Ex 1.1 Let A be the event that my friend and I are both chosen.

$$P[A] = \frac{\binom{1998}{118}}{\binom{2000}{120}} = \frac{\frac{1998!}{118!\,1880!}}{\frac{2000!}{120!\,1880!}} = \frac{357}{99950}$$

Ex 1.2 i) Since $A \subset B$, $A \cap B = A$

$B = (B/A) \cup AB = (B/A) \cup A$

$P[B] = P[B/A] + P[A]$

$P[B/A] = P[B] - P[A] \geq 0$

$P[A] \leq P[B]$

ii) If A and B are mutually exclusive, $A \cap B = \phi$

$P[A \cap B] = P[\phi] = 0.$

Since A and B are independent, $P[A \cap B] = P[A]P[B] > 0.$

Therefore $P[A \cap B] > 0.$

This is a contradiction with $P[A \cap B] = 0.$

Therefore, A and B are not mutually exclusive.

iii) $A \cup B = A \cup (B/(A \cap B))$

$P[A \cup B] = P[A] + P[B/(A \cap B)]$

Since $B = (B/A) \cup (B \cap A) = (B/(A \cap B)) \cup (A \cap B)$,

$P[B] = P[B/(A \cap B)] + P[A \cap B]$,

$P[B/(A \cap B)] = P[B] - P[A \cap B]$.

Then $P[A \cup B] = P[A] + P[B] - P[A \cap B]$.

Ex 1.3 i) It is impossible.

Suppose the probability of the head of the coin is $P[h] = p$.

$\in [0,1]$

Then the probability of the tail of the coin is $P[t] = 1 - p$.

Then $\begin{cases} p^2 = \frac{1}{3} \\ p(1-p) = \frac{1}{3} \\ (1-p)^2 = \frac{1}{3} \end{cases}$

Moreover, $p \in [0,1]$, we know that it can't be solved.
Therefore, it is impossible.

ii) It is impossible.
Suppose the probability of the head of one coin is $P[h_1] = p_1 \in [0,1]$
Then the probability of the tail of one coin is $P[t_1] = 1-p_1$.
Similarly, the probabilities of the head and tail of the other one coin are p_2 and $1-p_2$ respectively ($p_2 \in [0,1]$).
Then $\begin{cases} p_1 p_2 = \frac{1}{3} \\ p_1(1-p_2) + p_2(1-p_1) = p_1 + p_2 - 2p_1p_2 = \frac{1}{3} \\ (1-p_1)(1-p_2) = 1 + p_1p_2 - (p_1+p_2) = \frac{1}{3} \end{cases}$
Moreover, $p_1 . p_2 \in [0,1]$, we know that it can't be solved.
Therefore, it is impossible.

Ex 1.4 i). Suppose A is half of the subjects. Then $\neg A$ is the other half of the subject. Suppose B is those answering "yes".

$$P[B] = P[B|A]\,P[A] + P[B|\neg A]\,P[\neg A]$$
$$= 0.17 \times 0.5 + 0.03 \times 0.5 = 0.1 = 10\%$$

ii) $P[B|A] = 0.17 \neq P[B]$
Therefore it is not indepedent of being asked the first question.

Ex 1.5 Suppose A is chips being defective on the market, and B is chips being stolen.

$$P[A] = (1-1\%) \times 5\% + 1\% \times 50\% = 5.45\%$$

$P[B] = 1\%$ $\qquad P[\neg B] = 99\%$

$P[A|B] = 50\%$ $\qquad P[A|\neg B] = 5\%$

$$P[B|A] = \frac{P[A|B] \cdot P[B]}{P[A|B] \cdot P[B] + P[A|\neg B] \cdot P[\neg B]}$$

$$= \frac{0.5 \times 0.01}{0.5 \times 0.01 + 0.05 \times 0.99} = \frac{10}{109} = 9.17\%$$

Ex 1.6 Suppose X denotes the event that prisoner X will be excuted, where $X = A, B, C$. We denote the event "warden says B is not going to die" by B^*.

Then we know $P[A] = P[B] = P[C] = \frac{1}{3}$

$P[B^* | A] = \frac{1}{2}$ $\qquad P[B^* | B] = 0$ $\qquad P[B^* | C] = 1$

By Bayes's formula,

$$P[A|B^*] = \frac{P[B^*|A]P[A]}{P[B^*|A]P[A] + P[B^*|B]P[B] + P[B^*|C]P[C]}$$

$$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3}$$

Since $P[B|B^*] = 0$, this implies $P[C|B^*] = \frac{2}{3}$

Similarly, for C^*, we can get $P[A|C^*] = \frac{1}{3}$, $P[B|C^*] = \frac{2}{3}$ and $P[C|C^*] = 0$.

Therefore, both of them are wrong. The warden will give some information to A but after giving information, the probability that A will be excuted remains $\frac{1}{3}$.

Ex 1.7 There are two cases: A: one girl and one boy (the boy born in July).

B: two boys (at least one born in July).

$$P[A] = \frac{1}{2} \times \frac{1}{12} = \frac{1}{24}$$

$$P[B] = \frac{1}{4} \times \left(1 - \frac{11^2}{12^2}\right) = \frac{23}{576}$$

$$P[\text{the other child is girl}] = \frac{P[A]}{P[A] + P[B]} = \frac{24}{47}$$