VE281

Data Structures and Algorithms

Dynamic Programming

Learning Objectives:

- Understand the basic idea of dynamic programming
- Know under what situation dynamic programming could be applied

Outline

- Dynamic Programming
  - Motivation
  - Example: Matrix-Chain Multiplication
  - Summary

Limitation of Divide and Conquer

- Recursively solving subproblems can result in the same computations being repeated when the subproblems **overlap.**

- For example: computing the **Fibonacci sequence**

$$f_0 = 0;\ \ f_1 = 1;\ \ f_n = f_{n-1} + f_{n-2}, n \geq 2$$

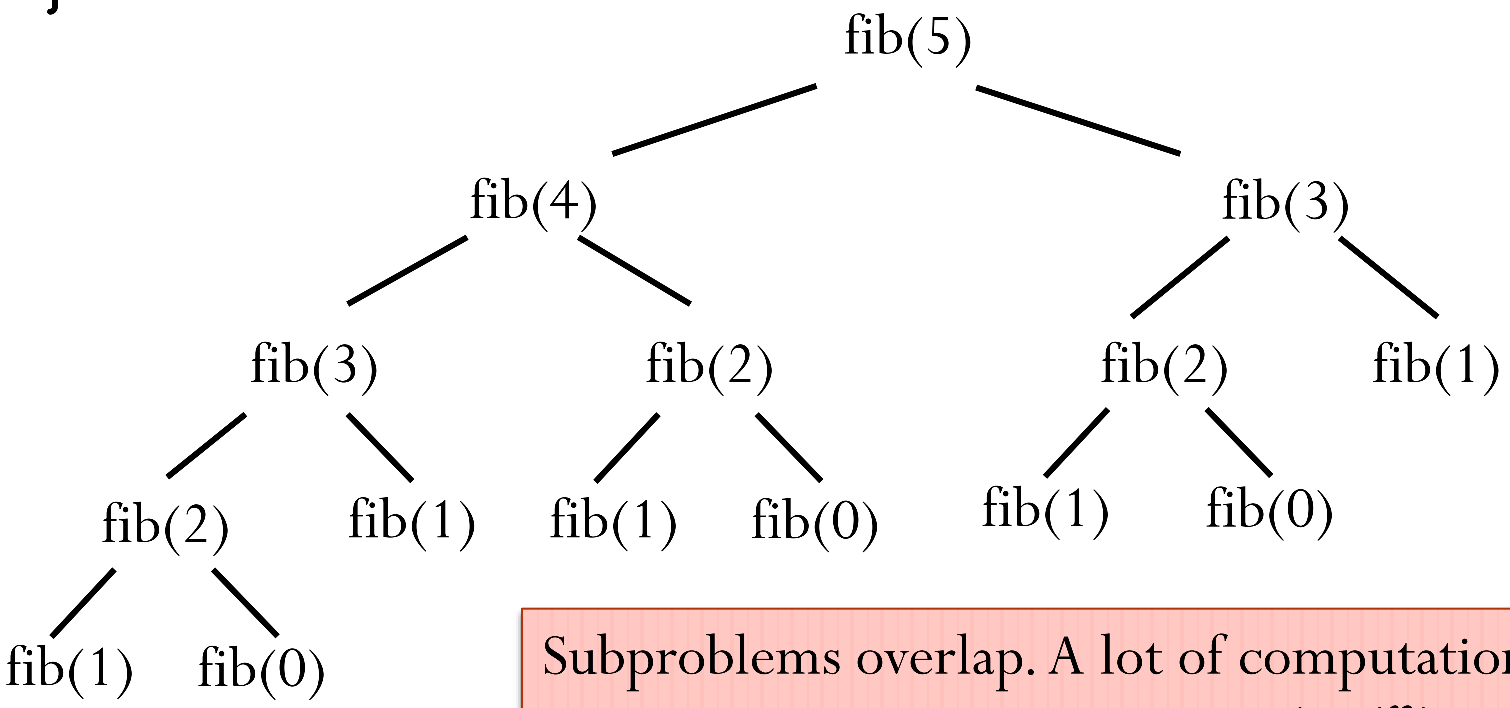
- Divide and conquer approach:

```
int fib(int n) {
   if(n <= 1) return n;
   return fib(n-1)+fib(n-2);
}
```

Fibonacci Sequence

Divide and Conquer Solution

```
int fib(int n) {
  if(n <= 1) return n;
  return fib(n-1)+fib(n-2);
}
```



Subproblems overlap. A lot of computation is wasted. Time complexity is $\Omega(1.5^n)$.

Fibonacci Sequence

Iterative Solution

- We can also compute the Fibonacci sequence in iterative way:

```
int fib(int n) {
   f[0] = 0; f[1] = 1;
   for(i = 2 to n)
      f[i] = f[i-1]+f[i-2];
   return f[n];
}
```

- Time complexity is $\Theta(n)$.

Dynamic Programming

- Used when a problem can be divided into subproblems that **overlap**.
  - Solve each subproblem **once** and store the solution in a table.
  - If a subproblem is encountered **again**, simply look up its solution in the table.
  - Reconstruct the solution to the original problem from the solutions to the subproblems.
- The more overlap the better, as this reduces the number of subproblems.
- Dynamic programming can be applied to solve **optimization problem**.

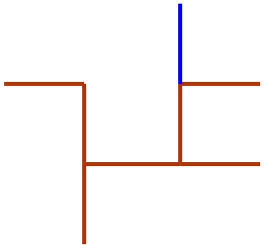
Optimization Problem

- Many problems we encounter are **optimization problems**:
  - A problem in which some function (called the **objective function**) is to be optimized (usually minimized or maximized) subject to some **constraints**.
- The solutions that satisfy the constraints are called **feasible solutions**.
- The number of feasible solutions is typically very large.
- We obtain the optimal solution by **searching** the feasible solution space.

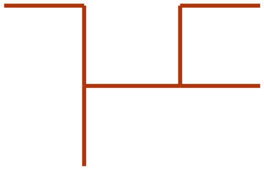
Optimization Problem

Example

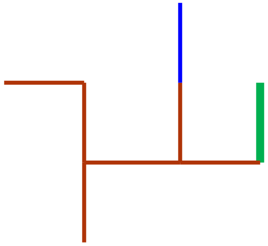
- Minimum spanning tree.
  - Objective function: the sum of all edge weights.
  - Constraints: a subgraph of a MST must also be MSTs.



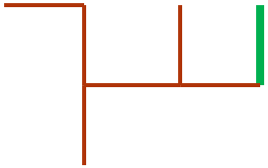
MST



MST



MST

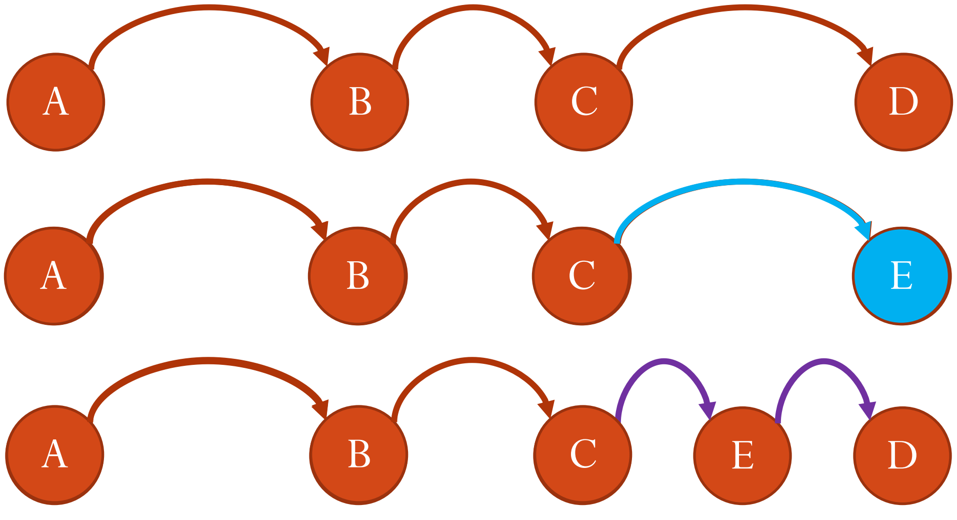


MST

Optimization Problem

Example

- Shortest path.
  - Objective function: the sum of all edge weights.
  - Constraints: a subgraph of a shortest path must also be shortest paths.



shortest path

shortest paths

shortest paths

Takeaway: Dynamic Programming is often linked with Induction!
Book-keep partial results to avoid redundant computation!

Outline

- Dynamic Programming
  - Motivation
  - Example: Matrix-Chain Multiplication
  - Summary

Matrix-Chain Multiplication

- What is the cost of multiplying two matrices A and B?
  - Suppose A is a $p \times q$ matrix and B is a $q \times r$ matrix.
  - Since the time to compute $C = AB$ is dominated by the number of **scalar multiplications**, we use the number of scalar multiplications as the complexity measure.

- $C_{ij} = \sum_{k=1}^{q} A_{ik} B_{kj}$.
  - We need q scalar multiplications to calculate C_{ij}.
  - C is of size $p \times r$.
- The number of scalar multiplications is pqr.

Matrix-Chain Multiplication

- Now how would you compute the multiplication of three matrices $A \times B \times C$?
  - Suppose A is of size 100×1, B is of size 1×100, and C is of size 100×1.

- If we multiply as $(A \times B) \times C$, the number of scalar multiplications is 20000.
- If we multiply as $A \times (B \times C)$, the number of scalar multiplications is 200.

Matrix-Chain Multiplication

- If we want to multiply a chain of matrices $A_1 \times A_2 \times \cdots \times A_n$, where A_i is of size $p_{i-1} \times p_i$, what is the best order of multiplication to minimize the number of scalar multiplications?

- This is an optimization problem.
- It can be proved that number of different orders on n matrices is $\Omega(4^n / n^{1.5})$.
- Instead of **enumerating** all of the orders, can we do better to solve the optimization problem?

Matrix-Chain Multiplication

- For simplicity, define the problem of finding the optimal order to multiply $A_i \times A_{i+1} \times \cdots \times A_j$ as $Q_{i\sim j}$. The minimal number of scalar multiplications is $m_{i\sim j}$.
  - We ultimately want to solve $Q_{1\sim n}$.

Matrix-Chain Multiplication

- Suppose in the optimal order for $A_i \times \cdots \times A_j$, the **last** multiplication is $(A_i \times \cdots \times A_k) \times (A_{k+1} \times \cdots \times A_j)$.
- Then the order of computing $A_i \times \cdots \times A_k$ in the **optimal** order of computing $A_i \times \cdots \times A_j$ must be an **optimal** order to compute $A_i \times \cdots \times A_k$.
  - Why?
    - If not, then we copy and paste the better order ➔ we have a better order for computing $A_i \times \cdots \times A_j$!
    - Similar conclusion for computing $A_{k+1} \times \cdots \times A_j$.

- If we know k, we can divide the problem $Q_{i\sim j}$ into two smaller instances: $Q_{i\sim k}$ and $Q_{(k+1)\sim j}$.

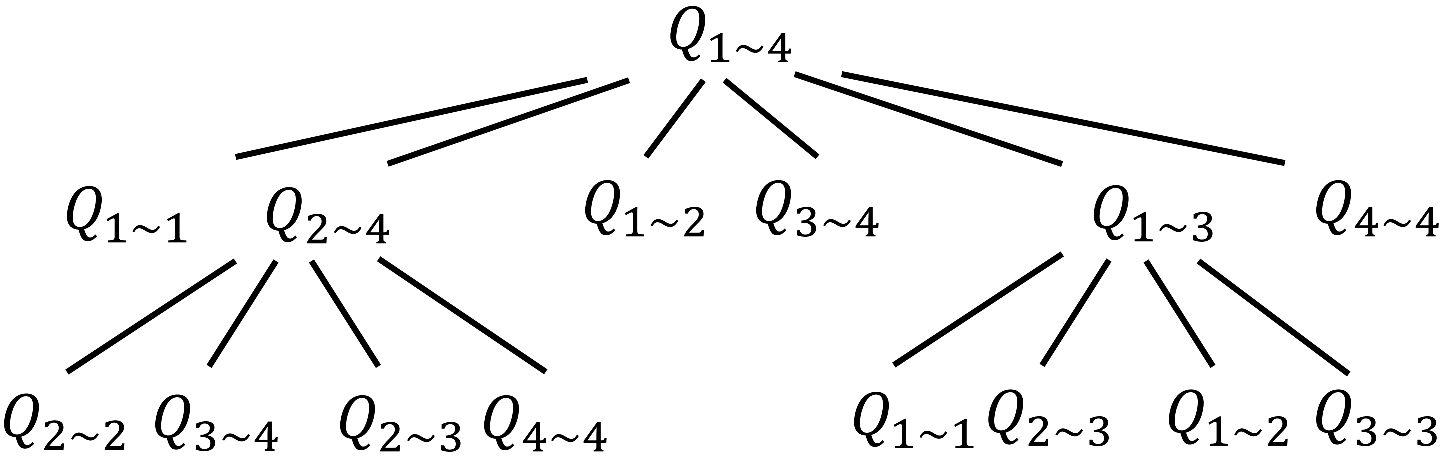
Matrix-Chain Multiplication

- Assume we have known the minimum number of scalar multiplications for $Q_{i\sim k}$ and $Q_{(k+1)\sim j}$ as $m_{i\sim k}$ and $m_{(k+1)\sim j}$.
  - Then $m_{i\sim j} = m_{i\sim k} + m_{(k+1)\sim j} + p_{i-1}p_k p_j$.

- **However, we don't know *k*!** We need to consider all possible divisions, i.e., all $i \leq k \leq j-1$.
- Thus, in order to solve $Q_{i\sim j}$, we need to consider all subproblems $Q_{i\sim k}$ and $Q_{(k+1)\sim j}$, for all $i \leq k \leq j-1$.
  - $m_{i\sim j} = \min_{i\leq k\leq j-1}(m_{i\sim k} + m_{(k+1)\sim j} + p_{i-1}p_k p_j)$

Matrix-Chain Multiplication

- In summary, we can divide the problem into subproblems of the same form.



$Q_{1\sim4}$

$Q_{1\sim1}$ $Q_{2\sim4}$ $Q_{1\sim2}$ $Q_{3\sim4}$ $Q_{1\sim3}$ $Q_{4\sim4}$

$Q_{2\sim2}$ $Q_{3\sim4}$ $Q_{2\sim3}$ $Q_{4\sim4}$ $Q_{1\sim1}$ $Q_{2\sim3}$ $Q_{1\sim2}$ $Q_{3\sim3}$

Many subproblems are overlapped.

Matrix-Chain Multiplication

- The straightforward recursive algorithm has exponential time complexity.
  - However, it will encounter each subproblem many times in different branches of the tree.

- The total number of different subproblems is not exponential.
  - They are $Q_{i\sim j}$, for $1 \leq i \leq j \leq n$.
  - The total number is $n(n+1)/2$.
- Instead, we use a **tabular**, **bottom-up** approach.

Matrix-Chain Multiplication

Bottom-up Approach

- Apply the recursive relation:

$$m_{i\sim j} = \min_{i\leq k\leq j-1}(m_{i\sim k} + m_{(k+1)\sim j} + p_{i-1}p_k p_j)$$

- Initial situation $m_{1\sim 1} = m_{2\sim 2} = \cdots = m_{n\sim n} = 0$.
- In the first round, we compute $m_{1\sim 2}, m_{2\sim 3}, \dots, m_{(n-1)\sim n}$.
- In the second round, we compute $m_{1\sim 3}, m_{2\sim 4}, \dots, m_{(n-2)\sim n}$.
- So on and so forth. In the l-th round, we compute $m_{1\sim (l+1)}, m_{2\sim (l+2)}, \dots, m_{(n-l)\sim n}$.
- Finally, we compute $m_{1\sim n}$.
- To obtain the multiplication order, we also record the partition k which gives the minimal $m_{i\sim j}$ as $s_{i\sim j}$.

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

j

i

m_{ij}	1	2	3	4
1	0			
2	–	0		
3	–	–	0	
4	–	–	–	0

j

i

k_{ij}	1	2	3	4
1	–			
2	–	–		
3	–	–	–	
4	–	–	–	–

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

j

m_{ij}		1	2	3	4
	1	0			
i	2	–	0		
	3	–	–	0	
	4	–	–	–	0

j

k_{ij}		1	2	3	4
	1	–			
i	2	–	–		
	3	–	–	–	
	4	–	–	–	–

$$\begin{aligned} m_{i\sim(i+1)} &= m_{i\sim i} + m_{(i+1)\sim(i+1)} + p_{i-1}p_ip_{i+1} \\ &= p_{i-1}p_ip_{i+1} \end{aligned}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
	1	0	100		
i	2	–	0	10	
	3	–	–	0	200
	4	–	–	–	0

k_{ij}			j		
		1	2	3	4
	1	–	1		
i	2	–	–	2	
	3	–	–	–	3
	4	–	–	–	–

$$\begin{aligned} m_{i\sim(i+1)} &= m_{i\sim i} + m_{(i+1)\sim(i+1)} + p_{i-1}p_ip_{i+1} \\ &= p_{i-1}p_ip_{i+1} \end{aligned}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
	1	0	100		
i	2	–	0	10	
	3	–	–	0	200
	4	–	–	–	0

k_{ij}			j		
		1	2	3	4
	1	–	1		
i	2	–	–	2	
	3	–	–	–	3
	4	–	–	–	–

$$m_{i\sim(i+2)} = \min\{m_{i\sim i} + m_{(i+1)\sim(i+2)} + p_{i-1}p_ip_{i+2},$$
$$m_{i\sim(i+1)} + m_{(i+2)\sim(i+2)} + p_{i-1}p_{i+1}p_{i+2}\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij} j

		1	2	3	4
	1	0	100		
i	2	−	0	10	
	3	−	−	0	200
	4	−	−	−	0

k_{ij} j

		1	2	3	4
	1	−	1		
i	2	−	−	2	
	3	−	−	−	3
	4	−	−	−	−

$$m_{1\sim3} = \min\{m_{1\sim1} + m_{2\sim3} + p_0p_1p_3,$$

$$m_{1\sim2} + m_{3\sim3} + p_0p_2p_3\} = \min\{20, 200\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

			j		
m_{ij}		1	2	3	4
	1	0	100	20	
i	2	−	0	10	
	3	−	−	0	200
	4	−	−	−	0

			j		
k_{ij}		1	2	3	4
	1	−	1	1	
i	2	−	−	2	
	3	−	−	−	3
	4	−	−	−	−

$$\begin{aligned} m_{1\sim 3} = \min\{ & m_{1\sim 1} + m_{2\sim 3} + p_0 p_1 p_3, \\ & m_{1\sim 2} + m_{3\sim 3} + p_0 p_2 p_3\} = \min\{20, 200\} \end{aligned}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij} j

i	1	2	3	4
1	0	100	20	
2	−	0	10	
3	−	−	0	200
4	−	−	−	0

k_{ij} j

i	1	2	3	4
1	−	1	1	
2	−	−	2	
3	−	−	−	3
4	−	−	−	−

$$m_{2\sim 4} = \min\{m_{2\sim 2} + m_{3\sim 4} + p_1 p_2 p_4,$$

$$m_{2\sim 3} + m_{4\sim 4} + p_1 p_3 p_4\} = \min\{400, 30\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

j

m_{ij}		1	2	3	4
i	1	0	100	20	
	2	–	0	10	30
	3	–	–	0	200
	4	–	–	–	0

j

k_{ij}		1	2	3	4
i	1	–	1	1	
	2	–	–	2	3
	3	–	–	–	3
	4	–	–	–	–

$$m_{2\sim 4} = \min\{m_{2\sim 2} + m_{3\sim 4} + p_1 p_2 p_4,$$

$$m_{2\sim 3} + m_{4\sim 4} + p_1 p_3 p_4\} = \min\{400, 30\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
	1	0	100	20	
i	2	—	0	10	30
	3	—	—	0	200
	4	—	—	—	0

k_{ij}			j		
		1	2	3	4
	1	—	1	1	
i	2	—	—	2	3
	3	—	—	—	3
	4	—	—	—	—

$$m_{i\sim(i+3)} = \min_{i\le k\le i+2}\left(m_{i\sim k} + m_{(k+1)\sim(i+3)} + p_{i-1}p_k p_{(i+3)}\right)$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
	1	0	100	20	
i	2	–	0	10	30
	3	–	–	0	200
	4	–	–	–	0

k_{ij}			j		
		1	2	3	4
	1	–	1	1	
i	2	–	–	2	3
	3	–	–	–	3
	4	–	–	–	–

$$m_{1\sim4} = \min_{1\le k\le 3}\left(m_{1\sim k} + m_{(k+1)\sim4} + p_0 p_k p_4\right)$$
$$= \min\{230, 2300, 220\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
i	1	0	100	20	220
	2	–	0	10	30
	3	–	–	0	200
	4	–	–	–	0

Optimal Value

k_{ij}			j		
		1	2	3	4
i	1	–	1	1	3
	2	–	–	2	3
	3	–	–	–	3
	4	–	–	–	–

$$m_{1\sim4} = \min_{1\le k\le 3}(m_{1\sim k} + m_{(k+1)\sim4} + p_0p_kp_4)$$
$$= \min\{230, 2300, 220\}$$

Matrix-Chain Multiplication

Constructing an Optimal Order

- We can construct an optimal order based on the records s_{ij}.

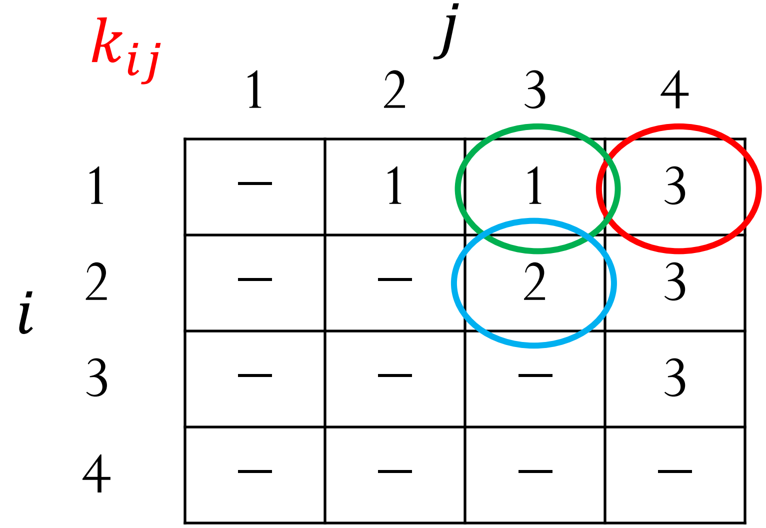
```
Print_Order(s, i, j) {
   if(i == j) cout << "A_i";
   else {
      cout << "(";
      Print_Order(s, i, s_ij);
      cout << "*";
      Print_Order(s, s_ij+1, j);
      cout << ")";
   }
}
```

- Initial call is **`Print_Order(s, 1, n);`**

Matrix-Chain Multiplication

Example

- Construct an optimal order
  - $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
  - $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$



k_{ij} j

i	1	2	3	4
1	–	1	1	3
2	–	–	2	3
3	–	–	–	3
4	–	–	–	–

$k_{14} = 3$ → $A_1 \times A_2 \times A_3 \times A_4 = (A_1 \times A_2 \times A_3) \times A_4$

$k_{13} = 1$ → $A_1 \times A_2 \times A_3 = A_1 \times (A_2 \times A_3)$

$k_{23} = 2$ → $A_2 \times A_3 = A_2 \times A_3$

$$A_1 \times A_2 \times A_3 \times A_4 = (A_1 \times (A_2 \times A_3)) \times A_4$$

Matrix-Chain Multiplication

Time Complexity

- Get the minimum number of scalar multiplications:
  - We need to obtain all m_{ij} and s_{ij}, for $1 \leq i \leq j \leq n$.
    - $O(n^2)$ records
    - Each $m_{i\sim j}$ is the minimum of $O(n)$ terms.
  - Total time complexity is $O(n^3)$.

- Obtain the optimal order:
  - $O(n)$

Matrix-Chain Multiplication

Summary

- Matrix-chain multiplication is an optimization problem.

- The solution is based on **dynamic programming**.
  - The original problem can be divided into same subproblems that **overlap**.
  - Each subproblem is solved once and stored in a table.
  - If a subproblem is encountered again, simply look up its solution in the table.
  - Reconstruct the solution to the original problem from the solutions to the subproblems.

Outline

- Dynamic Programming
  - Motivation
  - Example: Matrix-Chain Multiplication
  - Summary

Dynamic Programming for Optimization

- There are two key ingredients that an optimization problem must have in order for dynamic programming to apply:
  - Optimal substructure;
  - Overlapping subproblems.

Optimal Substructure

- An optimal solution to the problem contains **within** it **optimal solutions to subproblems**.
  - In matrix-chain multiplication, the optimal order on calculating $A_i \times \cdots \times A_j$ that splits the product between A_k and A_{k+1} contains within it optimal solutions to the problem of ordering $A_i \times \cdots \times A_k$ and $A_{k+1} \times \cdots \times A_j$.

- You can show optimal substructure property by supposing that each of the subproblem solutions is not optimal and then deriving a contradiction.

Overlapping Subproblems

- A recursive algorithm for the problem solves the same subproblems **over and over**, rather than always generating new subproblems.
  - E.g., subproblems of matrix-chain multiplication overlap.
  - In contrast, a problem for which a divide-and-conquer approach is suitable usually generates **brand-new** problems at each step of the recursion.
- Dynamic-programming algorithms take advantage of overlapping subproblems by
  - solving each subproblem once …
  - … and then storing the solution in a table where it can be looked up when needed.

Designing a Dynamic-Programming Algorithm

1. Characterize **the structure** of an optimal solution.
   - Usually, we need to define a **general** problem.
2. **Recursively** define the value of an optimal solution.
3. Compute the value of an optimal solution, typically in a **bottom-up** fashion.
4. Construct an optimal solution from computed information.