$\mathbb{R}^n$: $d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$ $\qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ $\qquad (x,y) = \sum_{i=1}^{n} x_i y_i$

l_∞: $d_p(x,y) = \sup_{i=\overline{1,\infty}} |x_i - y_i|$ $\qquad \|x\|_\infty = \sup_{i=\overline{1,\infty}} |x_i|$

l_p: $d_p(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}$ $\qquad \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$ $\qquad (x,y) = \sum_{i=1}^{\infty} x_i \overline{y_i}$

$C[a,b]$: $d_c(f,g) = \max_{t \in [a,b]} |f(t) - g(t)|$ $\qquad \|f\|_C = \max_{t \in [a,b]} |f(t)|$

$d(f,g) = \int_a^b |f(t) - g(t)|\,dt$ $\qquad \|f\|_1 = \int_a^b |f(t)|\,dt$ $\qquad (f,g) = \int_a^b f(t)\overline{g(t)}\,dt$

The parallelogram law: $2(\|x\|^2 + \|y\|^2) = \|x+y\|^2 + \|x-y\|^2$, where $\|x\| = \sqrt{(x,x)}$

Young's inequality: $p,q > 1$, $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow |xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}$

Let $x \geq 0, y \geq 0$ and $f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy$

$f'(x) = x^{p-1} - y \Rightarrow f_{min} = f(y^{\frac{1}{p-1}}) = 0 \Rightarrow$

$\Rightarrow f(x) \geq 0 \quad \forall x \geq 0$

Hölder's inequality: $\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{1/q}$, $p,q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

Let $\sum_{i=1}^{n} |x_i|^p = \sum_{i=1}^{n} |y_i|^q = 1 \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| \leq \frac{1}{p} \sum_{i=1}^{n} |x_i|^p + \frac{1}{q} \sum_{i=1}^{n} |y_i|^q = 1$

$\Rightarrow a_i = \frac{x_i}{\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}}$, $b_i = \frac{y_i}{\left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}} \Rightarrow \sum_{i=1}^{n} |a_i b_i| \leq 1 \Rightarrow$ H. ineq for finite sums

Minkowski's inequality: $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

$\sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \leq \underbrace{\sum_{i=1}^{n} |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| |x_i + y_i|^{p-1}}_{\text{apply Hölder's inequality}}$

Cauchy - Schwarz Inequality

$$|(x,y)| \le \sqrt{(x,x)} \cdot \sqrt{(y,y)}$$

Let $t \in \mathbb{R} \Rightarrow 0 \le (x+ty, x+ty) = (x,x) + t\underbrace{(y,x)}_{\overline{(x,y)}} + t(x,y) + t^2(y,y) =$

$$= t^2(y,y) + 2\mathrm{Re}(x,y)t + (x,x) \ge 0$$

$$\Rightarrow \mathcal{D} = \sqrt{4(\mathrm{Re}(x,y))^2 - 4(y,y)(x,x)} \le 0$$

$$\Rightarrow \quad |\mathrm{Re}(x,y)| \le \sqrt{(x,x)} \cdot \sqrt{(y,y)}$$

Let $\tilde{x} = xe^{-\varphi i}$, where $\varphi = \arg(x,y)$, $(x,y) = |(x,y)|e^{\varphi i}$

$\Rightarrow (\tilde{x},\tilde{x}) = (xe^{-\varphi i}, xe^{-\varphi i}) = e^{-\varphi i}e^{\varphi i}(x,x) = (x,x)$

$(\tilde{x},y) = (xe^{-\varphi i}, y) = e^{-\varphi i}(x,y) \Rightarrow e^{-\varphi i}(x,y) = e^{-\varphi i}|(x,y)|e^{\varphi i} = |(x,y)|$

$\Rightarrow (\tilde{x},y) \in \mathbb{R} \Rightarrow \quad |(\tilde{x},y)| \le \sqrt{(\tilde{x},\tilde{x})} \cdot \sqrt{(y,y)}$

$$\Downarrow$$

$$|(x,y)| \le \sqrt{(x,x)} \cdot \sqrt{(y,y)}$$

$\Rightarrow$ In $\mathbb{R}^n$, $\left|\sum_{i=1}^n x_i y_i\right| \le \left(\sum_{i=1}^n x_i^2\right)^{1/2} \cdot \left(\sum_{i=1}^n y_i^2\right)^{1/2}$

X, Y be linear spaces.

perator $A: X \to Y$ is linear if $A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2$,

$\forall \alpha, \beta \in \mathbb{K}$

$\forall x_1, x_2 \in X$

, Y be NLS.

erator norm $\|A\|$ is defined by $\|A\| = \sup\limits_{\substack{x \in X: \\ \|x\|_X \le 1}} \|Ax\|_Y$

$\|A\| < +\infty$ then A is a bounded operator.

linear operator $A: X \to Y$, X, Y are <u>finite</u> dimensional linear spaces

iven by a matrix and vice-versa

$A: \mathbb{R}^n \to \mathbb{R}^m \Rightarrow y = Ax$, $A = (a_{ij})_{m \times n}$

bounded iff $\exists C > 0$: $\|Ax\|_Y \le C \cdot \|x\|_X$ and if $C = \|A\|$

a matrix $A = (a_{ij})_{n \times n}$,

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \max_{1 \le j \le n} \sigma_{max}, \quad \sigma_{max} = \max |\sigma_i|,$$ σ_i are square root of eigenvalues of $A^T A$

$\|A\|_2 = 5.4650$

$= 7$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$