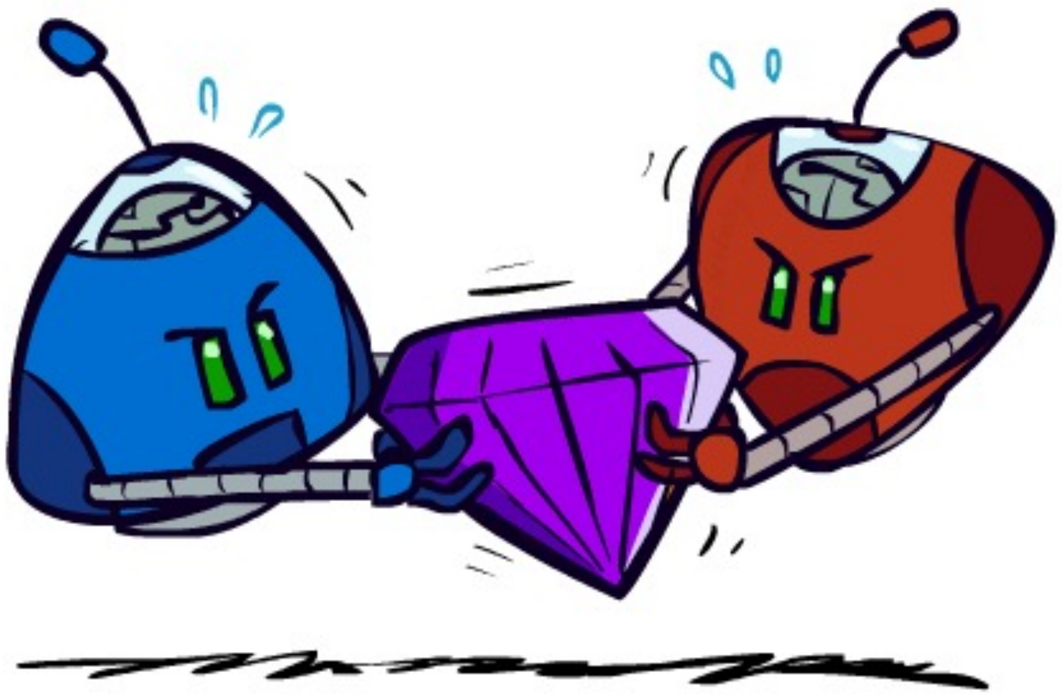
Ve492: Introduction to Artificial Intelligence
Game Theory



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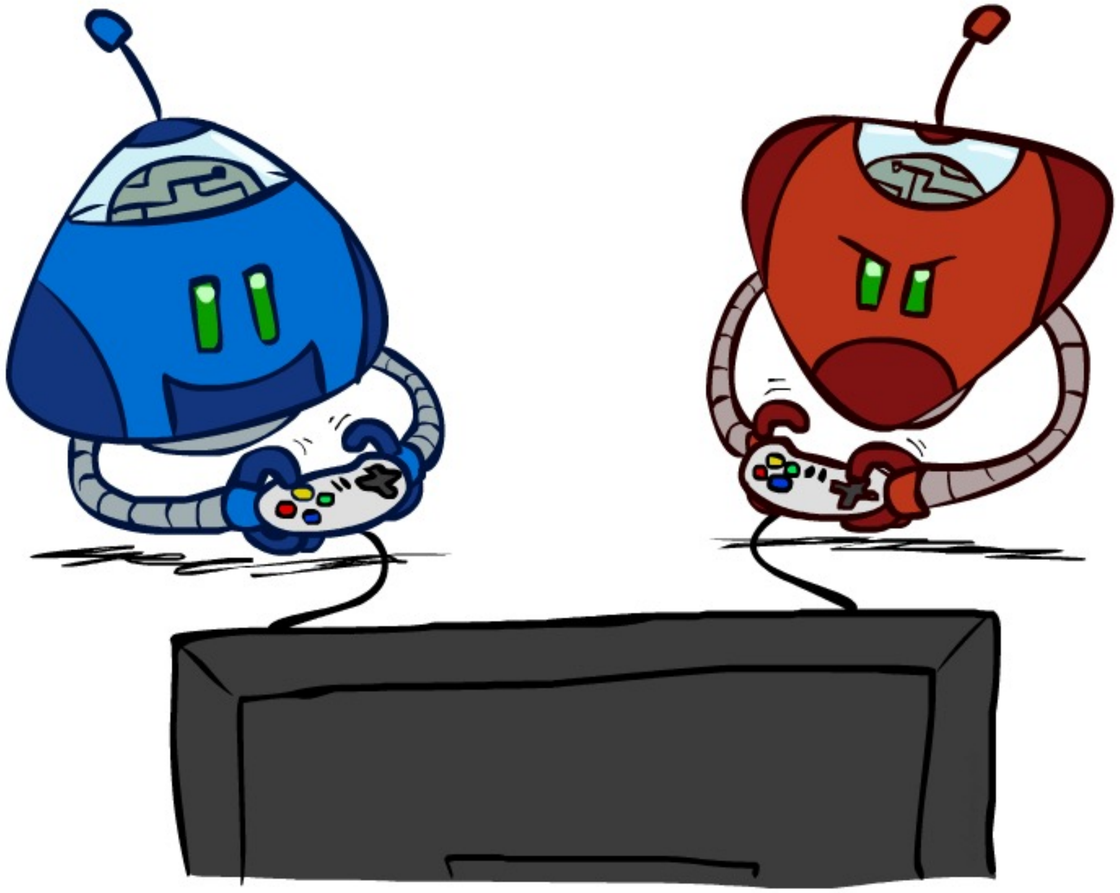
Slides adapted from http://ai.berkeley.edu, AIMA, UM, CMU

Announcements

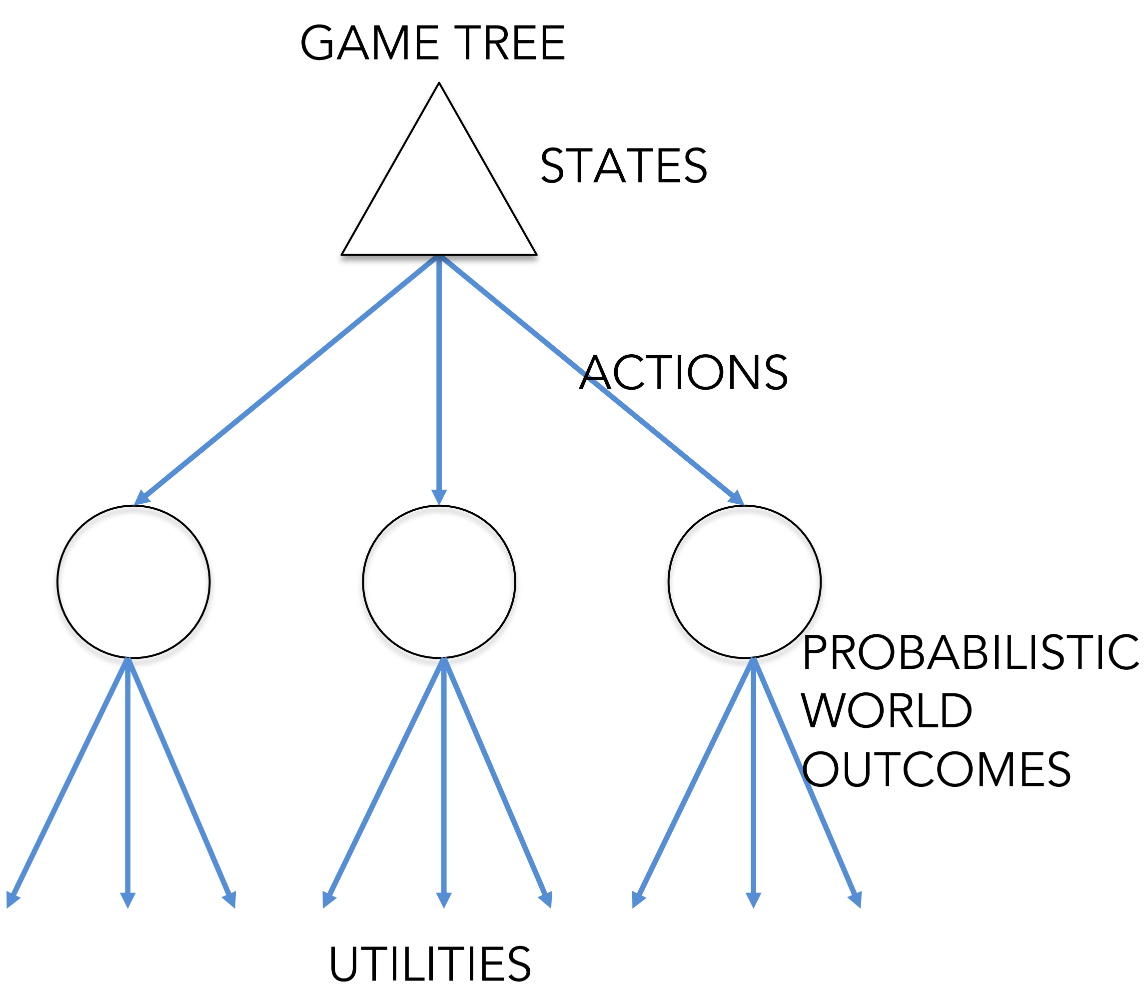
- P1 due May 31 at 11:59pm
- HW2 released today
- P2 to be released next week
- Mid-term exam June 25

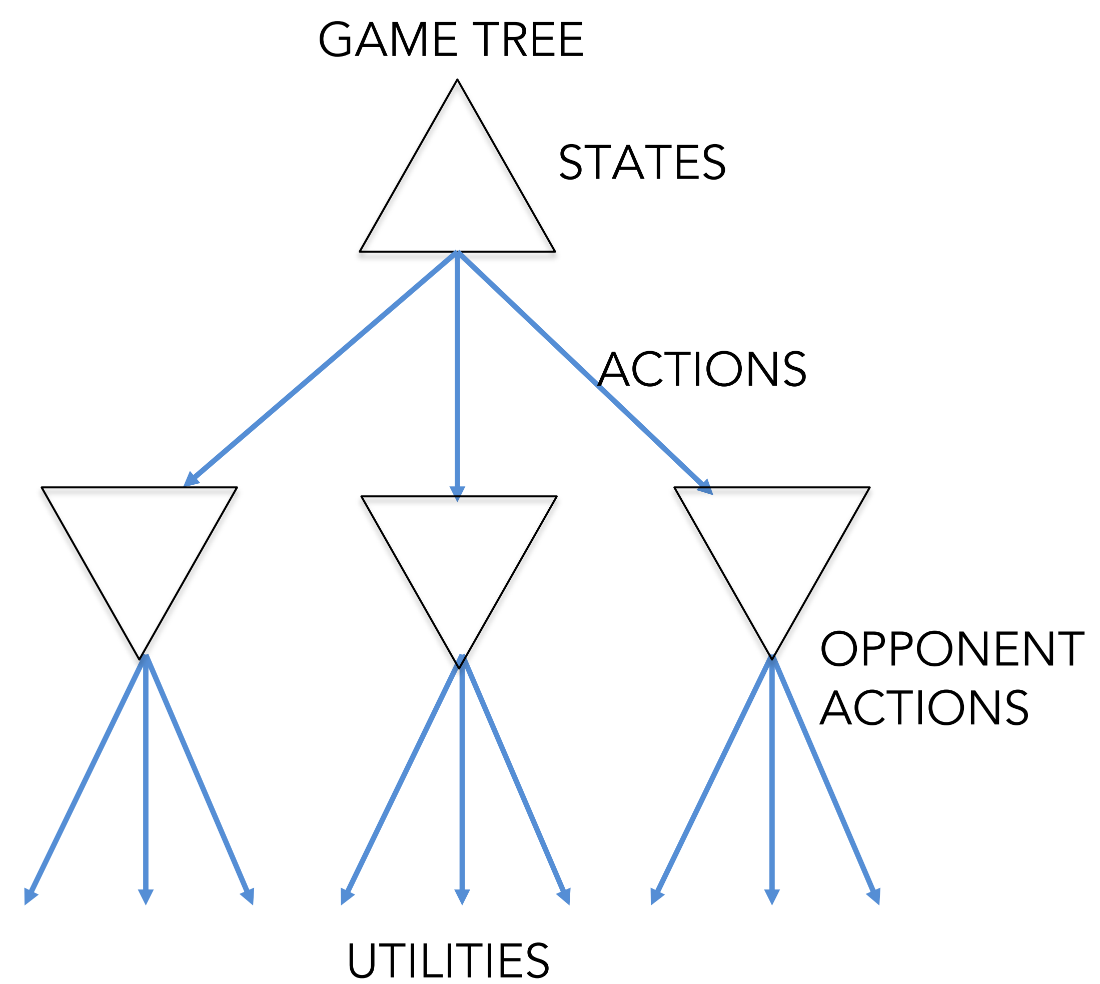
Outline

- Introduction
- Game Theory



Problems with Uncertainty vs Adversary





Decision Theory vs Game Theory

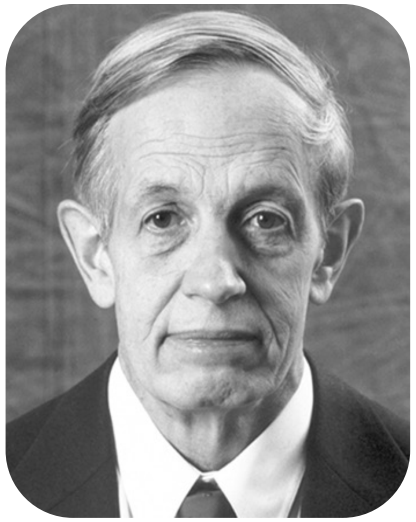
- Decision Theory: pick a strategy to maximize utility given world outcomes
- Game Theory: pick a strategy for player that maximizes her utility given the strategies of the other players
- Models are essentially the same
- Imagine the world is a player in the game!

History of Game Theory

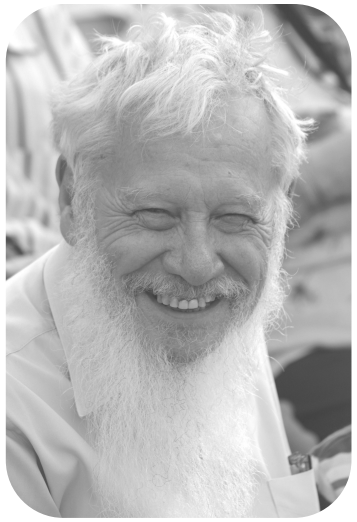
- Game theory is the study of strategic decision-making (of more than one player)
- Used in economics, political science etc.

John von Neumann

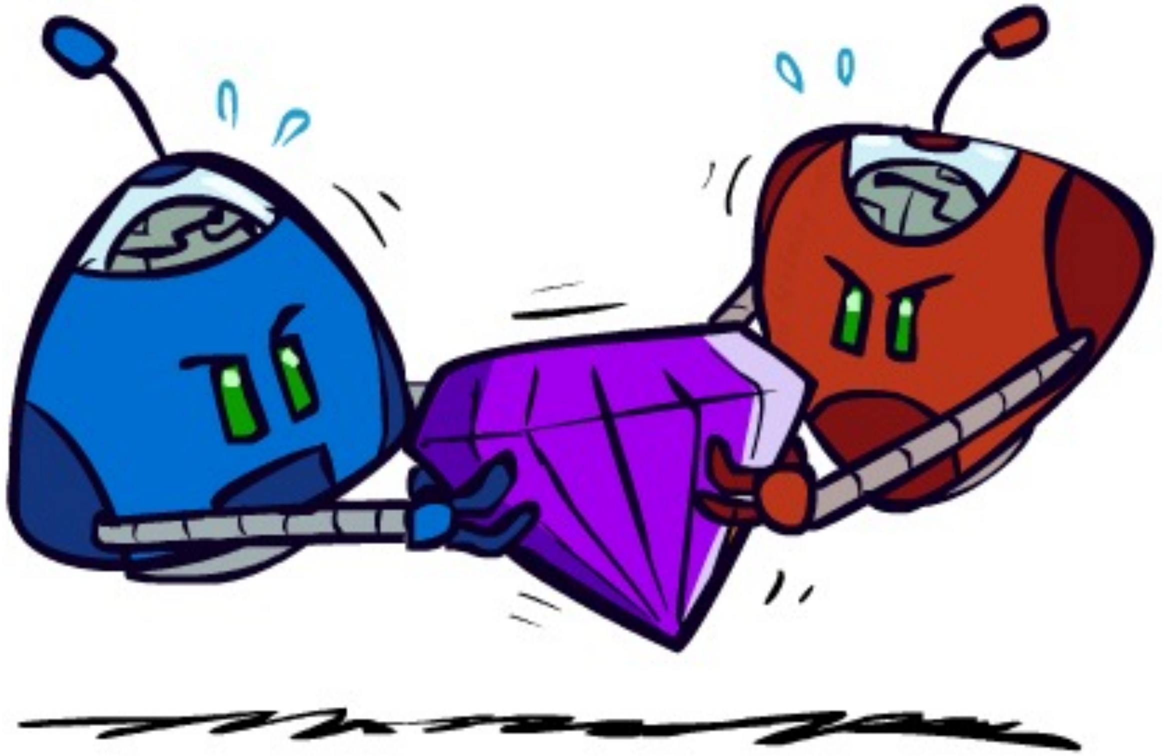
John Nash



Robert Aumann



Game Theory



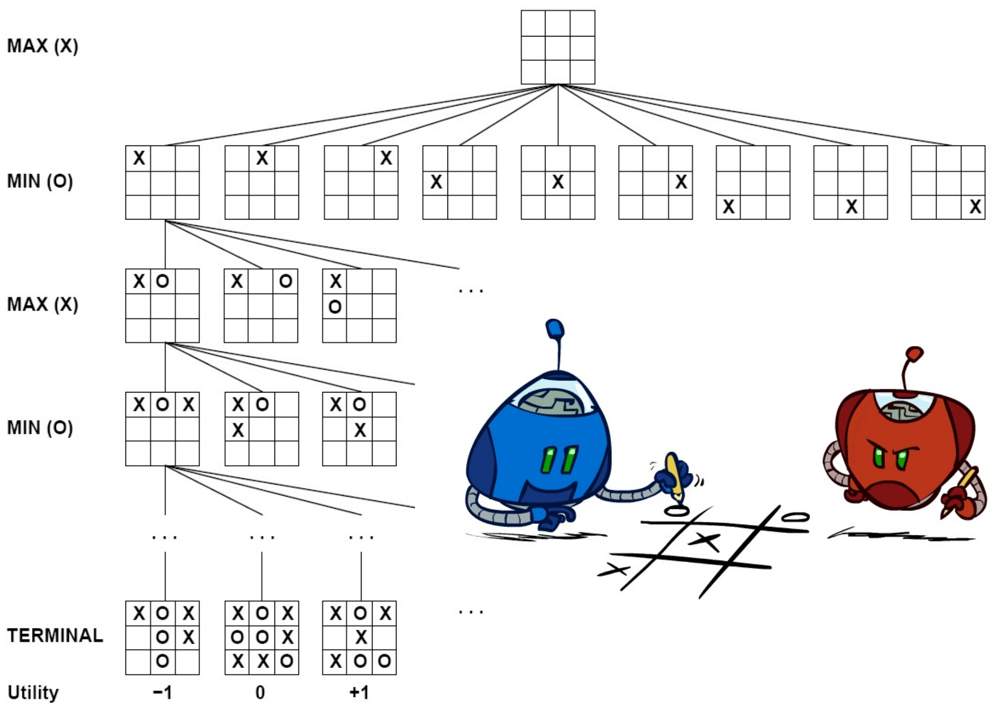
Important Notions

- Extensive Form vs Normal Form
- Strategy:
  - Pure strategy vs mixed strategy
  - Strategy profile
- Solution concepts
  - Nash equilibrium
  - Pareto optimal
  - Correlated equilibrium
- Famous games (e.g., Prisoner's dilemma)

Games: Extensive Form

- Representation:

  1. Set of all players of a game
  2. For every player, every opportunity they have to move
  3. What each player can do at each of their moves
  4. What each player knows/observes when making every move
  5. Payoffs received by everyone for all possible combo of moves



Alternative Representation: Normal Form

- Represent games as single-shot decision-making problems
- Represent only strategies (e.g., actions or policies) and utilities
- Easier to determine particular properties of games

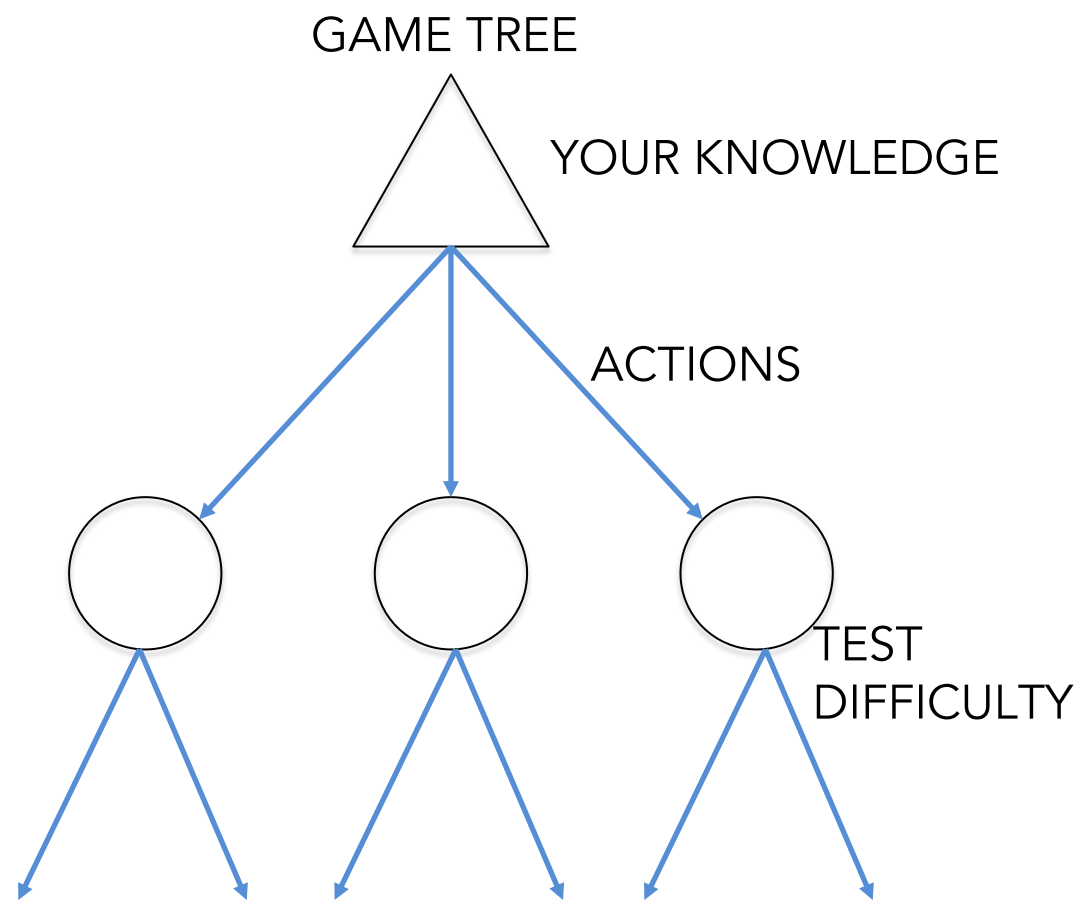
Studying – Normal Form Game

- Represent games as single shot
- Represent only strategies and utilities

STRATEGIES

	CRAM	DO HW	PLAY GAME
EASY	98	100	85
HARD	97	90	65

UTILITIES



Studying - Strategies and Utilities

- The world acts at the same time as you choose a strategy

World outcomes or adversary's strategies

STRATEGIES

	CRAM	DO HW	PLAY GAME
EASY	98	100	85
HARD	97	90	65

UTILITIES

Strategy/Utility Notations

- Strategy k for player $= \pi_k \in \Pi$ where Π is finite
- Utility $u(\pi_k, s)$ where s is a state of the world
- Which strategy should I adopt?
  - Maximize the expected utility based on state probabilities
- Is it beneficial to choose a strategy in a random way?

STRATEGIES

	CRAM	DO HW	PLAY GAME
EASY	98	100	85
HARD	97	90	65

UTILITIES

Mixed Strategies

- Pure strategies Π
- Mixed strategies $\Delta(\Pi)$ = set of probability distributions over Π
- Goal: Pick strategy that maximizes expected utility given exam probability

Pure Strategy: deterministic (p=1) selection of actions

STRATEGIES

	CRAM	DO HW	PLAY GAME
EASY	98	100	85
HARD	97	90	65

UTILITIES

Strategy 1: Always cram
Strategy 2: Always do HW
Strategy 3: Always play games
Strategy 4: ½ cram, ½ do HW
…

Mixed Strategy: randomized selection

Calculating Utilities of Pure Strategies

- What is the utility of pure strategy: CRAM?

$$u(CRAM) = P(Easy) \cdot u(CRAM, Easy) + P(Hard) \cdot u(CRAM, Hard)$$

- General formula:

$$u(\pi) = \sum_{s} P(s) \cdot u(\pi, s)$$

CRAM	DO HW	PLAY GAME
98	100	85
97	90	65

P(Easy) = .2
P(Hard) = .8

Calculating Utilities of Pure Strategies

- What is the utility of pure strategy: DO HW?
- What is the utility of pure strategy: PLAY GAME?

CRAM	DO HW	PLAY GAME
98	100	85
97	90	65

P(Easy) = .2
P(Hard) = .8

Calculating Utilities of Mixed Strategies

- What is the utility of mixed strategy: σ = (½ CRAM, ½ DO HW)?

$$u(\sigma) = \mathrm{P}_\sigma(\mathrm{CRAM})\Big(\sum_s \mathrm{P}(s)u(\mathrm{CRAM}, s)\Big) + \mathrm{P}_\sigma(\mathrm{DO\ HW})\Big(\sum_s \mathrm{P}(s)u(\mathrm{DO\ HW}, s)\Big)$$

- General formula:

$$u(\sigma) = \sum_k \sum_s \mathrm{P}(s)\mathrm{P}_\sigma(\pi_k)u(\pi_k, s) = \sum_k \mathrm{P}_\sigma(\pi_k) \sum_s \mathrm{P}(s)u(\pi_k, s)$$

CRAM	DO HW	PLAY GAME
98	100	85
97	90	65

P(Easy) = .2

P(Hard) = .8

Quiz: Grocery Shopping Transportation Decision

Suppose you want to decide how to get groceries from the store

	BIKE	WALK	BUS	DRIVE
SUN	1	2	1	1
RAIN	-2	-4	-1	0

1. How many pure strategies to do you have?

   A) 1 B) 2 C) 3 D) 4 E) Infinite

2. How many mixed strategies do you have?

   A) 4 B) 8 C) 16 D) 64 E) Infinite

3. What is your best pure strategy?

   A) Bike B) Walk C) Bus D) Drive E) It depends

Quiz: Grocery Shopping Transportation Decision

Suppose you want to decide how to get groceries from the store

	BIKE	WALK	BUS	DRIVE	
SUN	1	2	1	1	P=.5
RAIN	-2	-4	-1	0	P=.5

4. What is your best pure strategy?

   A) Bike B) Walk C) Bus D) Drive E) It depends

5. What is the utility of a ¼ walk, ¼ bike, and ½ drive strategy?

   A) -1/8 B) -1/4 C) -1/2 D) 1/8 E) 1/2

Game: Rock, Paper, Scissors

- Each player simultaneously picks rock, paper, or scissors
- Rock beats scissors, scissors beats paper, paper beats rock

P1's Strategies
$\Pi_1 = \{rock, paper, scissors\}$

P2's Strategies
$\Pi_2 = \{rock, paper, scissors\}$

Joint Utilities

- When both players choose their actions, they receive a utility based on both of their choices

P2's ACTIONS

		PLAYER 2		
		ROCK	PAPER	SCISSORS
PLAYER 1	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

JOINT UTILITIES

Normal Form Notation

- Players: $\{1, \ldots, N\}$
- Pure strategies for each player i
  - $\pi_{i,1}, \ldots, \pi_{i,n_i}$
- Utility functions that maps a strategy per player to a reward for player i
  - $u_i(\pi_1, \ldots, \pi_N) = u_i(\vec{\pi})$
- Strategy profile:
  - $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$ $\quad$ $\vec{\pi}_{-i} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_N)$

P2's ACTIONS

		PLAYER 2		
		ROCK	PAPER	SCISSORS
PLAYER 1	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

JOINT UTILITIES

Zero-Sum Games

- If each cell in the table sums to 0, the game is zero-sum:

$$\forall \vec{\pi}, \sum_i u_i(\vec{\pi}) = 0$$

- Is Rock, Paper, Scissors zero-sum?

P2's ACTIONS

		PLAYER 2		
		ROCK	PAPER	SCISSORS
PLAYER 1	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

JOINT UTILITIES

Solution Concepts

- Solution concept
  - Subset of outcomes of the games that are possibly interesting
  - Generally assumes that players are rational
- Minimax solution
- Nash equilibrium (NE)
  - Best response
  - Dominant strategies
  - With pure strategies vs with mixed strategies
  - Weak vs strict NE
- Pareto-optimal solutions
- Correlated equilibrium

Strategies for Games

- Best response against $\vec{\pi}_{-i}$
  - Strategy for player i that maximizes her utility given the strategy of the other players

Pure Strategies:

P2 always picks rock
P1 should ___________

P2 always picks paper
P1 should ___________

Mixed Strategies:

P2 randomly chooses between 50% rock and 50% paper
P1 should ___________

P2's ACTIONS

		PLAYER 2		
		ROCK	PAPER	SCISSORS
PLAYER 1	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

JOINT UTILITIES

Dominant Strategies

- A strategy $\pi_{i,k}$ for player i is strictly dominant if it is better than all other strategies for player i no matter any opponent's strategy:
  - $\forall k' \neq k, u_i(\pi_{i,k}, \vec{\pi}_{-i}) > u_i(\pi_{i,k'}, \vec{\pi}_{-i})$

	A	B	C	D	E
i	2,10	4,7	4,6	5,2	3,8
ii	3,8	6,4	5,2	1,3	2,6
iii	5,3	3,1	2,2	4,1	3,0
iv	6,7	9,5	7,5	8,5	3,5

Dominant Strategies

- A strategy $\pi_{i,k}$ for player i is strictly dominant if it is better than all other strategies for player i no matter any opponent's strategy:
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	A	B	C	D	E
i	2,10	4,7	4,6	5,2	3,8
ii	3,8	6,4	5,2	1,3	2,6
iii	5,3	3,1	2,2	4,1	3,0
iv	6,7	9,5	7,5	8,5	3,5

Is there always a dominant strategy?

- No! There is no dominant strategy in Rock, Paper, Scissors, for example.

P2's ACTIONS

		PLAYER 2		
		ROCK	PAPER	SCISSORS
PLAYER 1	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

JOINT UTILITIES

Prisoner's Dilemma

- 2 Players {1,2}
- Each as 2 strategies {Cooperate,Defect}
- Utilities in table:

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

- Is there a dominant strategy?
  - Yes!
- What is the best joint strategy for both prisoners?
  - Best joint strategy: prisoners cooperate

Measure of Social Welfare

- The sum of the utilities of the players is the social welfare

- SW(C,C) = -2
- SW(C,D) = -5
- SW(D,C) = -5
- SW(D,D) = -6

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

Prisoner's Dilemma

- Compute best responses

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

Prisoner's Dilemma

- Strategy profile (C, C) is not stable
- Each prisoner would profit by switching to defection assuming that the other prisoner continues to cooperate

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

Prisoner's Dilemma

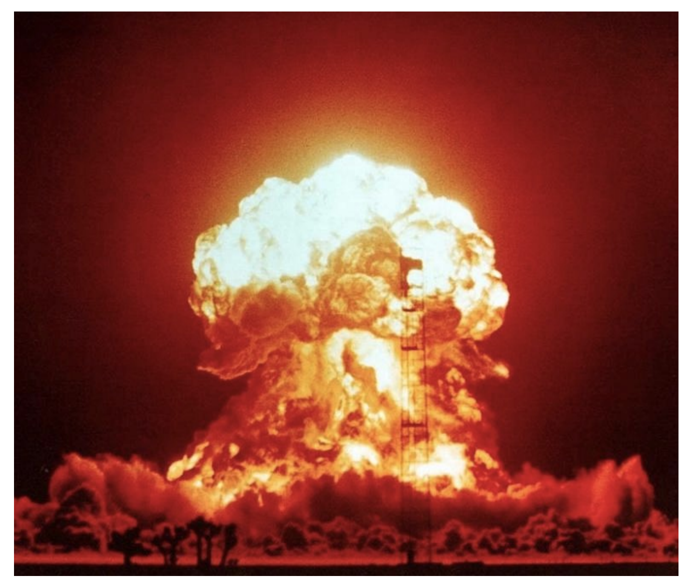
- If they both trust that the other prisoner will cooperate, each should defect. But both defecting results in lower scores!

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

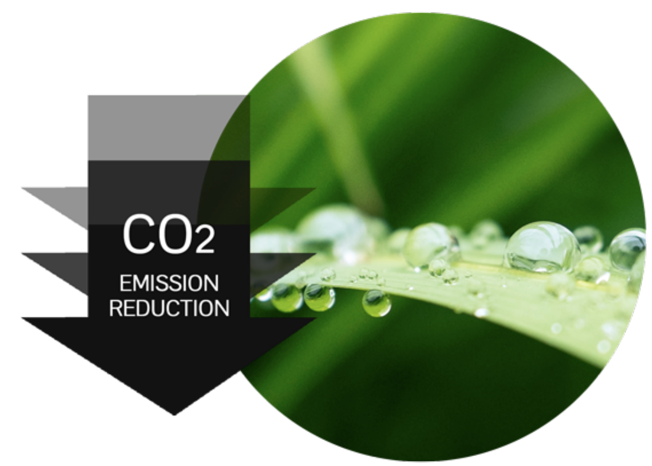
Tragedy of the Commons

- Individuals act in their own self-interest contrary to the common good

Political Ads



Nuclear Arms Race



CO2 Emissions

Nash Equilibrium

- Nash Equilibria: strategy profiles $\vec{\pi}$ where none of the participants benefit from unilaterally changing their decisions:

$$\forall i, u_i(\vec{\pi}) \geq u_i(\pi_i', \vec{\pi}_{-i})$$

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

Nash Equilibrium

- NOT A NASH EQUILIBRIUM - participants benefit from unilaterally changing their decision

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

Nash Equilibrium

- Strict Nash Equilibria are Nash Equilibria where the ”neighbor” strategy profiles have strictly less utility.

  - $\forall i, u_i(\vec{\pi}) > u_i(\pi_i', \vec{\pi}_{-i})$

		PRISONER 2	
		Cooperate	Defect
PRISONER 1	Cooperate	-1,-1	-5,0
	Defect	0,-5	-3,-3

Professor's Dilemma!

- What is / are the Nash equilibrium / equilibria?
- Which are strict Nash equilibria?

		Student	
		Study	Games
Professor	Effort	1000,1000	0,-10
	Slack	-10,0	0,0

Finding a Pure Nash Equilibrium

Pure Nash Equilibria are composed of pure strategies

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a dominant strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly dominated strategy and recurse

Finding a Pure Nash Equilibrium

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a dominating strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly dominated strategy and recurse

	L	C	R
U	10,3	1,5	5,4
M	3,1	2,4	5,2
D	0,10	1,8	7,0

	L	C
U	10,3	1,5
M	3,1	2,4
D	0,10	1,8

	L	C
U	10,3	1,5
M	3,1	2,4

	C
U	1,5
M	2,4

	C
M	2,4

Finding Nash Equilibrium Example 1

	A	B	C	D	E
i	2,10	4,7	4,6	5,2	3,8
ii	3,8	6,4	5,2	1,3	2,6
iii	5,3	3,1	2,2	4,1	3,0
iv	6,7	9,5	7,5	8,5	4,5

Finding Nash Equilibrium Example 2

	A	B	C	D	E
i	2,**4**	4,7	4,6	5,2	3,8
ii	3,8	6,4	5,2	1,3	2,6
iii	5,3	3,1	2,2	**9**,1	3,0
iv	6,7	9,5	**5**,5	8,5	4,5

No longer strict dominant strategies!

Finding Nash Equilibrium Example 2

	A	B	C	D	E
i	2,4	4,7	4,6	5,2	3,8
ii	3,8	6,4	5,2	1,3	2,6
iii	5,3	3,1	2,2	9,1	3,0
iv	6,7	9,5	5,5	8,5	4,5

>

D is strictly dominated by A

Finding Nash Equilibrium Example 2

	A	B	C	D	E
i	2,4	4,7	4,6	5,2	3,8
ii	3,8	6,4	5,2	1,3	2,6
iii	5,3	3,1	2,2	9,1	3,0
iv	6,7	9,5	5,5	8,5	4,5

D is weakly dominated by B

Finding Nash Equilibrium Example 2

	A	B	C	E
i	2,4	4,7	4,6	3,8
ii	3,8	6,4	5,2	2,6
iii	5,3	3,1	2,2	3,0
iv	6,7	9,5	5,5	4,5

Finding Nash Equilibrium Example 2

	A	B	C	E	
i	2,4	4,7	4,6	3,8	
ii	3,8	6,4	5,2	2,6	
iii	5,3	3,1	2,2	3,0	<
iv	6,7	9,5	5,5	4,5	

iii is strictly dominated by iv

Finding Nash Equilibrium Example 2

	A	B	C	E	
i	2,4	4,7	4,6	3,8	
ii	3,8	6,4	5,2	2,6	
iii	5,3	3,1	2,2	3,0	<
iv	6,7	9,5	5,5	4,5	

i is strictly dominated by iv

Finding Nash Equilibrium Example 2

	A	B	C		E
ii	3,8	6,4	5,2		2,6
iv	6,7	9,5	4,5		4,5

Finding Nash Equilibrium Example 2

	A	B	C		E
ii	3,8	6,4	5,2		2,6
iv	6,7	9,5	4,5		4,5

>

E is strictly dominated by A

Finding Nash Equilibrium Example 2

	A	B	C	E
ii	3,8	6,4	5,2	2,6
iv	6,7	9,5	4,5	4,5

>

C is strictly dominated by A

Finding Nash Equilibrium Example 2

	A	B	C	E
ii	3,8	6,4	5,2	2,6
iv	6,7	9,5	4,5	4,5

>

B is strictly dominated by A

Finding Nash Equilibrium Example 2

	A
ii	3,8
iv	6,7

ii is strictly dominated by iv

Finding Nash Equilibrium Example 2

	A
iv	6,7

Finding Nash Equilibrium Example 3 (Battle of Sexes)

	Opera	Football
Opera	(3, 2)	(0, 0)
Football	(0, 0)	(2, 3)

Finding Nash Equilibrium: Rock, Paper, Scissors

- Nash Equilibrium?
  - Not with pure strategies!

PLAYER 2

PLAYER 1		ROCK	PAPER	SCISSORS
	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

Nash Equilibria always exist in finite games

- Theorem (Nash, 1950)
  - If there are a finite number of players and each player has a finite number of actions, there always exists a Nash Equilibrium.
- The NE may be with pure or mixed strategies.

Calculating Utilities of Mixed Strategies

- Decision Theory Version:

$$u(\sigma) = \sum_k \sum_s P(s) P_\sigma(\pi_k) u(\pi_k, s)$$

- Game Theory Version:

$$u(\vec{\sigma}) = \sum_{\pi_1,\ldots,\pi_N} \prod_i P_{\sigma_i}(\pi_i) u(\pi_1, \ldots, \pi_N)$$

Example: Calculating Utilities

- What is u_1 for $\sigma_1 = (1/2,1/2,0)$ and $\sigma_2 = (0,1/2,1/2)$?
- Is $[\sigma_1 = (1/2,1/2,0), \sigma_2 = (0,1/2,1/2)]$ a mixed strategy equilibrium?

PLAYER 2

PLAYER 1		ROCK	PAPER	SCISSORS
	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

Finding the Mixed Strategy Nash Equilibrium

- What features of a mixed strategy profile qualify it as a NE?

- There is no reason for any player to deviate from their strategy, which occurs when the utilities of the weighted actions are equal and are as large as possible!

Finding the Mixed Strategy Nash Equilibrium

- P1
- P2

		PLAYER 2		
		ROCK	PAPER	SCISSORS
PLAYER 1	ROCK	0,0	-1,1	1,-1
	PAPER	1,-1	0,0	-1,1
	SCISSORS	-1,1	1,-1	0,0

Other Solution Concepts

- Correlated Equilibrium
- Pareto Optimal / Dominated

Correlated Equilibrium

- Suppose a mediator computes the best combined strategy $\sigma \in \Delta(\Pi_1 \times \Pi_2)$ for P1 and P2, samples a strategy profile (π_1, π_2), and shares π_1 with P1 and π_2 with P2

- The strategy is a CE if $\forall \pi_1' \in \Pi_1$
- $\sum_{\pi_2} P_\sigma(\pi_1, \pi_2) u_1(\pi_1, \pi_2) \geq \sum_{\pi_2} P_\sigma(\pi_1, \pi_2) u_1(\pi_1', \pi_2)$

- And the same for the other player

Game of Chicken

	Dare	Chicken out
Dare	(0, 0)	(7, 2)
Chicken out	(2, 7)	(6, 6)

Pareto Optimal and Pareto Dominated

- An outcome $u(\vec{\sigma}) = (u_1(\vec{\sigma}), \ldots, u_1(\vec{\sigma}))$ is Pareto optimal if there is no other outcome that all players would prefer, i.e., each player gets higher utility
  - At least one player would be disappointed in changing strategy
- An outcome $u(\vec{\sigma}) = (u_1(\vec{\sigma}), \ldots, u_1(\vec{\sigma}))$ is Pareto dominated by another outcome if all the players would prefer the other outcome

Summary

- Vocabulary
- Pure/Mixed Strategies (and calculating them)
- Zero-Sum Games
- Dominant vs Dominated Strategies
- Strict/Weak Nash Equilibrium
- Prisoner's dilemma, Tragedy of the commons
- Correlated Equilibrium
- Pareto Optimal/Dominated
- Social Welfare