VE401 Probabilistic Methods in Eng.
Midterm Review Part 1

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Two Methods of Estimating Parameters

Suppose $X_1, \ldots, X_n$ are samples for a random variable X.

- Method of moments. For any integer $k \geq 1$,

$$\widehat{\mathrm{E}[X^k]} = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

is an unbiased estimator for the kth moment of X.

- Maximum likelihood estimate.

$$\begin{aligned}\widehat{\theta} &= \arg\max_{\theta} \; L(\theta) = \arg\max_{\theta} \; \prod_{i=1}^{n} f_X(x_i) \\ &= \arg\max_{\theta} \; \ell(\theta),\end{aligned}$$

where $\ell(\theta) = \ln L(\theta)$.

Estimators

- Unbiased estimator for mean.

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

with

$$\mathrm{E}[\widehat{\mu}] = \mu, \qquad \mathrm{Var}\ \widehat{\mu} = \frac{\sigma^2}{n}.$$

- Unbiased estimator for variance.

$$\widehat{\sigma^2} = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Basic Distributions

Finding x such that $P[X \geq x] = p$.

- Standard normal distribution.
  `InverseCDF[NormalDistribution[0, 1], 1-p]`
- Chi-squared distribution with n degrees of freedom.
  `InverseCDF[ChiSquareDistribution[n], 1-p]`
- Student T-distribution with n degrees of freedom.
  `InverseCDF[StudentTDistribution[n], 1-p]`

Interval Estimation for Mean and Variance

Mean. Suppose we have a random sample of size n from a normal population with ***unknown*** mean μ and ***known*** variance σ^2.

- Statistic and distribution.

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}\,(0, 1)\,.$$

- $100(1-\alpha)\%$ two-sided confidence interval for μ.

$$\overline{X} \pm \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}.$$

- $100(1-\alpha)\%$ one-sided interval for μ.

$$L_u = \overline{X} + \frac{z_\alpha \cdot \sigma}{\sqrt{n}}, \qquad L_l = \overline{X} - \frac{z_\alpha \cdot \sigma}{\sqrt{n}}.$$

Interval Estimation for Mean and Variance

Variance. Suppose we have a random sample of size n from a normal population with ***unknown*** mean μ and ***unknown*** variance σ^2.

- Statistic and distribution.

$$\chi^2_{n-1} = \frac{(n-1)S^2}{\sigma^2} \sim \text{ChiSquared}\,(n-1)\,.$$

- $100(1-\alpha)\%$ two-sided confidence interval for σ^2.

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right].$$

- $100(1-\alpha)\%$ one-sided interval for σ^2.

$$L_u = \frac{(n-1)S^2}{\chi^2_{1-\alpha,n-1}}, \qquad L_l = \frac{(n-1)S^2}{\chi^2_{\alpha,n-1}}.$$

Interval Estimation for Mean and Variance

Mean. Suppose we have a random sample of size n from a normal population with ***unknown*** mean μ and ***unknown*** variance σ^2.

- Statistic and distribution.

$$T_{n-1} = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim \text{StudentT}\,(n-1)\,.$$

- $100(1-\alpha)\%$ two-sided confidence interval for μ.

$$\overline{X} \pm \frac{t_{\alpha/2,n-1} S}{\sqrt{n}}.$$

- $100(1-\alpha)\%$ one-sided interval for σ^2.

$$L_u = \overline{X} + \frac{t_{\alpha,n-1} S}{\sqrt{n}}, \qquad L_l = \overline{X} - \frac{t_{\alpha,n-1} S}{\sqrt{n}}.$$

Basic Principles of Counting

Suppose a set A of n objects is given.

- ***Permutation of k objects***: $\dfrac{n!}{(n-k)!}$ ways of choosing an ordered tuple of k objects from A.
- ***Combination of k objects***: $\dfrac{n!}{k!(n-k)!}$ ways of choosing an unordered set of k objects from A.
- ***Permutation of k indistinguishable objects***: $\dfrac{n!}{n_1!n_2!\dots n_k!}$ ways of partitioning A into k disjoint subsets $A_1, \dots, A_k$ whose union is A, where each A_i has n_i elements.

Note. It is important each outcome in the counting method is equally likely.

Counting Method

Example 1. Suppose that a deck of 52 cards containing four aces is shuffled thoroughly and the cards are then distributed among four players so that each player receives 13 cards. What is the probability that each player will receive one ace.

Counting Method

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Solution. Suppose we are allocating the 4 aces to 52 positions such that the $(13i-12)$th through $(13i)$th positions are allocated to the ith player. Then there are $\binom{52}{4}$ possible locations for the four cards, and among them 13^4 will lead to the desired result. Therefore,

$$p = \frac{13^4}{\binom{52}{4}} = 0.1055.$$

Conditional Probability

Definitions and Results.

- ***Conditional probability*** of "B occurs given A has occurred": $P[B|A] := \dfrac{P[B \cap A]}{P[A]}$.
- ***Independence*** of events A and B: $P[A \cap B] = P[A]P[B]$, which is equivalent to

$$\begin{aligned} P[A|B] &= P[A] \qquad \text{if } P[B] \neq 0, \\ P[B|A] &= P[B] \qquad \text{if } P[A] \neq 0. \end{aligned}$$

- ***Total probability*** for $P[B]$ on a sample space S, given events $A_1, \ldots, A_n \in S$ are mutually exclusive and $A_1 \cup \cdots \cup A_n = S$:

$$P[B] = \sum_{k=1}^{n} P[B|A_k] \cdot P[A_k].$$

Bayes's Theorem

Theorem. Let $A_1, \ldots, A_n \subset S$ be a set of pairwise mutually exclusive events whose union is S and who each have non-zero probability of occurring. Let $B \subset S$ be any event such that $P[B] \neq 0$. Then for any $A_k, k = 1, \ldots, n$,

$$P[A_k|B] = \frac{P[B \cap A_k]}{P[B]} = \frac{P[B|A_k] \cdot P[A_k]}{\sum_{j=1}^{n} P[B|A_j] \cdot P[A_j]}.$$

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1. Identifying sample space S.
2. What is the conditional probability of interest $P[A_k|B]$.
3. What are the conditional probabilities that we have $P[B|A_j]$.

Bayes's Theorem

Example 2 (assignment 1.5). It is reported that 50% of all computer chips produced are defective. Inspection assures that only 5% of the chips legally marketed are defective. Unfortunately, some chips are stolen before inspection. If 1% of all chips on the market are stolen, find the probability that a given chip is stolen given that it is defective.

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1. Sample space $\Rightarrow$ all chips that are marketed.
2. Suppose U denotes the sample space of all produced chips, and M denotes the sample space of all marketed chips.

$$P[D] = 50\% (\text{using} U) \quad \Rightarrow \quad P[D|S] = 50\% (\text{using} M),$$

since the stolen chips do not go through inspection.

Bayes's Theorem

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3. All other conditional probabilities are given in terms of M.

$$P[S] = 1\%, \qquad P[D|\neg S] = 5\%.$$

Then

$$P[S|D] = \frac{P[D|S] \cdot P[S]}{P[D|S] \cdot P[S] + P[D|\neg S] \cdot P[\neg S]}.$$

General Remarks

1. Be sure to have a working camera before the exam.
2. You need to use pencil and paper to sketch plots for histograms, stem-and-leaf diagrams and boxplots.
3. You need to upload your files by the end of the exam. You will not have extra time to do this...
4. Go over lecture slides, rc slides, assignments, etc.
5. Integrating by parts, substitution rule, etc.
6. Being familiar with distributions and their interpretations is sometimes helpful. (Poisson — exponential — gamma — failure density.)

Thanks for your attention!

Good luck for Midterm exam!