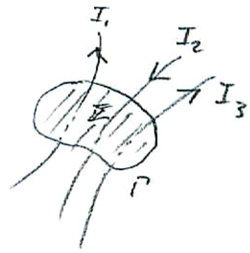
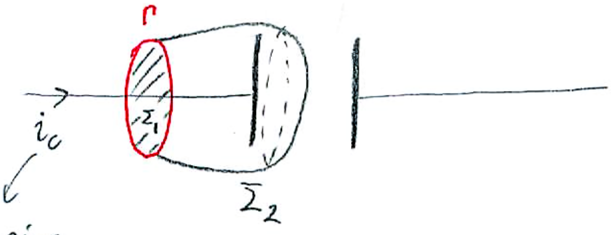
# Displacement current. Ampère's law revisited

Recall: $\oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 I_{encl}$ (Ampère's law (magnetostatics))




## Example

→ capacitor (parallel-plate)




$i_c$ → charging current (conduction current)

Apply Ampère's law with the surfaces chosen as $\Sigma_1, \Sigma_2$

$I_{encl,\Sigma_1} = i_c$

$I_{encl,\Sigma_2} = 0$ ? contradiction

Conclusion: Ampère's law is incomplete

What happens at $\Sigma_2$? $|E|$ and $\Phi_E$ increase as the capacitor is being charged

$\Phi_E$ ↳ flux through $\Sigma_2$

$$q = C\cdot v = \varepsilon_0 \frac{A}{d} Ed = \varepsilon_0 AE = \varepsilon_0 \Phi_E$$

$$q = \varepsilon_0 \Phi_E \Rightarrow \dot{q} = \varepsilon_0 \dot{\Phi}_E = i$$

$i$ → corresponding current

Define the displacement current through $\Sigma_2$

$$\boxed{i_D \overset{def}{=} \varepsilon_0 \frac{d\Phi_E}{dt}}$$

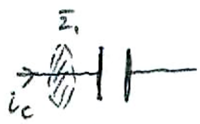
(Maxwell's idea)

Hence the complete Ampère's law

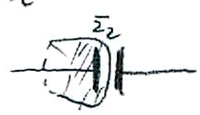
$$\boxed{\oint_\Gamma \bar{B}\cdot d\bar{l} = \mu_0 (i_c + i_D)_{encl}}$$

$i_D$ → displacement current

$i_c$ → conduction current



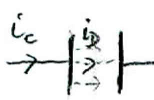

$\begin{cases} i_c \neq 0 \\ i_D = 0 \end{cases}$




$\begin{cases} i_c = 0 \\ i_D \neq 0 \end{cases}$

Notes: (1) corresponding displacement current density $j_D$ may be defined, $j_D = i_D/A$

(2) Kirchhof's junction rule is now fixed




$i_c - i_D = 0$

# How to detect the displacement current ?


circular-plate capacitor

**Idee:**

current ⇒ magnetic field

⇓

look for magnetic field between the plates

Use Ampère's law with loop Γ (circle with radius r)

1° $r < R$

$$\oint_\Gamma \bar{B}\cdot d\bar{l} \overset{\text{symmetry}}{=} 2\pi r\, B(r) = \mu_0 I_{encl} = \mu_0 \underbrace{\left(\frac{i_D}{\pi R^2}\right)}_{\text{current density}} \pi r^2 = \mu_0 \frac{i_D}{R^2} r^2$$

$$\boxed{B(r) = \frac{\mu_0}{2\pi} \frac{r}{R^2} i_D}$$

2° $r > R$

$$\oint_\Gamma \bar{B}\cdot d\bar{l} = 2\pi r\, B(r) = \mu_0 i_D \Rightarrow \boxed{B(r) = \frac{\mu_0}{2\pi r} i_D}$$

The displacement current may be detected by a magnetic field measurement ✓.

# Maxwell's equations (free space; integral form)

Gauss's law for $\bar{E}$ (1) $\oint_{\Sigma} \bar{E}\, d\bar{A} = \frac{Q_{encl}}{\varepsilon_0}$ any closed surface $\Sigma$

Gauss's law for $\bar{B}$ (2) $\oint_{\Sigma} \bar{B}\, d\bar{A} = 0$ any closed surface $\Sigma$

Ampère's law (3) $\oint_{\Gamma} \bar{B}\, d\bar{l} = \mu_0 \left( \underbrace{i}_{\text{conduction current}} + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{encl}$ any loop $\Gamma$

Faraday's law (4) $\oint_{\Gamma} \bar{E}\, d\bar{l} = -\frac{d\Phi_B}{dt}$ any stationary loop $\Gamma$

## Comment on the electric field

$$\bar{E} = \bar{E}_{charge} + \bar{E}_{induced}$$

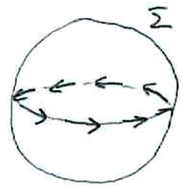
$\bar{E}_{charge}$: due to charges (conservative!)

$\bar{E}_{induced}$: due to time-dependent magnetic field (flux) (non-conservative!)

(*) $\oint_{\Gamma} \bar{E}\, d\bar{l} = \oint_{\Gamma} (\bar{E}_{charge} + \bar{E}_{induced}) = \underbrace{\oint_{\Gamma} \bar{E}_{charge}\, d\bar{l}}_{=0 \text{ always!}} + \oint_{\Gamma} \bar{E}_{induced}\, d\bar{l} =$

$= \oint_{\Gamma} \bar{E}_{induced}\, d\bar{l}$ only $\bar{E}_{induced}$ may contribute to circulation

(**) $\bar{E}_{induced}$




here $\oint_{\Sigma} \bar{E}_{induced}\, d\bar{A} = 0$ (no charge enclosed)

$\oint_{\Sigma} \bar{E}_{induced}\, d\bar{A}$ → flux

so $\oint_{\Sigma} \bar{E}\, d\bar{A} = \oint_{\Sigma} \bar{E}_{charge}\, d\bar{A}$

only $\bar{E}_{charge}$ contributes to flux

Maxwell's equations (free space ; differential form)

Apply Gauss-Ostrogradsky theorem and Stokes theorem to (1) - (4)

$$(1') \quad \operatorname{div} \bar{E} = \frac{\rho}{\varepsilon_0}$$

$$(2') \quad \operatorname{div} \bar{B} = 0$$

$$(3') \quad \operatorname{rot} \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$(4') \quad \operatorname{rot} \bar{E} = - \frac{\partial \bar{B}}{\partial t}$$

Comment (3) ⇒ (3')

$$\oint_\Gamma \bar{B}\, d\bar{l} = \mu_0 \left( i + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{encl}$$

lhs

$$\oint_\Gamma \bar{B} \cdot d\bar{l} \overset{\text{Stokes}}{=} \int_\Sigma \operatorname{rot} \bar{B}\, d\bar{A}$$

rhs

$$\mu_0 \left( i + \varepsilon_0 \frac{d\Phi_E}{dt} \right)_{encl} = \mu_0 \left( \int_\Sigma \bar{J}\, d\bar{A} + \varepsilon_0 \frac{d}{dt} \int_\Sigma \bar{E}\, d\bar{A} \right) =$$

$$= \mu_0 \int_\Sigma \left( \bar{J} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \right) d\bar{A}$$