vv256: Autonomous Systems of ODEs. Phase portraits. Stability.

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December 3, 2019

Autonomous Systems

A system of ordinary differential equations is called autonomous if it does not explicitly contain the independent variable.
We shall consider autonomous normal systems of the form

$$y_i' = f_i(y_1, \ldots, y_n),\ i = 1..n,$$

where the functions $f_i(y_1, \ldots, y_n)$ are well-defined in a certain domain Δ.

OR in the vector form

$$y' = f(y) \tag{1}$$

What is special about autonomous systems? The law of variation of the unknown functions described by the system does not change with time as is usually the case with physical laws.
Autonomous system $\rightarrow$ interpretation: geometric (the system of integral curves) and kinematic (the phase space)

Autonomous Systems: the Lotka-Volterra equations

Consider a a mathematical model of two spices in which one species (the predator) preys on the other species (the prey) while the prey lives on a different source of food.

- In the absence of the predator, the prey grows at a rate proportional to the current population
$$\frac{dy_1}{dt} = ay_1, a > 0 \quad \text{if } y_2 = 0.$$
- In the absence of the prey, the predator dies out
$$\frac{dy_2}{dt} = -cy_2,\ c > 0 \quad \text{if } y_1 = 0.$$
- The number of encounters between predator and prey is proportional to the product of their populations.

$$\frac{dy_1}{dt} = ay_1 - \alpha y_1 y_2,$$
$$\frac{dy_2}{dt} = -cy_2 + \gamma y_1 y_2.$$

Autonomous Systems

P1. If $y = (y_1, \ldots, y_n)$, $y_i = \varphi_i(t)$ is a solution of (1) then $z = (z_1, \ldots, z_n)$, $z_i = \varphi_i(t + C)$, $C = const$ is also a solution of the same system.

P2. To every solution $y = (y_1, \ldots, y_n)$, $y_i = \varphi_i(t)$ of (1), we make correspond the motion of a point in n-dimensional space defined by $y_i = \varphi_i \Rightarrow$ we obtain the trajectory of the motion.

If $y_i = \psi(t)$, $i = 1..n$ is another solution of (1), then the trajectories either do not intersect or coincide. Indeed, let $\exists t_1, t_2 \colon \varphi_i(t_1) = \psi_i(t_2)$, $i = 1..n$. Then

$$\varphi_i(t_1) = \varphi_i(t_1 - t_2 + t_2) = \varphi_i(t_2 + C) = \psi_i(t_2),\ C = t_1 - t_2$$

Therefore, the trajectories described by $y_i = \varphi_i$ and $y_i = \psi_i$ coincide but the first solution describes the trajectory with the time delay "C".

Autonomous Systems

P3. States of equilibrium and closed trajectories

Can a trajectory representing a solution intersect itself? Let $y = (y_1, \dots, y_n)$, $y_i = \varphi_i(t)$ be a solution of (1) with the maximal interval (m_1, m_2). If

$$\varphi_i(t_1) = \varphi_i(t_2),\ i = 1..n,\ t_1 \neq t_2,\ m_1 < t_1, t_2 < m_2$$

then

- $\forall\, \varphi_i(t) = a_i,\ i = 1..n \Rightarrow$ the point $(\varphi_1(t), \dots, \varphi_n(t))$ does not move with time. $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ is the state of the equilibrium of the system.
- $\exists\, T > 0 \colon \forall t \quad \varphi_i(t + T) = \varphi_i(t)$ but for $|\tau_1 - \tau_2| < T$ $\varphi_i(\tau_1) \neq \varphi_i(\tau_2)$ at least for one $i = 1..n$.
  The solution is periodic with period T, and the trajectory is closed $\Rightarrow$ cycle

Notice, that the maximal interval of existence $(m_1, m_2) = (-\infty, +\infty)$ in this case.

Autonomous Systems

P4. Phase Spaces

Since the system $y' = f(y)$ is defined in the domain Δ, so each point $(y_0^1, \ldots, y_0^n) \in \Delta$ corresponds to a sequence of n numbers

$$f_1(y_0^1, \ldots, y_0^n), \ldots, f_n(y_0^1, \ldots, y_0^n).$$

Consider these numbers as components of the vector $f(y_0^1, \ldots, y_0^n)$ in an n-dimensional space that starts at $(y_0^1, \ldots, y_0^n) \Rightarrow$ the autonomous system defines a vector field in the domain Δ. The solution $y_i = \varphi_i(t)$ satisfying the i.c $\varphi_i(t_0) = y_0^i$, $i = 1..n$ corresponds to the motion of a point along a certain trajectory that passes through $(y_0^1, \ldots, y_0^n)$ at $t = t_0$. The velocity of the point at that moment is $f(y_0^1, \ldots, y_0^n)$.

This n-dimensional vector field where solutions of the system are interpreted in the form of trajectories is called the phase space of the system. The trajectories are called phase trajectories (portraits).

The phase plane of a linear homogeneous system with constant coefficients

Construct phase trajectories on the phase plane of the system $y' = Ay$ or

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2, \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{cases}$$

The origin $(0, 0)$ is always the equilibrium state. Let the eigenvalues λ_1, λ_2 of A be real, distinct and nonzero. Expand the solution

$$y(t) = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$$

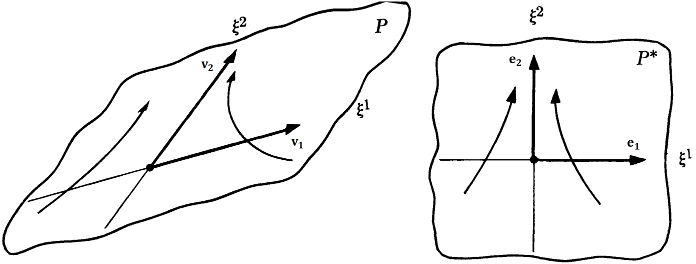
in terms of the basis (eigen)vectors v_1, v_2:

$$y = \xi_1 v_1 + \xi_2 v_2, \quad \xi_1 = C_1 e^{\lambda_1 t}, \xi_2 = C_2 e^{\lambda_2 t}$$

Attention: the coordinates ξ_1, ξ_2 on a phase plane P of the system are not rectangular $\Rightarrow$ make an affine mapping of the phase plane P onto an auxiliary plane P^* such that v_1, $v_2 \to e_1$, e_2

The phase plane of a linear homogeneous system with constant coefficients

The point $y = \xi_1 v_1 + \xi_2 v_2 \in P \to \underbrace{(\xi_1, \xi_2)}_{\text{rectangular coordinates}} \in P^*$.



Therefore, the trajectory defined by the parametric equations $\xi_1 = C_1 e^{\lambda_1 t}$, $\xi_2 = C_2 e^{\lambda_2 t}$ in P is mapped into a trajectory defined by the same equations in the rectangular coordinates of the plane P^*.

The phase plane of a linear homogeneous system with constant coefficients

- $C_1 = C_2 = 0$ the motion of a point that describes the state of equilibrium $(0, 0)$
- $C_1 = 0$, $C_2 > 0$ a motion which describes the positive semi axis of ordinates. Cases $\lambda_2 > 0$ and $\lambda_2 < 0$ correspond to the the motion away from the origin and toward to the origin.
- $C_2 = 0$, $C_1 > 0$ a motion which describes the positive semi axis of abscissas toward and away from the origin.
- There are also trajectories defined by $\xi_1 = C_1 e^{\lambda_1 t}$, $\xi_2 = -C_2 e^{\lambda_2 t}$, $\xi_1 = -C_1 e^{\lambda_1 t}$, $\xi_2 = C_2 e^{\lambda_2 t}$ which can be obtained by mirror reflections in corresponding axes. If trajectories are drawn in the first quadrant, then it is easy to construct the entire phase picture in the plane P^*.

The phase plane of a linear homogeneous system with constant coefficients

Theorem: Assume the eigenvalues of A are real. If a line l lies along an eigenvector of A, then in the phase plane any solution of $y' = Ay$ that starts at a point (y_1, y_2) on the line l remains on l for all t; as $t \to \infty$ it approaches the origin if the eigenvalue $\lambda_l < 0$, or moves away from the origin if $\lambda_l > 0$.

The phase plane of a linear homogeneous system with constant coefficients

Proof:

If $(y_1, y_2) \neq (0, 0)$ is any point on the line l , the position vector $y = (y_1, y_2)$ from $(0, 0)$ to (y_1, y_2) is some scalar multiple of the eigenvector v_l: $y = Cv_l$, $c = const$.

The tangent vector u to the solution curve through (y_1, y_2) satisfies

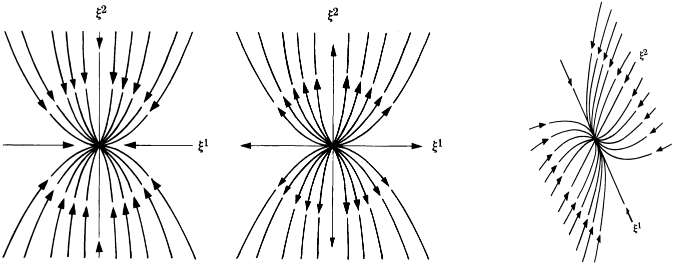
$$u = y' = Ay = A(cv_l) = c(Av_l) = c(\lambda_l v_l) = \lambda_l y$$

Therefore, for all values of t the tangent vector to the solution curve through (y_1, y_2) is a vector in the direction of y if $\lambda_l > 0$ or in the opposite direction if$\lambda_l < 0$.Since the tangent vector points along the line l for all t, this means that the solution must move along l. It will move away from $(0, 0)$ if $\lambda_l > 0$ or toward $(0, 0)$ if $\lambda_l < 0$.

Once the direction of the two eigenvectors is determined (they cannot have the same direction since they are linearly independent and cannot be constant multiples of each other), the geometry of the phase plane depends only on the signs of the two eigenvalues λ_1 and λ_2.

Classification of states of equilibrium

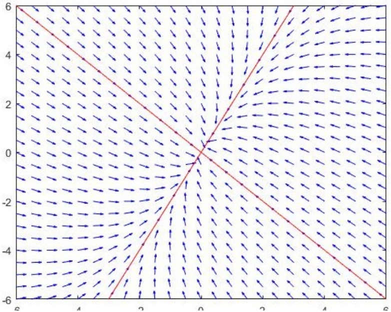
- $\lambda_2 < \lambda_1 < 0$ Stable Node
  1. the motion along positive semi axes goes toward the origin,
  2. the motion alone an arbitrary trajectory in the first quadrant is an asymptotic approach of the point toward the origin,
  3. as $t \to -\infty$ the motion goes in the direction of the axis of ordinates

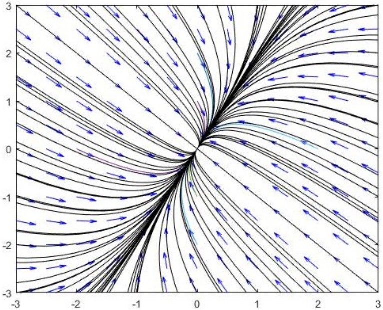


- $0 < \lambda_1 < \lambda_2$ Unstable Node

Classification of states of equilibrium

$$y' = \begin{pmatrix} -7 & 2 \\ 4 & -5 \end{pmatrix} \Rightarrow \lambda_1 = -9,\ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \lambda_2 = -3,\ v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$





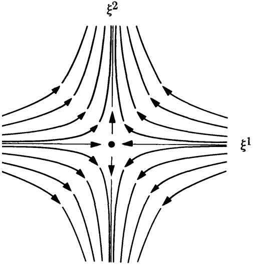
Matlab routine

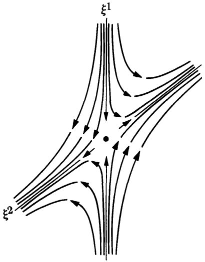
```matlab
function vectorfield(f,y1,y2,t)
if nargin==3
    t=0;
end
n1=length(y1);
n2=length(y2);
yp1=zeros(n2,n1);
yp2=zeros(n2,n1);
for i=1:n1
    for j=1:n2
        ypv=feval(f,t,[y1(i); y2(j)]);
        yp1(j,i)=ypv(1);
        yp2(j,i)=ypv(2);
    end
end
quiver(y1,y2,yp1,yp2,'r');
yplength=sqrt(yp1.^2+yp2.^2);
yp1n=yp1./yplength;
yp2n=yp2./yplength;
figure(1)
quiver(y1,y2,yp1n,yp2n,0.5);
axis tight;
```

```matlab
>> f = @(t,y) [-7*y(1)+2*y(2); 4*y(1)-5*y(2)]

f =

  function_handle with value:

    @(t,y)[-7*y(1)+2*y(2);4*y(1)-5*y(2)]

>> vectorfield(f,-6:.5:6,-6:.5:6)
hold on
    for p1=-5:2:5
      for p2=-5:2:5
        [ts,ys] = ode45(f,[0,10],[p1;p2]);
        plot(ys(:,1),ys(:,2))
      end
    end
```

Classification of states of equilibrium

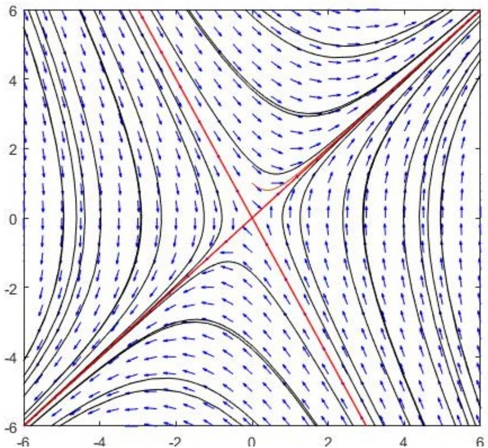
- $\lambda_1 < 0 < \lambda_2$ Saddle point
  1. the motion along the positive semi axis of abscissas is directed toward the origin,
  2. the motion along the positive semi axis of ordinates is directed away from the origin,
  3. the forms of the trajectories in the first quadrant resemble hyperbolas





Classification of states of equilibrium

$$y' = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \Rightarrow \lambda_1 = 2,\ v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix},\ \lambda_2 = 1,\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



Classification of states of equilibrium

- $\lambda_{1,2} = \mu \pm i\nu \Rightarrow$ the corresponding eigenvectors can be chosen to be complex conjugates: v, $\bar{v}$. Any solution

$$y = Cve^{\lambda t} + \bar{C}\bar{v}e^{\bar{\lambda}t}, \quad C \in \mathbb{C}.$$

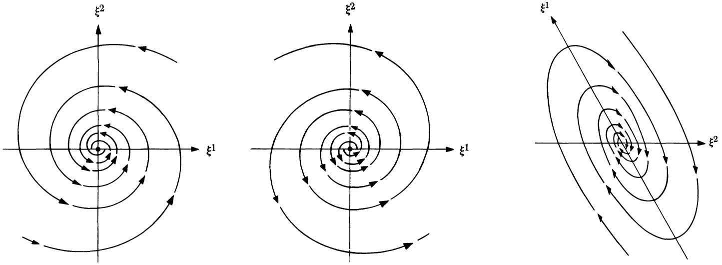
Denote $v = \frac{1}{2}(v_1 - iv_2)$, where v_1, v_2 are real vectors. Then v_1, v_2 forms the basis in the phase plane P Explain why
Let $\zeta = \xi_1 + i\xi_2 = Ce^{\lambda t} \Rightarrow y = \xi_1 v_1 + \xi_2 v_2$
We map the phase plane P to the auxiliary phase plane P^* in a such way that $v_1 \to 1$, $v_2 \to i$
The trajectory $y = \xi_1 v_1 + \xi_2 v_2$ will be mapped into a phase trajectory described by $\zeta = \xi_1 + i\xi_2 = Ce^{\lambda t}$
Rewrite the obtained equation in polar coordinates by letting $\zeta = \rho e^{i\varphi}$, $C = Re^{i\alpha}$ to obtain

$$\rho = Re^{\mu t} \qquad \text{the equation of the motion of}$$

$$\varphi = \alpha + \nu t \qquad \text{a point in the phase plane } P^*$$

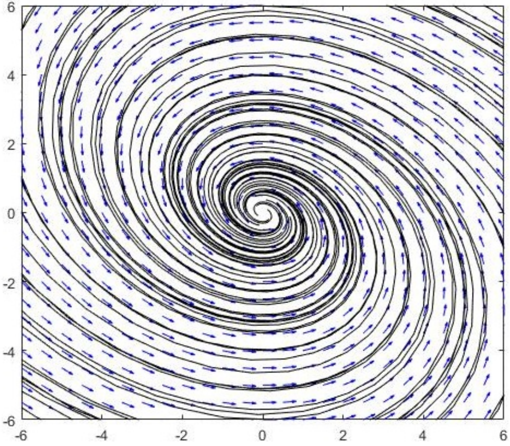
Classification of states of equilibrium

- $\mu \neq 0 \Rightarrow$ every trajectory is a logarithmic spiral. The corresponding image on the phase plane is called a focus.
  $\mu < 0$ the point approaches the origin as $t \to +\infty \Rightarrow$ stable focus
  $\mu > 0 \Rightarrow$ unstable focus



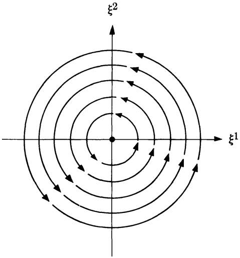
Classification of states of equilibrium

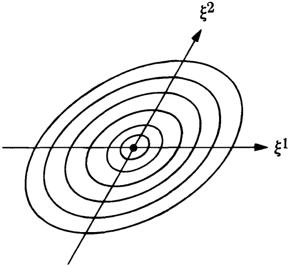
$$y' = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i$$



Classification of states of equilibrium

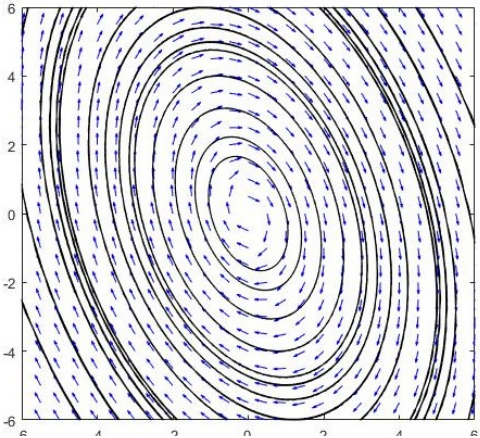
- $\mu = 0 \Rightarrow$ every phase trajectory except the state of equilibrium $(0, 0)$ is closed $\Rightarrow$ center.





Classification of states of equilibrium

$$y' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm 3i$$



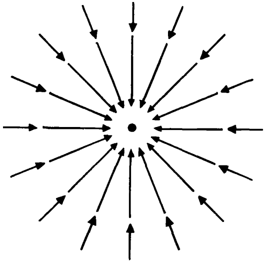
Classification of states of equilibrium

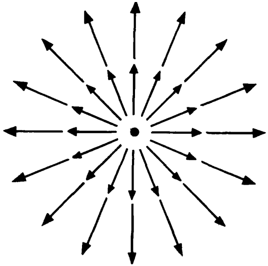
Degenerated Cases: $\lambda_1 = \lambda_2 = \lambda \neq 0$

- Two independent eigenvectors v_1, v_2:

$$\underbrace{y}_{\text{solution with i.c.}(0,y_0)} = C_1 v_1 e^{\lambda t} + C_2 v_2 e^{\lambda t} = y_0 e^{\lambda t}$$

A ray emanating from the origin: $\lambda < 0$ toward, $\lambda > 0$ away





Classification of states of equilibrium

Degenerated Cases: $\lambda_1 = \lambda_2 = \lambda \neq 0$

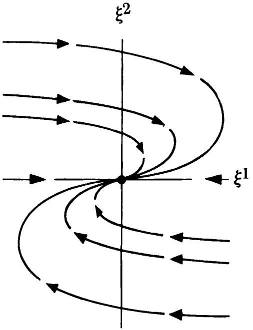
- One independent eigenvectors v_1:

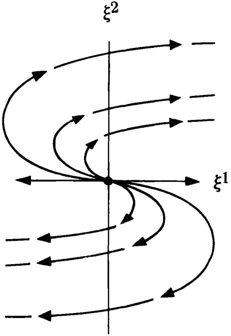
$$y = \xi_1 v_1 + \xi_2 v_2, \quad \xi_1 = e^{\lambda t}(C_1 + C_2 t), \, \xi_2 = C_2 e^{\lambda t}$$

Transform the phase plane P into the P^* such that $v_1, \, v_2 \to e_1, \, e_2 \Rightarrow$ the trajectories of P are transformed into trajectories of P^*, where the trajectories are already given in rectangular coordinates.
$\lambda < 0$: Change the sign of $C_1, \, C_2$ simultaneously $\Rightarrow$ a reflection w.r.t.the origin $\Rightarrow$ it is enough to consider trajectories in the upper half-plane.
$C_2 = 0, \, C_1 \neq 0$ positive $(C_1 > 0)$ and negative $(C_1 < 0)$ semi axes
$C_1 = 0, C_2 > 0$ $\xi_1 = C_2 e^{\lambda t} t$, $\xi_2 = C_2 e^{\lambda t}$ as t increases from zero, the point first moves to the right, then to the left, always descending toward the origin t_0

Classification of states of equilibrium

Stable degenerated node $\lambda < 0$ Unstable degenerated node $\lambda > 0$



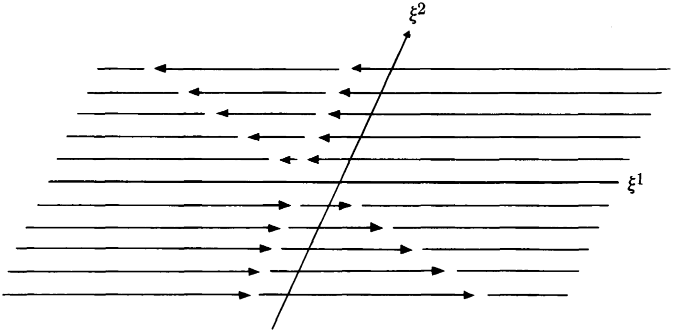


$\lambda > 0 \Rightarrow$ mirror reflection of the plane in the axis of ordinates

Classification of states of equilibrium

Degenerated Cases: $\lambda_1 = \lambda_2 = 0$

- Case 1: $y = y_0 \Rightarrow$ every point of the plane P is a state of equilibrium
- Case 2: $\xi_1 = C_1 + C_2 t$, $\xi_2 = C_2 \Rightarrow$ straight lines; all points of the line $\xi_2 = 0$ are equilibrium

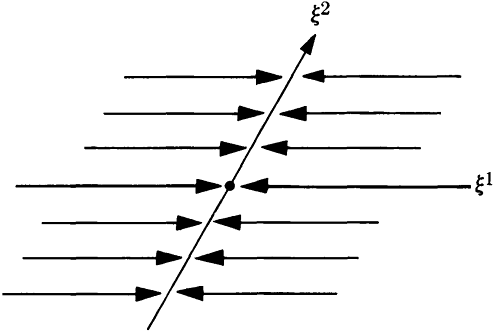


Classification of states of equilibrium

Degenerated Cases: $\lambda_1 \neq 0$, $\lambda_2 = 0$

$$y = \xi v_1 + \xi_2 v_2,\ \xi_2 = const,\ \xi_1 = C_1 e^{\lambda_1 t}$$

The motion is along the straight line $\xi_2 = const$ in the direction of the line $\xi_1 = 0$ All points of $\xi_1 = 0$ are states of equilibrium.

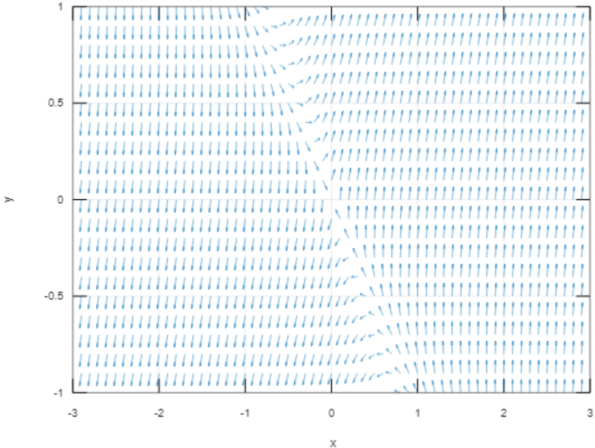


Example

Find and investigate the equilibrium states of

$$y' = \frac{dy}{dx} = \frac{2x + y}{3x + 4y}$$

See more solved examples in the class.



Stability

The idea of stability of the equilibrium state $a = (a_1, \dots, a_n)$ of the system $y' = f(y)$: any solution of the system starting at $t = 0$ from a point sufficiently close to a remains in the neighborhood of a during subsequent variation. Denote $\varphi(\xi, t)$ a solution with $\varphi(\xi, 0) = \xi$.

The equilibrium state a of the system $y' = f(y)$ is called stable (Lyapunov stable) if

1. $\exists \rho > 0$: for $|\xi - a| < \rho$ the solution $\varphi(\xi, t)$ is defined for all $t > 0$.
2. $\forall \varepsilon > 0 \exists 0 < \delta < \rho$: $|\xi - a| < \delta \Rightarrow |\varphi(\xi, t) - a| < \varepsilon$ for all $t > 0$.

An equilibrium state a is called asymptotically stable if it is stable and

$$\exists \sigma < \rho \colon |\xi - a| < \sigma \quad \lim_{t \to +\infty} |\varphi(\xi, t) - a| = 0$$

Stability

P1. Consider a linear homogeneous system with constant coefficients $y' = Ay$, and let $\psi(\xi, t)$ be its solution: $\psi(\xi, 0) = \psi$. If the eigenvalues of the matrix A have negative real parts then

$$\exists \alpha > 0,\ r > 0: \quad |\psi(\xi, t)| \leq r|\xi|e^{-\alpha t}, \quad t \geq 0.$$

Lyapunov and asymptotic stability of the equilibrium solution $y = 0$ follows automatically.

P2. Differentiation with respect to a system Let a smooth function

$$F(y_1, \ldots, y_n) = F(y)$$

be defined in the domain Δ, and $\varphi(t)$ be a such solution of the system (1) that $\varphi(t_0) = y$. The derivative $F'_{(1)}(y)$ w.r.t system (1) is defined by $F'_{(1)}(y) = \frac{d}{dt}F(\varphi(t))|_{t=t_0}$ or

$$F'_{(1)}(y) = \sum_{i=1}^{n} \frac{\partial F(y)}{\partial y_i} f_i(y)$$

it does not depend on the choice of φ but on the choice of y.

Stability

A function

$$W(y) = \sum_{i,j=1}^{n} w_{ij} y_i y_j, \quad w_{ij} = w_{ji} \in \mathbb{R}$$

is called a quadratic form of the vector $y = (y_1, \ldots, y_n)$.
The quadratic form $W(y)$ is called positive definite if $W(y) > 0$ for $y \neq 0$. Example: $W(y) = y_1^2 + y_2^2$

P3. For a positive definite quadratic form $W(y)$, there exist $\mu, \nu > 0$ such that

$$\mu |y|^2 \leq W(y) \leq \nu |y|^2 \quad \forall y$$

Proof: consider the function $W(\xi)$ on the unit ball $|\xi| = 1$ first.

Stability

P4. Consider a linear homogeneous system with constant coefficients

$$y_i' = \sum_{j=1}^{n} a_{ij} y_j, \quad j = 1..n \tag{2}$$

such that the eigenvalues of the coefficient matrix $A = (a_{ij})$ have negative real parts.
There exists a positive definite quadratic form $W(y)$ such that

$$W'_{(2)}(y) \leq -\beta W(y)$$

for an arbitrary vector y and a positive number β independent of y.
Proof: Consider the solution $\psi(\xi, t)$; $\psi(\xi, t) = \sum_{i=1}^{n} \xi_i \psi_i(t)$ and then define the function

$$W(\xi) = \int_0^\infty |\psi(\tau, \xi)|^2 \, d\tau = \sum_{i,j=1}^{n} \xi_i \xi_j \int_0^\infty (\psi_i(\tau), \psi_j(\tau)) \, d\tau$$

By **P1** the improper integral converges and $W(y)$ is a positive definite quadratic form of ξ. Find its derivative w.r.t.system (2).

Stability

P5. Lyapunov's Theorem Let $a = (a_1, \ldots, a_n)$ be the equilibrium state of the system (1). Define $y_i = a_i + \Delta y_i$, $i = 1..n$ and substitute it into (1) expanding the right sides into Taylor series

$$(\Delta y_i)' = \underbrace{f_i(a)}_{=0} + \sum_{j=1}^{n} \underbrace{\frac{\partial f_i(a)}{\partial y_j}}_{=a_{ij}} \Delta y_i \underbrace{+R_i}_{\text{second-order remainders}}, \ i = 1..n$$

We obtain a linearized system

$$(\Delta y_i)' = \sum_{j=1}^{n} a_{ij} \Delta y_j + R_i, \ i = 1..n$$

Theorem. If all eigenvelues of the Jacobian matrix $A = (a_{ij})$ have negative real parts, then the equilibrium state a of the system (1) is asymptotically stable.

Therefore, the Jacobian matrix may be used to analyze stability of equilibrium states for nonlinear systems. The theorem does not cover the case when at least one eigenvalue has zero real part.

Stability

The Lotka-Volterra system

Consider the predator-prey model described by the system

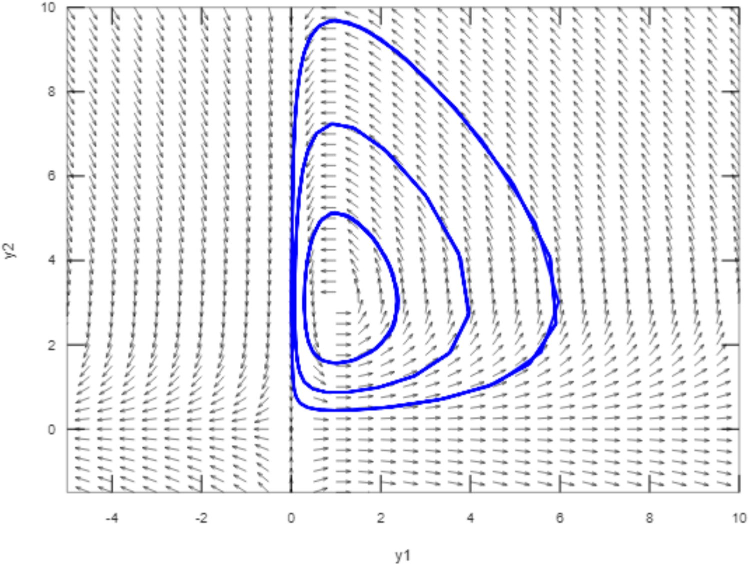
$$\begin{cases} \dfrac{dy_1}{dt} = \dfrac{3}{2} y_1 - \dfrac{1}{2} y_1 y_2, \\ \dfrac{dy_2}{dt} = -\dfrac{1}{2} y_2 + \dfrac{1}{2} y_1 y_2 \end{cases}$$

1. Find its equilibrium states $(0,0)$ and $(1,3)$.
2. Evaluate the Jacobian matrix $A = \begin{pmatrix} \frac{3}{2} - \frac{1}{2} y_2 & -\frac{1}{2} y_1 \\ \frac{1}{2} y_2 & -\frac{1}{2} + \frac{1}{2} y_1 \end{pmatrix}$ at the equilibrium points:

$A_{(0,0)} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \Rightarrow \lambda_1 = -1/2 < 0 < \lambda_2 = 3/2 \Rightarrow$ saddle unstable

$A_{(1,3)} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{3}{2} & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm i \frac{\sqrt{3}}{2} \Rightarrow$center stable

The Lotka-Volterra system



Stability: Lyapunov functions

P6. Lyapunov's Direct Method

Let E be an open subset of R^n containing the equilibrium state a of the system (1), and $V(y)$ be a real valued function such that $V(a) = 0$ and $V(y) > 0$ for all $y \neq a$.
If $V'_{(1)}(y) \leq 0$ for all $y \in E$ then V is called a Lyapunov function.

(a) if a Lyapunov function exists in the nbd of the equilibrium state a, then a is stable;

(b) if $V'_{(1)}(y) < 0$ for all $y \in E \setminus \{a\}$, then a is asymptotically stable;

(c) if $V'_{(1)}(y) > 0$ for all $y \in E \setminus \{a\}$, then a is unstable.

A rough idea why Lyapunov functions are so good: a Lyapunov function V decreases along the trajectories $\Rightarrow$ since $V > 0$, so all trajectories tend to zero (the minimum of V)

Stability: Examples

Example 1. Consider the system

$$\begin{cases} \dot{x} = -x + y + xy, \\ \dot{y} = x - y - x^2 - y^3. \end{cases}$$

The origin $(0,0)$ is the state of equilibrium.
The function $V(x,y) = x^2 + y^2$ is the Lyapunov one, and

$$V'(x,y) = 2x(-x+y+xy)+2y(x-y-x^2-y^3) = -2(x-y)^2-2y^4 < 0$$

$(0,0)$ is asymptotically stable

Example 2. The origin is the equilibrium point of the system

$$\begin{cases} \dot{x} = -x + x^2 - 2xy, \\ \dot{y} = -2y - 5xy + y^2. \end{cases}$$

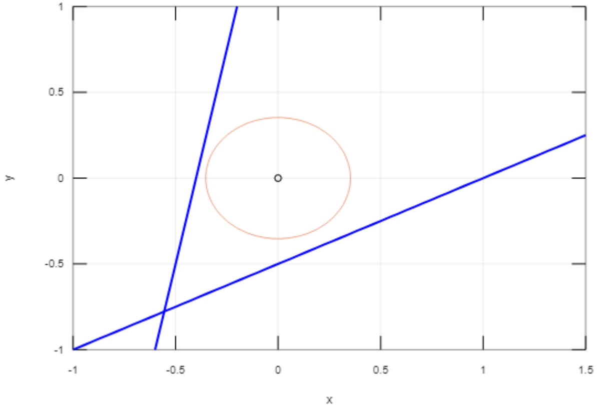
A standard guess for the Lyapunov function $V(x,y) = \frac{1}{2}(x^2 + y^2)$ gives

$$V'(x,y) = -x^2(1 - x + 2y) - y^2(2 + 5x - y)$$

If $x - 2y < 1$ and $5x - y > -2$ in some nbh of the origin, then V is a Lyapunov function.

Stability: Examples

Example 2.



The origin is asymptotically stable.

Stability: Examples

Example 3.

Consider the nonlinear differential equation

$$\ddot{z} - a\dot{z}(z^2 - 1) + z = 0.$$

Shall we set $y = \dot{z}$? A nice trick: just for fun! For a general second-order equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

we can transfer to Lienard coordinate by introducing

$$F(x) = \int^x f(\xi)\, d\xi \Rightarrow \frac{dF}{dt} = \dot{x}\frac{dF}{dx} = f(x)\dot{x}$$

Define $\quad y = \dot{x} + F(x) \Rightarrow \dot{y} = \ddot{x} + f(x)\dot{x} = -g(x)$ and obtain the system

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x). \end{cases}$$

Stability: Examples

Example 3. For the given nonlinear equation it will become

$$\begin{cases} \dot{x} = y + a(\frac{1}{3}x^3 - x), \\ \dot{y} = -x. \end{cases}$$

Use the trial Lyapunov function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ to get

$$V'(x, y) = x\dot{x} + y\dot{y} = \underbrace{a}_{>0} x^2(\frac{1}{3}x^2 - 1) \Rightarrow \underbrace{V' \leq 0}_{\text{for } x^2<3}$$

We need to know if $V' = 0$ in the nbd of $(0, 0)$.

- The largest open nbd of $(0, 0)$ which lies entirely in the region $\{x \colon x^2 < 3\}$ is $V_{\frac{3}{2}} = x^2 + y^2 < 3$.
- $V' = 0$ on $x = 0$.
- On $x = 0$, $\dot{x} = y \Rightarrow$ trajectories which intersect the line $x = 0$ remain on this line if $y = 0 \Rightarrow V' = 0$ only at $(0, 0) \Rightarrow$ origin is asymptotically stable

Stability

Example 4. A linear harmonic oscillator (spring mass) is described by the equation

$$\ddot{x} + kx = 0,\ k > 0.$$

Rewrite it in the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -kx \end{cases}$$

The origin $(0,0)$ is stable: usual analysis or direct Lyapunov method with $V(x,y) = \frac{1}{2}kx^2 + \frac{1}{2}y^2$.
What happens if we add a damping to the system?

$$\begin{cases} \dot{x} = y \\ \dot{y} = -kx - \varepsilon y^3(1 + x^2) \end{cases}$$

Attention! Jacobian $\begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$ does not reflect the damping, and linear analysis does not help here.

Stability

Apply the direct Lyapunov approach with the same function

$$V(x,y) = \frac{1}{2}kx^2 + \frac{1}{2}y^2 : \quad V(0,0) = 0,\ V(x,y) \neq 0,\ (x,y) \neq (0,0)$$

$$\dot{V} = kxy + y(-kx - \varepsilon y^3(1+x^2)) = -\varepsilon y^4(1+x^2)$$

$\varepsilon > 0 \Rightarrow$ stable (asymptotically)
$\varepsilon < 0 \Rightarrow$ unstable
How to construct a Lyapunov function?

Construction of a Lyapunov Function: an example

- Consider the equilibrium point $(x, y) = (0, 0)$ for the system

$$\begin{cases} \dot{x} = 2x - 5y + x^2 - 4xy, \\ \dot{y} = 2x - 4y + 2x^2 - 3xy + 8y^2. \end{cases}$$

- The Jacobian matrix evaluated at $(0, 0)$ is

$$A = \begin{pmatrix} 2 & -5 \\ 2 & -4 \end{pmatrix}$$

- with the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

- the eigenvalues of A are $-1 \pm i \Rightarrow$ the origin is asymptotically stable.

Construction of a Lyapunov Function: an example

- Find eignenvectors of A:

$$v_1 = \begin{pmatrix} 5 \\ 3 - i \end{pmatrix}, \quad v_2 = \bar{v}_1 = \begin{pmatrix} 5 \\ 3 + i \end{pmatrix}$$

- Vectors f_1, f_2 such that

$$A^T f_i = \bar{\lambda}_i f_i$$

  are called adjoint eigenvectors of A.
  In this example,

$$f_1 = \begin{pmatrix} 2 \\ -3 - i \end{pmatrix}, \quad f_2 = \bar{f}_1$$

- Following the proof of the Lyapunov theorem, we define

$$V(w) = C_1 \overline{(w, f_1)}(w, f_1) + C_2 \overline{(w, f_2)}(w, f_2),\ w = \begin{pmatrix} x \\ y \end{pmatrix}, C_1,\ C_2 > 0$$

Construction of a Lyapunov Function: an example

- Since $\bar{f}_1 = f_2 \Rightarrow \overline{(w, f_1)} = (w, f_2)$, and

$$V(w) = C\overline{(w, f_1)}(w, f_1) = C|(w, f_1)|^2$$

- $C = C_1 + C_2 > 0$ is simply a scale factor (of no importance in this case)
- Calculate

$$(w, f_1) = 2x + (-3 + i)y \Rightarrow C|(w, f_1)|^2 = 4x^2 - 12xy + 10y^2$$

- Thus, the Lyapunov function is

$$V(x, y) = 2x^2 - 6xy + 5y^2$$

Exercise: Check that the constructed function is indeed a Lyapunov function.

Proof of Lyapunov's Theorem: review of linear algebra

- Consider a real matrix A (recall, that any matrix corresponds to a linear operator in a finite dimensional space).
  We say that a matrix B is adjoint of A if

$$(Ax, y) = (x, By) \quad \forall x, y$$

- Easy to see that $B = \bar{A}^T \Rightarrow B = A^T$ for the real matrix A.
- A and A^T have the same eigenvalues, say λ_i, let λ_i be all distinct.
- There exist unique eigenvectors $\{v_i\}$, $\{f_i\}$ such that

$$Av_i = \lambda_i v_i, \quad Bf_i = \bar{\lambda}_i f_i$$

- The eigenvectors $\{f_i\}$ of the adjoint matrix are called adjoint eigenvectors.
- Since $(Av_i, f_j) = \lambda_i(v_i, f_j)$ and $(v_i, Bf_j) = \lambda_i(f_j, v_i)$, so

$$\lambda_i(v_i, f_j) = \lambda_j(v_i, f_j) \Rightarrow (v_i, f_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j \end{cases}$$

Proof of Lyapunov's Theorem: review of linear algebra

- Suppose that we write any vector y in the basis of eigenvectors of A:
$$y = \sum_{i=1}^{n} y_i v_i$$
- Take the inner product of y with f_j
$$(y, f_j) = \sum_{i=1}^{n} y_i (v_i, f_j) = y_j$$
- Therefore,
$$y = \sum_{i=1}^{n} (y, f_i) v_i$$

Proof of Lyapunov's Theorem

- Let A be the Jacobian matrix of $y' = f(y)$, and let the equilibrium point be the origin $y = 0$.
- Substitute the obtained representation

$$y = \sum_{i=1}^{n} y_i v_i = \sum_{i=1}^{n} (y, f_i) v_i$$

into the system

$$\frac{dy}{dt} = \sum_{i=1}^{n} \frac{dy_i}{dt} v_i = \sum_{i=1}^{n} \frac{d}{dt} (y, f_i) v_i$$

- Since $\frac{dy}{dt} = Ay + R,\ R = o(|y|)$ and

$$Ay = \sum_{i=1}^{n} y_i (Av_i) = \sum_{i=1}^{n} \lambda_i (y, f_i) v_i$$

- $$\frac{d}{dt}(y, f_i) = \lambda_i (y, f_i) + o(|y|)$$

Proof of Lyapunov's Theorem

- Define a Lyapunov function by

$$V(y) = \sum_{i=1}^{n} C_i \overline{(y, f_i)}(y, f_i), \quad C_i > 0$$

- The defined function V is differentiable and positive definite.
- Differentiate V w.r.t. time

$$V' = \sum_{i=1}^{n} C_i \left[\frac{d\,\overline{(y, f_i)}}{dt}(y, f_i) + \overline{(y, f_i)} \frac{d(y, f_i)}{dt} \right]$$

$$V' = \sum_{i=1}^{n} C_i \underbrace{(\bar{\lambda}_i + \lambda_i)}_{<0} \overline{(y, f_i)}(y, f_i) + o(|y|^2)$$

- There is a nbd E of $y = 0$ in which the sum dominates the $o(|y|^2) \Rightarrow V' < 0$ in $E \setminus \{0\}$

Limit Cycles

Consider a nonlinear system

$$\begin{cases} \dot{y}_1 = & y_1 + y_2 - y_1(y_1^2 + y_2^2), \\ \dot{y}_2 = & -y_1 + y_2 - y_2(y_1^2 + y_2^2) \end{cases}$$

- The only equilibrium point is $(0, 0)$, and the Jacobian matrix is

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- The eigenvalues of A are $1 \pm i \Rightarrow$ the origin is unstable focus.
- Do all solutions different from the equilibrium move away from the origin? since there are no other equilibrium points
- It is not the case.

Limit Cycles

- Transfer the equation to polar coordinates by

$$y_1 = r\cos\varphi,\ y_2 = r\sin\varphi \Rightarrow r^2 = y_1^2 + y_2^2$$

- Differentiating

$$2r\frac{dr}{dt} = 2y_1\dot{y}_1 + 2y_2\dot{y}_2$$

- and using the original equations

$$y_1\dot{y}_1 + y_2\dot{y}_2 = y_1(y_1 + y_2 - y_1(y_1^2 + y_2^2)) + y_2(-y_1 + y_2 - y_2(y_1^2 + y_2^2)) =$$

$$= y_1^2 + y_2^2 - (y_1^2 + y_2^2)^2 = r^2(1 - r^2),$$

- we obtain

$$r\frac{dr}{dt} = r^2(1 - r^2)$$

- There are two critical points of this equation: $r = 0$ and $r = 1$

Limit Cycles

- The critical point $r = 1$ defines a circle $y_1^2 + y_2^2 = 1$ in the phase plane.
- Since

$$\frac{dr}{dt} = r(1 - r^2) \begin{cases} > 0 & r < 1 \quad \text{motion away from } r = 1, \\ < 0 & r > 1 \quad \text{motion toward to } r = 1. \end{cases}$$

- The circle $r = 1$ is the limiting trajectory for the system.
- Differentiating $y_1 = r\cos\varphi$, $y_2 = r\sin\varphi$, obtain

$$\underbrace{y_2\dot{y_1} - y_1\dot{y_2}}_{y_1^2+y_2^2=r^2} = -r^2\frac{d\varphi}{dt} \Rightarrow \frac{d\varphi}{dt} = -1 \Rightarrow \varphi = -t + t_0$$

Limit Cycles

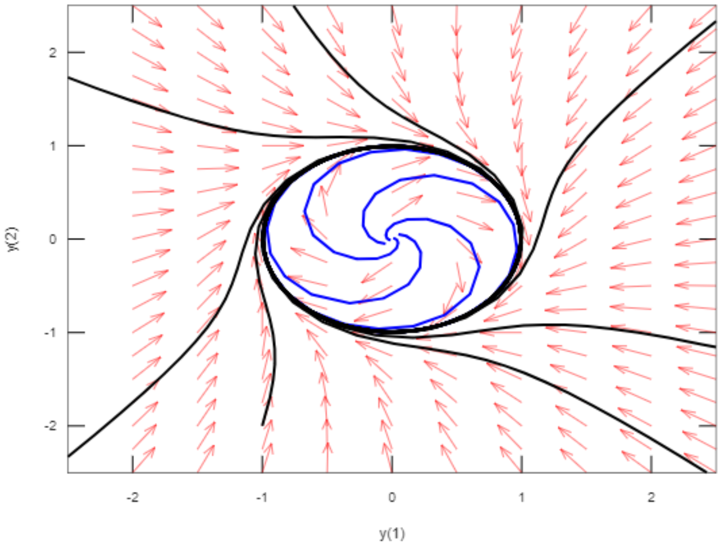
- Thus, one of the solutions of the system is

$$r = 1, \quad \varphi = -t + t_0.$$

  As t increases, a point satisfying the equations moves around the circle.
- Other solutions are

$$r = \frac{1}{\sqrt{1 + Ce^{-2t}}}, \quad \varphi = -t + t_0.$$

Limit Cycles



Limit Cycles

- A closed trajectory is called a limit cycle of all other nonclosed trajectories spiral toward it.
- The limit cycle is asymptotically stable if all other trajectories that start close to the closed trajectory spiral toward the closed trajectory as $t \to \infty$.
- The limit trajectory is not an equilibrium point, it is an orbit.
- However, closed trajectories may be isolated, and other solutions neither approach or depart from.

Limit Cycles

Theorem 1

If the functions $f_1(y_1, y_2)$, $f_2(y_1, y_2)$ have continuous first partial derivatives in a domain D of the y_1y_2-plane, then a closed trajectory of the system

$$\frac{dy_1}{dt} = f_1(y_1, y_2), \quad \frac{dy_2}{dt} = f_2(y_1, y_2)$$

must necessarily enclose at least one equilibrium point. If it encloses only one equilibrium point, the equilibrium point cannot be a saddle point. Use this statement as a negative test.

Theorem 2

If the functions $f_1(y_1, y_2)$, $f_2(y_1, y_2)$ have continuous first partial derivatives in a simply connected domain D of the y_1y_2-plane, and $\frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_2}$ is always positive or always negative in D, then there is no a closed trajectory of the system in D.

Limit Cycles

The Poincaré–Bendixson Theorem

Let R be a bounded region of the phase plane together with its boundary, and assume that R does not contain any equilibrium point of the system

$$\frac{dy_1}{dt} = f_1(y_1, y_2), \quad \frac{dy_2}{dt} = f_2(y_1, y_2).$$

If $C = [y_1(t), y_2(t)]$ is a trajectory that lies in R for some t_0 and remains in R for all $t > t_0$, then C is either itself a closed path or it spirals toward a closed path as $t \to \infty$. Thus in either case the system has a closed path in R.

Van der Pol Equation

- The equation

$$\ddot{u} - \mu(1 - u^2)\dot{u} + u = 0$$

  is called the van der Pol equation (it describes the current u in a triode oscillator)
- Let $x = u$, $y = \dot{u} \Rightarrow \dot{x} = y$, $\dot{y} = -x + \mu(1 - x^2)y$
- The Jacobian matrix at the equlibrium point $(0, 0)$ is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \Rightarrow \lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

- Does the van der Pol equation have limit cycles?

Van der Pol Equation

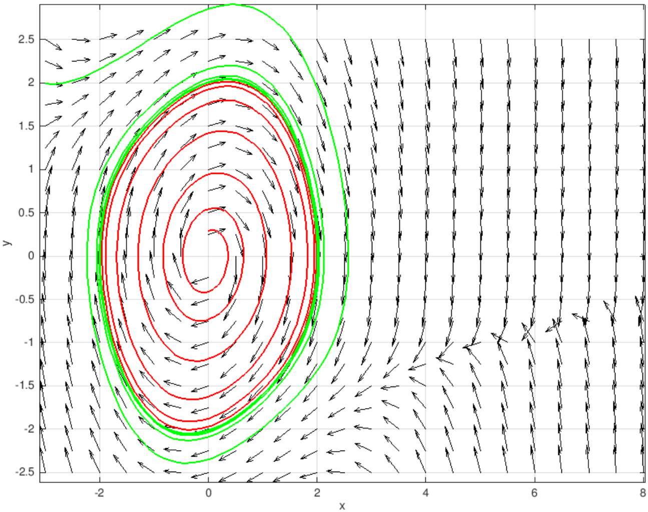
- The limit cycle should enclose $(0,0)$.
- Closed trajectories are not contained in $|x| < 1$:

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = \mu(1 - x^2) > 0 \quad \text{if } |x| < 1$$

- In polar coordinates,

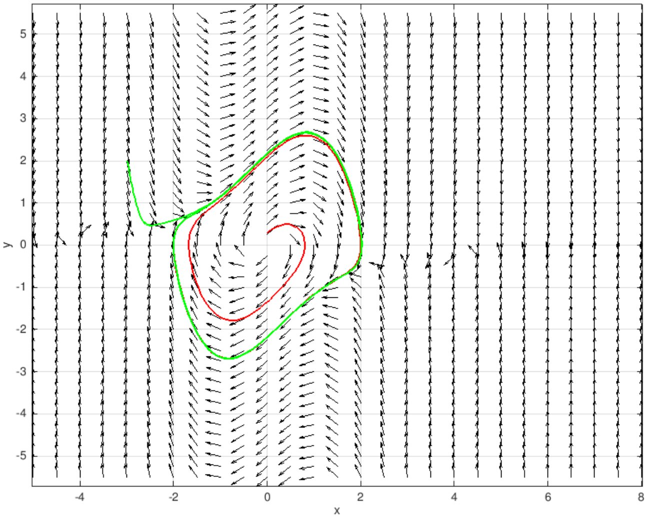
$$r' = \mu(1 - r^2 \cos^2\theta) r \sin^2\theta$$

Van der Pol Equation



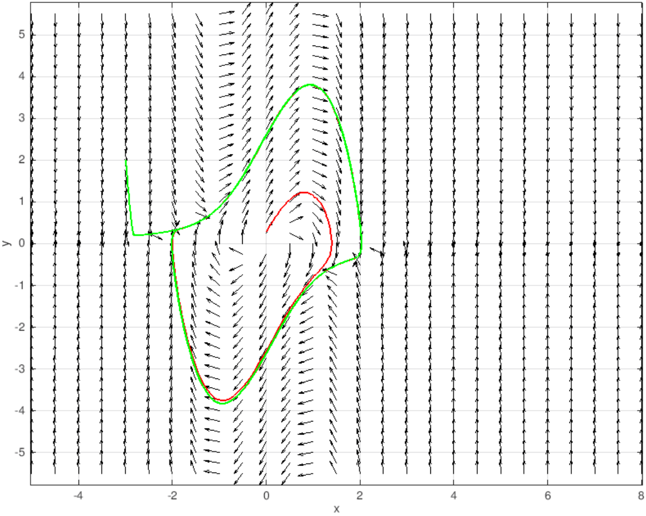
$\mu = 0.2$

Van der Pol Equation



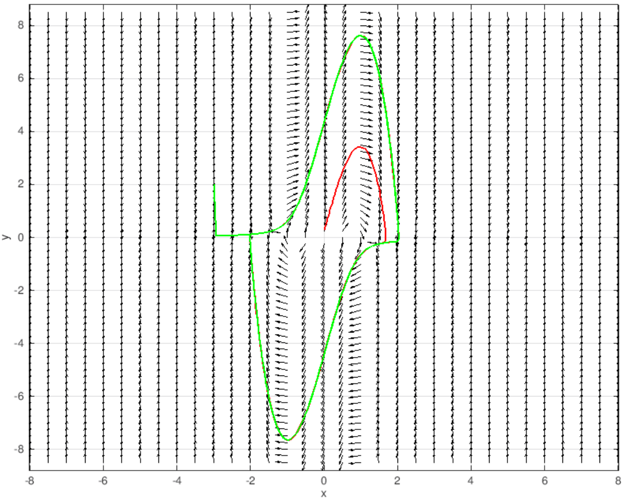
$\mu = 1$

Van der Pol Equation



$\mu = 2$

Van der Pol Equation



$\mu = 5$