Homework 4 Solution

Problems:

1. (a) $x(t) = 2\text{rect}\left(\frac{t-2}{4}\right)$ so $\boxed{X(\omega) = 8e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)}$ by time scaling and shifting property.

   (b) $x(t) = e^{-3t}\text{rect}\left(\frac{t-2}{4}\right) = e^{-3t}u(t) - e^{-3(t-4)}u(t-4)e^{-12}$ so $\boxed{X(\omega) = \frac{1}{3+j\omega}\left[1 - e^{-12}e^{-4j\omega}\right]}$

   (c) $x(t) = t\text{rect}\left(\frac{t-2}{4}\right) = ty(t)$ where from part (a): $Y(\omega) = 4e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)$.

   Thus $X(\omega) = \frac{d}{d\omega}jY(\omega) = \frac{d}{d\omega}j2e^{-2j\omega}\frac{\sin(2\omega)}{\omega} = \boxed{\frac{e^{-4j\omega}(1+4j\omega)-1}{\omega^2}}$

   (d) $x(t) = \cos(4\pi t)y(t)$ where from part (a): $Y(\omega) = 4e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)$.

   Hence $X(\omega) = \frac{1}{2}(Y(\omega - 4\pi) + Y(\omega + 4\pi)) = \boxed{2e^{-2jw}\left(\text{sinc}\left(\frac{2\omega - 8\pi}{\pi}\right) + (\text{sinc}\left(\frac{2\omega + 8\pi}{\pi}\right)\right)}$

2. From the FT table, $\frac{1}{\pi}\text{rect}\left(\frac{t}{\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\frac{\omega}{2}\right)$ and $\frac{1}{\pi}\text{tri}\left(\frac{t}{\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} \text{sinc}^2\left(\frac{\omega}{2}\right)$ Hence $F(\omega) = \text{sinc}^3(\omega/2) \stackrel{\mathcal{F}}{\longleftrightarrow}$ $\frac{1}{\pi}\text{rect}\left(\frac{t}{\pi}\right) * \frac{1}{\pi}\text{tri}\left(\frac{t}{\pi}\right)$. By using graphical convolution,

$$\text{rect}\left(\frac{t}{\pi}\right) * \text{tri}\left(\frac{t}{\pi}\right) = \begin{cases} \frac{1}{2\pi}\left(\frac{3}{2}\pi - t\right)^2 & \frac{1}{2}\pi < t < \frac{3}{2}\pi \\ \frac{1}{2\pi}\left(\frac{3}{2}\pi + t\right)^2 & -\frac{3}{2}\pi < t < -\frac{1}{2}\pi \\ \frac{3}{4}\pi - \frac{t^2}{\pi} & -\frac{1}{2}\pi < t < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}$$

and it follows that

$$\boxed{f(t) = \begin{cases} \frac{1}{2\pi^3}\left(\frac{3}{2}\pi - |t|\right)^2 & \frac{1}{2}\pi < |t| < \frac{3}{2}\pi \\ \frac{3}{4\pi} - \frac{t^2}{\pi^3} & -\frac{1}{2}\pi < t < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}}$$

The matlab code is given by:

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t = linspace(-2, 2, 301);
rect = inline('abs(t) ¡= 1/2');
f = (3/4 − t.^2). ∗ rect(t) + (t.^2/2 − 3 ∗ abs(t)/2 + 9/8). ∗ rect(abs(t) − 1);
plot(t, f)
xlabel('t'), ylabel('f(t)'), title('HW 7:2')
print('h0702', '-deps')
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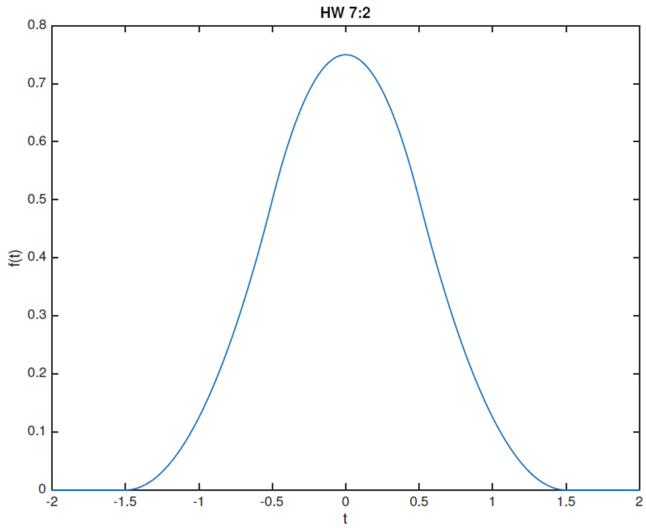

3. According to the FT table,
$$e^{-t^2/4} \overset{\mathcal{F}}{\longleftrightarrow} 2\sqrt{\pi}e^{-\omega^2} = F_1(\omega)$$
and also
$$(-jt)^2 e^{-t^2/4} \overset{\mathcal{F}}{\longleftrightarrow} \frac{d^2}{d\omega^2}F_1(\omega) = -4\sqrt{\pi}e^{-\omega^2}(1-2\omega^2)$$
Thus $F(\omega) = -\dfrac{d^2}{d\omega^2}F_1(\omega) = \boxed{4\sqrt{\pi}e^{-\omega^2}(1-2\omega^2)}$

4. If $f(t)$ is odd, $f(t) = -f(-t)$.
$$\left.\begin{array}{r} f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega) \\ f(-t) \overset{\mathcal{F}}{\longleftrightarrow} F(-\omega) \end{array}\right\} \Rightarrow F(\omega) = -F(-\omega) \Rightarrow F(\omega) \text{ is odd.}$$
If $f(t)$ is real, $f(t) = f^*(t)$.
$$\left.\begin{array}{r} f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega) \\ f^*(t) \overset{\mathcal{F}}{\longleftrightarrow} F^*(-\omega) \end{array}\right\} \Rightarrow F(\omega) = -F(-\omega) = F^*(-\omega) \Rightarrow F(\omega) = -F^*(\omega) \Rightarrow F(\omega) \text{ is purely imaginary.}$$

5.
$$f_e(t) = \frac{1}{2}[f(t)+f(-t)] \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega)+F(-\omega)]$$
$$f_o(t) = \frac{1}{2}[f(t)-f(-t)] \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega)-F(-\omega)]$$
Since $f(t)$ is real, $F(-\omega) = F^*(-\omega)$, and it follows that
$$f_e(t) \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega)+F^*(\omega)] = \text{real}\{F(\omega)\}$$
$$f_o(t) \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega)-F^*(\omega)] = j\text{imag}\{F(\omega)\}$$

6.
$$x(t) = t\text{sinc}^2(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = j\frac{d}{d\omega}\text{tri}\left(\frac{\omega}{2\pi}\right) = \pm j\frac{1}{2\pi}\text{rect}\left(\frac{\omega}{4\pi}\right)$$

So $E = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \dfrac{1}{2\pi}\int_{-2\pi}^{2\pi}\dfrac{1}{4\pi^2}d\omega = \boxed{\dfrac{1}{2\pi^2}}$

7. $f(t) = e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega) = \dfrac{1}{j\omega+1}$, so the fraction in the band [-7, 7] is

$$\frac{\int_{-7}^{7}|F(\omega)|^2 d\omega}{\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega} = \frac{\int_{-7}^{7}\frac{1}{1+\omega^2}d\omega}{\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}d\omega} = \frac{\tan^{-1}\omega\big|_{-7}^{7}}{\tan^{-1}\omega\big|_{-\infty}^{\infty}} = \frac{2\tan^{-1}7}{\pi} = \boxed{90.97\%}$$

8. (a)
$$H(s) = \frac{s^2+s+1}{(s^2+6s+25)(s^2+2)} = \frac{s^2+s+1}{(s+2)(s+3-4j)(s+3+4j)} = \frac{r_1}{s+2} + \frac{r_2}{s+3-4j} + \frac{r_2^*}{s+3+4j}$$

with $r_1 = \left.\dfrac{s^2+s+1}{s^2+6s+25}\right|_{s=-2} = \dfrac{3}{17}$ and $r_2 = \left.\dfrac{s^2+s+1}{(s+3+4j)(s+2)}\right|_{s=-3+4j} = \dfrac{7}{17} + \dfrac{71}{136}j = 0.665e^{j0.903}$.

Thus
$$H(j\omega) = \frac{3/17}{j\omega+2} + \frac{0.665e^{j0.903}}{j\omega+3-4j} + \frac{0.665e^{-j0.903}}{j\omega+3+4j}$$

Taking the inverse FT yields

$$\boxed{h(t) = \frac{3}{17}e^{-2t}u(t) + 1.33e^{-3t}\cos(4t+0.903)u(t)}$$

(b) Expanding the denominator polynomial:
$$\frac{Y(\omega)}{X(\omega)} = \frac{(j\omega)^2+j\omega+1}{(j\omega)^3+8(j\omega)^2+37j\omega+50}$$

so by cross multiplying

$$\boxed{50y(t) + 37\frac{d}{dt}y(t) + 8\frac{d^2}{dt^2}y(t) + \frac{d^3}{dt^3}y(t) = x(t) + \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)}$$

9. (a)
$$c_k = \frac{1}{4}\int_0^2 \sin(\pi t)e^{-jkt\pi/2} = \frac{(-1)^k-1}{4\pi(k^2/4-1)}$$

$$c_0 = 0$$

$$c_2 = \frac{1}{4j}, c_{-2} = \frac{-1}{4j}$$

$$\boxed{x(t) = \frac{1}{2}\sin(\pi t) + \sum_{k=1,k\neq 2}^{\infty} 2\cos(\frac{k\pi t}{2})\frac{(-1)^k-1}{4\pi(k^2/4-1)}}$$

(b)

$$c_k = \frac{1}{2}\int_{-1}^{1} te^{-jk\pi t}dt = \frac{j(-1)^k}{k\pi}, k \neq 0$$

$$c_0 = 0$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=1}^{\infty} \frac{j(-1)^k}{k\pi}e^{jk\pi t} + \frac{j(-1)^k}{-k\pi}e^{-jk\pi t} = \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\pi} 2\sin(k\pi t)}$$

10. (a) Method 1. The generator signal(the time-limited signal when n=0)is

$$g(t) = x(t)\text{rect}(t/6) = 2\delta(t) - \delta(t-2) - \delta(t+2)$$

so its FT is

$$G(\omega) = 2 - e^{j2\omega} - e^{-j2\omega} = 2(1 - \cos 2\omega)$$

Thus the FT of the period signal $x(t)$ is

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T_0} G(\omega)|_{\omega=k\omega_0} 2\pi\delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \omega_0 G(k\omega_0)\delta(\omega - k\omega_0)$$

$$\boxed{= \sum_{k=-\infty}^{\infty} (\pi/3)2(1 - \cos 2k\pi/3)\delta(\omega - k\pi/3)}$$

$$= \sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\pi/3)$$

where $a_0 = 0, a_{\pm1} = \pi, a_{\pm2} = \pi, a_{\pm3} = 0...$

(b) Method 2.

$$x(t) = 2x_1(t) - x_2(t) - x_3(t)$$

$$x_1(t) = \sum_{n=-\infty}^{\infty} \delta(t - T_0 n)$$

$$x_2(t) = \sum_{n=-\infty}^{\infty} \delta(t - T_0 n - 2)$$

$$x_3(t) = \sum_{n=-\infty}^{\infty} \delta(t - T_0 n + 2)$$

Where $T_0 = 6$. The FS of $x_1(t)$ is

$$x_1(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}$$

And by time shift property of FS we can get,

$$x_2(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{-jk\omega_0 2} e^{jk\omega_0 t}$$

$$x_3(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 2} e^{jk\omega_0 t}$$

By linear property of FS,

$$x(t) = 2\sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t} - \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{-jk\omega_0 2} e^{jk\omega_0 t} - \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 2} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{2}{T_0}(1-\cos(k\omega_0 2)) e^{jk\omega_0 t}$$

The FT of $x(t)$ is

$$x(t) \xrightarrow{\mathcal{F}} X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2}{T_0}(1-\cos(k\omega_0 2))\delta(\omega - k\omega_0)$$

$$\boxed{= \sum_{k=-\infty}^{\infty} (\pi/3)2(1-\cos 2k\pi/3)\delta(\omega - k\pi/3)}$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\pi/3)$$

where $a_0 = 0, a_{\pm 1} = \pi, a_{\pm 2} = \pi, a_{\pm 3} = 0...$

11. [(a)]

$$[e^{-at}\cos\omega_0 t]u(t) = \frac{1}{2}e^{-at}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-at}e^{-j\omega_0 t}u(t)$$

$$\boxed{X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} + \frac{1}{2(\alpha + j\omega_0 + j\omega)} = \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_0^2}}$$

(b)

$$x(t) = x_1(t) + x_2(t)$$

where,

$$x_1(t) = e^{-3t}sin(2t)u(t) \xrightarrow{\mathcal{F}} X_1(j\omega) = \frac{1/(2j)}{3 - j2 + j\omega} - \frac{1/(2j)}{3 + j2 + j\omega} = \frac{2}{(3 + j\omega)^2 + 4}$$

$$x_2(t) = e^{3t}sin(2t)u(-t) \xrightarrow{\mathcal{F}} X_2(j\omega) = -X_1(-j\omega) = -\frac{1/(2j)}{3 - j2 - j\omega} + \frac{1/(2j)}{3 + j2 - j\omega} = \frac{-2}{(3 - j\omega)^2 + 4}$$

Thus,

$$\boxed{X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{3j}{9+(\omega+2)^2} - \frac{3j}{9+(\omega-2)^2}}$$

12. [(a)]

$$X(j\omega) = \frac{e^{(4\omega+\pi/3)j} + e^{-(4\omega+\pi/3)j}}{2}$$

$$\boxed{x(t) = \frac{1}{2}e^{-j\pi t/12}\delta(t+4) + \frac{1}{2}e^{j\pi t/12}\delta(t-4)}$$

(b) From the given figure we can get

$$X(j\omega) = \begin{cases} \omega e^{-3j\omega} & ,|\omega| < 1 \\ 0 & ,|\omega| > 1 \end{cases}$$

$$\boxed{x(t) = \frac{\cos(t-3) - 1 - (t-3)sin(t-3)}{\pi(t-3)^2}}$$

13. From the figure we can get the frequency response of the lowpass differentiator is,

$$H(j\omega) = \begin{cases} \frac{j\omega}{3\pi} & ,-3\pi \le \omega \le 3\pi \\ 0 & ,otherwise \end{cases}$$

(a) Since $x(t) = \cos(2\pi t + \theta)$, $X(j\omega) = e^{-j\omega\theta/2\pi}\pi[\delta(\omega-2\pi)+\delta(\omega+2\pi)]$. It is zero outside the region $-3\pi \le \omega \le 3\pi$. Thus,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{j\omega}{3\pi}X(j\omega)$$

$$\boxed{y(t) = \frac{1}{3\pi}\frac{dx(t)}{dt} = -\frac{2}{3}\sin(2\pi t + \theta)}$$

(b) Since $x(t) = cos(4\pi t + \theta)$, $X(j\omega) = e^{-j\omega\theta/4\pi}\pi[\delta(\omega-4\pi)+\delta(\omega+4\pi)]$. The nonzero portions of $X(j\omega)$ lie outside the region $-3\pi \le \omega \le 3\pi$. This implies,

$$Y(j\omega) = X(j\omega)H(j\omega) = 0$$

$$\boxed{y(t) = 0}$$

(c) The Fourier series coefficients of the signal $x(t)$ are given by

$$c_k = \frac{1}{1}\int_0^{0.5} sin(2\pi t)e^{-jk2\pi t}dt$$

Also we have

$$X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)$$

Where $\omega_0 = 2\pi$. Thus, we can see that only the portions when $k = 0, \pm 1$ lie in the region $-3\pi \leq \omega \leq 3\pi$.

$$c_0 = \frac{1}{\pi}, c_1 = c_{-1}^* = \frac{-1}{4j}$$

The portion that can pass the filter is $x_{lp}(t) = \frac{1}{\pi} + \frac{1}{2}\sin(2\pi t)$. Finally, $\boxed{y(t) = \frac{1}{3\pi}\frac{dx_{lp}(t)}{dt} = \frac{1}{3}\cos(2\pi t)}$

14. $P_y(\omega) = P_f(\omega)|H(\omega)|^2$, from the diagram $P_f(w) = 2\delta(\omega-150)+2\delta(\omega+150)+0.5\delta(\omega-200)+0.5\delta(\omega+200)$. Note $|H(150)| = 18$ and $|H(200)| = 20$, so

    $\boxed{P_y(\omega) = 648\delta(\omega-150) + 648\delta(\omega+150) + 200\delta(\omega-200) + 200\delta(\omega+200)}$.

    The sketch is shown below.

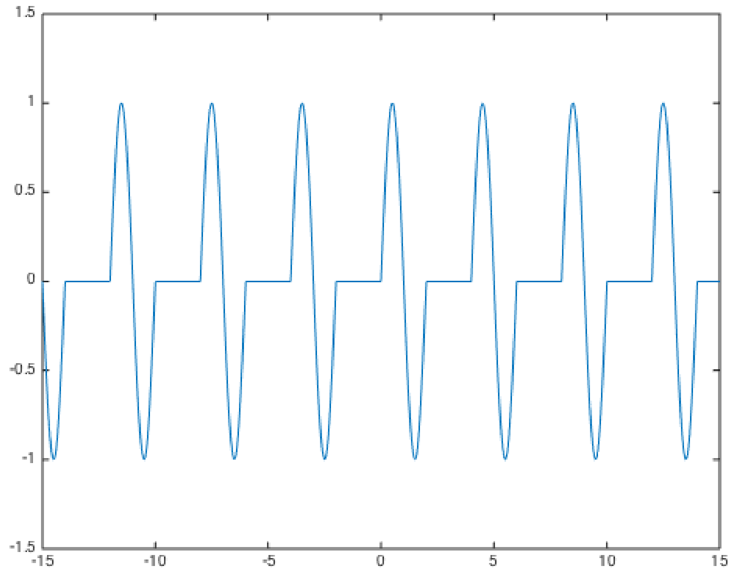

Figure 1: Problem 1

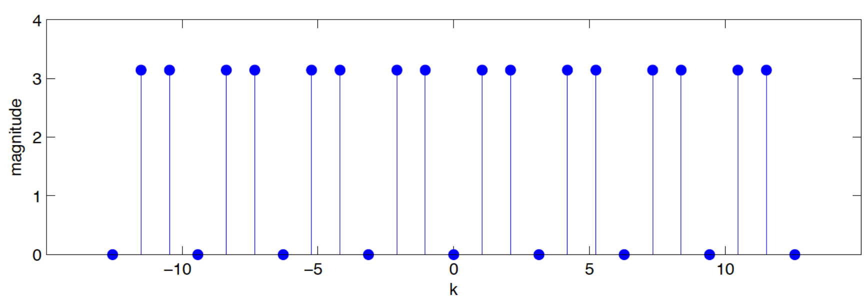

Figure 2: Problem 2

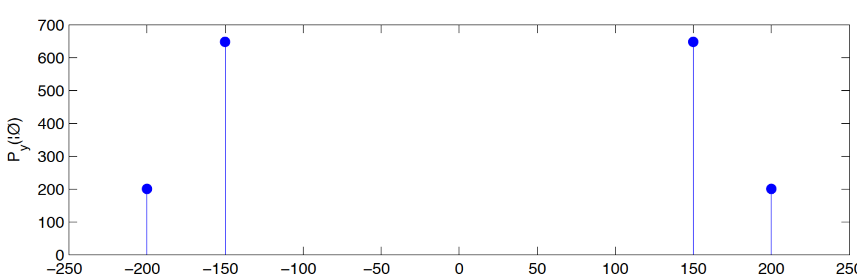

Figure 3: Problem 6