VE401 Recitation 6

Single Sample Tests and Comparison of Parameter

Test Statistics

From now on, you will learn a lot of tests. For each test, you need to pay attention to:

- What parameter are we testing?
- What parameter do we know?
- What is the distribution of test statistics?

Denoting the distribution of test statistics as $\mathcal{D}$, and d_α as the number such that $\int_{-\infty}^{d_\alpha} f_{\mathcal{D}}(x)\,dx = 1-\alpha$, we reject at significance level α,

- $H_0 : \theta = \theta_0$ if $\mathcal{D} > d_{\alpha/2}$ or $\mathcal{D} < d_{1-\alpha/2}$. (Test statistics is too small or too big)
- $H_0 : \theta \leq \theta_0$ if $\mathcal{D} > d_\alpha$. (Test statistics is too big)
- $H_0 : \theta \geq \theta_0$ if $\mathcal{D} < d_{1-\alpha}$. (Test statistics is too small)

Single Sample Tests for Mean (Z-Test and T-Test)

Single Sample Tests for Variance (Chi-Squared Test)

Single Sample Tests for Median (Non-Parametric)

Estimation of Proportion

Single Sample Test of Proportion (Z-Test)

Comparing Two Proportions (Z-Test, Pooled Test)

F Distribution

Comparing Two Variance (F-Test)

Comparing Two Means (Z-test, Pooled/Paired T-test)

Example: I am a soft drink manufacturer and I have two machines that produce bottles of soft drink. I would like to know if there is a difference in the products of the two machines. I sampled 10 bottles from each of the two machines.

In[410]:=
```
Data1 = RandomVariate[NormalDistribution[499, 2], 10]
Data2 = RandomVariate[NormalDistribution[500, 2], 10]
```

Out[410]=
```
{500.225, 497.209, 502.43, 499.553,
 498.25, 498.235, 498.463, 496.606, 498.849, 498.312}
```

Out[411]=
```
{500.439, 497.685, 498.616, 499.87,
 501.796, 500.566, 499.231, 497.896, 499.774, 498.836}
```

In[412]:=
```
Mean[Data1]
Mean[Data2]
```

Out[412]= 498.813

Out[413]= 499.471

In[414]:=
```
√Variance[Data1]
√Variance[Data2]
```

Out[414]= 1.63617

Out[415]= 1.27599

I would like to test $H_0 : \mu_1 \geq \mu_2$, and $H_1 : \mu_1 \leq \mu_2 - 1$.

Testing Parameter	Distribution of the two RVs	Sample size $n_1,\ n_2$	Variance σ_1^2, σ_2^2	Test Statistics	OC curve x−axis (for two−tailed test)
$\mu_1 - \mu_2$	Normal	any	known	$Z = \dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$	$d = \dfrac{\lvert\mu_1 - \mu_2\rvert}{\sqrt{\sigma_1^2 + \sigma_2^2}}$ where $n = \dfrac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2/n_1 + \sigma_2^2/n_2}$
		large	unknown	$Z = \dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)_0}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}$	
		small	unknown with $\sigma = \sigma_1 = \sigma_2$	$T_{n_1+n_2-2} = \dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)_0}{\sqrt{S_p^2(1/n_1 + 1/n_2)}}$ where $S_p^2 = \dfrac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2}$	In case of $n = n_1 = n_2$, $d = \dfrac{\lvert\mu_1 - \mu_2\rvert}{2\,\sigma}$ where we use modified sample size $n^* = 2\,n - 1$, and σ can be estimated.
		small	unknown	$T_\gamma = \dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)_0}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}$ where $\gamma = \dfrac{\left(S_1^2/n_1 + S_2^2/n_2\right)^2}{\frac{\left(S_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(S_2^2/n_2\right)^2}{n_2 - 1}}$ (we take the floor)	No simple OC Curves

Suppose I know that $\sigma_1 = \sigma_2 = 2$, which test would you use? What is the test statistics?

The Z-test with known variance. The test statistics is $Z = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} = \frac{498.813 - 499.471}{\sqrt{8/10}} = -0.73$.

This gives us a P-value of 0.23. This is large, so we failed to reject H_0.

In[419]:=
```
CDF[NormalDistribution[], (Mean[Data1] - Mean[Data2])/Sqrt[8/10]]
```

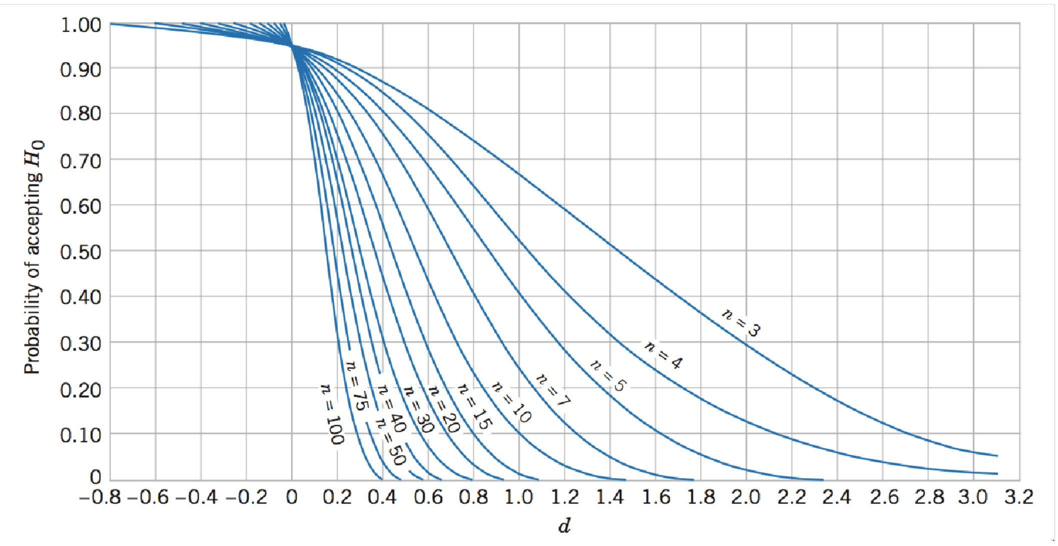
Out[419]= 0.230966

Suppose we don't know about the true variance, but suppose we know they are equal. which test would you use? What is the test statistics?

The T-test with equal variance. We have pooled variance $S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{9(1.64)^2 + 9(21.28)^2}{18} = 2.15$, so the test statistics is $T_{10+10-2} = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)_0}{\sqrt{S_p^2(1/n_1 + 1/n_2)}} = \frac{498.813 - 499.471}{\sqrt{2.15 \times 2/19}} = -1.0$. This gives us a P-value of 0.16. So there is no evidence to reject H_0.

In[426]:=
```
CDF[StudentTDistribution[18], (Mean[Data1] - Mean[Data2])/Sqrt[(Variance[Data1] + Variance[Data2])/2 * 2/10]]
```

Out[426]= 0.16461525213843



In[521]:=
```
Sqrt[Variance[Data1]]
Sqrt[Variance[Data2]]
```

Out[521]= 1.63617

Out[522]= 1.27599

Suppose we know that $\mu_1 = 499$, $\mu_2 = 500$. What is the power of the T-test for unknown variance if $\alpha = 0.05$?

We can use the estimated variance $S_p^2 = 2.15$, and calculate $d = \frac{\mu_2-\mu_1}{2\sigma} \approx \frac{1}{2\sqrt{2.15}} = 0.34$. Choosing $n^* = 2n - 1 = 19 \approx 20$, we get a β of approximately 0.56, so the power is approximately 0.44.

Suppose we don't know anything about the true variance. which test would you use? What is the test statistics?

Welch's T-test with unknown variance. We have the degrees of freedom $\gamma = \left\lfloor \frac{(S_1^2/n_1+S_2^2/n_2)^2}{\frac{(S_1^2/n_1)^2}{n_1-1}+\frac{(S_2^2/n_2)^2}{n_2-1}} \right\rfloor = 16$. $T_{16} = \frac{\overline{X}_1-\overline{X}_2-(\mu_1-\mu_2)_0}{\sqrt{S_p^2(1/n_1+1/n_2)}} = -1.0$. This gives us again a P-value of 0.17. So there is no evidence to reject H_0.

In[524]:=
```
Floor[(Variance[Data1] / 10 + Variance[Data2] / 10)^2 / ((Variance[Data1]/10)^2/9 + (Variance[Data2]/10)^2/9)]
```

Out[524]= 16

Comparing Locations (Non-parametric)

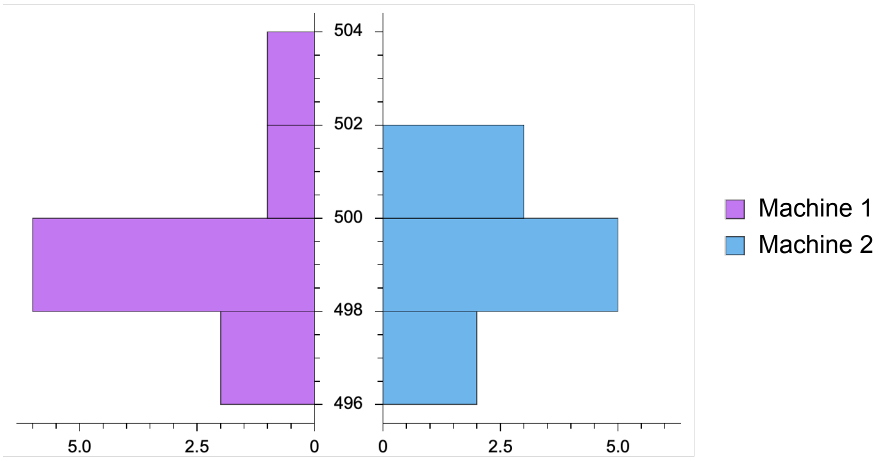
Testing Parameter	Distribution of X, Y	Sample Size	Hypothesis H_0	Test Statistics
$P[X > Y]$	continuous	m samples of X n samples of Y $m \leq n$.	$P[X > Y] = 1/2$ $P[X > Y] \geq 1/2$ $P[X > Y] \leq 1/2$	W_m

where W_m is the sum of ranks of X_1, ..., X_m when we rank the samples X_1, ..., X_m, Y_1, ..., Y_n from the smallest to the largest. When there is a tie, we take the average value. For example, if the first, second and third smallest value is a group of 3 ties, we rank them all as $\frac{1+3}{2} = 2$.

For small n, you can refer to a table of critical values here: https://www.stat.auckland.ac.nz/~wild/ChanceEnc/Ch10.wilcoxon.pdf. For $n \geq 20$, we can approximate the test statistics with normal distribution with $\mathrm{E}[W] = \frac{m(m+n+1)}{2}$, $\mathrm{Var}[W] = \frac{mn(m+n+1)}{12}$. If there are many ties, then for each group of t ties, the denominator of variance is reduced by $\frac{t^3+t}{12}$.

In[427]:=
```
PairedHistogram[Data1, Data2, ChartLegends → {{"Machine 1", "Machine 2"}},
 ChartStyle → {"Pastel", None, None}]
```

Out[427]=



In[428]:=
```
Sort[Data1]
```

Out[428]=
```
{496.606, 497.209, 498.235, 498.25,
 498.312, 498.463, 498.849, 499.553, 500.225, 502.43}
```

In[429]:=
```
Sort[Data2]
```

Out[429]=
```
{497.685, 497.896, 498.616, 498.836,
 499.231, 499.774, 499.87, 500.439, 500.566, 501.796}
```

Use Wilcoxon Rank Sum test to test the hypothesis $H_0 : P[X > Y] \geq 1/2$.

The test statistics $W_m = 1 + 2 + 5 + 6 + 7 + 8 + 11 + 13 + 16 + 20 = 89$. From the table, we can see that the P-value lies between 0.1 and 0.2. There is no strong evidence to reject $P[X > Y] \geq 1/2$.

		Lower Tail						Upper Tail					
		prob						prob					
n_A	n_B	.005	.01	.025	.05	.10	.20	.20	.10	.05	.025	.01	.005
4	4			10	11	13	14	22	23	25	26		
	5		10	11	12	14	15	25	26	28	29	30	
	6	10	11	12	13	15	17	27	29	31	32	33	34
	7	10	11	13	14	16	18	30	32	34	35	37	38
	8	11	12	14	15	17	20	32	35	37	38	40	41
	9	11	13	14	16	19	21	35	37	40	42	43	45
	10	12	13	15	17	20	23	37	40	43	45	47	48
	11	12	14	16	18	21	24	40	43	46	48	50	52
	12	13	15	17	19	22	26	42	46	49	51	53	55
5	5	15	16	17	19	20	22	33	35	36	38	39	40
	6	16	17	18	20	22	24	36	38	40	42	43	44
	7	16	18	20	21	23	26	39	42	44	45	47	49
	8	17	19	21	23	25	28	42	45	47	49	51	53
	9	18	20	22	24	27	30	45	48	51	53	55	57
	10	19	21	23	26	28	32	48	52	54	57	59	61
	11	20	22	24	27	30	34	51	55	58	61	63	65
	12	21	23	26	28	32	36	54	58	62	64	67	69
6	6	23	24	26	28	30	33	45	48	50	52	54	55
	7	24	25	27	29	32	35	49	52	55	57	59	60
	8	25	27	29	31	34	37	53	56	59	61	63	65
	9	26	28	31	33	36	40	56	60	63	65	68	70
	10	27	29	32	35	38	42	60	64	67	70	73	75
	11	28	30	34	37	40	44	64	68	71	74	78	80
	12	30	32	35	38	42	47	67	72	76	79	82	84
7	7	32	34	36	39	41	45	60	64	66	69	71	73
	8	34	35	38	41	44	48	64	68	71	74	77	78
	9	35	37	40	43	46	50	69	73	76	79	82	84
	10	37	39	42	45	49	53	73	77	81	84	87	89
	11	38	40	44	47	51	56	77	82	86	89	93	95
	12	40	42	46	49	54	59	81	86	91	94	98	100
8	8	43	45	49	51	55	59	77	81	85	87	91	93
	9	45	47	51	54	58	62	82	86	90	93	97	99
	10	47	49	53	56	60	65	87	92	96	99	103	105
	11	49	51	55	59	63	69	91	97	101	105	109	111
	12	51	53	58	62	66	72	96	102	106	110	115	117
9	9	56	59	62	66	70	75	96	101	105	109	112	115
	10	58	61	65	69	73	78	102	107	111	115	119	122
	11	61	63	68	72	76	82	107	113	117	121	126	128
	12	63	66	71	75	80	86	112	118	123	127	132	135
10	10	71	74	78	82	87	93	117	123	128	132	136	139
	11	73	77	81	86	91	97	123	129	134	139	143	147
	12	76	79	84	89	94	101	129	136	141	146	151	154
11	11	87	91	96	100	106	112	141	147	153	157	162	166
	12	90	94	99	104	110	117	147	154	160	165	170	174
12	12	105	109	115	120	127	134	166	173	180	185	191	195

Test for Mean of Difference of two RVs (Paired T-test)

For our data, we can say: we do 10 times of sampling. For each sampling, we take one sample each from the two machines. We can pair the data and get a new data $D_i := X_i - Y_i$.

In[433]:=
```
d = Data1 - Data2
```

Out[433]=
```
{-0.213956, -0.476581, 3.8135, -0.317297, -3.54625,
 -2.33138, -0.768637, -1.29035, -0.925138, -0.523963}
```

In[435]:=
```
Mean[d]
Variance[d]
```

Out[435]=
```
-0.658004
```

Out[436]=
```
3.55379
```

Testing Parameter	Distribution of (X, Y)	Sample Size	Test Statistics
μ_D where $D = X - Y$	joint bivariate normal distribution	$n = n_X = n_Y$	$T_{n-1} = \frac{\overline{D}-(\mu_D)_0}{\sqrt{S_D^2/n}}$
M_D where $D = X - Y$	independent, same distribution but different location $(D$ symmetric$)$	any	W_+ (Wilcoxon signed rank statistics)

Test $H_0: \mu_D \geq 0$ using paired T-test.

We have $T_{n-1} = \frac{\overline{D}-(\mu_D)_0}{\sqrt{S_D^2/n}} = \frac{-0.65}{\sqrt{3.55/10}} = -1.10$. The P-value is then 0.15. We don't have enough evidence to reject H_0.

```
In[438]:= CDF[StudentTDistribution[9], Mean[d] / Sqrt[Variance[d] / 10]]

Out[438]= 0.14916234489980
```

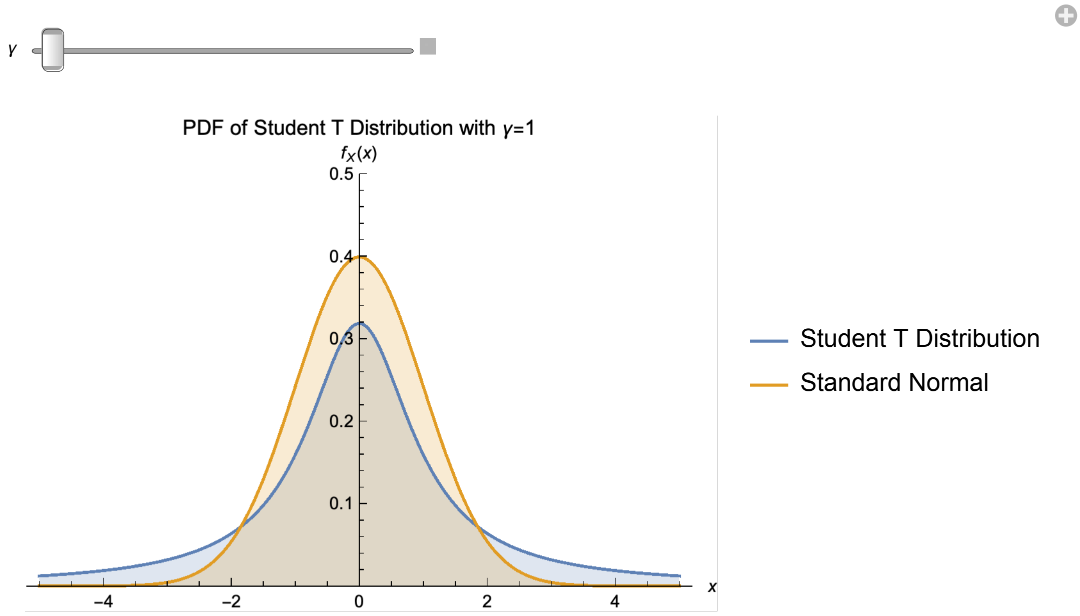
Recall that the pooled T-test gives us a P-value of 0.17. In general, pairing with the absence of correlation is unnecessary and actually makes a test less powerful.

Comparison Between Pooled and Paired T-test

If $n_X = n_Y = n$, we can use both

Testing Parameter	Distribution of (X, Y)	Test Statistics
$\mu_X - \mu_Y$	joint bivariate normal distribution	$T_{2n-2} = \frac{\overline{X}-\overline{Y}}{\sqrt{2\,S_p^2/n}}$ where $S_p^2 = \frac{S_1^2+S_2^2}{2}$
		$T_{n-1} = \frac{\overline{D}-(\mu_D)_0}{\sqrt{S_D^2/n}}$

Out[439]=



- When the test statistics is the same, pooled T-test is more powerful because of more degrees of freedom.
- We have $\frac{S_D^2}{n} \approx \sigma_{\overline{D}}^2 = \frac{2\sigma^2}{n}(1-\rho_{XY}) \approx \frac{2S_p^2}{n}(1-\rho_{XY})$. So, when n is large enough (so that difference in degrees of freedom has less impact), polled T-test will have $\begin{cases} \text{smaller power} & \text{if } \rho_{XY} > 0 \\ \text{larger power} & \text{if } \rho_{XY} \leq 0 \end{cases}$.

Estimation of Correlation Coefficient

We use method of moments to get an estimator for ρ_{XY}:

$$R = \hat{\rho} = \frac{\sum (X_i - \overline{X})(Y_i - \overline{Y})}{\sqrt{\sum (X_i - \overline{X})^2}\sqrt{\sum (Y_i - \overline{Y})^2}}$$

Single Sample Test for Correlation Coefficient (Z-test)

Testing Parameter	Sample Size	Test Statistics
ρ	large	$Z = \frac{\sqrt{n-3}}{2}\left(\ln\left(\frac{1+R}{1-R}\right) - \ln\left(\frac{1+\rho_0}{1-\rho_0}\right)\right) = \sqrt{n-3}\,(\text{artanh}(R) - \text{artanh}(\rho_0))$

The $100(1-\alpha)\%$ confidence interval for ρ is $\tanh\left(\text{artanh}(R) \pm \frac{z_{\alpha/2}}{\sqrt{n-3}}\right)$

Calculate a 95% confidence interval of the correlation coefficient between the bottle volumes produced by the two machines.

In[549]:=
```
Covariance[Data1, Data2] / Sqrt[Variance[Data1] Variance[Data2]]
```

Out[549]= 0.179961

We have $R = \frac{\sum(X_i - \overline{X})(Y_i - \overline{Y})}{\sqrt{\sum(X_i - \overline{X})^2}\sqrt{\sum(Y_i - \overline{Y})^2}} = 0.180.$

Then $\tanh\left(\operatorname{artanh}(R) \pm \frac{z_{\alpha/2}}{\sqrt{n-3}}\right) = \tanh(0.180 \pm 0.741) = [-0.51, 0.73].$

In[553]:=
```
Tanh[ArcTanh[Covariance[Data1, Data2] / Sqrt[Variance[Data1] Variance[Data2]]] +
  InverseCDF[NormalDistribution[], 1 - 0.025] / Sqrt[7]]
```

Out[553]= 0.727191

Categorical Data

Multinomial Distribution

A ***multinomial trial*** with parameter $p_1, \ldots, p_k$ is a trial that can result in one of k outcomes, with the probability of getting i^{th} outcome is p_i.

We define random vector $(X_1, \ldots, X_k) : S \to \Omega = \{0, 1, \ldots, n\}^k$ and PDF $f_{X_1 X_2 \ldots X_k}(x_1, \ldots, x_k) = \frac{n!}{x_1! \ldots x_k!} p_1^{x_1} \ldots p_k^{x_k}$ to have a ***multinomial distribution*** with parameters n and $p_1, \ldots, p_k$.

Properties:

- For $i = 1, \ldots, k$, $\mathrm{E}[X_i] = n\, p_i$,
- $\mathrm{Var}[X_i] = n\, p_i (1 - p_i)$.
- For $1 \le i < j \le k$, $\mathrm{Cov}[X_i, X_j] = -n\, p_i\, p_j$.

Pearson Statistics

Let $\left((X_1, \ldots, X_k), f_{X_1 X_2 \ldots X_k}(x_1, \ldots, x_k)\right)$ be multinomial random variable with parameters n and $p_1, \ldots, p_k$. For large n,

$$\sum_{i=1}^{k} \frac{(X_i - n\,p_i)^2}{n\,p_i} = \sum_{i=1}^{k} \frac{\left(X_i - \mathrm{E}[X_i]\right)^2}{\mathrm{E}[X_i]} =: \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

follows an approximate chi-squared distribution with $k-1$ degrees of freedom. Here O_i is the observed value and E_i is the expected value. For the approximation we require

$$\mathrm{E}[X_i] = n\,p_i \geq 1 \qquad \text{for all } i = 1, \ldots, k$$

$$\mathrm{E}[X_i] = n\,p_i \geq 5 \qquad \text{for } 80\,\% \text{ of all } i = 1, \ldots, k$$

Chi-Squared Goodness-of-Fit Test

Purpose: To fit a distribution on the data.

Testing Parameter	Sample Size	Test Statistics
$p_1, \ldots, p_k$	large	$X^2_{k-1-m} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$ where k is the number of categories and m is number of parameters we estimate

If the test statistics is too large ($X^2_{k-1-m} > \chi^2_{\alpha,k-1-m}$), we reject $H_0 : p_i = p_{i_0}$.

Goodness-of-Fit Test for Discrete Distribution

Steps:

- Usually in discrete distribution case, the categories are set.
- Calculate the expected value for each category given hypothesis is true. Sometimes the categories is incomplete, e.g. $\{0, 1, 2, 3, 4, 5\} \subset \mathbb{N}$. In this case the boundary category should also cover the probability that the value is greater than itself. Namely we need to calculate $P[X = 0], \ldots, P[X = 4], P[X \geq 5]$.
- Determine degrees of freedom.
- Calculate critical region.
- Calculate Pearson statistics to see if we can reject H_0.

Raindrops keep falling on my head at an unknown rate. I record the number of raindrops that fall on my head in every minute, and found that in a 42 minute time period,

Raindrops per minute	no. of minutes	p_i	E_i
0	3		
1	14		
2	9		
3	7		
4	5		
5	4		

.

(1) Suppose this is a Poisson distribution with parameter k, what is the estimation of k?
(2) Test whether this is a Poisson distribution with parameter $\hat{k}$.
(3) Is this a proper categorization? What is the degree of freedom of test statistics?

Suppose this is a Poisson distribution, then we have $\hat{k} = \overline{X} = \frac{0\times3+1\times14+2\times9+3\times7+4\times5+5\times4}{42} = 2.21$. Then we calculate the expected value for each category. For example for $X = 0$, $P[X = 0] = \frac{e^{-2.21}\,2.21^0}{0!} = 0.11$, so $E_i = 42\,P[X = 0] = 4.6$.

Out[545]=

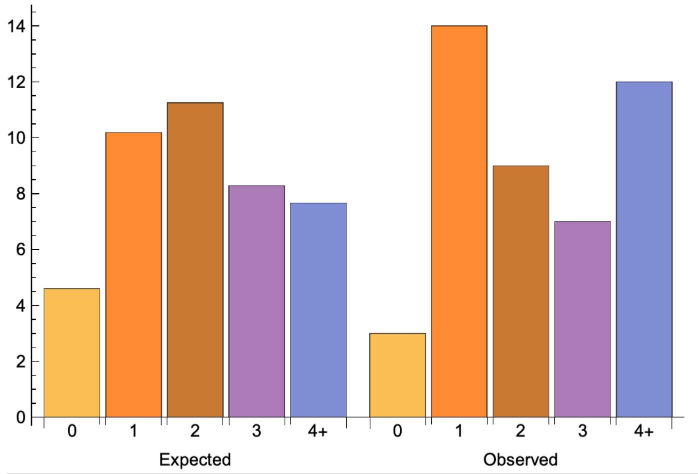
	0	1	2	3	4	5 or more
Expected	4.60743	10.1824	11.2516	8.28865	4.57948	3.09045
Observed	3	14	9	7	5	4

And we found that this violates the Cochran's rule, so we adjust to merge the last two categories:

Out[546]=

	0	1	2	3	4 or more
Expected	4.60743	10.1824	11.2516	8.28865	7.66994
Observed	3	14	9	7	9

Out[547]=



This time it's good. The test statistics is then

$$\begin{aligned}\chi^2_{5-1-1} &= \frac{(4.61-3)^2}{4.61} + \frac{(10.18-14)^2}{10.18} + \frac{(11.25-9)^2}{11.25} + \frac{(8.29-7)^2}{8.29} + \frac{(7.67-9)^2}{7.67} \\ &= 2.877\end{aligned}$$

The critical value is

In[478]:=
```
InverseCDF[ChiSquareDistribution[3], 0.95]
```

Out[478]= 7.81473

Since $2.877 < 7.81$, we cannot reject H_0 : The data follows Poisson distribution with $k = 2.21$.

Goodness-of-Fit Test for Continuous Distribution

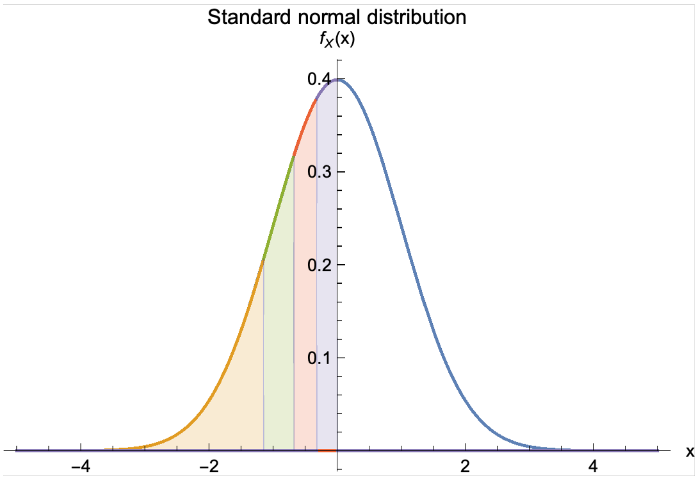
Steps:

- Usually in continuous distribution case, you need to set the categories. This is by dividing $\mathbb{R}$ into k equally likely segments. e.g. For normal distribution and $k = 8$,

$(a_0, a_1) = (-\infty, -1.15)$, $[a_1, a_2) = [-1.15, -0.675)$, $[a_2, a_3) = [-0.675, -0.32)$, $[a_3, a_4) = [-0.32, 0)$

$[a_4, a_5) = [0, 0.32)$, $[a_5, a_6) = [0.32, 0.675)$, $[a_6, a_7) = [0.675, 1.15)$, $[a_7, a_8) = [1.15, \infty)$

Out[497]=



- Calculate the expected value for each category, which should be n/k for every category. Categorize your data according to those intervals to get the observed values.
- Determine degrees of freedom.
- Calculate critical region.
- Calculate Pearson statistics to see if we can reject H_0.

Independence of Categorizations

Testing Parameter	Sample Size	Test Statistics
$p_1, \ldots, p_k$	large	$X^2_{(r-1)(c-1)} = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(O_{ij}-E_{ij})^2}{E_{ij}}$ where r is the number of rows and c is the number of columns

Steps:

- Sum the rows and columns to get the ***marginal row and column sums*** $n_{i\cdot}$, $n_{\cdot j}$. These are used to estimate ***marginal densities*** $\hat{p}_{i\cdot} = \frac{n_{i\cdot}}{n}$, $\hat{p}_{\cdot j} = \frac{n_{\cdot j}}{n}$.
- For each grid (i, j), multiply the estimated marginal densities together and calculate the expected number $E_{ij} = \frac{n_{i\cdot}\, n_{\cdot j}}{n}$.
- Determine degrees of freedom.
- Calculate critical region.
- Calculate Pearson statistics to see if we can reject H_0.

We conducted a survey. In the survey, people are asked to fill in their gender and age. It is

shown as follows:

	Male		Female		$n_{i\cdot}$
	O_i	E_i	O_i	E_i	
Under 18	75		25		
18 to 29	1242		60		
30 to 50	1531		68		
above 50	1040		39		
$n_{\cdot j}$					

. Are the age and gender of participants independent?

	Male		Female		$n_{i\cdot}$
	O_i	E_i	O_i	E_i	
Under 18	75	95.29	25	4.706	100
18 to 29	1242	1241	60	61.27	1302
30 to 50	1531	1524	68	75.25	1599
above 50	1040	1028	39	50.78	1079
$n_{\cdot j}$	3888		192		4080

Calculating the test statistics, $X^2_{(r-1)(c-1)} = \sum_{i=1}^{r} \sum_{i=1}^{c} \frac{(O_{ij}-E_{ij})^2}{E_{ij}} = 95.34$. The degrees of freedom is $(r-1)(c-1) = 3$. The test statistics is far larger than the critical value, so we reject the null hypothesis that the gender and age are independent.

Problems in Assignments

Testing Hypotheses on an Exponential Variable

Let X be an exponential random variable with parameter β. Devise a test statistic for testing $H_0 : \beta = \beta_0$ and $H_0 : \beta \leq \beta_0$.

Solution

We know that $Y = X_1 + \ldots + X_n$ follows a Gamma distribution with $\alpha = n$ and β, so,

$$m_Y(t) = \left(1 - \frac{t}{\beta}\right)^{-n} \Rightarrow m_{2\beta Y}(t) = m_Y(2\beta t) = (1-2t)^{-n}$$

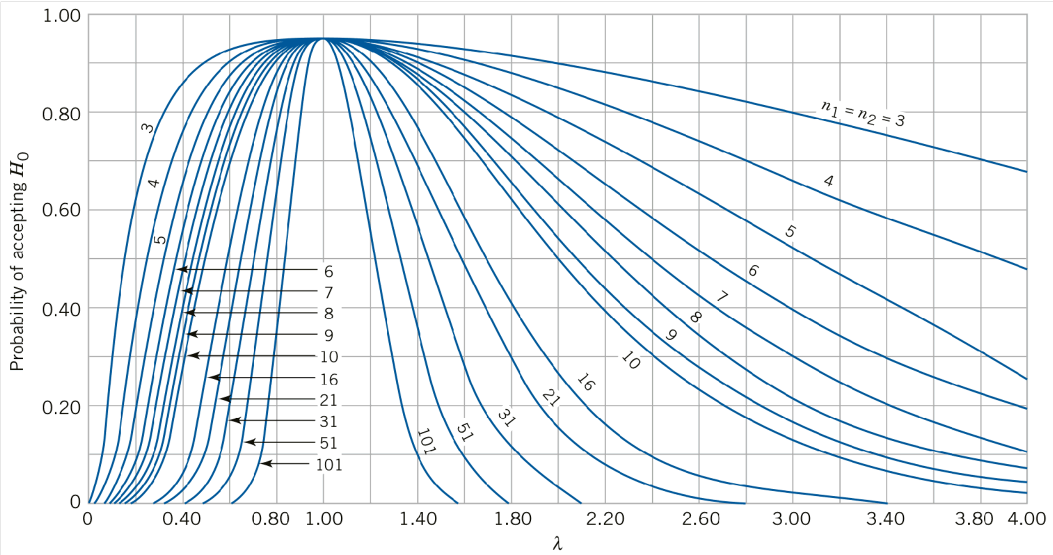
which is a moment generating function of Chi-squared distribution with $\gamma = 2n$. So we conclude that $2\beta Y = 2\beta \sum_{i=1}^{n} X_i = 2n\beta \overline{X}$ follows Chi-squared distribution with $\gamma = 2n$. Therefore we set our test statistics to be $X^2_{2n} = 2n\beta_0 \overline{X}$.

We note that when the true β is larger, $\overline{X}$, which is the estimator for the mean $1/\beta$, will be smaller. So we reject

- $H_0 : \beta = \beta_0$ if $X^2_{2\,n} > \chi^2_{0.025,2\,n}$ or $X^2_{2\,n} < \chi^2_{0.975,2\,n}$.
- $H_0 : 1/\beta \geq 1/\beta_0$ if $X^2_{2\,n} < \chi^2_{1-0.05,2\,n}$.

OC Curve For F-test

Calculate the power of F-test when $n_1 \neq n_2$.



Solution

$$
\begin{aligned}
P[\text{fail to reject } H_0] &= P[\text{Sample statistics lies outside critical region}] \\
&= P\left[f_{0.975,n_1-1,n_2-1} < \frac{S_1^2}{S_2^2} < f_{0.025,n_1-1,n_2-1} \,\middle|\, \lambda = \frac{\sigma_1}{\sigma_2}\right] \\
&= P\left[\frac{1/\sigma_1^2}{1/\sigma_2^2}\, f_{0.975,n_1-1,n_2-1} < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < \frac{1/\sigma_1^2}{1/\sigma_2^2}\, f_{0.025,n_1-1,n_2-1} \,\middle|\, \lambda = \frac{\sigma_1}{\sigma_2}\right] \\
&= P\left[\frac{1}{\lambda^2}\, f_{0.975,n_1-1,n_2-1} < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < \frac{1}{\lambda^2}\, f_{0.025,n_1-1,n_2-1} \,\middle|\, \lambda = \frac{\sigma_1}{\sigma_2}\right] \\
&= P\left[\frac{1}{\lambda^2}\, f_{0.975,n_1-1,n_2-1} < F_{n_1-1,n_2-1} < \frac{1}{\lambda^2}\, f_{0.025,n_1-1,n_2-1}\right]
\end{aligned}
$$

In[535]:=
```
n1 = 8; n2 = 16;
f[λ_] := CDF[FRatioDistribution[n1 - 1, n2 - 1],
    1/λ^2 InverseCDF[FRatioDistribution[n1 - 1, n2 - 1], 0.975]] -
   CDF[FRatioDistribution[n1 - 1, n2 - 1],
    1/λ^2 InverseCDF[FRatioDistribution[n1 - 1, n2 - 1], 0.025]];
Plot[f[λ], {λ, 0, 5}, PlotLabel → "OC Curve of F test when n1=" <>
   ToString[n1] <> ",n2=" <> ToString[n2], AxesLabel → {"λ", "P[Accept H0]"}]
```

Out[537]=