Example

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Determine the signal $x(t)$ that has the following spectrum:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \mathrm{rect}\left(\frac{\omega - k4\pi}{2\pi}\right).$$

Solution (1)

The spectrum is periodic, so it must be the spectrum of some "sampled" signal. So we write it in the form of the FT sampling property:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\frac{\omega - k4\pi}{2\pi}\right) = 2 \sum_{k=-\infty}^{\infty} F(\omega - k4\pi)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(\omega - k\omega_s)$$

where $\omega_s = 4\pi$, $T_s = 1/2$ *and*

$$F(\omega) = \frac{1}{2} \operatorname{rect}\left(\frac{\omega}{2\pi}\right)$$

Solution (1)

Taking the inverse FT:

$$F(\omega) = \frac{1}{2}\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right) \xleftrightarrow{\mathcal{F}} f(t) = \frac{1}{2}\,\mathrm{sinc}(t)$$

By the sampling property of the FT,

$$x(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} f(n/2)\delta(t - n/2)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} \frac{1}{2}\,\mathrm{sinc}(n/2)\,\delta(t - n/2).}$$

Example (1)

Example

Show that

$$\frac{1}{a}\,\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right) = \mathrm{tri}\left(\frac{t}{a}\right)$$

where $a > 0$.

Hint: You may use the fact that $\mathrm{rect}(t) * \mathrm{rect}(t) = \mathrm{tri}(t)$

Solution (1)

Use the convolution definition and variable exchange

$$\mathrm{tri}(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$$

$$\mathrm{tri}(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\,\mathrm{rect}(t-\tau)\; d\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\,\mathrm{rect}\left(\frac{t}{a}-\tau\right)\; d\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \int_{-\infty}^{\infty} \mathrm{rect}\left(a\frac{\tau}{a}\right)\mathrm{rect}\left(\frac{t}{a}-a\frac{\tau}{a}\right)\; d\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{\tau'}{a}\right)\mathrm{rect}\left(\frac{t}{a}-\frac{\tau'}{a}\right)\frac{1}{a}\,d\tau', \quad \tau' = a\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \frac{1}{a}\,\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right)$$

Solution (2)

Use the convolution definition and integration

$$y(t) = \mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right)$$

- *For $t < -a$, the integral is 0.*
- *For* $-a < t < 0$
$$\int_{-a/2}^{t+a/2} d\tau = a + t$$
- *For* $0 < t < a$
$$\int_{t-a/2}^{a/2} d\tau = a - t$$
- *For $t > a$, the integral is 0.*
- *Combining*

$$y(t) = \begin{cases} a + t, & -a < t < 0 \\ a - t, & 0 < t < a \\ 0, & \textit{otherwise} \end{cases} = a\,\mathrm{tri}(t/a)\,.$$

Solution (3)

Show that the LHS and RHS have the same Fourier transform.

$$\mathrm{tri}(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right).$$

Using FT time transform property, we have

$$\mathrm{tri}\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} a\,\mathrm{sinc}^2\left(a\frac{\omega}{2\pi}\right).$$

Using FT convolution property, LHS becomes

$$\frac{1}{a}\,\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} a\,\mathrm{sinc}^2\left(a\frac{\omega}{2\pi}\right).$$

Therefore

$$\frac{1}{a}\,\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right) = \mathrm{tri}\left(\frac{t}{a}\right)$$