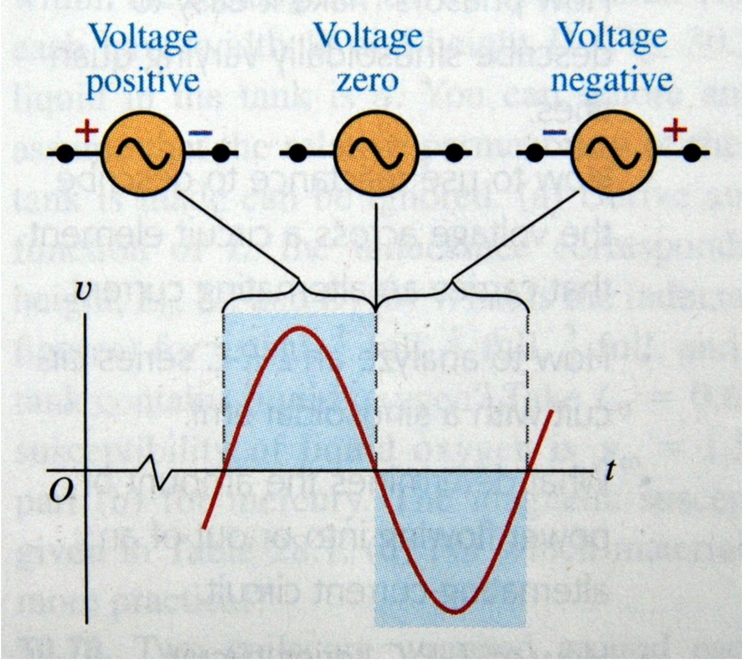
AC circuits

Alternating Current

SETUP

- source supplying a sinusoidally varying voltage

$$v(t) = V\cos(\omega t + \phi)$$



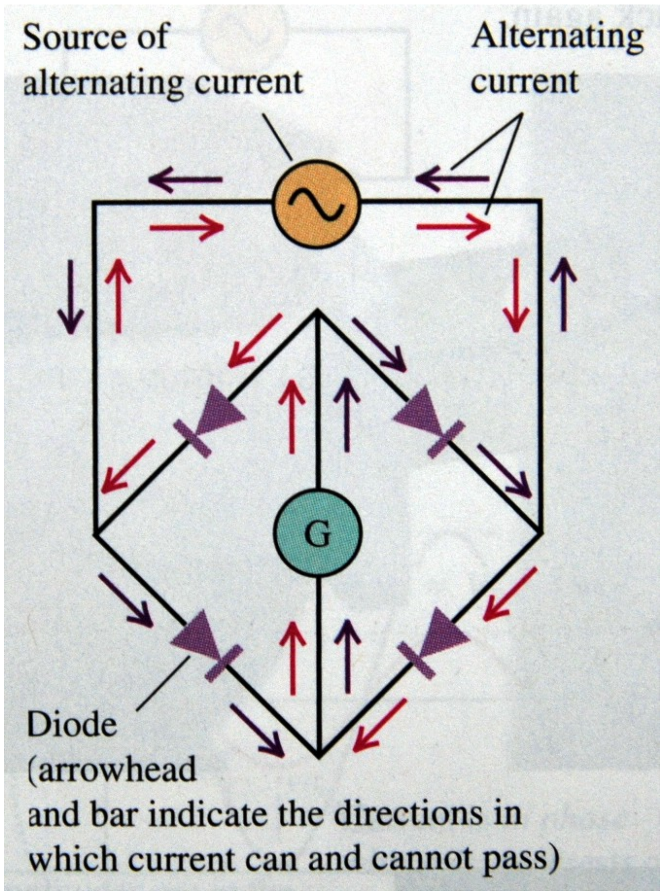
amplitude and angular frequency

$$V = \text{const} \qquad \omega = \frac{2\pi}{T}$$

- circuit elements: resistor *R*, inductor *L*, capacitor *C*

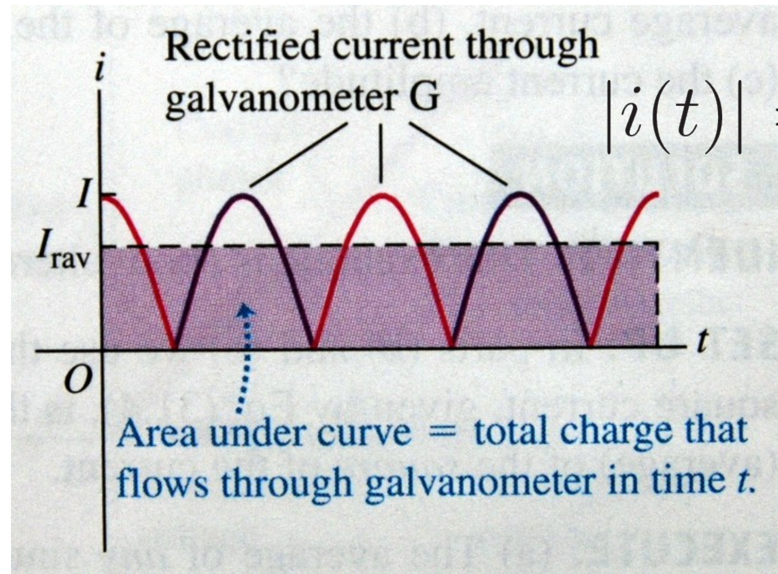
their response to AC? current in the circuit?

How to measure AC?



Graetz bridge
(full-wave rectifier)

rectified current



$$|i(t)| = |I\cos(\omega t)|$$

rectified average current
(produces equivalent total charge flow)

$$I_{\mathrm{rav}} = \frac{1}{T}\int_{t_1}^{t_1+T} |i(t)|\,dt = \frac{2}{\pi} I$$

Root-Mean-Square (rms) Values

time-dependent physical (scalar) quantity $s(t)$

$$S_{\mathrm{rms}} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s^2(t)\, dt}$$ (average over time)

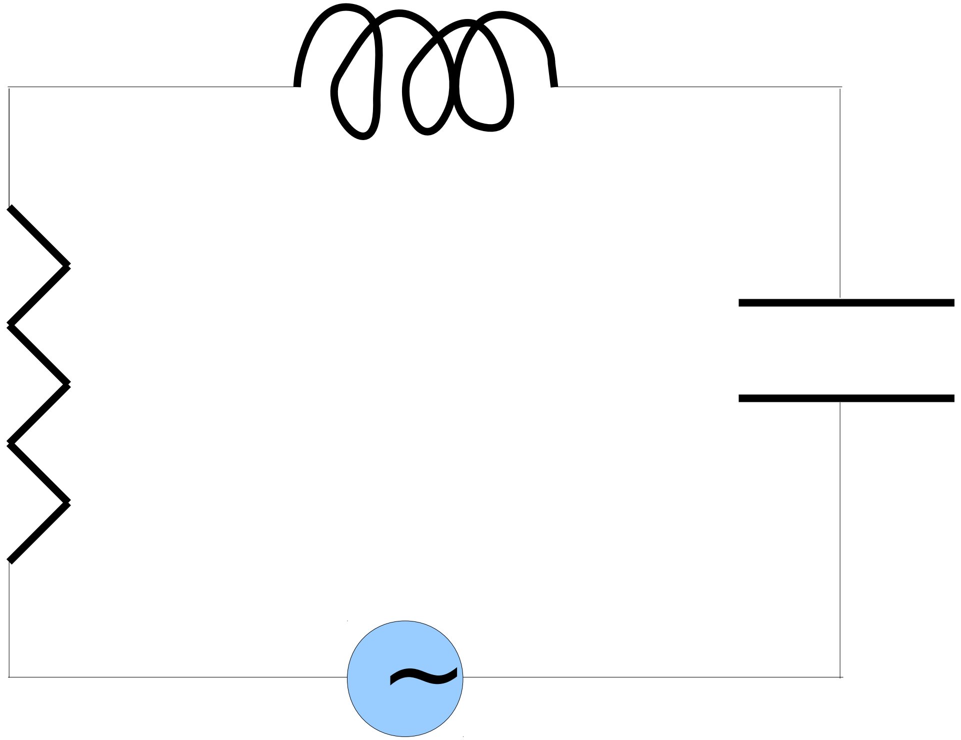
rms of a sinusoidonal AC $i(t) = I \cos \omega t$

$$I_{\mathrm{rms}} = \sqrt{\frac{1}{T} \int_{t_1}^{t_1+T} i^2(t)\, dt} = \frac{I}{\sqrt{2}}$$ (again, over one full cycle)

$$I_{\mathrm{rav}} < I_{\mathrm{rms}} < I$$

AC Circuits

circuit elements:
AC source, resistor *R*, inductor *L*, capacitor *C*



useful tool
complex numbers

Review: Complex Numbers

imaginary unit *j* $$j^2 \overset{def}{=} -1$$

algebraic form $$z = x + jy$$
(*x*, *y* are real numbers)

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y$$

real part imaginary part

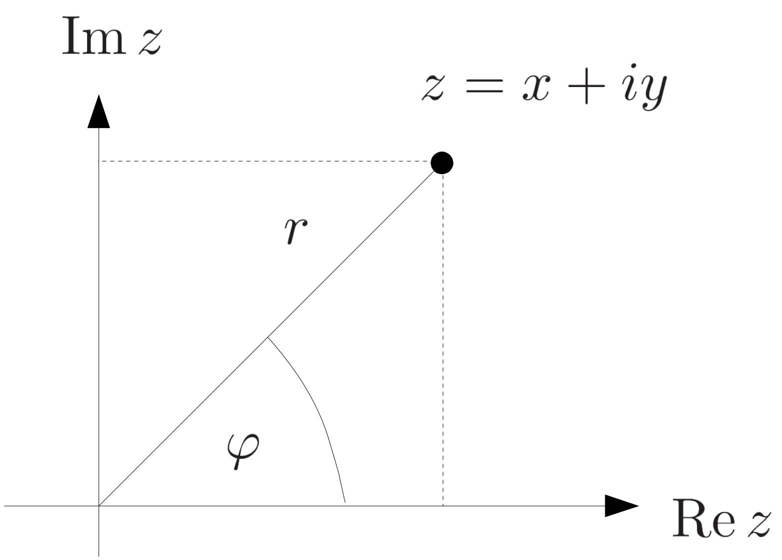
polar form $$z = re^{j\varphi}$$

$$r = \sqrt{x^2 + y^2}, \qquad \varphi = \arctan y/x$$

Euler's formula $$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$z = r\cos\varphi + jr\sin\varphi$$

complex plane



absolute value

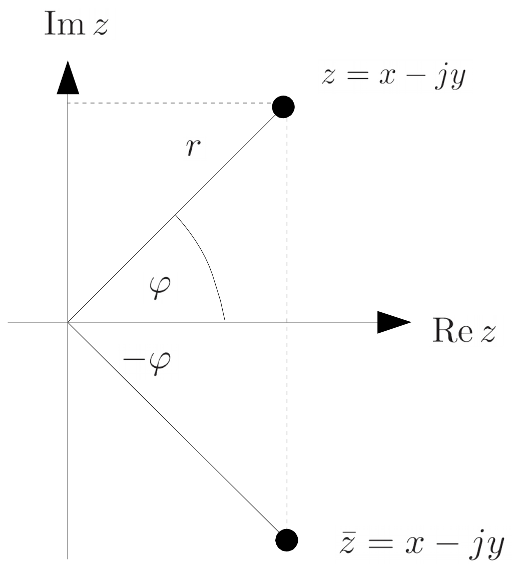
$$|z| = \sqrt{x^2 + y^2} = r$$

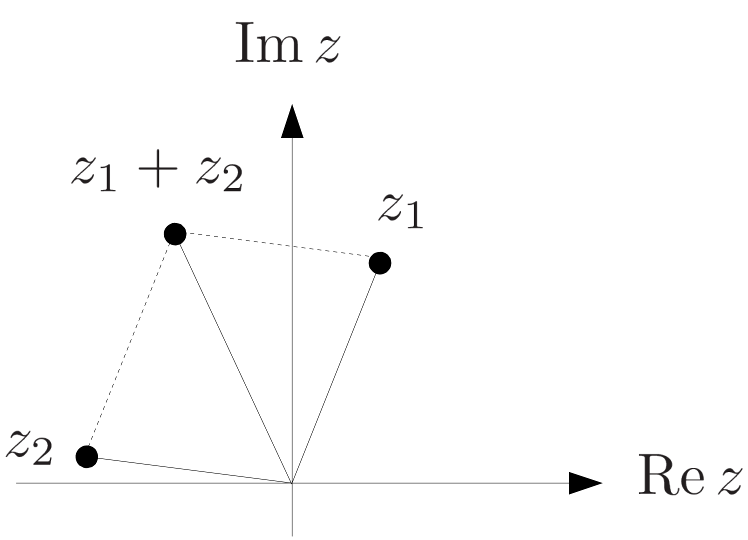
Basic Operations: conjugation and addition

conjugation

$$\bar{z} = x - jy$$

$$\bar{z} = re^{-j\varphi}$$

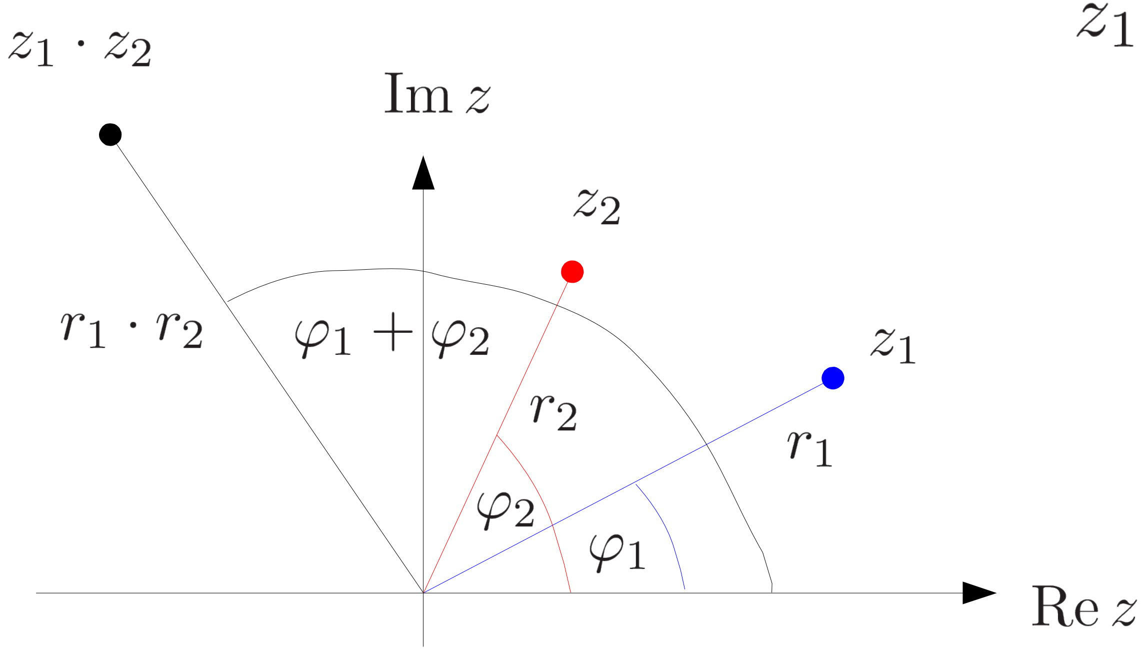




addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Basic Operations: multiplication



$$z_1 \cdot z_2 = r_1 r_2 e^{j(\varphi_1 + \varphi_2)}$$

multiplication by the imaginary unit $\quad j = e^{j\frac{\pi}{2}}$

↓

rotation by $\quad \pi/2$

Note

$$z \cdot \bar{z} = (x + jy)(x - jy) = x^2 + y^2 = |z|^2$$

Analysis of AC Circuits: Idea

At any instant of time represent the AC by a complex number

$$\tilde{i}(t) = Ie^{j\omega t}$$

Use the complex current in Ohm's law, formulas for potential difference across inductor/capacitor, Kirchhoff's rules etc.

All physical (measurable) time-dependent quantities will be associated with the real parts of their complex counterparts, e.g.

$$i(t) = \mathrm{Re}\,\tilde{i}(t) = I\cos\omega t$$

We will show that this time dependence of the current in the circuit corresponds with sinusoidally varying voltage supplied by source

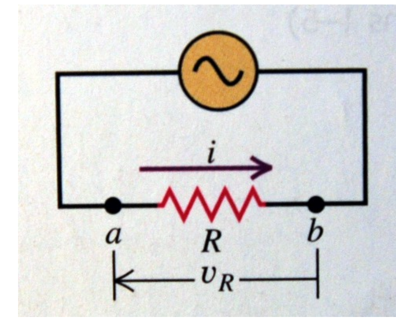
$$v(t) = V\cos(\omega t + \phi)$$

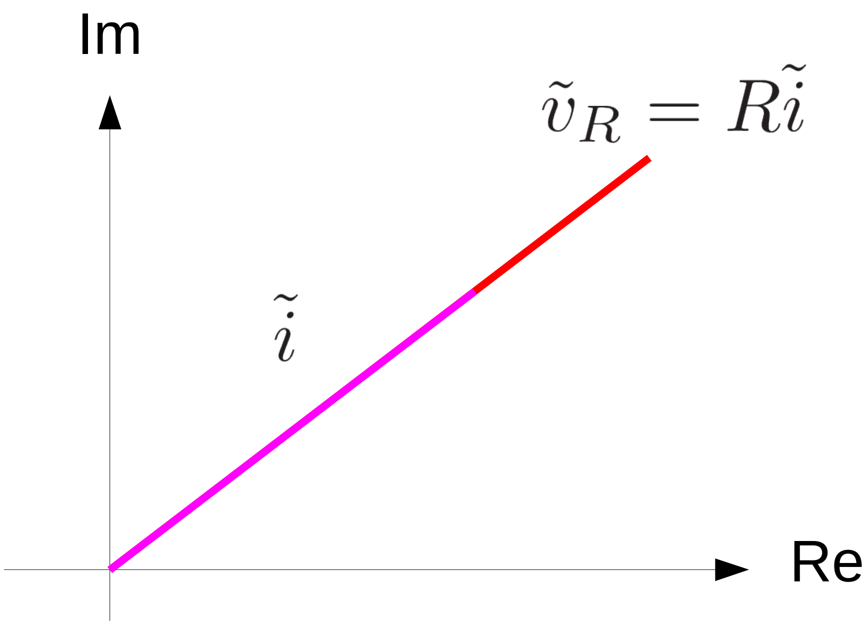
Why can we use this representation?

Resistor in an AC circuit

Ohm's law (macroscopic)

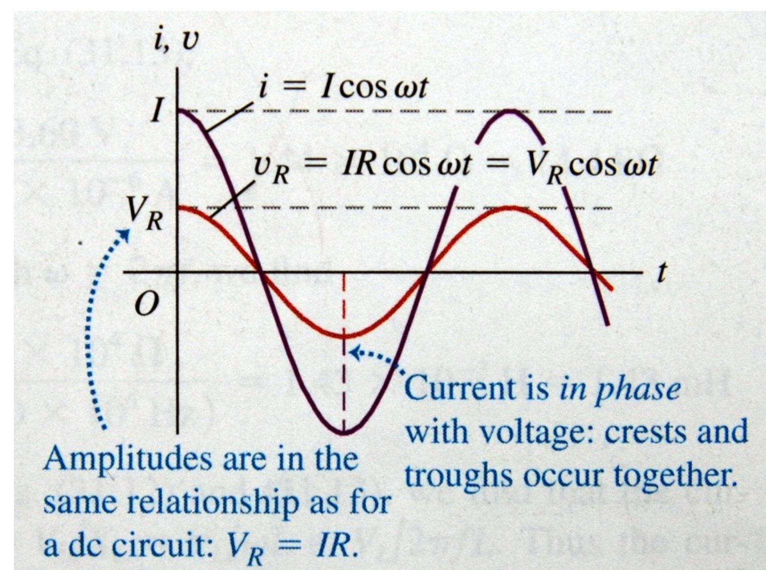
$$\tilde{v}_R = \tilde{i}R = IRe^{j\omega t}$$





current and voltage *in phase*

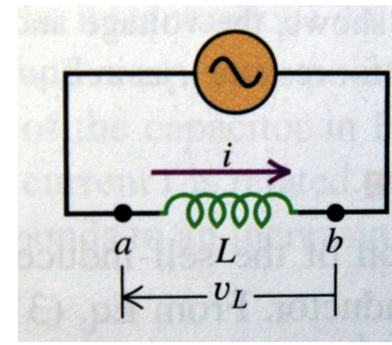
real parts



Inductor in an AC circuit

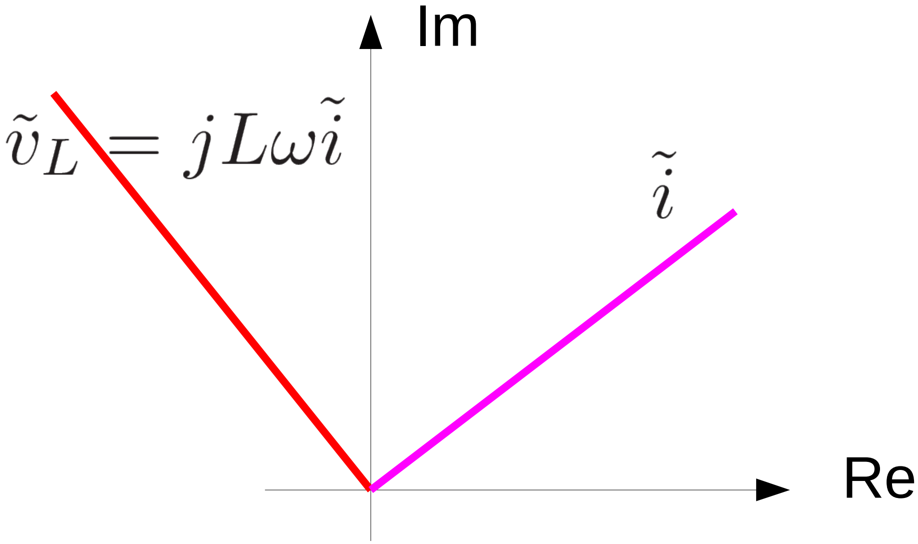
potential difference across inductor (watch the sign!)

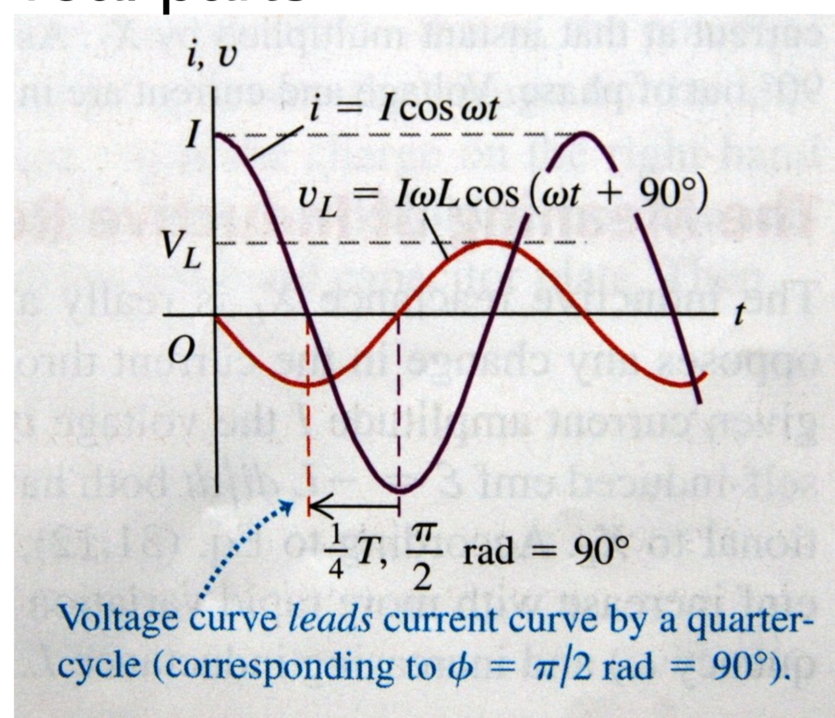
$$\tilde{v}_L(t) = +L\frac{d\tilde{i}(t)}{dt}$$



derivative $$\frac{d\tilde{i}}{dt} = \frac{d}{dt}\left(Ie^{j\omega t}\right) = j\omega I e^{j\omega t} = j\omega\tilde{i}$$

real parts



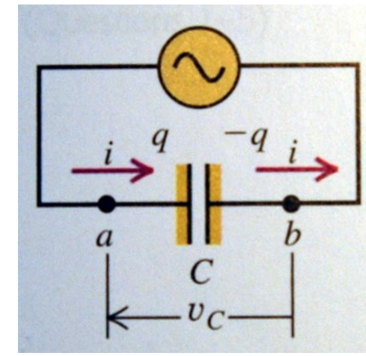


voltage leads current by $\phi = \pi/2$
(multiplication by *j*)

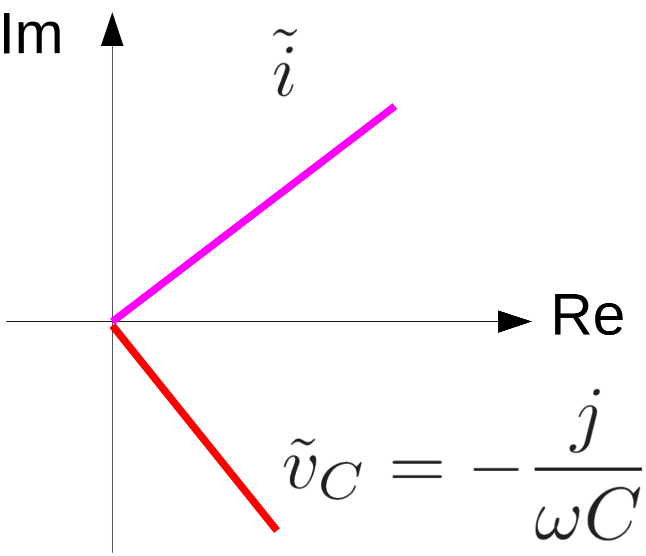
Capacitor in an AC circuit

complex quantity representing charge

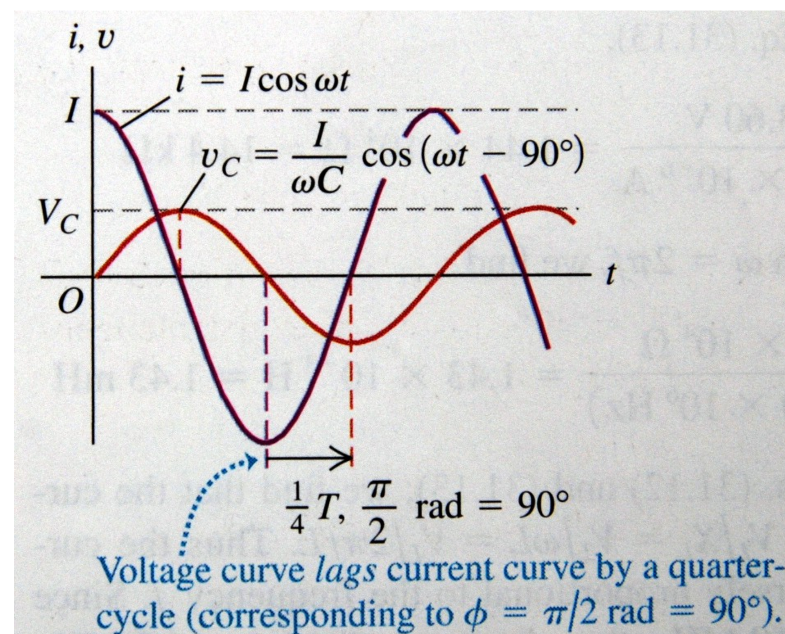
$$\tilde{q}(t) = \int \tilde{i}(t)\,dt = \int I e^{j\omega t}\,dt = -\frac{j}{\omega} I e^{j\omega t}$$



corresponding (complex) voltage $\tilde{v}_C(t) = \frac{\tilde{q}(t)}{C} = -\frac{j}{\omega C} I e^{i\omega t}$



$$\tilde{v}_C = -\frac{j}{\omega C}\tilde{i}$$



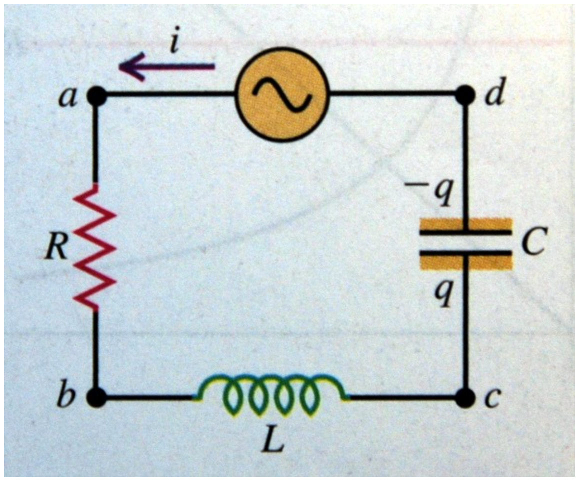
voltage *lags* current by
(multiplication by -*j*)

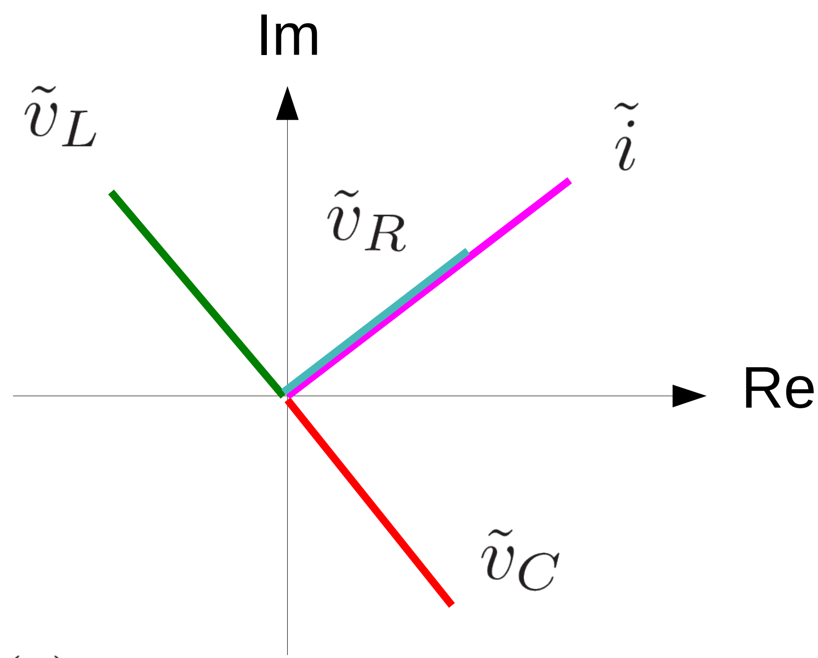
$$\phi = -\pi/2$$

Summary
(of what we know so far)

element	*voltage*	*phase w.r.t. the current*
resistor	$\tilde{v}_R = R\tilde{i}$	0
inductor	$\tilde{v}_L = jL\omega\tilde{i}$	$+\pi/2$
capacitor	$\tilde{v}_C = -\dfrac{j}{\omega C}\tilde{i}$	$-\pi/2$

RLC series circuit





Kirchhoff's loop rule

$$\tilde{v}(t) = \tilde{v}_{ad}(t) = \tilde{v}_R(t) + \tilde{v}_L(t) + \tilde{v}_C(t)$$

$$\tilde{v}(t) = \underbrace{\left(R + j\omega L - \frac{j}{\omega C}\right)}_{\tilde{Z}=Ze^{j\phi}} Ie^{j\omega t} = IZe^{j(\omega t+\phi)}$$

complex impedance

where $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}, \qquad \tan\phi = \left(\omega L - \frac{1}{\omega C}\right)/R$

Summary

Complex voltage and its real (measurable) part

$$\tilde{v}(t) = IZe^{j(\omega t+\phi)}, \qquad v(t) = \mathrm{Re}\,\tilde{v}(t) = \underbrace{IZ}_{V}\cos(\omega t+\phi)$$

Our initial guess for the form of the current was correct.

Relation between current and voltage
(and between their amplitudes)

$$\tilde{v} = \tilde{i}\tilde{Z}, \qquad V = IZ$$

where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

(real) impedance

inductive reactance (X_L)

capacitive reactance (X_C)

Why does the representation with complex numbers work here?

All relations (voltages across elements of the circuit) that we use here are linear functions of the current, or involve the time-derivative of the current (differentiation is a linear operation as well).

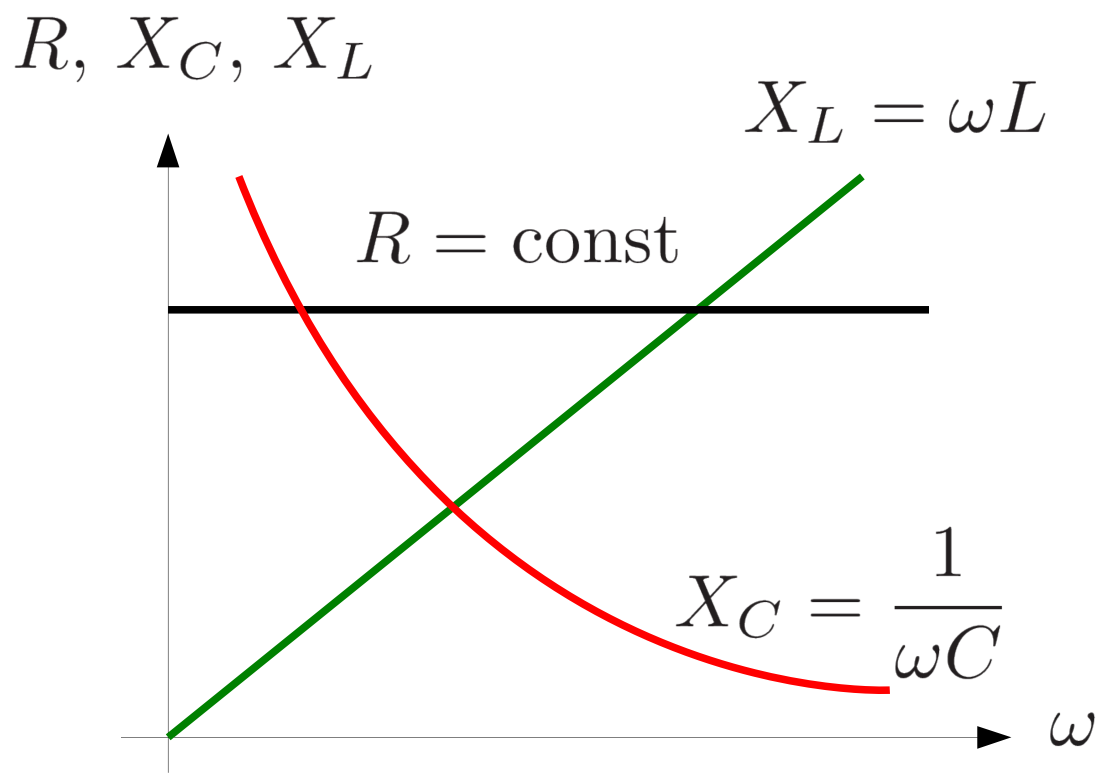
↓

The real part (physical, measurable) and the imaginary part **do not mix.**

Inductive Reactance and Capacitive Reactance

Notice that, the higher the frequency is, the smaller the capacitive reactance, and the larger the inductive reactance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$



Z is minimum when

$$\omega = \omega_0 = 1/\sqrt{LC}$$

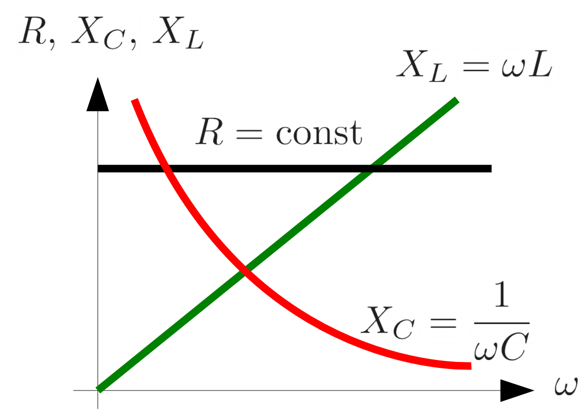
compare

DC current	$V = IR$
AC current	$V = IZ$

(holds also for rms values)

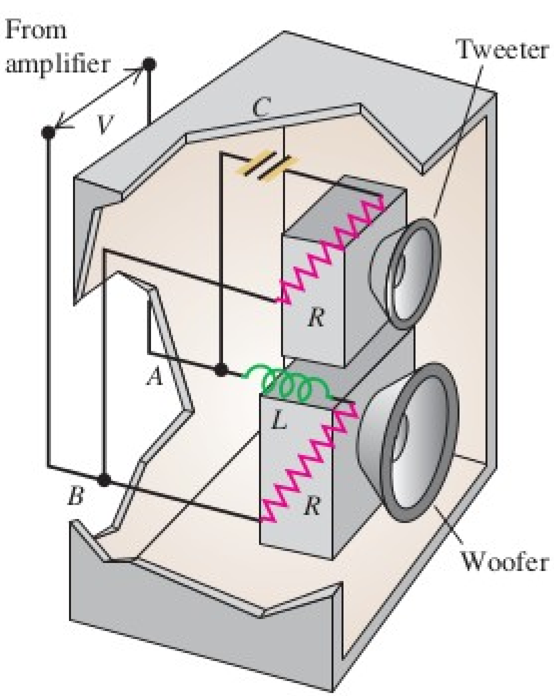
Discussion and Application

In the low-frequency limit, the main contribution to reactance is due to the capacitor; whereas in the high- frequency limit, it is mostly the inductor that contributes to reactance.



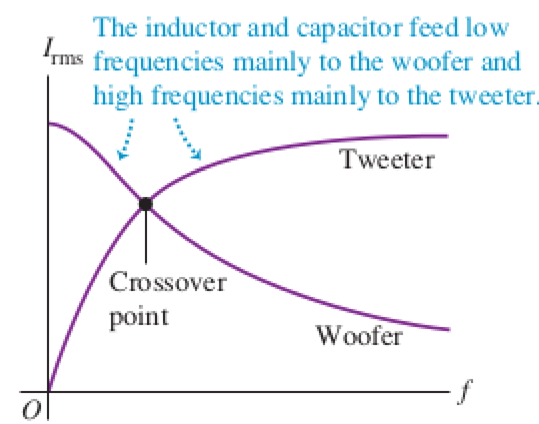
Loudspeaker system

(a) A crossover network in a loudspeaker system



In order to route signals of different frequency to the appropriate speaker, the woofer and tweeter are connected in parallel across the amplifier.

(b) Graphs of rms current as functions of frequency for a given amplifier voltage

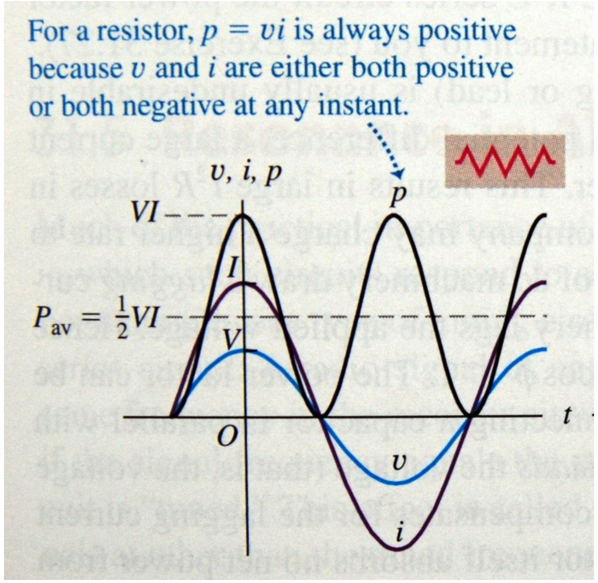


The capacitor in the tweeter branch blocks the low-frequency components of sound, but passes the higher frequencies; the inductor in the woofer branch does the opposite.

Power in AC Circuits: single element

Instantaneous power delivered to a single circuit element

$$p(t) = v(t)i(t)$$



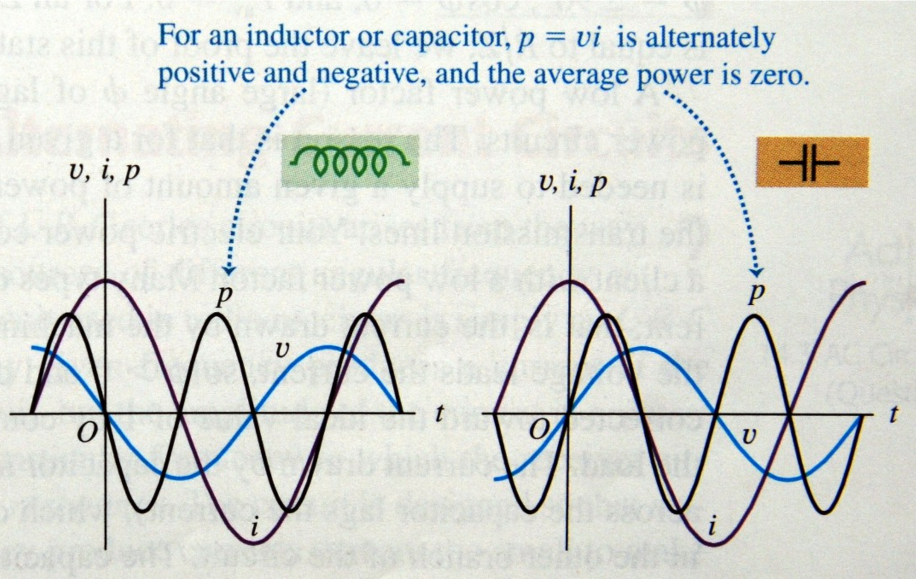
Resistive element only:
current and voltage *in phase*

The average power delivered to the resistor

$$P_{\rm av} = \frac{VI}{2} = \frac{VI}{\sqrt{2}\sqrt{2}} = V_{\rm rms}I_{\rm rms} = I_{\rm rms}^2 R$$

For pure resistance $v(t)i(t)$ is always positive

Power in AC Circuits: single element



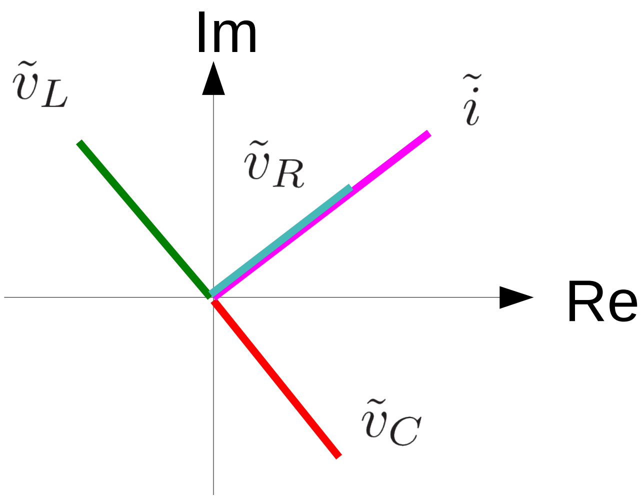
One element (either inductor or capacitor) only:

The current and the voltage *out of phase* by $\pm\pi/2$

The average power delivered to inductor/capacitor is zero

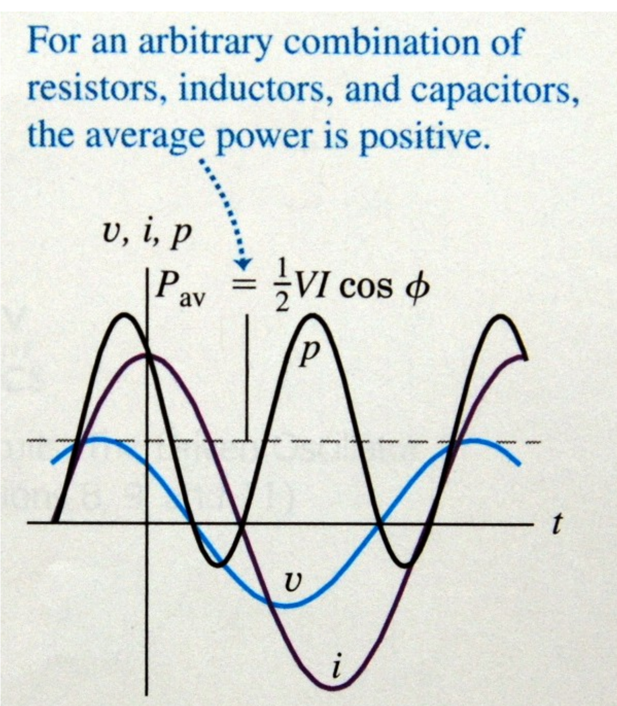
$$P_{\text{av}} = \frac{1}{T}\int_{t}^{t+T} V\cos\left(\omega t \pm \frac{\pi}{2}\right) I\cos\omega t\,dt = 0$$

Power in AC Circuits: RLC



If all elements are present, then the current and the voltage *out of phase* by

$$\phi = \arctan\left(\omega L - \frac{1}{\omega C}\right)/R$$



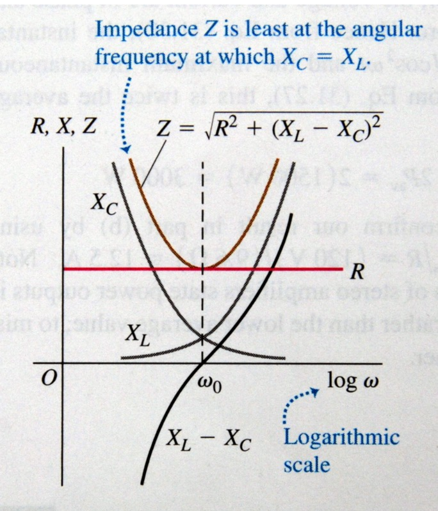
The average power delivered into the RLC circuit

$$P_{\mathrm{av}} = \frac{1}{T}\int_{t}^{t+T} V\cos(\omega t + \phi) I \cos\omega t \, dt$$

power factor

$$P_{\mathrm{av}} = \frac{VI}{2}\cos\phi = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos\phi$$

Resonance in AC series LRC circuits



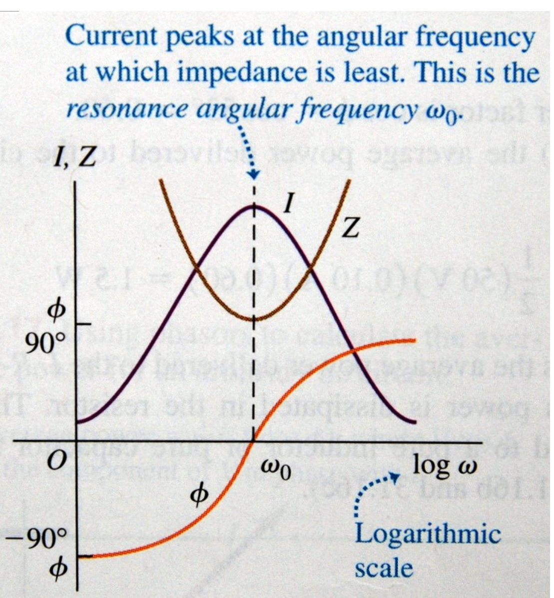
Impedance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and phase lag

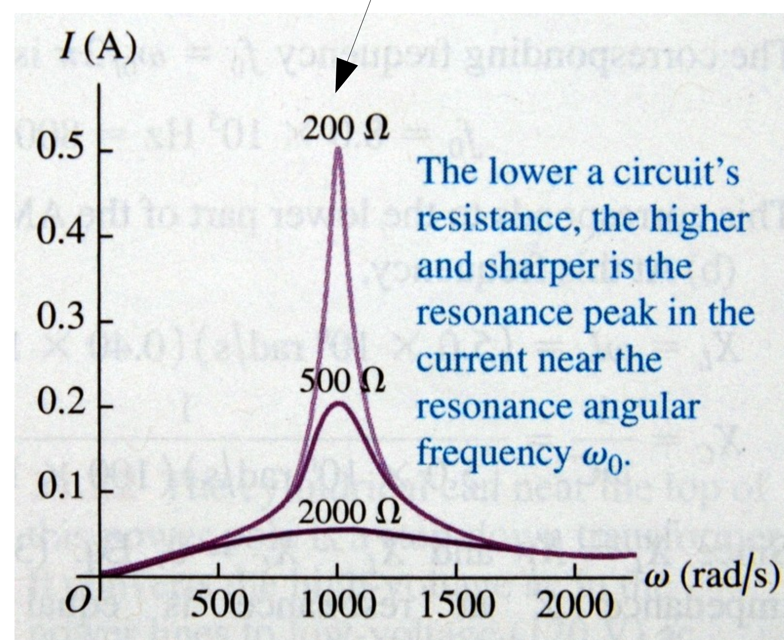
$$\tan\phi = \left(\omega L - \frac{1}{\omega C}\right) / R$$

resonance peak

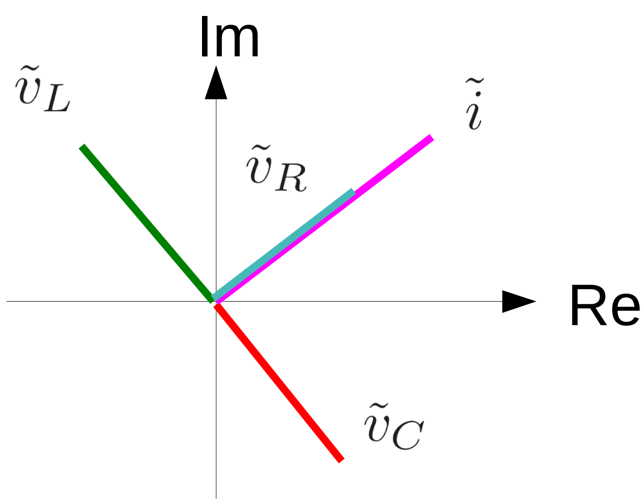


Z is minimum when

$$\omega = \omega_0 = 1/\sqrt{LC}$$



What happens at resonance frequency?



For $\omega_0 L = \dfrac{1}{\omega_0 C}$

we have $V_L = \omega_0 L I = \dfrac{I}{\omega_0 C} = V_C$

At the resonance frequency: voltage amplitudes across inductor and capacitor equal, <u>but</u> the phases differ by π

Conclusion: The circuit behaves as if the capacitor and inductor were not present. There is no phase lag between the current and the voltage.

Conclusions

Driving a series LRC circuit with a sinusoidally-varying voltage source, produces a sinusoidal current with the same, frequency, but (in general) a non-zero phase shift.

The non-zero phase shift between the driving voltage and the current is introduced by the capacitor and the inductor. However, if the circuit is in resonance, the amplitudes of the voltage across these elements are equal with a half-cycle phase shift. Therefore, their contributions to the circuit's impedance cancel and the current is in phase with the driving voltage.

Strong current response to a sinusoidally varying voltage at resonance frequency – an AC LRC circuit can be precisely tuned to respond to signals of specific frequency.