

2. For  $n$  items, each item  $I_i$  has a value  $v_i$  and size  $S_i$ . The capacity is  $B$ . First, we can re-index them such that  $\frac{v_1}{S_1} \geq \frac{v_2}{S_2} \geq \dots \geq \frac{v_n}{S_n}$

We can write the greedy solution as  $G = (x_1, x_2, x_3, \dots, x_n)$ , where  $x_i$  indicates fraction of item  $I_i$  taken.

We can write any optimal solution as  $O = (y_1, y_2, y_3, \dots, y_n)$ , where  $y_i$  indicates fraction of item  $I_i$  taken.

We can get  $\sum_{i=1}^n x_i S_i = \sum_{i=1}^n y_i S_i = B$

For the first item  $I_a$  where the two solutions differ from each other, we can have  $x_a > y_a$  since greedy always takes as much as it can.

Then we consider a new solution  $O' = (y'_1, y'_2, y'_3, \dots, y'_n)$ .

For  $j < a$ , we let  $y'_j = y_j$ .

For  $j = a$ , we let  $y'_j = x_j$ .

For  $j > a$ , we remove items of total size  $(x_a - y_a) S_a$  and reset  $y'_j$ .

The total value of solution  $O'$  is no less than  $O$ .

Since  $O$  is an optimal solution, the total value of  $O'$  and  $O$  must equal to each other.

Therefore,  $O'$  is also an optimal solution.

If we continue this process, we can finally convert  $O$  into  $G$  without changing the total value.

Therefore,  $G$  is an optimal solution.