Homework 5

Answers:

1. [10]

(a) Using PFE (letting $s = j\omega$), we have

$$H(s) = \frac{2s+1}{s^3+5s^2+8s+4} = \boxed{\tfrac{1}{s+2} + \tfrac{3}{(s+2)^2} + \tfrac{-1}{s+1}}.$$

For PFE in Matlab, we directly use `[r, p, k] = residue([2 1], [1 5 8 4])`. The output conforms with our result. (*Note that `r = [1 3 -1]` from Matlab while the first two elements of p are both -2.*)

(b)

$$h(t) = \boxed{(3t+1)e^{-2t}u(t) - e^{-t}u(t)}.$$

Sample code for the two plots are given below:

```
clc;
clear all;
close all;
t = linspace(0,7);
h = ((3*t + 1) .* exp(-2 * t) - exp(-t)) .* (t>0);
subplot(211), plot(t, h)
xlabel('t'), ylabel('h(t)')
subplot(212), impulse([2 1], [1 5 8 4])
```

And the plots:

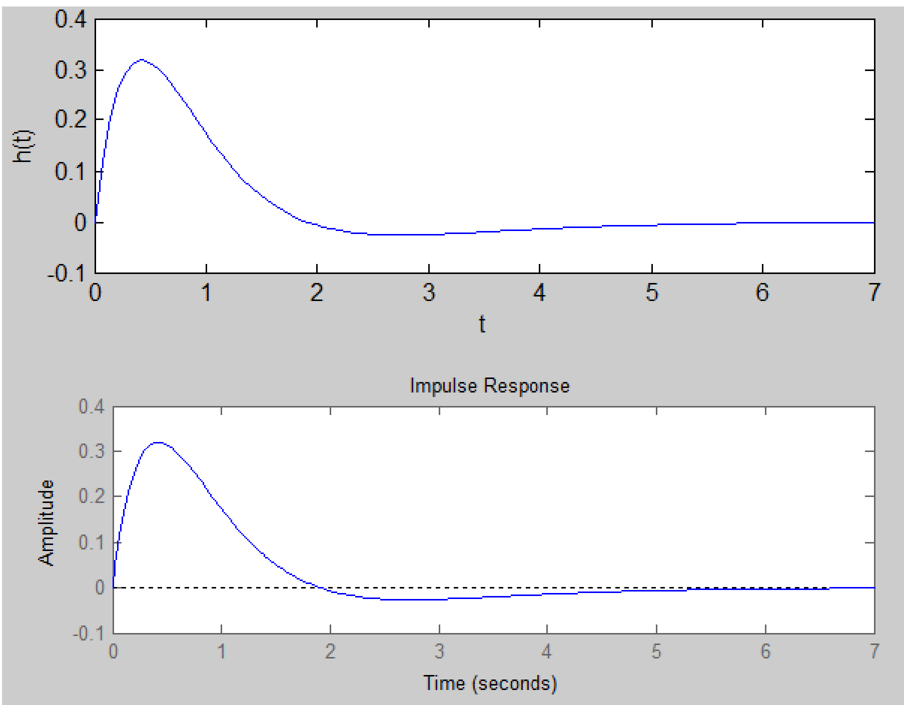


Figure 1: HW5-1(b)

2. [10]

(a) The spectrum of p(t):

$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi k}{T}) = 6\pi\sum_{k=-\infty}^{\infty}\delta(\omega - 6\pi k).$$

And the spectrum of $\cos(\omega_0 t)$ is $\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$.
Hence

$$X_p(\omega) = \frac{1}{2\pi}P(\omega) * [\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)],$$

and the four sketches are given below:

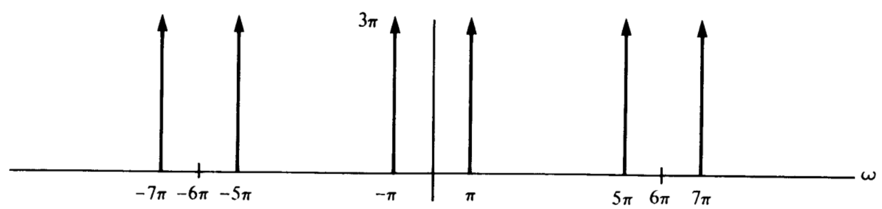


Figure 2: HW5-2-(a)-i: $\omega_0 = \pi$

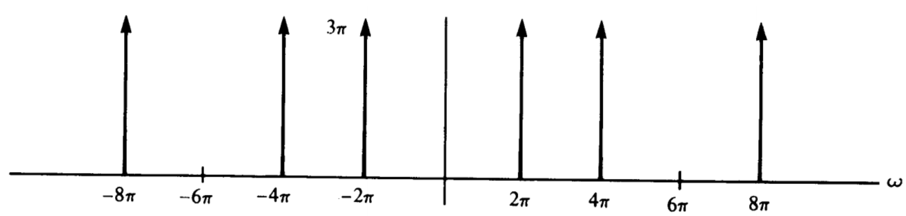


Figure 3: HW5-2-(a)-ii: $\omega_0 = 2\pi$

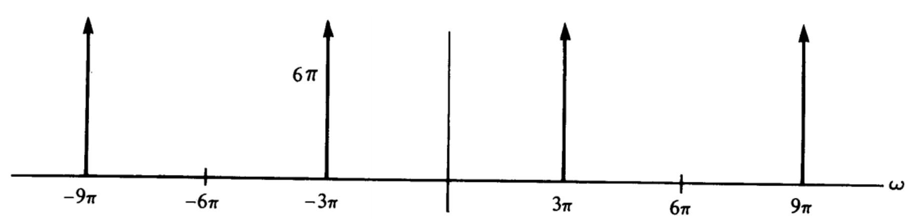


Figure 4: HW5-2-(a)-iii: $\omega_0 = 3\pi$

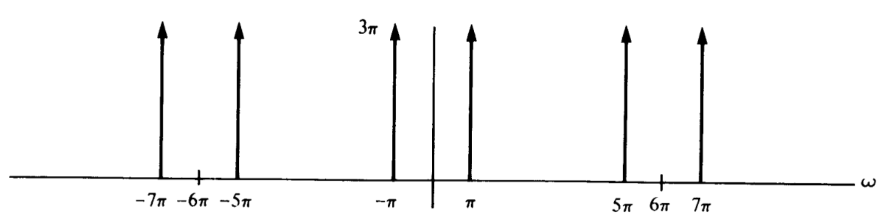


Figure 5: HW5-2-(a)-iv: $\omega_0 = 5\pi$

(b) (i) and (iv) are identical. We will not be able to reconstruct (iv) from $x_p(t)$.

3. [5] Since the sampled signal $x_p(t)$ is indistinguishable for certain values of ω, the output should be indistinguishable for certain values of ω as well. Hence, $Q(\omega)$ should be periodic in ω. So only $\boxed{\text{Figure 0503(c)}}$ is a possible candidate.

4. [5] When $x(t)$ is composited with $n(t)$, its bandwidth doubles to $4W$. However, we can alias the noise region (the bands for $n(t)$) to get a bigger T (lower sampling frequency) since we only need to reconstruct $x(t)$ and don't care about $n(t)$.

   If the bands for $n(t)$ is fully aliased, the sampling frequency will be $3W$, and $\boxed{T_{max} = \frac{2\pi}{3W}}$, as illustrated below:

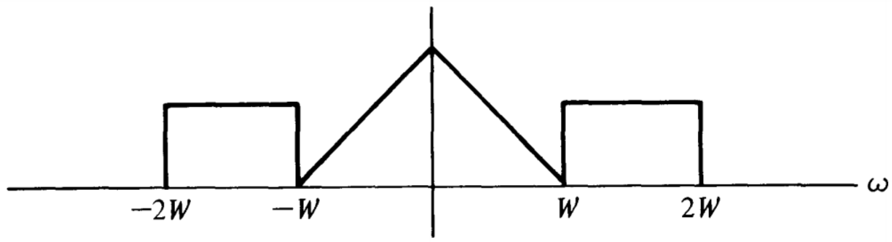


Figure 6: HW5-4-1: Composed spectrum.

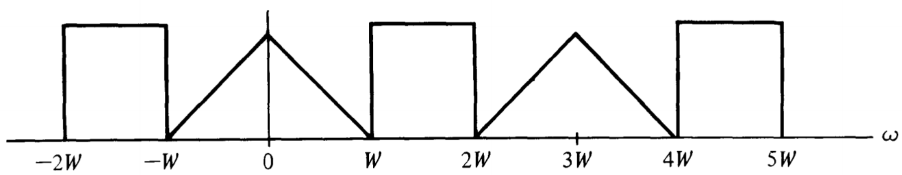


Figure 7: HW5-4-2: Spectrum after sampling allowing aliasing.

5. [5] As given below, note that the amplitude of both input should be T.

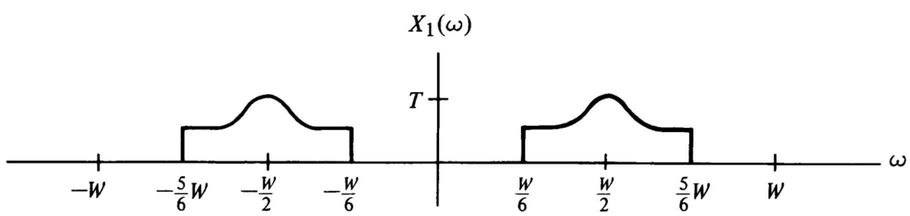


Figure 8: HW5-5-1.

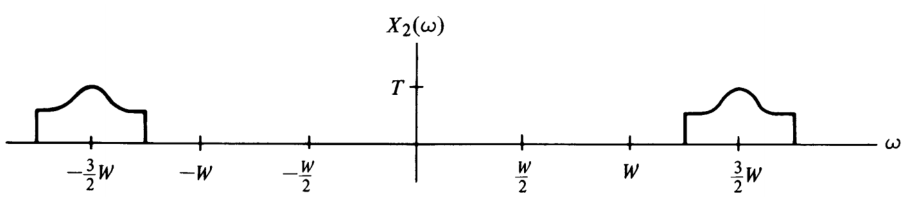


Figure 9: HW5-5-2.

6. [12]

(a) Time shift doesn't affect bandwidth. $X(\omega) = 0$ for $|\omega| > 2W$. Hence, $T_{max} = \frac{\pi}{2W}$, $A = T$, $2W < W_c < 2\pi/T - 2W$.

(b) $X(\omega) = 0$ for $|\omega| > 3W$. Hence, $T_{max} = \frac{\pi}{3W}$, $A = T$, $3W < W_c < 2\pi/T - 3W$.

(c) $X(\omega) = 0$ for $|\omega| > W$. Hence, $T_{max} = \frac{\pi}{W}$, $A = T$, $W < W_c < 2\pi/T - W$.

(d) $X(\omega) = 1/2(X(\omega - 2W) + X(\omega + 2W))$.
$X(\omega) = 0$ for $|\omega| > 3W$. Hence, $T_{max} = \frac{\pi}{3W}$, $A = T$, $3W < W_c < 2\pi/T - 3W$.

7. [5] It is more convenient to solve this problem in time domain than in frequency domain. First note that $x_p(t)$ is the sampled version of $x(t)$, as shown in Figure below.

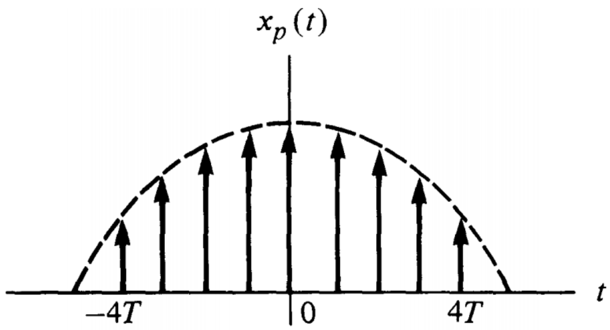


Figure 10: HW5-7-1: sketch of $x_p(t)$.

Carrying out the convolution with $h(t)$ in the time domain, we get $y(t)$:

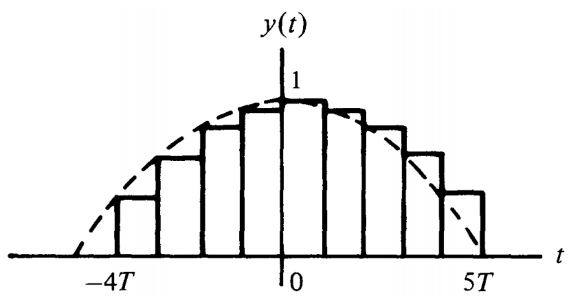


Figure 11: HW5-7-2: sketch of $y(t)$.

Finally for $w(t)$, we first compose the two blocks containing $h(t)$. Note that $h(t)$ convolve with $(1/T)h(t)$ give us the following result:

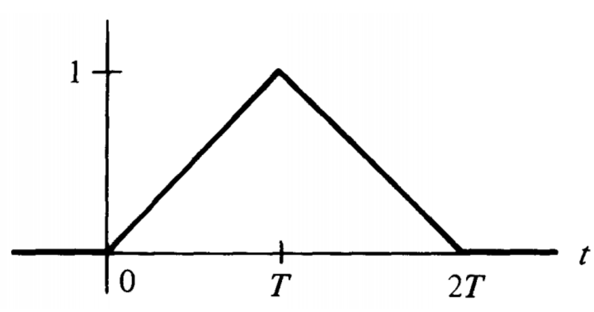


Figure 12: HW5-7-3: sketch of $(1/T)h(t) * h(t)$.

$x_p(t)$ convolving with the figure above give us the following decomposed response:

We compose all the "triangles" to see that this is actually the first-order interpolation between the samples of $x(t)$ (with a T time delay). Note that the ideal system realizing linear interpolation with NO time delay is actually non-casual.

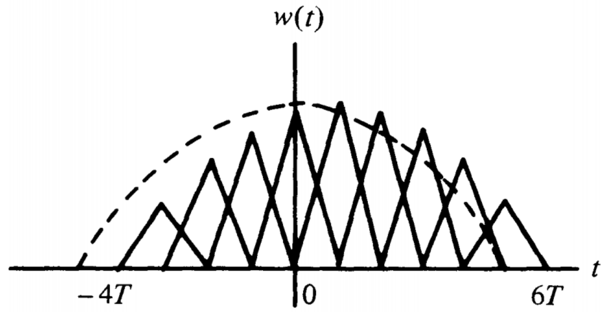


Figure 13: HW5-7-4: sketch of $w(t)$ in a decomposed version.

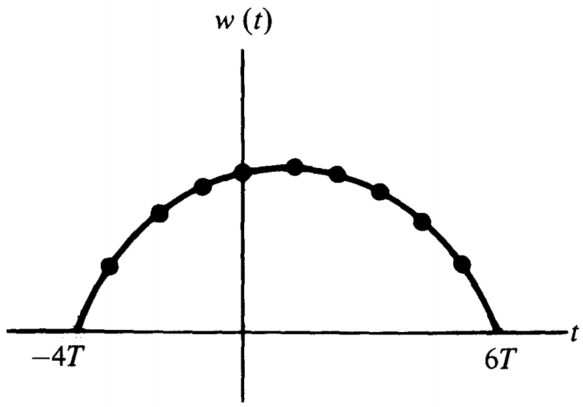


Figure 14: HW5-7-5: sketch of $w(t)$.

8. [10]

   (a)

$$Y(\omega) = \frac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)]$$

$$\begin{aligned}
W(\omega) &= \frac{1}{4}\left[X(\omega - \omega_c - \omega_d) + X(\omega - \omega_c + \omega_d) + X(\omega + \omega_c - \omega_d) + X(\omega + \omega_c + \omega_d)\right] \\
&= \frac{1}{4}\left[X(\omega - \Delta\omega - 2\omega_c) + X(\omega + \Delta\omega) + X(\omega - \Delta\omega) + X(\omega + \Delta\omega + 2\omega_c)\right]
\end{aligned}$$

Since $\omega_M + \Delta\omega < \omega_{co} < 2\omega_c + \Delta\omega - \omega_M$, the parts $X(\omega - \Delta\omega - 2\omega_c)$ and $X(\omega + \Delta\omega + 2\omega_c)$ will be filtered out. Therefore the output of the filter would be

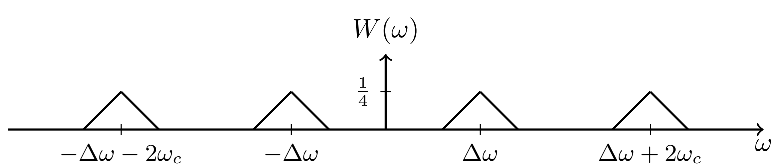
$$Z(\omega) = \frac{1}{4}\left[X(\omega + \Delta\omega) + X(\omega - \Delta\omega)\right]$$

and it follows that

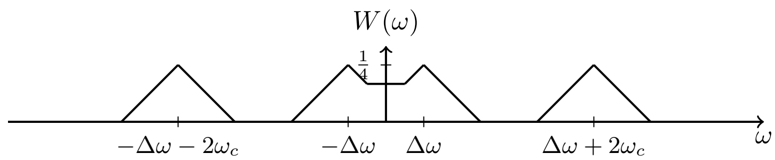
$$z(t) = \frac{1}{2}x(t)\cos(\Delta\omega t)$$

which is proportional to $x(t)\cos(\Delta\omega t)$.

   (b) The output of the demodulator is as below,



If we consider aliasing, then the output will be



9. [5] Given

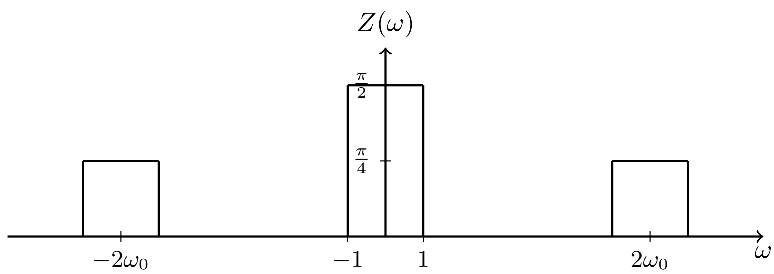
$$x(t) = \text{sinc}\left(\frac{t}{\pi}\right) \to \mathcal{F}X(\omega) = \pi\text{rect}\left(\frac{\omega}{2}\right)$$

By modulation property,

$$\begin{aligned} Y(\omega) &= \frac{1}{2}\left(X(\omega - \omega_0) + X(\omega + \omega_0)\right) \\ &= \frac{\pi}{2}\left(\text{rect}\left(\frac{\omega - \omega_0}{2}\right) + \text{rect}\left(\frac{\omega + \omega_0}{2}\right)\right) \end{aligned}$$

Then

$$\begin{aligned} Z(\omega) &= \frac{1}{2}\left(Y(\omega - \omega_0) + Y(\omega + \omega_0)\right) \\ &= \frac{\pi}{4}\left(\text{rect}\left(\frac{\omega - 2\omega_0}{2}\right) + 2\text{rect}\left(\frac{\omega}{2}\right) + \text{rect}\left(\frac{\omega + 2\omega_0}{2}\right)\right) \end{aligned}$$



10. [10] After filtering,

$$X_1(\omega) = X(\omega)H(\omega) = \left(1 - \left|\frac{\omega}{2\pi}\right|\right)\text{rect}\left(\frac{\omega}{4\pi}\right)\text{rect}\left(\frac{\omega}{2\pi}\right) = \left(1 - \left|\frac{\omega}{2\pi}\right|\right)\text{rect}\left(\frac{\omega}{2\pi}\right)$$

After modulation,

$$X_2(\omega) = \frac{1}{2}\left(X_1(\omega + 8\pi) + X_1(\omega - 8\pi)\right)$$

Passing through an intergrator,

$$Y(\omega) = X_2(\omega) \cdot \frac{1}{j\omega} + \pi X_2(0) \cdot \delta(\omega) = \frac{1}{j\omega}X_2(\omega) = \frac{1}{2j\omega}\left(X_1(\omega + 8\pi) + X_1(\omega - 8\pi)\right)$$

11. [8]

(a) The maximum value of $x(t)$ is 1. Therefore, $m = 1/A$.

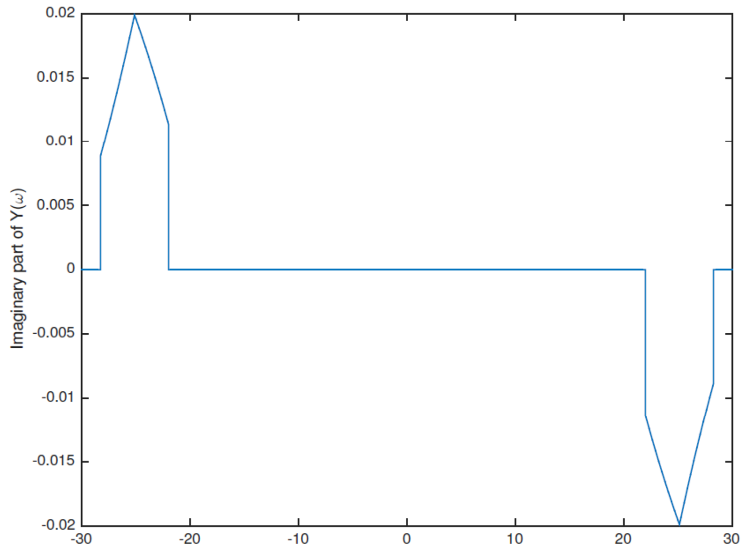
$$y(t) = A\cos(\omega_c t + \theta_c) + \frac{1}{2}\cos((\omega_c + \omega_M)t + \theta_c) + \frac{1}{2}\cos((\omega_c - \omega_M)t + \theta_c)$$

Therefore, $y(t)$ consists of three sinusoids. From Parseval's theorem, we know that total power in $y(t)$ is the sum of powers in each of sinusoids. Now note the power in a sinusoid of frequency ω_0 is

$$\frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \cos^2(\omega_0 t)dt = \frac{1}{2}$$

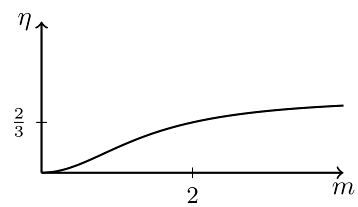
Therefore,

$$P_y = \frac{A^2}{2} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2m^2} + \frac{1}{4}$$



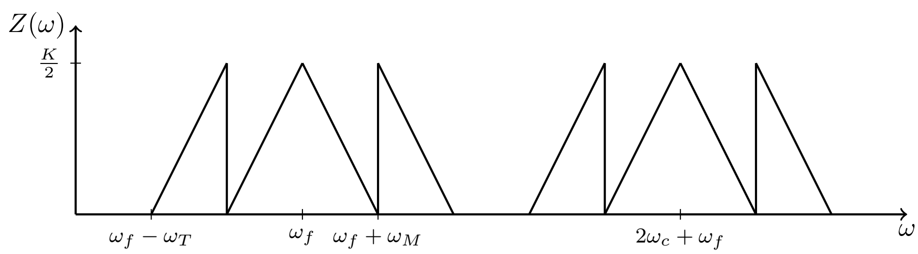
(b) The power in the sidebands is given by $\frac{1}{2}\cos((\omega_c + \omega_M)t + \theta_c) + \frac{1}{2}\cos((\omega_c - \omega_M)t + \theta_c)$, so the efficiency is

$$\eta = \frac{1/4}{1/(2m^2) + 1/4} = \frac{m^2}{2 + m^2}$$



12. [15]

(a) The spectrum $Z(\omega)$:



(b) From the figure above, we know that to avoid aliasing,

$$\omega_f + \omega_M \leq 2\omega_c + \omega_f - \omega_T \quad \Rightarrow \quad \omega_T \leq 2\omega_c - \omega_M$$

and in case $\omega_f - \omega_T$ is negative, we also have to make sure

$$-\omega_f + \omega_T \leq \omega_f - \omega_M \quad \Rightarrow \quad \omega_T \leq 2\omega_f - \omega_M$$

(c)

$$G = \frac{2}{K}, \qquad \alpha = \omega_f - \omega_M, \qquad \beta = \omega_f + \omega_M$$