vv256: Week 2-3.
Intervals of Existence. Direction fields. Autonomous equations.Singular solutions. Linear spaces.

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Outline

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1. Lecture 4: Intervals of Existence of a solution. Direction fields. Autonomous equations.
   - Direction fields
   - Intervals of Existence of a solution.
   - Autonomous Equations
2. Lecture 5: Intervals of Existence. Gronwall's and Bihari's inequalities.
3. Lecture 6: Implicit first-order ODEs. Singular solutions.
   - Implicit first-order ODEs
   - Singular Solutions

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- Choose a lattice/grid in the (t, y)-plane and draw short segments of the line with the slope $f(t, y)$ at each node.
- Thus obtained picture is called a direction/slope field.

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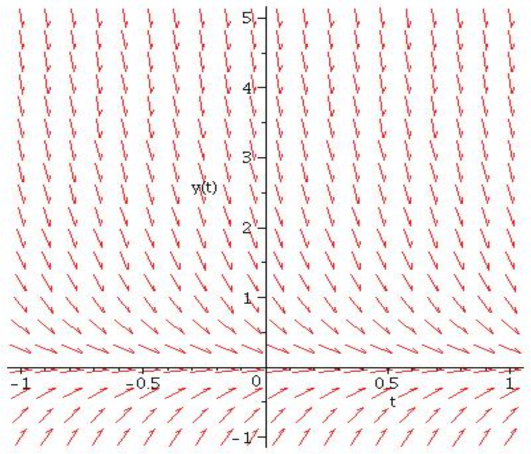

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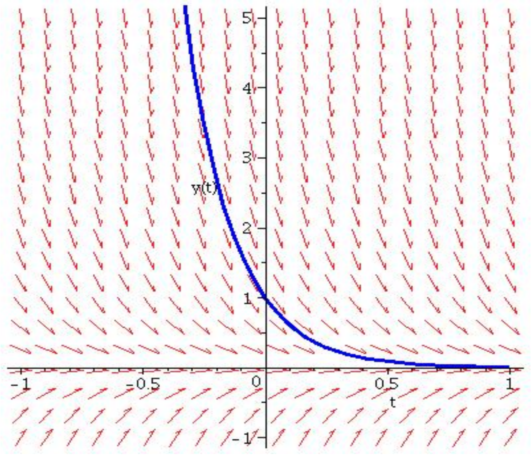

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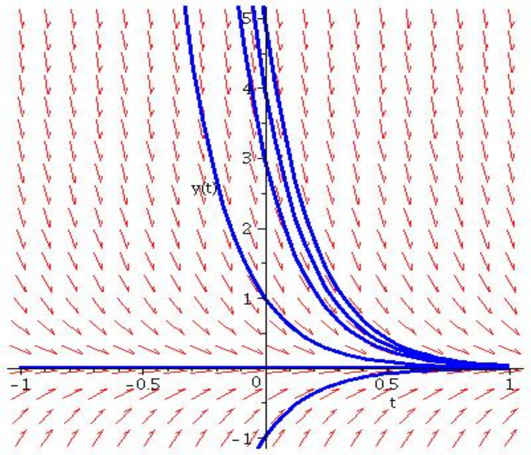

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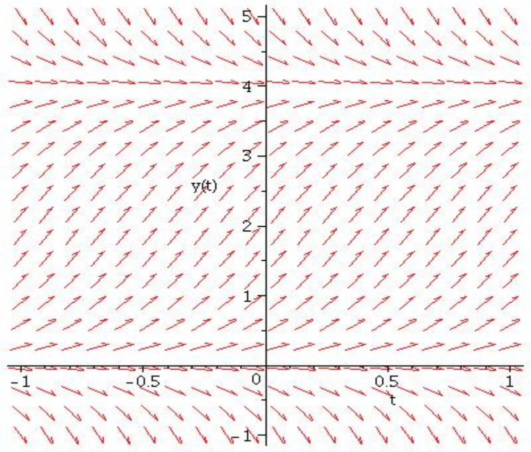

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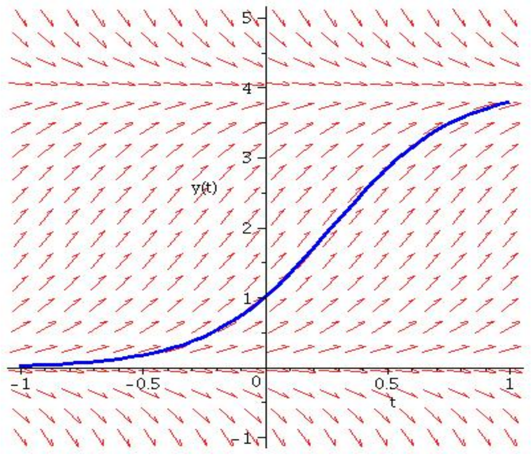

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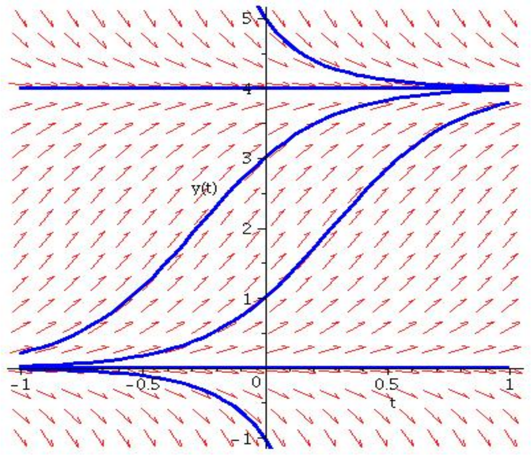

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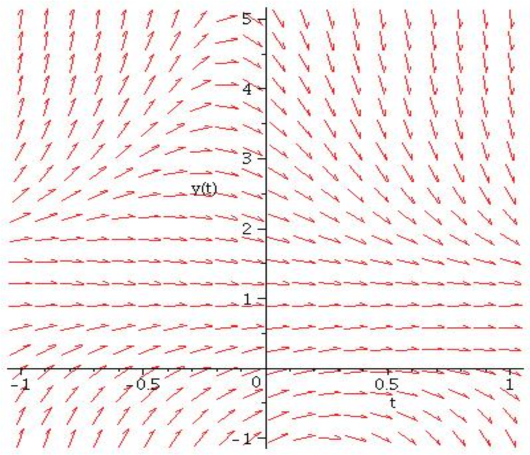

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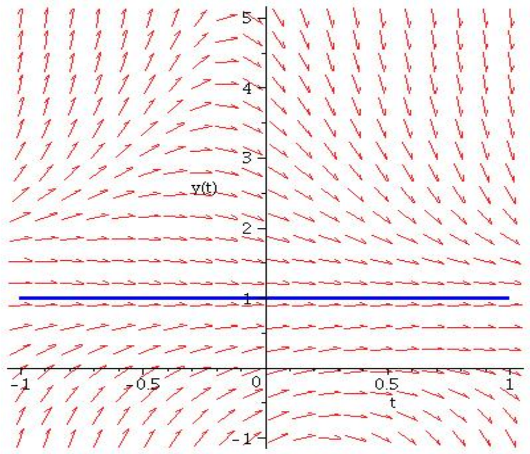

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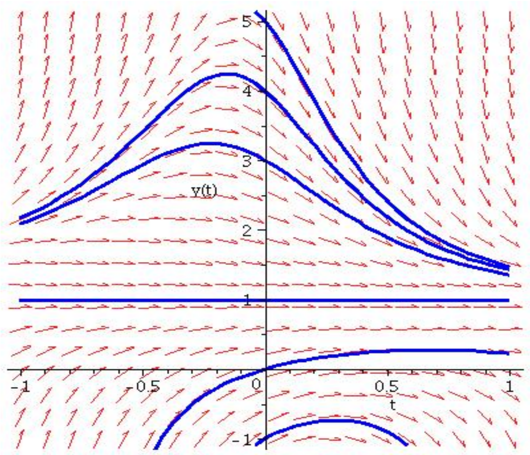

Intervals of existence

Theorem: Let J be an open interval of the form $a < t < b$ and t_0 be a point in J. Consider the IVP

$$y' + p(t)y = q(t),\ y(t_0) = y_0,$$

where y_0 is a given initial value.
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If p and q are continuous on J, then the IVP has a unique solution on J for any y_0.
Definition: The largest open interval J on which an IVP has a unique solution is called the maximal interval of existence for that solution.

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$$p(t) = \frac{1}{t+1},\ q(t) = \frac{e^{2t}}{t+1} \text{ and hence,}$$

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The IC is given at $t_0 = 2 > -1$, therefore the IVP has a unique solution in the interval $-1 < t < +\infty$.

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- If $t_0 = y_0 = 0$ then the IVP has infinitely many solutions, given by $y(t) = Ct$ with any constant C on the entire real line.

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Theorem. Consider the IVP

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where f, f_y are continuous in an open rectangle R functions, $R = \{(t, y) : a < t < b,\ c < y < d\}$. If $(t_0, y_0) \in R$ then the IVP has a unique solution in some open interval J of the form $t_0 - h < t < t_0 + h$ contained in the interval $a < t < b$.
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Definition: We say that a mathematical model is well-posed (correctly formulated) if there exists its unique solution.

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$t_0 = 2 \ \Rightarrow$ the maximal interval of existence $1 < t < +\infty$.

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- This function is well-defined on $-\infty < t < -1$ and $0 < t < +\infty$.
- Thus, there are two different solutions with the maximal interval of existence $0 < t < +\infty$. Explain!

Intervals of existence: Exercises

Find the largest open interval on which the conditions of existence and uniqueness are satisfied, without solving the IVP itself.

- $(2t+1)y' - 2y = sint$, $y(0) = -2$
- $(t^2 - 3t + 2)y' + ty = e^t$, $y(3/2) = -1$
- $(t^2 - t - 2)^{1/2}y' + 3y = (t-3)^{1/2}$, $y(4) = 1$

Solve the DE with each of the given ICs and find the maximum interval of existence of the solution

- $y' = 4ty^2\, y(0) = 2$, $y(-1) = -2$, $y(3) = -1$
- $(y-2)y' = t$, $y(2) = 3\, y(-2) = 1$, $y(0) = 1$, $y(1) = 2$
- $y' = 4(y-1)^{1/2}$, $y(0) = 5$, $y(-1) = 2$

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What are limitations of this model?

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What are limitations of this model?
We need to make the model realistic!

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$f(y) = r(1-y/B)y \Rightarrow y = 0,\ y = B \leftarrow$ critical points/equilibrium sol.

Population with logistic growth

$$u(y) = r - ay, \quad a = const > 0.$$

We obtain the logistic equation of the form

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The solution of the logistic equation is

$$\frac{y}{B - y} = Ce^{rt}, \quad C = const \neq 0.$$ Obtain it!

Population with logistic growth

If $y(0) = y_0 > 0$ then the solution is

$$y(t) = \frac{y_0 B}{y_0 + (B - y_0)e^{-rt}}.$$

Is the function $y(t)$ well-defined?

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What does it mean?
B is the size of the largest population that the environment can sustain long term $\Rightarrow$ B is the environmental carrying capacity.

Population with logistic growth

An equilibrium solution y_0 is called

- **stable** if any other solution starting close to y_0 remains close to y_0 for all time.
- **asymptotically stable** if it is stable and any solution starting close to y_0 becomes arbitrarily close to y_0 as t increases.

An equilibrium solution that is not stable is called **unstable**.

Are equilibrium solutions $y_0 = 0$ and $y_0 = B$ stable or unstable?

$y_0 = B$ is asymptotically stable and $y_0 = 0$ is unstable.

Population with logistic growth: Example

Consider the IVP

$$y' = 400y - 4y^2, \quad y(0) = y_0.$$

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Sketch the graph of the obtained solution:

- $y' > 0$ for $0 < y < 100$, and $y' < 0$ when $y > 100$,
- $y'' = 32y(50 - y)(100 - y)$ and hence, $y'' > 0$, $0 < y < 50$, $y > 100$ and $y'' < 0$, $50 < y < 100$,

and let y_0 be $25, 75, 125$ to obtain the particular solutions

$$y_1(t) = \frac{100}{1 + 3e^{-400t}}, \quad y_2(t) = \frac{300}{3 + e^{-400t}}, \quad y_3(t) = \frac{500}{5 - e^{-400t}}$$

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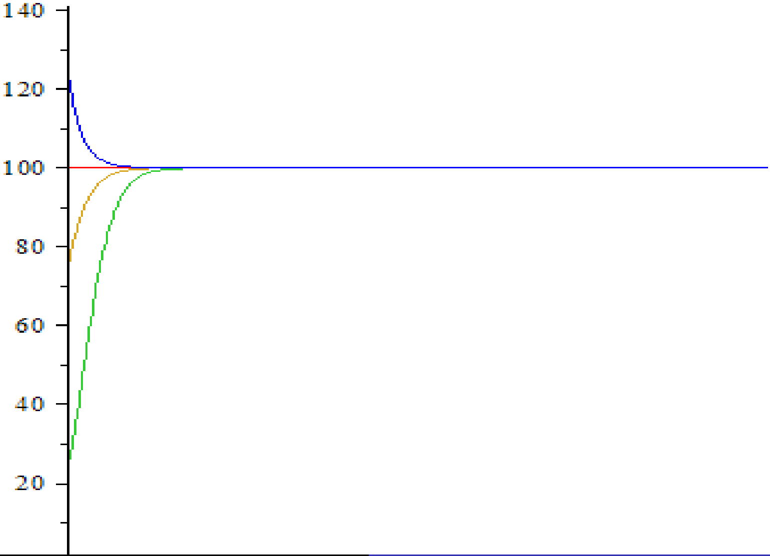

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Any IC of the form $y(0) = y_0$ yields $C = (y_0 - 2)/(y_0 - 5)$ and

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$$y(t) = \frac{5(y_0 - 2) - 2(y_0 - 5)e^{-12t}}{y_0 - 2 - (y_0 - 5)e^{-12t}}.$$

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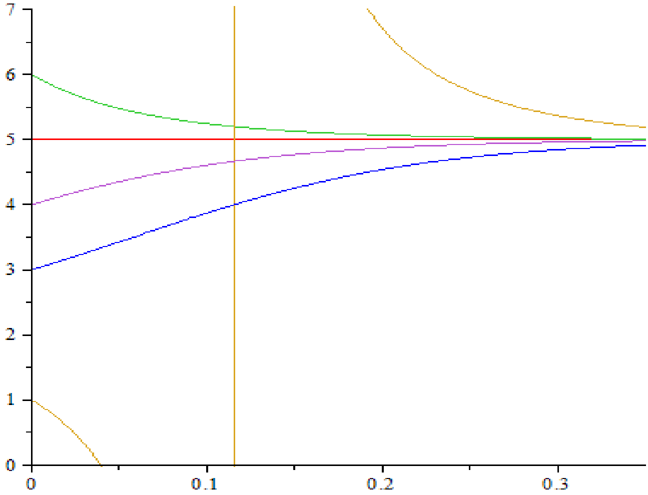
Populations starting with a size less than 2 are not large enough to survive and die out in finite time. Check what happens with the solution at other points t!!! We plot the particular solutions for $y_0 = 1, 3, 4, 6$.

Logistic population with harvesting: Example



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What happens when $y_0 < B$ and $y_0 > B$?

- If $y_0 < B$, then the solution is well-defined for all $t > 0$ and $\lim\limits_{t\to\infty} y(t) = 0$.

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- If $y_0 < B$, then the solution is well-defined for all $t > 0$ and $\lim\limits_{t\to\infty} y(t) = 0$.
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What does it mean? A population that starts with a size above the value of B grows without bound in finite time.

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$$y' = -40\left(1 - \frac{y}{20}\right)y = 2y(y-20).$$

The equilibrium solutions are $y(t) = 0$, $y(t) = 20$.

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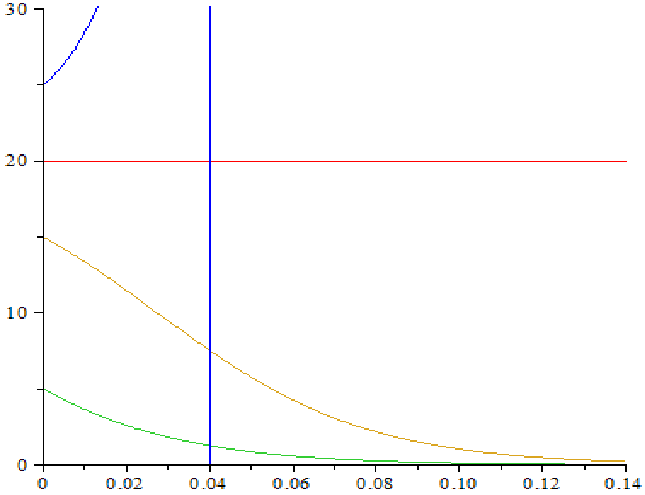
Sketch the particular solutions for the cases $y_0 = 5, 15, 25$:

$$y_1(t) = \frac{20}{1+3e^{40t}},\ y_1(t) = \frac{60}{3+e^{40t}}\ y_1(t) = \frac{100}{5-3e^{40t}}.$$

Population with a critical threshold: Example



Exercises

Find the critical points and equilibrium solutions of the given DE and solve the DE with each of the prescribed ICs. Sketch the graphs of the solutions obtained and comment on the stability/instability of the equilibrium solutions. Identify the model governed by the IVP, if any, and describe its main elements.

- $y' = 300y - 2y^2, y(0) = 50, y(0) = 100, y(0) = 200$
- $y' = 240y - 3y^2, y(0) = 20, y(0) = 60, y(0) = 100$
- $y' = 15y - y2/2, y(0) = 10, y(0) = 20, y(0) = 40$
- $y' = 8y - 2y^2 - 6, y(0) = 1/2, y(0) = 3/2, y(0) = 5/2, y(0) = 7/2$
- $y' = y^2 + y - 6, y(0) = -4, y(0) = -2, y(0) = 1, y(0) = 3$

Outline

Lemma 1: Gronwall-Bellman

Let $u(t) \geq 0$, $f(t) \geq 0$ for all $t \geq t_0$, $u(t)$, $f(t) \in C[t_0, +\infty)$, and

$$\forall t \geq t_0 \quad u(t) \leq C + \int_{t_0}^{t} f(t_1) u(t_1)\, dt_1,$$

where C is a positive constant.
Then

$$\forall t \geq t_0 \quad u(t) \leq C \exp\left(\int_{t_0}^{t} f(t_1)\, dt_1\right). \tag{1}$$

Proof.

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$$\frac{u(t)}{C + \int_{t_0}^{t} f(t_1)u(t_1)\,dt_1} \leq 1 \Rightarrow \frac{f(t)u(t)}{C + \int_{t_0}^{t} f(t_1)u(t_1)\,dt_1} \leq f(t)$$

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Integrate the last inequality with respect to t from t_0 to t.

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Integrate the last inequality with respect to t from t_0 to t.Since

$$\frac{d}{dt}\left[C + \int_{t_0}^{t} f(t_1)u(t_1)\, dt_1\right] = f(t)u(t),$$

Lemma 1: Gronwall-Bellman

we obtain

$$\ln \left| C + \int_{t_0}^{t} f(t_1)u(t_1)\,dt_1 \right| - \ln C \leq \int_{t_0}^{t} f(t)\,dt.$$

Therefore,

$$u(t) \leq C + \int_{t_0}^{t} f(t_1)u(t_1)\,dt_1 \leq C \exp\left(\int_{t_0}^{t} f(t_1)\,dt_1 \right).$$

Lemma 1: Gronwall-Bellman

Corollary: Let $u(t)$ be a positive continuous function satisfying

$$u(t) \leq u(\tau) + \int_{\tau}^{t} f(t_1)u(t_1)\,dt_1 \quad \forall t, \tau \in (a,b),$$

where $f(t) \in C(a,b)$ and $f(t) \geq 0 \,\forall t \in (a,b)$.
Then for all $a < t_0 \leq t < b$

$$u(t_0)\exp\left[-\int_{t_0}^{t} f(t_1)\,dt_1\right] \leq u(t) \leq u(t_0)\exp\left[\int_{t_0}^{t} f(t_1)\,dt_1\right].$$

Proof: Leave as an exercise.

Lemma 2: Bihari- LaSalle

Let $u(t) \geq 0$, $f(t) \geq 0$ for all $t \geq t_0$, $u(t)$, $f(t) \in C[t_0, +\infty)$, and

$$u(t) \leq C + \int_{t_0}^{t} f(t_1)\Phi(u(t_1))\,dt_1,$$

where C is a positive constant, $\Phi(u)$ is a positive non-decreasing continuous function for all $0 < u < \bar{u}\,(\bar{u} \leq \infty)$. Define

$$\Psi(u) = \int_{C}^{u} \frac{du_1}{\Phi(u_1)},\ 0 < u < \bar{u}.$$

If $\displaystyle\int_{t_0}^{t} f(t_1)\,dt_1 < \Psi(\bar{u} - 0)$, $t_0 \leq t < \infty$, then

$$u(t) \leq \Psi^{-1}\left[\int_{t_0}^{t} f(t_1)\,dt_1\right] \quad \forall t_0 \leq t < \infty \tag{2}$$

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Integrating with respect to t from t_0 to t, we have

$$\int_{t_0}^{t} \frac{w'(t)dt}{\Phi(w(t))} = \int_{w(t_0)}^{w(t)} \frac{dw}{\Phi(w(t))} \leq \int_{t_0}^{t} f(t_1)\,dt_1,$$

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where $w(t) = C + \displaystyle\int_{t_0}^{t} f(t_1)\Phi(u(t_1))\,dt_1$. Since $w(t_0) = C > 0$

and $w(t) \geq C > 0$ then $\Psi(w(t)) \leq \displaystyle\int_{t_0}^{t} f(t_1)\,dt_1$.

Bihari-LaSalle: Proof

Recall that $\Psi'(u) = \dfrac{1}{\Phi(u)} > 0$ for all $0 < u < \bar{u}$.

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Recall that $\Psi'(u) = \dfrac{1}{\Phi(u)} > 0$ for all $0 < u < \bar{u}$.Therefore, the function $v = \Psi(u)$ has a unique inverse function $u = \Psi^{-1}(v)$ (continuous and monotonically increasing) defined on $\Psi(+0) < v < \Psi(\bar{u} - 0)$, where $\Psi(+0) < 0$.

Bihari-LaSalle: Proof

Recall that $\Psi'(u) = \dfrac{1}{\Phi(u)} > 0$ for all $0 < u < \bar{u}$.Therefore, the function $v = \Psi(u)$ has a unique inverse function $u = \Psi^{-1}(v)$ (continuous and monotonically increasing) defined on $\Psi(+0) < v < \Psi(\bar{u} - 0)$, where $\Psi(+0) < 0$.We can express $w(t)$ from the last inequality provided $\displaystyle\int_{t_0}^{t} f(t_1)\,dt_1 < \Psi(\bar{u} - 0)$ as follows

$$w(t) \le \Psi^{-1}\left[\int_{t_0}^{t} f(t_1)\,dt_1\right]$$

It gives

$$u(t) \le C + \int_{t_0}^{t} f(t_1)\Phi(u(t_1))\,dt_1 = w(t) \le \Psi^{-1}\left[\int_{t_0}^{t} f(t_1)\,dt_1\right].$$

Bihari-LaSalle: intervals of existence

Find an interval of existence of the solution for the following IVP

$$y' = t + y^3, \quad y(0) = 0.$$

Solution.

We have $f(t, y) = t + y^3$, $\dfrac{\partial f}{\partial y}(t, y) = 3y^2$, $(t_0, y_0) = (0, 0)$. What are the conditions for existence of the unique solution? f, f_y are continuous in any rectangular R that contains $(t_0, y_0) = (0, 0)$, for example $R = \{(t, y) \in \mathbb{R}^2 \colon |t| \leq a, |y| \leq b\}$. $\Rightarrow$ there exists a unique solution of the problem defined on $0 - h < t < 0 + h$. How can we find the interval of existence without solving the equation? Use Bihari's inequality.

The equation implies that

$$y(t) = \int_0^t (t_1 + y^3(t_1))\, dt_1 = \frac{t^2}{2} + \int_0^t y^3(t_1)\, dt_1$$

Bihari-LaSalle: intervals of existence

$$|y(t)| \le \frac{a^2}{2} + \int_0^t |y(t_1)|^3 \, dt_1.$$

Denote $u(t) = |y(t)|$, $C = a^2/2$, $f(t) = 1$, $\Phi(u) = u^3$. Then

$$v = \Psi(u) = \int_C^u \frac{du_1}{u_1^3} = \frac{1}{2}\left(\frac{1}{C^2} - \frac{1}{u^2}\right)$$

$$u = \Psi^{-1}(v) = \frac{C}{\sqrt{1 - 2C^2 v}}$$

Then the Bihari inequality implies that

$$u(t) \le \Psi^{-1}\left[\int_0^t 1 \, dt_1\right] = \frac{C}{\sqrt{1 - 2C^2 t}}.$$

That is,

$$|y(t)| \le \frac{C}{\sqrt{1 - 2C^2 t}}, \ \forall t > 0.$$

Bihari-LaSalle: intervals of existence

What is the restriction for t?

$$0 < t < \frac{1}{2C^2}$$

From the equation $a = \frac{1}{2C^2}$ find that $\max a = \sqrt[5]{2}$. Then the solution exists on the interval $[0, \sqrt[5]{2})$.

Can we also extend the interval of existence along the negative part of the t-axis? Yes, change t to $-t$ $(t \geq 0)$ in the equation and proceed with the same procedure.

We obtain $u(t) \leq \frac{C}{\sqrt{1+2C^2 t}}$, $t \leq 0$. $\Rightarrow$ the solution exists for $-\sqrt[5]{2} < t < 0$ as well.

Answer: We can guarantee the existence of the unique solution in the interval $-\sqrt[5]{2} < t < \sqrt[5]{2}$.

Bihari-LaSalle: intervals of existence

Using Bihari's lemma, find an interval of existence of the solution for the following IVPs:

$$1.\ y' = 2y^2 - t,\ y(1) = 1.$$

$$2. \begin{cases} y_1' = y_2^2, & y_1(0) = 1, \\ y_2' = y_1^2, & y_2(0) = 2. \end{cases}$$

Outline

Implicit first-order ODEs

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where F is a known smooth function. Consider the following cases

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$$\Phi_1(t, y, C) = 0,\ \Phi_2(t, y, C) = 0, \ldots, \Phi_k(t, y, C) = 0$$

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be general solutions of (4).What is the general solution of the equation (3)?

$$\Phi_1(t, y, C) \cdot \Phi_2(t, y, C) \cdot \ldots \cdot \Phi_k(t, y, C) = 0$$

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$$\frac{1}{p} = \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial p} \cdot \frac{dp}{dy} \Rightarrow \frac{dy}{dp} = \frac{p\frac{\partial \varphi}{\partial p}}{1 - p\frac{\partial \varphi}{\partial y}}$$

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If a general solution of this equation has the representation $y = \Theta(p, C)$, where Θ is known and C is a constant, then

$$\begin{cases} t = \varphi(\Theta(p, C), p) \\ y = \Theta(p, C) \end{cases}$$

is the general solution of the equation $t = \varphi(y, y')$ in the parametric form. Non-parametric form? $\Rightarrow$ Eliminate p.
Exercise: Find the general solution if $t = \varphi(y')$.

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If a general solution $t = \Theta(p, C)$ of this equation exists then

$$\begin{cases} t = \Theta(p, C) \\ y = \psi(\Theta(p, C), p) \end{cases}$$

is the general solution of the equation $y = \psi(t, y')$ in the parametric form.
What happens if $y = \psi(y')$?

Implicit first-order ODEs

What is common in cases 2 and 3?

- In both cases equations are explicit with respect to either t or y, and
- we differentiate w.r.t. another variable.
- New equations are explicit w.r.t. corresponding derivatives

However, new explicit equations may not have analytical representation of the solution!!!
We are to consider two types of equations for which the approach described above works and explicit equations are solvable.

Lagrange Equation

The equation

$$y = t\varphi(y') + \psi(y')$$

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$$(\varphi(p) - p)\frac{dt}{dp} + \varphi'(p)t + \psi'(p) = 0.$$

What is the type of this equation? Linear $\Rightarrow$ Find its solution $t = \Phi(p, C)$ and obtain the general solution of the Lagrange equation in the form

$$\begin{cases} t = \Phi(p, C) \\ y = \Phi(p, C)\varphi(p) + \psi(p) \end{cases}$$

Attention! The Lagrange equation may also have special solutions of the form $y = \varphi(c)t + \psi(c)$, where c is the root of the equation $\varphi(c) = 0$. We will consider the question of special solutions later.

Clairaut equation

If $\varphi(y') = y'$ in the Lagrange equation then the equation

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Therefore, plugging C instead of y' in Clairaut's equation we immediately obtain the general solution. How we can get a singular solution from the general one? Differentiate w.r.t C.

Examples

1. Find the solution of the equation $F(y') = 0$.

Let $y' = p$ and $F(p_0) = 0$. Then $F(\frac{y-C}{t}) = 0$.

2. Solve the equation $(y')^2 - (t+y)y' + ty = 0$.

We have $y' = t$ and $y' = y$. Then $y = \frac{t^2}{2} + C$, $y = Ce^t$ and the general solution is $(y - \frac{t^2}{2} - C)(y - Ce^t) = 0$.

3. Consider the equation $y = y' + (y')^2 e^{y'}$.

Case 2 $\Rightarrow$ $y' = p$ and $y = p + p^2 e^p$. Therefore, $dt = \frac{1+(p^2+2p)e^p}{p} dp$ and $t = \ln|p| + (p+1)e^p + C$. The general solution has the form

$$\begin{cases} t = \ln|p| + (p+1)e^p + C \\ y = p + p^2 e^p \end{cases}$$

Moreover, we need to complement it with the obvious solution $y = 0$.

Exercises

Solve the following ODEs:

$$1.(y')^2 - 2ty' - 8t^2 = 0.$$

$$2.t^2(y')^2 + 3tyy' + 2y^2 = 0.$$

$$3.(y')^3 - y(y')^2 - t^2y' + t^2y = 0.$$

$$4.t = \ln y' + \sin y'.$$

$$5.y = \sin^{-1} y' + \ln(1 + (y')^2).$$

$$6.y = ty' + y' + \sqrt{y'}.$$

$$7.y = y' \ln y'.$$

$$y = 3/2ty' + e^{y'}.$$

Singular Solutions

We have already considered ODEs with singular solutions (check examples above). Intuitively, a singular solution is a special solution that is not contained in the general solution for any values of the constant C including $C = \pm\infty$.What do singular solution mean geometrically? How can we plot them?

Definition. A solution $y = \varphi(t)$ of the differential equation

$$F(t, y, y') = 0 \tag{5}$$

is called singular if the uniqueness property does not hold at any of its points, that is,

- there is another solution of the same ODEs passing through each point (t_0, y_0) of the singular solution, and
- both solutions have the same tangent at the point (t_0, y_0) but
- another non-singular solution is different form the singular one in any arbitrary small neighborhood of the point (t_0, y_0).

Singular Solutions

Does a singular solution satisfies the equation (5)? Yes. Moreover, if $F(t, y, y')$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial y'}$ are continuous with respect to all arguments t, y, y' then any singular solution satisfies the equation

$$\frac{\partial F(t, y, y')}{\partial y'} = 0. \tag{6}$$

How can we find a singular solutions from (5) and (6)? $\Rightarrow$ Eliminate y'. Elimination gives us an equation

$$\psi_p(t, y) = 0$$

which is called p-discriminant of the equation (5), and the integral curve corresponding p-discriminant is called the p-discriminant integral curve.

Is a p-discriminant curve unique? Does it define a singular solution? In general, no. $\Rightarrow$ Double-check.

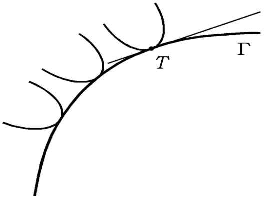
Singular Solutions

Consider the equation

$$\Phi(t, y, C) = 0$$

with a parameter C and continuous Φ_t, Φ_y, Φ_C. It defines a family of curves depending on one parameter.

An **envelope** of the family of curves with a parameter is a smooth curve Γ that touches one curve of the family at any of its points and any its segment is touched by an infinite number of curves from the family. What does it mean if curves touch? A common tangent.



Singular Solutions

Does the envelope satisfies the definition of a singular integral curve? Yes. $\Rightarrow$ the envelope defines a singular solution.

An envelope is a part of a C-discriminant curve defined a by

$$\begin{cases} \Phi(t, y, C) = 0 \\ \frac{\partial \Phi(t,y,C)}{\partial C} = 0 \end{cases}$$

To make sure that a branch of a C-discriminant curve is an envelope, we check the following conditions.

- there exist bounded partial derivatives $\left|\dfrac{\partial \Phi}{\partial t}\right| \leq M, \left|\dfrac{\partial \Phi}{\partial y}\right| \leq N, \; M, N = const,$
- $\dfrac{\partial \Phi}{\partial t} \neq 0$, or, $\dfrac{\partial \Phi}{\partial y} \neq 0$

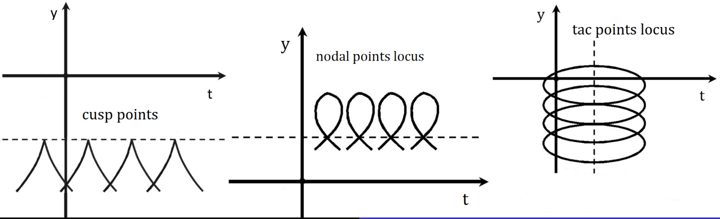
Are these condition are necessary or sufficient? Sufficient. $\Rightarrow$ if they are not satisfied on a branch of the C-discriminant curve, it can still be an envelope.

Singular Solutions

The equations of p-discriminant and C-discriminant have a certain structure

$$\psi_p(t, y) = E \cdot C \cdot T^2 = 0,$$
$$\psi_C(t, y) = E \cdot N^2 \cdot C^3 = 0,$$

where $E = 0$ is the equation of the envelope, $C = 0$ is the equation of the cusp locus, $N = 0$ is the equation of nodal locus, $T = 0$ is the equation of the tac locus. Attention! Over all locus points only the envelope is a singular solution.



Singular Solutions: Example 1

Consider the equation

$$ty' + (y')^2 - y = 0.$$

1. Find its p-discriminant curve

$$y = -\frac{t^2}{4}.$$

2. Check if it is a solution of the DE.
3. Check if it is a singular solution. Find the general solution of the equation $y = Ct + C^2$. Why?Check the type of the equation. If two curves $y = y_1(t)$ and $y = y_2(t)$ touch at the point $t = t_0$ then

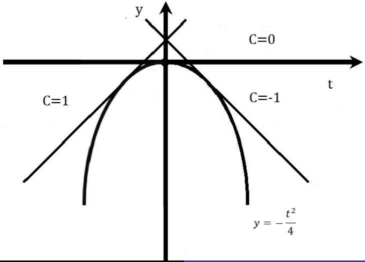
$$y_1(t_0) = y_2(t_0),\ y_1'(t_0) = y_2'(t_0).$$

Singular Solutions: Example 1

It gives us

$$-\frac{t_0^2}{4} = Ct_0 + C^2, \ -\frac{t_0}{2} = C.$$

and hence, $-\frac{t_0^2}{4} = -\frac{t_0^2}{4} \quad \Rightarrow$ at each point of the curve $y = -\frac{t^2}{4}$, another curve of the form $y = Ct + C^2$ touches it, with $C = -\frac{t_0}{2}$. Therefore, $y = -\frac{t^2}{4}$ is a singular solution.



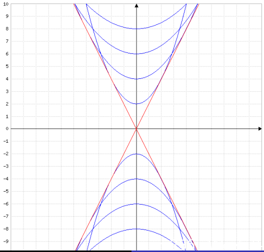
Singular Solutions: Example 2

Find singular solutions of the equation

$$t(y')^2 - 2yy' + 4t = 0,\ t > 0$$

with the general solution $t^2 = C(y - C)$.

1. Find the C-discriminant curve $y = \pm 2t$
2. Verify that both functions are solutions of the equation.
3. Prove that each function is a singular solution. Use the sufficient condition.

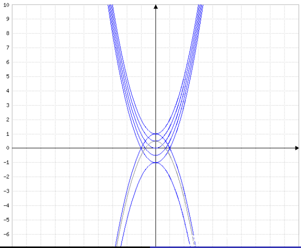


Singular Solutions: Example 3

Find a singular solution of the equation

$$(y')^2 = 4t^2.$$

1. p-discriminant is $2y' = 0, (y')^2 = 4t^2$ and hence, $t^2 = 0$.
2. Is $t = 0$ an integral curve of the equation? No. But it may be tac points locus. Why?
3. The general solution of the equation is $y = \pm t^2 + C$. Therefore, the line $t = 0$ is indeed the tac points locus.

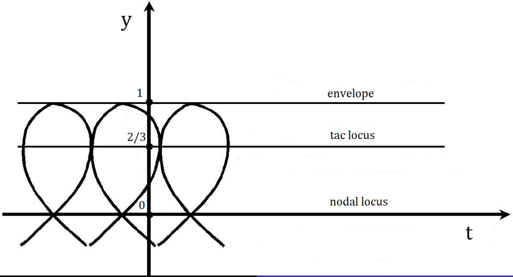


Singular Solutions: Example 4

Find singular solution of the equation

$$(y')^2(2-3y)^2 = 4(1-y).$$

1. Find the p-discriminant $(2-3y)^2(1-y) = 0$. What are the conclusions?
2. Find the general solution of the equation $y^2(1-y) = (t-C)^2$.
3. Find the C-discriminant curve $y^2(1-y) = 0$.



Singular Solutions: Exercises

For the following equations, find singular solutions if they exist.
1.$(1+(y')^2)y^2-4yy'-4t=0$,
2.$(y')^2-4y=0$,
3.$(y')^3-4tyy'+8y^2=0$,
4.$(y')^2-y^2=0$,
5.$(ty'+y)^2+3t^5(ty'-2y)=0$.
Use C-discriminant to find singular solutions for the following equations 1.$y=(y')^2-ty'+t^2/2$, $y=Ct+C^2+t^2/2$,
2.$(ty'+y)^2=y^2y'$, $y(C-t)=C^2$,
3.$y^2(y')^2+y^2=1$, $(x-C)^2+y^2=1$,
4.$(y')^2-yy'+e^t=0$, $y=Ce^t+1/C$.