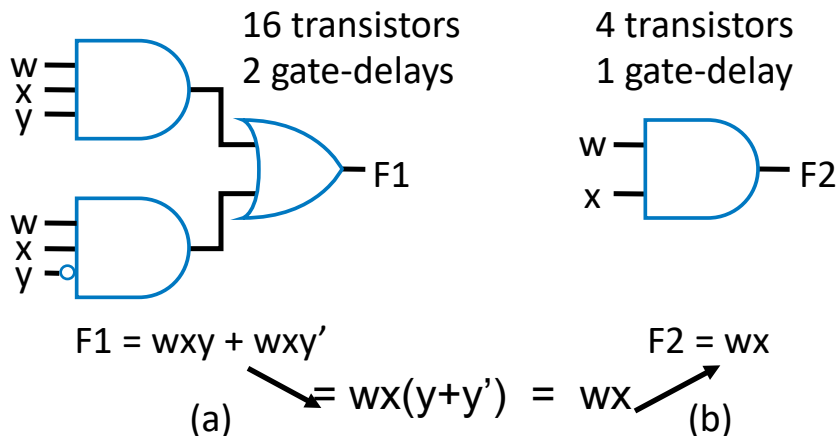


Topic 4

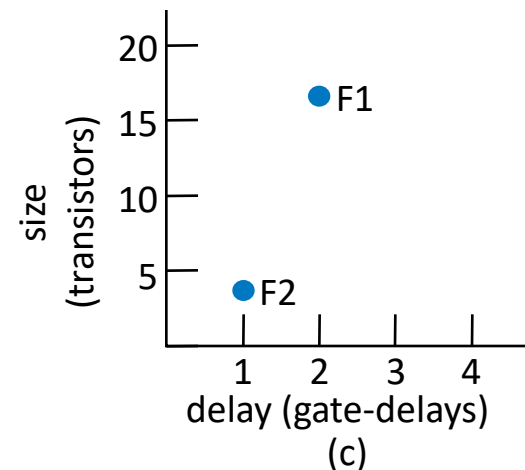
Logic Optimization

Simplification and Optimization

- How can we build better circuits?
- Let's consider two important design criteria
 - **Delay** – the time from input change to correct stable output response
 - **Size** – the number of transistors
 - For quick estimation, assume
 - Every gate has delay of “1 gate-delay”
 - Every gate *input* requires 2 transistors
 - Ignore inverters for simplicity



Transforming F1 to F2 represents an **optimization**: Better in all criteria of interest



Logic Optimization

- **Two-level size optimization using algebraic methods**
 - Goal: circuit with only two levels (**AND-OR network**), with minimum transistors
 - Sum-of-products yields two levels
 - $F = abc + abc'$ is sum-of-products
 - $G = w(xy + z)$ is not

Example

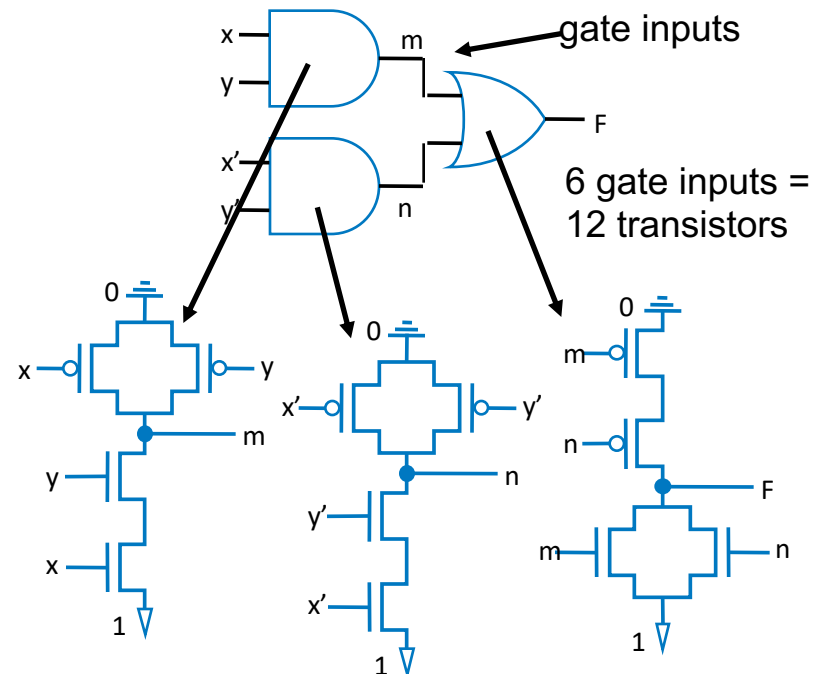
$$F = xyz + xyz' + x'y'z' + x'y'z$$

$$F = xy(z + z') + x'y'(z + z')$$

$$F = xy*1 + x'y'*1$$

$$F = xy + x'y'$$

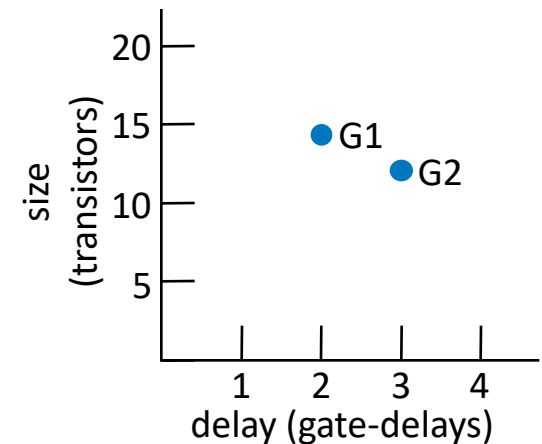
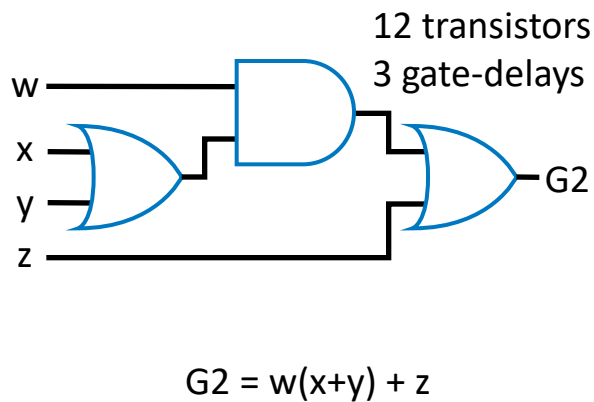
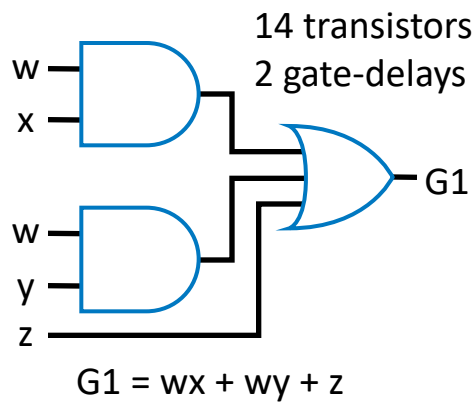
4 literals + 2 terms = 6 gate inputs



Logic Optimization

- **Multi-level optimization**
 - Improves some, but worsens other

Transforming G1 to G2 represents a **tradeoff**. Some criteria better, others worse



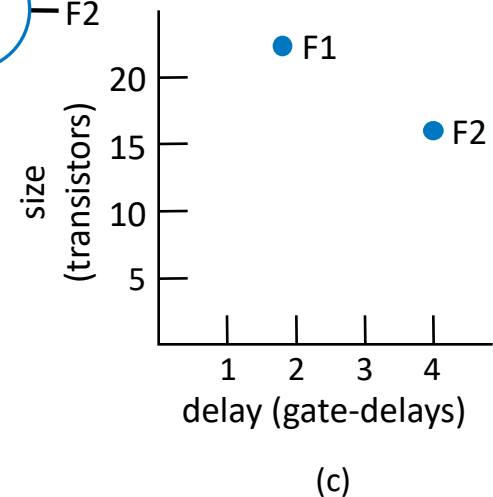
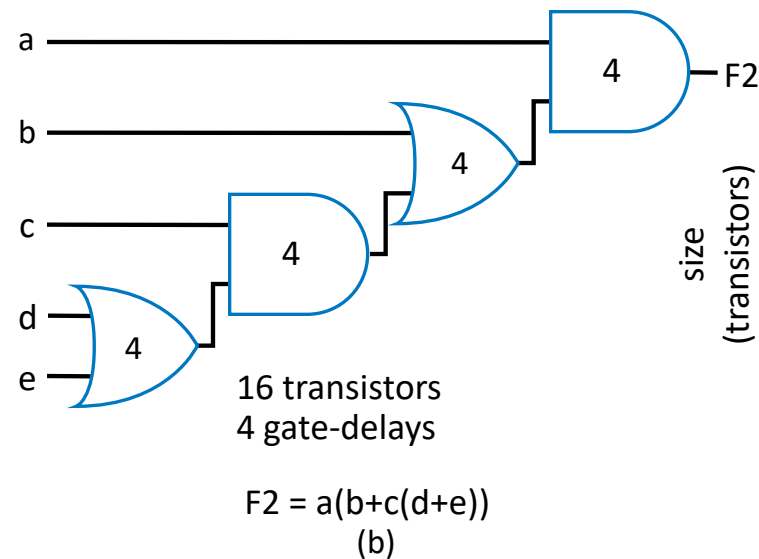
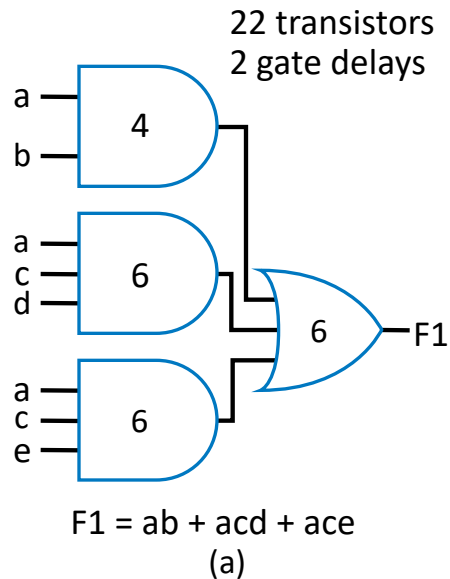
Performance/Size Tradeoffs

- **Delay & Size tradeoff**

- We don't always need the speed of two level logic
- Multiple levels may yield fewer gates
- Example

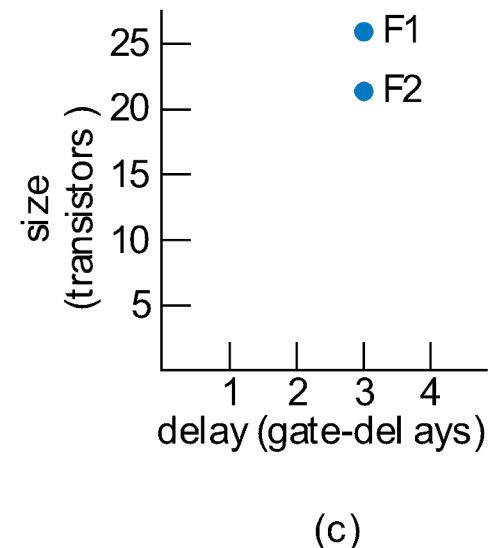
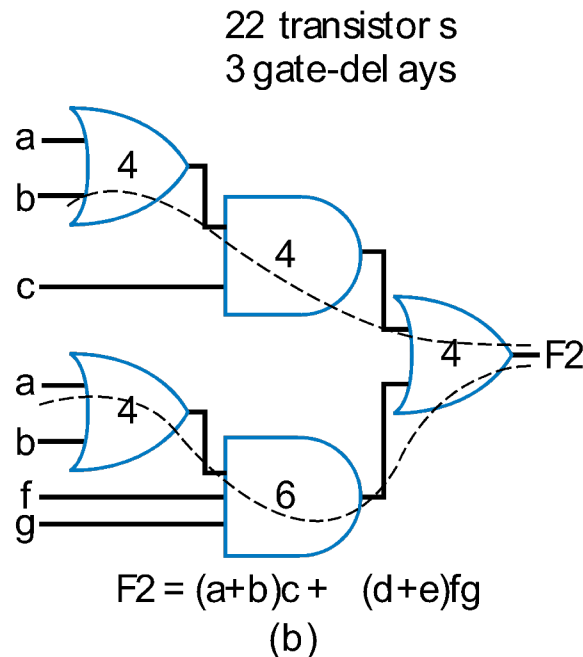
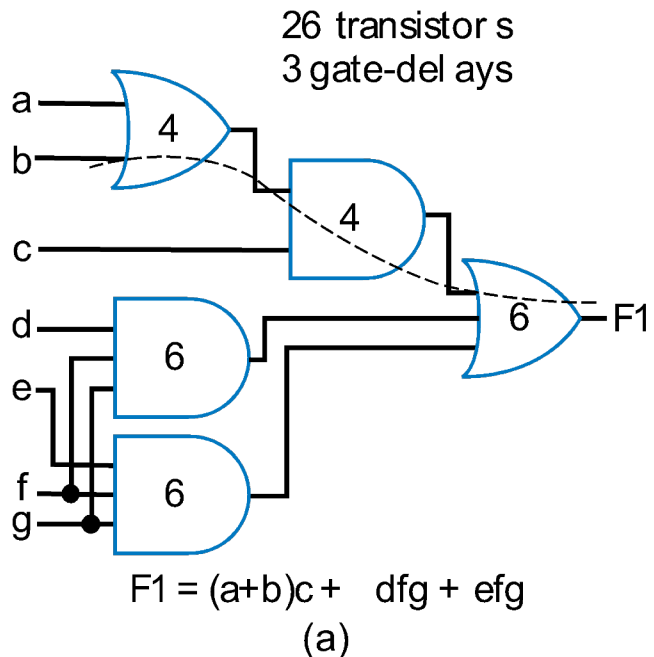
- $F1 = ab + acd + ace \rightarrow F2 = ab + ac(d + e) = a(b + c(d + e))$

- General technique: Factor out literals



Critical Path

- **Critical path: longest delay path from an input to output**
- **Optimization**
 - Reduce delay by shortening length of critical path
 - Reduce size by using multiple levels on non-critical paths
 - But may make non-critical path become critical path



Logic Optimization

- **Optimization using Boolean Algebra**
 - To obtain Boolean equations with fewer literals (optimization in size)
 - To obtain circuit with shorter delay
- **Optimization using other techniques**
 - Karnaugh-map (to achieve simplified two-level circuit)
 - Quine-McCluskey (not in this class)

Karnaugh Map (K-map) Technique

- A graphical technique used to *simplify* a logic equation
- A way to show the *relationship* between the logic inputs and corresponding output
 - Like truth table
- Much cleaner and more *procedural* than algebraic simplification by theorems of Boolean algebra
- Theoretically, it can be used for any number of input variables,
 - BUT is only practical for less than six, we will limit our discussion to logic equations with *five or less* variables

Building a K-map

- **K-map can be filled up directly from a truth table**
 - Each minterm corresponds to a cell in the K-map
- **K-map cells are labeled so that both horizontal and vertical movement differ only in one variable**

F	Y		Y'		Y
	X				
	X'	m0		m1	
	X	m2		m3	

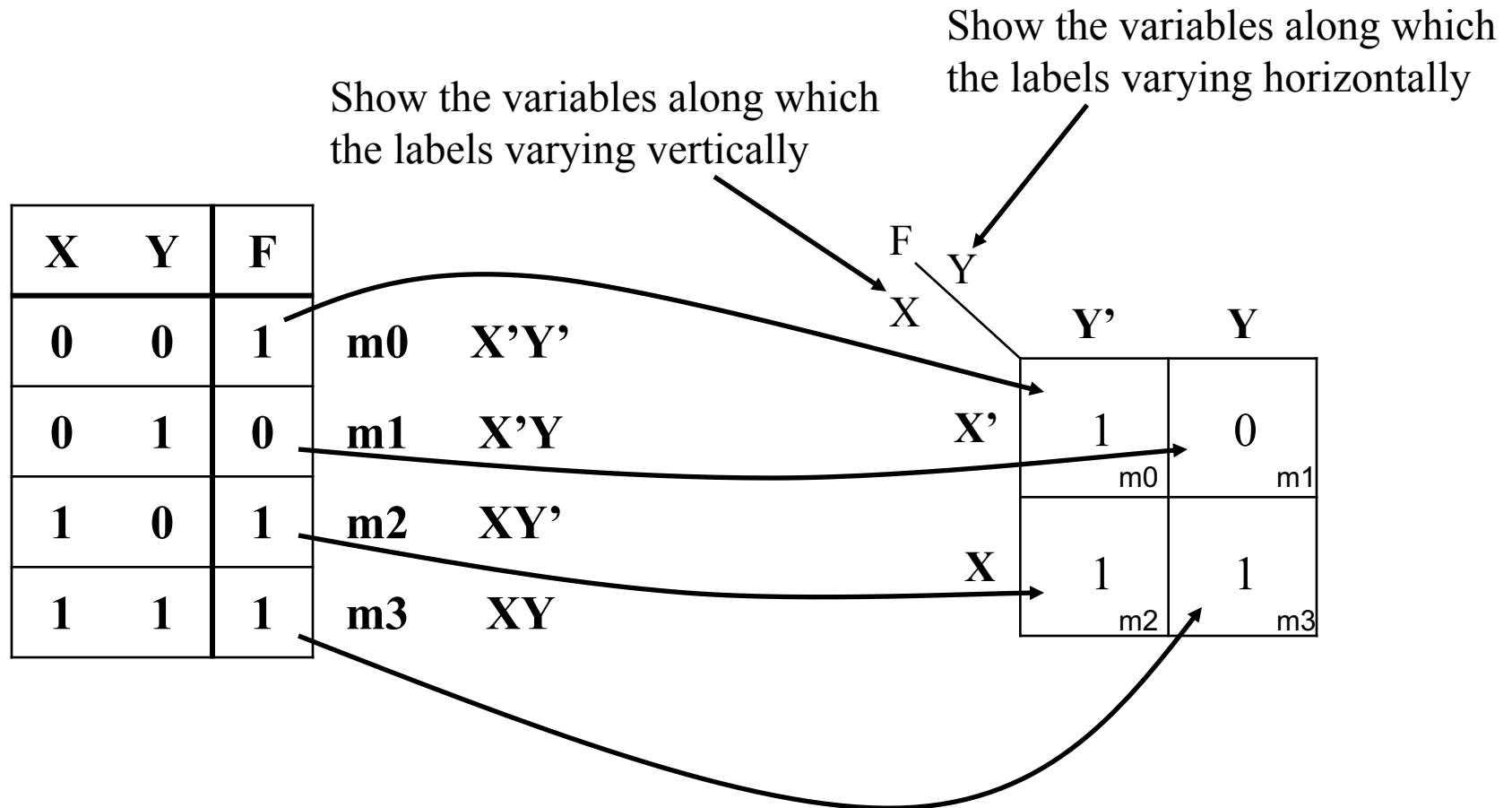
F	YZ		Y'Z		YZ		Y'Z'	
	X							
	X'	m0	m1	m3	m2			
	X	m4	m5	m7	m6			

F	YZ		Y'Z		YZ		Y'Z'	
	WX							
	W'X'	m0	m1	m3	m2			
	W'X	m4	m5	m7	m6			
	WX	m12	m13	m15	m14			
	WX'	m8	m9	m11	m10			

- **Since the adjacent cells differ in only one variable, they can be grouped to create simpler terms in the sum-of-product expression.**

Two-Variable K-map

- There are four minterms – 2 by 2 square map



Three-Variable K-map

- There are $2^3 = 8$ minterms – 2 by 4 rectangular map

Show the variables along which
the labels varying vertically

Show the variables along which
the labels varying horizontally

X	Y	Z	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

m0 $X'Y'Z'$
m1 $X'Y'Z$
m2 $X'YZ'$
m3 $X'YZ$
m4 $XY'Z'$
m5 $XY'Z$
m6 XYZ'
m7 XYZ

		YZ			
	X	Y'Z'	Y'Z	YZ	YZ'
X'		1 m0	0 m1	1 m3	1 m2
X		0 m4	0 m5	1 m7	0 m6

Four-Variable K-map

- There are $2^4=16$ minterms – 4 by 4 square map

W	X	Y	Z	F
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

m0	$W'X'Y'Z'$
m1	$W'X'Y'Z$
m2	$W'X'YZ'$
m3	$W'X'YZ$
m4	$W'XY'Z'$
m5	$W'XY'Z$
m6	$W'XYZ'$
m7	$W'XYZ$
M8	$WX'Y'Z'$
m9	$WX'Y'Z$
M10	$WX'YZ'$
m11	$WX'YZ$
m12	$WXY'Z'$
m13	$WXY'Z$
m14	$WXYZ'$
m15	$WXYZ$

Show the variables along which the labels varying vertically or horizontally

	$Y'Z'$	$Y'Z$	YZ	YZ'
$W'X'$	1 m0	0 m1	1 m3	1 m2
$W'X$	0 m4	0 m5	1 m7	0 m6
WX	1 m12	0 m13	1 m15	0 m14
WX'	1 m8	1 m9	0 m11	0 m10

Label the Rows and Columns by 0 and 1

		F	
		Y	Y'
X		0	1
X'	0	1 m0	0 m1
X	1	1 m2	1 m3

		F			
		YZ	Y'Z'	Y'Z	YZ'
X		00	01	11	10
X'	0	1 m0	0 m1	1 m3	1 m2
X	1	0 m4	0 m5	1 m7	0 m6

		F			
		YZ	Y'Z'	Y'Z	YZ'
WX		00	01	11	10
W'X'	00	1 m0	0 m1	1 m3	1 m2
W'X	01	0 m4	0 m5	1 m7	0 m6
WX	11	1 m12	0 m13	1 m15	0 m14
WX'	10	1 m8	1 m9	0 m11	0 m10

0 represents the primed form
1 represents the unprimed form

Simplify – Grouping and Canceling

- Group is in shape of rectangle or square
- Group the adjacent 1's until all the 1's are grouped

		F		Y	
		X		0	1
X'Y'	0	1	0		
	1	0	0		

No adjacent 1's, the minterm cannot be further simplified:
 $F = X'Y'$

		F		Y	
		X		0	1
X'	0	1	1		
	1	0	0		

Two adjacent 1's:
 $F = X'Y' + X'Y$
 $= X'(Y' + Y)$
 $= X' \cdot 1$
 $= X'$

If both primed and unprimed forms of a letter appear in the same group, the letter can be canceled

A group corresponds to a Sum-of-Minterm expression

Grouping and Canceling

- **No zeros in the group**

F \ X \ YZ		00	01	11	10
0		0	1	1	0
1		0	0	1	0

F \ X \ YZ		00	01	11	10
0		0	1	1	0
1		0	0	1	0

- **The number of 1's in the group should be 2^N , $N = 0, 1, 2, \dots$**

F \ X \ YZ		00	01	11	10
0		1	1	1	0
1		0	0	0	0

F \ X \ YZ		00	01	11	10
0		1	1	1	0
1		0	0	0	0

Grouping and Canceling

- Group as many adjacent 1's as possible

Redundant term

YZ

X'Z

	YZ	00	01	11	10
X	0	0	1	1	0
1		0	0	1	0

$$\begin{aligned}
 F &= X'Y'Z + X'YZ + XYZ \\
 &= (X'Y'Z + X'YZ) + (X'YZ + XYZ) \\
 &= X'Z(Y' + Y) + YZ(X' + X) \\
 &= X'Z + YZ
 \end{aligned}$$

YZ

X'Z

Redundant terms

	YZ	00	01	11	10
X	0	1	1	1	1
1		0	0	1	1

Y

$$\begin{aligned}
 F &= X'Y'Z' + X'Y'Z + X'YZ + X'YZ' + XYZ + XYZ' \\
 &= (X'Y'Z' + X'Y'Z + X'YZ + X'YZ') + \\
 &\quad (X'YZ + X'YZ' + XYZ + XYZ') \\
 &= (X'Y' + X'Y) + (X'Y + XY) \\
 &= X' + Y
 \end{aligned}$$

Grouping and Canceling

- Group as many adjacent 1's as possible

F YZ
 WX

	00	01	11	10
00	1	1	1	1
01	1	1	1	1
11	1	0	1	0
10	1	0	0	0

W' , because
 horizontally, both Y and Y' appear, Y cancels
 both Z and Z' are included, Z cancels;
 vertically, both X and X' appear, X cancels.
 That leaves the 0 for W – primed W .

XYZ , because
 only W and W' appear, W cancels.
 The remaining term 111 implies XYZ

$Y'Z'$, because
 both W and W' appear, and both X and X' appear,
 So W and X cancel.
 That leaves the 00 for YZ – primed Y and primed Z .

Grouping and Canceling

- Edges wrap around

F \ YZ		X			
		00	01	11	10
X	0	1	0	0	1
	1	1	0	0	1

$$F = Z'$$

F \ YZ		WX			
		00	01	11	10
WX	00	1	0	0	1
	01	1	0	0	1
	11	0	0	0	0
	10	1	0	0	1

$$F = \mathbf{W'Z'} + \mathbf{X'Z'}$$

Grouping and Canceling

- **Summary**

- Group is in shape of rectangle or square
- Group the adjacent 1's until all the 1's are grouped
- The number of 1's in the group should be 2^N , $N = 0, 1, 2, \dots$
- Collect as many 1's as possible in the same group
- No zeros in the group
- Edges wrap around
- If both primed and unprimed forms of a letter appear in a same group, the letter cancels
- The simplified result will be a sum-of-product form; the number of the product terms is decided by the number of the groups

Group Patterns of 2-Variable Map

	0	1
0	1	0
1	0	1

	0	1
0	1	1
1	0	0

	0	1
0	0	1
1	0	1

	0	1
0	1	1
1	1	1

- **Summary**

- A group of one cell represents a minterm, giving a term of two literals
- A group of two cells represents a term of one literal
- A group of all the four cells gives a logic 1

Group Patterns of 3-Variable Map

	00	01	11	10
0	1	0	1	0
1	0	1	0	1

	00	01	11	10
0	1	1	1	0
1	0	0	1	1

	00	01	11	10
0	1	1	0	1
1	1	1	1	

	00	01	11	10
0	1	1	1	1
1	1	0	0	1

	00	01	11	10
0	1	1	1	1
1	1	1	1	0

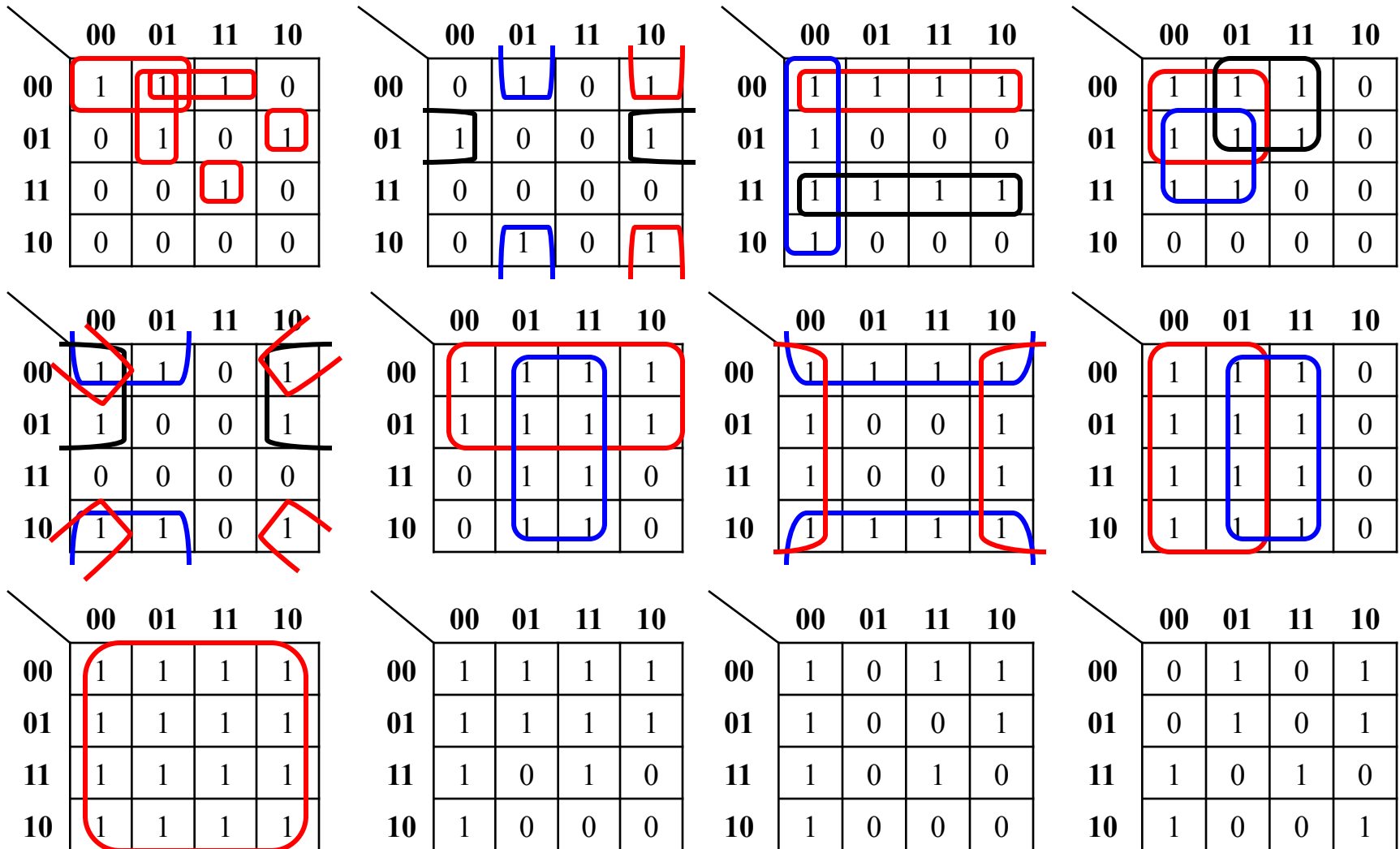
	00	01	11	10
0	1	1	1	1
1	1	1	1	1

Group Patterns of 3-Variable Map

- **Summary**

- A group of one cell represents a minterm, giving a term of three literals
- A group of two cells represents a term of two literals
- A group of four cells represents a term of one literal
- A group of all the eight cells gives a logic 1

Group Patterns of 4-Variable Map



Group Patterns of 4-Variable Map

- **Summary**
 - A group of one cell represents a minterm, giving a term of four literals
 - A group of two cells represents a term of three literals
 - A group of four cells represents a term of two literals
 - A group of eight cells represents a term of one literal
 - A group of all the sixteen cells gives a logic 1
- **The more the number of cells in one group, the less the number of literals that group represents, hence cheaper to implement using logic gates**

Don't Care Conditions

- The possible input combinations might not be all valid or not for consideration for a device
 - Hence we don't care what the corresponding outputs are under those conditions
 - Called **don't care** conditions
 - Mark the corresponding outputs by **X**

A	B	C	D	F
<i>0</i>	<i>0</i>	<i>0</i>	<i>0</i>	<i>X</i>
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
<i>1</i>	<i>0</i>	<i>0</i>	<i>1</i>	<i>X</i>
1	0	1	0	1
1	0	1	1	1
<i>1</i>	<i>1</i>	<i>0</i>	<i>0</i>	<i>X</i>
<i>1</i>	<i>1</i>	<i>0</i>	<i>1</i>	<i>X</i>
<i>1</i>	<i>1</i>	<i>1</i>	<i>0</i>	<i>X</i>
1	1	1	1	1

Don't Care Conditions

- By employing **don't care** conditions, logic equations can be further simplified

- Example:

$$F(A, B, C, D) = \sum m(2, 3, 5, 6, 7, 10, 11, 15) + \sum d(0, 9, 12, 13, 14)$$

- Fill out the K-map with 1's and X's
 - Each "X" can be either 0 or 1 depending upon the needs of simplification
 - Not all X's have to be considered
- Apply the same grouping and canceling rules before using 'X': $F = A'C + A'BD + B'C + CD$
after: $F = C + BD$ (Essential PI?)

F AB \ CD		AB			
		00	01	11	10
CD	00	X ₀	0 ₁	1 ₃	1 ₂
	01	0 ₄	1 ₅	1 ₇	1 ₆
	11	X ₁₂	1 ₁₃	1 ₁₅	1 ₁₄
	10	0 ₈	X ₉	1 ₁₁	1 ₁₀

Prime Implicants

- Implicant: is a product term
- A **prime implicant (PI)** is a group that cannot be entirely contained by another implicant

F \ WX \ YZ		YZ			
		00	01	11	10
WX	00	1	1	1	1
	01	1	1	1	1
	11	0	0	0	0
	10	0	0	0	0

— Prime implicant

..... Not prime implicants

F \ WX \ YZ		YZ			
		00	01	11	10
WX	00	1	1	1	0
	01	0	0	1	0
	11	0	0	0	0
	10	0	0	0	1

Essential Prime Implicants

- A **prime implicant (PI)** is **essential** if a cell is covered **ONLY** by that PI
- The **essential PIs** can be found by
 - looking at each cell marked as 1 and not covered by any other essential PI
 - and checking the number of PIs that cover it

F		YZ			
WX		00	01	11	10
00	1	0	1	1	
01	0	1	1	0	
11	0	1	1	0	
10	1	1	1	1	

Essential Prime Implicants

- Check each cell marked as 1, only if it has not been covered by an essential PI

Karnaugh map for function F with variables WX and YZ:

	YZ	00	01	11	10
WX	00	1	0	1	1
	01	0	1	1	0
	11	0	1	1	0
	10	1	1	1	1

Red lines indicate prime implicants: $X'Z'$ (covering cells (00,00), (00,10), (10,00), (10,10)), $X'Z$ (covering cells (00,10), (10,10)), XZ' (covering cells (10,00), (10,10)), and XZ (covering cells (10,10)).

Essential PI: $X'Z'$

Karnaugh map for function F with variables WX and YZ:

	YZ	00	01	11	10
WX	00	1	0	1	1
	01	0	1	1	0
	11	0	1	1	0
	10	1	1	1	1

Red lines indicate prime implicants: $X'Z'$ (covering cells (00,00), (00,10), (10,00), (10,10)), $X'Z$ (covering cells (00,10), (10,10)), XZ' (covering cells (10,00), (10,10)), and XZ (covering cells (10,10)). Blue lines indicate prime implicants: $X'Y$ (covering cells (00,00), (00,10), (10,00), (10,10)), $X'Z$ (covering cells (00,10), (10,10)), XZ' (covering cells (10,00), (10,10)), and XZ (covering cells (10,10)).

No essential PIs found

Essential Prime Implicants

F \ WX \ YZ		YZ			
		00	01	11	10
WX	00	1	0	1	1
	01	0	1	1	0
	11	0	1	1	0
	10	1	1	1	1

Essential PI: XZ

F \ WX \ YZ		YZ			
		00	01	11	10
WX	00	1	0	1	1
	01	0	1	1	0
	11	0	1	1	0
	10	1	1	1	1

No essential PIs found

Essential Prime Implicants

F		YZ			
WX		00	01	11	10
00	1	0	1	1	
01	0	1	1	0	
11	0	1	1	0	
10	1	1	1	1	

No essential PIs found

Essential Prime Implicants

F		YZ			
		WX	00	01	11
WX	00	1	0	1	1
	01	0	1	1	0
	11	0	1	1	0
	10	1	1	1	1

Essential PIs

F		YZ			
		WX	00	01	11
WX	00	1	0	1	1
	01	0	1	1	0
	11	0	1	1	0
	10	1	1	1	1

Non essential PIs

- **Essential PIs have to be used in the simplified equation**
- **Cells not covered by essential PIs can be represented by any PIs covering them**

$$F = X'Z' + XZ + WX'(\text{or } WZ) + X'Y(\text{or } YZ)$$

Product-of-Sum Simplification – An Alternate Method

- Redraw the K-map for F' by switching 1's and 0's

W	X	Y	Z	F	F'
0	0	0	0	1	0
0	0	0	1	1	0
0	0	1	0	1	0
0	0	1	1	1	0
0	1	0	0	0	1
0	1	0	1	0	1
0	1	1	0	0	1
0	1	1	1	1	0
1	0	0	0	1	0
1	0	0	1	1	0
1	0	1	0	0	1
1	0	1	1	0	1
1	1	0	0	1	0
1	1	0	1	0	1
1	1	1	0	0	1
1	1	1	1	0	1

		F YZ			
		00	01	11	10
WX	00	1	1	1	1
	01	0	0	1	0
	11	1	0	0	0
	10	1	1	0	0

		F' YZ			
		00	01	11	10
WX	00	0	0	0	0
	01	1	1	0	1
	11	0	1	1	1
	10	0	0	1	1

Product-of-Sum Simplification – An Alternate Method

- Two forms of the same truth table

F		YZ			
WX		00	01	11	10
00		1	1	1	1
01		0	0	1	0
11		1	0	0	0
10		1	1	0	0

Sum-of-Product form:

$$F = \underline{W'X'} + \underline{X'Y'} + \underline{WY'Z'} + \underline{W'YZ}$$

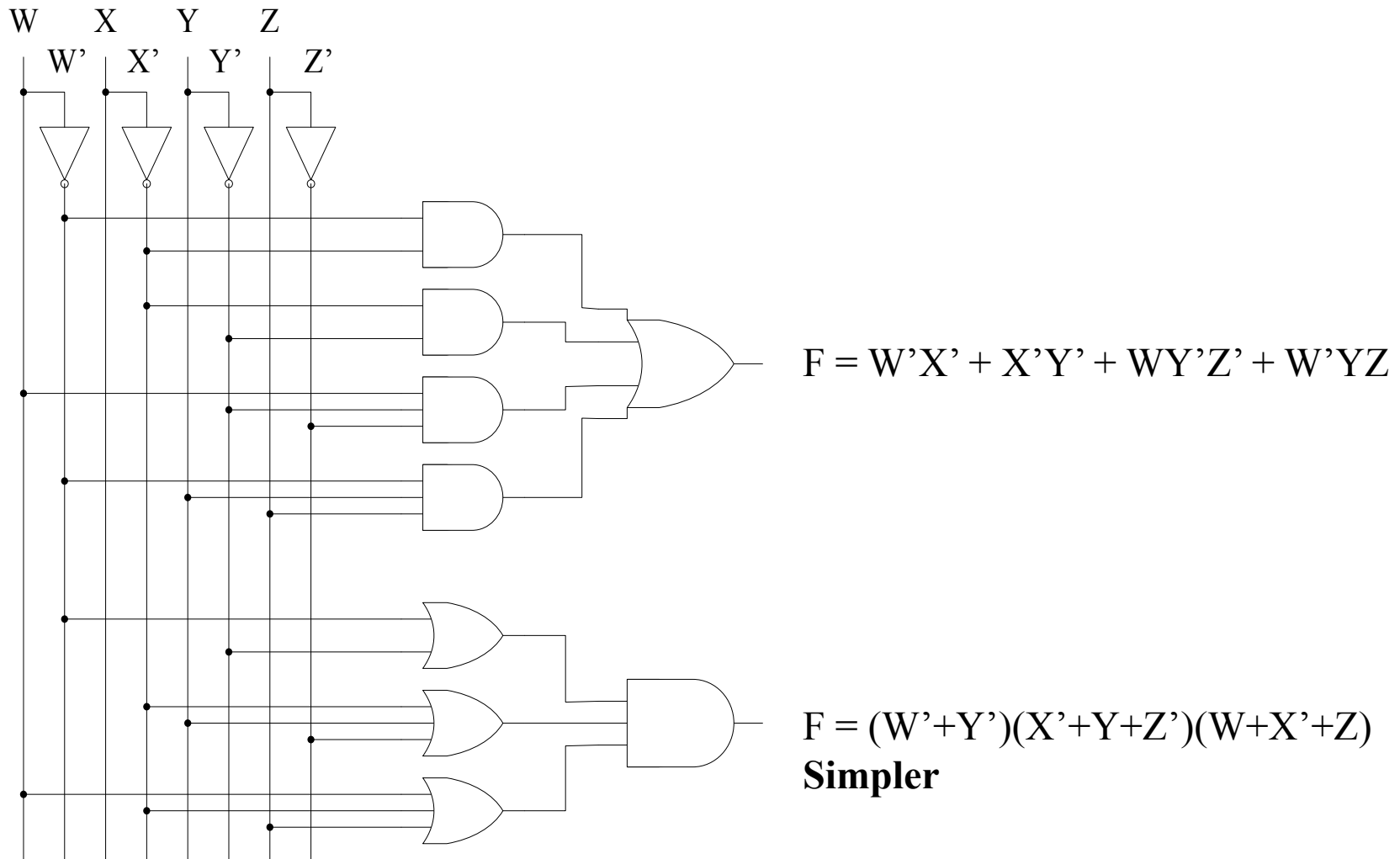
F'		YZ			
WX		00	01	11	10
00		0	0	0	0
01		1	1	0	1
11		0	1	1	1
10		0	0	1	1

$$F' = \underline{WY} + XY'Z + W'XZ'$$

Product-of-Sum form:

$$\begin{aligned}
 F &= (F')' && \text{(DeMorgan's Law)} \\
 &= (WY + XY'Z + W'XZ')' \\
 &= (WY)' (XY'Z)' (W'XZ')' \\
 &= (W' + Y')(X' + Y + Z')(W + X' + Z)
 \end{aligned}$$

Product-of-Sum Simplification – An Alternate Method



Simplify Any Standard Sum-of-Product Form

Method 1: fill out the table directly

- $F = A'C + A'BD + AB'C + BCD$



A	B	C	D	F	
0	0	0	0	0	m0
0	0	0	1	0	m1
0	0	1	0	1	m2
0	0	1	1	1	m3
0	1	0	0	0	m4
0	1	0	1	1	m5
0	1	1	0	1	m6
0	1	1	1	1	m7
1	0	0	0	0	m8
1	0	0	1	0	m9
1	0	1	0	1	m10
1	0	1	1	1	m11
1	1	0	0	0	m12
1	1	0	1	0	m13
1	1	1	0	0	m14
1	1	1	1	1	m15

F	CD			
	00	01	11	10
AB				
00	0	0	1	1
01	0	1	1	1
11	0	0	1	0
10	0	0	1	1



Simplify Any Standard Sum-of-Product Form

- **$F = A'C + A'BD + AB'C + BCD$**
 - **Method 2: convert any form of equation to sum-of-minterm**
 - AND with sum of the primed and unprimed forms of the missing literal, one at a time until all the missing literals are considered
 - Remove the duplicated minterms

$$\begin{aligned} F &= A'C + A'BD + AB'C + BCD \\ &= A'C (B+B') + A'BD (C+C') + AB'C (D+D') + BCD (A+A') \\ &= A'BC + A'B'C + A'BCD + A'BC'D + AB'CD + \\ &\quad AB'CD' + ABCD + A'BCD \\ &= A'BC (D+D') + A'B'C (D+D') + A'BCD + A'BC'D + AB'CD + \\ &\quad AB'CD' + ABCD + A'BCD \\ &= A'BCD + A'BCD' + A'B'CD + A'B'CD' + A'BCD + A'BC'D + \\ &\quad AB'CD + AB'CD' + ABCD + A'BCD \\ &= \Sigma m(7, 6, 3, 2, 7, 5, 11, 10, 15, 7) \\ &= \Sigma m(2, 3, 5, 6, 7, 10, 11, 15) \end{aligned}$$

Simplify Any Standard Sum-of-Product Form

- $F = A'C + A'BD + AB'C + BCD$

A	B	C	D	F	
0	0	0	0	0	m0
0	0	0	1	0	m1
0	0	1	0	1	m2
0	0	1	1	1	m3
0	1	0	0	0	m4
0	1	0	1	1	m5
0	1	1	0	1	m6
0	1	1	1	1	m7
1	0	0	0	0	m8
1	0	0	1	0	m9
1	0	1	0	1	m10
1	0	1	1	1	m11
1	1	0	0	0	m12
1	1	0	1	0	m13
1	1	1	0	0	m14
1	1	1	1	1	m15

F \ CD		AB			
		00	01	11	10
00	0	0	0	1	1
01	0	0	1	1	1
11	0	0	0	1	0
10	0	0	0	1	1

After simplification:

$$F = A'C + A'BD + B'C + CD$$

(Essential PI?)

Dealing with Five Variables

<i>E</i>	A	B	C	D	F
0	0	0	0	0	X
0	0	0	0	1	0
0	0	0	1	0	1
0	0	0	1	1	1
0	0	1	0	0	0
0	0	1	0	1	1
0	0	1	1	0	1
0	0	1	1	1	1
0	1	0	0	0	0
0	1	0	0	1	X
0	1	0	1	0	1
0	1	0	1	1	1
0	1	1	0	0	X
0	1	1	0	1	X
0	1	1	1	0	X
0	1	1	1	1	1

<i>E</i>	A	B	C	D	F
1	0	0	0	0	1
1	0	0	0	1	1
1	0	0	1	0	0
1	0	0	1	1	X
1	0	1	0	0	X
1	0	1	0	1	1
1	0	1	1	0	0
1	0	1	1	1	0
1	1	0	0	0	1
1	1	0	0	1	X
1	1	0	1	0	1
1	1	0	1	1	0
1	1	1	0	0	1
1	1	1	0	1	1
1	1	1	1	0	X
1	1	1	1	1	0

E'

F \ CD \ AB	00	01	11	10
00	X ₀	0 ₁	1 ₃	1 ₂
01	0 ₄	1 ₅	1 ₇	1 ₆
11	X ₁₂	X ₁₃	1 ₁₅	X ₁₄
10	0 ₈	X ₉	1 ₁₁	1 ₁₀

E

F \ CD \ AB	00	01	11	10
00	1 ₀	1 ₁	X ₃	0 ₂
01	X ₄	1 ₅	0 ₇	0 ₆
11	1 ₁₂	1 ₁₃	0 ₁₅	X ₁₄
10	1 ₈	X ₉	1 ₁₁	0 ₁₀

$$F = E'(C+BD) + E(C'+B'D)$$

Power Optimization

- **Power is another important design criteria**
 - Measured in Watts (energy/second)
 - Rate at which energy is consumed
- **Increasingly important as more transistors on a chip**
 - Power not scaling down at same rate as size
 - cooling is difficult
 - CMOS technology: Switching a wire from 0 to 1 consumes power (known as *dynamic power*)
 - $P = k * CV^2f$
 - k: constant; C: capacitance of wires; V: voltage; f: switching frequency
 - Power reduction methods
 - Reduce voltage: But slower, and there's a limit
 - What else?

Using Low-Power Gates on Non-Critical Paths

- **Another method: Use low-power gates**
 - Multiple versions of gates may exist
 - Fast/high-power, and slow/low-power, versions
 - Use **slow/low-power gates on non-critical paths**
 - Reduces power, without increasing delay

