vv256: Week 1-2. Introduction to solutions of differential equations.First-order ODEs.

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UM-SJTU Joint Institute

September 20, 2019

Outline

1. Lecture 1: Introduction
   - Course Info
   - Differential Equations and Their Solutions
   - Classification of DE
2. Lecture 2: Separable and linear ordinary differential equations
   - Separable Equations
   - Linear equations
3. Lecture 3: Other first-order ordinary differential equations
   - Homogeneous Polar Equations
   - Bernoulli Equations
   - Riccati Equations
   - Exact Equations

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Recommended Literature

- George F. Simmons. Differential equations with applications and historical notes, third edition, CRC Press, 2017.
- William E.Boyce, Richard C.DiPrima, Douglas B. Meade. Elementary Differential Equations and Boundary Value Problems, 11th edition, Wiley, 2017.
- Kam Tim Chau. Theory of differential equations in engineering and mechanics, CRC Press, 2017.
- Yuefan Deng. Lectures, problems and solutions for ordinary differential equations, second addition, World Scientific, 2018.
- Marcelo Epstein. Partial differential equations. Mathematical techniques for engineers, Springer, 2017.

Course Assessment

Final Grade (100%)

Assignments (20%)+Midterm Exam I (20%) + Midterm Exam 2 (20%)+Final Exam (40%)

Differential Equations and Their Solutions

Physical (chemical, biological etc) process $\to$ mathematical model $\to$ differential equation

Definition. A differential equation (DE) is a relation that contains an unknown function, say y, and one or more of its derivatives.

If $y = y(x)$ then the general form of an ordinary differential equation (ODE) is

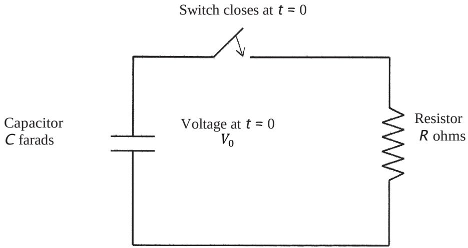
$$F(y, y', y'', \ldots, y^{(n)}) = 0.$$

If $y = y(x_1, x_2)$ then the general representation of a partial differential equation (PDE) is

$$T\left(y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial^2 y}{\partial x_1 \partial x_2}, \frac{\partial^2 y}{\partial x_1^2}, \frac{\partial^2 y}{\partial x_2^2}, \ldots, \frac{\partial^n y}{\partial x_1^n}, \frac{\partial^n y}{\partial x_2^2}, \frac{\partial^n y}{\partial x_1^{n-1} \partial x_2}\right) = 0.$$

Example 1: RC Circuit

Consider the circuit with resistor and capacitor as shown below. Kirchhoff's voltage law $\rightarrow$ the sum of the voltage drops around a closed circuit is zero.



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It is the initial-value problem (IVP) with the solution

$$V(t) = V_0 e^{-\frac{1}{RC}t}.$$

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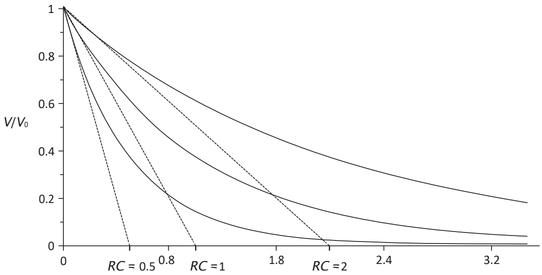
Graph of $V(t) = V_0 exp(-\frac{t}{RC})$ for $RC = 0.5, 1,$ and 2.



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What else? There is also a force due to the resistance of the ambient medium to motion, its proportional to the velocity and acts in upward (negative direction), it equals $-\gamma g$.

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$$m\frac{dv}{dt} = mg - \gamma v$$

Example 2: Free fall in gravity

Exercise: Let an object of mass $m = 10$ dropped into a liquid-filled reservoir reach the bottom with velocity $v_b = 24.5$. If $g = 9.8$ and the motion resistance coefficient of the liquid is $\gamma = 2$, compute the depth of the reservoir.

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The position $y(t) = v'(t)$ and $v(0) = v_0 = 0$.

Example 3: Population growth

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If the size of the population at $t = 0$ is $P(0) = P_0$ then $P(t) = P_0 e^{(\beta-\delta)t}$.

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If $\beta = \beta(t), \delta = \delta(t)$ then we obtain the same model with the solution

$$P(t) = P_0 e^{\int_0^t (\beta(\tau) - \delta(\tau))\, d\tau}, \quad t > 0.$$

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Exercise: Suppose that the annual birth and death rates in a population of the initial size $P_0 = 100$ are $\beta(t) = 2t + 1$ and $\delta(t) = 4t + 4$. Determine how long it will take for the size of the population to decrease to 40.

Example 4: Radioactive decay

If $k = const > 0$ is the rate of decay of a radioactive isotope (the number of decaying atoms per unit of atom 'population' per unit time), then the approximate change ΔN in the number $N(t)$ of atoms during a very short time interval Δt is

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After separating the variables, integrating, and using an IC of the form $N(0) = N_0$, we find the solution

$$N(t) = N_0 e^{-kt}, \quad t > 0.$$

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$$\frac{1}{2} N_0 = N_0 e^{-kt^*}.$$

Divide both sides by N_0 and take logarithms to find that

$$t^* = \frac{ln 2}{k}.$$

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Newton's law of cooling $\rightarrow$ the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature.
Let $T(t)$ be the temperature at time $t > 0$ of an object immersed in an outside medium of temperature θ and k is the heat transfer coefficient of the object material.

$$T' = -k(T - \theta), \quad k > 0$$

Why do we need negative sign on the right-hand of the equation?

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Problem: Take a pot with boiling water and put it in a sink with running water with the constant temperature $5^\circ C$. The temperature of hot water goes to $60^\circ C$ in 10 minutes. How long will it take to cool hot water to $20^\circ C$?

Classification of Differential Equations

- ODEs and PDEs (number of independent variables)
- Order of a DE
- Linear and nonlinear DEs
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- Initial-value problems (propagation problems)
- Boundary-value problems (equilibrium problems)

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Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous functions in some open rectangle $R = \{(x, y) \colon |x - x_0| < a, |y - y_0| < b\}$, $a, b > 0$, and hence, there exist $K, L > 0$ such that

$$(a)\, |f(x, y)| \leq K, \quad (b)\, \left|\frac{\partial f}{\partial y}\right| \leq L \quad \forall (x, y).$$

Then the IVP has a unique solution in the interval $|x - x_0| \leq \alpha$, where $\alpha = \min\{a, \dfrac{b}{K}\}$.

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where f and g are given functions.
Remark. If there is any value y_0 such that $g(y_0) = 0$, then $y = y_0$ is a solution (equilibrium solution). To find other (non-constant) solutions, we assume that $g(y) \neq 0$.

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$$\frac{1}{g(y)}dy = f(x)dx, \quad \text{and}$$

- Integrate each side with respect to its variable to obtain

$$G(y) = F(x) + C, \tag{2}$$

where F and G are anti-derivatives of f and $1/g$, and C is an arbitrary constant.

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- The solution of (2) also satisfies (1):

$$\frac{d}{dx}G(y(x)) = \frac{d}{dy}G(y)\frac{dy}{dx} = \frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

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where $C_1 = \pm e^C$ is a non-zero constant (it generates all non zero solutions). If we allow $C_1 = 0$ then the solution $y = C_1 e^{-4x^2}$ includes also the equilibrium solution.

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Integrate by part on the right-hand side:

$$-e^{-y} = 2xe^{-2x} - \int 2e^{-2x}dx = (2x+1)e^{-2x} + C.$$

Change the signs of both sides, take logarithms to produce the general solution

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$$y(x) = -\ln[-(2x+1)e^{-2x} - C].$$

Since $y(0) = 0$, $C = -2$ and the solution of the IVP is

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Answer: $y = x - 2$.

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$$\int \frac{2y\,dy}{y^2+2} = \int \frac{dx}{x+1}.$$

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$$\int \frac{2y\,dy}{y^2+2} = \int \frac{dx}{x+1}.$$

$$\ln(y^2+2) = \ln|x+1| + C, \quad C = const.$$

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Applying IC, we obtain $y^2 = x - 1$, or $y = \pm(x-1)^{1/2}$. However, $y = -(x-1)^{1/2}$ does not satisfy the IC.

Separable equations: Example 4

Consider the IVP

$$(5y^4 + 3y^2 + e^y)y' = \cos x, \quad y(0) = 0.$$

Applying separation of variables, we obtain the family of all solution curves for the differential equation.

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$$y^5 + y^3 + e^y = \sin x + C, \quad C = const.$$

The IC yields $C = 1$, so the solution curve passing through the point $(0, 0)$ has the equation

$$y^2 + y^3 + e^y = \sin x + 1.$$

Separable equations: Exercises

Solve the following IVP:

1. $y' = -4xy^2$, $y(0) = 1$
2. $y' = (3-2x)y$, $y(2) = e^6$
3. $y' = (x-3)(y^2+1)$, $y(0) = 1$
4. $3(x^2+2)y^2y' = 4x$, $y(1) = (\ln 9)^{1/3}$
5. $y' = xe^{2x}/(y^4+2y)$, $y(0) = -1$

Linear Equations

A general form of a first-order linear DE is

$$y' + p(t)y = q(t),$$

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- We need just one such function, let $C = 1$ and hence,$\mu = e^{\int p\,dt}$.
- Substitute μ into $\mu y' + \mu p y = \mu q \Rightarrow \quad (\mu y)' = \mu q$ and,
- finally find

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)\,dt + C\right).$$

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Integrate $(\mu y)' = \mu q$ from t_0 to t, then

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(\tau)q(\tau)\, d\tau$$

and

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(\tau)q(\tau)\, d\tau + \mu(t_0)y(t_0)\right).$$

Linear Equations: Example 1

Consider the IVP

$$y' - 3y = 6,\ y(0) = -1.$$

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$$\mu(t) = e^{\int (-3)\, dt} = e^{-3t}, \quad y(t) = e^{3t}\left(\int 6e^{-3t}\, dt + C\right) = Ce^{3t} - 2.$$

Applying the IC, we obtain $C = 1$ and hence, $y(t) = e^{3t} - 2$.

Linear Equations: Example 2

Consider the IVP

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$$= \frac{2}{3}(t^2+1)^2.$$

Linear Equations: Exercises

Solve the following IVP:

1. $(t-1)y' + y = (t-1)e^t$, $y(2) = 3$ Ans: $\dfrac{(t-2)e^t + 3}{t-1}$
2. $y' + 4y + 16 = 0$, $y(0) = -2$ Ans: $y = 2e^{-4t} - 4$
3. $ty' + 4y = 6t^2$, $y(1) = 4$ Ans: $t^2 + 3t^{-4}$
4. $y' = (2+y)\sin t$, $y(\pi/2) = -3$ Ans: $-e^{-\cos t} - 2$
5. $2ty' - y = 2/\sqrt{t}$, $y(1) = 1$

Outline

1. Lecture 1: Introduction
   - Course Info
   - Differential Equations and Their Solutions
   - Classification of DE
2. Lecture 2: Separable and linear ordinary differential equations
   - Separable Equations
   - Linear equations
3. Lecture 3: Other first-order ordinary differential equations
   - Homogeneous Polar Equations
   - Bernoulli Equations
   - Riccati Equations
   - Exact Equations

Homogeneous Polar Equations

What is a homogeneous polar equation?

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$$y'(x) = f\left(\frac{y}{x}\right), \quad x \neq 0,$$

where f is a given one-variable function.

- Make the substitution $y(x) = xv(x)$. What is the derivative of y?

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- The new unknown function v satisfies the DE

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What is the type of this equation?

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- For $f(v) - v \neq 0$ $\quad \displaystyle\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$
- What happens when $f(v) - v = 0$? $v' = 0$ The equation has singular solutions of the form $y = cx$, $c = const$.

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Is it a homogeneous polar equation?

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Is it a homogeneous polar equation? Yes!

$$y' = 1 + 2\frac{y}{x}$$

Then $f(v) = 1 + 2v$ and $f(v) - v = v + 1$. For $v \neq -1$,

$$\int \frac{dv}{v+1} = \int \frac{dx}{x} \quad \Rightarrow \quad v + 1 = Cx, \; C = const \neq 0.$$

$$y(x) = Cx^2 - x$$

The case $v = -1$ is equivalent to $y = -x$ and it is covered by the solution above if we allow $C = 0$. Applying the IC we obtain $y(x) = 4x^2 - x$.

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Is f well-defined?

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Apply the IC and find that $C = 4 + \ln 2$. Therefore, the solution is

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$$2\frac{y}{x} + \ln\left|\frac{y}{2x^2}\right| = 4.$$

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Is $\bar{y}(x)$ a solution?
This type of 'solution' would be caused by the algebraic structure of the equation and not by its differential nature. In general, correctly formulated mathematical models are not expected to exhibit such anomalies.

Homogeneous Polar Equations: Remark

Consider the IVP

$$(xy-3x^2)y' = 2y^2 - 5xy - 3x^2, \quad y(1) = 3.$$

Then

$$f_1(x,y) = xy - 3x^2 = x(y-3x),$$
$$f_2(x,y) = 2y^2 - 5xy - 3x^2 = (x+2y)(y-3x),$$

and the function $y(x) = 3x$ satisfies it. We need to clean the DE algebraically, that is, assume that $y \neq 3x$ and divide both parts by $y - 3x$. We obtain the IVP $xy' = x + 2y$, $y(1) = 3$ with the solution $y = 4x^2 - x$.

The discarded function $y(x) = 3x$ also satisfies the prescribed IC, so, at first glance, it would appear that the IVP does not have a unique solution. This situation is unacceptable in mathematical modeling, where $y(x) = 3x$ is normally considered a spurious 'solution' and ignored.

Bernoulli Equations

The general form of a Bernoulli equation is

$$y' + p(t)y = q(t)y^n, \quad n \neq 1.$$

Is it a linear equation?

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Is it a linear equation? Yes. $\Rightarrow$ Apply the known methods to solve it.

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we have

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Exercise: Solve this IVP and the IVP for the Bernoulli equation.
Answer: $y(t) = (t^2 + 3t^{-4})^2$.

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where q_0, q_1, and q_2 are given functions, with $q_0, q_2 \neq 0$.

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We obtain the linear equation $\quad w' + (q_1 + 2q_2y_1)w = -q_2.$

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Consider the IVP for the Riccati equation

$$y' = -1 - t^2 + 2(t^{-1} + t)y - y^2, \; y(1) = 10/7$$

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Guess a particular solution.

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$$y = y_1 + \frac{1}{w} = t + \frac{3t^2}{t^3 + 3C}.$$

$$y(1) = 10/7 \quad \Rightarrow \quad C = 2 \quad \Rightarrow \quad y(t) = \frac{t^4 + 3t^2 + 6t}{t^3 + 6}.$$

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Therefore,

$$y(t) = \left(1 + \frac{1}{t + C}\right)\cos t, \quad C = -1, \quad y(t) = \frac{t\cos t}{t - 1}.$$

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We can integrate either A. the first equation with respect to x or B. the second equation with respect to y.

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where $g(y)$ is an arbitrary function of y.

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What is the value of C? $C = 1$.

$$xy^2 - 2x^2y^3 + 2x = 1$$

Exercise: Try the alternative B.

Exercises

Solve the following IVP:

1. $6xy^{-1}+8x^{-3}y^3+(4y-3x^2y^{-2}-12x^{-2}y^2)y'=0,\ y(1)=1/2$
   Ans: $3x^2y^{-1}-4x^{-2}y^3+2y^2=6$
2. $x\sin(2y)-3x^2+(y+x^2\ cos(2y))y'=0,\ y(1)=\pi$
   Ans: $y^2+x^2\sin 2y-2x^3=\pi^2-2$
3. $y'+y=-y^3,\ y(0)=1$ Ans: $y(t)=(2e^{2t}-1)^{-1/2}$
4. $y'-3y=-e^{-4t}y^2,\ y(0)=-1/2$
5. $xy'=3y-x,\ y(1)=1$ Ans: $y(x)=(x^3+x)/2$
6. $(x+y)y'=2x-y,\ y(1)=-1-\sqrt{2}$
7. $y'=t^{-2}+3t^{-1}-(4t^{-1}+3)y+2y^2,\ y(1)=5/2,\ y_1(t)=1/t$
   Ans: $y(t)=(3t+2)/(2t)$
8. $y'=2-4t-4t^2e^{-t}+(2+4te^{-t})ye^{-t}y^2,\ y(1)=2+e,\ y_1(t)=2t$