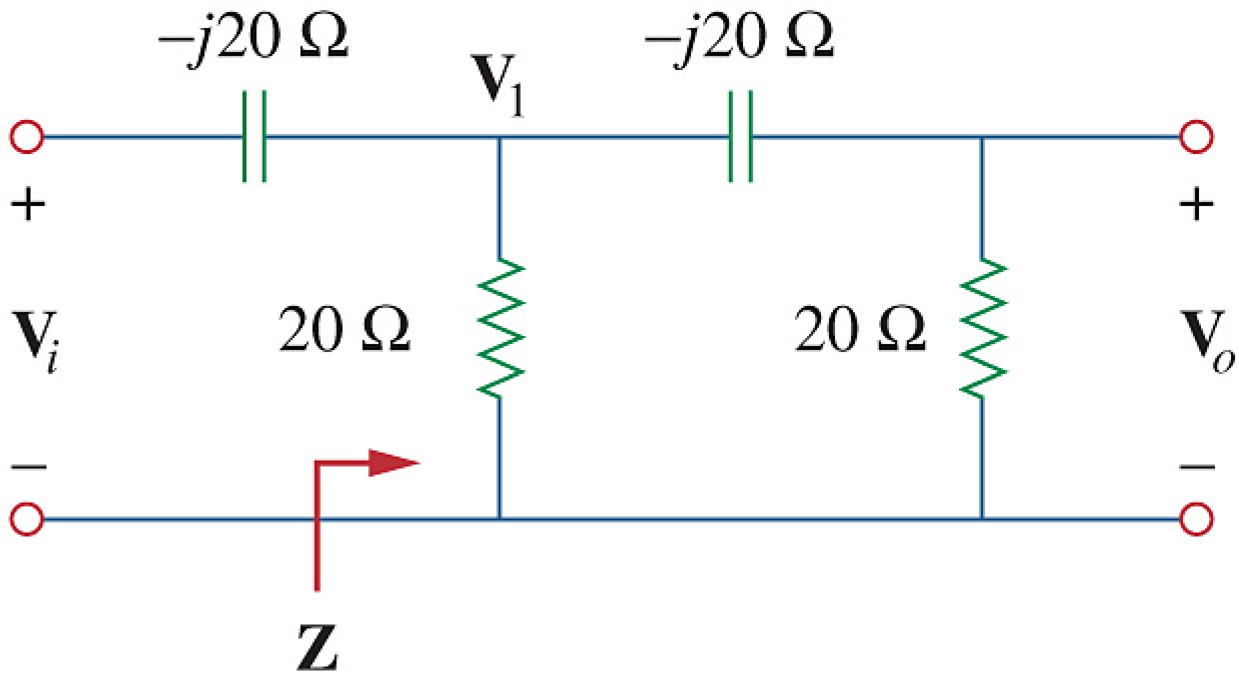


Figure 9.33