Transformation of Random Variables and Reliability

Transformation of Variables

The following theorem allows us to perform transformations of random variables and obtain the densities of the transformed variables.

10.1. Theorem. Let $(\boldsymbol{X}, f_{\boldsymbol{X}})$ be a continuous multivariate random variable and let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable, bijective map with inverse φ^{-1}. Then $\boldsymbol{Y} = \varphi \circ \boldsymbol{X}$ is a continuous multivariate random variable with density

$$f_{\boldsymbol{Y}}(y) = f_{\boldsymbol{X}} \circ \varphi^{-1}(y) \cdot |\det D\varphi^{-1}(y)|,$$

where $D\varphi^{-1}$ is the Jacobian of φ^{-1}.

We will not prove this theorem, which is based on the substitution rule for multivariable integrals.

Transformation of Variables

We can use transformation of bivariate random variables to obtain densities of sums and products of random variables, as the following example shows:

10.2. Lemma. Let $((X, Y), f_{XY})$ be a continuous bivariate random variable. Let $U = X/Y$. Then the density f_U of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(uv, v) \cdot |v| \, dv.$$

Proof.
Consider the transformation $\varphi\colon (X, Y) \mapsto (U, V)$ where

$$\varphi(x, y) = \begin{pmatrix} x/y \\ y \end{pmatrix}.$$

Then

$$\varphi^{-1}(u, v) = \begin{pmatrix} uv \\ v \end{pmatrix}.$$

Transformation of Variables

Proof.

We calculate

$$D\varphi^{-1}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}$$

so

$$|\det D\varphi^{-1}(u,v)| = |v|.$$

Then

$$f_{UV}(u,v) = f_{XY}(uv,v)|v|.$$

The marginal density f_U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_{-\infty}^{\infty} f_{XY}(uv,v)\cdot|v|\,dv. \qquad \square$$

The Chi Random Variable

Consider the following problem: a point $z = (z_1, \dots, z_n)$ in $\mathbb{R}^n$ is randomly selected in such a way that every coordinate value z_i, $i = 1, \dots, n$, is determined independently of the other coordinates by a random variable Z_i. Suppose that each Z_i follows a standard normal distribution.

We are interested in the distribution function of the random variable

$$\chi_n := \sqrt{\sum_{i=1}^{n} Z_i^2}$$

which describes the distance of the selected point from the origin. For instance, while the expected value of each coordinate is $\mathrm{E}[Z_i] = 0$, we do not know the expected distance from the origin, $\mathrm{E}[\chi_n]$.

We say that χ_n is a ***chi random variable*** and that it follows a ***chi distribution with n degrees of freedom.***

The Chi Distribution

To find the density f_{χ_n}, we consider the cumulative distribution function F_{χ_n},

$$F_{\chi_n}(y) = P[\chi_n \le y].$$

Clearly, $F_{\chi_n}(y) = 0$ for $y < 0$. For $y \ge 0$,

$$\begin{aligned} F_{\chi_n}(y) &= P[\chi_n \le y] = P[\chi_n^2 \le y^2] = P\left[\sum_{k=1}^{n} Z_k^2 \le y^2\right] \\ &= \int_{\sum_{k=1}^{n} z_k^2 \le y^2} f_{Z_1 \dots Z_n}(z_1, \dots, z_n)\, dz_1 \dots dz_n \end{aligned}$$

Note that the n independent standard normal variables $Z_1, \dots, Z_n$ have joint density

$$f_{Z_1 \dots Z_n}(z_1, \dots, z_n) = \frac{1}{(2\pi)^{n/2}} e^{-\sum_{k=1}^{n} z_k^2/2}.$$

The Chi Distribution

We hence obtain

$$F_{\chi_n}(y) = \int\limits_{\sum_{k=1}^{n} z_k^2 \le y^2} (2\pi)^{-n/2} e^{-\sum_{k=1}^{n} z_k^2/2} \, dz_1 \dots dz_n.$$

It becomes convenient to introduce polar coordinates $(r, \theta_1, \dots, \theta_{n-1})$ with $r > 0$, $0 < \theta_{n-1} < 2\pi$ and $0 < \theta_k < \pi$ for $k = 1, \dots, n-2$ as follows:

$$\begin{aligned}
x_1 &= r \cos\theta_1 \\
x_2 &= r \sin\theta_1 \cos\theta_2 \\
x_3 &= r \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\vdots \\
x_{n-1} &= r \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2} \cos\theta_{n-1} \\
x_n &= r \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2} \sin\theta_{n-1}.
\end{aligned}$$

$x_n^2 + x_{n-1}^2 + \dots + x_2^2 + x_1^2 = r^2$

The Chi Distribution

The integral becomes

$$F_{\chi_n}(y) = \int_0^{2\pi} \int_{\cancel{-\pi/2}\ 0}^{\cancel{\pi/2}\ \pi} \cdots \int_{\cancel{-\pi/2}\ 0}^{\cancel{\pi/2}\ \pi} \int_0^y (2\pi)^{-n/2} e^{-r^2/2}\, r^{n-1} \times D(\theta_1, \dots, \theta_{n-1})\, dr\, d\theta_1 \dots d\theta_{n-2}\, d\theta_{n-1}$$

where $D(\theta_1, \dots, \theta_{n-1})$ is independent of r. Writing

$$C_n = (2\pi)^{-n/2} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} D(\theta_1, \dots, \theta_{n-1})\, d\theta_1 \dots d\theta_{n-2}\, d\theta_{n-1}$$

we have

$$F_{\chi_n}(y) = C_n \int_0^y e^{-r^2/2}\, r^{n-1}\, dr.$$

The Chi Distribution

We determine C_n from

$$1 = \lim_{y\to\infty} F_{\chi_n}(y) = C_n \int_0^\infty e^{-r^2/2}\, r^{n-1}\, dr = C_n \Gamma\left(\frac{n}{2}\right) 2^{n/2-1},$$

where we have substituted $\rho = r^2/2$ in the integral to obtain the gamma function. It follows that

$$F_{\chi_n}(y) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2-1}} \int_0^y e^{-r^2/2}\, r^{n-1}\, dr.$$

and the density of χ_n is given by

$$f_{\chi_n}(y) = F'_{\chi_n}(y) = \frac{2}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} y^{n-1} e^{-y^2/2}. \tag{10.1}$$

for $y \geq 0$ (and $f_{\chi_n}(y) = 0$ for $y < 0$).

The Chi-Squared Distribution

In statistics, we will be particularly interested in the ***chi-squared random variable*** with n degrees of freedom,

$$\chi_n^2 = \sum_{i=1}^{n} Z_i^2. \tag{10.2}$$

where again $Z_1, \dots, Z_n$ are independent standard normal random variables. Hence, a chi-squared random variable represents the sum of the squares of independent standard normal variables.

We obtain the density of χ_n^2 by again considering the cumulative distribution function: For $y \geq 0$,

$$\begin{aligned} F_{\chi_n^2}(y) &= P[\chi_n^2 \leq y] = P[-\sqrt{y} \leq \chi_n \leq \sqrt{y}] \\ &= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2-1}} \int_0^{\sqrt{y}} e^{-r^2/2}\, r^{n-1}\, dr \end{aligned}$$

The Chi-Squared Distribution

Differentiating and applying the chain rule, we have

$$\begin{aligned} f_{\chi_n^2}(y) = F'_{\chi_n^2}(y) &= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2-1}} \frac{d}{dy} \int_0^{\sqrt{y}} e^{-r^2/2}\, r^{n-1}\, dr \\ &= \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} y^{n/2-1} e^{-y/2}. \end{aligned}$$

Now if $y < 0$,

$$F_{\chi_n^2}(y) = P[\chi_n^2 < y] \leq P[\chi_n^2 < 0] = 0,$$

so differentiation yields $f_{\chi_n^2}(y) = 0$ for $y < 0$.

The density $f_{\chi_n^2}$ is called a ***chi-squared distribution***. We have already remarked that it is a gamma distribution with $\beta = 2$ and $\alpha = n/2$.

The Sum of Independent Chi-Squared Variables

Suppose we have two independent chi-squared random variables with m and n degrees of freedom, χ^2_m and χ^2_n. Then we can write

$$\chi^2_m = \sum_{i=1}^{m} X_i^2, \qquad \chi^2_n = \sum_{j=1}^{n} Y_j^2$$

where the X_i and Y_j, $i = 1, \dots, m$, $j = 1, \dots, n$, are independent standard normal random variables. Now the sum

$$\chi^2_{m+n} := \chi^2_m + \chi^2_n = \sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{n} Y_j^2$$

is clearly the sum of $m + n$ squares of independent standard normal random variables. Therefore, it also follows a chi-squared distribution, but with $m + n$ degrees of freedom.

The Sum of Independent Chi-Squared Variables

We have the following general result:

10.3. Lemma. Let $\chi^2_{\gamma_1}, \dots, \chi^2_{\gamma_n}$ be n independent random variables following chi-squared distributions with $\gamma_1, \dots, \gamma_n$ degrees of freedom, respectively. Then

$$\chi^2_\alpha := \sum_{k=1}^{n} \chi^2_{\gamma_k}$$

is a chi-squared random variable with $\alpha = \sum_{k=1}^{n} \gamma_k$ degrees of freedom.

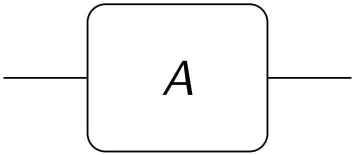
Question. The sum of two chi-squared random variables is again a chi-squared random variable. What about the difference of two such variables?

(1) The difference is also a chi-squared random variable.

(2) The difference follows some other distribution. √ 差可能为负数

A Black Box System

Consider a "black box" unit A:



We don't care what the unit A does or what it looks like inside. We simply assume that at time $t = 0$ the unit A is working. Then at any time $t > 0$, either

- A is working or
- A has failed.

When A fails, it fails completely and can not be repaired.

Failure Density

The time when A fails is random; we describe it by the continuous random variable T_A. The density of T_A is called the

failure density f_A.

The cumulative distribution function of T_A is denoted by F_A.

We note that

$$\begin{aligned} f_A(t) &= \lim_{\Delta t \to 0} \frac{P[t \le T \le t + \Delta t]}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F_A(t + \Delta t) - F_A(t)}{\Delta t} \end{aligned} \tag{10.3}$$

Reliability Function

In practice, one often works with the

reliability function R_A.

The reliability function gives the probability that A is working at time $t \geq 0$.

By our assumption, $R_A(0) = 1$ and

$$P[T \geq t] = R_A(t) = 1 - P[\text{component } A \text{ fails before time } t] \quad P[T < t]$$
$$= 1 - \int_0^t f_A(s)\,ds$$
$$= 1 - F_A(t).$$

Hazard Rate

For practical purposes, an important quantity is the

hazard rate ϱ_A.

defined by

$$\varrho_A(t) := \lim_{\Delta t \to 0} \frac{P[t \le T \le t + \Delta t \mid \boxed{t \le T}]}{\Delta t}$$

(compare with (10.3)). We see that

$$\begin{aligned}\varrho_A(t) &= \lim_{\Delta t \to 0} \frac{P[t \le T \le t + \Delta t]}{P[T \ge t] \cdot \Delta t} \\ &= \frac{f_A(t)}{R_A(t)}.\end{aligned}$$

Question. The hazard rate is often directly observable in practice, while the failure density f is not. Why is this so?

Interpretation of the Hazard Rate

The hazard rate function ϱ can be interpreted qualitatively as follows:

(i) If ϱ is decreasing over an interval, then as time goes by a failure is less likely to occur than it was earlier in the time interval. This happens in situations in which defective systems tend to fail early. As time goes by, the hazard rate for a well-made system decreases.

(ii) A steady hazard rate is expected over the useful life span of a component. A failure tends to occur during this period due mainly to random factors.

(iii) If ϱ is increasing over an interval, then as time goes by a failure is more likely to occur. This normally happens for systems that begin to fail primarily due to wear.

A typical component may exhibit all these behaviors over its lifetime, giving rise to a so-called ***bathtub curve***.

Finding the Reliability Function

Often one has information on ϱ, but not of the failure density f or reliability function R.

10.4. Theorem. Let X be a random variable with failure density f, reliability function R and hazard rate ϱ. Then

$$R(t) = e^{-\int_0^t \varrho(x)\,dx}.$$

Proof.
Since $R(x) = 1 - F(x)$ we have $R'(x) = -F'(x)$. Therefore,

$$\varrho(x) = \frac{f(x)}{R(x)} = \frac{F'(x)}{R(x)} = -\frac{R'(x)}{R(x)}$$

so

$$R'(x) = -\varrho(x)R(x).$$

Solving this equation with $R(0) = 1$ (why?), we obtain the result. $\square$

The Weibull Density

10.5. Example. One hazard function in widespread use is the function

$$\varrho(t) = \alpha\beta t^{\beta-1}, \qquad t > 0, \qquad \alpha, \beta > 0$$

- If $\beta = 1$, the hazard rate is constant
- If $\beta > 1$, the hazard rate is increasing
- If $\beta < 1$, the hazard rate is decreasing

The reliability function is given by

$$R(t) = e^{-\int_0^t \alpha\beta x^{\beta-1}\,dx} = e^{-\alpha t^{\beta}}.$$

The failure density is given by

$$f(t) = \varrho(t)R(t) = \alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}.$$

This density is called the ***Weibull density***, named after W. Weibull who introduced it in 1951.

Weibull Distribution

Waloddi Weibull (1887-1970).Abernethy, R. B., Waloddi Weibull- História, Extract from The New Weibull Handbook.

10.6. Definition. A random variable (X, f_X) is said to have a Weibull distribution with parameters α and β if its density is given by

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, & x > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \alpha, \beta > 0.$$

$\beta = 1 \Rightarrow$ exp

10.7. Theorem. Let X be a Weibull random variable with parameters α and β. The mean and variance of X are given by

$$\mu = \alpha^{-1/\beta}\Gamma(1 + 1/\beta)$$

and

$$\sigma^2 = \alpha^{-2/\beta}\Gamma(1 + 2/\beta) - \mu^2.$$

The Uniform Distribution

10.8. Example. Consider a uniform failure density of

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, failure is equally likely at any time between $x = 0$ and $x = 1$. In Mathematica, the uniform distribution is implemented as follows:

```
PDF[UniformDistribution[{0, 1}], x]
```

$$\begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{True} \end{cases}$$

The Uniform Distribution

Question. Use the Mathematica commands **`SurvivalFunction`** and **`HazardFunction`** to find the reliability function and the hazard rate for the uniform distribution on $[0, 1]$.

Systems in Series and Parallel Configurations

Components in multiple-component systems can be installed in the system in various ways. Many systems are arranged in "series" configuration, some are in "parallel" and others are combinations of the two designs.

10.9. Definition.

(i) A system whose components are arranged in such a way that the system fails whenever any of its components fail is called a ***series*** system.

(ii) A system whose components are arranged in such a way that the system fails only if all of its components fail is called a ***parallel*** system.

Reliability of Series and Parallel Systems

Assuming the components are independent of each other, the reliability of a series system with k components is given by

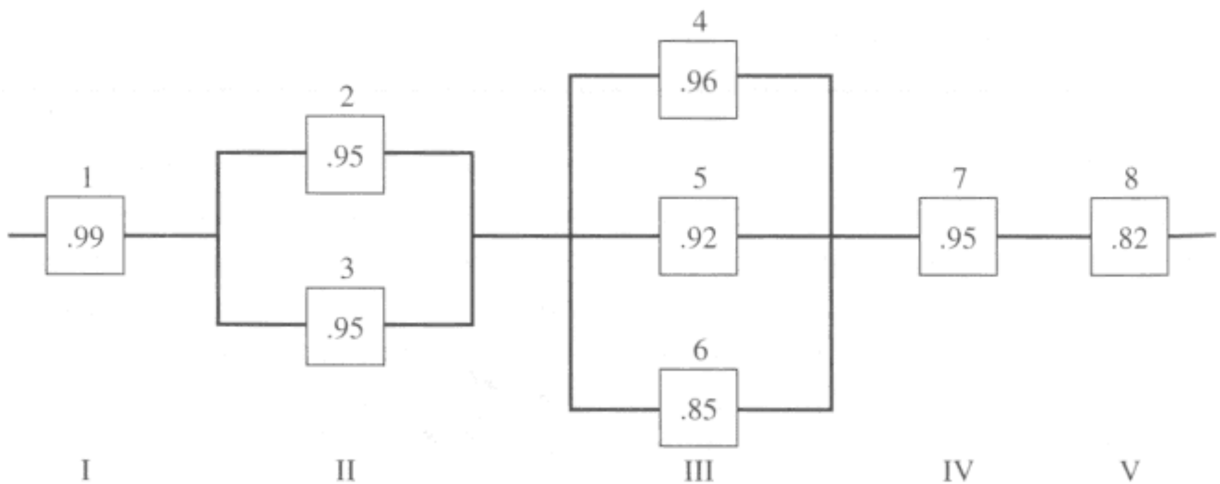
$$R_s(t) = \prod_{i=1}^{k} R_i(t),$$

where R_i is the reliability of the ith component. The reliability of a parallel system is given by

$$\begin{aligned} R_p(t) &= 1 - P[\text{all components fail before } t] \\ &= 1 - \prod_{i=1}^{k} (1 - R_i(t)). \end{aligned}$$

Reliability of Series and Parallel Systems

10.10. Example. Consider a system consisting of eight independent components, connected as shown below:



The numbers shown are the reliabilities $R(t_0)$ for fixed $t_0 > 0$. The reliability of the entire system is the product of assemblies I-V, working out to 0.77.