vv256: Fourier Series.

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Inner product

Let X be a linear space.
A complex-valued function $(\cdot, \cdot)\colon X \times X \to \mathbb{C}$ satisfying

1. $(x, x) \geq 0 \quad \forall x \in X$,
2. $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z), \quad \forall x,\ y,\ z \in X, \forall \alpha, \beta \in \mathbb{K}$,
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Examples: $\mathbb{R}^n : (x,y) = \sum_{i=1}^{n} x_i y_i$, $\mathbb{C}^n : (x,y) = \sum_{i=1}^{n} x_i \bar{y}_i$,

$$l_2 : (x,y) = \sum_{i=1}^{\infty} x_i \bar{y}_i, \quad \underbrace{C[a,b]}_{\text{incomplete}} : (x,y) = \int_a^b x(t) y(t)\, dt$$

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4. Parallelogram identity

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$$

Hint: Use it to verify if a NLS is an inner product space.

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A system $\{e_i\}$ is said to be complete if the equality $(e_i, x) = 0$ for all $i = 1..\infty$, implies that $x = 0$.

Fourier series

A functional series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

is called a Fourier series of the function $f(x)$.

It is a real form of the Fourier series.

Theorem 1

A Fourier series of a periodic ($\omega = 2\pi$), piecewise continuous bounded function $f(x)$ converges at all points $x \in \mathbb{R}$ and its sum equals

$$S(x) = \frac{f(x-0) + f(x+0)}{2}.$$

Remark: $S(x) = f(x)$ at the points where $f(x)$ is continuous, and $S(x)$ equals to the average of left-hand side and right-hand side limits at the points where $f(x)$ has jump discontinuities.

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Find a Fourier series expansion of the periodic ($T = 2\pi$) function

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Substitute the obtained coefficients into the Fourier series:

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi(2n-1)^2}\cos(2n-1)x + \frac{(-1)^{n-1}}{n}\sin nx\right)$$

The series converges to $f(x)$ at all $x \neq (2n-1)\pi$.

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The sum of the Fourier series equals $(\pi + 0)/2 = \frac{\pi}{2}$ at the points $x = (2n-1)\pi$.

A Fourier series of a periodic function $y = f(x)$ with $T = 2l$ has the following representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n}{l} x + b_n \sin \frac{\pi n}{l} x \right)$$

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Even and odd functions

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- If a periodic function $y = f(x)$ is even then its Fourier series is a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n}{l}x$$

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- $a_n = \begin{cases} -8/(\pi n)^2 & n = 2k - 1 \\ 0 & n = 2k \end{cases}$

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$$a_0 = \frac{2}{2}\int_0^2 |x|\, dx = 2$$

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- $a_n = \begin{cases} -8/(\pi n)^2 & n = 2k-1 \\ 0 & n = 2k \end{cases}$

The Fourier series

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi}{2}x$$

equals to the function $f(x)$ on $[-2, 2]$.

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- The sum $S(x)$ is $f(x)$ in (a, b), $S(a) = f(a)/2$, $S(b) = f(b)/2$

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$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{\pi n}{2} x$$

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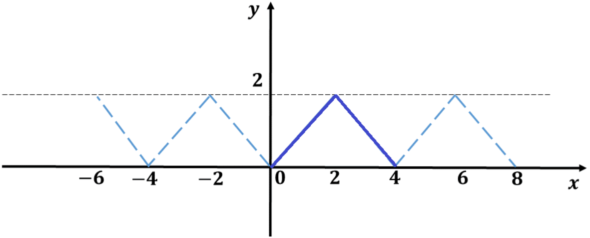
Find the Fourier cosine series of the function shown below



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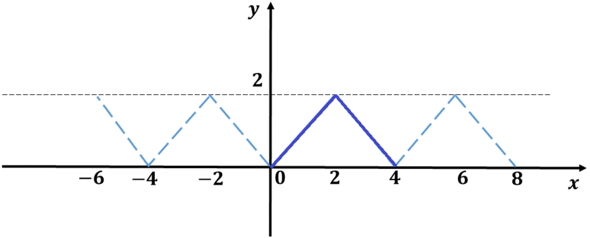

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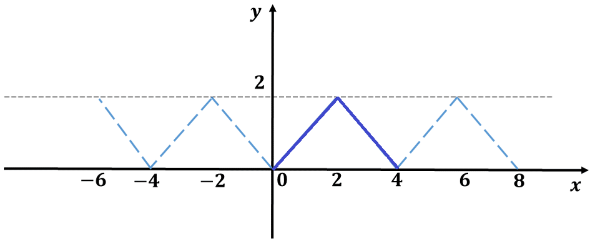

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$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi}{2}x$$

Remark

- A Fourier series converges to the value of the corresponding function at the points of continuity
  $\Rightarrow$ we may use Fourier series to find sums of series.
- For example, let $x = 2$ in the Fourier series (Example 5):

$$2 = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example 6

Expand the function $y = x^2$ in cosine Fourier series on the interval $[0, \pi]$, and use thus obtained series to find sums of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$$

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$$= -\frac{4}{\pi n} \left(-\frac{x}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right) = \frac{4(-1)^n}{n^2}$$

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- Take $x = \pi$:

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Gibbs Phenomenon

- Explore the behavior of the Fourier series at the points of discontinuity of the square wave function

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- The Fourier series (over/under) shoots the actual value of $x(t)$ at points of discontinuity.

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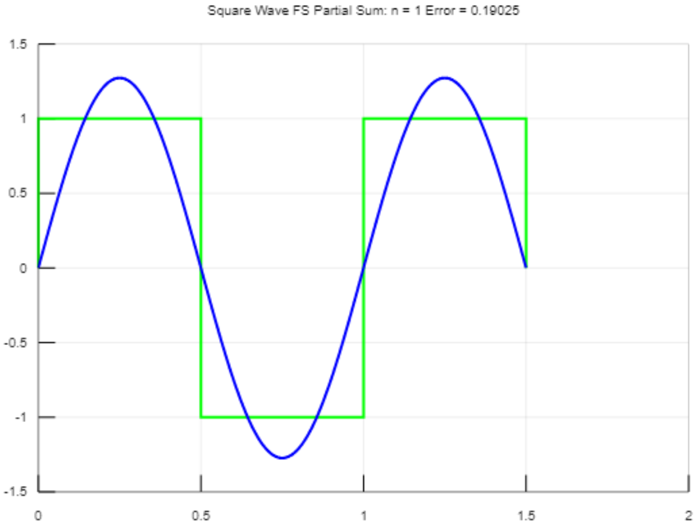

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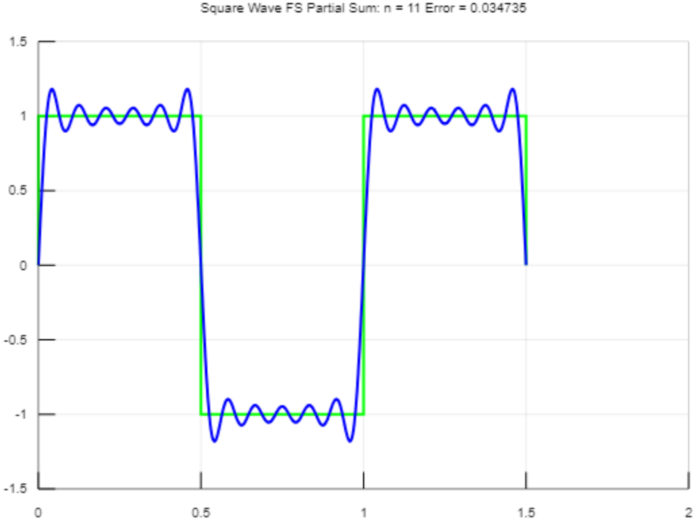

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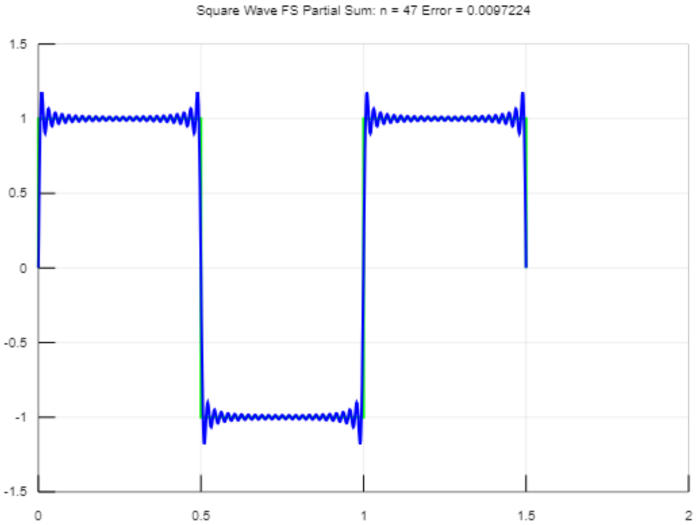

The Complex Exponential Form

- Consider a piecewise continuous periodic ($T = 2\pi$) real-valued function $f(x)$. It's Fourier series expansion is

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- The system $\{\frac{e^{inx}}{\sqrt{2\pi}}\}$ is orthonormal check it! $\Rightarrow$ the obtained series is also a partial case of the general Fourier series

$$f(x) = \sum_{i=1}^{\infty} (f(x), e_i)\, e_i, \quad \{e_i\} \text{ is orhonormal and complete}$$

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- Express the coefficients of the real form of the Fourier series from the coefficients of the complex form

$$\begin{aligned} a_n - ib_n &= 2c_n, \\ a_n + ib_n &= 2c_{-n} \end{aligned} \quad \Rightarrow$$

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$$f(x) = c_0 + \sum_{n=1}^{\infty} 2|c_n|cos(nx + \varphi_n)$$

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- Express the coefficients of the real form of the Fourier series from the coefficients of the complex form

$$\begin{array}{l} a_n - ib_n = 2c_n, \\ a_n + ib_n = 2c_{-n} \end{array} \Rightarrow \begin{array}{l} a_n = c_n + c_{-n} \\ ib_n = c_{-n} - c_n \end{array}$$

- If $f(x)$ is real then $c_{-n} = \overline{c_n} \Rightarrow$

$$c_n = |c_n| e^{i\varphi_n},\ c_{-n} = |c_n| e^{-i\varphi_n}$$

- The real form becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} |c_n| \left(e^{i(nx+\varphi_n)} + e^{-i(nx+\varphi_n)} \right)$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} 2|c_n| cos(nx + \varphi_n)$$

- Derive that

$$2|c_n| = \sqrt{a_n^2 + b_n^2},\ \varphi_n = -\tan^{-1}(b_n/a_n)$$

Multiplication

- Can we multiply Fourier series? What happens then? Let

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n e^{inx}, \, g(x) = \sum_{n=-\infty}^{+\infty} g_n e^{inx}$$

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- The Fourier coefficients

$$h_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{+\infty} f_k e^{ikx} g(x)e^{-inx}\,dx =$$

$$= \sum_{k=-\infty}^{+\infty} f_k \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-k)x} g(x)\,dx}_{g_{n-k}} = \sum_{k=-\infty}^{+\infty} f_k g_{n-k}$$

Parceval's Identity

- Notice that
$$\overline{f_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)e^{-inx}}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-i(-n)x}\,dx = g_{-n}$$

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Exercise: Derive Parseval's identity in the form
$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Mean-Square Error Approximation

- Consider a problem of approximation a periodic function with the Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n e^{inx}$$

by a finite sum, say

$$f_N(x) = \sum_{n=-N}^{N} \alpha_n e^{inx}$$

- What approximation is a good approximation? How can we define an approximation error?
- The mean-square error ε is

$$\varepsilon_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f_N(x)|^2\, dx = \underbrace{\sum_{n=-N}^{N} |f_n - \alpha_n|^2 + \sum_{|n|>N} |f_n|^2}_{\text{we applied Parseval's identity}}$$

Mean-Square Error Approximation

- How to minimize the error? Let $\alpha_n = f_n$ for all $|n| \le N$.

$$\varepsilon_N = \sum_{|n|>N} |f_n|^2 \qquad = \underbrace{\sum_{n=-\infty}^{\infty} |f_n|^2 -}_{\text{apply Parseval's identity}} \qquad \sum_{n=-N}^{N} |f_n|^2 =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \sum_{n=-N}^{N} |f_n|^2 \to 0 \; as \; N \to \infty$$