vv256: Bessel's equation. Series solutions

Dr.Olga Danilkina

UM-SJTU Joint Institute

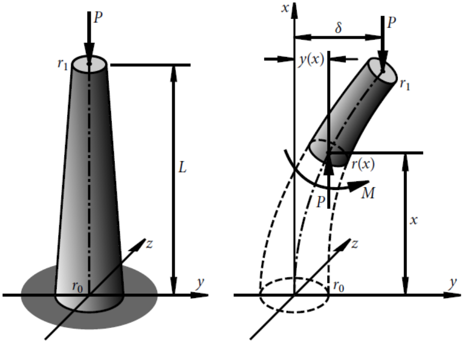
October 29, 2019

Motivating Examples: Buckling of a Tapered Column

Consider the stability of a tapered column of length L fixed at the base $x = 0$ and free at the top $x = L$. The column is subjected to an axial compressive load P at the top.

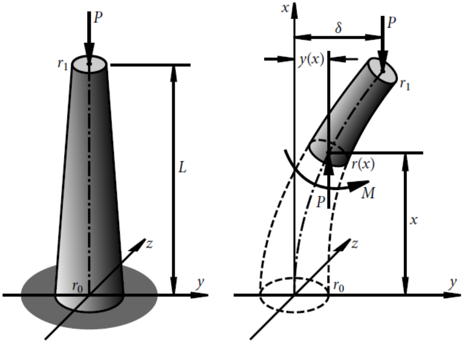
Motivating Examples: Buckling of a Tapered Column

Consider the stability of a tapered column of length L fixed at the base $x = 0$ and free at the top $x = L$. The column is subjected to an axial compressive load P at the top.



Motivating Examples: Buckling of a Tapered Column

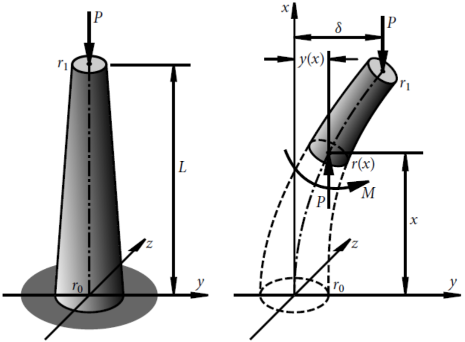
Consider the stability of a tapered column of length L fixed at the base $x = 0$ and free at the top $x = L$. The column is subjected to an axial compressive load P at the top.



The cross-section of the column is of circular shape, with radii r_0 at the base and $r_1 < r_0$ at the top, respectively, varying linearly along the length x.

Motivating Examples: Buckling of a Tapered Column

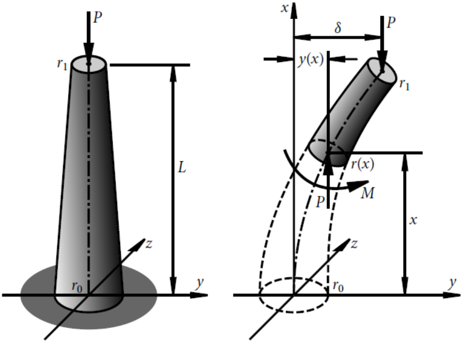
Consider the stability of a tapered column of length L fixed at the base $x = 0$ and free at the top $x = L$. The column is subjected to an axial compressive load P at the top.



The cross-section of the column is of circular shape, with radii r_0 at the base and $r_1 < r_0$ at the top, respectively, varying linearly along the length x. The modulus of elasticity for the column material is E.

Motivating Examples: Buckling of a Tapered Column

Consider the stability of a tapered column of length L fixed at the base $x = 0$ and free at the top $x = L$. The column is subjected to an axial compressive load P at the top.



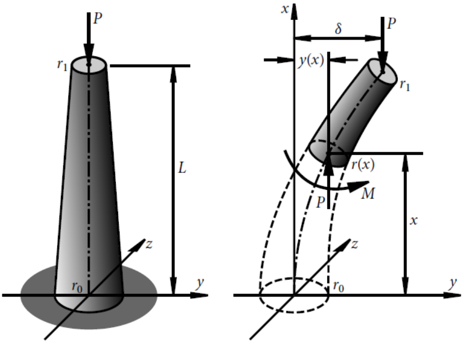
The cross-section of the column is of circular shape, with radii r_0 at the base and $r_1 < r_0$ at the top, respectively, varying linearly along the length x. The modulus of elasticity for the column material is E. Determine the buckling load P_{cr} when the column loses its stability.

Motivating Examples: Buckling of a Tapered Column

- Consider the equilibrium of a segment between x and L.

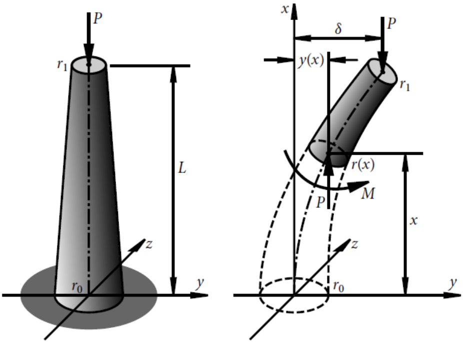
Motivating Examples: Buckling of a Tapered Column

- Consider the equilibrium of a segment between x and L.
- The lateral deflection of the column at x is $y(x)$.



Motivating Examples: Buckling of a Tapered Column

- Consider the equilibrium of a segment between x and L.
- The lateral deflection of the column at x is $y(x)$.



- The bending moment at x is

$$M(x) = P(\delta - y(x)),$$

where δ is the deflection at the free end of the column.

Motivating Examples: Buckling of a Tapered Column

- The moment-curvature relation requires that

$$EI(x)y''(x) = M(x) = P(\delta - y(x)),$$

Motivating Examples: Buckling of a Tapered Column

- The moment-curvature relation requires that

$$EI(x)y''(x) = M(x) = P(\delta - y(x)),$$

where $I(x)$ is the moment of inertia of the circular cross-section at x given by

$$I(x) = \frac{\pi}{4}r^4(x) = \frac{\pi}{4}\left(r_0\left(1 - \frac{r_0 - r_1}{r_0}\cdot\frac{x}{L}\right)\right)^4 = I_0(1 - k_1\bar{x})^4$$

with

$$I_0 = \frac{\pi r_0^4}{4},\ k_1 = \frac{r_0 - r_1}{r_0},\ \bar{x} = \frac{x}{L}$$

Motivating Examples: Buckling of a Tapered Column

- Let $\eta = \frac{y}{L}$, $\bar{\delta} = \frac{\delta}{L}$

Motivating Examples: Buckling of a Tapered Column

- Let $\eta = \frac{y}{L}$, $\bar{\delta} = \frac{\delta}{L}$
- We obtain a second-order differential equation

$$EI_0(1 - k_1\bar{x})^4 \cdot \frac{1}{L}\frac{d^2\eta}{d\bar{x}^2} + P[L\eta(\bar{x})] = P(L\bar{\delta})$$

Motivating Examples: Buckling of a Tapered Column

- Let $\eta = \frac{y}{L}$, $\bar{\delta} = \frac{\delta}{L}$
- We obtain a second-order differential equation

$$EI_0(1 - k_1\bar{x})^4 \cdot \frac{1}{L}\frac{d^2\eta}{d\bar{x}^2} + P[L\eta(\bar{x})] = P(L\bar{\delta})$$

$$(1 - k_1\bar{x})^4 \frac{d^2\eta}{d\bar{x}^2} + k^2\eta(\bar{x}) = k^2\bar{\delta}, \quad k^2 = \frac{PL^2}{EI_0}$$

Motivating Examples: Buckling of a Tapered Column

- Let $\eta = \frac{y}{L}$, $\bar{\delta} = \frac{\delta}{L}$
- We obtain a second-order differential equation

$$EI_0(1 - k_1\bar{x})^4 \cdot \frac{1}{L}\frac{d^2\eta}{d\bar{x}^2} + P[L\eta(\bar{x})] = P(L\bar{\delta})$$

$$(1 - k_1\bar{x})^4 \frac{d^2\eta}{d\bar{x}^2} + k^2\eta(\bar{x}) = k^2\bar{\delta}, \quad k^2 = \frac{PL^2}{EI_0}$$

- The general solution is

$$\eta(\bar{x}) = \eta_C(\bar{x}) + \eta_P(\bar{x}),$$

where $\eta_P(\bar{x})$ is a particular solution and $\eta_C(\bar{x})$ is the complementary solution, which is the solution of the homogeneous equation

$$(1 - k_1\bar{x})^4 \frac{d^2\eta}{d\bar{x}^2} + k^2\eta(\bar{x}) = 0.$$

Motivating Examples: Buckling of a Tapered Column

- Introduce a new variable $\xi = 1 - k_1\bar{x}$, $k \neq 0$ to transform the equation to the form

Motivating Examples: Buckling of a Tapered Column

- Introduce a new variable $\xi = 1 - k_1\bar{x}$, $k \neq 0$ to transform the equation to the form

$$\xi^4 k_1^2 \frac{d^2\eta(\xi)}{d\xi^2} + k^2\eta(\xi) = 0$$

Motivating Examples: Buckling of a Tapered Column

- Introduce a new variable $\xi = 1 - k_1\bar{x}$, $k \neq 0$ to transform the equation to the form

$$\xi^4 k_1^2 \frac{d^2\eta(\xi)}{d\xi^2} + k^2\eta(\xi) = 0$$

$$\Rightarrow \frac{d^2\eta(\xi)}{d\xi^2} + K^2\xi^{-4}\eta(\xi) = 0,\ K = \frac{k}{k_1}$$

Motivating Examples: Buckling of a Tapered Column

- Introduce a new variable $\xi = 1 - k_1\bar{x}$, $k \neq 0$ to transform the equation to the form

$$\xi^4 k_1^2 \frac{d^2\eta(\xi)}{d\xi^2} + k^2\eta(\xi) = 0$$

$$\Rightarrow \frac{d^2\eta(\xi)}{d\xi^2} + K^2\xi^{-4}\eta(\xi) = 0,\ K = \frac{k}{k_1}$$

- The equation is of the form

$$\frac{d^2\eta(\xi)}{d\xi^2} + \frac{1-2\alpha}{\xi}\frac{d\eta(\xi)}{d\xi} + \left[(\beta\rho\xi^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{\xi^2}\right]\eta(\xi) = 0$$

with $\alpha = 1/2$, $\beta = K$, $\rho = -1$, $\nu = 1/2$.

Motivating Examples: Buckling of a Tapered Column

- Introduce a new variable $\xi = 1 - k_1\bar{x}$, $k \neq 0$ to transform the equation to the form

$$\xi^4 k_1^2 \frac{d^2\eta(\xi)}{d\xi^2} + k^2\eta(\xi) = 0$$

$$\Rightarrow \frac{d^2\eta(\xi)}{d\xi^2} + K^2\xi^{-4}\eta(\xi) = 0,\ K = \frac{k}{k_1}$$

- The equation is of the form

$$\frac{d^2\eta(\xi)}{d\xi^2} + \frac{1-2\alpha}{\xi}\frac{d\eta(\xi)}{d\xi} + \left[(\beta\rho\xi^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{\xi^2}\right]\eta(\xi) = 0$$

with $\alpha = 1/2$, $\beta = K$, $\rho = -1$, $\nu = 1/2$.

- Transform the equation into the Bessel equation (see the hint in the end).

Solutions of Bessel's Equation

Bessel's equation

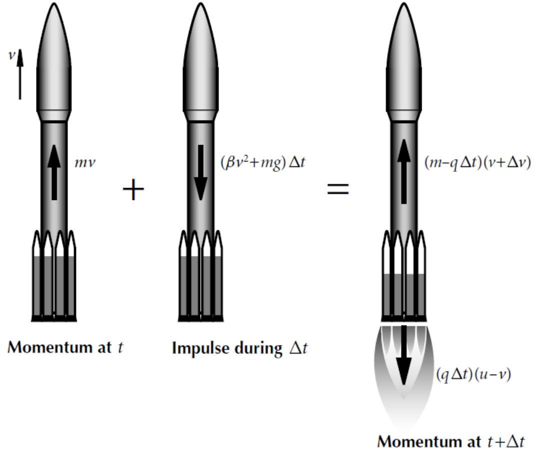
The equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,\ x > 0,$$

$\nu = const \geq 0$, is called Bessel's equation

Motivating Examples: Ascending Motion of a Rocket

Consider the ascending motion of a rocket of initial mass m_0 (including shell and fuel). The fuel is consumed at a constant rate $q = -dm/dt$ and is expelled at a constant speed u relative to the rocket. At time t, the mass of the rocket is $m(t) = m_0 - qt$. If the velocity of the rocket is $v = v_0$ at $t = t_0$, determine the velocity $v(t)$.



Motivating Examples: Ascending Motion of a Rocket

- The aerodynamic drag force F_d, which is opposed to the direction of motion, is

$$F_d = \frac{1}{2}\rho\nu^2 C_d A,$$

Motivating Examples: Ascending Motion of a Rocket

- The aerodynamic drag force F_d, which is opposed to the direction of motion, is

$$F_d = \frac{1}{2}\rho\nu^2 C_d A,$$

  - A is the the frontal (or projected) area of the rocket,
  - ρ is the density of the air,
  - C_d is the dimensionless drag coefficient, a number used to model the complex dependencies of drag on shape inclination and some flow conditions.

Motivating Examples: Ascending Motion of a Rocket

- The aerodynamic drag force F_d, which is opposed to the direction of motion, is

$$F_d = \frac{1}{2}\rho\nu^2 C_d A,$$

  - A is the the frontal (or projected) area of the rocket,
  - ρ is the density of the air,
  - C_d is the dimensionless drag coefficient, a number used to model the complex dependencies of drag on shape inclination and some flow conditions.
- Note that the drag force F_d depends not on the velocity but on the velocity squared.

Motivating Examples: Ascending Motion of a Rocket

- The aerodynamic drag force F_d, which is opposed to the direction of motion, is

$$F_d = \frac{1}{2}\rho\nu^2 C_d A,$$

  - A is the the frontal (or projected) area of the rocket,
  - ρ is the density of the air,
  - C_d is the dimensionless drag coefficient, a number used to model the complex dependencies of drag on shape inclination and some flow conditions.
- Note that the drag force F_d depends not on the velocity but on the velocity squared.
- If the fluid properties are considered constant, the drag force can be written as

$$F_d = \beta\nu^2,$$

  where β is the damping coefficient.

Motivating Examples: Ascending Motion of a Rocket

- apply the Impulse-Momentum Principle between time t and time $t + \Delta t$:

Motivating Examples: Ascending Motion of a Rocket

- apply the Impulse-Momentum Principle between time t and time $t + \Delta t$:

  Momentum at $t+$ Impulse during $\Delta t =$ Momentum at $(t+\Delta t)$

Motivating Examples: Ascending Motion of a Rocket

- apply the Impulse-Momentum Principle between time t and time $t + \Delta t$:

  Momentum at $t+$ Impulse during $\Delta t =$ Momentum at $(t+\Delta t)$

$$
\begin{aligned}
\text{Momentum at } t = \quad & m(t)\nu(t), \\
\text{Impulse during } \Delta t = \quad & -(\beta\nu^2(t) + m(t)g)\Delta t \\
\text{Momentum at } (t + \Delta t) = \quad & m(t + \Delta t)\nu(t + \Delta t) - \\
& -(q\Delta t)(u - \nu(t))
\end{aligned}
$$

Motivating Examples: Ascending Motion of a Rocket

- apply the Impulse-Momentum Principle between time t and time $t + \Delta t$:

  Momentum at $t+$ Impulse during $\Delta t =$ Momentum at $(t+\Delta t)$

$$\begin{array}{ll} \text{Momentum at } t = & m(t)\nu(t), \\ \text{Impulse during } \Delta t = & -(\beta\nu^2(t) + m(t)g)\Delta t \\ \text{Momentum at} \, (t + \Delta t) = & m(t+\Delta t)\nu(t+\Delta t)- \\ & -(q\Delta t)(u - \nu(t)) \end{array}$$

$$\begin{aligned} & m(t)\nu(t) - [\beta\nu^2(t) + m(t)g]\Delta t = \\ & = [m(t) - q\Delta t][\nu(t) + \Delta\nu] - (q\Delta t)[u - \nu(t)] \end{aligned}$$

Motivating Examples: Ascending Motion of a Rocket

- apply the Impulse-Momentum Principle between time t and time $t + \Delta t$:

  Momentum at $t+$ Impulse during $\Delta t =$ Momentum at $(t+\Delta t)$

  $$\begin{aligned} \text{Momentum at } t = \quad & m(t)\nu(t), \\ \text{Impulse during } \Delta t = \quad & -(\beta\nu^2(t) + m(t)g)\Delta t \\ \text{Momentum at } (t + \Delta t) = \quad & m(t + \Delta t)\nu(t + \Delta t) - \\ & -(q\Delta t)(u - \nu(t)) \end{aligned}$$

  $$\begin{aligned} & m(t)\nu(t) - [\beta\nu^2(t) + m(t)g]\Delta t = \\ & = [m(t) - q\Delta t][\nu(t) + \Delta\nu] - (q\Delta t)[u - \nu(t)] \end{aligned}$$

- Letting $\Delta t \Rightarrow 0$, we obtain the equation of motion of a rocket moving upward at high speed during the propelled phase

Motivating Examples: Ascending Motion of a Rocket

- apply the Impulse-Momentum Principle between time t and time $t + \Delta t$:

  Momentum at $t+$ Impulse during $\Delta t =$ Momentum at $(t+\Delta t)$

$$\begin{aligned}
\text{Momentum at } t = \quad & m(t)\nu(t), \\
\text{Impulse during } \Delta t = \quad & -(\beta\nu^2(t) + m(t)g)\Delta t \\
\text{Momentum at } (t + \Delta t) = \quad & m(t + \Delta t)\nu(t + \Delta t)- \\
& -(q\Delta t)(u - \nu(t))
\end{aligned}$$

$$\begin{aligned}
& m(t)\nu(t) - [\beta\nu^2(t) + m(t)g]\Delta t = \\
& = [m(t) - q\Delta t][\nu(t) + \Delta\nu] - (q\Delta t)[u - \nu(t)]
\end{aligned}$$

- Letting $\Delta t \Rightarrow 0$, we obtain the equation of motion of a rocket moving upward at high speed during the propelled phase

$$\underbrace{m(t)\frac{d\nu(t)}{dt}}_{\textit{Inertia force}} + \underbrace{\beta\nu^2(t)}_{\textit{Drag force}} + \underbrace{m(t)g}_{\textit{Gravity}} - \underbrace{qu}_{\textit{Thrust force}} = 0$$

Motivating Examples: Ascending Motion of a Rocket

- apply the Impulse-Momentum Principle between time t and time $t + \Delta t$:

  Momentum at $t+$ Impulse during $\Delta t =$ Momentum at $(t+\Delta t)$

  $$\begin{aligned} \text{Momentum at } t = \quad & m(t)\nu(t), \\ \text{Impulse during } \Delta t = \quad & -(\beta\nu^2(t) + m(t)g)\Delta t \\ \text{Momentum at } (t + \Delta t) = \quad & m(t + \Delta t)\nu(t + \Delta t) - \\ & -(q\Delta t)(u - \nu(t)) \end{aligned}$$

  $$\begin{aligned} & m(t)\nu(t) - [\beta\nu^2(t) + m(t)g]\Delta t = \\ & = [m(t) - q\Delta t][\nu(t) + \Delta\nu] - (q\Delta t)[u - \nu(t)] \end{aligned}$$

- Letting $\Delta t \Rightarrow 0$, we obtain the equation of motion of a rocket moving upward at high speed during the propelled phase

  $$\underbrace{m(t)\frac{d\nu(t)}{dt}}_{\textit{Inertia force}} + \underbrace{\beta\nu^2(t)}_{\textit{Drag force}} + \underbrace{m(t)g}_{\textit{Gravity}} - \underbrace{qu}_{\textit{Thrust force}} = 0$$

a first-order nonlinear differential equation with variable coefficients

Motivating Examples: Ascending Motion of a Rocket

- Use the change of variables

$$\nu(t) = \frac{m(t)\dot{V}(t)}{\beta V(t)},$$

where $V(t)$ is the new transformed "velocity".

Motivating Examples: Ascending Motion of a Rocket

- Use the change of variables

$$\nu(t) = \frac{m(t)\dot{V}(t)}{\beta V(t)},$$

where $V(t)$ is the new transformed "velocity".

- And then change the time t to the dimensionless variable τ

$$\tau = \frac{2}{q}\sqrt{\beta g m(t)}$$

Motivating Examples: Ascending Motion of a Rocket

- Use the change of variables

$$\nu(t) = \frac{m(t)\dot{V}(t)}{\beta V(t)},$$

where $V(t)$ is the new transformed "velocity".

- And then change the time t to the dimensionless variable τ

$$\tau = \frac{2}{q}\sqrt{\beta g m(t)}$$

to transform the equation of the motion into the Bessel equation

$$\tau^2 \frac{d^2 V(\tau)}{d\tau^2} + \tau \frac{dV(\tau)}{d\tau} + (\tau^2 - v^2)V(\tau) = 0, \; v = 2\sqrt{\frac{\beta u}{q}}$$

Series Solution about an Ordinary Point

So, we need to know how to solve Bessel's equation!

- We introduce a concept of a series solution for a general nth-order equation with variable coefficients.
- We shall distinguish between the cases of ordinary, singular, regular singular and irregular points in series solutions.

Series Solution about an Ordinary Point

We shall consider two motivating examples to demonstrate that power series can be used to solve ODEs.

Example 1. Let a solution of the equation

$$y' - y = 0$$

Series Solution about an Ordinary Point

We shall consider two motivating examples to demonstrate that power series can be used to solve ODEs.

Example 1. Let a solution of the equation

$$y' - y = 0$$

be in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < r, \quad r > 0,$$

where the constants a_n to be determined.

Series Solution about an Ordinary Point

We shall consider two motivating examples to demonstrate that power series can be used to solve ODEs.

Example 1. Let a solution of the equation

$$y' - y = 0$$

be in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \, |x| < r, \quad r > 0,$$

where the constants a_n to be determined.

- Differentiate $y(x)$ with respect to x

Series Solution about an Ordinary Point

We shall consider two motivating examples to demonstrate that power series can be used to solve ODEs.

Example 1. Let a solution of the equation

$$y' - y = 0$$

be in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < r, \quad r > 0,$$

where the constants a_n to be determined.

- Differentiate $y(x)$ with respect to x

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} =$$

Series Solution about an Ordinary Point

We shall consider two motivating examples to demonstrate that power series can be used to solve ODEs.
Example 1. Let a solution of the equation

$$y' - y = 0$$

be in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < r, \quad r > 0,$$

where the constants a_n to be determined.

- Differentiate $y(x)$ with respect to x

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m$$

change the index of summation

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n =$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

x^0 :

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$x^0: \qquad a_1 - a_0 = 0$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{aligned} x^0 &: \quad a_1 - a_0 = 0 \Rightarrow \quad a_1 = a_0, \\ x^1 &: \end{aligned}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{aligned} x^0 :& \qquad a_1 - a_0 = 0 \Rightarrow & \qquad a_1 = a_0, \\ x^1 :& \qquad 2a_2 - a_1 = 0 & \end{aligned}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{array}{lll} x^0 : & a_1 - a_0 = 0 \Rightarrow & a_1 = a_0, \\ x^1 : & 2a_2 - a_1 = 0 \Rightarrow & a_2 = \dfrac{1}{2}a_1 = \end{array}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{aligned}
x^0 &: \quad & a_1 - a_0 = 0 \Rightarrow \quad & a_1 = a_0, \\
x^1 &: \quad & 2a_2 - a_1 = 0 \Rightarrow \quad & a_2 = \frac{1}{2}a_1 = \frac{1}{2!}a_0, \\
x^2 &:
\end{aligned}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{aligned}
x^0 : &\qquad a_1 - a_0 = 0 \Rightarrow &\qquad a_1 = a_0, \\
x^1 : &\qquad 2a_2 - a_1 = 0 \Rightarrow &\qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2!}a_0, \\
x^2 : &\qquad 3a_3 - a_2 = 0 &
\end{aligned}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{aligned}
x^0 :& \qquad a_1 - a_0 = 0 \Rightarrow & a_1 &= a_0, \\
x^1 :& \qquad 2a_2 - a_1 = 0 \Rightarrow & a_2 &= \frac{1}{2}a_1 = \frac{1}{2!}a_0, \\
x^2 :& \qquad 3a_3 - a_2 = 0 \Rightarrow & a_3 &= \frac{1}{3}a_2 =
\end{aligned}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{array}{lcr} x^0 : & a_1 - a_0 = 0 \Rightarrow & a_1 = a_0, \\ x^1 : & 2a_2 - a_1 = 0 \Rightarrow & a_2 = \frac{1}{2}a_1 = \frac{1}{2!}a_0, \\ x^2 : & 3a_3 - a_2 = 0 \Rightarrow & a_3 = \frac{1}{3}a_2 = \frac{1}{3} \cdot \frac{1}{2!}a_0, \\ \ldots & & \\ x^n : & & \end{array}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{aligned}
x^0 &: & a_1 - a_0 = 0 \Rightarrow & \quad a_1 = a_0, \\
x^1 &: & 2a_2 - a_1 = 0 \Rightarrow & \quad a_2 = \frac{1}{2}a_1 = \frac{1}{2!}a_0, \\
x^2 &: & 3a_3 - a_2 = 0 \Rightarrow & \quad a_3 = \frac{1}{3}a_2 = \frac{1}{3}\cdot\frac{1}{2!}a_0, \\
\ldots & & & \\
x^n &: & (n+1)a_{n+1} - a_n = 0 &
\end{aligned}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{array}{lcr}
x^0: & a_1 - a_0 = 0 \Rightarrow & a_1 = a_0, \\
x^1: & 2a_2 - a_1 = 0 \Rightarrow & a_2 = \frac{1}{2}a_1 = \frac{1}{2!}a_0, \\
x^2: & 3a_3 - a_2 = 0 \Rightarrow & a_3 = \frac{1}{3}a_2 = \frac{1}{3} \cdot \frac{1}{2!}a_0, \\
\ldots & & \\
x^n: & (n+1)a_{n+1} - a_n = 0 \Rightarrow & a_{n+1} = \frac{1}{n+1}a_n =
\end{array}$$

Series Solution about an Ordinary Point

- Substitute it into the differential equation

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$$

- For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero:

$$\begin{aligned}
x^0 :& \quad a_1 - a_0 = 0 \Rightarrow & a_1 &= a_0, \\
x^1 :& \quad 2a_2 - a_1 = 0 \Rightarrow & a_2 &= \frac{1}{2}a_1 = \frac{1}{2!}a_0, \\
x^2 :& \quad 3a_3 - a_2 = 0 \Rightarrow & a_3 &= \frac{1}{3}a_2 = \frac{1}{3} \cdot \frac{1}{2!}a_0, \\
\ldots & & & \\
x^n :& \quad (n+1)a_{n+1} - a_n = 0 \Rightarrow & a_{n+1} &= \frac{1}{n+1}a_n = \\
& & &= \frac{1}{n+1}\frac{1}{n!}a_0 = \frac{1}{(n+1)!}a_0
\end{aligned}$$

Series Solution about an Ordinary Point

- The solution of the equation is

$$y(x) = a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \ldots + \frac{1}{n!} a_0 x^n + \ldots =$$

Series Solution about an Ordinary Point

- The solution of the equation is

$$y(x) = a_0 + a_0 x + \frac{1}{2!}a_0 x^2 + \frac{1}{3!}a_0 x^3 + \ldots + \frac{1}{n!}a_0 x^n + \ldots =$$

$$= a_0 \underbrace{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + \ldots\right)}$$

Series Solution about an Ordinary Point

- The solution of the equation is

$$y(x) = a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \ldots + \frac{1}{n!} a_0 x^n + \ldots =$$

$$= a_0 \underbrace{\left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots + \frac{1}{n!} x^n + \ldots \right)}_{e^x} =$$

Series Solution about an Ordinary Point

- The solution of the equation is

$$y(x) = a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \ldots + \frac{1}{n!} a_0 x^n + \ldots =$$

$$= a_0 \underbrace{\left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots + \frac{1}{n!} x^n + \ldots\right)}_{e^x} =$$

$$= a_0 e^x, \quad a_0 \text{ is a constant}$$

Series Solution about an Ordinary Point

Example 2. Consider the equation

$$y'' + y = 0.$$

Series Solution about an Ordinary Point

Example 2. Consider the equation

$$y'' + y = 0.$$

- Suppose that the solution is in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,\ |x| < r, \quad r > 0, \text{ and}$$

Series Solution about an Ordinary Point

Example 2. Consider the equation

$$y'' + y = 0.$$

- Suppose that the solution is in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < r, \quad r > 0, \text{ and}$$

- Differentiate $y(x)$ twice w.r.t. x

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} =$$

Series Solution about an Ordinary Point

Example 2. Consider the equation

$$y'' + y = 0.$$

- Suppose that the solution is in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < r, \quad r > 0, \text{ and}$$

- Differentiate $y(x)$ twice w.r.t. x

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m$$

Series Solution about an Ordinary Point

Example 2. Consider the equation

$$y'' + y = 0.$$

- Suppose that the solution is in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < r, \quad r > 0, \text{ and}$$

- Differentiate $y(x)$ twice w.r.t. x

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m$$

- Substitute the derivative into the equation to obtain

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n =$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

x^0 :

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n \right] x^n = 0$$

$$x^0 : \qquad 2 \cdot 1 a_2 + a_0 = 0$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

$$\begin{aligned} x^0: &\qquad 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow &\qquad a_2 = -\tfrac{1}{2!} a_0, \\ x^1: & & \end{aligned}$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

$$x^0 : \qquad 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow \qquad a_2 = -\tfrac{1}{2!} a_0,$$

$$x^1 : \qquad 3 \cdot 2 a_3 + a_1 = 0$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n \right] x^n = 0$$

$$\begin{aligned}
x^0 : \quad & 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow & a_2 &= -\tfrac{1}{2!} a_0, \\
x^1 : \quad & 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow & a_3 &= -\frac{1}{3!} a_1, \\
x^2 : \quad & 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow & a_4 &= -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\
x^3 : \quad &
\end{aligned}$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n \right] x^n = 0$$

$$\begin{aligned}
x^0 : &\qquad 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow &\qquad a_2 &= -\tfrac{1}{2!} a_0, \\
x^1 : &\qquad 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow &\qquad a_3 &= -\frac{1}{3!} a_1, \\
x^2 : &\qquad 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow &\qquad a_4 &= -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\
x^3 : &\qquad 5 \cdot 4 a_5 + a_3 = 0
\end{aligned}$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

$$\begin{aligned}
x^0: &\quad 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow & a_2 &= -\tfrac{1}{2!} a_0, \\
x^1: &\quad 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow & a_3 &= -\frac{1}{3!} a_1, \\
x^2: &\quad 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow & a_4 &= -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\
x^3: &\quad 5 \cdot 4 a_5 + a_3 = 0 \Rightarrow & a_5 &= -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1, \\
x^4: &
\end{aligned}$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

$$\begin{aligned}
x^0 : &\quad 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow & a_2 &= -\tfrac{1}{2!} a_0, \\
x^1 : &\quad 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow & a_3 &= -\frac{1}{3!} a_1, \\
x^2 : &\quad 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow & a_4 &= -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\
x^3 : &\quad 5 \cdot 4 a_5 + a_3 = 0 \Rightarrow & a_5 &= -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1, \\
x^4 : &\quad 6 \cdot 5 a_6 + a_4 = 0 & &
\end{aligned}$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

$$\begin{aligned}
x^0 :&\quad 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow &\quad a_2 &= -\tfrac{1}{2!} a_0, \\
x^1 :&\quad 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow &\quad a_3 &= -\frac{1}{3!} a_1, \\
x^2 :&\quad 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow &\quad a_4 &= -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\
x^3 :&\quad 5 \cdot 4 a_5 + a_3 = 0 \Rightarrow &\quad a_5 &= -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1, \\
x^4 :&\quad 6 \cdot 5 a_6 + a_4 = 0 \Rightarrow &\quad a_6 &= -\frac{1}{6 \cdot 5} a_4 = -\frac{1}{6!} a_0, \\
x^5 :&
\end{aligned}$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n \right] x^n = 0$$

$$\begin{aligned}
x^0 : \quad & 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow & a_2 &= -\tfrac{1}{2!} a_0, \\
x^1 : \quad & 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow & a_3 &= -\frac{1}{3!} a_1, \\
x^2 : \quad & 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow & a_4 &= -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\
x^3 : \quad & 5 \cdot 4 a_5 + a_3 = 0 \Rightarrow & a_5 &= -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1, \\
x^4 : \quad & 6 \cdot 5 a_6 + a_4 = 0 \Rightarrow & a_6 &= -\frac{1}{6 \cdot 5} a_4 = -\frac{1}{6!} a_0, \\
x^5 : \quad & 7 \cdot 6 a_7 + a_5 = 0
\end{aligned}$$

Series Solution about an Ordinary Point

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

$$\begin{aligned}
x^0 : &\quad 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow & a_2 &= -\tfrac{1}{2!} a_0, \\
x^1 : &\quad 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow & a_3 &= -\frac{1}{3!} a_1, \\
x^2 : &\quad 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow & a_4 &= -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\
x^3 : &\quad 5 \cdot 4 a_5 + a_3 = 0 \Rightarrow & a_5 &= -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1, \\
x^4 : &\quad 6 \cdot 5 a_6 + a_4 = 0 \Rightarrow & a_6 &= -\frac{1}{6 \cdot 5} a_4 = -\frac{1}{6!} a_0, \\
x^5 : &\quad 7 \cdot 6 a_7 + a_5 = 0 \Rightarrow & a_7 &= -\frac{1}{7 \cdot 6} a_5 = -\frac{1}{7!} a_1,
\end{aligned}$$

Series Solution about an Ordinary Point

- In general,

$$a_{2k} = (-1)^k \frac{1}{(2k)!} a_0, \quad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!} a_1$$

Series Solution about an Ordinary Point

- In general,

$$a_{2k} = (-1)^k \frac{1}{(2k)!} a_0, \quad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!} a_1$$

- The solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}_{\cos x} + a_1 \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{\sin x} =$$

$$= a_0 \cos x + a_1 \sin x, \quad a_0, a_1 \text{ are constants}$$

Series Solution about an Ordinary Point

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

A point x_0 is called an ordinary point of the given differential equation if each of the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ and $f(x)$ are analytic at $x = x_0$, that is

Series Solution about an Ordinary Point

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

A point x_0 is called an ordinary point of the given differential equation if each of the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ and $f(x)$ are analytic at $x = x_0$, that is $p_i(x)$, $i = 1 \ldots n-1$ and $f(x)$ can be expressed as power series about x_0 that are convergent for $|x - x_0| < r$, $r > 0$:

$$p_i(x) = \sum_{n=0}^{\infty} p_{i,n}(x - x_0)^n, \quad f(x) = \sum_{n=0}^{\infty} f_n(x - x_0)^n.$$

Theorem: Series Solution about an Ordinary Point

Theorem

If x_0 is an ordinary point of nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x),$$

then any solution of the equation can be expressed as a power series in $x - x_0$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \ |x - x_0| < R$$

and this representation is unique. Here $R \geq r$ is the radius of convergence.

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0, \; p > 0.$$

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,\ p > 0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0,\ p_1(x) = -\frac{2x}{1-x^2},\ p_0(x) = \frac{p(p+1)}{1-x^2}$$

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,\ p > 0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0,\ p_1(x) = -\frac{2x}{1-x^2},\ p_0(x) = \frac{p(p+1)}{1-x^2}$$

Both $p_1(x)$ and $p_0(x)$ can be expanded in power series:

$$p_1(x) = -\frac{2x}{1-x^2} =$$

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,\ p > 0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0,\ p_1(x) = -\frac{2x}{1-x^2},\ p_0(x) = \frac{p(p+1)}{1-x^2}$$

Both $p_1(x)$ and $p_0(x)$ can be expanded in power series:

$$p_1(x) = -\frac{2x}{1-x^2} = -2x\sum_{n=0}^{\infty}(x^2)^n =$$

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,\ p > 0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0,\ p_1(x) = -\frac{2x}{1-x^2},\ p_0(x) = \frac{p(p+1)}{1-x^2}$$

Both $p_1(x)$ and $p_0(x)$ can be expanded in power series:

$$p_1(x) = -\frac{2x}{1-x^2} = -2x\sum_{n=0}^{\infty}(x^2)^n = -2\sum_{n=0}^{\infty}x^{2n+1},\ |x| < 1,$$

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0, \; p > 0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0, \; p_1(x) = -\frac{2x}{1-x^2}, \; p_0(x) = \frac{p(p+1)}{1-x^2}$$

Both $p_1(x)$ and $p_0(x)$ can be expanded in power series:

$$p_1(x) = -\frac{2x}{1-x^2} = -2x\sum_{n=0}^{\infty}(x^2)^n = -2\sum_{n=0}^{\infty}x^{2n+1}, \; |x| < 1,$$

$$p_0(x) = p(p+1)\frac{1}{1-x^2} =$$

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,\ p > 0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0,\ p_1(x) = -\frac{2x}{1-x^2},\ p_0(x) = \frac{p(p+1)}{1-x^2}$$

Both $p_1(x)$ and $p_0(x)$ can be expanded in power series:

$$p_1(x) = -\frac{2x}{1-x^2} = -2x\sum_{n=0}^{\infty}(x^2)^n = -2\sum_{n=0}^{\infty}x^{2n+1},\ |x| < 1,$$

$$p_0(x) = p(p+1)\frac{1}{1-x^2} = p(p+1)\sum_{n=0}^{\infty}(x^2)^n =$$

Example: Legendre Equation

Find the power series solution in x of the **Legendre equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,\ p > 0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0,\ p_1(x) = -\frac{2x}{1-x^2},\ p_0(x) = \frac{p(p+1)}{1-x^2}$$

Both $p_1(x)$ and $p_0(x)$ can be expanded in power series:

$$p_1(x) = -\frac{2x}{1-x^2} = -2x\sum_{n=0}^{\infty}(x^2)^n = -2\sum_{n=0}^{\infty}x^{2n+1},\ |x| < 1,$$

$$p_0(x) = p(p+1)\frac{1}{1-x^2} = p(p+1)\sum_{n=0}^{\infty}(x^2)^n = p(p+1)\sum_{n=0}^{\infty}x^{2n},\ |x| < 1.$$

Legendre Equation

Therefore, $x = 0$ is an ordinary point and unique power series solution exists

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < 1.$$

Legendre Equation

Therefore, $x = 0$ is an ordinary point and unique power series solution exists

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < 1.$$

To determine the coefficients a_n, $n = 0, 1, \ldots$, substitute the series solution into the equation to obtain

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Legendre Equation

Therefore, $x = 0$ is an ordinary point and unique power series solution exists

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < 1.$$

To determine the coefficients a_n, $n = 0, 1, \ldots$, substitute the series solution into the equation to obtain

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Changing the index of summation in the one part of the first term

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} =$$

Legendre Equation

Therefore, $x = 0$ is an ordinary point and unique power series solution exists

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,\ |x| < 1.$$

To determine the coefficients a_n, $n = 0, 1, \ldots$, substitute the series solution into the equation to obtain

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Changing the index of summation in the one part of the first term

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m,$$

we obtain

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$

$$-2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$
$$-2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

As in the examples above, this equation is true if the coefficients of x^n, $n = 0, 1, \ldots,$ equal zero

x^0 :

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$

$$-2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

As in the examples above, this equation is true if the coefficients of x^n, $n = 0, 1, \ldots,$ equal zero

$$x^0: \qquad 2 \cdot 1 a_2 + p(p+1)a_0 = 0$$

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$
$$-2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

As in the examples above, this equation is true if the coefficients of x^n, $n = 0, 1, \ldots,$ equal zero

$$x^0: \qquad 2 \cdot 1 a_2 + p(p+1)a_0 = 0 \Rightarrow \qquad a_2 = -\frac{p(p+1)}{2!}a_0,$$

x^1 :

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$
$$-2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

As in the examples above, this equation is true if the coefficients of x^n, $n = 0, 1, \ldots,$ equal zero

$$x^0: \quad 2 \cdot 1 a_2 + p(p+1)a_0 = 0 \Rightarrow \quad a_2 = -\frac{p(p+1)}{2!}a_0,$$

$$x^1: \quad 3 \cdot 2 a_3 - 2a_1 + p(p+1)a_1 = 0$$

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$
$$-2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

As in the examples above, this equation is true if the coefficients of x^n, $n = 0, 1, \ldots,$ equal zero

$$\begin{aligned} x^0: &\quad 2\cdot 1 a_2 + p(p+1)a_0 = 0 \Rightarrow &\quad a_2 = -\frac{p(p+1)}{2!}a_0, \\ x^1: &\quad 3\cdot 2a_3 - 2a_1 + p(p+1)a_1 = 0 \Rightarrow &\quad a_3 = -\frac{(p-1)(p+2)}{3!}a_1, \end{aligned}$$

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$

$$-2\sum_{n=1}^{\infty} na_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

As in the examples above, this equation is true if the coefficients of x^n, $n = 0, 1, \ldots,$ equal zero

$$x^0: \qquad 2\cdot 1a_2 + p(p+1)a_0 = 0 \Rightarrow \qquad a_2 = -\frac{p(p+1)}{2!}a_0,$$

$$x^1: \qquad 3\cdot 2a_3 - 2a_1 + p(p+1)a_1 = 0 \Rightarrow \quad a_3 = -\frac{(p-1)(p+2)}{3!}a_1,$$

For all $n \geq 2$, the coefficient of x^n is

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0$$

Legendre Equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n -$$

$$-2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

As in the examples above, this equation is true if the coefficients of x^n, $n = 0, 1, \ldots,$ equal zero

$$x^0: \qquad 2\cdot 1 a_2 + p(p+1)a_0 = 0 \Rightarrow \qquad a_2 = -\frac{p(p+1)}{2!}a_0,$$

$$x^1: \qquad 3\cdot 2a_3 - 2a_1 + p(p+1)a_1 = 0 \Rightarrow \quad a_3 = -\frac{(p-1)(p+2)}{3!}a_1,$$

For all $n \geq 2$, the coefficient of x^n is

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0$$

$$a_{n+2} = -\frac{(p-n)(p+(n+1))}{(n+2)(n+1)}a_n$$

Legendre Equation

$$x^2: \quad a_4 = -\frac{(p-2)(p+3)}{4\cdot 3}a_2 = (-1)^2\frac{p(p+1)(p-2)(p+3)}{4!}a_0$$

$$x^3: \quad a_5 = -\frac{(p-3)(p+4)}{5\cdot 4}a_3 = (-1)^2\frac{(p-1)(p+2)(p-3)(p+4)}{5!}a_1$$

Legendre Equation

$$x^2: \qquad a_4 = -\frac{(p-2)(p+3)}{4\cdot 3}a_2 = (-1)^2\frac{p(p+1)(p-2)(p+3)}{4!}a_0$$

$$x^3: \quad a_5 = -\frac{(p-3)(p+4)}{5\cdot 4}a_3 = (-1)^2\frac{(p-1)(p+2)(p-3)(p+4)}{5!}a_1$$

Are you ready to derive the general formula for the coefficients a_n?
Let us calculate two more of them

Legendre Equation

$$x^2: \qquad a_4 = -\frac{(p-2)(p+3)}{4 \cdot 3} a_2 = (-1)^2 \frac{p(p+1)(p-2)(p+3)}{4!} a_0$$

$$x^3: \quad a_5 = -\frac{(p-3)(p+4)}{5 \cdot 4} a_3 = (-1)^2 \frac{(p-1)(p+2)(p-3)(p+4)}{5!} a_1$$

Are you ready to derive the general formula for the coefficients a_n?
Let us calculate two more of them

$$x^4: \qquad a_6 = -\frac{(p-4)(p+5)}{6 \cdot 5} a_4$$

Legendre Equation

$$x^2: \qquad a_4 = -\frac{(p-2)(p+3)}{4 \cdot 3} a_2 = (-1)^2 \frac{p(p+1)(p-2)(p+3)}{4!} a_0$$

$$x^3: \quad a_5 = -\frac{(p-3)(p+4)}{5 \cdot 4} a_3 = (-1)^2 \frac{(p-1)(p+2)(p-3)(p+4)}{5!} a_1$$

Are you ready to derive the general formula for the coefficients a_n?
Let us calculate two more of them

$$x^4: \qquad a_6 = -\frac{(p-4)(p+5)}{6 \cdot 5} a_4 = (-1)^3 \frac{p(p+1)(p-2)(p+3)(p-4)(p+5)}{6!} a_0$$

Legendre Equation

$$x^2: \qquad a_4 = -\frac{(p-2)(p+3)}{4\cdot 3} a_2 = (-1)^2 \frac{p(p+1)(p-2)(p+3)}{4!} a_0$$

$$x^3: \quad a_5 = -\frac{(p-3)(p+4)}{5\cdot 4} a_3 = (-1)^2 \frac{(p-1)(p+2)(p-3)(p+4)}{5!} a_1$$

Are you ready to derive the general formula for the coefficients a_n?
Let us calculate two more of them

$$x^4: \qquad a_6 = -\frac{(p-4)(p+5)}{6\cdot 5} a_4$$

$$= (-1)^3 \frac{p(p+1)(p-2)(p+3)(p-4)(p+5)}{6!} a_0$$

$$x^5: \qquad a_7 = -\frac{(p-5)(p+6)}{7\cdot 6} a_5$$

$$=$$

Legendre Equation

$$x^2: \qquad a_4 = -\frac{(p-2)(p+3)}{4 \cdot 3} a_2 = (-1)^2 \frac{p(p+1)(p-2)(p+3)}{4!} a_0$$

$$x^3: \quad a_5 = -\frac{(p-3)(p+4)}{5 \cdot 4} a_3 = (-1)^2 \frac{(p-1)(p+2)(p-3)(p+4)}{5!} a_1$$

Are you ready to derive the general formula for the coefficients a_n? Let us calculate two more of them

$$x^4: \qquad a_6 = -\frac{(p-4)(p+5)}{6 \cdot 5} a_4$$

$$= (-1)^3 \frac{p(p+1)(p-2)(p+3)(p-4)(p+5)}{6!} a_0$$

$$x^5: \qquad a_7 = -\frac{(p-5)(p+6)}{7 \cdot 6} a_5$$

$$= (-1)^3 \frac{(p-1)(p+2)(p-3)(p+4)(p-5)(p+6)}{7!} a_1$$

Legendre Equation

In general,

$$a_{2k} = \quad (-1)^k \frac{p(p+1)(p-2)(p+3)\ldots(p-2k+2)(p+2k-1)}{(2k)!} a_0$$

Legendre Equation

In general,

$$\begin{aligned} a_{2k} = &\quad (-1)^k \frac{p(p+1)(p-2)(p+3)\ldots(p-2k+2)(p+2k-1)}{(2k)!} a_0 \\ &= \frac{(-1)^k}{(2k)!} \prod_{i=1}^{k} ((p-2i+2)(p+2i-1)) a_0 \end{aligned}$$

Legendre Equation

In general,

$$
\begin{aligned}
a_{2k} = & \quad (-1)^k \frac{p(p+1)(p-2)(p+3)\ldots(p-2k+2)(p+2k-1)}{(2k)!} a_0 \\
& = \frac{(-1)^k}{(2k)!} \prod_{i=1}^{k} ((p-2i+2)(p+2i-1)) a_0 \\
a_{2k+1} = & \quad (-1)^k \frac{(p-1)(p+2)(p-3)\ldots(p-2k+1)(p+2k)}{(2k+1)!} a_0
\end{aligned}
$$

Legendre Equation

In general,

$$
\begin{aligned}
a_{2k} = &\quad (-1)^k \frac{p(p+1)(p-2)(p+3)\ldots(p-2k+2)(p+2k-1)}{(2k)!} a_0 \\
&= \frac{(-1)^k}{(2k)!} \prod_{i=1}^{k} ((p-2i+2)(p+2i-1)) a_0 \\
a_{2k+1} = &\quad (-1)^k \frac{(p-1)(p+2)(p-3)\ldots(p-2k+1)(p+2k)}{(2k+1)!} a_0 \\
&= \frac{(-1)^k}{(2k+1)!} \prod_{i=1}^{k} ((p-2i+1)(p+2i)) a_1
\end{aligned}
$$

Legendre Equation

In general,

$$
\begin{aligned}
a_{2k} = \quad & (-1)^k \frac{p(p+1)(p-2)(p+3)\dots(p-2k+2)(p+2k-1)}{(2k)!} a_0 \\
& = \frac{(-1)^k}{(2k)!} \prod_{i=1}^{k} ((p-2i+2)(p+2i-1)) a_0 \\
a_{2k+1} = \quad & (-1)^k \frac{(p-1)(p+2)(p-3)\dots(p-2k+1)(p+2k)}{(2k+1)!} a_0 \\
& = \frac{(-1)^k}{(2k+1)!} \prod_{i=1}^{k} ((p-2i+1)(p+2i)) a_1
\end{aligned}
$$

Therefore, the power series solution of Legendre equation is

$$
\begin{aligned}
y(x) = \quad & a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \prod_{i=1}^{k} ((p-2i+2)(p+2i-1)) x^{2k} \\
& + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \prod_{i=1}^{k} ((p-2i+1)(p+2i)) x^{2k+1}, \ |x| < 1.
\end{aligned}
$$

Series Solution about an Ordinary Point

Example. Find the power series solution in x of the equation

$$xy'' + y\ln(1-x) = 0,\ |x| < 1.$$

Series Solution about an Ordinary Point

Example. Find the power series solution in x of the equation

$$xy'' + y\ln(1-x) = 0,\ |x| < 1.$$

- Write the equation in the form $y'' + p_1(x)y' + p_0(x)y = 0$ and hence, $p_1(x) = 0,\ p_0(x) = \dfrac{\ln(1-x)}{x}$

Series Solution about an Ordinary Point

Example. Find the power series solution in x of the equation

$$xy'' + y\ln(1-x) = 0,\ |x| < 1.$$

- Write the equation in the form $y'' + p_1(x)y' + p_0(x)y = 0$ and hence, $p_1(x) = 0,\ p_0(x) = \dfrac{\ln(1-x)}{x}$
- Since $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n,\ |x| < 1$, integrating both sides of the equation w.r.t x yields

$$\ln(1-x) = -\int \frac{1}{1-x}dx =$$

Series Solution about an Ordinary Point

Example. Find the power series solution in x of the equation

$$xy'' + y\ln(1-x) = 0,\ |x| < 1.$$

- Write the equation in the form $y'' + p_1(x)y' + p_0(x)y = 0$ and hence, $p_1(x) = 0$, $p_0(x) = \dfrac{\ln(1-x)}{x}$
- Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$, integrating both sides of the equation w.r.t x yields

$$\ln(1-x) = -\int \frac{1}{1-x}dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1},\ |x| < 1$$

Series Solution about an Ordinary Point

Example. Find the power series solution in x of the equation

$$xy'' + y\ln(1-x) = 0, \ |x| < 1.$$

- Write the equation in the form $y'' + p_1(x)y' + p_0(x)y = 0$ and hence, $p_1(x) = 0, \ p_0(x) = \dfrac{\ln(1-x)}{x}$
- Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1$, integrating both sides of the equation w.r.t x yields

$$\ln(1-x) = -\int \frac{1}{1-x}dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \ |x| < 1$$

$$p_0(x) = \frac{\ln(1-x)}{x} =$$

Series Solution about an Ordinary Point

Example. Find the power series solution in x of the equation

$$xy'' + y\ln(1-x) = 0, \; |x| < 1.$$

- Write the equation in the form $y'' + p_1(x)y' + p_0(x)y = 0$ and hence, $p_1(x) = 0$, $p_0(x) = \dfrac{\ln(1-x)}{x}$
- Since $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, $|x| < 1$, integrating both sides of the equation w.r.t x yields

$$\ln(1-x) = -\int \frac{1}{1-x}dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \; |x| < 1$$

$$p_0(x) = \frac{\ln(1-x)}{x} = -\sum_{n=0}^{\infty} \frac{x^n}{n+1}, \; |x| < 1$$

$x = 0$ is an ordinary point!

Series Solution about an Ordinary Point

- The solution of the differential equation can be expressed in a power series
$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < 1.$$

Series Solution about an Ordinary Point

- The solution of the differential equation can be expressed in a power series
$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < 1.$$
- Substituting it into the differential equation results in
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} \frac{x^n}{n+1} \sum_{n=0}^{\infty} a_n x^n = 0$$

Series Solution about an Ordinary Point

- The solution of the differential equation can be expressed in a power series
$$y(x) = \sum_{n=0}^{\infty} a_n x^n,\ |x| < 1.$$
- Substituting it into the differential equation results in
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} \frac{x^n}{n+1} \sum_{n=0}^{\infty} a_n x^n = 0$$
- $\displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} =$

Series Solution about an Ordinary Point

- The solution of the differential equation can be expressed in a power series
$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < 1.$$
- Substituting it into the differential equation results in
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} \frac{x^n}{n+1} \sum_{n=0}^{\infty} a_n x^n = 0$$
- $\displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m$ and

Series Solution about an Ordinary Point

- The solution of the differential equation can be expressed in a power series
$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \; |x| < 1.$$
- Substituting it into the differential equation results in
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} \frac{x^n}{n+1} \sum_{n=0}^{\infty} a_n x^n = 0$$
- $\displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$ and
$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} \cdot \sum_{n=0}^{\infty} a_n x^n =$$

Series Solution about an Ordinary Point

- The solution of the differential equation can be expressed in a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ |x| < 1.$$

- Substituting it into the differential equation results in

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} \frac{x^n}{n+1} \sum_{n=0}^{\infty} a_n x^n = 0$$

- $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m$ and

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} \cdot \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{x^m}{m+1} a_{n-m} x^{n-m} \right) =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{a_{n-m}}{m+1} \right) x^n$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - \sum_{m=0}^{n} \frac{a_{n-m}}{m+1} \right] x^n = 0$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}\right]x^n=0$$

- For this equation to be true, the coefficient of $x_n, n=0,1,\ldots$ must be zero

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - \sum_{m=0}^{n} \frac{a_{n-m}}{m+1} \right] x^n = 0$$

- For this equation to be true, the coefficient of $x_n, n = 0, 1, \ldots$ must be zero

$$a_{n+2} = \frac{1}{(n+2)(n+1)} \sum_{m=0}^{n} \frac{a_{n-m}}{m+1}$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}\right]x^n=0$$

- For this equation to be true, the coefficient of $x_n, n = 0, 1, \ldots$ must be zero

$$a_{n+2}=\frac{1}{(n+2)(n+1)}\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}$$

- Therefore,

$$n = 0 :$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}\right]x^n=0$$

- For this equation to be true, the coefficient of $x_n, n=0,1,\ldots$ must be zero

$$a_{n+2}=\frac{1}{(n+2)(n+1)}\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}$$

- Therefore,

$$n=0: \quad a_2=\frac{a_0}{2}$$

$$n=1:$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}\right]x^n=0$$

- For this equation to be true, the coefficient of $x_n, n = 0, 1, \ldots$ must be zero

$$a_{n+2}=\frac{1}{(n+2)(n+1)}\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}$$

- Therefore,

$$n=0: \quad a_2=\frac{a_0}{2}$$

$$n=1: \quad a_3=\frac{1}{3\cdot 2}(a_1+\frac{a_0}{2})=$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}\right]x^n=0$$

- For this equation to be true, the coefficient of $x_n, n=0,1,\ldots$ must be zero

$$a_{n+2}=\frac{1}{(n+2)(n+1)}\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}$$

- Therefore,

$$\begin{aligned}
n=0: &\quad a_2=\frac{a_0}{2}\\
n=1: &\quad a_3=\frac{1}{3\cdot 2}(a_1+\frac{a_0}{2})=\frac{a_0}{12}+\frac{a_1}{6}\\
n=2: &
\end{aligned}$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - \sum_{m=0}^{n}\frac{a_{n-m}}{m+1}\right]x^n = 0$$

- For this equation to be true, the coefficient of $x_n, n = 0, 1, \ldots$ must be zero

$$a_{n+2} = \frac{1}{(n+2)(n+1)}\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}$$

- Therefore,

$$\begin{aligned}
n = 0: &\quad a_2 = \frac{a_0}{2} \\
n = 1: &\quad a_3 = \frac{1}{3 \cdot 2}(a_1 + \frac{a_0}{2}) = \frac{a_0}{12} + \frac{a_1}{6} \\
n = 2: &\quad a_4 = \frac{1}{4 \cdot 3}(a_2 + \frac{a_1}{2} + \frac{a_0}{3}) =
\end{aligned}$$

Series Solution about an Ordinary Point

- The equation becomes

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}\right]x^n=0$$

- For this equation to be true, the coefficient of $x_n, n=0,1,\ldots$ must be zero

$$a_{n+2}=\frac{1}{(n+2)(n+1)}\sum_{m=0}^{n}\frac{a_{n-m}}{m+1}$$

- Therefore,

$$\begin{aligned}
n=0: &\quad a_2=\frac{a_0}{2}\\
n=1: &\quad a_3=\frac{1}{3\cdot 2}(a_1+\frac{a_0}{2})=\frac{a_0}{12}+\frac{a_1}{6}\\
n=2: &\quad a_4=\frac{1}{4\cdot 3}(a_2+\frac{a_1}{2}+\frac{a_0}{3})=\frac{5a_0}{72}+\frac{a_1}{24}
\end{aligned}$$

Series Solution about an Ordinary Point

$$n = 3: \quad a_5 = \frac{1}{5 \cdot 4}(a_3 + \frac{a_2}{2} + \frac{a_1}{3} + \frac{a_0}{4}) =$$

Series Solution about an Ordinary Point

$$n = 3: \quad a_5 = \frac{1}{5 \cdot 4}\left(a_3 + \frac{a_2}{2} + \frac{a_1}{3} + \frac{a_0}{4}\right) = \frac{7a_0}{240} + \frac{a_1}{40}$$

$$n = 4: \quad a_6 = \frac{1}{6 \cdot 5}\left(a_4 + \frac{a_3}{2} + \frac{a_2}{3} + \frac{a_1}{4} + \frac{a_0}{5}\right) =$$

Series Solution about an Ordinary Point

$$n = 3: \quad a_5 = \frac{1}{5 \cdot 4}\left(a_3 + \frac{a_2}{2} + \frac{a_1}{3} + \frac{a_0}{4}\right) = \frac{7a_0}{240} + \frac{a_1}{40}$$

$$n = 4: \quad a_6 = \frac{1}{6 \cdot 5}\left(a_4 + \frac{a_3}{2} + \frac{a_2}{3} + \frac{a_1}{4} + \frac{a_0}{5}\right) = \frac{43a_0}{2700} + \frac{a_1}{80}$$

Series Solution about an Ordinary Point

$$n = 3: \quad a_5 = \frac{1}{5 \cdot 4}(a_3 + \frac{a_2}{2} + \frac{a_1}{3} + \frac{a_0}{4}) = \frac{7a_0}{240} + \frac{a_1}{40}$$
$$n = 4: \quad a_6 = \frac{1}{6 \cdot 5}(a_4 + \frac{a_3}{2} + \frac{a_2}{3} + \frac{a_1}{4} + \frac{a_0}{5}) = \frac{43a_0}{2700} + \frac{a_1}{80}$$

- The solution of the equation is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \frac{a_0}{2}x^2 + \left(\frac{a_0}{12} + \frac{a_1}{6}\right)x^3 +$$

$$+ \left(\frac{5a_0}{72} + \frac{a_1}{24}\right)x^4 + \left(\frac{7a_0}{240} + \frac{a_1}{40}\right)x^5 + \left(\frac{43a_0}{2700} + \frac{a_1}{80}\right)x^6 + \dots$$

Series Solution about a Regular Singular Point

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

Series Solution about a Regular Singular Point

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

A point x_0 is called a singular point of the given differential equation if it is not an ordinary point, that is, not all of the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are analytic at $x = x_0$.

Series Solution about a Regular Singular Point

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

A point x_0 is called a singular point of the given differential equation if it is not an ordinary point, that is, not all of the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are analytic at $x = x_0$.
A point x_0 is called a regular singular point of the given differential equation if it is not an ordinary point, BUT all of $(x - x_0)^{n-k} p_k(x)$ are analytic for $k = 0, 1, \ldots, n-1$.

Series Solution about a Regular Singular Point

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

A point x_0 is called a singular point of the given differential equation if it is not an ordinary point, that is, not all of the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are analytic at $x = x_0$.
A point x_0 is called a regular singular point of the given differential equation if it is not an ordinary point, BUT all of $(x - x_0)^{n-k}p_k(x)$ are analytic for $k = 0, 1, \ldots, n-1$.
A point x_0 is irregular point of the given differential equation if it is neither an ordinary point nor a regular singular point.

Fuchs' Th.: Series Solution about a Regular Singular Point

Fuch's Theorem

For the second-order linear homogeneous ordinary differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0,$$

if $x = 0$ is a regular singular point then

$$xP(x) = \sum_{n=0}^{\infty} P_n x^n, \quad x^2 Q(x) = \sum_{n=0}^{\infty} Q_n x^n, \; |x| < r.$$

Let the **indicial equation**

$$\alpha(\alpha - 1) + \alpha P_0 + Q_0 = 0$$

has two real roots $\alpha_1 \geq \alpha_2$. Then the DE has at least one Frobenius series solution given by

Fuchs' Th.: Series Solution about a Regular Singular Point

Fuch's Theorem

$$y_1(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n, \; a_0 \neq 0, \; 0 < x < r,$$

where the coefficients a_n can be determined by substituting $y_1(x)$ into the differential equation. A second linearly independent solution is obtained as follows:

1. If $\alpha_1 - \alpha_2$ is not equal to an integer, then a second **Frobenius series solution** is given by

$$y_2(x) = x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < r,$$

in which the coefficients b_n can be determined by substituting $y_2(x)$ into the differential equation.

Fuchs' Th.: Series Solution about a Regular Singular Point

Fuch's Theorem

2. If $\alpha_1 = \alpha_2 = \alpha$, then

$$y_2(x) = y_1(x) \ln x + x^{\alpha} \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < r,$$

where b_n can be determined by substituting $y_2(x)$ into the differential equation, once $y_1(x)$ is known. In this case, the second solution $y_2(x)$ is not a Frobenius series solution.

3. If $\alpha_1 - \alpha_2$ is a positive integer, then

$$y_2(x) = a y_1(x) \ln x + x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < r,$$

where b_n and a can be determined by substituting y_2 into the differential equation. The parameter a may be zero, in which case the second solution $y_2(x)$ is also a Frobenius series solution.

Solutions of Bessel's Equation

Bessel's equation

The equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,\ x > 0,$$

$\nu = const \geq 0$, is called Bessel's equation

Solutions of Bessel's Equation

Bessel's equation is of the form

$$y'' + P(x)y' + Q(x)y = 0, \; P(x) = \frac{1}{x}, \; Q(x) = \frac{x^2 - \nu^2}{x^2}.$$

$x = 0$ is not an ordinary point (Why?)

Solutions of Bessel's Equation

Bessel's equation is of the form

$$y'' + P(x)y' + Q(x)y = 0,\ P(x) = \frac{1}{x},\ Q(x) = \frac{x^2 - \nu^2}{x^2}.$$

$x = 0$ is not an ordinary point (Why?) Since

$$xP(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots \Rightarrow$$

Solutions of Bessel's Equation

Bessel's equation is of the form

$$y'' + P(x)y' + Q(x)y = 0,\; P(x) = \frac{1}{x},\; Q(x) = \frac{x^2 - \nu^2}{x^2}.$$

$x = 0$ is not an ordinary point (Why?) Since

$$xP(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots \Rightarrow P_0 = 1,$$

Solutions of Bessel's Equation

Bessel's equation is of the form

$$y'' + P(x)y' + Q(x)y = 0, \; P(x) = \frac{1}{x}, \; Q(x) = \frac{x^2 - \nu^2}{x^2}.$$

$x = 0$ is not an ordinary point (Why?) Since

$$xP(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots \Rightarrow P_0 = 1,$$

$$x^2 Q(x) = x^2 - \nu^2 = -\nu^2 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \ldots \Rightarrow$$

Solutions of Bessel's Equation

Bessel's equation is of the form

$$y'' + P(x)y' + Q(x)y = 0,\ P(x) = \frac{1}{x},\ Q(x) = \frac{x^2 - \nu^2}{x^2}.$$

$x = 0$ is not an ordinary point (Why?) Since

$$xP(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots \Rightarrow P_0 = 1,$$

$$x^2 Q(x) = x^2 - \nu^2 = -\nu^2 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \ldots \Rightarrow Q_0 = -\nu^2$$

Both $xP(x)$, $x^2 Q(x)$ are analytic at $x = 0$ and can be expanded as power series convergent for $|x| < \infty$.

Solutions of Bessel's Equation

Bessel's equation is of the form

$$y'' + P(x)y' + Q(x)y = 0,\ P(x) = \frac{1}{x},\ Q(x) = \frac{x^2 - \nu^2}{x^2}.$$

$x = 0$ is not an ordinary point (Why?) Since

$$xP(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots \Rightarrow P_0 = 1,$$

$$x^2 Q(x) = x^2 - \nu^2 = -\nu^2 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \ldots \Rightarrow Q_0 = -\nu^2$$

Both $xP(x)$, $x^2 Q(x)$ are analytic at $x = 0$ and can be expanded as power series convergent for $|x| < \infty$.Hence, $x = 0$ is a regular singular point.

The indicial equation is $\alpha(\alpha - 1) + \alpha P_0 + Q_0 = 0$:

$$\alpha(\alpha - 1) + \alpha \cdot 1 - \nu^2 = 0 \Rightarrow$$

Solutions of Bessel's Equation

Bessel's equation is of the form

$$y'' + P(x)y' + Q(x)y = 0,\ P(x) = \frac{1}{x},\ Q(x) = \frac{x^2 - \nu^2}{x^2}.$$

$x = 0$ is not an ordinary point (Why?) Since

$$xP(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots \Rightarrow P_0 = 1,$$

$$x^2 Q(x) = x^2 - \nu^2 = -\nu^2 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \ldots \Rightarrow Q_0 = -\nu^2$$

Both $xP(x)$, $x^2 Q(x)$ are analytic at $x = 0$ and can be expanded as power series convergent for $|x| < \infty$.Hence, $x = 0$ is a regular singular point.

The indicial equation is $\alpha(\alpha - 1) + \alpha P_0 + Q_0 = 0$:

$$\alpha(\alpha - 1) + \alpha \cdot 1 - \nu^2 = 0 \Rightarrow \alpha_1 = \nu,\ \alpha_2 = -\nu$$

Solutions of Bessel's Equation

Bessel's equation has a Frobenius series solution of the form

$$y_1(x) = x^\nu \sum_{n=0}^{\infty} a_n x^n =$$

Solutions of Bessel's Equation

Bessel's equation has a Frobenius series solution of the form

$$y_1(x) = x^{\nu}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\nu},\ a_0 \neq 0,\ 0 < x < \infty$$

Solutions of Bessel's Equation

Bessel's equation has a Frobenius series solution of the form

$$y_1(x) = x^{\nu} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\nu},\ a_0 \neq 0,\ 0 < x < \infty$$

Differentiate y_1 twice and substitute y_1, y_1', y_1'' into Bessel's equation to obtain

Solutions of Bessel's Equation

Bessel's equation has a Frobenius series solution of the form

$$y_1(x) = x^{\nu} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\nu},\ a_0 \neq 0,\ 0 < x < \infty$$

Differentiate y_1 twice and substitute $y_1, y_1',\ y_1''$ into Bessel's equation to obtain

$$x^2 \sum_{n=0}^{\infty} a_n (n+\nu)(n+\nu-1) x^{n+\nu-2} \quad + x \sum_{n=0}^{\infty} a_n (n+\nu) x^{n+\nu-1} +$$

$$+(x^2 - \nu^2) \sum_{n=0}^{\infty} a_n x^{n+\nu} = 0$$

Solutions of Bessel's Equation

Bessel's equation has a Frobenius series solution of the form

$$y_1(x) = x^\nu \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\nu},\ a_0 \neq 0,\ 0 < x < \infty$$

Differentiate y_1 twice and substitute $y_1, y_1',\ y_1''$ into Bessel's equation to obtain

$$x^2 \sum_{n=0}^{\infty} a_n (n+\nu)(n+\nu-1) x^{n+\nu-2} \quad + x \sum_{n=0}^{\infty} a_n (n+\nu) x^{n+\nu-1} + \\ +(x^2 - \nu^2) \sum_{n=0}^{\infty} a_n x^{n+\nu} = 0$$

Change the index of summation in the first part of the last term

$$\sum_{n=0}^{\infty} a_n x^{n+\nu+2} \Rightarrow \sum_{m=2}^{\infty} a_{m-2} x^{m+\nu} = \sum_{n=2}^{\infty} a_{n-2} x^{n+\nu}$$

Solutions of Bessel's Equation

$$x^{\nu}\left(\sum_{n=0}^{\infty}\left[(n+\nu)(n+\nu-1)+(n+\nu)-\nu^2\right]a_n x^n + \right.$$
$$\left. + \sum_{n=2}^{\infty} a_{n-2}x^n\right) = 0 \qquad\qquad x^{\nu} \neq 0$$

$$\sum_{n=0}^{\infty} n(n+2\nu)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots,$ must be zero:

$$\begin{array}{lcr} x^0: & 0\cdot(0+2\nu)a_0 = 0 \Rightarrow & a_0 \neq 0 \text{ is arbitrary,} \\ x^1: & 1\cdot(1+2\nu)a_1 = 0 \Rightarrow & a_1 = 0, \\ x^n: & n\cdot(n+2\nu)a_n + a_{n-2} = 0 \Rightarrow & a_n = -\dfrac{a_{n-2}}{n(n+2\nu)},\ n \geq 2 \end{array}$$

Solutions of Bessel's Equation

Therefore, $a_{2n+1} = 0$, $n = 0, 1, \dots$ and

$$a_2 = -\frac{a_0}{2(2+2\nu)} = -\frac{a_0}{2^2 \cdot 1(1+\nu)}$$
$$a_4 = -\frac{a_2}{4(4+2\nu)} = -\frac{a_2}{2^2 \cdot 2(1+\nu)} = (-1)^2 \frac{a_0}{2^4 \cdot 2!(1+\nu)(2+\nu)}$$
$$\dots$$
$$a_{2n} = (-1)^n \frac{a_0}{2^{2n} \cdot n!(1+\nu)(2+\nu)\dots(n+\nu)}$$

and

$$y_1(x) = a_0 x^\nu \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(1+\nu)(2+\nu)\dots(n+\nu)} \left(\frac{x}{2}\right)^{2n},$$
$$0 < x < \infty$$

Properties of the Gamma function

The Gamma function

Define the Gamma function by

$$\Gamma(\nu+1)=\int_0^\infty t^\nu e^{-t}\,dt,\ \nu>0.$$

Properties of the Gamma function

The Gamma function

Define the Gamma function by

$$\Gamma(\nu+1) = \int_0^\infty t^\nu e^{-t}\,dt,\ \nu > 0.$$

Using integration by parts,

$$\begin{aligned}\Gamma(\nu+1) = -\int_0^\infty t^\nu d(e^{-t}) = & -t^\nu e^{-t}\Big|_{t=0}^{\infty} + \int_0^\infty e^{-t}\nu t^{\nu-1}\,dt \\ & = nu\int_0^\infty e^{-t}t^{\nu-1}\,dt = \nu\Gamma(\nu)\end{aligned}$$

Properties of the Gamma function

The Gamma function

Define the Gamma function by

$$\Gamma(\nu+1) = \int_0^\infty t^\nu e^{-t}\,dt,\ \nu > 0.$$

Using integration by parts,

$$\begin{aligned}\Gamma(\nu+1) = -\int_0^\infty t^\nu d(e^{-t}) = -t^\nu e^{-t}\Big|_{t=0}^{\infty} + \int_0^\infty e^{-t}\nu t^{\nu-1}\,dt \\ = nu\int_0^\infty e^{-t}t^{\nu-1}\,dt = \nu\Gamma(\nu)\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu) = (n+\nu)(n+\nu-1)\Gamma(n+\nu-1)\ldots \\ = (n+\nu)(n+\nu-1)\ldots(1+\nu)\Gamma(1+\nu)\end{aligned}$$

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1,$$

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1,\ \Gamma(2) = 1 \cdot \Gamma(1) = 1,$$

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1,\ \Gamma(2) = 1{\cdot}\Gamma(1) = 1,\ \Gamma(3) = 2{\cdot}\Gamma(2) = 2{\cdot}1 = 2!, \ldots$$

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1,\ \Gamma(2) = 1\cdot\Gamma(1) = 1,\ \Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!, \ldots$$

$$\Gamma(k+1) = k \cdot \Gamma(k) = k!$$

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1,\ \Gamma(2) = 1{\cdot}\Gamma(1) = 1,\ \Gamma(3) = 2{\cdot}\Gamma(2) = 2{\cdot}1 = 2!, \ldots$$

$$\Gamma(k+1) = k \cdot \Gamma(k) = k!$$

Let $a_0 = [2^\nu \Gamma(1+\nu)]^{-1} \Rightarrow$ the first Frobenius solution is

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1,\ \Gamma(2) = 1{\cdot}\Gamma(1) = 1,\ \Gamma(3) = 2{\cdot}\Gamma(2) = 2{\cdot}1 = 2!, \ldots$$

$$\Gamma(k+1) = k \cdot \Gamma(k) = k!$$

Let $a_0 = [2^\nu \Gamma(1+\nu)]^{-1} \Rightarrow$ the first Frobenius solution is

$$y_1(x) = J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu},$$

$$0 < x < \infty$$

Solutions of Bessel's Equation

If $\nu = k$ is an integer, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1,\ \Gamma(2) = 1 \cdot \Gamma(1) = 1,\ \Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2!, \ldots$$

$$\Gamma(k+1) = k \cdot \Gamma(k) = k!$$

Let $a_0 = [2^\nu \Gamma(1+\nu)]^{-1} \Rightarrow$ the first Frobenius solution is

$$y_1(x) = J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu},$$

$$0 < x < \infty$$

The function $J_\nu(x)$ is called the Bessel function of the first kind of order ν.

What about the second linearly independent solution? Fuch's theorem!

Case 1. $\alpha_1 - \alpha_2 = 2\nu$ is not an integer

A second Frobenius solution is

$$y_2(x) = x^{-\nu} \sum_{n=0}^{\infty} b_n x^n,\ 0 < x < \infty.$$

Case 1. $\alpha_1 - \alpha_2 = 2\nu$ is not an integer

A second Frobenius solution is

$$y_2(x) = x^{-\nu} \sum_{n=0}^{\infty} b_n x^n,\ 0 < x < \infty.$$

We can follow the same procedure to show that

$$a_{2n-1} = 0,\ a_{2n} = (-1)^n \frac{b_0}{2^{2n} n!(1-\nu)(2-\nu)\dots(n-\nu)},$$

$$y_2(x) = b_0 x^{-\nu} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(1-\nu)(2-\nu)\dots(n-\nu)} \left(\frac{x}{2}\right)^{2n},$$

$$0 < x < \infty.$$

Letting $b_0 = \left[2^{-\nu}\Gamma(1-\nu)\right]^{-1}$,

$$y_2(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu} = J_{-\nu}(x)$$

Case 1. $\alpha_1 - \alpha_2 = 2\nu$ is not an integer

A second Frobenius solution is

$$y_2(x) = x^{-\nu} \sum_{n=0}^{\infty} b_n x^n,\ 0 < x < \infty.$$

We can follow the same procedure to show that

$$a_{2n-1} = 0,\ a_{2n} = (-1)^n \frac{b_0}{2^{2n} n!(1-\nu)(2-\nu)\dots(n-\nu)},$$

$$y_2(x) = b_0 x^{-\nu} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(1-\nu)(2-\nu)\dots(n-\nu)} \left(\frac{x}{2}\right)^{2n},$$

$$0 < x < \infty.$$

Case 1. $\alpha_1 - \alpha_2 = 2\nu$ is not an integer

A second Frobenius solution is

$$y_2(x) = x^{-\nu} \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty.$$

We can follow the same procedure to show that

$$a_{2n-1} = 0, \; a_{2n} = (-1)^n \frac{b_0}{2^{2n} n! (1-\nu)(2-\nu) \dots (n-\nu)},$$

$$y_2(x) = b_0 x^{-\nu} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(1-\nu)(2-\nu)\dots(n-\nu)} \left(\frac{x}{2}\right)^{2n}, \quad 0 < x < \infty.$$

Letting $b_0 = \left[2^{-\nu}\Gamma(1-\nu)\right]^{-1}$,

$$y_2(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu} = J_{-\nu}(x)$$

Case 1. $\alpha_1 - \alpha_2 = 2\nu$ is not an integer

Then the general solution is

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$

Case 1. $\alpha_1 - \alpha_2 = 2\nu$ is not an integer

Then the general solution is

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$

Note, that the general solution can be also written in the form

$$y(x) = D_1 J_\nu(x) + D_2 Y_\nu(x),$$

where

$$Y_\nu(x) = \frac{J_\nu \cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi}$$

is the Bessel function of the second kind of order ν.

Case 2. $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$

The first Frobenius series solution is simplified as

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \; 0 < x < \infty$$

Case 2. $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$

The first Frobenius series solution is simplified as

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \; 0 < x < \infty$$

A second linearly independent solution is

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty.$$

Case 2. $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$

The first Frobenius series solution is simplified as

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \; 0 < x < \infty$$

A second linearly independent solution is

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty.$$

Substituting y_2, y_2', y_2'' into Bessel's equation results in

$$(x^2 y_1'' + x y_1' + x^2 y_1) \ln x + 2x y_1' + \sum_{n=2}^{\infty} n(n-1) b_n x^n + \\ + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=0}^{\infty} b_n x^{n+2} = 0.$$

Case 2. $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$

The first Frobenius series solution is simplified as

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \; 0 < x < \infty$$

A second linearly independent solution is

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty.$$

Substituting y_2, y_2', y_2'' into Bessel's equation results in

$$(x^2 y_1'' + x y_1' + x^2 y_1) \ln x + 2x y_1' + \sum_{n=2}^{\infty} n(n-1) b_n x^n + \\ + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=0}^{\infty} b_n x^{n+2} = 0.$$

Since $x^2 y_1'' + x y_1' + x^2 y_1 = 0$ and

Case 2. $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$

$$2xy_1' = 2x\sum_{n=1}^{\infty}(-1)^n\frac{1}{(n!)^2}\frac{2n\cdot x^{2n-1}}{2^{2n}} = \sum_{n=1}^{\infty}(-1)^n\ \frac{4n}{(n!)^2}\left(\frac{x}{2}\right)^{2n},$$

we obtain

$$\sum_{n=1}^{\infty}(-1)^n\frac{4n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}+\sum_{n=2}^{\infty}n(n-1)b_nx^n+\sum_{n=1}^{\infty}nb_nx^n+\sum_{n=0}^{\infty}b_nx^{n+2}=0$$

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$\begin{aligned} x^1: &\qquad 1\cdot b_1 = 0 \Rightarrow b_1 = 0,\\ x^n,\ n\geq 1: &\qquad b_{2n+1} = 0 \text{ and} \end{aligned}$$

$$(-1)^n\frac{4n}{(n!)^2}\left(\frac{1}{2}\right)^{2n} + [2n(2n-1)+2n]b_{2n} + b_{2n-2} = 0$$

and

$$b_{2n} = (-1)^{n+1}\frac{1}{n(n!)^2}\left(\frac{1}{2}\right)^{2n} - \frac{b_{2n-2}}{(2n)^2}$$

Case 2. $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$

For simplicity, take $b_0 = 0$. Using mathematical induction, it can be shown that

$$b_{2n} = (-1)^{n+1} \frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{(n!)^2} \left(\frac{1}{2}\right)^{2n}$$

Hence, a second linearly independent solution is

$$y_2(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

or, in terms of the Bessel function of the second kind of order 0, $Y_0(x)$,

$$y_2(x) = \frac{\pi}{2} Y_0(x) + (\ln 2 - \gamma) J_0(x),\ 0 < x, \infty$$

Case 2. $\alpha_1 = \alpha_2 \Rightarrow \nu = 0$

The Bessel function of the second kind of order 0 is defined as

$$Y_0(x) = \frac{2}{\pi}\left((\ln\frac{x}{2} + \gamma)J_0(x) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right)$$

in which

$$\gamma = 0.57721566490153\ldots = \lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k} - \ln n)$$

is the Euler constant.
The general solution is

$$y(x) = C_1 J_0(x) + C_2 Y_0(x).$$

Case 3. ν is a positive integer

The first Frobenius series solution is simplified as

$$y_1(x) = J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+\nu)!} \left(\frac{x}{2}\right)^{2n+\nu}, \; 0 < x < \infty$$

Case 3. ν is a positive integer

The first Frobenius series solution is simplified as

$$y_1(x) = J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+\nu)!} \left(\frac{x}{2}\right)^{2n+\nu}, \; 0 < x < \infty$$

A second linearly independent solution is

$$y_2(x) = a y_1(x) \ln x + x^{-\nu} \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty$$

Case 3. ν is a positive integer

The first Frobenius series solution is simplified as

$$y_1(x) = J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+\nu)!} \left(\frac{x}{2}\right)^{2n+\nu}, \; 0 < x < \infty$$

A second linearly independent solution is

$$y_2(x) = a y_1(x) \ln x + x^{-\nu} \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty$$

$$y_2'(x) = a(y_1'(x) \ln x + \frac{y_1}{x}) + \sum_{n=0}^{\infty} (n-\nu) b_n x^{n-\nu-1},$$

Case 3. ν is a positive integer

The first Frobenius series solution is simplified as

$$y_1(x) = J_\nu(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n+\nu}, \; 0 < x < \infty$$

A second linearly independent solution is

$$y_2(x) = a y_1(x)\ln x + x^{-\nu}\sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty$$

$$y_2'(x) = a(y_1'(x)\ln x + \frac{y_1}{x}) + \sum_{n=0}^{\infty}(n-\nu)b_n x^{n-\nu-1},$$

$$y_2''(x) = a\left(y_1''(x)\ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2}\right) + \sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)b_n x^{n-\nu-2}.$$

Case 3. ν is a positive integer

Substituting into Bessel's equation results in

$$a[x^2y_1''+xy_1'+(x^2-\nu^2)y_1]\ln x+2axy_1'+\sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)b_nx^{n-\nu}+$$

$$+\sum_{n=0}^{\infty}(n-\nu)b_nx^{n-\nu}+\sum_{n=0}^{\infty}b_nx^{n-\nu+2}-\sum_{n=0}^{\infty}\nu^2b_nx^{n-\nu}=0$$

Case 3. ν is a positive integer

Substituting into Bessel's equation results in

$$a[x^2y_1''+xy_1'+(x^2-\nu^2)y_1]\ln x+2axy_1'+\sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)b_nx^{n-\nu}+$$

$$+\sum_{n=0}^{\infty}(n-\nu)b_nx^{n-\nu}+\sum_{n=0}^{\infty}b_nx^{n-\nu+2}-\sum_{n=0}^{\infty}\nu^2b_nx^{n-\nu}=0$$

y_1 is a solution of the Bessel's equation $\Rightarrow x^2y_1''+xy_1'+x^2y_1=0$ and

Case 3. ν is a positive integer

Substituting into Bessel's equation results in

$$a[x^2y_1''+xy_1'+(x^2-\nu^2)y_1]\ln x+2axy_1'+\sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)b_nx^{n-\nu}+$$

$$+\sum_{n=0}^{\infty}(n-\nu)b_nx^{n-\nu}+\sum_{n=0}^{\infty}b_nx^{n-\nu+2}-\sum_{n=0}^{\infty}\nu^2b_nx^{n-\nu}=0$$

y_1 is a solution of the Bessel's equation $\Rightarrow x^2y_1''+xy_1'+x^2y_1=0$ and

$2axy_1'=$

Case 3. ν is a positive integer

Substituting into Bessel's equation results in

$$a[x^2y_1''+xy_1'+(x^2-\nu^2)y_1]\ln x+2axy_1'+\sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)b_nx^{n-\nu}+$$

$$+\sum_{n=0}^{\infty}(n-\nu)b_nx^{n-\nu}+\sum_{n=0}^{\infty}b_nx^{n-\nu+2}-\sum_{n=0}^{\infty}\nu^2b_nx^{n-\nu}=0$$

y_1 is a solution of the Bessel's equation $\Rightarrow x^2y_1''+xy_1'+x^2y_1=0$ and

$$2axy_1'=2ax\sum_{n=1}^{\infty}(-1)^n\frac{(2n+\nu)\cdot x^{2n+\nu-1}}{n!(n+\nu)!2^{2n+\nu}}=$$

Case 3. ν is a positive integer

Substituting into Bessel's equation results in

$$a[x^2y_1''+xy_1'+(x^2-\nu^2)y_1]\ln x+2axy_1'+\sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)b_nx^{n-\nu}+$$

$$+\sum_{n=0}^{\infty}(n-\nu)b_nx^{n-\nu}+\sum_{n=0}^{\infty}b_nx^{n-\nu+2}-\sum_{n=0}^{\infty}\nu^2b_nx^{n-\nu}=0$$

y_1 is a solution of the Bessel's equation $\Rightarrow x^2y_1''+xy_1'+x^2y_1=0$ and

$$2axy_1'=2ax\sum_{n=1}^{\infty}(-1)^n\frac{(2n+\nu)\cdot x^{2n+\nu-1}}{n!(n+\nu)!2^{2n+\nu}}=\sum_{n=1}^{\infty}(-1)^n\frac{2a(2n+\nu)}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n\cdots}$$

Case 3. ν is a positive integer

Substituting into Bessel's equation results in

$$a[x^2y_1''+xy_1'+(x^2-\nu^2)y_1]\ln x+2axy_1'+\sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)b_nx^{n-\nu}+$$

$$+\sum_{n=0}^{\infty}(n-\nu)b_nx^{n-\nu}+\sum_{n=0}^{\infty}b_nx^{n-\nu+2}-\sum_{n=0}^{\infty}\nu^2b_nx^{n-\nu}=0$$

y_1 is a solution of the Bessel's equation $\Rightarrow x^2y_1''+xy_1'+x^2y_1=0$ and

$$2axy_1'=2ax\sum_{n=1}^{\infty}(-1)^n\frac{(2n+\nu)\cdot x^{2n+\nu-1}}{n!(n+\nu)!2^{2n+\nu}}=\sum_{n=1}^{\infty}(-1)^n\frac{2a(2n+\nu)}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n-}$$

and multiplying the equation by x^{ν}

$$\sum_{n=1}^{\infty}(-1)^n\frac{2^{\nu+1}a(2n+\nu)}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2(n+\nu)}+\sum_{n=0}^{\infty}n(n-2\nu)b_nx^n+\sum_{n=2}^{\infty}b_{n-2}x^n=0$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$x^0 :$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$x^0 : \qquad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \dots$ must be zero.

$$\begin{aligned} x^0 : &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\ x^1 : &\quad \end{aligned}$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$\begin{aligned}
x^0: &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } \ b_0 = 1, \\
x^1: &\quad 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow
\end{aligned}$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$\begin{aligned} x^0: &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\ x^1: &\quad 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \end{aligned}$$

x^n, $2 \leq n < 2\nu$:

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$
\begin{aligned}
x^0: &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\
x^1: &\quad 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0,
\end{aligned}
$$

$$
x^n,\ 2 \leq n < 2\nu: \qquad n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow
$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$\begin{aligned}
x^0 : &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\
x^1 : &\quad 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \le n < 2\nu : &\quad n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \frac{b_{n-2}}{n(2\nu - n)} \\
n = 2 : &
\end{aligned}$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$
\begin{aligned}
x^0: &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } \; b_0 = 1, \\
x^1: &\quad 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \le n < 2\nu: &\quad n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \frac{b_{n-2}}{n(2\nu - n)} \\
n = 2: &\quad b_2 = \tfrac{b_0}{2(2\nu - 2)} =
\end{aligned}
$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$\begin{aligned}
x^0 : &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\
x^1 : &\quad 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \leq n < 2\nu : &\quad n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \frac{b_{n-2}}{n(2\nu - n)} \\
n = 2 : &\quad b_2 = \tfrac{b_0}{2(2\nu-2)} = \tfrac{1}{2^2 \cdot 1(\nu-1)} \\
n = 4 : &
\end{aligned}$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$
\begin{array}{cc}
x^0 : & 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } \; b_0 = 1, \\
x^1 : & 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\, 2 \le n < 2\nu : & n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \dfrac{b_{n-2}}{n(2\nu - n)} \\
n = 2 : & b_2 = \frac{b_0}{2(2\nu - 2)} = \frac{1}{2^2 \cdot 1(\nu - 1)} \\
n = 4 : & b_4 = \frac{b_2}{4(2\nu - 4)} =
\end{array}
$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \dots$ must be zero.

$$\begin{aligned}
x^0 &: & 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\
x^1 &: & 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \le n < 2\nu &: & n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \frac{b_{n-2}}{n(2\nu - n)} \\
n = 2 &: & b_2 = \tfrac{b_0}{2(2\nu-2)} = \tfrac{1}{2^2 \cdot 1(\nu-1)} \\
n = 4 &: & b_4 = \tfrac{b_2}{4(2\nu-4)} = \tfrac{1}{2^4 \cdot 2!(\nu-1)(\nu-2)} \\
&\dots & \\
n = 2k &: &
\end{aligned}$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$
\begin{aligned}
x^0 : &\quad 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } \; b_0 = 1, \\
x^1 : &\quad 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \le n < 2\nu : &\quad n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \frac{b_{n-2}}{n(2\nu - n)} \\
n = 2 : &\quad b_2 = \tfrac{b_0}{2(2\nu-2)} = \tfrac{1}{2^2 \cdot 1(\nu-1)} \\
n = 4 : &\quad b_4 = \tfrac{b_2}{4(2\nu-4)} = \tfrac{1}{2^4 \cdot 2!(\nu-1)(\nu-2)} \\
\ldots & \\
n = 2k : &\quad b_{2k} = \tfrac{b_{2k-2}}{2^{2k}(2\nu-2k)} =
\end{aligned}
$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$
\begin{array}{cc}
x^0 : & 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\
x^1 : & 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \leq n < 2\nu : & n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \dfrac{b_{n-2}}{n(2\nu - n)} \\
n = 2 : & b_2 = \frac{b_0}{2(2\nu-2)} = \frac{1}{2^2 \cdot 1(\nu-1)} \\
n = 4 : & b_4 = \frac{b_2}{4(2\nu-4)} = \frac{1}{2^4 \cdot 2!(\nu-1)(\nu-2)} \\
\ldots & \\
n = 2k : & b_{2k} = \frac{b_{2k-2}}{2^{2k}(2\nu-2k)} = \frac{1}{2^{2k} k!(\nu-1)(\nu-2)\ldots(\nu-k)} = \frac{(\nu-k-1)!}{2^{2k} k!(\nu-1)
\end{array}
$$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$
\begin{array}{rc}
x^0: & 0 \cdot (0 - 2\nu) b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\
x^1: & 1 \cdot (1 - 2\nu) b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \le n < 2\nu: & n(n - 2\nu) b_n + b_{n-2} = 0 \Rightarrow b_n = \dfrac{b_{n-2}}{n(2\nu - n)} \\
n = 2: & b_2 = \frac{b_0}{2(2\nu-2)} = \frac{1}{2^2 \cdot 1(\nu-1)} \\
n = 4: & b_4 = \frac{b_2}{4(2\nu-4)} = \frac{1}{2^4 \cdot 2!(\nu-1)(\nu-2)} \\
\ldots & \\
n = 2k: & b_{2k} = \frac{b_{2k-2}}{2^{2k}(2\nu-2k)} = \frac{1}{2^{2k} k!(\nu-1)(\nu-2)\ldots(\nu-k)} = \frac{(\nu-k-1)!}{2^{2k} k!(\nu-1)}
\end{array}
$$

and $b_{2k=1} = 0,\ n = 0, 1, 2, \ldots.$

Case 3. ν is a positive integer

For this equation to be true, the coefficient of x_n, $n = 0, 1, \ldots$ must be zero.

$$
\begin{aligned}
x^0: &\quad 0\cdot(0-2\nu)b_0 = 0 \Rightarrow b_0 = \text{ is arbitrary, take } b_0 = 1, \\
x^1: &\quad 1\cdot(1-2\nu)b_1 = 0 \Rightarrow b_1 = 0, \\
x^n,\ 2 \le n < 2\nu: &\quad n(n-2\nu)b_n + b_{n-2} = 0 \Rightarrow b_n = \frac{b_{n-2}}{n(2\nu - n)} \\
n = 2: &\quad b_2 = \tfrac{b_0}{2(2\nu-2)} = \tfrac{1}{2^2\cdot 1(\nu-1)} \\
n = 4: &\quad b_4 = \tfrac{b_2}{4(2\nu-4)} = \tfrac{1}{2^4\cdot 2!(\nu-1)(\nu-2)} \\
\ldots & \\
n = 2k: &\quad b_{2k} = \tfrac{b_{2k-2}}{2^{2k}(2\nu-2k)} = \tfrac{1}{2^{2k}k!(\nu-1)(\nu-2)\ldots(\nu-k)} = \tfrac{(\nu-k-1)!}{2^{2k}k!(\nu-1)
\end{aligned}
$$

and $b_{2k=1} = 0$, $n = 0, 1, 2, \ldots$.

From the coefficient of $x^{2\nu}$,

$$
\frac{2^{\nu+1}a\nu}{\nu!}\left(\frac{x}{2}\right)^{2\nu} + b_{2\nu-2} = 0 \Rightarrow a = -\frac{1}{2^{\nu-1}(\nu-1)!}
$$

The value of $b_{2\nu}$ is arbitrary, for simplicity, take $b_{2\nu} = 0$.

Case 3. ν is a positive integer

From the coefficient of $x^{2(\nu+n)}$, $n \geq 1$,

$$(-1)^{n+1}\frac{2^{\nu+1}a(2n+\nu)}{n!(n+\nu)!}\left(\frac{1}{2}\right)^{2(n+\nu)}+(2n+2\nu)(2n)b_{2\nu+2n}+b_{2(n-1+\nu)}=0$$

Using mathematical induction, it can be shown that

$$b_{2(n+\nu)}=(-1)^{n+1}\frac{2^{\nu-1}aA_n}{n!(n+\nu)!}\left(\frac{1}{2}\right)^{2(n+\nu)},$$

where

$$A_n=\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)+\left(\frac{1}{1+\nu}+\frac{1}{2+\nu}+\ldots+\frac{1}{n+\nu}\right)$$

Case 3. ν is a positive integer

Hence, a second linearly independent solution is

$$y_2(x) = aJ_\nu(x)\ln x + x^{-\nu}\left\{\sum_{n=0}^{\nu-1}\frac{(\nu-n-1)!}{n!(\nu-1)!}\left(\frac{x}{2}\right)^{2n} + \right.$$

$$\left. + a\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{\nu-1}A_n}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2(n+\nu)}\right\}$$

Case 3. ν is a positive integer

Hence, a second linearly independent solution is

$$y_2(x) = aJ_\nu(x)\ln x + x^{-\nu}\left\{\sum_{n=0}^{\nu-1}\frac{(\nu-n-1)!}{n!(\nu-1)!}\left(\frac{x}{2}\right)^{2n} + \right.$$

$$\left. +a\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{\nu-1}A_n}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2(n+\nu)}\right\}$$

Using the notation

$$1+\frac{1}{2}+\ldots+\frac{1}{n} = \psi(n+1)+\gamma, \quad \psi(1) = -\gamma,$$

where $\psi(n) = \Gamma'(n)/\Gamma(n)$ is the psi function, $y_2(x)$ can be expressed in terms of the Bessel function of the second kind of order ν, $Y_\nu(x)$

Case 3. ν is a positive integer

$$y_2(x) = a\left\{\frac{\pi}{2}Y_\nu(x) - \frac{1}{2}[\gamma - \psi(\nu+1) - 2\ln 2]J_\nu(x)\right\},$$

where, for $0 < x < \infty$

$$Y_\nu(x) = \frac{2}{\pi}J_\nu \ln\frac{x}{2} - \frac{1}{\pi}\sum_{n=0}^{\nu-1}\frac{(\nu-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-\nu} - .$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\psi(n+1)+\psi(n+\nu+1)}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n+\nu}$$

The general solution is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x).$$

This is the case that, when $\alpha_1 - \alpha_2$ is a positive integer, the second solution CONTAINS the logarithmic term $\ln x$.

Case 4. $\nu = k + 1/2$, $k = 0, 1, \ldots,$ and $\alpha_1 - \alpha_2 = 2k + 1$ ia a positive integer

The first Frobenius series solution becomes

$$y_1(x) = J_{k+1/2}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n+k+\frac{3}{2})} \left(\frac{x}{2}\right)^{2n+k-\frac{1}{2}}$$

and a second linearly independent solution is

$$y_2(x) = a y_1(x) \ln x + x^{-(k+\frac{1}{2})} \sum_{n=0}^{\infty} b_n x^n, \; 0 < x < \infty$$

Show that

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu} = J_{-\nu}(x),$$

that is the same as in the Case 1 when 2ν is not an integer.

This is the case that, when $\alpha_1 - \alpha_2$ is a positive integer, the second solution DOES NOT CONTAIN the logarithmic term $\ln x$.

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{x^2}\right] y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2 \frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - \nu^2)\eta = 0,\ \xi > 0?$$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{x^2}\right]y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2\frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - \nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{x^2}\right] y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2 \frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - \nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

$$\xi\frac{dy}{d\xi} =$$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{x^2}\right] y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2 \frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - \nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

$$\xi\frac{dy}{d\xi} = \xi\frac{dy/dx}{d\xi/dx} =$$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2-\nu^2\rho^2}{x^2}\right]y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2\frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2-\nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

$$\xi\frac{dy}{d\xi} = \xi\frac{dy/dx}{d\xi/dx} = \beta x^\rho \frac{1}{\beta\rho x^{\rho-1}}\frac{dy}{dx} =$$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{x^2}\right] y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2 \frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - \nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

$$\xi\frac{dy}{d\xi} = \xi\frac{dy/dx}{d\xi/dx} = \beta x^\rho \frac{1}{\beta\rho x^{\rho-1}}\frac{dy}{dx} = \frac{x}{\rho}\frac{dy}{dx} \Rightarrow$$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{x^2}\right]y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2\frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - \nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

$$\xi\frac{dy}{d\xi} = \xi\frac{dy/dx}{d\xi/dx} = \beta x^\rho\frac{1}{\beta\rho x^{\rho-1}}\frac{dy}{dx} = \frac{x}{\rho}\frac{dy}{dx} \Rightarrow \xi\frac{d\cdot}{d\xi} = \frac{x}{\rho}\frac{d\cdot}{dx}$$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2-\nu^2\rho^2}{x^2}\right]y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2\frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2-\nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

$$\xi\frac{dy}{d\xi} = \xi\frac{dy/dx}{d\xi/dx} = \beta x^\rho\frac{1}{\beta\rho x^{\rho-1}}\frac{dy}{dx} = \frac{x}{\rho}\frac{dy}{dx} \Rightarrow \xi\frac{d\cdot}{d\xi} = \frac{x}{\rho}\frac{d\cdot}{dx}$$

Then

$$\xi\frac{d}{d\xi}\left(\xi\frac{d\eta}{d\xi}\right) + (\xi^2-\nu^2)\eta = 0 \Rightarrow$$

Remark

As we have already seen, Bessel's equation may not appear in the standard form in practice. How to transform the equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2-\nu^2\rho^2}{x^2}\right]y = 0,\ x > 0$$

to the Bessel equation

$$\xi^2\frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2-\nu^2)\eta = 0,\ \xi > 0?$$

Change the variable $\xi = \beta x^\rho$

$$\xi\frac{dy}{d\xi} = \xi\frac{dy/dx}{d\xi/dx} = \beta x^\rho \frac{1}{\beta\rho x^{\rho-1}}\frac{dy}{dx} = \frac{x}{\rho}\frac{dy}{dx} \Rightarrow \xi\frac{d\cdot}{d\xi} = \frac{x}{\rho}\frac{d\cdot}{dx}$$

Then

$$\xi\frac{d}{d\xi}\left(\xi\frac{d\eta}{d\xi}\right) + (\xi^2-\nu^2)\eta = 0 \Rightarrow \frac{x}{\rho}\frac{d}{dx}\left(\frac{x}{\rho}\frac{d\eta}{dx}\right) + (\beta^2x^{2\rho}-\nu^2)\eta = 0$$

Remark

Make the transformation $\eta = x^{-\alpha} y$,

Remark

Make the transformation $\eta = x^{-\alpha} y$,

$$x \frac{d\eta}{dx} = x^{1-\alpha} \frac{dy}{dx} - \alpha x^{-\alpha} y,$$

Remark

Make the transformation $\eta = x^{-\alpha} y$,

$$x \frac{d\eta}{dx} = x^{1-\alpha} \frac{dy}{dx} - \alpha x^{-\alpha} y,$$

$$x \frac{d}{dx} \left(x \frac{d\eta}{dx} \right) =$$

Remark

Make the transformation $\eta = x^{-\alpha} y$,

$$x \frac{d\eta}{dx} = x^{1-\alpha} \frac{dy}{dx} - \alpha x^{-\alpha} y,$$

$$x \frac{d}{dx} \left(x \frac{d\eta}{dx} \right) = x^{2-\alpha} \frac{d^2 y}{dx^2} + (1 - 2\alpha) x^{1-\alpha} \frac{dy}{dx} + \alpha^2 x^{-\alpha} y$$

Remark

Make the transformation $\eta = x^{-\alpha} y$,

$$x \frac{d\eta}{dx} = x^{1-\alpha} \frac{dy}{dx} - \alpha x^{-\alpha} y,$$

$$x \frac{d}{dx} \left(x \frac{d\eta}{dx} \right) = x^{2-\alpha} \frac{d^2 y}{dx^2} + (1 - 2\alpha) x^{1-\alpha} \frac{dy}{dx} + \alpha^2 x^{-\alpha} y$$

And the differential equation becomes

$$x^{2-\alpha} \frac{d^2 y}{dx^2} + (1 - 2\alpha) x^{1-\alpha} \frac{dy}{dx} + \alpha^2 x^{-\alpha} y + (\beta^2 x^{2\rho} - \nu^2) \rho^2 x^{-\alpha} y = 0$$

and it is the considered equation.

Summary: Solutions of Bessel's Equation

Denote the solution of Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,\ x > 0,\ \nu = const \geq 0$$

as $y(x) = \mathcal{B}_\nu(x)$ where

$$\mathcal{B}_\nu(x) = C_1(x)J_\nu(x) + C_2Y_\nu(x),$$

$J_\nu(x),\ Y_\nu(x)$ are the Bessel functions of the first and second kinds of order ν.
If $\nu \neq 0, 1, 2, \ldots$, the solution can be written as

$$\mathcal{B}_\nu(x) = C_1(x)J_\nu(x) + C_2J_{-\nu}(x).$$

Summary: Solutions of Bessel's Equation

Denote the solution of Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,\ x > 0,\ \nu = const \geq 0$$

as $y(x) = \mathcal{B}_\nu(x)$ where

$$\mathcal{B}_\nu(x) = C_1(x)J_\nu(x) + C_2Y_\nu(x),$$

$J_\nu(x),\ Y_\nu(x)$ are the Bessel functions of the first and second kinds of order ν.

If $\nu \neq 0, 1, 2, \ldots$, the solution can be written as

$$\mathcal{B}_\nu(x) = C_1(x)J_\nu(x) + C_2J_{-\nu}(x).$$

The solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left[(\beta\rho x^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{x^2}\right]y = 0,\ x > 0$$

is given by

$$y(x) = x^\alpha \mathcal{B}_\nu(\beta x^\rho)$$

Properties of the Bessel function of the first kind

- $$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x),$$

- $$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x) + \frac{\nu}{x} J_\nu(x) =$$
  $$= \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)],$$

- $$\left(\frac{d}{xdx}\right)^m [x^\nu J_\nu(x)] = x^{\nu-m} J_{\nu-m}(x),$$

- $$\left(\frac{d}{xdx}\right)^m [x^{-\nu} J_\nu(x)] = (-1)^m x^{-\nu-m} J_{\nu+m}(x),$$

Example: Buckling of a Tapered Column

Recall, that buckling of a tapered column is described by the equation with the general solution

$$\eta(\bar{x}) = \eta_C(\bar{x}) + \eta_P(\bar{x}),$$

Example: Buckling of a Tapered Column

Recall, that buckling of a tapered column is described by the equation with the general solution

$$\eta(\bar{x}) = \eta_C(\bar{x}) + \eta_P(\bar{x}),$$

where $\eta_P(\bar{x})$ is a particular solution and $\eta_C(\bar{x})$ is the solution of the equation

$$\frac{d^2\eta(\xi)}{d\xi^2} + \frac{1-2\alpha}{\xi}\frac{d\eta(\xi)}{d\xi} + \left[(\beta\rho\xi^{\rho-1})^2 + \frac{\alpha^2-\nu^2\rho^2}{\xi^2}\right]\eta(\xi) = 0,$$

$\alpha = 1/2$, $\beta = K$, $\rho = -1$, $\nu = 1/2$.

Example: Buckling of a Tapered Column

Recall, that buckling of a tapered column is described by the equation with the general solution

$$\eta(\bar{x}) = \eta_C(\bar{x}) + \eta_P(\bar{x}),$$

where $\eta_P(\bar{x})$ is a particular solution and $\eta_C(\bar{x})$ is the solution of the equation

$$\frac{d^2\eta(\xi)}{d\xi^2} + \frac{1-2\alpha}{\xi}\frac{d\eta(\xi)}{d\xi} + \left[(\beta\rho\xi^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{\xi^2}\right]\eta(\xi) = 0,$$

$\alpha = 1/2$, $\beta = K$, $\rho = -1$, $\nu = 1/2$.
Therefore,

$$\eta_C(\xi) = \xi^{\alpha}\mathcal{B}_{\nu}(\beta\xi^{\rho}) = \xi^{1/2}\mathcal{B}_{\frac{1}{2}}(K\xi^{-1}).$$

Example: Buckling of a Tapered Column

Recall, that buckling of a tapered column is described by the equation with the general solution

$$\eta(\bar{x}) = \eta_C(\bar{x}) + \eta_P(\bar{x}),$$

where $\eta_P(\bar{x})$ is a particular solution and $\eta_C(\bar{x})$ is the solution of the equation

$$\frac{d^2\eta(\xi)}{d\xi^2} + \frac{1-2\alpha}{\xi}\frac{d\eta(\xi)}{d\xi} + \left[(\beta\rho\xi^{\rho-1})^2 + \frac{\alpha^2 - \nu^2\rho^2}{\xi^2}\right]\eta(\xi) = 0,$$

$\alpha = 1/2$, $\beta = K$, $\rho = -1$, $\nu = 1/2$.
Therefore,

$$\eta_C(\xi) = \xi^{\alpha}\mathcal{B}_{\nu}(\beta\xi^{\rho}) = \xi^{1/2}\mathcal{B}_{\frac{1}{2}}(K\xi^{-1}).$$

And the deflection of the column is

$$\eta(\xi) = \xi^{1/2}[C_1 J_{\frac{1}{2}}(K\xi^{-1}) + C_2 J_{-\frac{1}{2}}(K\xi^{-1})] + \bar{\delta}.$$

Example: Buckling of a Tapered Column

Determine $C_1, C_2, \bar{\delta}$ using boundary conditions

$$\text{at}\, x = 0\, \text{or}\ \xi = 1: \quad \eta(\xi) = 0,\, \eta'(\xi) = 0,$$

$$\text{at}\, x = L\, \text{or}\ \xi = 1 - k_1: \quad \eta(\xi) = \bar{\delta}.$$

Note that

$$J'_{\frac{1}{2}}(x) = J_{\frac{1}{2}-1}(x) - \frac{\frac{1}{2}}{x} J_{\frac{1}{2}}(x) = J_{-\frac{1}{2}}(x) - \frac{1}{2x} J_{\frac{1}{2}}(x),$$

$$J'_{-\frac{1}{2}}(x) = -J_{-\frac{1}{2}+1}(x) + \frac{-\frac{1}{2}}{x} J_{-\frac{1}{2}}(x) = -J_{\frac{1}{2}}(x) - \frac{1}{2x} J_{-\frac{1}{2}}(x),$$

Example: Buckling of a Tapered Column

Differentiate the solution $\eta(\xi)$:

$$\eta'(\xi) = \frac{1}{2\sqrt{\xi}}\left[C_1 J_{\frac{1}{2}}\left(\frac{K}{\xi}\right) + C_2 J_{-\frac{1}{2}}\left(\frac{K}{\xi}\right)\right] +$$

$$\sqrt{\xi}\left\{C_1\left[J_{-\frac{1}{2}}\left(\frac{K}{\xi}\right) - \frac{\xi}{2K}J_{\frac{1}{2}}\left(\frac{K}{\xi}\right)\right] + C_2\left[-J_{\frac{1}{2}}\left(\frac{K}{\xi}\right) - \frac{\xi}{2K}J_{-\frac{1}{2}}\left(\frac{K}{\xi}\right)\right]\right\}$$

Therefore,

$$\eta'(1) = C_1\left[J_{\frac{1}{2}}(K) - KJ_{-\frac{1}{2}}(K)\right] + C_2\left[J_{-\frac{1}{2}}(K) + KJ_{\frac{1}{2}}(K)\right] = 0$$

At $x = L$ or $\xi = 1 - k_1$

$$\eta(1-k_1) = \sqrt{1-k_1}\left[C_1 J_{\frac{1}{2}}(K_L) + C_2 J_{-\frac{1}{2}}(K_L)\right] + \bar{\delta} = \bar{\delta},\ K_L = \frac{K}{1-k_1}.$$

Thus, the second equation to determine C_1, C_2 is

$$C_1 J_{\frac{1}{2}}(K_L) + C_2 J_{-\frac{1}{2}}(K_L) = 0$$

Example: Buckling of a Tapered Column

The determinant of the linear homogeneous system for C_1, C_2 should be zero to have nontrivial solutions

$$\begin{vmatrix} J_{\frac{1}{2}}(K) - KJ_{-\frac{1}{2}}(K) & J_{-\frac{1}{2}}(K) + KJ_{\frac{1}{2}}(K) \\ J_{\frac{1}{2}}(K_L) & J_{-\frac{1}{2}}(K_L) \end{vmatrix} = 0$$

The equation

$$J_{-\frac{1}{2}}(K_L)[J_{\frac{1}{2}}(K) - KJ_{-\frac{1}{2}}(K)] - J_{\frac{1}{2}}(K_L)[J_{-\frac{1}{2}}(K) + KJ_{\frac{1}{2}}(K)] = 0$$

is called the buckling equation.

For a given value of $k_1 = (r_0 - r_1)/r_0$, the roots of this algebraic equation $K_n, n = 1, 2, \ldots$, can be determined, from which the nth buckling load can be found.

$$K = \frac{k}{k_1},\ k^2 = \frac{PL^2}{EI_0} \Rightarrow P_n = (p_n\pi)^2 \frac{EI_0}{L^2},\ p_n = \frac{k_1 K_n}{\pi},\ n = 1, 2, \ldots$$

Motivating Examples: Ascending Motion of a Rocket

Recall that the equation of ascending motion of a rocket is Bessel's equation

$$\tau^2 \frac{d^2 V(\tau)}{d\tau^2} + \tau \frac{dV(\tau)}{d\tau} + (\tau^2 - v^2) V(\tau) = 0,\ v = 2\sqrt{\frac{\beta u}{q}}.$$

The solution of the Bessel's equation is

$$V(\tau) = C_1 J_\nu(\tau) + C_2 Y_\nu(\tau),$$

$$\frac{dV(\tau)}{d\tau} = C_1 \left(\frac{\nu}{\tau} J_\nu(\tau) - J_{\nu+1}(\tau) \right) + C_2 \left(\frac{\nu}{\tau} Y_\nu(\tau) - Y_{\nu+1}(\tau) \right)$$

The velocity of the rocket is

$$\nu(\tau) = \frac{m(\tau) \dot{V}(\tau)}{\beta V(\tau)} = \frac{m(\tau)(-\frac{2\beta g}{q\tau} \frac{dV(\tau)}{d\tau})}{\beta V(\tau)} =$$

$$= \frac{\frac{q^2\tau^2}{4\beta g}(-\frac{2\beta g}{q\tau}) \left(\frac{\nu}{\tau}[C_1 J_\nu(\tau) + C_2 Y_\nu(\tau)] - [C_1 J_{\nu+1}(\tau) + C_2 Y_{\nu+1}(\tau)] \right)}{\beta(C_1 J_\nu(\tau) + C_2 Y_\nu(\tau))}$$

Exercises

A. Show that the general solution of the Airy equation

$$y'' - xy = 0$$

is, for $|x| < \infty$, $y(x) =$

$$= a_0 \left(1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(3k-2)}{(3n)!} x^{3n} \right) + a_1 \left(x + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(3k-1)}{(3n+1)!} x^{3n+1} \right)$$

B. Determine the general solution of the following differential equations in terms of power series about $x = 0$.

$$1.\ y''' + xy = 0, \quad 2.(1-x^2)y'' + y = 0, \quad 3.y'' - 2x^2 y = 0$$

$$4.y'' - 2x^2 y' + xy = 0, \quad 5.(x^2-1)y'' + (4x-1)y' + 2y = 0$$

C. Determine two linearly independent solutions of the following equations using the series solution approach

$$1.x^2y'' + (x - 2x^2)y' - xy = 0 \quad 2.x^2y'' - (2x + x^2)y' + 2y = 0$$

$$3.x^2y'' + (\frac{1}{2}x + x^2)y' + xy = 0$$