VE 216 Summer 2020
Lab 3: Feedback Control
Part I: Intro & Pre-lab Assignment

(Note: Borrowed from UMich EECS 216 Lab 4)

1 Introduction

In this laboratory project we will develop the idea of feedback control, the process of adjusting the input to a system as a function of a measured variable so as to obtain a desired response. Stated like this, the concept seems very vague, almost as vague as the notion of a system seemed to you in the first few weeks of this course. We will try to make the notion of feedback control more precise by looking at a particular example, the idle speed control loop in your car. After that, we'll discuss different kinds of control.

When you press on the accelerator pedal of your car, the engine produces more power. When you take your foot off the pedal, the engine does not stop[1], rather, it runs at a fixed idle speed, such as 750 RPM (revolutions per minute). Moreover, the engine maintains a constant idle speed even when the power load on the engine changes, such as when the air conditioner cycles on or off, or, assuming you have your foot on the brake, when you shift the transmission from park to neutral to drive. How is it possible for the engine to idle at a fixed speed under a variety of loads? The answer is feedback control. A sensor measures the speed of the engine and a "mechanism", called a feedback controller, automatically adjusts the power requested[2] from the engine so as to maintain a constant idle speed.

The engine system illustrates all the basic components of what we call a *feedback control system*. The key elements are: (i) A measured Quantity that is to be regulated to a desired value (e.g., engine idle speed); this is typically defined to be the *system output*; (ii) An *input* that can be varied so as to change the value of the output (e.g., throttle position); (iii) An element that determines how to adjust the input so as to drive the output to a desired value; this element is called the *controller*. The overall system is shown in Fig. 1.0.1, where the comparator computes the error between the desired output value and the current output value, and the controller seeks to drive this error to zero. Such a feedback control system is also known as a *closed-loop control system* due to the closed signal path that connects all four components (e.g., Comparator to Controller to Plant to Measurement to Comparator).

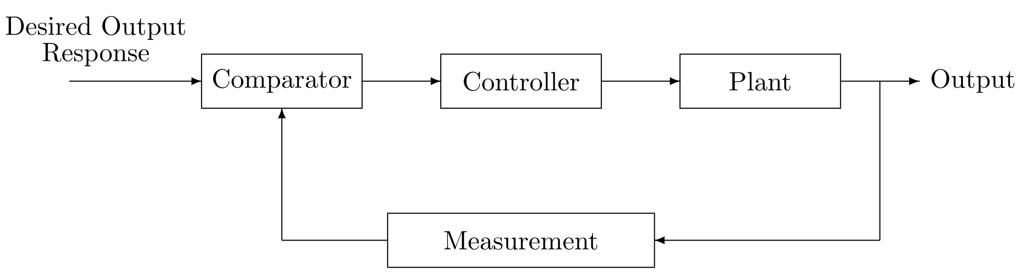

Figure 1.0.1: Closed-Loop Control System. The plant is the generic term for the system to be controlled.

[1] It may if you have a hybrid electric vehicle, or HEV.
[2] Power is varied by opening and closing the throttle, which regulates air flow and hence engine power, and/or by changing the spark timing, which regulates power as well.

In a feedback control system, information from the observed output of the system is used to modify the system's input and, consequently, alter the system output. The controller is designed to adjust the input to the plant, based on the current output signal, in order to achieve a desired output signal. Feedback control systems are widely used and can be found in almost every consumer product that involves electronics, including ovens, computer hard drives, CD players, cellphones, automobiles, etc. Of course, there are many non-electrical control systems, too. The human body contains a host of complex biologically-based feedback control systems. These systems control everything from your heart rate to your ability to breathe, swallow and walk.

Control systems do not always require feedback. Sometimes one just needs to regulate the sequence in which events occur, such as when coins inserted into a vending machine cause a selected product to be delivered and the correct change to be computed and returned to the customer. This type of control is called *sequential control* or *discrete-event control*.

Another type of non-feedback control is used in situations where measurements are hard to make, or the economic cost of adding the necessary sensor to the system is too high. In these situations, the controller computes the input to the plant on the basis of the desired output alone, by using a *mathematical model* of the system to determine the correct input. Such a control method is called[3] *open-loop control* for the obvious reason that the loop depicted in Fig. 1.0.1 is no longer present because nothing is measured! You may be surprised to learn that the icon of high-tech industries, semiconductor chip manufacturing, is largely based on open-loop manufacturing processes. For example, the amount of microwave power applied to a reactive ion etcher is typically determined ahead of time by careful experimentation to deliver a given etch rate[4] on the assumption that the machines being used are extremely repeatable from one manufacturing run to the next.

Although not all control systems utilize feedback control, many do. In an environment where conditions change over time, disturbances are present, or an accurate model of the system is not available for pre-computing the required control signal, it can be very difficult, if not impossible, to precisely regulate the system's output using an open-loop control method. In these cases, adjusting the input as a function of a measured variable is more effective. *The use of feedback control reduces the need for a very accurate system model.*

Another reason for using feedback control is *stability*. Civilian aircraft are designed to be open-loop stable, meaning that even in the absence of feedback, the aircraft is BIBO stable, so that small changes to the control surfaces produce small changes in the aircraft's pitch, for example. There are cases, however, where it would be useful if small changes in the control surfaces produce large changes in pitch. In combat situations for fighter aircraft, speed of response is often the difference between life and death. The F-16 was the first military craft[5] to be deliberately designed to be open-loop *unstable*. In fact, to achieve the desired response speeds require, it is so unstable that human reflexes are too slow to achieve stability. A sophisticated digital feedback control system makes the system BIBO stable as well as highly maneuverable. On the downside, if a key actuator or sensor fails[6], the plane tumbles and disintegrates in less than a second!

In summary, feedback control allows (1) the output of a system to be regulated to a precise value in the absence of a precise system model or when the characteristics of the system change over time, and (2) an unstable system to be stabilized.

In the present laboratory, we will consider the control of LTI systems using LTI controllers. Furthermore, we will restrict our attention to systems that have only a single input and a single output and which operate in the absence of large external disturbances. We will find that Laplace transform (i.e., s-domain) tools are ideally suited to the analysis and design of linear feedback-control systems, and that the response of the system will be determined by the location of the poles and zeros of the closed-loop system transfer function. Basic concepts will be illustrated by constructing a PD (proportional-differential) feedback controller using operational amplifiers and by comparing the measured performance of the closed-loop system against theory.

[3] It is also known as feedforward control

[4] This is key to Intel's "copy exactly" manufacturing philosophy.

[5] What was the first civilian aircraft to be designed to be open-loop unstable? The Wright brothers' 1903 aircraft that they successfully flew at Kitty Hawk, North Carolina! The pitch mode was unstable and had to stabilized by the pilot acting as the feedback controller.

[6] Redundancy is used to improve reliability and safety.

2 Preliminary Information

2.1 A Closed-Loop Feedback Model

The block diagram of a generic closed-loop feedback control system is shown in Fig. 1.0.1. In this figure, the "plant" is the system whose output we wish to control. A measurement is performed on the output by the measurement sub-system, which is then compared with the desired output, i.e., the input to the control system. The output of the comparator, which computes the difference between the desired and the actual output, is fed into a controller whose output serves as the input to the plant. If the comparator and measurement system were removed from the above diagram then one would be left with an open-loop (i.e., no feedback) controller.

2.2 Closed-Loop Transfer Function

As mentioned earlier, we will restrict our discussion to the case where all of the blocks shown in Fig. 1.0.1 are LTI systems. Thus the input/output relationship of each of these blocks can be described by a system transfer function. Furthermore, we will realize the comparator as an adder and change the gain on the measurement to -1. With these changes, the block diagram of Fig. 2.2.1 becomes:

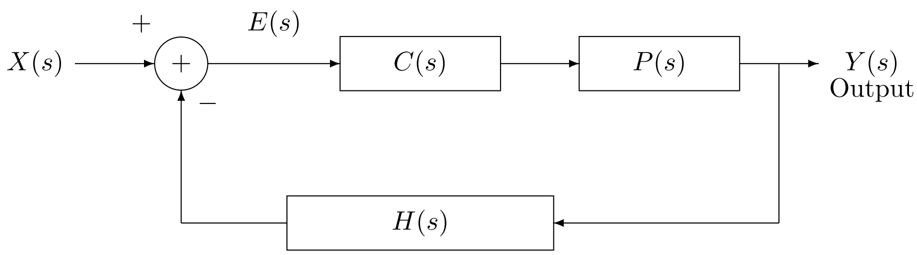

Figure 2.2.1: Closed-Loop Feedback Control System. (C: Controller, P: Plant, M: Measurement)

It is a simple matter of algebra to compute the closed-loop transfer function, $G_{cl}(s) = Y(s)/X(s)$, of the systems shown in Fig. 1.0.1 as indicated below.

$$Y(s) = E(s)C(s)P(s) \tag{2.2.1}$$
$$E(s) = X(s) - H(s)Y(s) \tag{2.2.2}$$

Combining Eqs. (2.2.1) and (2.2.2) yields

$$G_{cl}(s) = \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)} \tag{2.2.3}$$
$$\frac{E(s)}{X(s)} = \frac{1}{1 + C(s)P(s)H(s)} \tag{2.2.4}$$

2.3 Examples of Feedback Control Systems

2.3.1 DC Motor Model

Let us assume that we want to control the shaft position of a DC motor. To help design a controller for this DC motor, we mathematically model the angular position, $\theta(t)$ of the shift by the following differential equation

$$\frac{d^2\theta(t)}{dt^2} + \frac{d\theta(t)}{dt} = V(t) \tag{2.3.1}$$

where $V(t)$ is the voltage applied to the motor. Thus the plant (i.e., the motor) is a LTI system that has the following system transfer function

$$P(s) = \frac{1}{s(s+1)} \tag{2.3.2}$$

This transfer function corresponds to a second-order LTI system with no zeros, and a pair of real poles located at the origin and at -1.

2.3.2 No Controller

Suppose we want the shaft of the motor to rotate by one radian per second by using a unit step as the input, i.e., $V(t) = u(t)$. Then

$$\theta(s) = V(s)P(s) = \frac{1}{s}\frac{1}{s(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \tag{2.3.3}$$

where the right-most side of Eq. (2.3.3) is obtained by a partial fraction expansion. Consequently,

$$\theta(t) = (t - 1 + e^{-t})u(t) \tag{2.3.4}$$

which is clearly not going to achieve the desired rotation by one radian per second!

2.3.3 Controller Without Feedback (i.e., Open-Loop Control)

Shown below is the block diagram of an open-loop controller. We choose a differentiator as the controller,

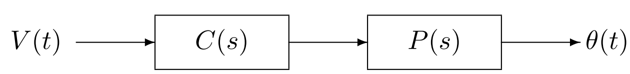

Figure 2.3.1: Open-Loop Controller

i.e.,

$$C(s) = s \tag{2.3.5}$$

of a unit-step input. Thus

$$\theta(s) = V(s)C(s)P(s) = \frac{1}{s}s\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \tag{2.3.6}$$

Therefore

$$\theta(t) = (1 - e^{-t})u(t) \tag{2.3.7}$$

Clearly, $\lim_{t\to\infty} \theta(t) = 1$, so that this controller results in no steady-state error (i.e., $e(t) = V(t) - \theta(t) = 0$ as $t \to \infty$) when the input is a unit step. The response of the DC motor to the unit step, however, is rather slow. It takes approximately 2.3 s for the shaft angle to reach 90% of its final value of 1 radian.

We might ask how well this same controller would work if the input were a ramp rather than a step. Under such circumstances

$$V(s) = \frac{1}{s^2} \tag{2.3.8}$$

$$\theta(s) = V(s)C(s)P(s) = \frac{1}{s^2}s\frac{1}{s(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \tag{2.3.9}$$

Therefore

$$\theta(t) = (t - 1 + e^{-t})u(t) \tag{2.3.10}$$

and the steady-state error is 1 radian/second. Thus the angular position of the shift deviates from the desired position by 1 radian as $t \to \infty$, and this differentiator-controller is inadequate for a ramp input.

2.3.4 Sensitivity of an Open-Loop Controller to Plant Changes

Suppose that overheating changes the transfer function of the mtoro by a small amount to become

$$P(s) = \frac{1}{(s+0.01)(s+1)} \tag{2.3.11}$$

i.e., one of the poles has moved along the real axis from the origin into the left-hand plane (LHP). The step response of the motor now becomes

$$\theta(s) = V(s)C(s)P(s) = \frac{1}{s}s\frac{1}{(s+0.01)(s+1)} = \frac{100/99}{s+0.01} - \frac{100/99}{s+1} \tag{2.3.12}$$

Therefore

$$\theta(t) = (100/99)(e^{-0.01t} - e^{-t})u(t) \tag{2.3.13}$$

and in steady state $\lim_{t\to\infty}\theta(t) = 0$, yielding a steady state error of one radian. It is clear that this open-loop controller is quite sensitive to variations in the plant transfer function.

2.3.5 Feedback (or Closed-Loop) Control

Consider now the feedback control system shown in Fig. 2.2.1 with $H(s) = 1$, $C(s) = Ks$ (i.e., a differential controller) and $P(s) = 1/[s(s+1)]$ as before. Then according to Eq. (2.2.3) the closed-loop transfer function is given by

$$G_{cl}(s) = \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1+C(s)P(s)H(s)} = \frac{KsP(s)}{1+KsP(s)} = \frac{K}{s+(K+1)} \tag{2.3.14}$$

and the step response becomes

$$\theta(s) = V(s)G_{cl}(s) = \frac{1}{s}\frac{K}{s+(K+1)} = \frac{K/(K+1)}{s} - \frac{K/(K+1)}{s+(K+1)} \tag{2.3.15}$$

or equivalently

$$\theta(t) = \frac{K}{K+1}(1-e^{-(K+1)t})u(t). \tag{2.3.16}$$

If K is sufficiently large, then the steady-state error, $\lim_{t\to\infty}(V(t)-\theta(t)) = 1/(K+1)$ will be small. Also note that the response time is much improved over that obtained by an open-loop controller. In particular, the 90% rise-time has been reduced from 2.3 s to 2.3/(K+1) s.

2.3.6 Sensitivity of the Closed-Loop Controller to Plant Changes

As before we will consider the situation where the transfer function of the motor changes by a small amount to become that given by Eq. (2.3.11). Using the same feedback controller described above, the closed-loop response becomes

$$G_{cl}(s) = \frac{Ks}{s^2+(1.01+K)s+0.01}. \tag{2.3.17}$$

Thus the step response is given by

$$\theta(s) = \frac{K}{s^2+(1.01+K)s+0.01} = \frac{K/(a_+-a_-)}{s+a_+} - \frac{K/(a_+-a_-)}{s+a_-} \tag{2.3.18}$$

where

$$a_\pm = \frac{(1.01+K)\mp\sqrt{(1.01+K)^2-0.04}}{2}. \tag{2.3.19}$$

Therefore for K large $a_+ \approx 0$ and $a_- \approx K$, yielding the step response

$$\theta(t) \approx (1-e^{-Kt})u(t) \tag{2.3.20}$$

Notice that with this closed-loop controller the step response is relatively insensitive to small changes in the plant.

2.3.7 Using Feedback to Stabilize Unstable Systems

Consider a plant that has the following transfer function

$$P(s) = \frac{1}{s-1} \tag{2.3.21}$$

This system is not bounded-input/bounded-output (BIBO) stable, since its transfer function has a pole in the right-half plane. It is easy to verify, for example, that the unit step response of this system is given by $(-1+e^t)u(t)$ and hence is not bounded. By implementing the feedback system shown in Fig. 2.2.1 with $H(s) = 1$ and $C(s) = K$, the closed-loop transfer function becomes

$$G_{cl}(s) = \frac{C(s)P(s)}{1+C(s)P(s)} = \frac{1}{s+(K-1)} \tag{2.3.22}$$

Thus for $K > 1$, the pole has been moved into the LHP and the system has become BIBO stable.

3 Pre-Lab Homework

3.1 Plant

In this laboratory experiment we have decided to construct our own plant so that we will know its characteristics (i.e., $P(s)$). It is convenient for illustrative purposes to construct this plant using an op-amp. The circuit we have chosen is shown below in Fig. 3.1.1.

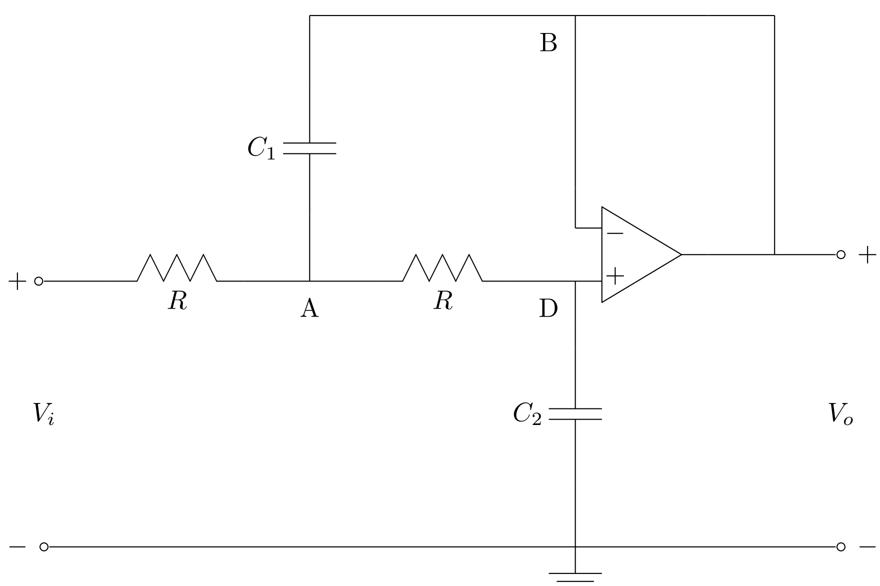

Figure 3.1.1: Op-Amp Circuit for Plant

3.1.1

Using nodal analysis along with the golden rules of op-amps, show that the transfer function of this circuit is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{a_1}{s^2+a_2 s+a_3} \tag{3.1.1}$$

and express the values of a_1,a_2,a_3 in terms of R, C_1, and C_2.

3.1.2

With $C_1 = 100\mu\text{F}$ and $R = 10\ \text{k}\Omega$ find the value of C_2 so that the plant has a pair of complex conjugated poles located at $-1 \pm j\sqrt{399}$ (i.e., $\approx -1 \pm j20$).

3.1.3

With the above values for R, C_1, and C_2 find and plot (Matlab) the unit step response of the plant.

3.2 PID Feedback Controller

If we consider the block diagram shown in Fig. 1.0.1, there are many possible choices for the controller, $C(s)$. Controllers of the following form

$$C(s) = K_p + \frac{K_I}{s} + K_D s \tag{3.2.1}$$

are known as proportional-integral-differential (PID) controllers and are widely used. The name of this controller is quite descriptive, since in the time-domain the K_p term corresponds to multiplication by a constant, whereas the terms $1/s$ and s correspond to integration and differentiation, respectively. Each of these terms can be implemented in hardware using op-amp circuits indicated below:

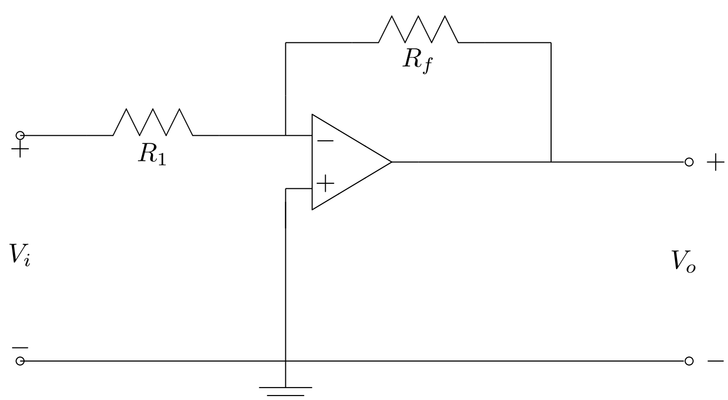

Figure 3.2.1: Proportional Controller

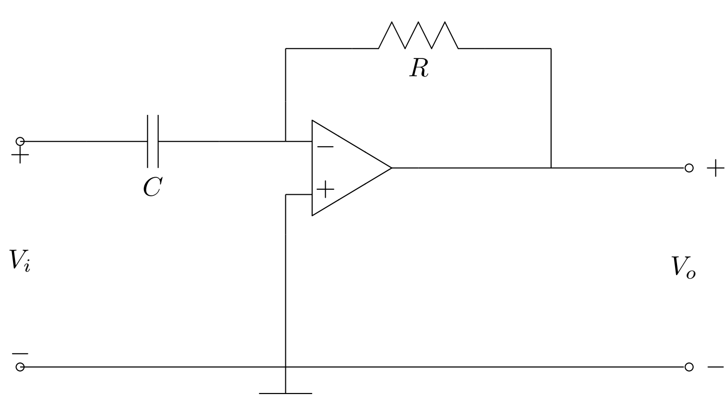

Figure 3.2.2: Differential Controller

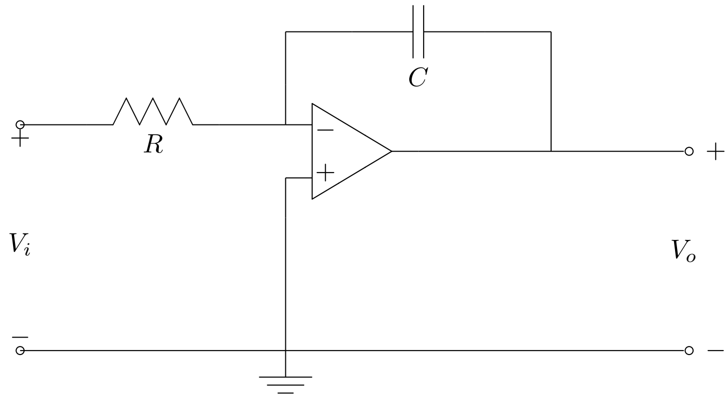

Figure 3.2.3: Integral Controller

Applying the Golden Rules for op-amps along with a simple nodal analysis, **verify** the relationships given by Eqs. (3.2.2)-(3.2.4) below.

- Proportional controller (Fig. 3.2.1):

$$\frac{V_o(s)}{V_i(s)} = K_p \text{ where } K_p = -\frac{R_f}{R_1} \tag{3.2.2}$$

- Differentiator controller (Fig. 3.2.2):

$$\frac{V_o(s)}{V_i(s)} = K_D s \text{ where } K_D = -RC \tag{3.2.3}$$

- Integral controller (Fig. 3.2.3):

$$\frac{V_o(s)}{V_i(s)} = \frac{K_I}{s} \text{ where } K_I = -\frac{1}{RC} \tag{3.2.4}$$

Combining the outputs of the three circuits above using a summing op-amp allows one to realize a general PID transfer function.

In the present lab, we will implement and analyze a PD controller to control the plant described previously in Section 3.1. Although we established analytically that a single differentiator is sufficient to control the plant, a practical issue prevents the use of an ideal differentiator (i.e., a term $K_D s$) as the only component of the controller sub-system in Fig. 2.2.1. This issue can be appreciated by considering the case where the output of the plant is initially zero and a unit step is applied as the input to the controller, i.e., $x(t) = u(t)$. The controller (i.e., $C(s)$) has a term which attempts to differentiate this step input. The derivative of a step, however, is a delta function, and the large voltage generated at the output of $C(s)$ saturates the device (an op-amp, for example) producing a response that is no longer linear.

3.3 PD Controller With 2-Degrees of Freedom

In order to avoid the problem of saturation, we will implement the control system shown below. This type of controller is known as a controller with 2-degrees of freedom.

3.3.1

Derive the closed-loop response, $G_{cl}(s) = Y(s)/X(s)$, of the controller in Fig. 3.3.1 in terms of $C_1(s)$, $C_2(s)$ and $P(s)$.

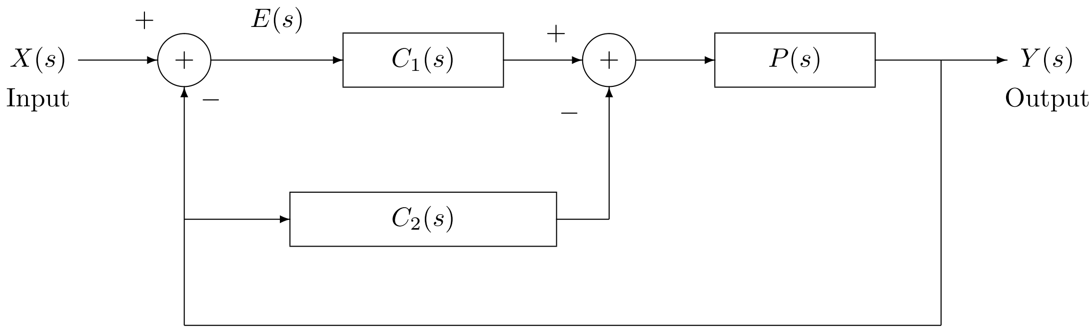

Figure 3.3.1: Two-Degree of Freedom Controller

3.3.2

Find the closed-loop response for a PD controller (i.e., $C_1(s) = K_p$, $C_2(s) = K_D s$) with 2-degrees of freedom as shown in Fig. 3.3.1. Express your answer in terms of K_p, K_D and $P(s)$.

3.3.3

Find the closed-loop response for the PD controller (i.e., $C_1(s) = K_p$, $C_2(s) = K_D s$) with 2-degrees of freedom shown in Fig. 3.3.1 assuming the plant described in Section 3.1 (using the R & C values given and computed in that section). Express your answer in terms of K_p and K_D.

3.4 Unit Step Response of PD Controller With 2-Degrees of Freedom

3.4.1

The closed-loop transfer function derived in question 3.3.3 of Section 3.3 has a pair of poles and no zeros. Find expressions for the poles, s_1 and s_2, in terms of K_P and K_D. Find the conditions on K_D and K_P such that the poles are (a) distinct and real-valued, (b) identical and real-valued and (c) a pair of complex conjugate numbers. Case (a) is referred to as an *over-damped* system, case (b) as a *critically-damped* system and case (c) as an *under-damped* system.

3.4.2

Show that the unit step response of the over-damped system can be written as

$$y(t) = (A_1 + A_2 e^{s_1 t} + A_3 e^{s_2 t})u(t). \tag{3.4.1}$$

Express A_1, A_2 and A_3 in terms of the poles s_1, s_2 and K_P. Show that $y(t)$ is a monotonically increasing function of t (i.e., $dy/dt > 0$ for all t), $0 \leq y(t) < 1$ for all t, and

$$\lim_{t\to\infty} y(t) = \frac{K_p}{K_p + 1}. \tag{3.4.2}$$

3.4.3

Show that the unit step response of the critically-damped system can be written as

$$y(t) = (A_1 + A_2 e^{s_1 t} + A_3\, \mathrm{t} e^{s_1 t})u(t). \tag{3.4.3}$$

Express A_1, A_2 and A_3 in terms of the pole s_1 and K_P. Show that $y(t)$ is a monotonically increasing function of t (i.e., $dy/dt > 0$ for all t), $0 \leq y(t) < 1$ for all t, and

$$\lim_{t\to\infty} y(t) = \frac{K_p}{K_p + 1}. \tag{3.4.4}$$

3.4.4

Show that the unit step response of the under-damped system (i.e., poles are complex and $s_2 = s_1^*$) can be written as

$$y(t) = (A_1 + B_1 e^{at}\cos(bt + \theta))u(t) \tag{3.4.5}$$

where $a = \mathrm{Re}\{s_1\}$, $b = \mathrm{Im}\{s_1\}$. Express A_1, B_1 and θ in terms of a, b and K_P. Show that $y(t)$ achieves a maximum value of

$$\frac{K_p}{K_p + 1}[1 + e^{\pi a/b}] \tag{3.4.6}$$

and

$$\lim_{t\to\infty} y(t) = \frac{K_p}{K_p + 1}. \tag{3.4.7}$$

Observe that the unit step response of a second-order, over-damped system rises smoothly and monotonically from 0 up to some final value that is less than 1. Conversely, the unit step response of the system when it is under-damped will overshoot its final value, which is less than 1, and will exhibit a damped oscillatory behavior. The oscillatory behavior will finally decay away, yielding a $\lim_{t\to\infty}(x(t) - y(t))$ steady-state value that is less than 1. A system that is critically damped lies on the boundary between overdamped and under-damped. The solution is very similar to the over-damped case, but the rise-time (i.e., the time needed to achieve some fraction of the final value) is minimized. Note that the response of this control system is determined and easily analyzed from knowledge of the pole location of the closed-loop system transfer function. The location of these poles can be controlled by varying the parameters K_P and K_D. When the poles of the closed-loop system are both real, the system does not exhibit any oscillatory behavior in its step response. Furthermore, the rise-time will be primarily be determined by the pole of smallest absolute value. If our goal is to get the fastest rise-time then we need to chose the pole locations to maximize the minimum of the absolute values of the two poles, i.e.,

$$\text{rise time} \propto \frac{1}{\max\min\{|s_1|, |s_2|\}}.$$

When the poles are complex-conjugate pairs, the step response will exhibit a decaying oscillatory behavior. The oscillation frequency will be determined by the imaginary value of the pole, while the decay rate will be determined by the real value of the pole.

In the analysis presented in this section, we have found that the steady-state error (i.e., $\lim_{t\to\infty}(x(t) - y(t))$) is given by $1/(K_p+1)$. Thus once we have decided upon an acceptable value for this error, for example 2%, K_P is fixed (i.e., ≈ 50). We are now free to vary K_D in order to obtain an acceptable response time (note with K_P fixed, K_D alone will determine the location of the poles). In control theory the location of the poles is often plotted as a function of some system parameter. Such plots are known as root locus plots. The root locus plot of our closed-loop system with K_P set to 50 is shown in Fig. 3.4.1. The vertical axis is the imaginary part of the pole, while the horizontal axis is the real part. The root locus contours are shown in this plot as K_D increases from a value of 0.5 up to 1. Note that as K_D increases from 0.5 up to 0.709, the contours in this plot consist of a pair of branches, one which is drawn as a solid line and the other dashed. The points on these branches are the complex conjugate poles. For example, when $K_D = 0.5$ there is a pole on the lower branch at $-100 - j100$ and a second, complex conjugate pole, $-100 + j100$, on the upper branch. The branches merge together at $K_D = 0.709$ where there is a single pole. This merge point corresponds to a critically damped system. As K_D continues to increase beyond 0.709, a pair of real-valued poles occur that move in opposite directions along the real axis. One pole approaches $-\infty$ while the other approaches 0.

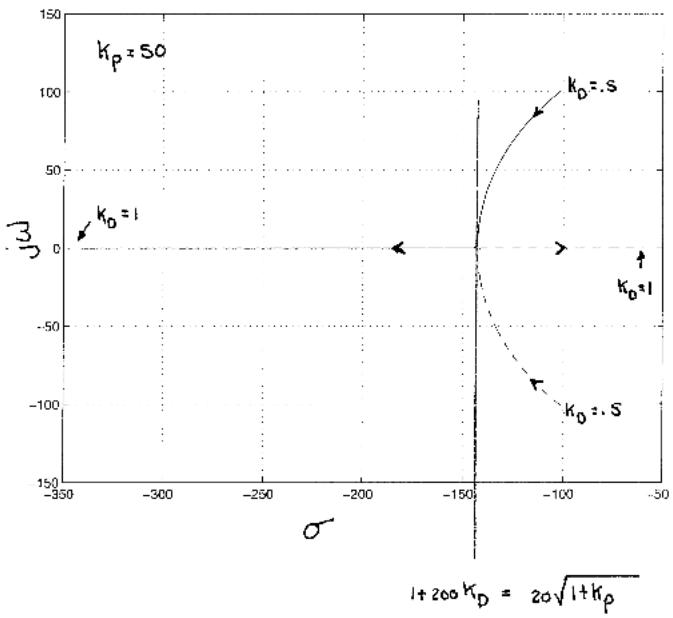

Figure 3.4.1: Root-locus plot of PD controller with 2 degrees of freedom

3.5 Op-Amp Realization of PD Controller with 2-Degrees of Freedom

A PD controller realized using the block diagram shown in Fig. 3.3.1 can be implemented in hardware as follows:

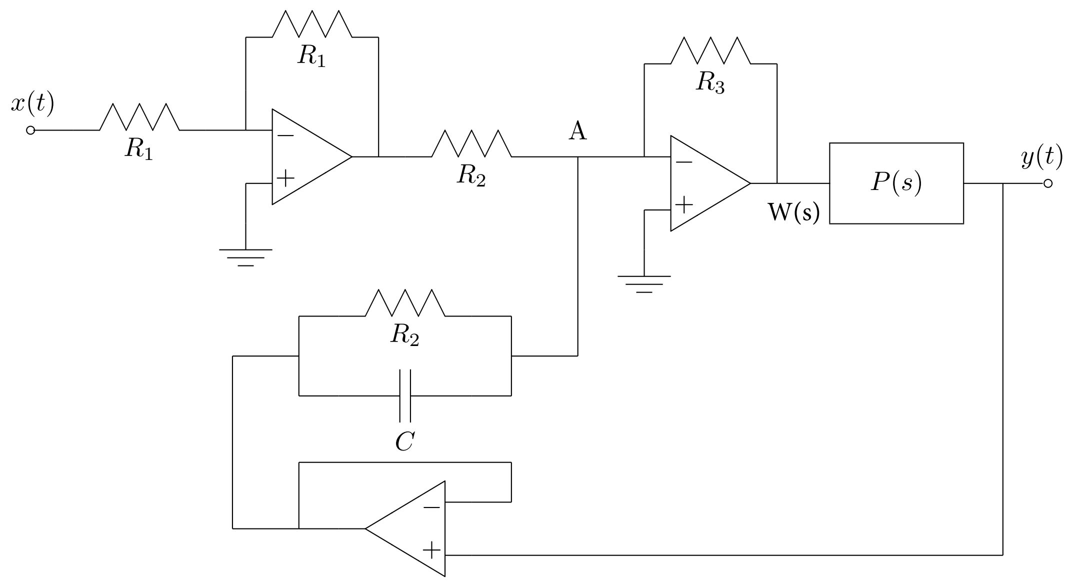

Figure 3.5.1: Op-Amp Realization of PD Controller with 2-Degrees of Freedom

3.5.1

Show that in Fig. 3.5.1 $W(s)$ can be related to $X(s)$ and $Y(s)$ by the following equation:

$$W(s) = K_p(X(s) - Y(s)) - K_D sY(s) \tag{3.5.1}$$

where K_p and K_D are constants. Express the constants K_p and K_D in terms of R_2, R_3 and C.