LEC011 Maximum Flow and Linear Program

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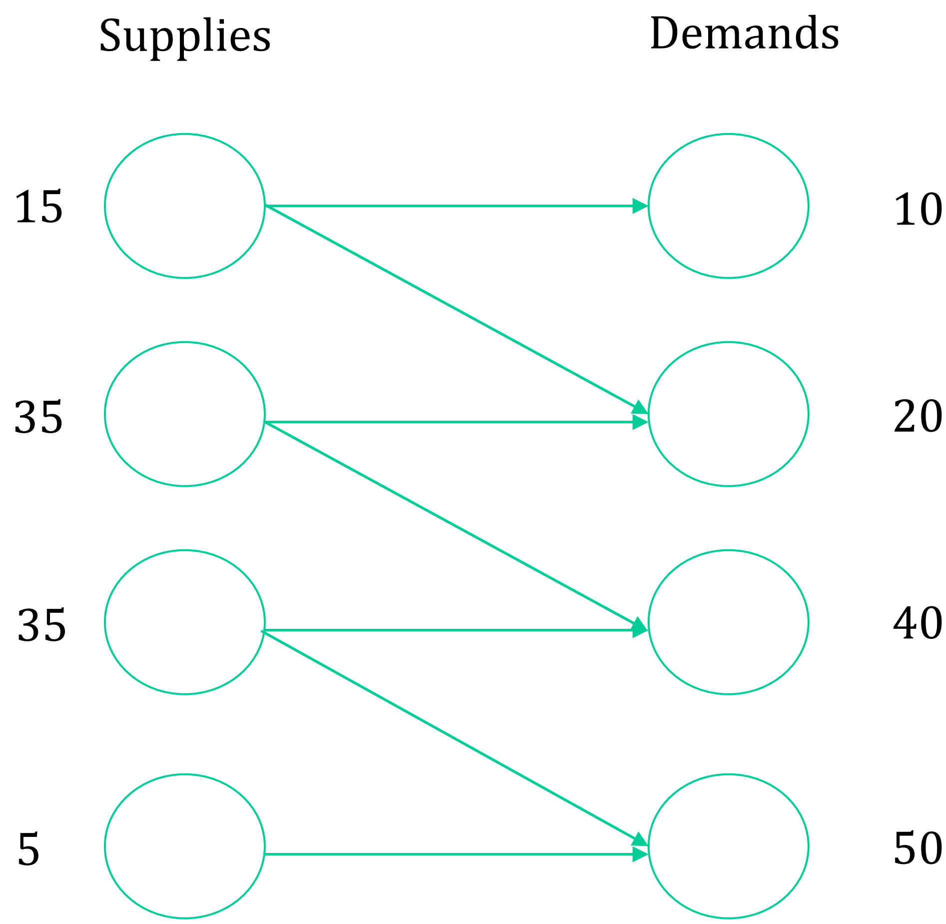
Max Flow Applications



Linear Programming

- Comparison to systems of linear equations

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{gathered}
$$

Linear Programming

- Ingredients of a Linear Program

1. Decision variables $x_1, \ldots, x_n \in \mathbb{R}$

2. Linear constraints, each of the form

$$\sum_{j=1}^{n} a_{ij} x_j \quad (*) \quad b_i$$

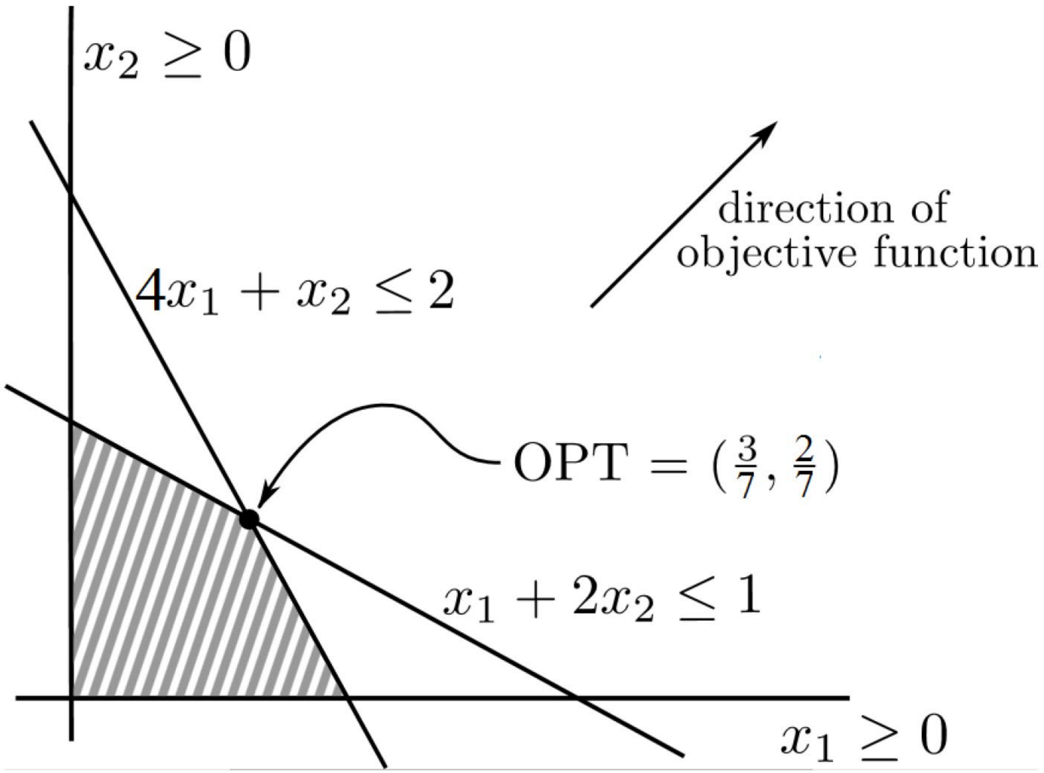
   where $(^*)$ could be $\leq, \geq$, or $=$

3. A linear objective function, of the form

$$\max \sum_{j=1}^{n} c_j x_j \qquad \text{or} \qquad \min \sum_{j=1}^{n} c_j x_j$$

A Simple Example

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Python + Gurobi Time!

MaxFlow is a LP

- Decision variables

$$\{f_e\}_{e \in E}$$

- Constraints $(2m + n - 2)$

$$\underbrace{\sum_{e \in \delta^-(v)} f_e}_{\text{flow in}} - \underbrace{\sum_{e \in \delta^-(v)} f_e}_{\text{flow out}} = 0$$

$$f_e \leq u_e$$

- Objectives

$$f_e \geq 0$$

$$\max \textstyle\sum_{e \in \delta^+(s)} f_e$$

Generalization of MaxFlow

- Min-Cost MaxFlow

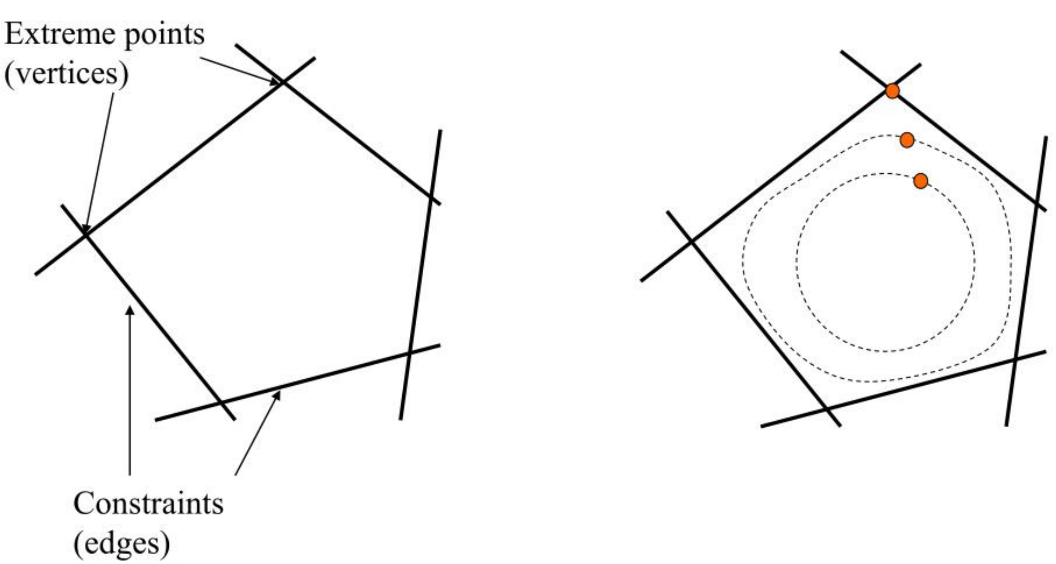
$$\min \sum_{e \in E} c_e f_e$$

- Easy to change the LP formulation!

Solution Approaches

- Simplex methods
- Interior point methods

Interior point methods



Simplex: search from vertex to vertex along the edges

Interior-point methods: go through the inside of the feasible space

14

LP Duality

- Question: given a feasible solution, how can we know whether it is optimal or close to optimal?

$$\boxed{\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & 4x_1 + x_2 \le 2 \\ & x_1 + 2x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{array}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \le 4x_1 + x_2 \le \underbrace{2}_{\text{upper bound}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \le x_1 + 2x_2 \le \underbrace{1}_{\text{upper bound}}$$

Can we get an even better upper bound?

$$x_1 + x_2 \le \frac{1}{7}\underbrace{(4x_1 + x_2)}_{\le 2 \text{ by } (2)} + \frac{3}{7}\underbrace{(x_1 + 2x_2)}_{\le 1 \text{ by } (3)} \le \frac{1}{7} \cdot 2 + \frac{3}{7} \cdot 1 = \frac{5}{7}$$

Deriving the Dual LP

Compact form

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

"Multipliers"

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j & & \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{1j} x_j \leq b_1 & \longrightarrow \quad & y_1 \geq 0 \\ & \sum_{j=1}^{n} a_{2j} x_j \leq b_2 & \longrightarrow \quad & y_2 \geq 0 \\ & \vdots \leq \vdots & & \\ & \sum_{j=1}^{n} a_{mj} x_j \leq b_m & \longrightarrow \quad & y_m \geq 0 \\ & x_1, \ldots, x_n \geq 0 & & \end{aligned}$$

The idea is to "dominates the objective coefficients": $\sum_{i=1}^{m} y_i a_{ij} \geq c_j$

More compactly, Find $\mathbf{y} \geq 0$ such that $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$

Deriving the Dual LP

$$\boxed{\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}}$$

Find $\mathbf{y} \ge 0$ such that $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$

$$\begin{aligned}
\underbrace{\sum_{j=1}^{n} c_j x_j}_{\text{x's obj fn}} &\le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j \\
&= \sum_{i=1}^{m} y_i \cdot \left(\sum_{j=1}^{n} a_{ij} x_j \right) \\
&\le \underbrace{\sum_{i=1}^{m} y_i b_i}_{\text{upper bound}}
\end{aligned}$$

More compactly, $\mathbf{c}^T\mathbf{x} \le \left(\mathbf{A}^T\mathbf{y}\right)^T\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \le \mathbf{y}^t\mathbf{b}$

Deriving the Dual LP

$$\boxed{\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}}$$

Find $\mathbf{y} \geq 0$ such that $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$

$$\mathbf{c}^T\mathbf{x} \leq \left(\mathbf{A}^T\mathbf{y}\right)^T\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \leq \mathbf{y}^t\mathbf{b}$$

$$\boxed{\begin{array}{ll} \min & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}}$$

A Simple Example

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \mathbf{b}^T \mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \le 2 \\ & x_1 + 2x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & 2y_1 + y_2 \\ \text{s.t.} \quad & 4y_1 + y_2 \ge 1 \\ & y_1 + 2y_2 \ge 1 \\ & y_1, y_2 \ge 0 \end{aligned}$$

$$x_1^* = 3/7, x_2^* = 2/7$$

$$y_1^* = 1/7, y_2^* = 3/7$$

Optimal objectives are the same!

Weak and Strong Duality

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

We have

- Weak Duality $\quad \mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$ for any feasible solutions $\mathbf{x}$ and $\mathbf{y}$
- Strong Dualiy $\quad \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$ at optimality

Proof = Separating hyperplance theorem + Farkas's Lemma

More General Form

Primal	Dual
variables $x_1, \ldots, x_n$	n constraints
m constraints	variables $y_1, \ldots, y_m$
objective function $\mathbf{c}$	right-hand side $\mathbf{c}$
right-hand side $\mathbf{b}$	objective function $\mathbf{b}$
$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
constraint matrix $\mathbf{A}$	constraint matrix $\mathbf{A}^T$
i-th constraint is "$\leq$"	$y_i \geq 0$
i-th constraint is "$\geq$"	$y_i \leq 0$
i-th constraint is "$=$"	$y_i \in \mathbb{R}$
$x_j \geq 0$	j-th constraint is "$\geq$"
$x_j \leq 0$	j-th constraint is "$\leq$"
$x_j \in \mathbb{R}$	j-th constraint is "$=$"

Back to MaxFlow

- Let the decision variable be flow on an s-t path

$$
\begin{aligned}
\max \quad & \sum_{P \in \mathcal{P}} f_P \\
\text{s.t.} \quad & \underbrace{\sum_{P \in \mathcal{P} : e \in P} f_P}_{\text{total flow on } e} \le u_e \ \text{ for all } e \in E \\
& f_P \ge 0 \text{ for all } P \in \mathcal{P}
\end{aligned}
$$

Question: how to write its dual and what is its interpretation?