# Ve401 Probabilistic Methods in Engineering

# Spring 2020 — Assigment 3

**Date Due: 11:00 PM, Friday, the 27th of March 2020**

This assignment has a total of **(57 Marks)**.

**Exercise 3.1 Maxwell-Boltzmann Statistics**

The distribution function of the speed (modulus of the velocity) $V$ of a gas molecule is described by the Maxwell-Boltzmann law

$$f_V(v) = \begin{cases} \left(\dfrac{2}{\pi}\right)^{1/2} \left(\dfrac{m}{kT}\right)^{3/2} v^2 e^{-\frac{m}{kT}v^2/2} & v > 0 \\ 0 & v \le 0 \end{cases}$$

where $m > 0$ is the mass of the molecule, $T > 0$ is its temperature and $k > 0$ is the Boltzmann constant.

i) Find the mean and variance of $V$.
**(2 Marks)**

ii) Find the mean of the kinetic energy $E = mV^2/2$.
**(2 Marks)**

iii) Find the probability density $f_E$ of $E$.
**(3 Marks)**

**Exercise 3.2 Half-Integer Values of the Gamma Function**

Calculate $\Gamma((2n + 1)/2)$, $n \in \mathbb{N}$, where $\Gamma$ denotes the Euler gamma function.
**(3 Marks)**

**Exercise 3.3 Finding Probabilities with the Normal Distribution**

The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter.

i) What is the probability that a samples strength is less than 6250 kg / cm$^2$?
**(1 Mark)**

ii) What is the probability that a samples strength is between 5800 and 5900 kg / cm$^2$?
**(1 Mark)**

iii) What strength is exceeded by 95% of the samples?
**(2 Marks)**

(This exercise appeared in the first midterm exam in the Fall Term of 2012.)

**Exercise 3.4 A Tricky Question involving the Binomial Distribution**

A mathematics textbook has 200 pages on which typographical errors in the equations could occur. Suppose there are in fact five errors randomly dispersed among these 200 pages.

i) What is the probability that a random sample of 50 pages will contain at least one error?
**(2 Marks)**

ii) How large must the random sample be to assure that at least three errors will be found with 90% probability? (You may use a normal approximation to the binomial distribution.)
**(3 Marks)**

(This exercise appeared in the first midterm exam in the Fall Term of 2012.)

**Exercise 3.5 Cauchy Distribution**

Suppose that $X$ and $Y$ follow independent standard normal distributions. Find the density of $U = X/Y$. Does the expectation of $U$ exist?
**(3 Marks)**

**Exercise 3.6 Sum of Two Continuous Random Variables**
Let $X$ and $Y$ be continuous random variables with parameters with joint density $f_{XY}$. Let $U = X + Y$ and prove that the density of $U$ is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(u - v, v)\, dv.$$

*Hint:* Consider the transformation $(x, y) \mapsto (x + y, y)$.
**(2 Marks)**

**Exercise 3.7 Sum of Two Exponential Distributions**
Let $X$ and $Y$ be independent exponentially distributed random variables with parameters $\beta_1 = 1/3$ and $\beta_2 = 1$, respectively. Let $U = X + Y$ and show that

$$f_U(u) = \begin{cases} (e^{-u/3} - e^{-u})/2 & u > 0 \\ 0 & u \leq 0 \end{cases}$$

**(2 Marks)**

**Exercise 3.8 Linear Combination of Two Normal Distributions**
Let $X_1$ and $X_2$ be independent normal distributions with means $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$, respectively. Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Show that the linear combination

$$Y = \lambda_1 X_1 + \lambda_2 X_2$$

follows a normal distribution and find the mean and variance of $Y$.
**(4 Marks)**

**Exercise 3.9 Bivariate Normal Distribution**
Let $((X_1, X_2), f_{X_1X_2})$ be a continuous bivariate random variable[1] following the *bivariate normal distribution* given by

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}}\, e^{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\varrho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

with parameters $\sigma_1, \sigma_2 > 0$, $\mu_1, \mu_2 \in \mathbb{R}$ and $|\varrho| < 1$.

i) Verify that the marginal density for $X_1$ is that of a normal distribution with mean $\mu_1$ and variance $\sigma_1^2$.
**(3 Marks)**

ii) Show that $\varrho$ is the coefficient of correlation between $X_1$ and $X_2$.
**(3 Marks)**

iii) Show that $X_1$ and $X_2$ are independent if and only if $\varrho = 0$. Is this property true for a bivariate random variable with an arbitrary distribution? Why or why not?
**(2 Marks)**

iv) Prove that

$$\mu_{X_2|x_1} = \mu_2 + \varrho\frac{\sigma_2}{\sigma_1}(x - \mu_1).$$

where $X_1 \mid x_2$ is the conditional random variable $X_1$ in the case $X_2 = x_2$.
**(3 Marks)**

v) The life $X_1$ of a light bulb and its filament diameter $X_2$ follow a bivariate normal random variable with the parameters $\mu_1 = 2000$ hours, $\mu_2 = 0.1$ inch, $\sigma_1^2 = 2500\,\text{hours}^2$, $\sigma_2^2 = 0.01\,\text{inch}^2$ and $\varrho = 0.87$.

The quality-control manager wishes to determine the life of each bulb by measuring the filament diameter. If a filament diameter is 0.098 inch, what is the probability that the tube will last 1950 hours or longer?
**(2 Marks)**

[1] This exercise was part of the first midterm exam in the fall term of 2008. You should not be afraid of evaluating integrals!

**Exercise 3.10 Bivariate Normal Distribution as a Mixture of Independent Normal Distributions**
Let $X = (X_1, X_2)$ be a random vector. Then we define the expectation vector and the variance-covariance matrix as follows:

$$\mathrm{E}[X] := \begin{pmatrix} \mathrm{E}[X_1] \\ \mathrm{E}[X_2] \end{pmatrix}, \qquad \operatorname{Var} X := \begin{pmatrix} \operatorname{Var} X_1 & \operatorname{Cov}(X_1, X_2) \\ \operatorname{Cov}(X_2, X_1) & \operatorname{Var} X_2 \end{pmatrix}.$$

Let $A$ be a constant $2 \times 2$ matrix and $Y = (Y_1, Y_2) = AX$.

i) Show that $\mathrm{E}[AX] = A\,\mathrm{E}[X]$.
**(1 Mark)**

ii) Show that $\operatorname{Var}(AX) = A(\operatorname{Var} X)A^T$.
**(2 Marks)**

iii) Suppose that $X_1$ and $X_2$ follow independent normal distributions with means $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$, respectively. Show that the joint density is given by

$$f_X(x) = f_X(x_1, x_2) = \frac{1}{2\pi\sqrt{\det \varSigma_X}} e^{-\frac{1}{2}\langle x-\mu_X, \varSigma_X^{-1}(x-\mu_X)\rangle}$$

where $\mu_X = (\mu_1, \mu_2)$ and $\varSigma_X = \operatorname{diag}(\sigma_1^2, \sigma_2^2)$ is the $2 \times 2$ matrix with the variances on the diagonal and all other entries vanishing.
**(1 Mark)**

iv) Suppose that $X_1$ and $X_2$ follow independent normal distributions with means $\mu_1, \mu_2 \in \mathbb{R}$ and variances $\sigma_1^2, \sigma_2^2 > 0$, respectively. Let $Y = AX$ where $A$ is an invertible $n \times n$ matrix. Show that

$$f_Y(y) = \frac{1}{2\pi\sqrt{|\det \varSigma_Y|}} e^{-\frac{1}{2}\langle y-\mu_Y, \varSigma_Y^{-1}(y-\mu_Y)\rangle} \tag{$*$}$$

where $\mu_Y = \mathrm{E}[Y]$, $\varSigma_Y = \operatorname{Var} Y$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$.
**(2 Marks)**

v) Show that $(*)$ can be written as

$$f_Y(y_1, y_2) = \frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\varrho^2}}\, e^{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\varrho\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right) + \left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]} \tag{$**$}$$

where $\mu_{Y_i}$ is the mean and $\sigma_{Y_i}^2$ the variance of $Y_i$, $i = 1, 2$, and $\varrho$ is the correllation of $Y_1$ and $Y_2$.
**(2 Marks)**

*Remark:* The above statements (except v), of course) generalize to $n$-dimensional random vectors $(X_1, \ldots, X_n)$.

**Exercise 3.11 Reliability of a System**
A system consists of two independent components connected in series. The life span (in hours) of the first component follows a Weibull distribution with $\alpha = 0.006$ and $\beta = 0.5$; the second has a lifespan in hours that follows the exponential distribution with $\beta = 1/25000$.

i) Find the reliability of the system at 2500 hours.
**(2 Marks)**

ii) Find the probability that the system will fail before 2000 hours.
**(2 Marks)**

iii) If the two components are connected in parallel, what is the system reliability at 2500 hours?
**(2 Marks)**