LEC012 Facility Location

VG441 SS2021

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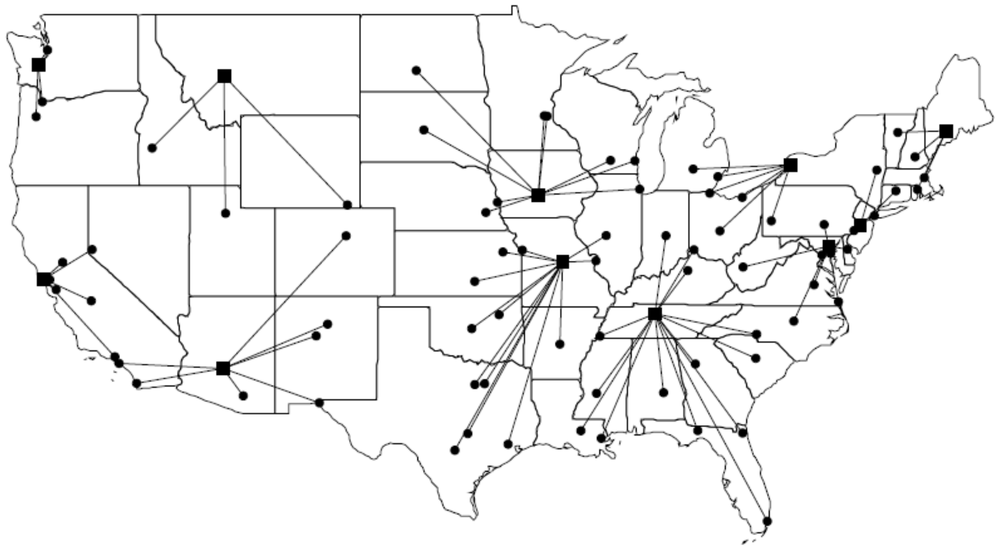
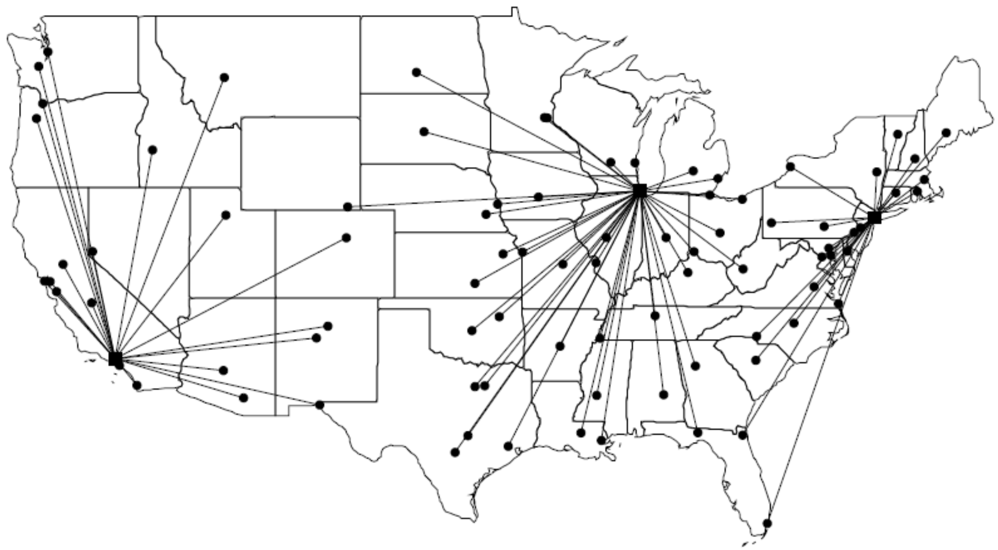
Configurations



(a) Many facilities open.



(b) Few facilities open.

Formulation

- Sets

$$I = \text{set of customers}$$
$$J = \text{set of potential facility locations}$$

- Parameters

$$h_i = \text{annual demand of customer } i \in I$$
$$c_{ij} = \text{cost to transport one unit of demand from facility } j \in J \text{ to customer } i \in I$$
$$f_j = \text{fixed annual cost to open a facility at site } j \in J$$

- Decision Variables

$$x_j = 1 \text{ if facility } j \text{ is opened, } 0 \text{ otherwise}$$
$$y_{ij} = \text{the fraction of customer } i \text{ 's demand that is served by facility } j$$

Transportation costs c_{ij}

- Euclidean distance

$$\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

- Manhattan or rectilinear metric

$$|a_1 - a_2| + |b_1 - b_2|$$

- Great circle

$$r \arccos\left(\sin\alpha_1 \sin\alpha_2 + \cos\alpha_1 \cos\alpha_2 \cos(\Delta\beta)\right)$$

- GPS/MAP distance

MILP Formulation

Decision variables:

$x_j = 1$ if facility j is opened, 0 otherwise
$y_{ij} =$ the fraction of customer i 's demand that is served by facility j

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} \\
\text{subject to} \quad & \sum_{j \in J} y_{ij} = 1, \qquad \forall i \in I \\
& y_{ij} \leq x_j, \qquad \forall i \in I, \forall j \in J \\
& x_j \in \{0,1\}, \qquad \forall j \in J \\
& y_{ij} \geq 0, \qquad \forall i \in I, \forall j \in J
\end{aligned}
$$

Assignment Constraint

Linking Constraint

Python + Gurobi Time!

jupyter

GUROBI
OPTIMIZATION

Lagrangian Relaxation

- Introduce a penalty term: $\sum_{i \in I} \lambda_i \left(1 - \sum_{j \in J} y_{ij}\right)$

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} + \sum_{i \in I} \lambda_i \left(1 - \sum_{j \in J} y_{ij}\right) \\
& = \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} (h_i c_{ij} - \lambda_i)\, y_{ij} + \sum_{i \in I} \lambda_i \\
\text{subject to} \quad & y_{ij} \le x_j, \qquad \forall i \in I, \forall j \in J \\
& x_j \in \{0,1\}, \qquad \forall j \in J \\
& y_{ij} \ge 0, \qquad \forall i \in I, \forall j \in J
\end{aligned}
$$

Lagrangian Relaxation

- It turns out that this relaxiation is easy to solve:

$$
\begin{aligned}
\text{minimize} \quad & = \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} (h_i c_{ij} - \lambda_i)\, y_{ij} + \sum_{i \in I} \lambda_i \\
\text{subject to} \quad & y_{ij} \leq x_j, \qquad \forall i \in I, \forall j \in J \\
& x_j \in \{0, 1\}, \qquad \forall j \in J \\
& y_{ij} \geq 0, \qquad \forall i \in I, \forall j \in J
\end{aligned}
$$

- Observe that if we open facility j, i.e., set $x_j = 1$

$$\beta_j = \sum_{i \in I} \min \{0, h_i c_{ij} - \lambda_i\} \longrightarrow \text{“benefit”}$$

- So we should open j if and only if $\beta_j + f_j < 0$

Lagrangian Relaxation

$$\begin{array}{lll} \text{minimize} & = \sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J} (h_i c_{ij} - \lambda_i)\, y_{ij} + \sum_{i\in I} \lambda_i & \\ \text{subject to} & y_{ij} \le x_j, & \forall i \in I, \forall j \in J \\ & x_j \in \{0,1\}, & \forall j \in J \\ & y_{ij} \ge 0, & \forall i \in I, \forall j \in J \end{array}$$

Solution:

$$\bar{x}_j = \begin{cases} 1, & \text{if } \beta_j + f_j < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\bar{y}_{ij} = \begin{cases} 1, & \text{if } \bar{x}_j = 1 \text{ and } h_i c_{ij} - \lambda_i < 0 \\ 0, & \text{otherwise} \end{cases}$$

Objective value:

$$z_{\mathrm{LR}}(\lambda) = \sum_{j\in J} \min\{0, \beta_j + f_j\} + \sum_{i\in I} \lambda_i$$

Now Construct UB

- Use the “opened” facility in the relaxed solution
- Assign each customer to its nearest open facility

Updating Multipliers

- Adjust the penalty multipliers:

If $\sum_{j \in J} y_{ij} = 0$, then λ_i is too small; it should be increased.
If $\sum_{j \in J} y_{ij} > 1$, then λ_i is too large; it should be decreased.
If $\sum_{j \in J} y_{ij} = 1$, then λ_i is just right; it should not be changed.

- Then a natural updating rule becomes

$$\lambda_i^{t+1} = \lambda_i^t + \Delta^t \left(1 - \sum_{j \in J} y_{ij}\right)$$

where the step-size is

$$\Delta^t = \frac{\left(\mathrm{UB} - z_{\mathrm{LR}}\left(\lambda^t\right)\right)}{\sum_{i \in I}\left(1 - \sum_{j \in J} y_{ij}\right)^2}$$