

## Solution for Homework 1

1. [6!]

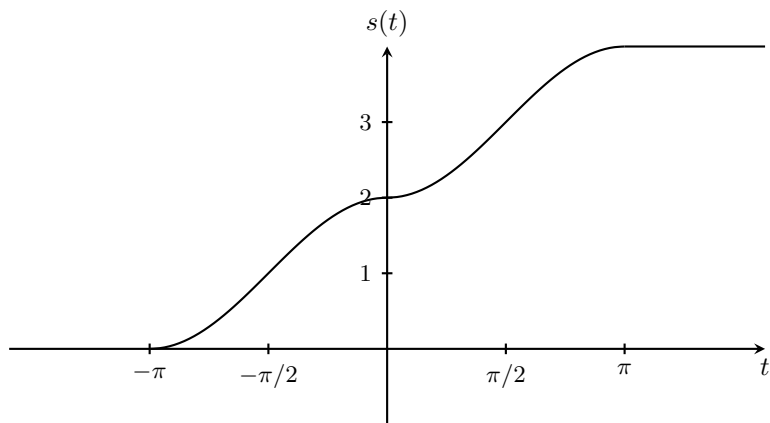
(a)  $x(t) = |\sin(t)|$

(b)  $x(t)$  is a power signal.

(c)

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

$$= \begin{cases} 0 & t < -\pi, \\ \cos(t) + 1 & -\pi \leq t \leq 0, \\ -\cos(t) + 3 & 0 < t \leq \pi, \\ 4 & \pi < t. \end{cases}$$



2. [12!]  $E_\infty$  and  $P_\infty$  denote the average power and energy, respectively.

(a)  $E_\infty = \int_0^\infty e^{-2t} dt = \frac{1}{2}$ .  $P_\infty = 0$ , because  $E_\infty < \infty$ .

(b)  $x_2(t) = e^{j(2t+\pi/4)}$ ,  $|x_2(t)| = 1$ . Therefore,

$$E_\infty = \int_{-\infty}^\infty |x_2(t)|^2 dt = \int_{-\infty}^\infty dt = \infty.$$

$$P_\infty = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x_2(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dt = \lim_{T \rightarrow \infty} 1 = 1.$$

(c)  $x_3(t) = \cos(t)$ . Therefore,

$$E_\infty = \int_{-\infty}^\infty |x_3(t)|^2 dt = \int_{-\infty}^\infty \cos^2(t) dt = \infty.$$

$$P_\infty = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x_3(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos^2(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left( \frac{1 + \cos(2t)}{2} \right) dt = \frac{1}{2}.$$

(d)  $x_1[n] = (\frac{1}{2})^n u[n]$ ,  $|x_1[n]|^2 = (\frac{1}{4})^n u[n]$ . Therefore,

$$E_\infty = \sum_{n=-\infty}^\infty |x_1[n]|^2 = \sum_{n=0}^\infty \left( \frac{1}{4} \right)^n = \frac{4}{3}.$$

$P_\infty = 0$ , because  $E_\infty < \infty$ .

(e)  $x_2[n] = e^{j(\pi/2n+\pi/8)}$ ,  $|x_2[n]|^2 = 1$ . Therefore,

$$E_\infty = \sum_{n=-\infty}^\infty |x_2[n]|^2 = \sum_{n=-\infty}^\infty 1 = \infty.$$

$$P_\infty = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x_2[n]|^2 = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N 1 = 1.$$

(f)  $x_3[n] = \cos(\frac{\pi}{4}n)$ . Therefore,

$$E_\infty = \sum_{n=-\infty}^\infty |x_3[n]|^2 = \sum_{n=-\infty}^\infty \cos^2\left(\frac{\pi}{4}n\right) = \infty.$$

$$P_\infty = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x_3[n]|^2 = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \cos^2\left(\frac{\pi}{4}n\right) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \left( \frac{1 + \cos(\frac{\pi}{2}n)}{2} \right) = \frac{1}{2}.$$

3. [3!]

$$\begin{aligned}
 A &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_0^T e^{-t} dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} (1 - e^{-T}) \\
 &= \boxed{0}
 \end{aligned}$$

$$\begin{aligned}
 P &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_0^T e^{-2t} dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{4T} (1 - e^{-2T}) \\
 &= \boxed{0}
 \end{aligned}$$

$$\begin{aligned}
 E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\
 &= \int_0^{\infty} e^{-2t} dt \\
 &= \boxed{\frac{1}{2}}
 \end{aligned}$$

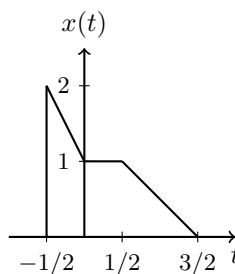
4. [6!]

(a)

$$x(t) = (t+2)\text{rect}\left(\frac{t+1}{2}\right) + 2\text{rect}(t-1/2) + 2t\text{rect}(t-3/2)$$

Or,

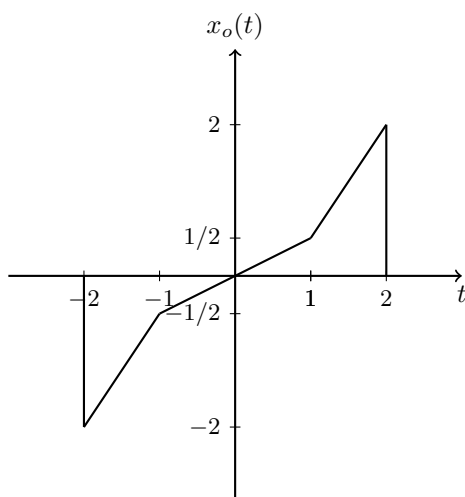
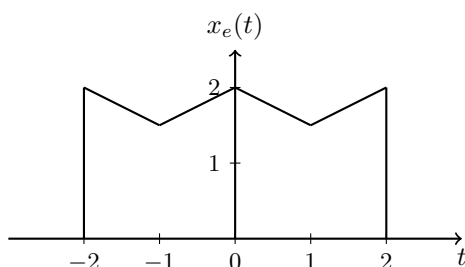
$$x(t) = \begin{cases} t+2 & -2 < t < 0, \\ 2 & 0 \leq t \leq 1, \\ 2t & 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The graph of  $s(t)$  is as below,

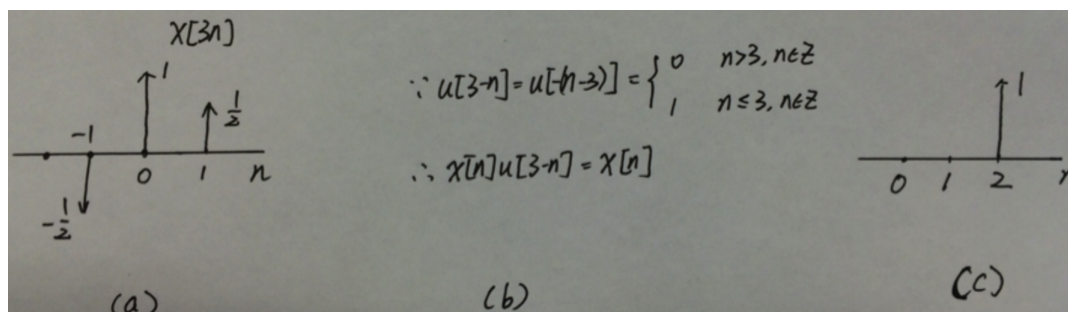
(c)

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \begin{cases} -t/2 + 1 & -2 < t \leq -1, \\ t/2 + 2 & -1 < t \leq 0, \\ -t/2 + 2 & 0 < t \leq 1, \\ t/2 + 1 & 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} = \begin{cases} 3t/2 + 1 & -2 < t \leq -1, \\ t/2 & -1 < t \leq 1, \\ 3t/2 - 1 & 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}$$



5. [6!] Solutions are as follows:



6. [9!]

- (a) If  $T_1/T_2$  is rational, then there exists some integers  $n_1$  and  $n_2$  such that  $T_1/T_2 = n_2/n_1$ . Let  $T = n_1T_1 = n_2T_2$ . Then

$$x(t+T) = x_1(t+T) + x_2(t+T) = x_1(t+n_1T_1) + x_2(t+n_2T_2) = x_1(t) + x_2(t) = x(t)$$

Thus  $x(t)$  is periodic with period  $T$ .

- (b) Similarly, if  $T_1/T_2$  is rational, then there exists some integers  $n_1$  and  $n_2$  such that  $T_1/T_2 = n_2/n_1$ . Let  $T = n_1T_1 = n_2T_2$ . Then

$$x(t+T) = x_1(t+T)x_2(t+T) = x_1(t+n_1T_1)x_2(t+n_2T_2) = x_1(t)x_2(t) = x(t)$$

Thus  $x(t)$  is periodic with period  $T$ . Moreover, if we take  $n_1$  and  $n_2$  to be co-prime, then  $T$  is the least common multiple of  $T_1$  and  $T_2$ .

- (c) i. The periods of  $\sin(\pi t/3)$ ,  $\cos(\pi t/4)$ ,  $\sin(\pi t/5)$ , and  $\sin(\pi t/2)$  are 6, 8, 10, and 4, respectively. Then least common multiple is 120, so one period of  $x(t)$  is 120.  
 ii. The ratio of two periods is not rational, so the sum of the two signals is not periodic.

7. [4!] The solution is as follows:

**We first decompose  $x(t)$  and  $y(t)$  into sums of exponentials. Thus,**

$$x(t) = \frac{1}{2} e^{j(2\pi t/3)} + \frac{1}{2} e^{-j(2\pi t/3)} + \frac{e^{j(16\pi t/3)}}{j} - \frac{e^{-j(16\pi t/3)}}{j},$$

$$y(t) = \frac{e^{j\pi t}}{2j} - \frac{e^{-j\pi t}}{2j}$$

**Multiplying  $x(t)$  and  $y(t)$ , we get**

$$z(t) = \frac{1}{4j} e^{j(5\pi/3)t} - \frac{1}{4j} e^{-j(\pi/3)t} + \frac{1}{4j} e^{j(\pi/3)t} - \frac{1}{4j} e^{-j(5\pi/3)t}$$

$$- \frac{1}{2} e^{j(19\pi/3)t} + \frac{1}{2} e^{j(13\pi/3)t} + \frac{1}{2} e^{-j(13\pi/3)t} - \frac{1}{2} e^{-j(19\pi/3)t}$$

**We see that all complex exponentials are powers of  $e^{j(\pi/3)t}$ . Thus, the fundamental period is  $2\pi/(\pi/3) = 6$  s.**

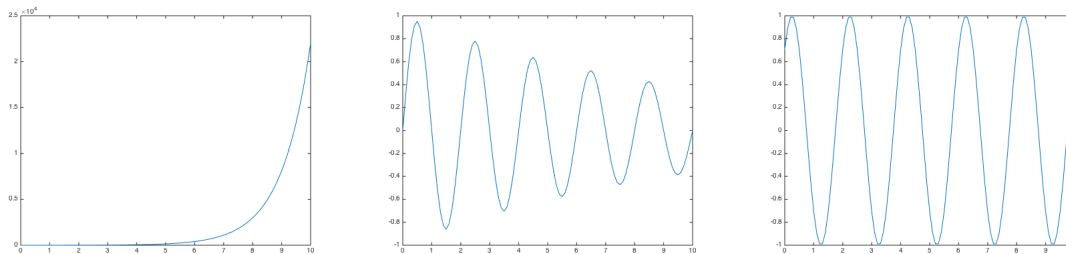
8. [3!] Matlab

- (a) `t = 0:0.1:10;`  
`y = exp(t);`  
`plot(t, y)`  
 (b) `t = 0:0.1:10;`  
`y = exp(-0.1.*t).*sin(pi.*t);`  
`plot(t, y)`  
 (c) `t = 0:0.1:10;`  
`y = sin(pi.*t+pi/4);`  
`plot(t, y)`

9. [4!]

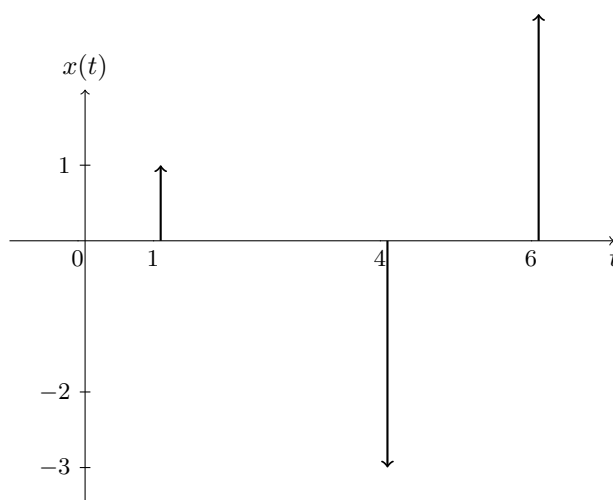
- (a)

$$x(t) = \boxed{u(t-1) - 3u(t-4)u(6-t)}$$

Figure 1:  $y(t) = e^t$ ,  $y(t) = e^{-0.1t} \sin(\pi t)$ , and  $y(t) = \sin(\pi t + \frac{\pi}{4})$ .

(b)

$$\frac{dx(t)}{dt} = \boxed{\delta(t-1) - 3\delta(t-4) + 3\delta(t-6)}$$



10. [10!]

(a) The system is linear and stable.

$$y_1(t) = x_1(t-2) + x_1(2-t)$$

$$y_2(t) = x_2(t-2) + x_2(2-t)$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = x(t-2) + x(2-t)$$

$$y(t) = \alpha x_1(t-2) + \beta x_2(t-2) + \alpha x_1(2-t) + \beta x_2(2-t)$$

$$y(t) = \alpha(x_1(t-2) + x_1(2-t)) + \beta(x_2(t-2) + x_2(2-t)) = \alpha y_1(t) + \beta y_2(t)$$

It is obviously not Memoryless or Causal.It is not Time-invariant,

$$y_d(t) = x(t-2-d) + x(2-t-d) = x((t-d)-2) + x(2-(t+d)) \neq y(t-d)$$

(b) The system is Linear, Stable.It is not time-invariant because,

$$y_d(t) = x\left(\frac{t}{3} - d\right) = x\left(\frac{t-3d}{3}\right) = y(t-3d)$$

it is not Memoryless or Causal, consider  $t = \pm 1$ , when  $t = -1$ , the output depend on future input and when  $t = 1$ , the output depend on past input.

- (c) The system is Memoryless thus Causal. It is also Stable because the absolute value of the output of  $\cos$  function is bounded at 1.

It is Time invariant,

$$y_d(t) = \cos(x(t-d)) = y(t-d)$$

It is not linear,

$$y_1(t) = \cos(x_1(t))$$

$$y_2(t) = \cos(x_2(t))$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = \cos(x(t)) = \cos(\alpha x_1(t))\cos(\beta x_2(t)) - \sin(\alpha x_1(t))\sin(\beta x_2(t)) \neq \alpha y_1(t) + \beta y_2(t)$$

- (d) it is not Memoryless, Causal, Stable or Time invariant. But it is Linear. It is not Time invariant because,

$$y_d(t) = \int_{-\infty}^{t/2} x(\tau-d)d\tau = \int_{-\infty}^{t/2-d} x(\tau-d)d(\tau-d) = \int_{-\infty}^{(t-2d)/2} x(s)ds = y(t-2d)$$

- (e) It is not memoryless by the definition of derivative,

$$\frac{dx(t)}{dt} = \frac{x(t) - x(t-\Delta t)}{\Delta t}$$

It is Time invariant and Linear.

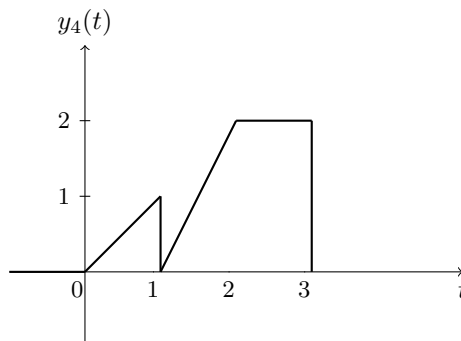
For causality, both yes and no are OK. It is causal practically, but if you consider the mathematical requirement that left and right derivatives must be defined and match for  $x(t)$  to be differentiable,

it is not causal

It is not stable.

11. [8!]

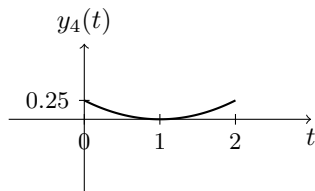
- (a) No. Causal system's output depends only on values of the input at present and in the past. Look at the second pair, during the period (0, 1), if the system is causal, the output during this period should remain 0. Because the input left to 1 are all the same(all 0).
- (b) No. Look at the first pair and the third pair.  $x_2(t) = x_1(t-1)$ , but  $y_2(t) \neq y_1(t-1)$ , so it is not time invariant.
- (c) No. We already proved that the system is not causal in (a), so it is not memoryless, either.
- (d) Use the linearity of the system, cascade the input  $x_1(t)$  and  $x_3(t)$  to form  $x_4(t)$ . Then we can get the output  $y_4(t)$  based on  $y_1(t), y_3(t)$ .



12. [3!] Let  $y(t) = x_1(t)x_2(t)$  where  $x_1(-t) = -x_1(t)$  and  $x_2(-t) = -x_2(t)$ . So  $y(-t) = x_1(-t)x_2(-t) = (-x_1(t))(-x_2(t)) = x_1(t)x_2(t) = y(t)$ .  $y(t)$  is even.

13. [6!]

(a) The sketch is shown below.



- (b) By the scaling property of  $\delta(t)$ ,  $x(t) = \delta(t - \frac{1}{2}) + \delta(t - 2) - \frac{1}{3}\delta(t - 4/3)$ .  $s(1/2) = \frac{1}{16}$ ,  $s(2) = 0$ ,  $s(4/3) = \frac{1}{36}$ . By Shifting property,  $\int_{-\infty}^{\infty} s(t)x(t)dt = s(1/2) + s(2) - \frac{1}{3}s(4/3) = \boxed{\frac{23}{432}}$
14. [2!] Assumption: If  $x(t) = 0$  for  $t < t_0$ , then  $y(t) = 0$  for  $t < t_0$ . To prove that: The system is causal. Let us consider an arbitrary signal  $x_1(t)$ . Then, let us consider another signal  $x_2(t)$  which is the same as  $x_1(t)$  for  $t < t_0$ . But for  $t > t_0$ ,  $x_2(t) \neq x_1(t)$ . Since the system is linear,

$$x_1(t) - x_2(t) \rightarrow y_1(t) - y_2(t).$$

Since  $x_1(t) - x_2(t) = 0$  for  $t < t_0$ , by our assumption  $y_1(t) - y_2(t) = 0$  for  $t < t_0$ . This implies that  $y_1(t) = y_2(t)$  for  $t < t_0$ . In other words, the output is not affected by input values for  $t \geq t_0$ . Therefore, the system is causal.

Assumption: The system is causal. To prove that: If  $x(t) = 0$  for  $t < t_0$ , then  $y(t) = 0$  for  $t < t_0$ .

Let us assume that the signal  $x(t) = 0$  for  $t < t_0$ . Then we may express  $x(t)$  as  $x(t) = x_1(t) - x_2(t)$ , where  $x_1(t) = x_2(t)$  for  $t < t_0$ . Since the system is linear, the output to  $x(t)$  will be  $y(t) = y_1(t) - y_2(t)$ . Now, since the system is causal,  $x_1(t) = x_2(t)$  for  $t < t_0$  implies that  $y_1(t) = y_2(t)$  for  $t < t_0$ . Therefore,  $y(t) = 0$  for  $t < t_0$ .

15. [6!]

- (a) Yes
- (b) No
- (c) No
- (d) Yes
- (e) Yes

16. [8!]

- (a) True. if  $x[n] = x[n + N]$ , then  $y_1[n] = y_1[n + N_0]$ . ie: periodic with period  $N_0 = N/2$  if  $N$  is even, and periodic with period  $N_0 = N$  if  $N$  is odd.
- (b) False.  $y_1[n]$  is periodic doesn't imply  $x[n]$  is periodic. Say  $x[n] = g[n] + h[n]$  where,

$$g[n] = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$h[n] = \begin{cases} 0, & n \text{ even} \\ (1/2)^n, & n \text{ odd} \end{cases}$$

Then  $y_1[n] = x[2n]$  is periodic but  $x[n]$  is clearly not periodic.

- (c) True.  $x[n] = x[n + N]$ ;  $y_2[n + N_0] = y_2[n]$  where  $N_0 = 2N$



(d) True.  $y_2[n + N] = y_2[n]$ ;  $x[n + N_0] = x[n]$  where  $N_0 = N/2$

17. [4!] Time-shift transformation will not change the energy or power of a signal.

(a)

$$\begin{aligned} E[x(-at + b)] &= E[x(at)] = \int_{-\infty}^{\infty} |x(at)|^2 dt \\ &= \int_{-\infty}^{\infty} |x(u)|^2 \frac{1}{a} du \\ &= \int_{-\infty}^{\infty} |x(u)|^2 \frac{1}{a} du \\ &= \boxed{\frac{1}{a} E_x} \end{aligned}$$

(b)

$$\begin{aligned} P[x(-at + b)] &= P[x(at)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(at)|^2 dt \\ &= \lim_{aT \rightarrow \infty} \frac{1}{2T} \int_{-aT}^{aT} |x(u)|^2 \frac{1}{a} du \\ &= \lim_{T' \rightarrow \infty} \frac{a}{2T'} \int_{-T'}^{T'} |x(u)|^2 \frac{1}{a} du \\ &= \boxed{P_x} \end{aligned}$$