LEC014 Traveling Salesman Problem (TSP)

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Traveling Salesman Problem (TSP)

- A set of cities $V = \{1, 2, \ldots, n\}$
- A distance function (*metric*) $d : V \times V \to R_+$
  - Symmetry:

$$d(u, v) = d(v, u), \forall u, v \in V$$

  - Triangular inequality:

$$d(u, w) \leq d(u, v) + d(v, w), \forall u, v, w \in V$$

- Consider a complete graph

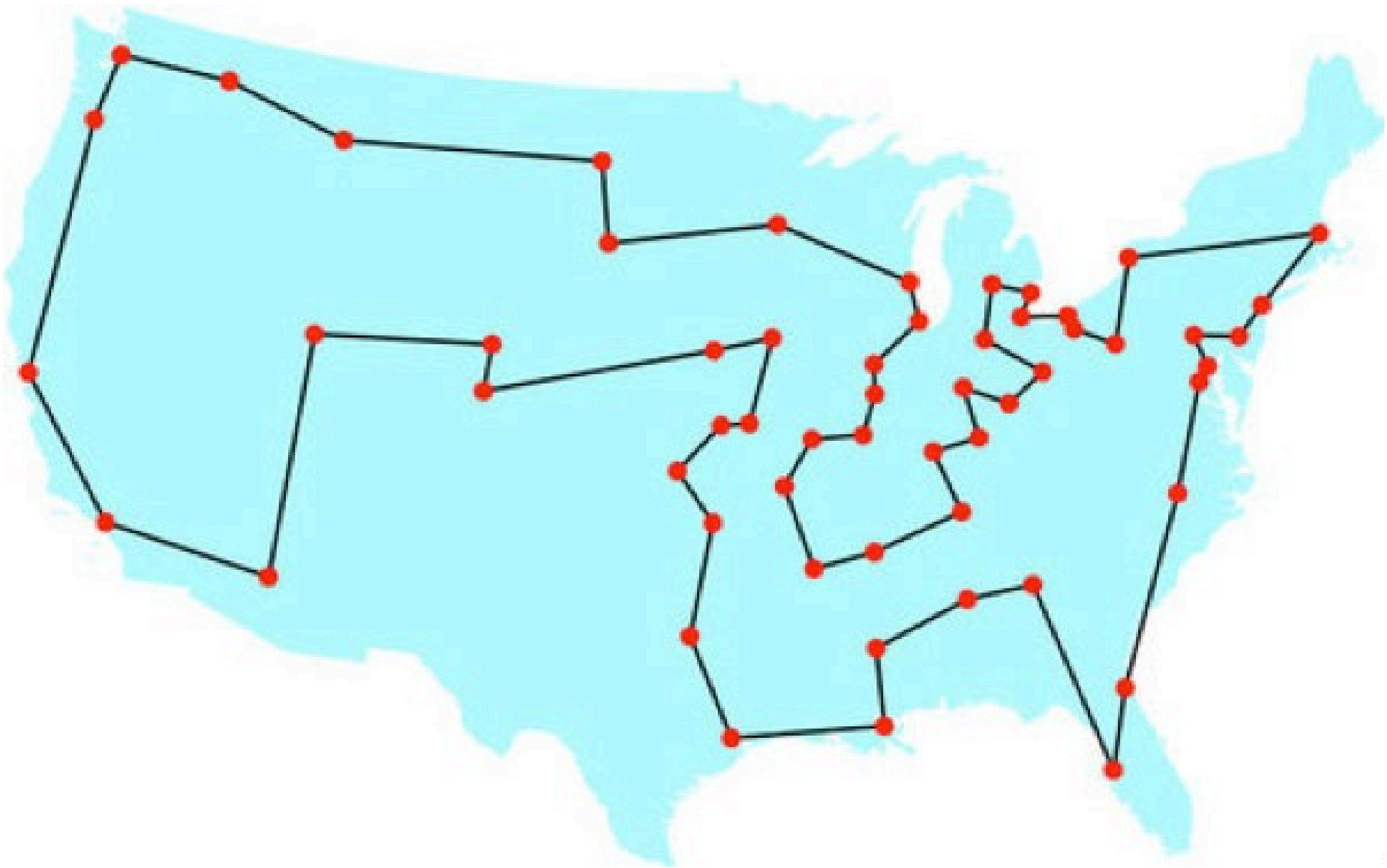
$$(V, d)$$

vertices   edge costs

Objective: find a tour of minimum distance that visits each city exactly once and return to its starting point

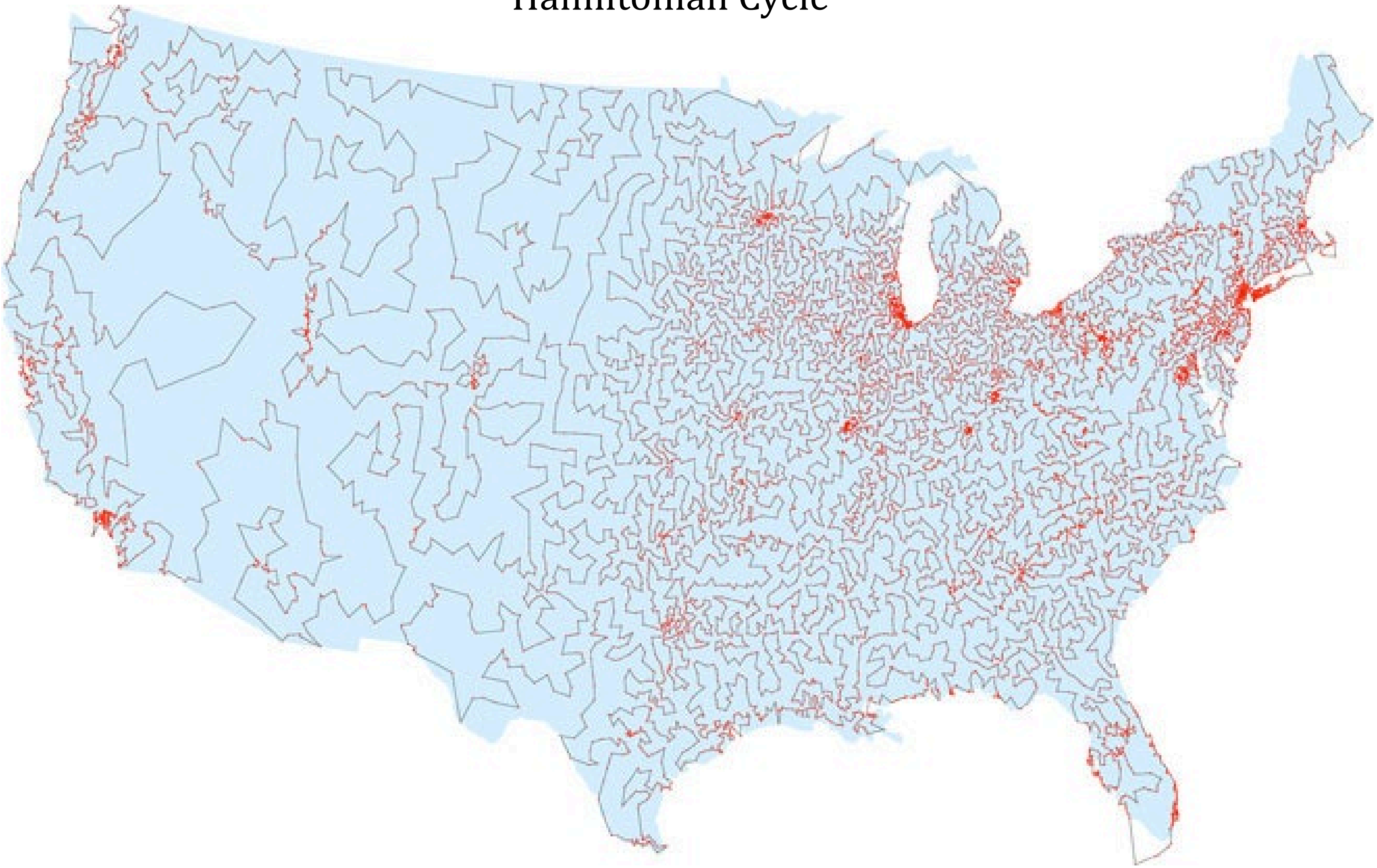
Metric TSP

Hamiltonian Cycle

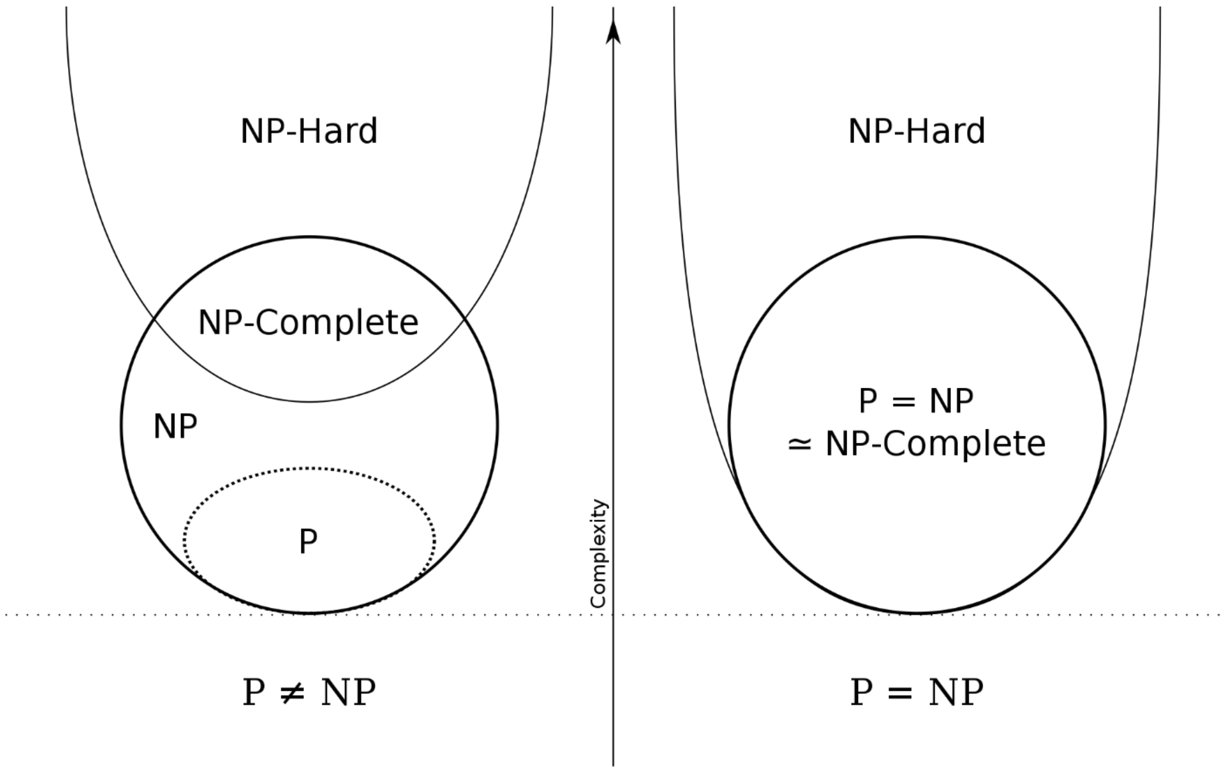


Metric TSP

Hamiltonian Cycle



Metric TSP



Metric TSP is NP-Hard!

Notion of Approximation Algo

TSP

Lemma: For any instance I to the traveling salesman problem, the cost of optimal tour it at least the cost of the minimum spanning tree on I, i.e., $MST(I) \leq TSP(I)$

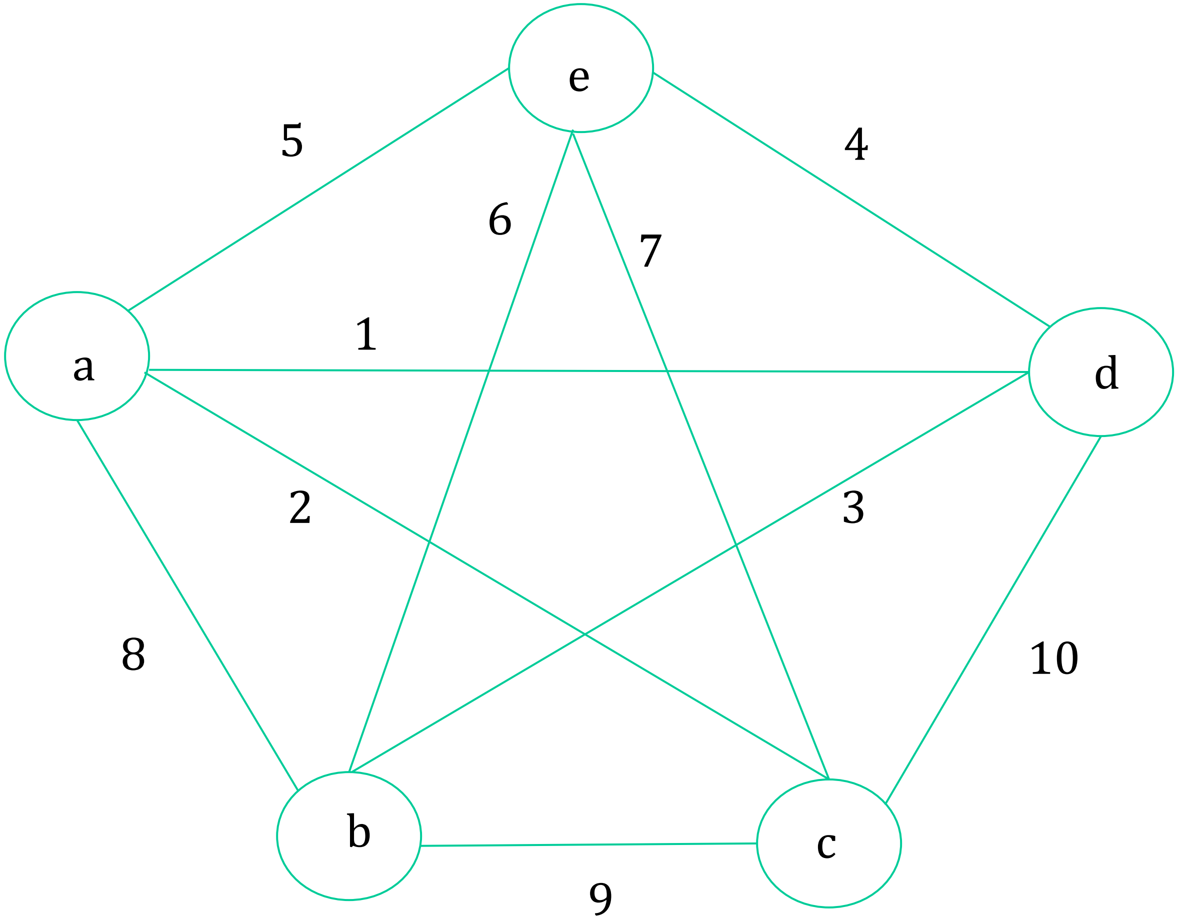
Proof: We assume instance I has $n \geq 2$ cities. Start with the optimal TSP tour of cost $TSP(I)$. If your remove one edge from the tour (break the cycle), the result is a spanning tree $ST(I)$ with a cost at most $TSP(I)$. since the minimum spanning tree (MST) is the one with the minimum cost over all spanning trees, it follows that $MST(I) \leq TSP(I)$.

TSP Double-Tree Algorithm

Algorithm (Double-tree algorithm)

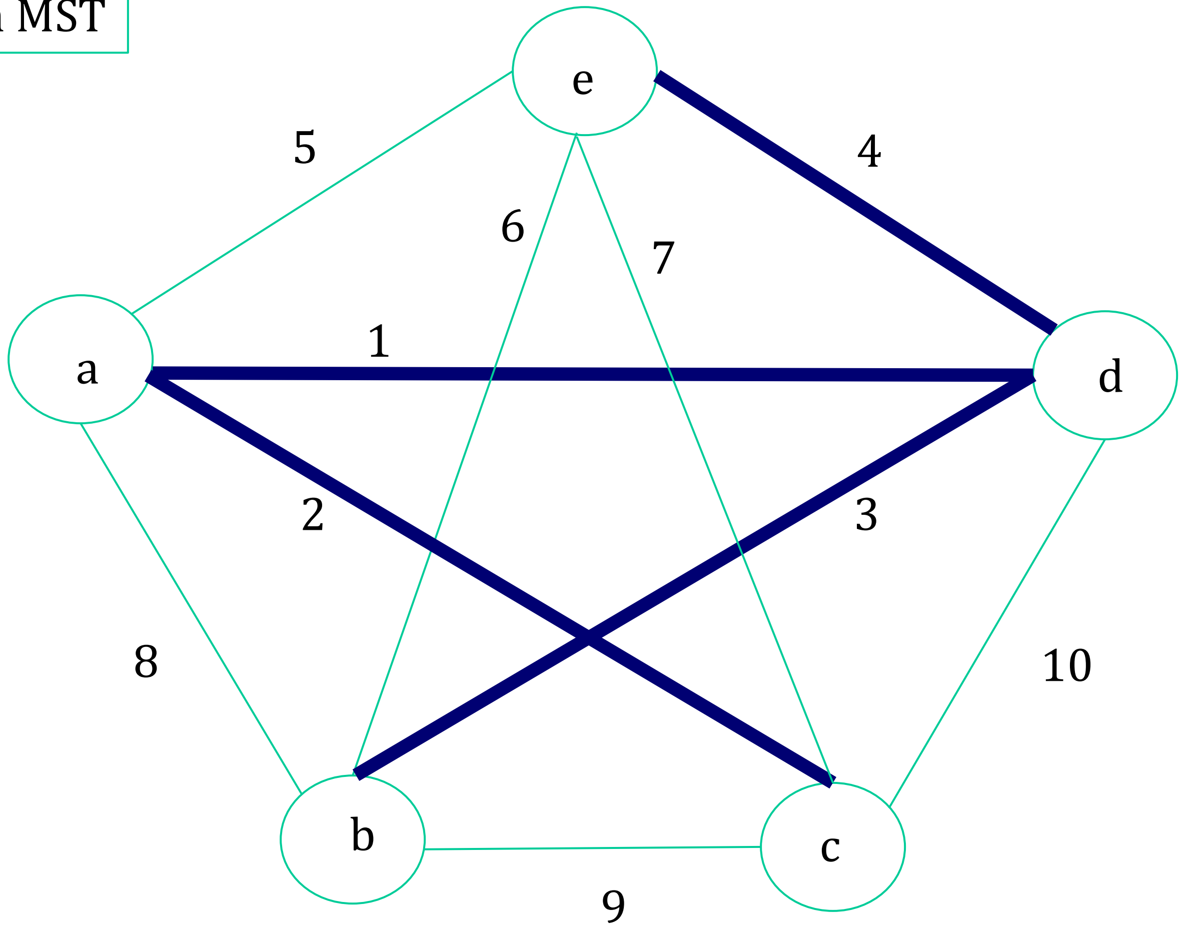
1. Compute the minimum spanning tree M on (V, d).
2. Double all edges of M and call the resulting graph D.
3. Find a walk W that uses each edge of D exactly once.
4. Shortcut W by skipping vertices that are re-visited to get a valid TSP tour T.

Example



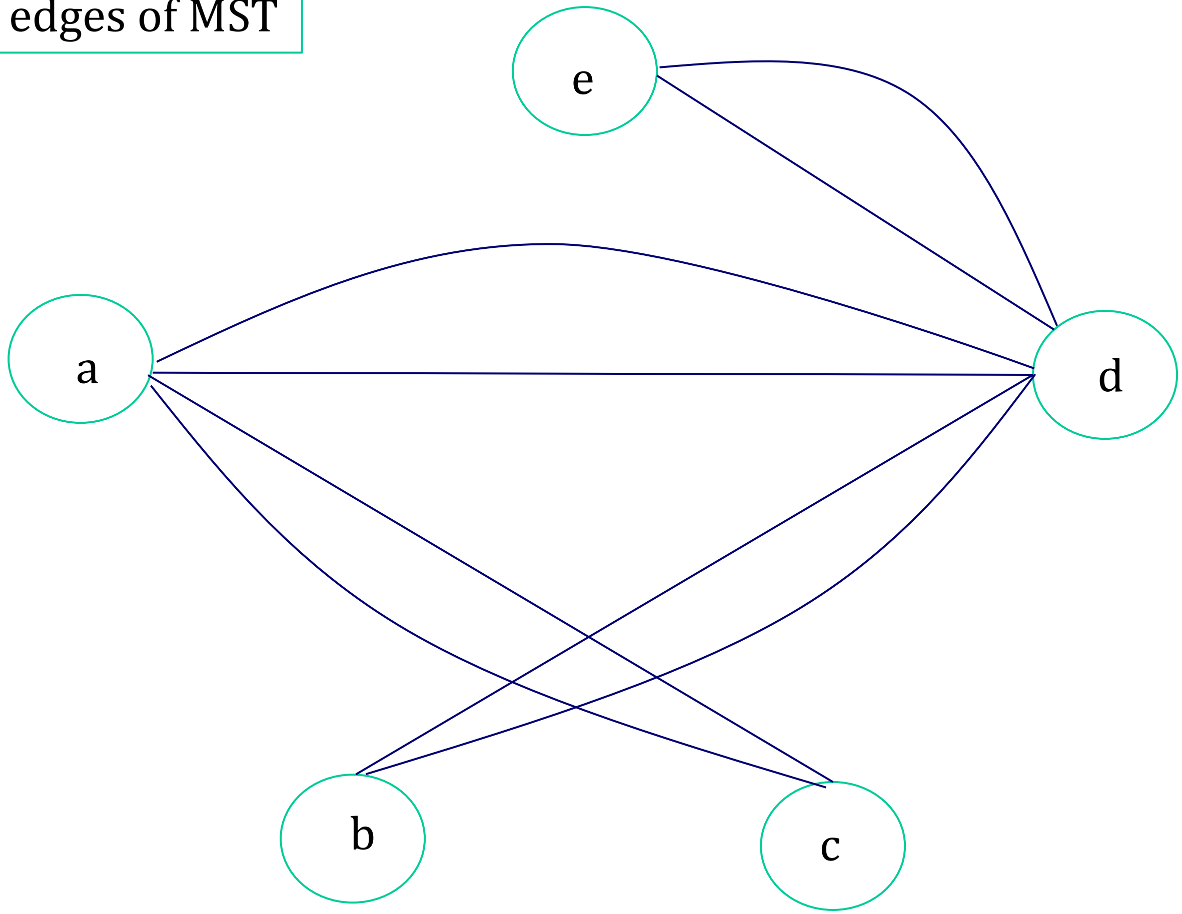
Example

Find an MST



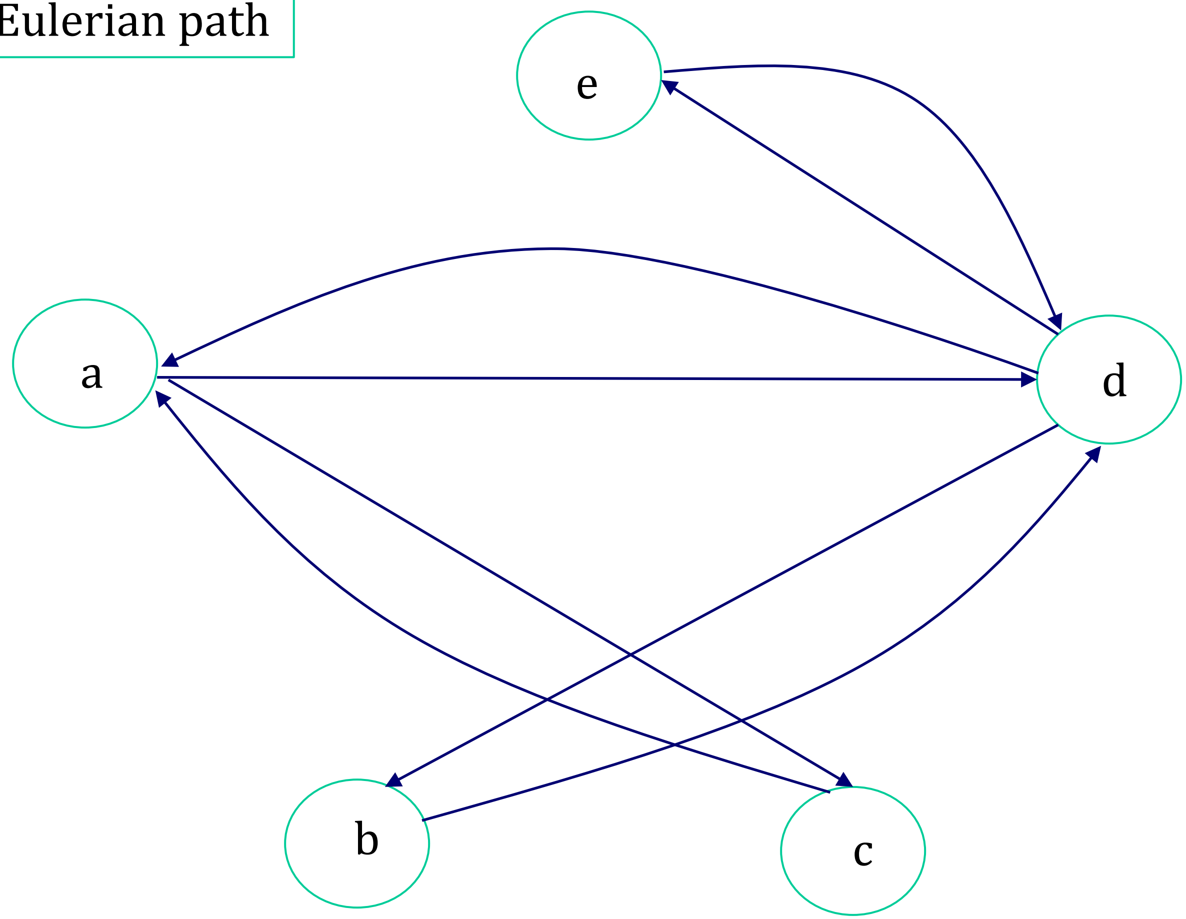
Example

Double edges of MST



Example

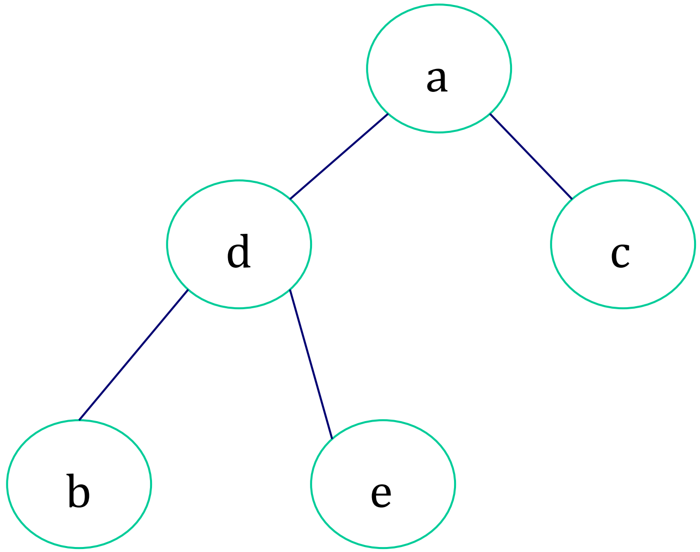
Find a Eulerian path



a → d → b → d → e → d → a → c → a

Example

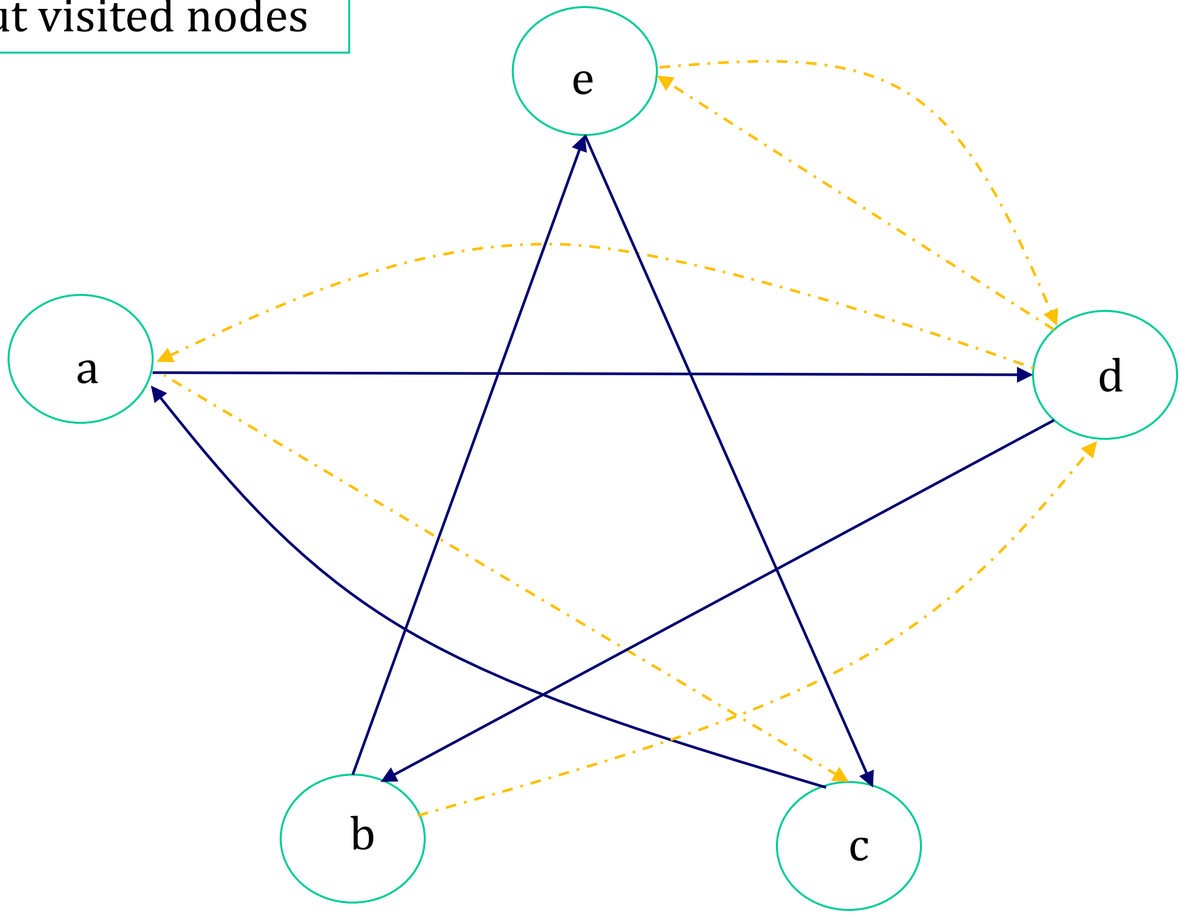
Find a Eulerian path (use DFS traversal)



a → d → b → d → e → d → a → c → a

Example

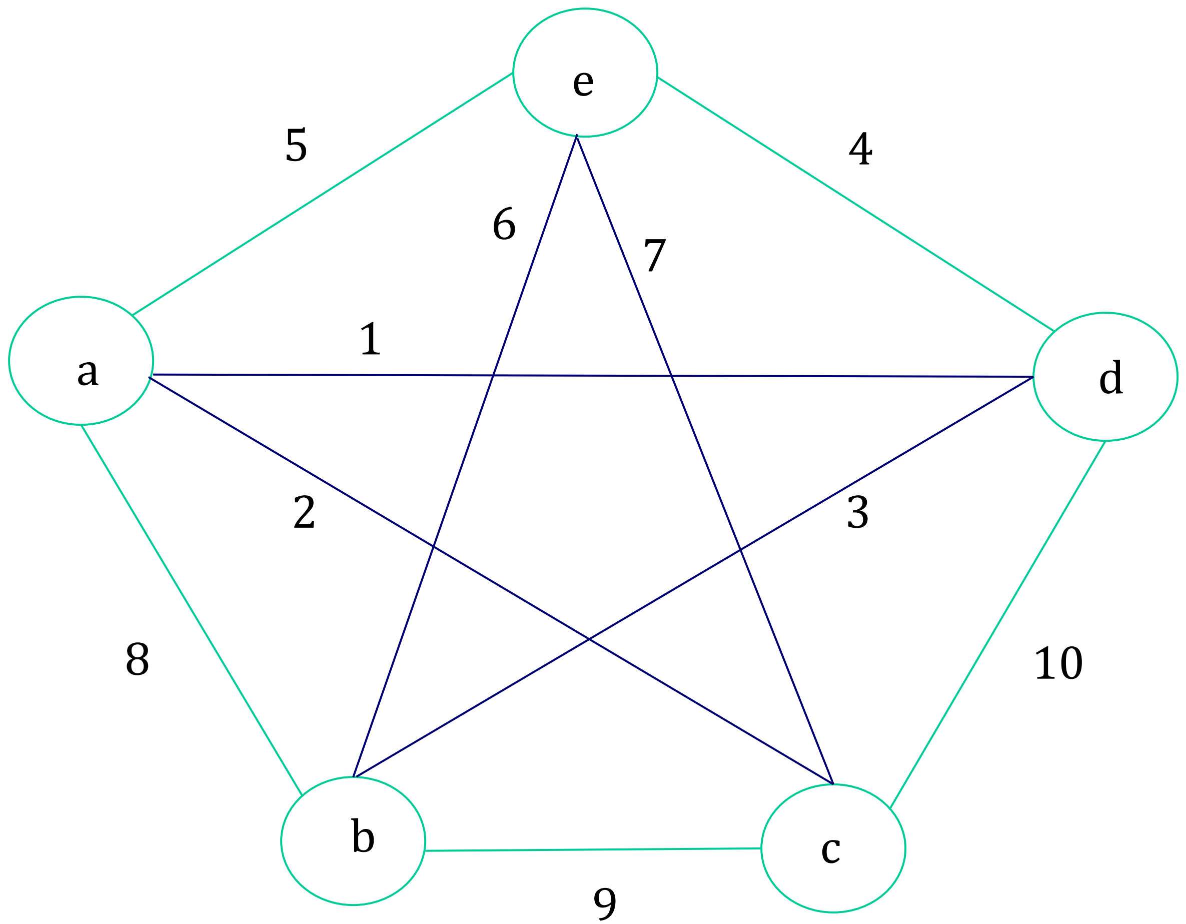
Shortcut visited nodes



Before shortcutting: a → d → b → d → e → d → a → c → a

After shortcutting: a → d → b → e → c → a

Example



Output: a → d → b → e → c → a with cost 19
OPT: a → d → b → e → c → a with cost 19

Double Tree = 2-Approximation

Theorem: The double-tree algorithm for TSP is a 2 -approximation algorithm.

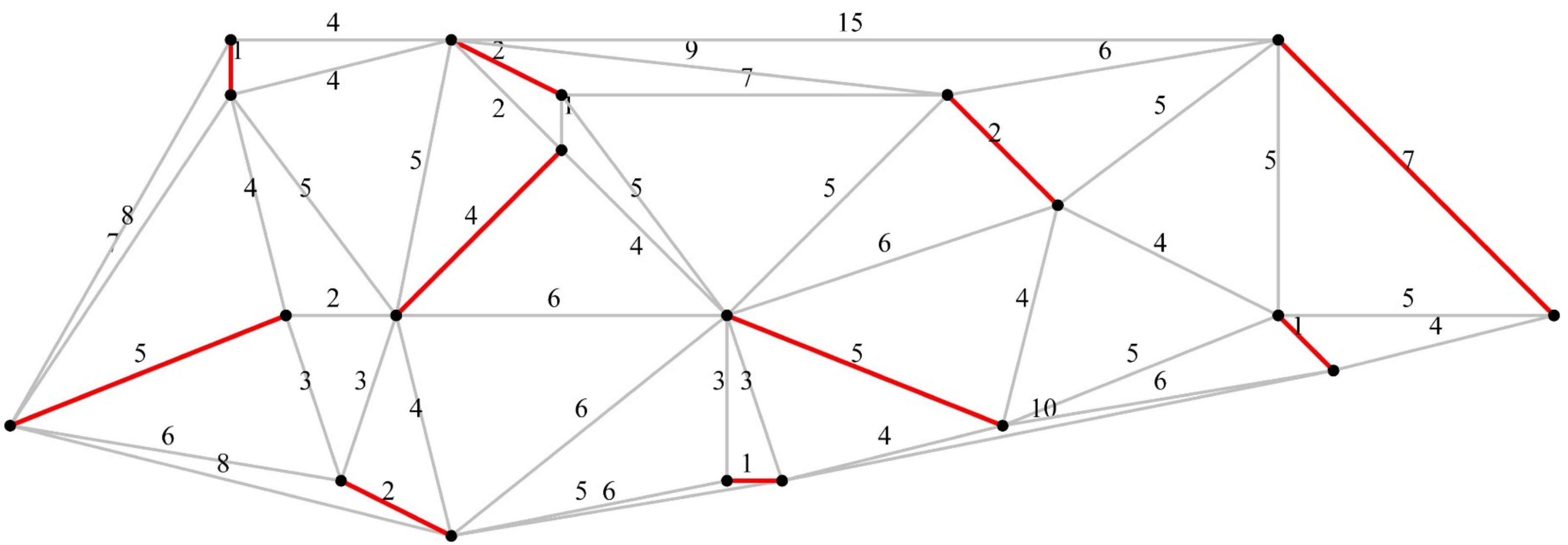
Proof: Let OPT be the cost of the optimal TSP tour.

- The cost of the minimum spanning tree M is at most OPT.
- We then double each edge (replace it with two copies) of M and the cost of the resulting graph D is at most $2OPT$. Also D is Eulerian by construction and a walk W of cost at most $2OPT$.
- Let W be the sequence $i_0, i_1, \ldots, i_k$ of cities where there may be repetitions. To get a tour T, we removing all but the first occurrence of each city in this sequence. This tour T contains each city exactly once (starts at i_0 and returns to i_0). We now show that the cost of T is at most that of W. Consider two consecutive cities in T : i_ℓ and i_m (we omitted $i_{\ell+1}, \ldots, i_{m-1}$ since these cities were already visited earlier in T). It then follows from the triangle inequality (and induction) that the distance d_{i_ℓ, i_m} is upper bounded by the total distance of the edges $(i_\ell, i_{\ell+1}), \ldots, (i_{m-1}, i_m)$ Adding up over all edges in T, the cost of T is at most the cost of W which is at most $2OPT$.

A Better Approximation Algorithm?

- Yes, the celebrated Christofides' algorithm for TSP

Matching: The input is a graph $G = (U, E)$ with even number of vertices U and distance function $d : U \times U \to R_+$. The goal is to find edges $K \subseteq E$ such that each vertex has exactly one end-point in K with minimum cost $\sum_{e \in K} d(e)$



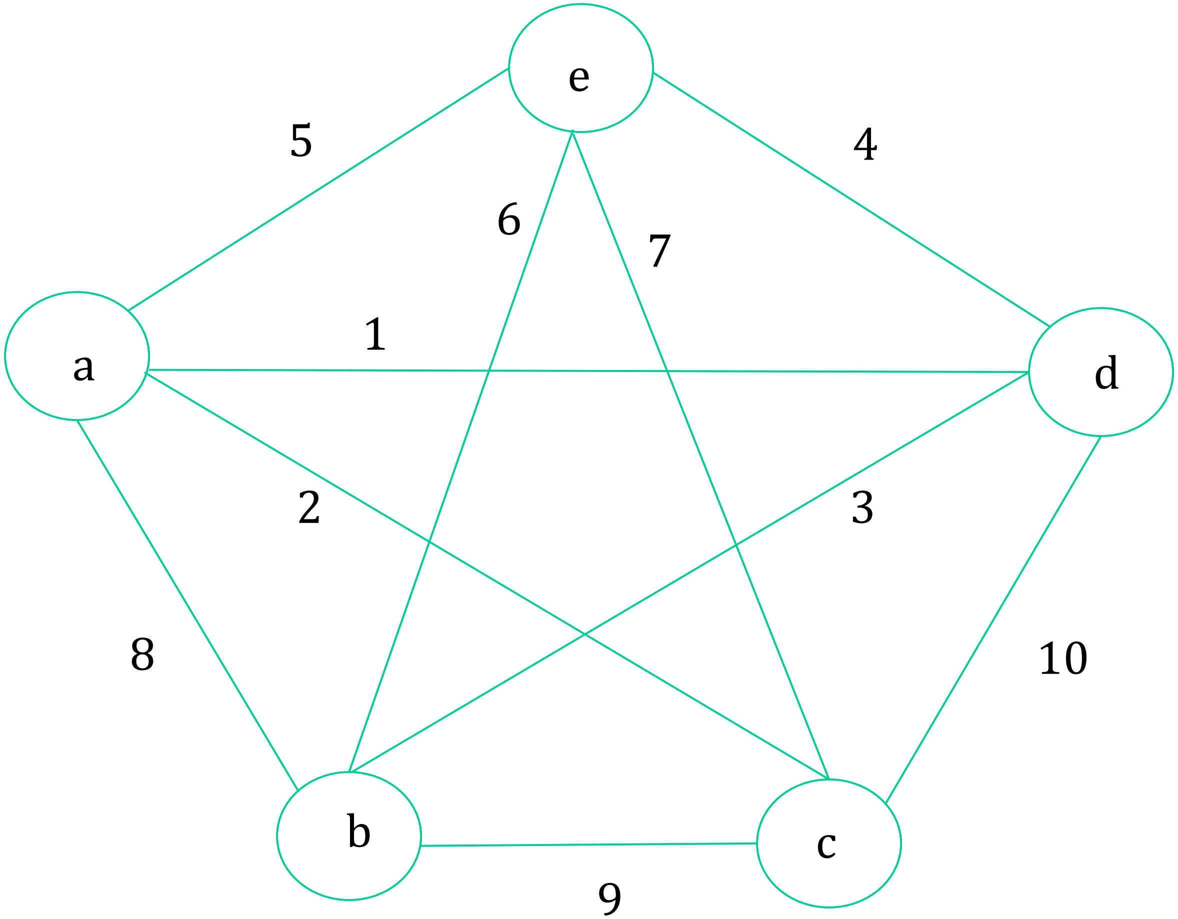
"Minimum-weight-perfect-matching" can be efficiently solved in $O(nmlogn)$
https://www.math.uwaterloo.ca/~bico/papers/match_ijoc.pdf

Christofides' algorithm

Algorithm: (Christofides' algorithm for TSP)

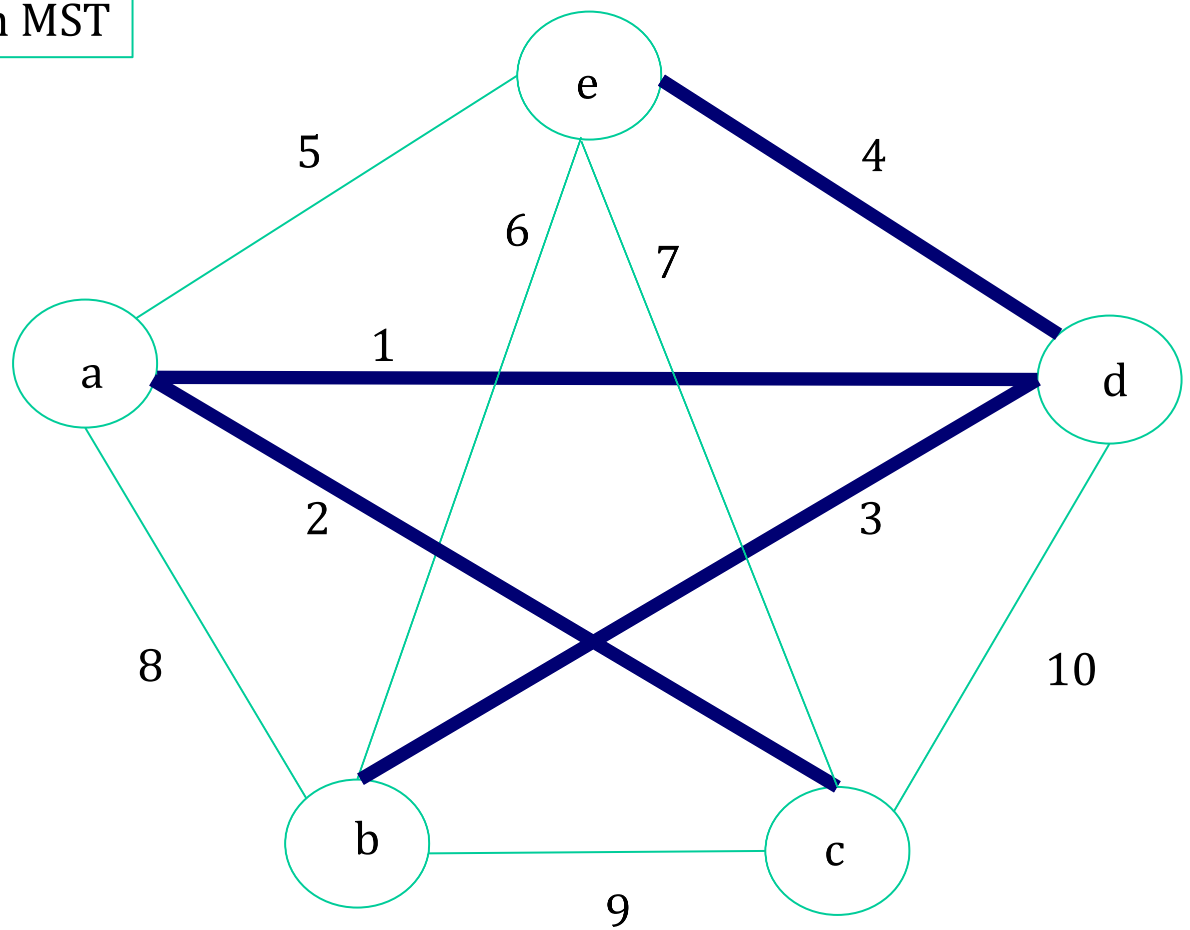
1. Compute the minimum spanning tree M on (V, d)
2. Compute the minimum cost matching K on odd degree vertices of M
3. Add the edges of K to M to obtain an Eulerian graph D'
4. Find a walk W' that uses each edge of D' exactly once.
5. Shortcut W' by skipping vertices that are re-visited to get a valid TSP tour T'

Example



Example

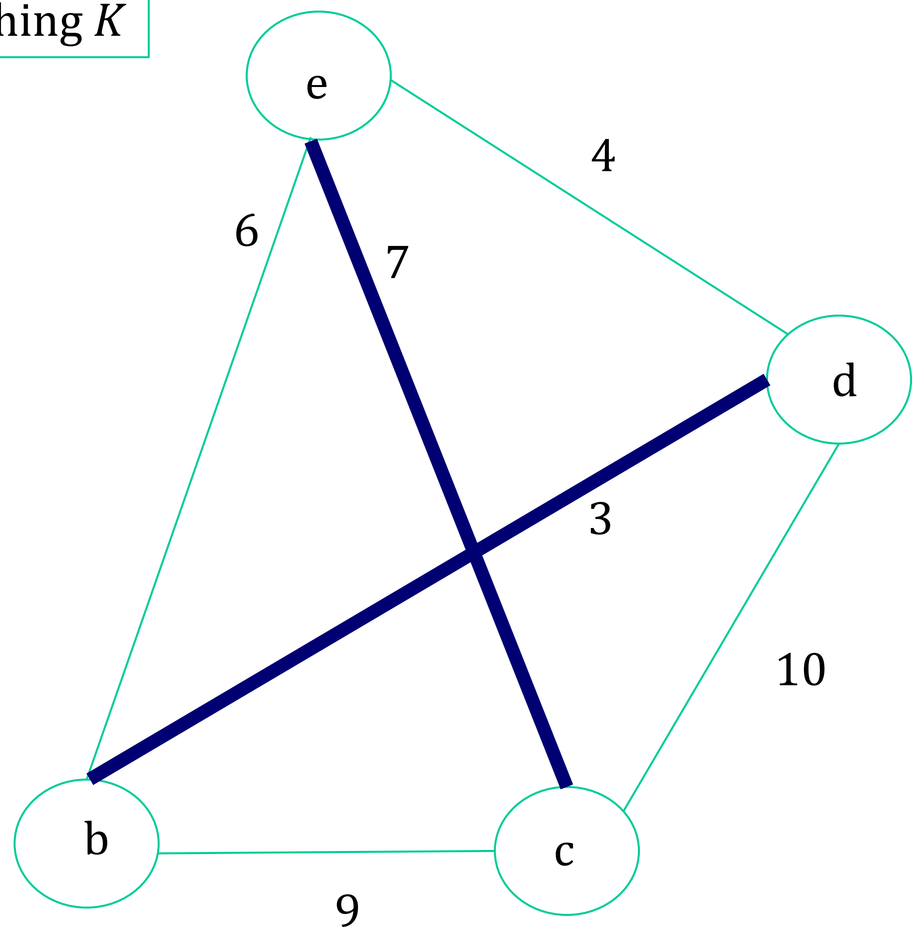
Find an MST



Find the set of odd degree vertices in M:
$U = \{e, d, b, c\}$

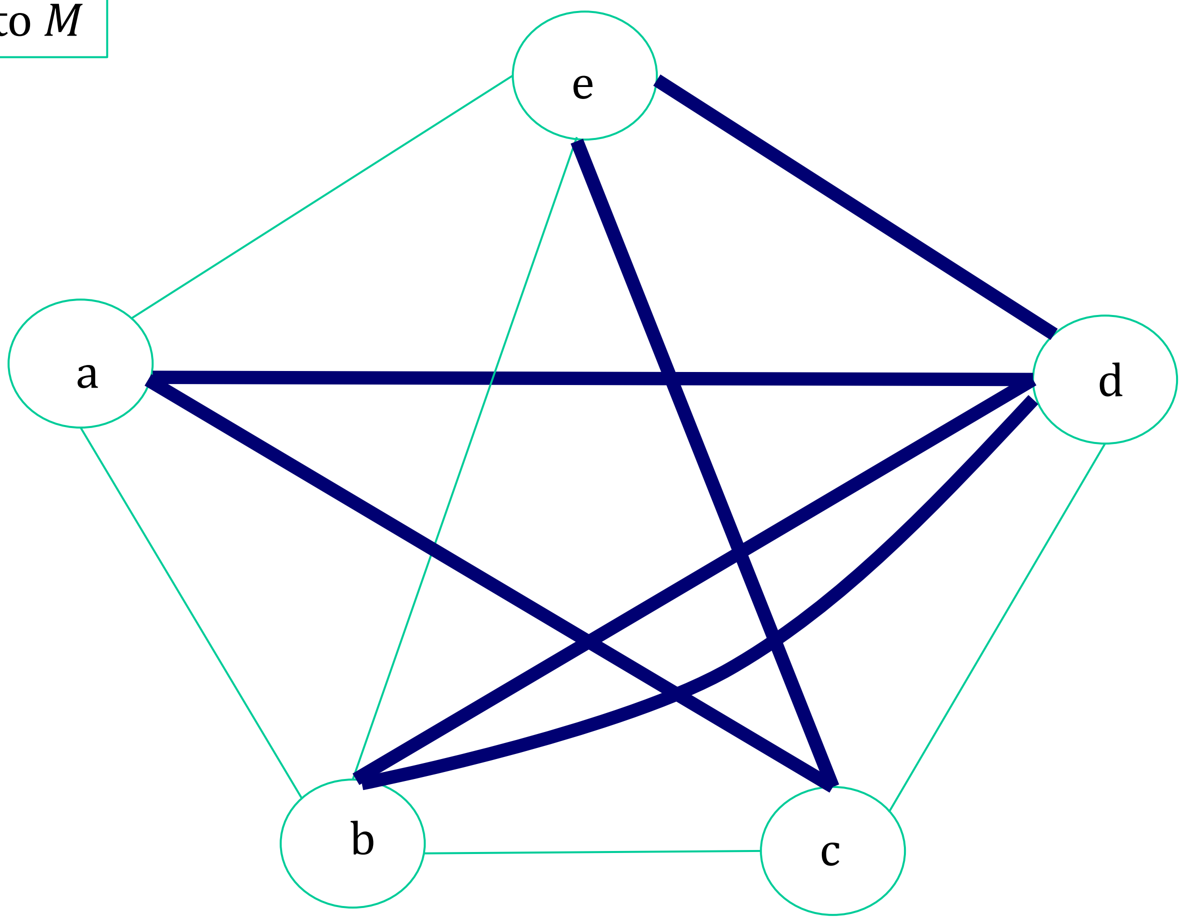
Example

Find min-weight-matching K



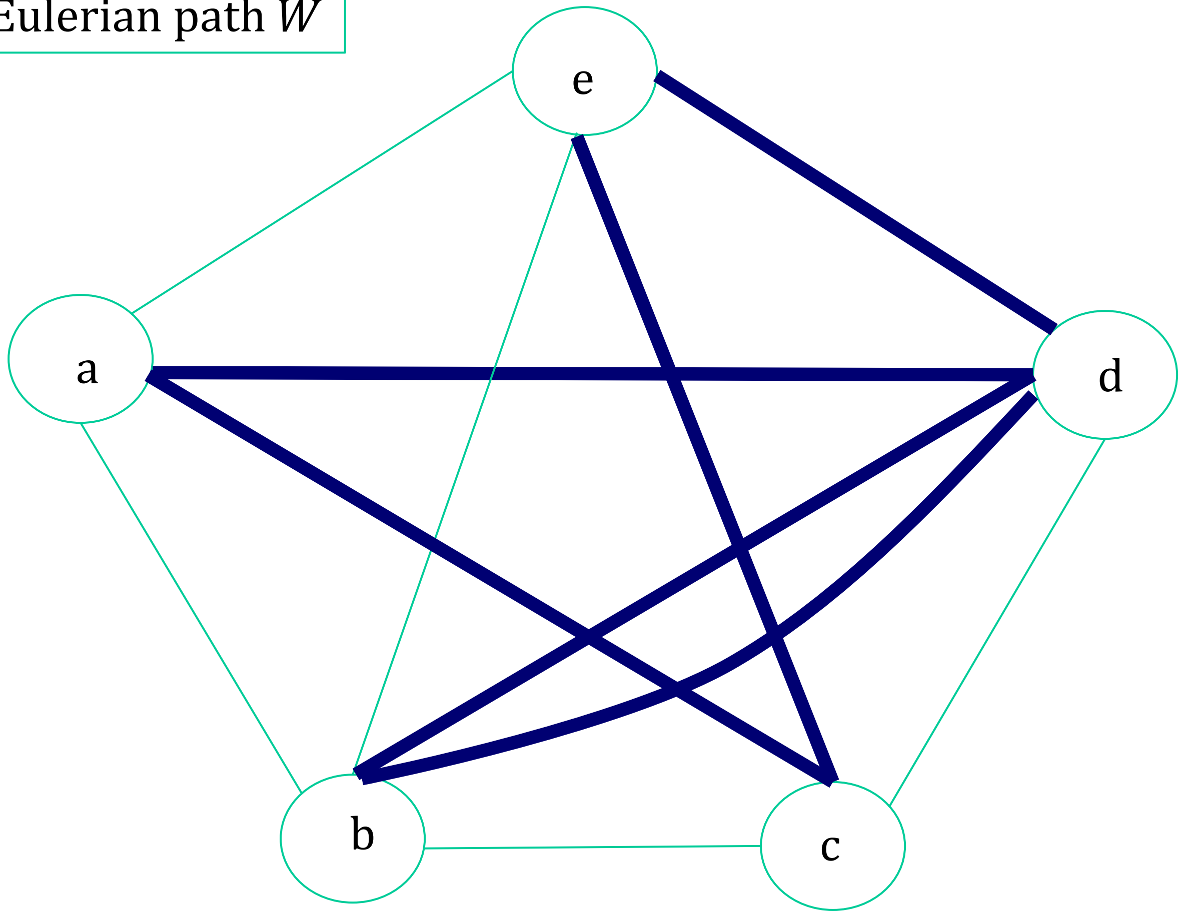
Example

Add K to M



Example

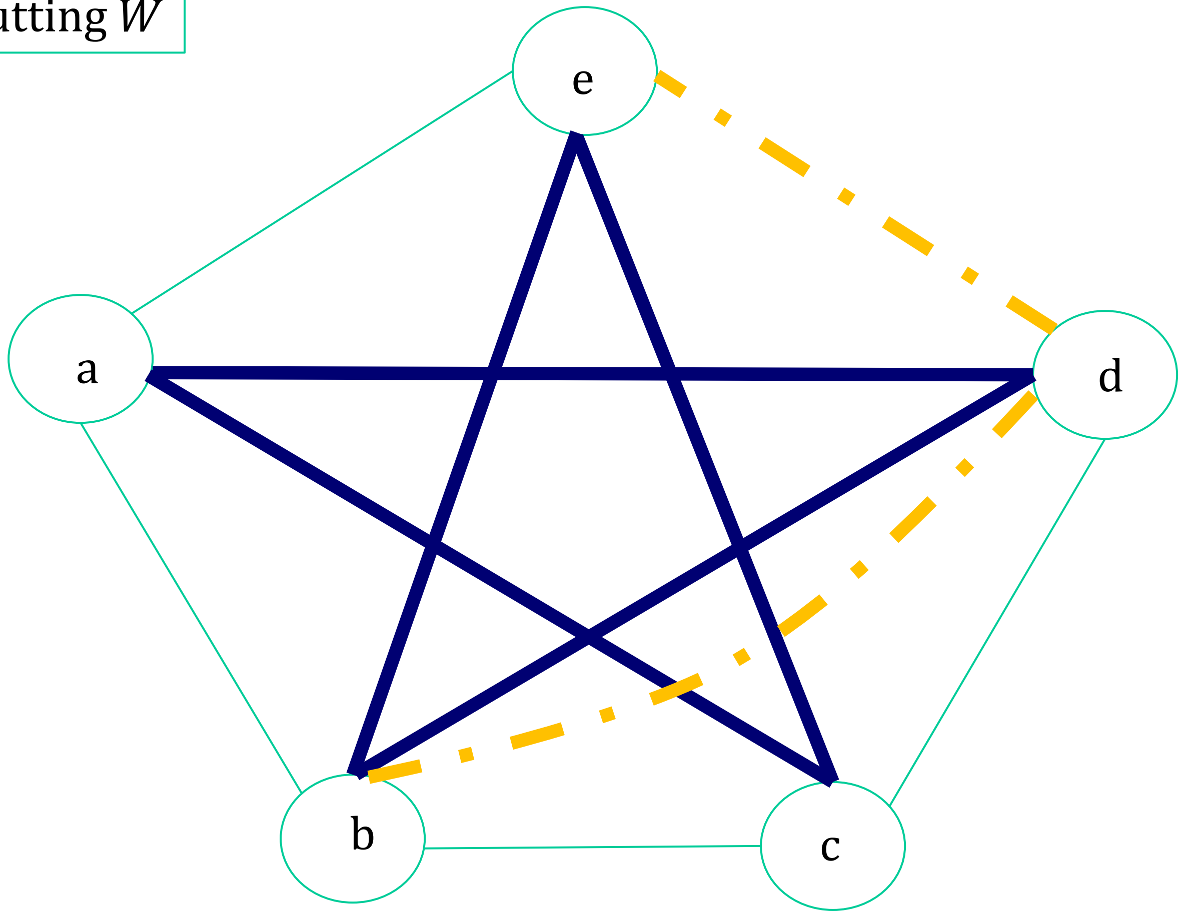
Find a Eulerian path W



a → d → b → d → e → c → a

Example

Shortcutting W



Before shortcutting: a → d → b → d → e → c → a

After shortcutting: a → d → b → e → c → a

1.5-Approximation

Lemma: The number of odd degree vertices in M is even.

Proof: Let $V_{\text{even}} \subset V$ and $V_{\text{odd}} \subset V$ be the subsets of even and odd degree vertcies in M, respectively

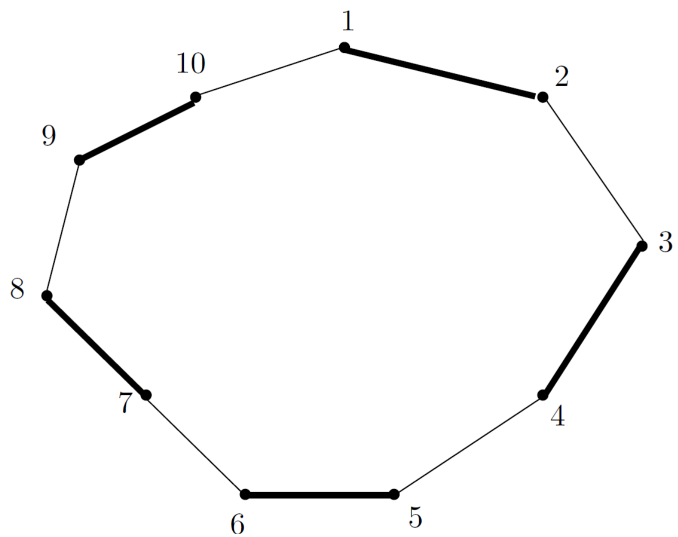
$$2|E| = \sum_{\vartheta \in V} \deg(\vartheta) = \sum_{\vartheta \in V_{odd}} \deg(\vartheta) + \underbrace{\sum_{\vartheta \in V_{even}} \deg(\vartheta)}_{\text{even}} = \text{even}$$

Hence $|V_{\text{odd}}|$ is even.

1.5-Approximation

Lemma: The minimum cost matching on any set U (even number of vertices) is at most $\frac{1}{2}OPT$, where OPT is the cost of the optimal TSP tour.

Proof: Consider the optimal TSP tour O and shortcut over all vertices not in U to obtain cycle O' containing vertices U. By triangle inequality, the cost of O' is at most that of O which is OPT. We define two candidate matchings on U using O'. By renumbering vertices let O' be the sequence $1, 2, \cdots, |U|, 1$ of vertices. Let M_1 be the matching that pairs vertices as $(1,2), (3,4) \cdots (|U|-1, |U|)$ and M_2 be $(|U|, 1), (2,3) \cdots (|U|-2, |U|-1)$. Then $\text{cost}\,(M_1) + \text{cost}\,(M_2) = \text{cost}\,(O') \leq OPT$. So $\min\left(\text{cost}\,(M_1), \text{cost}\,(M_2)\right) \leq OPT/2$.



1.5-Approximation

Theorem: Christofides' algorithm for TSP is a 3/2-approximation algorithm.

Proof: We know that the Cost$(MST) \leq OPT$, and that the min-cost matching has Cost$(K) \leq OPT/2$. So the cost of D' (and hence T') is at most $\frac{3}{2}OPT$.

Python Time

- pip install Christofides

Use the compute() function which takes as input a distance_matrix and returns a Christofides solution as follows:

```
from Christofides import christofides
TSP = christofides.compute(distance_matrix)
```

The Distance Matrix is an upper Triangular matrix with distance from a node on to itself 0, since Christofides algorithm could only be applied for undirected graphs. Also the distance between a node on to itself is practically 0. Example for distance_matrix is as follows, distance_matrix =

```
[[0,45,65,15],
 [0,0,56,12],
 [0,0,0,89],
 [0,0,0,0]]
```