Homework2 Solutions

1. Let $d = t_1 - t_0$. When $x(t) = \delta(t-t_0)$, $y(t) = f(t)x(t) = f(t_0)\delta(t-t_0)$. So we have $y(t-d) = f(t_0)\delta(t-t_0-d) = f(t_0)\delta(t-t_1)$. On the other hand, Let $x_1(t) = x(t-d) = \delta(t-t_0-d) = \delta(t-t_1)$, the corresponding $y_1(t) = f(t)x_1(t) = f(t_1)\delta(t-t_1)$. Since we have $f(t_0) \neq f(t_1)$, thus $y_1(t) \neq y(t-d)$ at $t = t_1$. $\boxed{\text{The system is not time-invariant}}$.
(*A proof without using counterexamples is also OK.*)

2. (a) This system is $\boxed{\text{time-invariant}}$ as $y_1(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{s} x_1(\tau-5)d\tau\right]ds = \int_{-\infty}^{t}\left[\int_{-\infty}^{s} x(\tau-d-5)d\tau\right]ds$
$= \int_{-\infty}^{t}\left[\int_{-\infty}^{s-d} x(\tau-d-5)d(\tau-d)\right]ds = \int_{-\infty}^{t}\left[\int_{-\infty}^{s-d} x(\tau^*-5)d\tau^*\right]ds = \int_{-\infty}^{t-d}\left[\int_{-\infty}^{s} x(\tau-5)d\tau\right]ds$
$= y(t-d)$.
We can get its impulse response by substituting $x(t)$ with $\delta(t)$: $h(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{s}\delta(\tau-5)d\tau\right]ds = \int_{-\infty}^{t} u(s-5)ds = \boxed{(t\text{-}5)u(t\text{-}5)}$.

   (b) (We may try transforming it into convolution form: $y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)^2} rect((\tau-1)/4)x(\tau)d\tau$, but we cannot "extract" a $g(t)$ from it.) In fact, the system is $\boxed{\text{not TI}}$, and a counterexample can be found by letting $x(t) = \delta(t)$, and $x_1(t) = \delta(t-5)$.

   (c) This system is $\boxed{\text{TI}}$ because we could write it in the exact form of convolution:
$y(t) = \int_{-\infty}^{\infty}(\tau^2 rect(\tau/6))x(t-\tau)d\tau + \int_{-\infty}^{\infty}((t-\tau+3)^{-2}u(t+1-\tau))x(\tau)d\tau$.
The impulse response is given by $\boxed{h(t) = t^2 rect(t/6) + (t+3)^{-2}u(t+1)}$.

3. .

(a) Correct statement: If $y(t) = h(t) * x(t)$ then $y(t-3) = h(t) * x(t-3)$.
Proof: $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$ so $y(t-3) = \int_{-\infty}^{\infty} h(\tau)x(t-3-\tau)\,d\tau = h(t) * x(t-3)$.

(b) Incorrect statement: If $y(t) = h(t) * x(t)$ then $y(t-3) \overset{not!}{=} h(t-3) * x(t-3)$. Example: $h(t) = x(t) = \delta(t)$. Then $y(t) = \delta(t)$ so $y(t-3) = \delta(t-3)$. But $h(t-3) * x(t-3) = \delta(t-3) * \delta(t-3) = \delta(t-6) \neq \delta(t-3)$.
(c) Correct statement: If $y(t) = h(t) \cdot x(t)$ then $y(t-3) = h(t-3) \cdot x(t-3)$, by definition of the product of two signals. But $y(t-3) \neq h(t) \cdot x(t-3)$ again by considering $h(t) = x(t) = \delta(t)$.

4. .

(a) $h(t-\tau)$ is nonzero over $t-a < \tau < t-b$ and $x(\tau)$ is nonzero over $c < \tau < d$. The integral of their product is nonzero when $t-a > c$ and $t-b < d$, so that the intervals overlap. Thus $\boxed{a+c < t < b+d}$ is the range of values of t for which $y(t)$ is possibly nonzero.
(b) $a = 1, b = 3, c = -5, d = -1$ so $\boxed{-4 < t < +2.}$
For $t-1 > -5$ but $t-3 < -5$, $y(t) = \int_{-5}^{t-1} 1\,dt = t+4$. For $t-3 > -5$ but $t-1 < -1$, $y(t) = \int_{t-3}^{t-1} 1\,dt = 2$.
For $t-3 < -1$ but $t-1 > -1$, $y(t) = \int_{t-3}^{-1} 1\,dt = 2-t$. So $\boxed{y(t) = \begin{cases} t+4, & -4 < t < -2 \\ 2, & -2 < t < 0 \\ 2-t, & 0 < t < 2 \\ 0, & \text{otherwise.} \end{cases}}$

5. (a) When $t \geq 0$, $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau}e^{-\beta(t-\tau)}u(\tau)u(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau$;
when $t \leq 0$, from above we have $y(t) = 0$.
When $t \geq 0$, and when $\alpha = \beta$, $y(t) = \int_0^t e^{-\beta t}d\tau = te^{-\beta t}$.
When $t \geq 0$, and when $\alpha \neq \beta$, $y(t) = e^{-\beta t}\int_0^t e^{(\beta-\alpha)t} = e^{-\beta t}\frac{e^{(\beta-\alpha)t}-1}{\beta-\alpha}$.

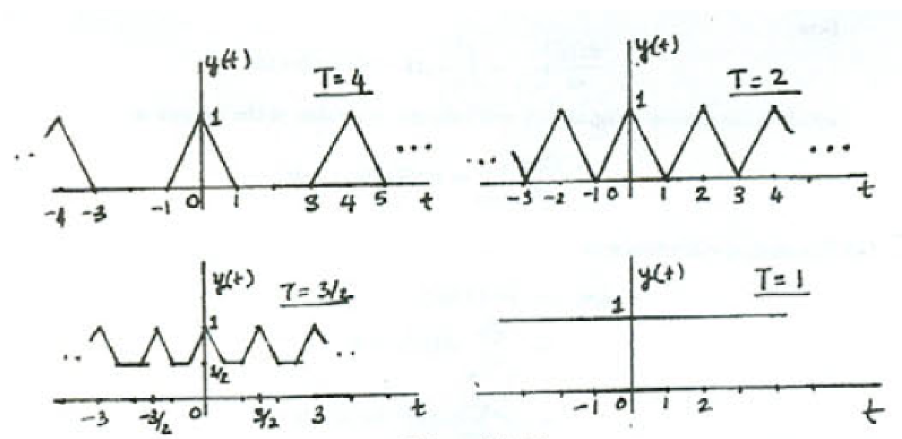
In summary,

$$y(t) = \boxed{\begin{cases} e^{-\beta t}\frac{e^{(\beta-\alpha)t}-1}{\beta-\alpha}u(t) & \alpha \neq \beta \\ te^{-\beta t}u(t) & \alpha = \beta. \end{cases}}$$

(b) $y(t) = h(t) * x(t) = (\frac{4}{3}\text{rect}(t-0.5)) * (at+b) - \frac{1}{3}(a(t-2)+b) = \int_{t-1}^{t} \frac{4}{3}(a\tau+b)d\tau - \frac{1}{3}(a(t-2)+b) = \frac{4}{3}\left[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)\right] - \frac{1}{3}(a(t-2)+b) = \boxed{\text{at + b}}$.

6. (a) $\boxed{\text{Not time-invariant}}$. A simple counterexample is $x(t) = \sin(t)$, $x_d(t) = \sin(t-1)$ (when $T \neq 1/m$, $m \in Z \setminus \{0\}$).

   (b) The sketches are shown below:



7. (a) $\int_{-\infty}^{\infty} y(t)dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right]dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)dtd\tau = \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t-\tau)dt\right]d\tau = \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t^*)dt^*\right]d\tau = \left[\int_{-\infty}^{\infty} x(t)dt\right]\left[\int_{-\infty}^{\infty} h(\tau)d\tau\right]$.

   (b) $\frac{d}{dt}y(t) = \frac{d}{dt}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau = h(t) * (\frac{d}{dt}x(t))$.
   The other part of the statement was very similar.

8. (a) Trivial, $\boxed{tu(t)}$.

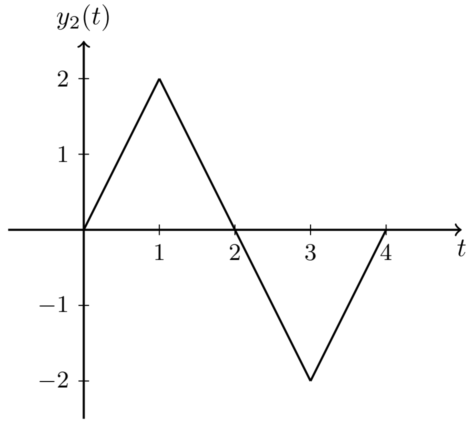
   (b) $u(t) * \int_{-\infty}^{t} u(s)ds = u(t) * (tu(t)) = \int_{-\infty}^{t} su(s)ds = \frac{t^2}{2}u(t)$.
   $u(t) * \int_{-\infty}^{t} su(s)ds = u(t) * \frac{t^2}{2}u(t) = \int_{-\infty}^{t} \frac{s^2}{2}u(s)ds = \frac{t^3}{6}u(t)$.
   From the linearity of convolution, we have $u(t) * t^2u(t) = \boxed{\frac{t^3}{3}u(t)}$

9. $h(t) = \frac{d}{dt}s(t)$. Sketch $s(t)$ and we easily get from graph that $h(t) = \boxed{3\delta(t) - \delta(t-2) - \text{rect}(\frac{t-1}{2})}$.

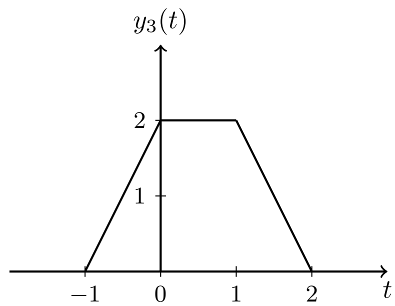
10. (a) $y(t) = x(\sin(t))$: $\boxed{\text{linear, stable, non-causal, dynamic, time-variant}}$
    When $t = -\pi$, $y(-\pi) = x(0)$, so it is not causal or memoryless.
    $y_d(t) = x_d(\sin(t)) = x(\sin(t) - t_0) \neq y(t - t_0)$, so time-variant.

    (b) $y(t) = \frac{d}{dt}\{e^{-t}x(t)\}$: $\boxed{\text{linear, non-stable, non-causal/causal, dynamic, time-variant}}$
    When considering differentiation, both causal and non-causal are OK.
    Time-varying gain, so time-variant.

11. (a) $x_2(t) = x_1(t) - x_1(t-2) \Rightarrow y_2(t) = y_1(t) - y_1(t-2)$

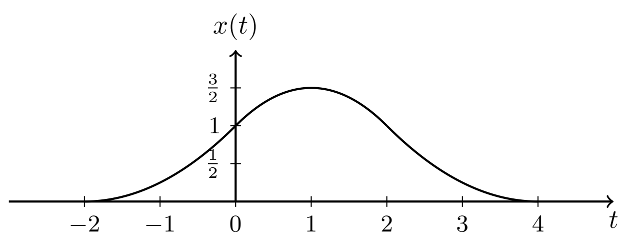


(b) $x_3(t) = x_1(t) + x_1(t+1) \Rightarrow y_3(t) = y_1(t) + y_1(t+1)$



12. The triangular pulse is defined as $\text{tri}(t) = (1 - |t|)\text{rect}(t/2)$.
Compute $x(t) = \text{tri}(t/2) * \text{rect}\left(\frac{t-1}{2}\right)$.Express your answer using braces, and carefully sketch. $x(\tau) = (1 - |\tau/2|)\text{rect}(\tau/4)$, so graphically,

$$x(t) = \begin{cases} \int_{-2}^{t}(1+\tau/2)d\tau & -2 < t < 0 \\ \int_{t-2}^{0}(1+\tau/2)d\tau + \int_{0}^{t}(1-\tau/2)d\tau & 0 < t < 2 \\ \int_{t-2}^{2}(1-\tau/2)d\tau & 2 < t < 4 \\ 0 & \text{otherwise} \end{cases} = \boxed{\begin{cases} 1+t+t^2/4 & -2 < t < 0 \\ 1+t-t^2/2 & 0 < t < 2 \\ 4-2t+t^2/4 & 2 < t < 4 \\ 0 & \text{otherwise} \end{cases}}$$



13. Find the impulse response of the following LTI systems and further determine whether they are causal, stable and static.

(a) $y(t) = \int_{-\infty}^{t}(t-\tau)e^{-(t-\tau)}x(\tau)d\tau = \int_{-\infty}^{\infty}(t-\tau)u(t-\tau)e^{-(t-\tau)}x(\tau)d\tau \Rightarrow \boxed{h(t) = te^{-t}u(t)}$
$\boxed{\text{Causal}}$: $h(t) = 0$ for $t < 0$.
$\boxed{\text{Stable}}$: $\int_{-\infty}^{\infty}|h(t)|dt = \int_{0}^{\infty}te^{-t}dt = -(t+1)e^{-t}\Big|_{0}^{\infty} = 1$.
$\boxed{\text{Dynamic}}$: $h(t) \neq 0$ for $t > 0$.

(b) $y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)}x(\tau)d\tau = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t-\tau}{2}\right) e^{-2(t-\tau)}x(\tau)d\tau \Rightarrow \boxed{h(t) = \text{rect}(t/2)e^{-2t}}$

$\boxed{\text{Non-causal}}$: $h(t) \neq 0$ for $t < 0$.

$\boxed{\text{Stable}}$: $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-1}^{1} e^{-2t}dt = \frac{e^2 - e^{-2}}{2}$.

$\boxed{\text{Dynamic}}$: $h(t) \neq 0$ for $t > 0$.

14.

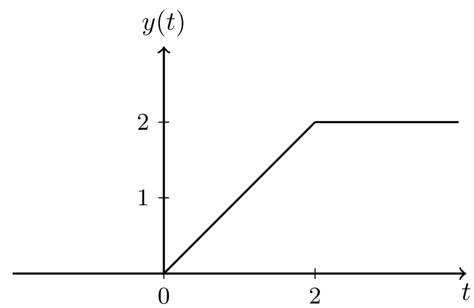
$$\frac{dx(t)}{dt} = -6e^{-3t}u(t-1) + 2e^{-3t}\delta(t-1) = -3x(t) + 2e^{-3}\delta(t-1) \rightarrow -3y(t) + e^{-2t}u(t)$$

Thus we know

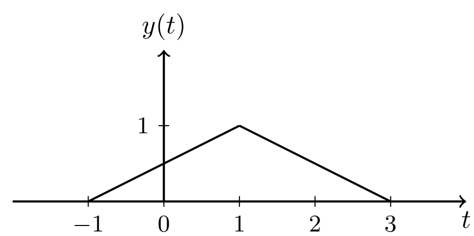
$$2e^{-3}\delta(t-1) \rightarrow e^{-2t}u(t) \quad \Rightarrow \quad \delta(t) \rightarrow \frac{1}{2}e^{-2t+1}u(t+1)$$

and it follows $\boxed{h(t) = \frac{1}{2}e^{-2t+1}u(t+1)}$.

15. (a) $\boxed{y(t) = 2y_0(t)}$

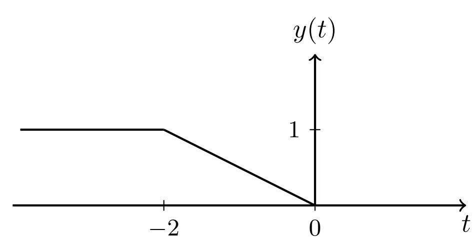


(b) $\boxed{y(t) = y_0(t+1) - y_0(t-1)}$

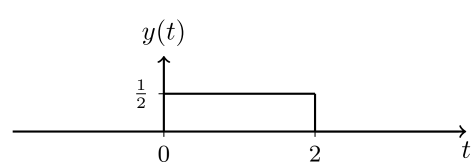


(c) $\boxed{y(t) \text{ cannot be determined.}}$

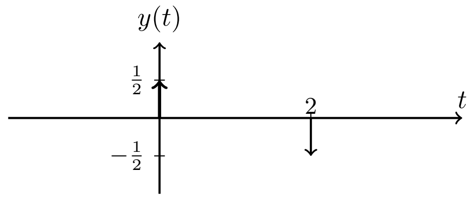
(d) $\boxed{y(t) = y_0(-t)}$



(e) $\boxed{y(t) = y_0'(t)}$



(f) $\boxed{y(t) = y_0''(t)}$



16. Plug $y(t) = Ce^{st}$ into the equation, and we get $s = -10$. Hence the homogeneous solution is

$$y_h(t) = Ce^{-10t}$$

For the particular solution, when $t > 0$

$$y_p(t) = P_0 x(t) + P_1 \frac{d}{dt} x(t) + \cdots = P_0$$

Thus

$$y(t) = y_h(t) + t_p(t) = Ce^{-10t} + P_0 \qquad \text{for } t > 0$$

Plugging into the differential equation, we get

$$-10Ce^{-10t} + 10Ce^{-10t} + 10P_0 = 2 \Rightarrow P_0 = \frac{1}{5}$$

Moreover, with the initial condition, we have

$$y(0) = C + P_0 = 1 \Rightarrow C = \frac{4}{5}$$

or

$$y(0) = C + P_0 = 0 \Rightarrow C = -\frac{1}{5}(initial\ condition = 0)$$

Thus $\boxed{y(t) = \left(\frac{4}{5}e^{-10t} + \frac{1}{5}\right) u(t)}$ or $\boxed{y(t) = \left(-\frac{1}{5}e^{-10t} + \frac{1}{5}\right) u(t)}$.