Ve401 Probabilistic Methods in Engineering

Spring 2020 — Assigment 2

Date Due: 11:00 PM, Friday, the 20$^{\text{th}}$ of March 2020

This assignment has a total of **(29 Marks)**.

Exercise 2.1 Discrete Uniform Distribution

A discrete random variable is said to be *uniformly distributed* if it assumes a finite number of values with each value occurring with the same probability. For example, in the generation of a single random digit taken from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ the number generated is uniformly distributed with each possible digit occurring with probabilty $1/10$.

In general, the density for a uniformly distributed random variable $X: S \to \{x_1, \dots, x_n\} \subset \mathbb{R}$, $n \in \mathbb{N}$, is given by

$$f(x_k) = \frac{1}{n}, \qquad\qquad k = 1, \dots, n.$$

i) Find the moment-generating function for a discrete uniform random variable.
(2 Marks)

ii) Use the moment-generating function to find $\mathrm{E}[X]$ and $\mathrm{Var}[X]$.
(2 Marks)

Exercise 2.2 Uniqueness of Moment Generating Functions - Simple Case

Suppose that two discrete random variables (X, f_X) and (Y, f_Y) both take on values $\{0, \dots, n\}$, $n \in \mathbb{N}$. Suppose that the moment-generating functions are equal in some neighborhood of zero, i.e., there exists some $\varepsilon > 0$ such that

$$m_X(t) = m_Y(t) \qquad \text{for all } t \in (-\varepsilon, \varepsilon).$$

Show that $f_X(x) = f_Y(x)$ for $x = 0, \dots, n$.
(4 Marks)

Exercise 2.3 Sums of Independent Discrete Random Variables

Two discrete random variables X and Y are said to be independent if

$$P[X = x \text{ and } Y = y] = P[X = x] \cdot P[Y = y] \qquad \text{for any } x \in \operatorname{ran} X \text{ and } y \in \operatorname{ran} Y.$$

i) Let $Z = X + Y$, where X and Y are assumed to be independent. Show that

$$P[Z = z] = \sum_{x+y=z} P[X = x] \cdot P[Y = y]$$

using the formula for total probability.
(3 Marks)

ii) Show that the sum of two independent and identical geometric random variables follows a Pascal distribution with $r = 2$.
(3 Marks)

Exercise 2.4 Density of the Poisson Approximation

The density $p_x(t)$, $x \in \mathbb{N}$, of the Poisson distribution is obtained iteratively from the following differential equations:

$$p_0' = -\lambda p_0, \qquad\qquad p_x' + \lambda p_x = \lambda p_{x-1}.$$

Use induction to prove that $p_x(t) = (\lambda t)^x e^{-\lambda t}/x!$. Justify the initial values that you apply in your derivation.
(4 Marks)

Exercise 2.5 Poisson Approximation to the Binomial Distribution

Consider the density f of the binomial distribution,

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}. \qquad (*)$$

Let k be fixed so that $np = k$ and set $p = k/n$. Replace p by k/n everywhere in $(*)$ and then let $n \to \infty$. Use Stirling's formula[1] to show that for every x, $f(x) \to (k^x/x!)e^{-k}$, the density of the Poisson distribution with parameter k.
(4 Marks)

Exercise 2.6 Continuous Uniform Distribution

A continuous random variable X is said to be *uniformly distributed* over an interval (a, b) if its density is given by

$$f(x) = \begin{cases} 1/(b-a) & \text{for } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

i) Show that this is a density for a continuous random variable.
(1 Mark)

ii) Sketch the graph of the density and shade the area of the graph hat represents $P[X \le (a+b)/2]$.
(1 Mark)

iii) Find the probability pictured in part ii).
(1 Mark)

iv) Let (c, d) and (e, f) be subintervals of (a, b) of equal length. What is the relationship between $P[c \le X \le d]$ and $P[e \le X \le f]$?
(1 Mark)

v) Find the cumulative distribution function F for a uniformly distributed random variable.
(1 Mark)

vi) Show that $\mathrm{E}[X] = (a+b)/2$ and $\operatorname{Var} X = (b-a)^2/12$.
(2 Marks)

[1]Stirling's formula states that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad \text{as } n \to \infty.$$

where $f(n) \sim g(n)$ as $n \to \infty$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.