Name and ID: ______________________________

1 Existence and Uniqueness Theorem for the IVP

Theorem 1. ***Lipschitz Condition***
Consider the Initial Value Problem (IVP)

$$y' = f(x, y), \quad y(x_0) = y_0$$

Suppose

1. $f(x, y), \dfrac{\partial f}{\partial y}(x, y)$ *are continuous functions in some open rectangle* $R = \{(x, y) : |x - x_0| < a, |y - y_0| < b\}$
2. $\exists K, L$, *s.t.*

$$\begin{cases} |f(x, y)| \leq K \\ |\dfrac{\partial f}{\partial y}| \leq L \end{cases}$$

Then the IVP has a unique solution in the interval

$$|x - x_0| \leq \alpha, \qquad \alpha = min\{a, \frac{b}{K}\}$$

And L here is called the Lipschitz constant.

Comments

Notice that Lipschitz only provides the sufficient condition for the existence of the unique solution.
Actually, it extends the concept of continuous function in the context of DE, namely, *Lipschitz continuous*.

Question1 (1 point)
Apply the *Lipschitz Condition*, investigate the maximal interval of existence for the IVP

$$y' = x + y^3, \quad y(0) = 0$$

Solution:

Possible Answer:

$$-\frac{1}{3}\sqrt[5]{36} < x < \frac{1}{3}\sqrt[5]{36}$$

Question2 (1 point)
Apply the *Lipschitz Condition*, investigate the maximal interval of existence for the IVP

$$y' = 2y^2 - x, \quad y(1) = 1$$

Solution:

Possible Answer:

$$0.87 \leq x \leq 1.13$$

$$(\text{Hint : Assume } a < 1)$$

2 Interval of Existence

Definition 1. ***Interval of Existence***
The largest open interval J on which an IVP has a unique solution is called the maximal interval of existence for that solution.

Theorem 2. ***First Order, Linear, IVP***
Consider the IVP

$$y' + p(t)y = q(t), \quad y(t_0) = y_0$$

If p, q are both continuous on an open interval J who contains the initial point, i.e.

$$J : a < t < b, \quad t_0 \in J$$

Then the IVP has a unique solution on J for any y_0.

Comments

Notice that the interval of existence cares nothing about the value of initial y. And that the continuity is the powerful condition not only in the field of Calculus but also in the discourage of DE.
It is the sufficient yet not necessary condition. But for the first order, linear case we could always apply the theorem directly instead of applying the lemma to estimate the maximal interval of existence.

Steps

1. If possible, transform the target equation into the first order, linear DE.
2. Test the continuity.
3. Test if the initial condition provides the additional information except the equation.

Question1 (1 point)
Consider the IVP

$$ty' - t = 0, \quad y(t_0) = y_0$$

Investigate the maximal interval of existence.

Solution:

Follow the steps that shown above, we can derive without solving the equation that

$$\begin{aligned}
&\textit{when } t_0 > 0, \quad J : (0, \infty) \\
&\textit{when } t_0 < 0, \quad J : (-\infty, 0) \\
&\textit{when } t_0 = y_0 = 0, \quad J : \textit{entire real line} \\
&\textit{when } t_0 = 0 \neq y_0, \quad J : \varnothing
\end{aligned}$$

Theorem 3. ***First Order, explicit, IVP***
Consider the IVP,

$$y' = f(t, y), \quad y(t_0) = y_0$$

If

1. $f, \dfrac{\partial f}{\partial y}$ *are both continuous in some open rectangle* $R = \{(x, y) : a < t < b, c < y < d\}$
2. *Initial point* $(t_0, y_0) \in R$

Then the IVP has a unique solution in some open interval J of the form

$$J : t_0 - h < t < t_0 + h, \quad J \subseteq (a < t < b)$$

Comments

Notice that the open rectangle in the case is no longer symmetric about the initial point, compared to the Lipschitz condition.
Actually, it is the reduced version of the Lipschitz condition mentioned above. Keep in mind that although it provides a much looser condition for the existence of the interval, it is also a sufficient yet not necessary condition.

3 Inequalities

Theorem 4. ***Gronwall-Bellman inequality***
`See the slide pages (35 - 37, Week 2) for proof.`

Corollary 5. ***Bihari-Lasalle inequality***
Given or if we can construct

$$u(t) \leq K + \int_{t_0}^{t} f(t_1)\Phi(u(t_1))dt_1$$

where

$$\begin{cases} u(t) \geq 0, f(t) \geq 0, \quad u(t), f(t) \in C\,[t_0, +\infty) \\ K > 0 \textit{ is a const} \\ \Phi(u) \textit{ positive, non-decreasing, continuous for all } 0 < u < \bar{u} \end{cases}$$

Define

$$v = \psi(u) = \int_{K}^{u} \frac{du_1}{d\Phi(u_1)}, 0 < u < \bar{u}$$

If

$$\int_{t_0}^{t} f(t_1)dt_1 < \lim_{u \leftarrow \bar{u}^-} \psi(u), \ t_0 < t < \infty$$

then

$$u(t) \leq \psi^{-1}[\int_{t_0}^{t} f(t_1)dt_1], \ \forall t_0 < t < \infty$$

Question1 (1 point)
Prove the Bihari-Lasalle inequality based on what you have learned so far.

Solution:

Hint: Consider the substitution

$$w(t) = K + \int_{t_0}^{t} f(t_1)\Phi(u(t_1))dt_1$$

`See the RC (week 2) notes for reference.`

4 Continuation of Solutions

Actually, when it comes to the context of IVP, often it is the case that we want to expand our solution from the given single point to a band over the number axes. In fact, the information of the IVP could be completely attributed in the proposition below.

Theorem 6. ***Bihari's lemma: Continuation on the right***
If either of the Theorem 1,2,3 is satisfied, and the inequality

$$y(t) \leq C + \int_{t_0}^{t} f(t_1) \cdot \ \Phi(y(t_1))dt, \qquad t_0 \leq t \leq a$$

holds where

$$y = y(t), f = f(t) \ \textit{non-negative, continuous}$$

$$C > 0$$

$$\Phi = \Phi(u), \ \textit{positive, non-decreasing, continuous}$$

If

$$\int_{t_0}^{t} f(t_1)dt_1 < \lim_{u \leftarrow \bar{u}^-} \psi(u), \ t_0 < t < \infty$$

then

$$y(t) \leq \psi^{-1}[\int_{t_0}^{t} f(t_1)dt_1], \ \forall t_0 < t < \infty$$

Comments

Bihari's lemma always provides us with the non-linear IVP's largest interval of existence in practice.

Steps

1. Ensure the existence of the interval of existence.
2. Integrate and apply relaxation to construct the Bihari's condition.
3. Apply the Bihari's inequality to estimate the interval.

Question1 (1 point)

Apply the *Bihari's lemma*, investigate the maximal interval of existence for the IVP

$$y' = 2y^2 - x, \quad y(1) = 1$$

Solution:

Possible Answer:

$$0.62 \leq x \leq 1.5$$

Hint: for the left hand side, consider the substitution

$$x = 1 - t, \quad t \geq 0$$

Question2 (2 points)

Apply the *Bihari's lemma*, investigate the maximal interval of existence for the linear system:

$$\begin{cases} y_1'(t) = y_2^2, & y_1(0) = 1, \\ y_2'(t) = y_1^2, & y_2(0) = 2, \end{cases} \tag{1}$$

Solution:

Possible answer:

$$-\frac{1}{3} \leq t \leq \frac{1}{3}$$

Hint: Apply the triangle inequality to combine the conditions derived from the linear system.