Problem Set 10

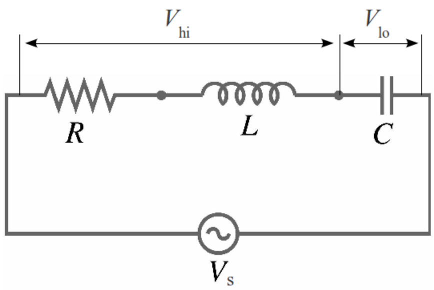
Due: 29 November 2019, 12.30 p.m.

Problem 1. One application of LRC series circuits is to high-pass or low-pass filters, which filter out either the low- or high-frequency components of a signal.

(a) In a high-pass filter the output voltage is taken across the LR combination (see the figure below). Derive an expression for $V_{\mathrm{hi}}/V_{\mathrm{s}}$, the ratio of the output and source amplitudes as a function of the angular frequency ω of the source. Show that when ω is small, this ratio is proportional to ω and thus is small, and show that the ratio approaches unity in the limit of large frequency.

(b) In a low-pass filter the output voltage is taken across the capacitor in an LRC circuit. Derive an expression for $V_{\mathrm{lo}}/V_{\mathrm{s}}$, the ratio of the output and source amplitudes as a function of the angular frequency ω of the source. Show that when ω is large this ratio is proportional to ω^{-2} and thus is small, and show that the ratio approaches unity in the limit of small frequency.

(*3 + 3 points*)



Problem 2. A resistor, inductor, and capacitor are connected in parallel to an AC source with voltage amplitude V and angular frequency ω. Let the source voltage be given by $v(t) = V\cos\omega t$.

(a) Argue that the instanteneous voltages v_R, v_L, and v_C at any instant are each equal to v and that $i = i_R + i_L + i_C$, where i is the current through the source and i_R, i_L, and i_C, are the currents through the resistor, the inductor, and the capacitor, respectively.

(b) What are the phases of i_R, i_L, and i_C with respect to v? Draw the corresponding diagram on the complex plane.

(c) Show that the current amplitude I for the current i throught the source is given by $I = \sqrt{I_R^2 + (I_C - I_L)^2}$ and this result can be written as $I = V/Z$, with $Z^{-1} = \sqrt{1/R^2 + (\omega C - 1/\omega L)^2}$.

(d) Show that at the angular frequency $\omega_0 = 1/\sqrt{LC}$, $I_C = I_L$ and I is a minimum. Since I is a minimum at resonance, is it to correct to say that the power delivered to the resistor is also a minimum at $\omega = \omega_0$? Explain.

(e) At resonance, what is the phase angle of the source current with respect to source voltage? How does this compare to the phase angle for an LRC series circuit at resonance?

(*1 + 1 + 2 + 1 + 1 points*)

Problem 3. Let $f_1(x,t) = Ae^{-k(x-vt)^2}$, $f_2(x,t) = A\sin[k(x-vt)]$, and $f_3(x,t) = Ae^{-k(kx^2+vt)}$, $f_4(x,t) = A\sin(kx)\cos(kvt)^3$. Check that the functions f_1 and f_2, satisfy the 1D classical wave equation, but f_3 and f_4 do not.

(4 × 1 point)

Problem 4. Show that (a) the *standing wave* $\xi(x,t) = A\sin(kx)\cos(\omega t)$ satisfies the wave equation and (b) express it as a sum of a wave traveling to the left and a wave traveling to the right.

(1 + 1 points)

Problem 5. A solar sailcraft uses a large, low-mass sail and the energy and momentum of sunlight for propulsion. (a) Should the sail be absorbing or reflective? Why? (b) The total power output of the sun is 3.9×10^{26} W. How large a sail is necessary to propel a 10-tone spacecraft against the gravitational force of the sun? Express your result in square kilometers. (c) Explain why your answer to part (b) is independent of the distance from the sun.

(1 + 3/2 + 3/2 points)

Problem 6. A cylindrical conductor with radius a and resistivity ρ carries a constant current I.

(a) What are the magnitude and direction of the electric field vector $\mathbf{E}$ and the magnetic field vector $\mathbf{B}$ at a point just outside the wire and at a distance a from the axis of the cylinder?

(b) What are the magnitude and direction of the Poynting vector $\mathbf{S}$ at the same point?

(c) Find the rate of flow of energy into the volume occupied by a length l of the conductor. *Hint.* What is the physical meaning of the Poynting vector integrated over a closed surface?

(d) How does your result compare to the rate of generation of thermal energy in the same volume? Why does the Poynting vector point in the direction found in (b)?

(2 + 2 + 1/2 + 1/2 + 1 points)