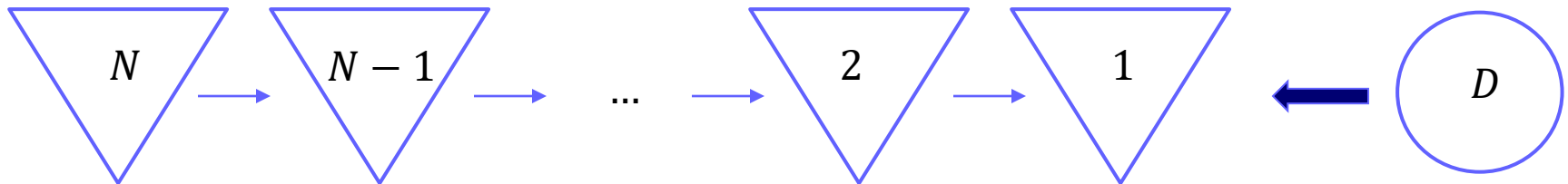# LEC009 Inventory Management IV

VG441 SS2021

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Multi-Echelon Problem

- N stages of a (serial) supply chain

- Demand rate is λ deterministically only at stage 1

- Stockouts are not allowed

$$k_j = \text{ fixed cost for orders of stage } j$$
$$h'_j = \text{ inventory holding-cost rate for stage } j$$
$$I'_j(t) = \text{ (local) inventory of stage } j \text{ at time } t$$
$$L'_j = \text{ leadtime for stage-}j \text{ orders } = 0 \text{ W.L.O.G.}$$

Question: what is the optimal ordering strategy?

$$k_j = \text{ fixed cost for orders of stage } j$$

$$h'_j = \text{ inventory holding-cost rate for stage } j$$

$$I'_j(t) = \text{ (local) inventory of stage } j \text{ at time } t$$
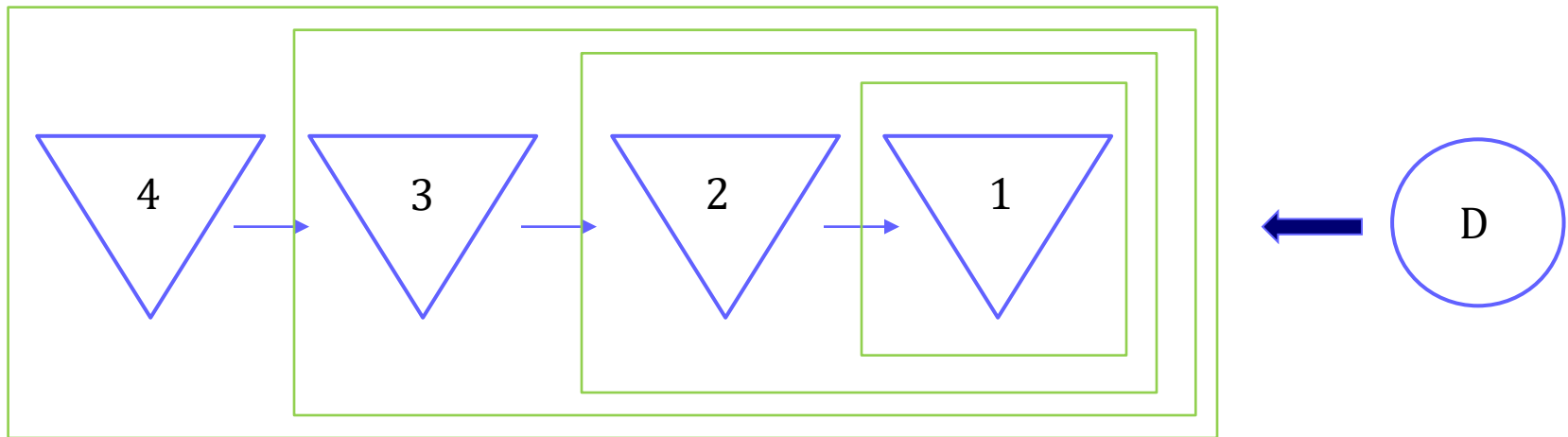
$$I_j(t) = \text{echelon inventory of stage } j \text{ at time } t = \Sigma_{i \leq j} I'_i(t)$$

$$h_j = \text{echelon-inventory holding-cost rate for stage } j = h'_j - h'_{j+1} \geq 0$$
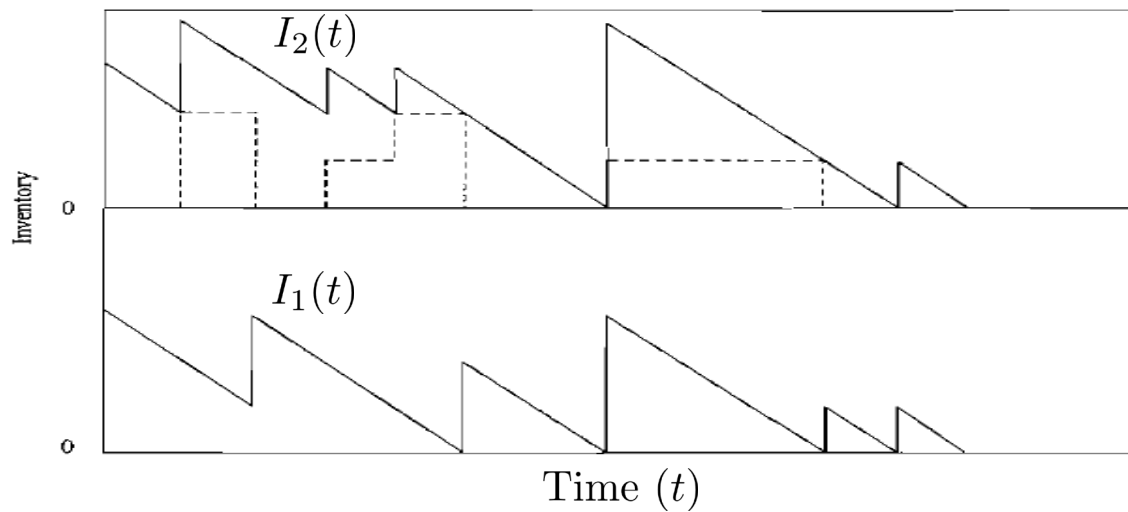
$$\sum_j h'_j I'_j(t) = \sum_j h_j I_j(t) \quad \text{for all } t$$

Local inventories over time

Echelon inventories over time

Policy Structures

- A policy is <mark>nested</mark> if for all j, whenever stage j orders, so does stages $j - 1$.

- A policy is <mark>ZIO</mark> (zero-inventory-ordering) if order occurs only when its echelon inventory is zero.

Convince yourself that nested and ZIO are optimal.

Stationary-Interval Policies

u_j = order interval for stage j
$\mathbf{u}$ = the vector $(u_j)_j$
$g_j = h_j \lambda$
$C(\mathbf{u})$ = average cost of the policy specified by $\mathbf{u}$

$$C(\mathbf{u}) = \sum_j \left[\frac{k_j}{u_j} + \frac{1}{2} g_j u_j \right]$$

Minimize $\quad C(\mathbf{u})$

subject to $\quad u_j = \xi_j u_{j-1}, \qquad$ for all $j = 2, \ldots, J$

$\qquad\qquad\quad \xi_j \in \mathbb{Z}^+, \qquad$ for all $j = 2, \ldots, J$

Caveat: This MILP may be difficult to solve.

Here is the Plan

- First, we solve a simpler "relaxed" problem.

- Solution of the relaxed problem is a lower bound.

- We round off this relaxed solution to obtain a feasible solution.

- We get an upper bound on this feasible solution.

- Show that the two bounds are close together.

Optimality Gap

○ Upper Bound of Feasible Solution

○ Feasible Solution (based on rounding)

☆ Optimal Solution (OPT)

○ Relaxed Solution

u_j = order interval for item j
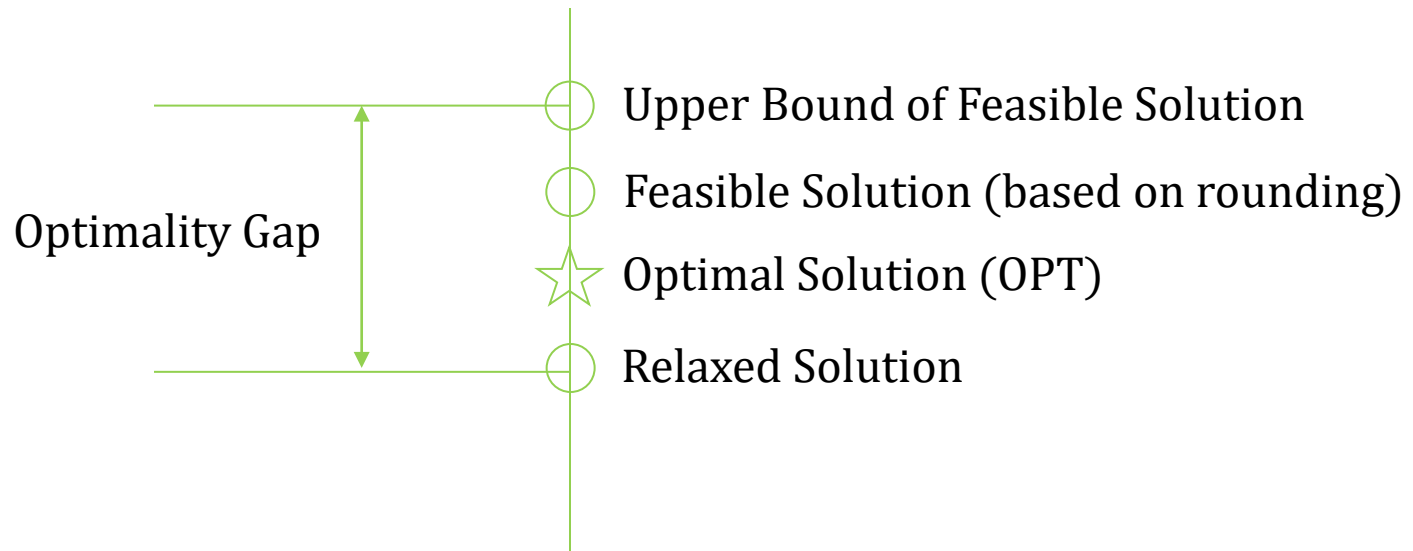$\mathbf{u}$ = the vector $(u_j)_j$
$g_j = h_j \lambda$
$C(\mathbf{u}) = \sum_j \left[\frac{k_j}{u_j} + \frac{1}{2} g_j u_j \right]$

$$\begin{aligned}
\text{Minimize} \quad & C(\mathbf{u}) \\
\text{subject to} \quad & u_j = \xi_j u_{j-1}, && \text{for all } j = 2, \ldots, J \\
& \xi_j \in \mathbb{Z}^+, && \text{for all } j = 2, \ldots, J
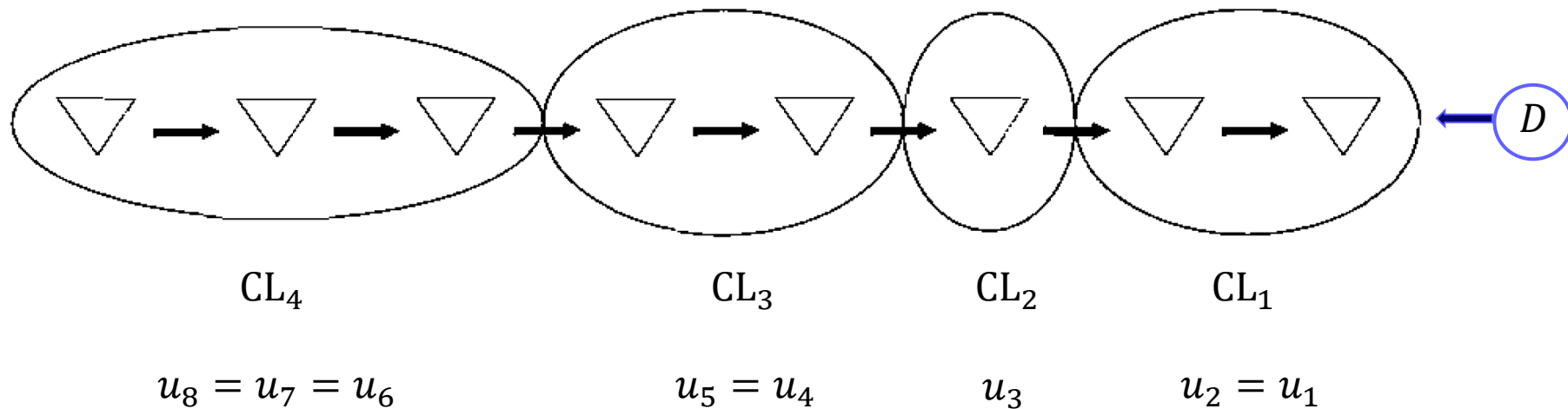\end{aligned}$$

Relax the constraints ...

$$\begin{aligned}
\text{Minimize} \quad & C(\mathbf{u}) \\
\text{subject to} \quad & u_j \geq u_{j-1}, && \text{for all } j = 2, \ldots, J
\end{aligned}$$

Relaxed and Feasible Solutions

$$\text{Minimize} \qquad C(\mathbf{u})$$

$$\text{subject to} \qquad u_j \geq u_{j-1}, \qquad \text{for all } j = 2, \ldots, J$$

Clusters.

$$u_8 = u_7 = u_6 \qquad\qquad u_5 = u_4 \qquad u_3 \qquad u_2 = u_1$$

Feasible solution: round to the nearest power-of-2

Gap between this feasible solution and the relaxed solution: ~1.06

Deterministic Demand (Nonstationary)

Input: T period demands $d_1, \ldots, d_T$
Decisions: $q_j(t), I_j(t)$

$$
\begin{aligned}
\min \quad & \sum_j \sum_t I_j(t) h_j + \sum_j \sum_t k_j \mathbb{1}\left(q_j(t) > 0\right) \\
\text{s.t.} \quad & I_j(t) = I_j(t-1) + q_j(t) - d_t && \forall t, \ \forall j \\
& I_j(t-1) + q_j(t) \geq I_{j-1}(t-1) + q_{j-1}(t) && \forall t, \ \forall j \\
& I_j(t) \geq 0, q_j(t) \geq 0 && \forall t, \ \forall j \\
& I_j(0) = 0 && \forall j
\end{aligned}
$$

Dynamic Programming

- Define $F(i, s, t)$ as the optimal cost of subproblem defined for stages $i, \dots, 1$ and periods $[s, t)$
- ZIO: $I_i(s-1) = I_i(t-1) = 0$
- Nested: $I_j(s-1) = I_j(t-1) = 0$, for all $j < i$
- Goal: $F(N, 1, T+1)$
- Boundary Conditions:
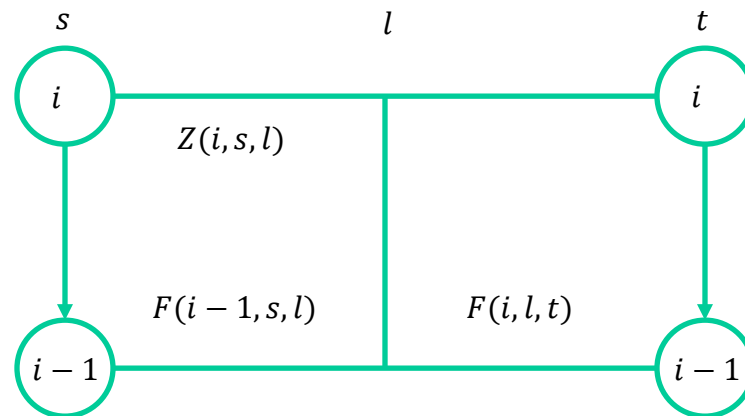
$$F(0, \cdot, \cdot) = 0$$

$$F(\cdot, T+1, \cdot) = 0$$

$$F(i, s, s+1) = \sum_{j=1}^{i} k_j \text{ for all } i$$

- For each time $s < t$ and stage i, the cost of covering $d_s, \dots, d_{t-1}$

$$Z_{s,t}^i = k_i + \sum_{a=s}^{t-2} h_a \sum_{b=a-1}^{t-1} d_b$$

- Dynamic Programming (on graphical models)

$$F(i,s,t) = \min_{s \le l \le t-1} \left\{ Z_{s,l}^i + F(i-1,s,l) + F(i,l,t) \right\}$$

Stochastic Model

- Stage 1 faces stochastic demand D per period and penalty p
- Lead times $L_1, \ldots, L_N$ and lead time demand $D_1, \ldots, D_N$
- <mark>Echelon base-stock</mark> policy is optimal.

Theorem: Let $\underline{g}_0(x) = (p + h'_1)\, x^-$. For $j = 1, \ldots, N$, let

$$\hat{g}_j(x) = h_j x + \underline{g}_{j-1}(x)$$

$$g_j(y) = \mathbb{E}\left[\hat{g}_j\left(y - D_j\right)\right]$$

$$S_j^* = \operatorname{argmin}\left\{g_j(y)\right\}$$

$$\underline{g}_j(x) = g_j\left(\min\left\{S_j^*, x\right\}\right)$$

Then $\mathbf{S}^* = \left(S_j^*\right)_{j=1}^{N}$ is the optimal base-stock vector and $g_N\left(S_N^*\right)$ is the corresponding optimal cost

Stochastic Model

- Use the following heuristics:

Theorem 6.4 (Shang and Song (2003)): For any j and y
(a) $g_j^l(y) \leq g_j(y) \leq g_j^u(y)$
(b) $S_j^l \leq S_j^* \leq S_j^u$

$$S_j^l = \tilde{F}_j^{-1}\left(\frac{p + \sum_{i=j+1}^N h_i}{p + \sum_{i=1}^N h_i}\right) \quad \leq \quad S_j^* \quad \leq \quad S_j^u = \tilde{F}_j^{-1}\left(\frac{p + \sum_{i=j+1}^N h_i}{p + \sum_{i=j}^N h_i}\right)$$

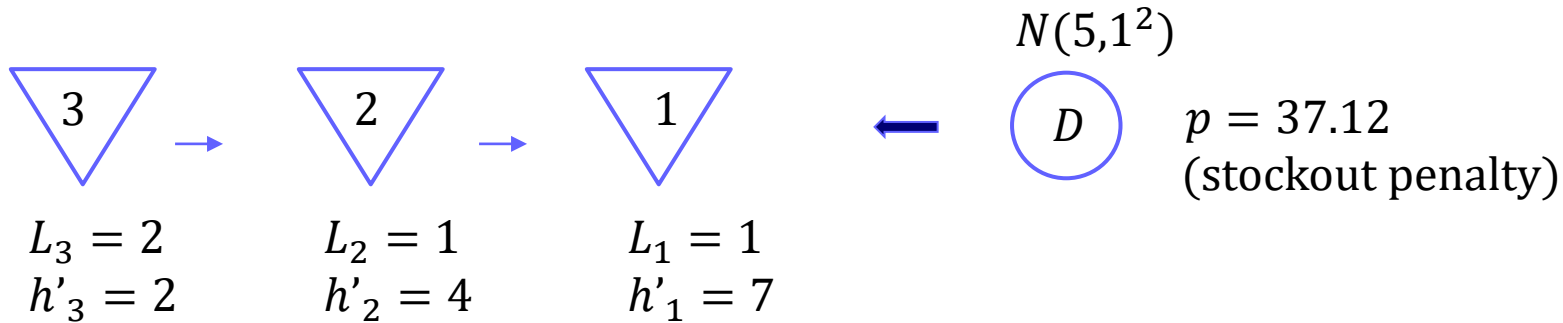$$\tilde{D}_j = \sum_{i=1}^j D_i$$

Lead-time demand with lead time $\sum_{i=1}^j L_i$

Computational optimality gap $\leq 1\%$

$$N(5,1^2)$$

$$\boxed{3} \rightarrow \boxed{2} \rightarrow \boxed{1} \quad \leftarrow \quad \bigcirc D$$

$p = 37.12$
(stockout penalty)

$L_3 = 2$ $L_2 = 1$ $L_1 = 1$
$h'_3 = 2$ $h'_2 = 4$ $h'_1 = 7$

$$\tilde{D}_1 \sim N\left(5 \cdot 1, 1^2 \cdot 1\right) = N(5,1)$$
$$\tilde{D}_2 \sim N\left(5 \cdot 2, 1^2 \cdot 2\right) = N(10,2)$$
$$\tilde{D}_3 \sim N\left(5 \cdot 4, 1^2 \cdot 4\right) = N(20,4)$$

We have $(h_1, h_2, h_3) = (3, 2, 2)$. Therefore:

$$S_1^u = \tilde{F}_1^{-1}\left(\tfrac{37.12+4}{37.12+7}\right) = 6.49 \qquad S_1^l = \tilde{F}_1^{-1}\left(\tfrac{37.12+4}{37.12+7}\right) = 6.49$$
$$S_2^u = \tilde{F}_2^{-1}\left(\tfrac{37.12+2}{37.12+4}\right) = 12.35 \qquad S_2^l = \tilde{F}_2^{-1}\left(\tfrac{37.12+2}{37.12+7}\right) = 11.71$$
$$S_3^u = \tilde{F}_3^{-1}\left(\tfrac{37.12+0}{37.12+2}\right) = 23.27 \qquad S_3^l = \tilde{F}_3^{-1}\left(\tfrac{37.12+0}{37.12+7}\right) = 22.00$$

Taking the mean, we have

$$\tilde{S}_1 = \tfrac{1}{2}(6.49 + 6.49) = 6.49$$
$$\tilde{S}_2 = \tfrac{1}{2}(12.35 + 11.71) = 12.03$$
$$\tilde{S}_3 = \tfrac{1}{2}(23.27 + 22.00) = 22.63$$

These values are very close to $S^* = (6.49, 12.02, 22.71)$ and indeed their costs are very similar: $g(\tilde{\mathbf{S}}) = 47.66$, compared to $g\left(\mathbf{S}^*\right) = 47.65$

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