VE281

Data Structures and Algorithms

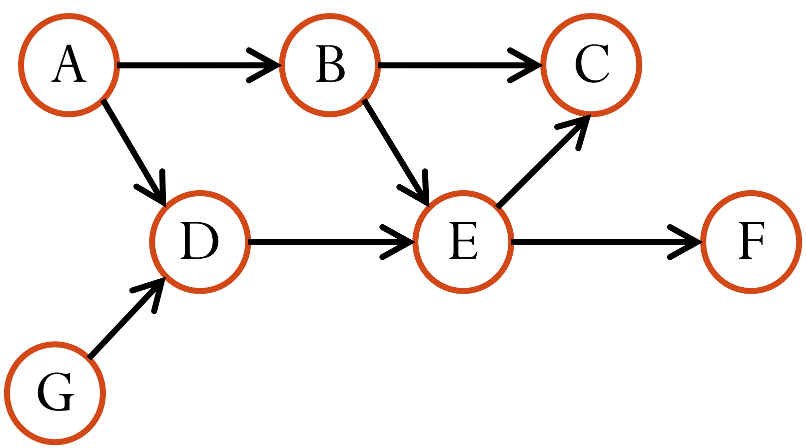
Topological Sorting

Learning Objectives:

- Know what a topological sorting is and why it is useful
- Know the topological sorting algorithm and its runtime complexity

Topological Sorting

- **Topological sorting** : an ordering on nodes of a **directed graph** so that **<u>for each</u>** edge (v_i, v_j) (means: an edge **from** v_i **to** v_j) in the graph, v_i is before v_j in the ordering.
  - Also known as **topological ordering**.

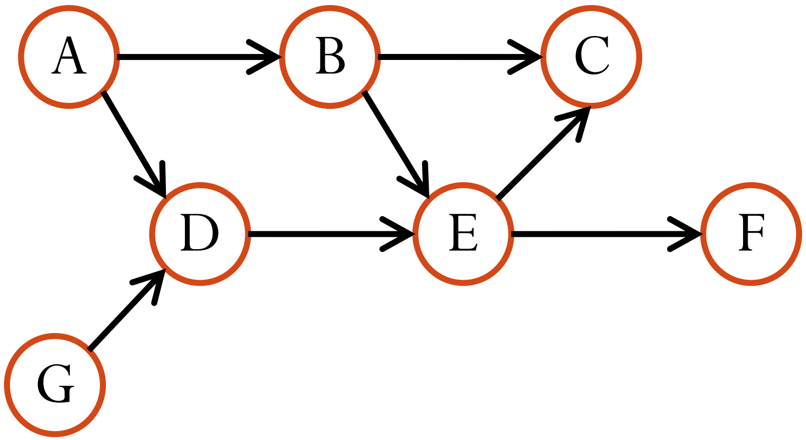


A topological sorting is: A, G, D, B, E, C, F

Which Graph Has Topological Sorting?

- Is there any "topological sorting" for directed graph **with cycles**?
  - In other words, can we order the nodes so that for each edge (v_i, v_j), v_i is before v_j in the ordering?
  - **Answer**: **No!** (Why?)
- How about **directed acyclic graph (DAG)**?
  - Yes! Guarantee to have a topological ordering.
  - Why? There is always a **source node** S in a DAG. Put S first. For the graph without S, again, there is a source node. Put it next …
- Next, we will focus on topological sorting on **DAG**.

Topological Sorting



- Topological sorting is not necessarily **unique**:
  - A, G, D, B, E, C, F and A, B, G, D, E, F, C are both topological sorting.
- Are the following orderings topological sorting?
  - A, B, E, G, D, C, F
  - A, G, B, D, E, F, C

Topological Sorting

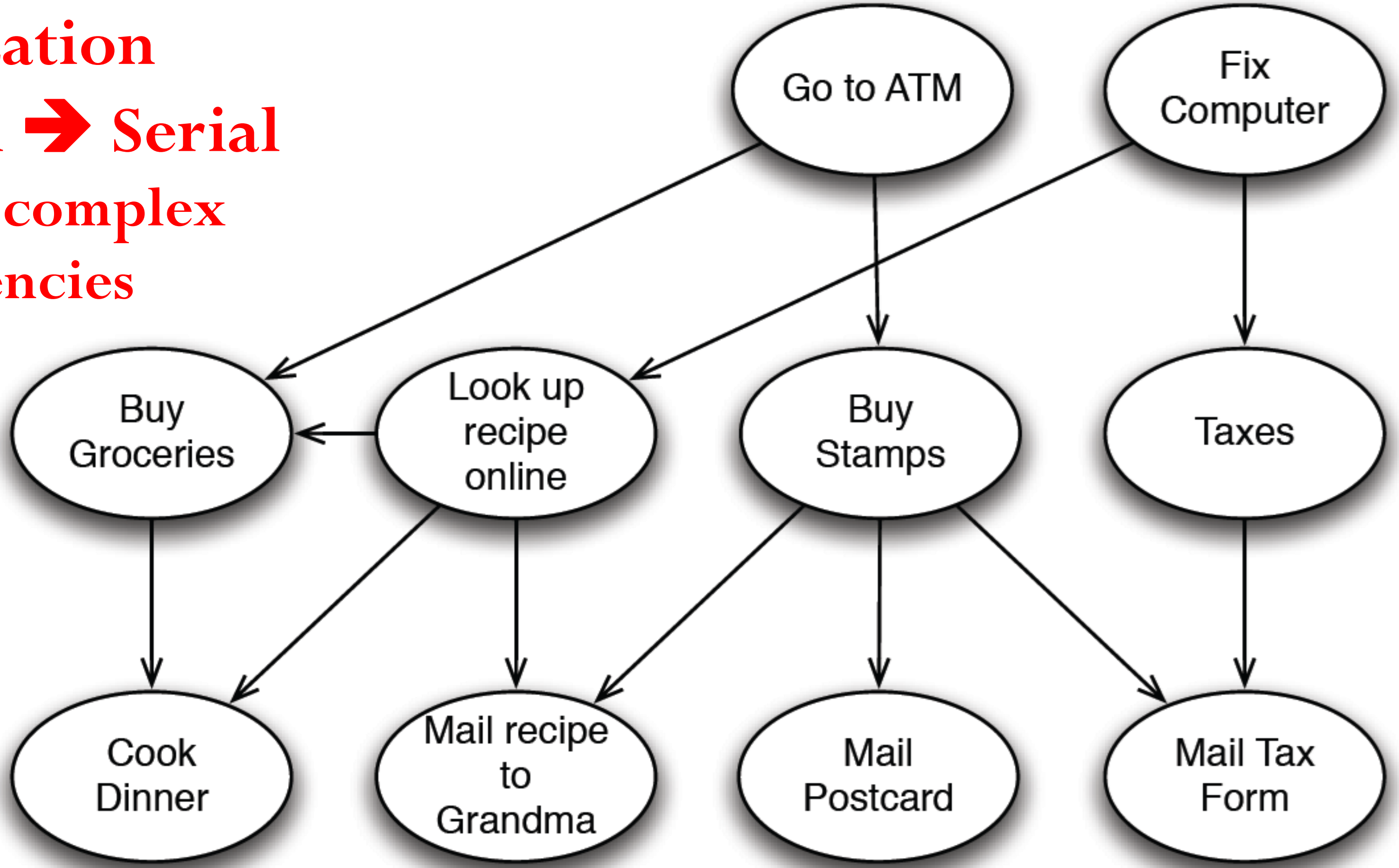
Applications

- Scheduling tasks when some tasks depend on other tasks being completed.

Serialization

Parallel ➔ Serial

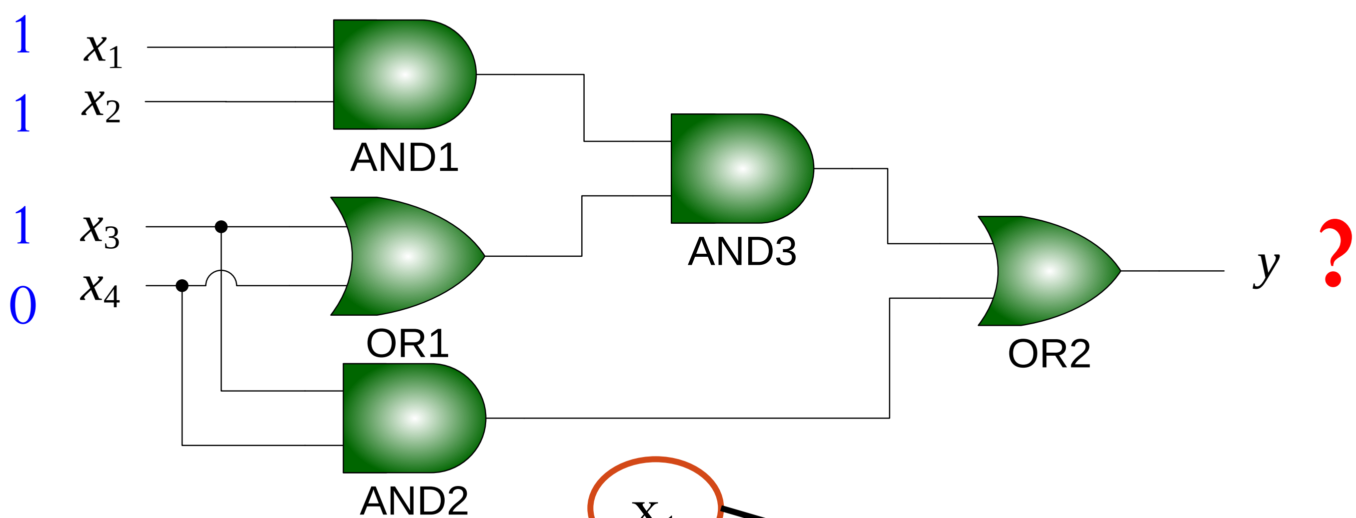
Sort out complex dependencies

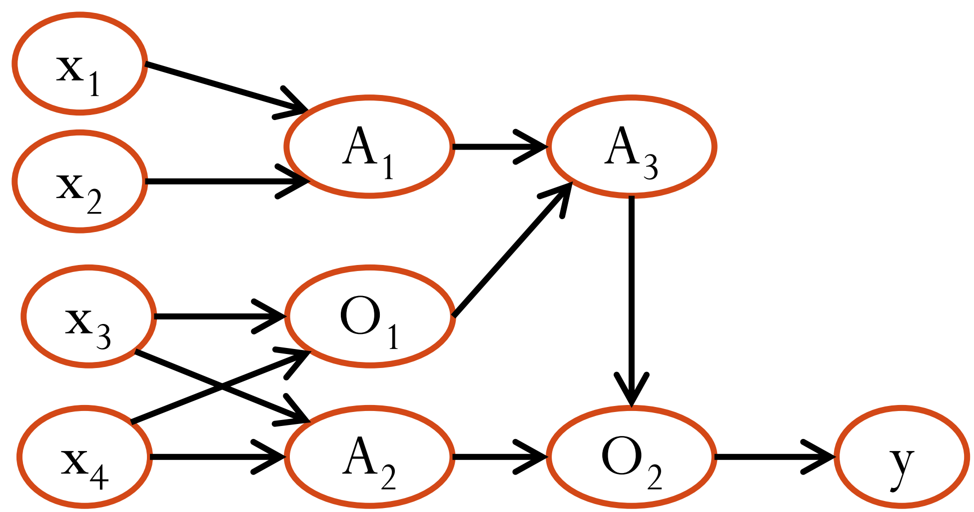


Topological Sorting

Applications

- Evaluating a combination logic circuit given a set of inputs.

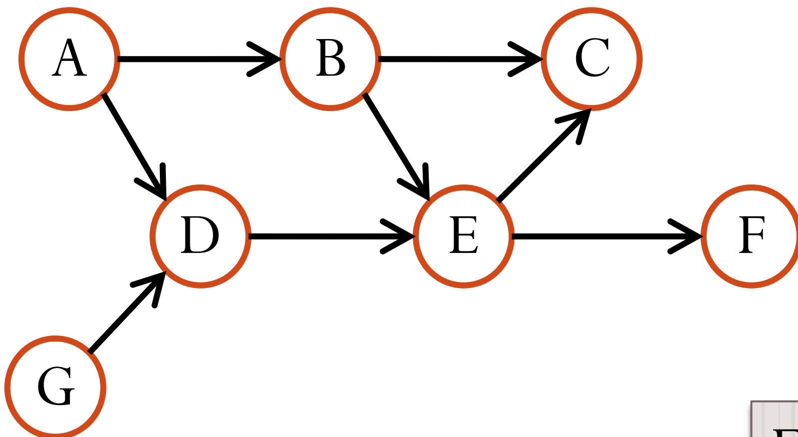




Topological Sorting: Algorithm

- Based on a **queue**.
- Algorithm:
  1. Compute the in-degrees of all nodes. (**in-degree**: number of **incoming** edges of a node.)
  2. **Enqueue** all in-degree 0 nodes into a queue.
  3. While queue is not empty
     1. **Dequeue** a node v from the queue and visit it.
     2. Decrement in-degrees of node v's neighbors.
     3. If any neighbor's in-degree becomes 0, **enqueue** it into the queue.

Topological Sorting Algorithm
Example



Queue

Enqueue A and G

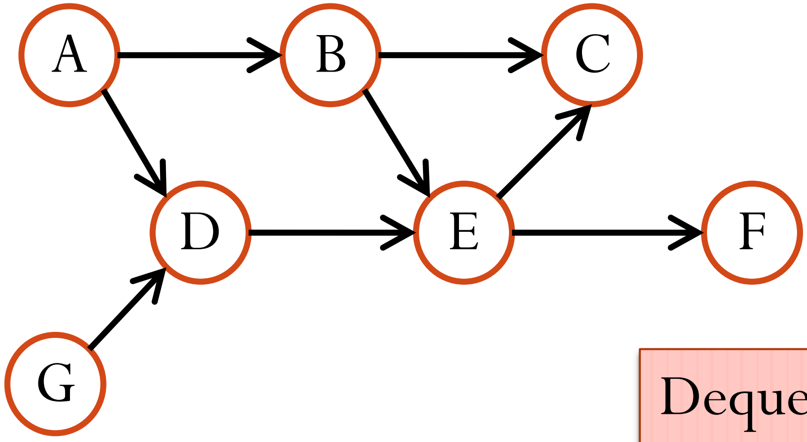
In-degrees

A	B	C	D	E	F	G
0	1	2	2	2	1	0

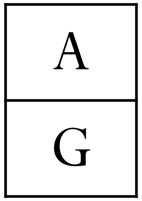
Order

Topological Sorting Algorithm

Example



Queue



Dequeue A, visit A, and decrement in-degrees of A's neighbors.

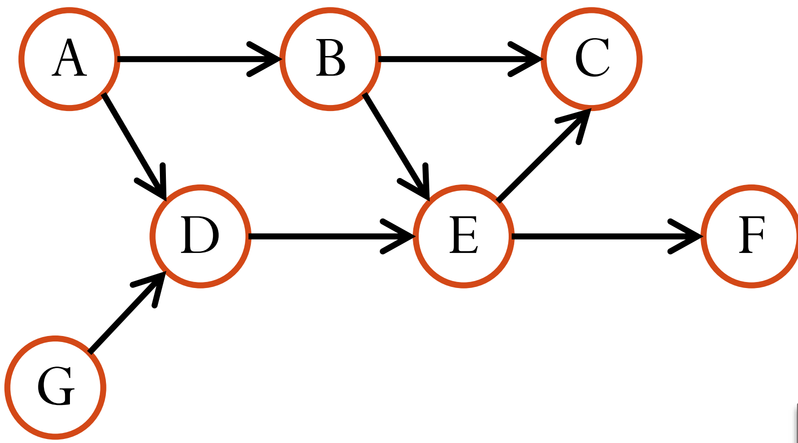
In-degrees

A	B	C	D	E	F	G
0	1	2	2	2	1	0

Order

Topological Sorting Algorithm

Example



Queue

G

Enqueue B

In-degrees

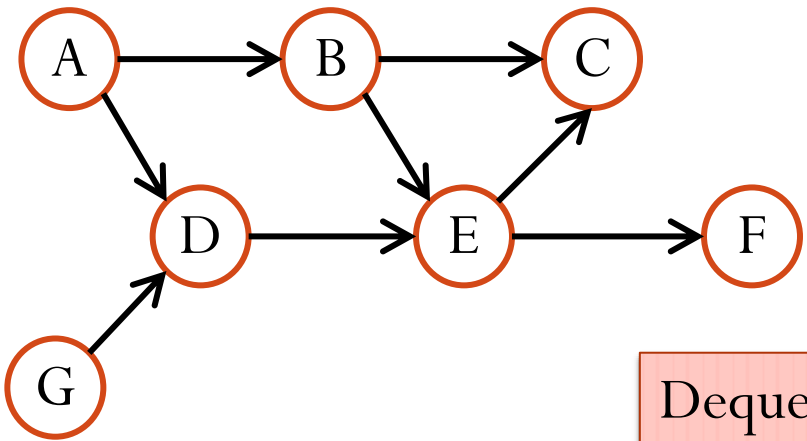
A	B	C	D	E	F	G
0	~~1~~ 0	2	~~2~~ 1	2	1	0

Order

A						

Topological Sorting Algorithm

Example



Queue

G
B

Dequeue G, visit G, and decrement in-degrees of G's neighbors.

In-degrees

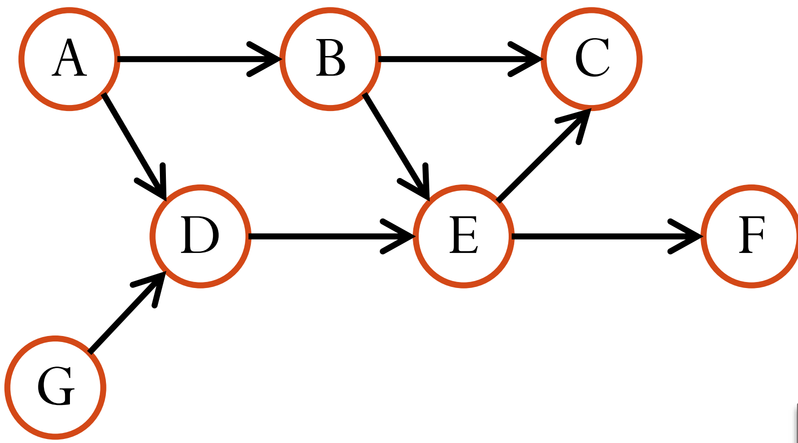
A	B	C	D	E	F	G
0	0	2	1	2	1	0

Order

A						

Topological Sorting Algorithm

Example



Queue

B

Enqueue D

In-degrees

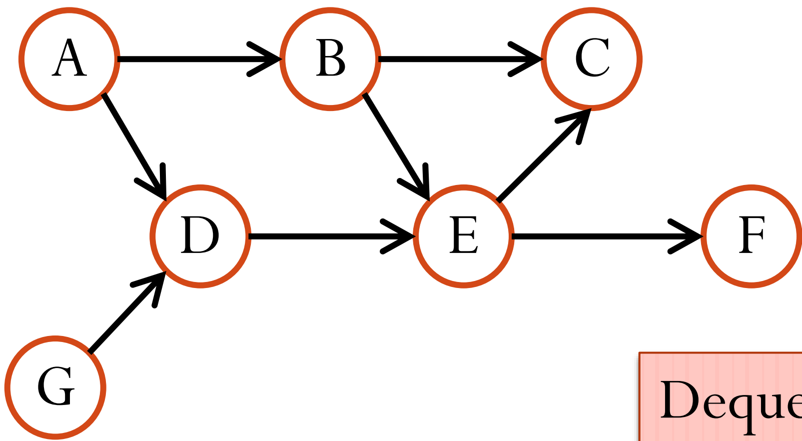
A	B	C	D	E	F	G
0	0	2	~~1~~ 0	2	1	0

Order

A	G					

Topological Sorting Algorithm

Example



Queue

B
D

Dequeue B, visit B, and decrement in-degrees of B's neighbors.

In-degrees

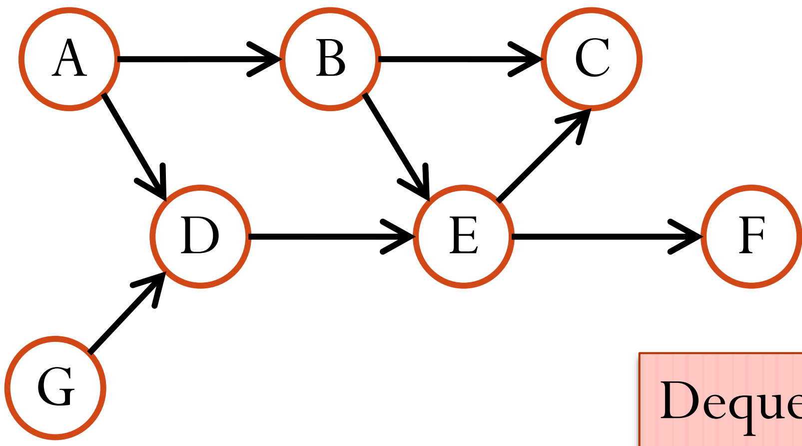
A	B	C	D	E	F	G
0	0	2	0	2	1	0

Order

A	G					

Topological Sorting Algorithm

Example



Queue

D

Dequeue D, visit D, and decrement in-degrees of D's neighbors.

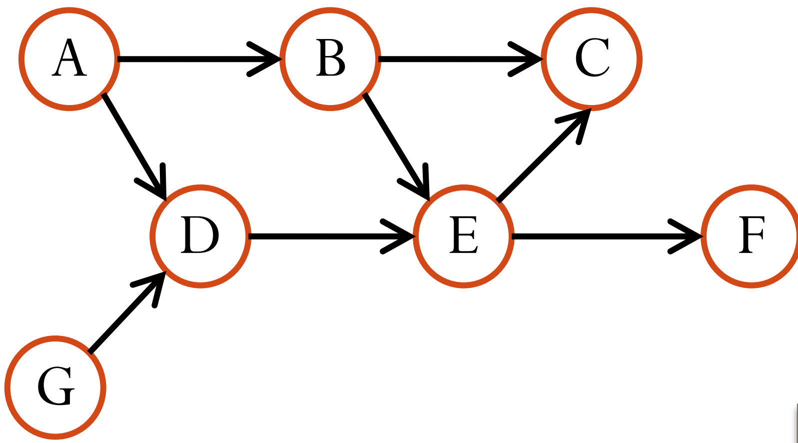
In-degrees

A	B	C	D	E	F	G
0	0	~~2~~ 1	0	~~2~~ 1	1	0

Order

A	G	**B**				

Topological Sorting Algorithm
Example



Queue

Enqueue E

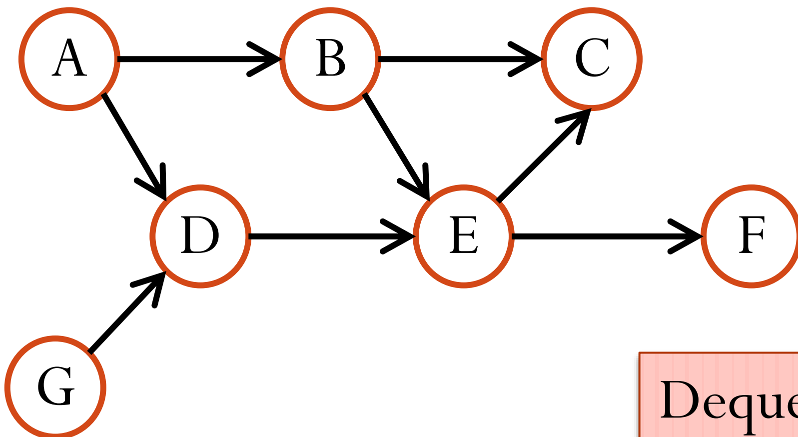
In-degrees

A	B	C	D	E	F	G
0	0	1	0	~~1~~ 0	1	0

Order

A	G	B	**D**			

Topological Sorting Algorithm
Example



Queue

E

Dequeue E, visit E, and decrement in-degrees of E's neighbors.

In-degrees

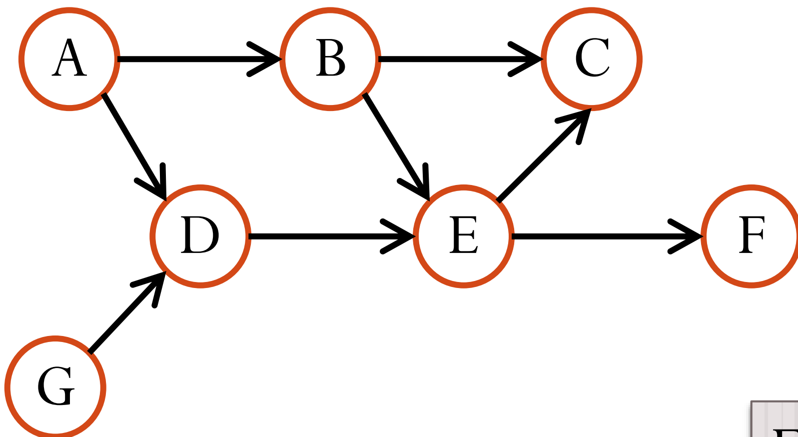
A	B	C	D	E	F	G
0	0	1	0	0	1	0

Order

A	G	B	D			

Topological Sorting Algorithm

Example



Queue

Enqueue C and F

In-degrees

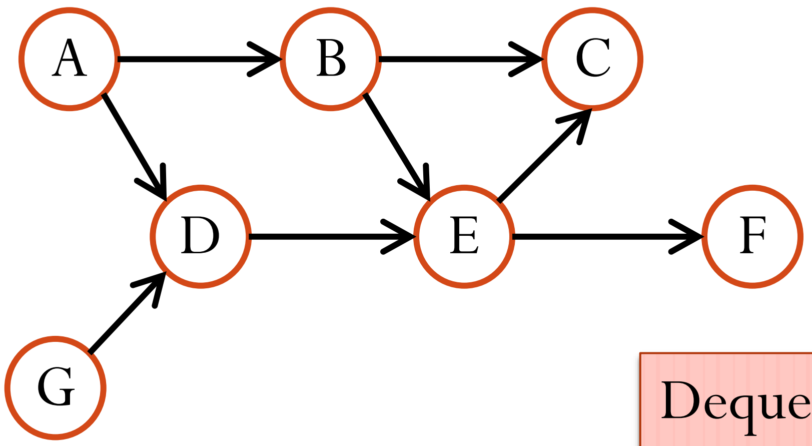
A	B	C	D	E	F	G
0	0	~~1~~ 0	0	0	~~1~~ 0	0

Order

A	G	B	D	**E**		

Topological Sorting Algorithm

Example



Queue

C
F

Dequeue C, visit C, and decrement in-degrees of C's neighbors.

In-degrees

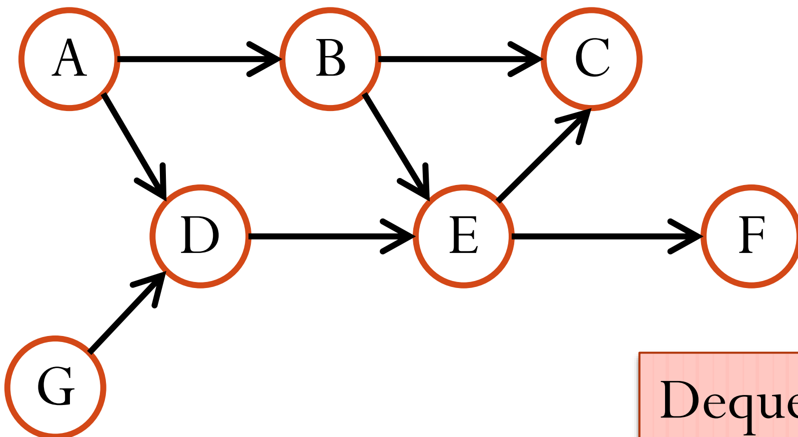
A	B	C	D	E	F	G
0	0	0	0	0	0	0

Order

A	G	B	D	E		

Topological Sorting Algorithm

Example



Queue

F

Dequeue F, visit F, and decrement in-degrees of F's neighbors.

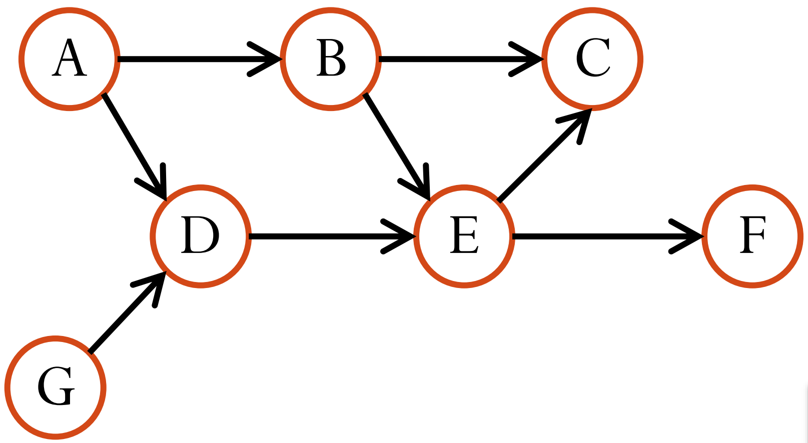
In-degrees

A	B	C	D	E	F	G
0	0	0	0	0	0	0

Order

A	G	B	D	E	C	

Topological Sorting Algorithm
Example



Queue

Queue is now empty. Done!

In-degrees

A	B	C	D	E	F	G
0	0	0	0	0	0	0

Order

A	G	B	D	E	C	F

Topological Sorting Algorithm

Time Complexity

Assume adjacency list representation

1. Compute the in-degrees of all nodes. $O(|V| + |E|)$ in total
2. Enqueue all in-degree 0 nodes into a queue. $O(|V|)$ in total
3. While queue is not empty
    1. Dequeue a node v from the queue and visit it. $O(|V|)$ in total
    2. Decrement in-degrees of node v's neighbors. $O(|E|)$ in total
    3. If any neighbor's in-degree becomes 0 … $O(|V|)$ in total
        - … place it in the queue.

Total running time is $O(|V| + |E|)$.