Continuous Random Variables

Continuous Random Variables

7.1. Definition. Let S be a sample space. A ***continuous random variable*** is a map $X: S \to \mathbb{R}$ together with a function $f_X: \mathbb{R} \to \mathbb{R}$ with the properties that

(i) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ and

(ii) $\int\limits_{-\infty}^{\infty} f_X(x)\,dx = 1$.

The integral of f_X is interpreted as the probability that X assumes values x in a given range, i.e.,

$$P[a \leq X \leq b] = \int_a^b f_X(x)\,dx$$

The function f_X is called the ***probability density function*** (or just density) of the random variable X.

The Probability Density Function

Notice that by the above definition,

$$P[X = x] = \int_x^x f_X(y)\,dy = 0,$$

i.e., the probability that X assumes any specific value is zero. We see that f_X no longer represents a probability, but is truly a *density*.

Cumulative Distribution

7.2. Definition. Let (X, f_X) be a continuous random variable. The cumulative distribution function for X is defined by $F_X\colon \mathbb{R} \to \mathbb{R}$,

$$F_X(x) := P[X \le x] = \int_{-\infty}^{x} f_X(y)\, dy$$

Notice that by the fundamental theorem of calculus we can easily obtain the density f_X from F_X:

$$f_X(x) = F_X'(x).$$

Expectation and Variance

We can define the expectation of a continuous random variable X analogously to that of discrete variables:

$$\mathrm{E}[X] := \int_{\mathbb{R}} x \cdot f_X(x)\, dx$$

It is the possible to prove (using some technical arguments in measure theory) that for any "reasonable" function $\varphi\colon \mathbb{R} \to \mathbb{R}$ we have

$$\mathrm{E}[\varphi \circ X] = \int_{-\infty}^{\infty} \varphi(x) \cdot f_X(x)\, dx,$$

similarly to the discrete case. As before,

$$\mathrm{Var}[X] = \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

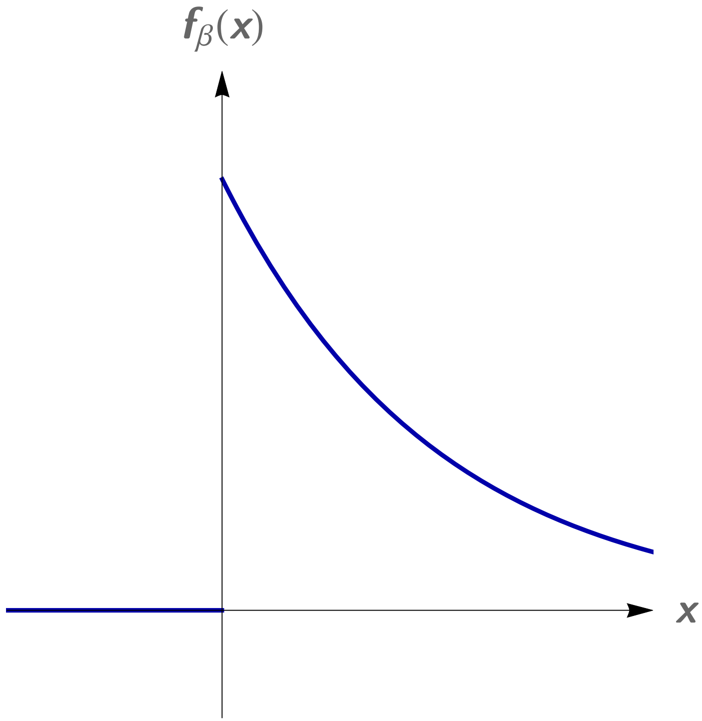
and all the previously established properties of the expectation and variance continue to hold in the continuous case.

The Exponential Distribution

7.3. Definition. Let $\beta \in \mathbb{R}$, $\beta > 0$. A continuous random variable (X, f_β) with density

$$f_\beta(x) = \begin{cases} \beta e^{-\beta x}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

is said to follow an ***exponential distribution*** with parameter β.



It is easy to verify that $f_\beta(x) \ge 0$ for all $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} f_\beta(x)\,dx = 1.$$

Expectation and Variance

Through integration by parts, we find the expectation and variance:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\, f_\beta(x)\, dx = \int_0^\infty \beta x e^{-\beta x}\, dx \\ &= -xe^{-\beta x}\Big|_0^\infty + \int_0^\infty e^{-\beta x}\, dx = \frac{1}{\beta}. \end{aligned}$$

$\int uv' = uv - \int u'v$

The second moment is

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_\beta(x)\, dx = \int_0^\infty \beta x^2 e^{-\beta x}\, dx \\ &= -x^2 e^{-\beta x}\Big|_0^\infty + 2\int_0^\infty x\, e^{-\beta x}\, dx = \frac{2}{\beta^2} \end{aligned}$$

and therefore,

$$\mathrm{Var}[X] = E[X^2] - E[X]^2 = \frac{1}{\beta^2}.$$

The Moment-Generating Function

We now see that

$$m_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

so the moment-generating function of a continuous random variable is (up to a sign) the ***bilateral Laplace transform*** of its density.

For the exponential distribution we have

$$\begin{aligned} m_X(t) = \mathrm{E}[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx}\, f_\beta(x)\,dx \\ &= \int_0^{\infty} \beta e^{-(\beta-t)x}\,dx \\ &= \frac{\beta}{(\beta-t)} \int_0^{\infty} e^{-y}\,dy \\ &= (1-t/\beta)^{-1}. \end{aligned}$$

Connection to the Poisson Distribution

The exponential distribution has a close relationship with the Poisson distribution.

Recall that for Poisson-distributed events (arrivals) the probability of x arrivals in the time interval $[0, t]$ is given by

$$p_x(t) = \frac{(\lambda t)^x}{x!} e^{-\lambda t}, \qquad\qquad x \in \mathbb{N}.$$

Then $p_0(t)$ is the probability of no arrivals in $[0, t]$. This can also be interpreted as the probability that the first arrival occurs at a time greater than t.

Denote by T the time of the first arrival (it is a continuous random variable). Then

$$P[T > t] = p_0(t) = e^{-\lambda t}, \qquad\qquad t \geq 0.$$

and $P[T > t] = 1$ for $t < 0$.

Connection to the Poisson Distribution

Hence, if we denote by F_T the cumulative distribution of the density of T, we have

$$F_T(t) = P[T \le t] = 1 - e^{-\lambda t}, \qquad t \ge 0,$$

and $F_T(t) = 0$ for $t < 0$. Since $f_T(t) = F_T'(t)$, the density is

$$f_T(t) = \lambda e^{-\lambda t}, \qquad t \ge 0.$$

and $f_T(t) = 0$ for $t < 0$.

Thus the time between successive arrivals of a Poisson-distributed random variable is exponentially distributed with parameter $\beta = \lambda$.

Connection to the Poisson Distribution

7.4. Example. An electronic component is known to have a useful life represented by an exponential density with failure rate of $\lambda = 10^{-5}$ failures per hour, i.e., $\beta = 10^{-5}$. The mean time to failure, $\mathrm{E}[X]$, is thus $1/\beta = 10^5$ hours.

Suppose we wanted to determine the fraction of such components that would fail before the mean or expected life:

$$P[T \leq 1/\beta] = \int_0^{1/\beta} \beta e^{-\beta x}\, dx = 1 - e^{-1} = 0.63212.$$

That is, 63.2% of the components will fail before the mean life time.

Observe that this result does not depend on the value of β.

Location of Continuous Distributions

This is a good opportunity to discuss the ***location*** of a random variable (X, f_X). The location is supposed to give the "center" of the distribution. There are three main ways of doing this:

(i) The ***median*** M_X, defined by $P[X \leq M_X] = 0.5$. In the context of Example 7.4, this is the time where half of the components will have failed.

(ii) The ***mean*** $\mathrm{E}[X]$.

(iii) The ***mode*** x_0, which is the location of the maximum of f_X (if there is a unique maximum location). In the context of Example 7.4, the mode gives the time with the greatest failure density, i.e., the time around which failure is most likely. For the exponential distribution, $x_0 = 0$.

Depending on the application, any of these three measures may be referred to as the location of a distribution.

Memoryless Property of the Exponential Distribution

The exponential distribution has an interesting and unique property: it is memoryless. In other words,

$$P[X > x + s \mid X > x] = P[X > s].$$

To see this, note that

$$P[X > x] = \int_x^\infty f(t)\,dt = \int_x^\infty \lambda e^{-\lambda t}\,dt = e^{-\lambda x}.$$

Then

$$\begin{aligned} P[X > x + s \mid X > x] &= \frac{P[(X > x + s) \cap (X > x)]}{P[X > x]} = \frac{P[X > x + s]}{P[X > x]} \\ &= \frac{e^{-\lambda(x+s)}}{e^{-\lambda x}} = e^{-\lambda s} = P[X > s]. \end{aligned}$$

Time to Several Arrivals

The exponential distribution describes the time to the first (or next) arrival in a Poisson process.

Generalization: the time T_r needed for $r \in \mathbb{N} \setminus \{0\}$ arrivals to occur.

The cumulative distribution function is given by

$$\begin{aligned} F_{T_r}(t) &= P[T_r < t] \\ &= 1 - P[T_r > t] \\ &= 1 - P[\text{strictly less than } r \text{ arrivals before } t] \\ &= 1 - \sum_{n=0}^{r-1} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \end{aligned}$$

for $t > 0$ and $F_{T_r}(t) = 0$ for $t < 0$.

Time to Several Arrivals

As before, we find, for $t \geq 0$,

$$\begin{aligned} f_{T_r}(t) &= F'_{T_r}(t) \\ &= \lambda e^{-\lambda t} \sum_{n=0}^{r-1} \frac{(\lambda t)^n}{n!} - \lambda e^{-\lambda t} \sum_{n=1}^{r-1} \frac{(\lambda t)^{n-1}}{(n-1)!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{r-1}}{(r-1)!} \\ &= \frac{\lambda^r}{(r-1)!} t^{r-1} e^{-\lambda t} \end{aligned}$$

and $f_{T_r}(t) = 0$ for $t < 0$.

The Gamma Distribution

7.5. Definition. Let $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$. A continuous random variable $(X, f_{\alpha,\beta})$ with density

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

is said to follow a ***gamma distribution*** with parameters α and β. Here

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z}\, dz, \qquad \alpha > 0,$$

is the ***Euler gamma function***. for $\alpha = 1$, it becomes exp distribution

Hence, the time needed for the next r arrivals in a Poisson process with rate λ is determined by a Gamma distribution with parameters

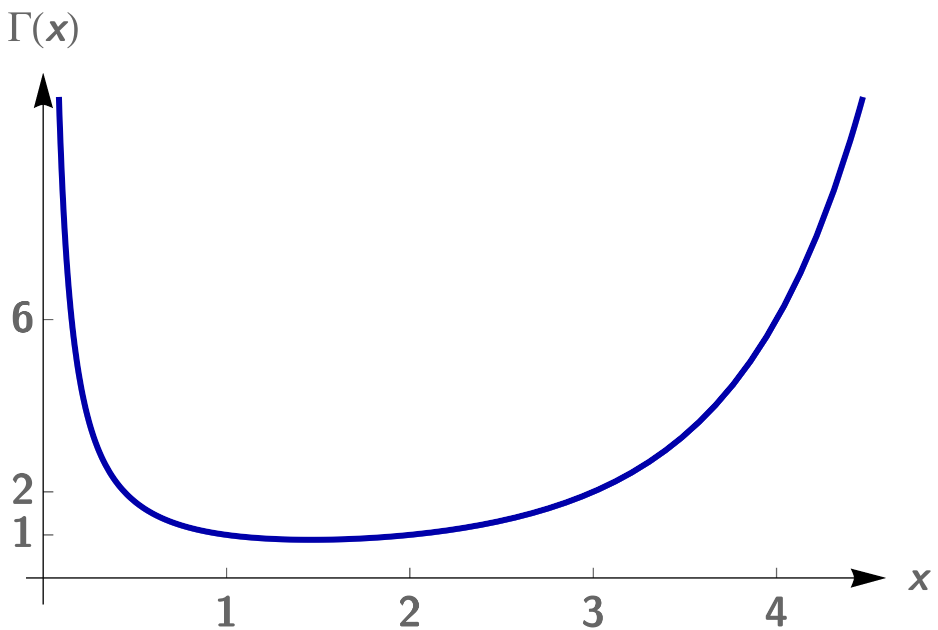
$$\alpha = r \qquad \text{and} \qquad \beta = \lambda.$$

Gamma Distribution

The gamma function satisfies $\Gamma(1) = 1$ and $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ if $\alpha > 1$. In other words,

$$n! = \Gamma(n+1) \qquad \text{for } n \in \mathbb{N}.$$

Hence it is a continuous extension of the factorial function to the positive real numbers. Below is its graph for $\alpha \in (0, 5)$.



Mean, Variance, Moment-Generating Function

7.6. Theorem. Let $(X, f_{\alpha,\beta})$ be a Gamma distributed random variable with parameters $\alpha, \beta > 0$.

(i) The moment-generating function of X is given by

$$m_X : (-\infty, \beta) \to \mathbb{R}, \qquad\qquad m_X(t) = (1 - t/\beta)^{-\alpha}.$$

(ii) $\mathrm{E}[X] = \alpha/\beta$.

(iii) $\mathrm{Var}[X] = \alpha/\beta^2$.

Mean, Variance, Moment-Generating Function

Proof.
We will verify the moment-generating function only.

$$\begin{aligned} m_X(t) = \mathrm{E}[e^{tX}] &= \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x(\beta-t)}\, dx \end{aligned}$$

Substituting $y = x(\beta - t)$, we have $dy = (\beta - t)dx$ and

$$\begin{aligned} m_X(t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} (\beta - t)^{-1} \int_0^\infty [y/(\beta - t)]^{\alpha-1} e^{-y}\, dy \\ &= \frac{(\beta - t)^{-\alpha} \beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y}\, dy \\ &= (1 - t/\beta)^{-\alpha}. \end{aligned}$$

□

The Chi-Squared Distribution

The Gamma distribution is popular for modeling applications, since its parameters allow it to be fitted to many situations.

An important example is the ***chi-squared distribution***.

7.7. Definition. Let $\gamma \in \mathbb{N}$. A continuous random variable (χ^2_γ, f_X) with density

$$f_\gamma(x) = \begin{cases} \frac{1}{\Gamma(\gamma/2)2^\alpha} x^{\gamma/2-1} e^{-x/2}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

is said to follow a chi-squared distribution with γ ***degrees of freedom***.

The chi-squared distribution is simply a gamma distribution with $\beta = 2$ and $\alpha = \gamma/2$. It is worth noting that

$$\mathrm{E}[\chi^2_\gamma] = \gamma, \qquad \mathrm{Var}[\chi^2_\gamma] = 2\gamma.$$

This distribution plays an important role in statistics.

Density of a Gamma Distribution with $\alpha = \beta = 2$