vv256: Laplace Transform

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General Integral Transforms

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$\alpha = 0$, $\beta = \infty$, $K(s,t) = e^{-st}$ $\Rightarrow$improper integral
We assume that $f(t)$ as a real-valued function defined for $t > 0$.

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The existence and uniqueness of the Laplace transform of a function $f(t)$ is guaranteed if there exist real K, M and a such that

1. $f(t)$ is piecewise continuous for $t > 0$,
2. $|f(t)| \leq Ke^{at}$ for $t \geq M$.

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Separate real and imaginary parts:

$$L\left[\cos at\colon\, t \to s\right] = \frac{s}{s^2 + a^2}, \quad L\left[\sin at\colon\, t \to s\right] = \frac{a}{s^2 + a^2}$$

Some General Properties

- Linearity.
  If $\bar{f}(s) = L[f(t) \colon t \to s]$ and $\bar{g}(s) = L[g(t) \colon t \to s]$ exist then $L[af(t) + bg(t) \colon t \to s]$ exists for all constants a and b and

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$$L\left[af(t) + bg(t) \colon t \to s\right] = a\bar{f}(s) + b\bar{g}(s)$$

- First shifting property.
  Suppose $\bar{f}(s) = L\left[f(t) \colon t \to s\right]$ exists and that a is a constant then $L\left[e^{at} f(t) \colon t \to s\right]$ exists and

$$L\left[e^{at} f(t) \colon t \to s\right] = \bar{f}(s - a)$$

or

$$L^{-1}\left[\bar{f}(s - a) \colon s \to t\right] = e^{at} f(t)$$

Laplace Transforms of Selected Functions

$f(t) = L^{-1}\left[\bar{f}(s)\colon s \to t\right]$	$\bar{f}(s) = L\left[f(t)\colon t \to s\right]$	
1	$\dfrac{1}{s}$	$Re(s) > 0$
e^{at}	$\dfrac{1}{s-a}$	$Re(s) > 0$
$t^n,\ n \in \mathbb{Z}_+$	$\dfrac{n!}{s^{n+1}}$	$Re(s) > 0$
$t^p,\ p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$	$Re(s) > 0$
$\cos at$	$\dfrac{s}{s^2+a^2}$	$Re(s) > 0$
$\sin at$	$\dfrac{a}{s^2+a^2}$	$Re(s) > 0$

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$\cosh at$	$\dfrac{s}{s^2 - a^2}$	$Re(s) > \lvert a\rvert$
$\sinh at$	$\dfrac{a}{s^2 - a^2}$	$Re(s) > \lvert a\rvert$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$	$Re(s) > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$	$Re(s) > a$
$t^n e^{at},\ n \in \mathbb{Z}_+$	$\dfrac{n!}{(s-a)^{n+1}}$	$Re(s) > a$

Convolution

Let $\quad \bar{f}(s) = L\left[f(t)\colon\, t \to s\right], \bar{g}(s) = L\left[g(t)\colon\, t \to s\right]$

be Laplace transforms of $f(t)$ and $g(t)$.

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whenever this integral exists.

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You can check the result $L[(t/2)\sin t \colon t \to s]$ by direct calculation as well

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$$f(t) = g(t) - L^{-1}\left[\frac{\bar{g}(s)}{s} \colon s \to t\right] = g(t) + \int_0^t g(x)\, dx$$

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- Both results are valid for $s > 0$.

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It immediately follows that

$$y(t) = L^{-1}\left[\frac{1}{s - 2} \colon s \to t\right] = e^{2t}.$$

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Thus

$$y(t) = \cos 2t + \sin 2t$$

As you can see, the method as useful as far we can take inverse Laplace transforms easily.

Example: electrical circuit

Example 3. Consider an electrical circuit with

- an inductor of inductance $L = 10^3$ Henry,
- a resistor of resistance $R = 6000\,Ohm$,
- an uncharged capacitor of capacitance $C = (1/9)10^3$ Farad and
- a battery of electromotive force $U(t) = 250\cos t$ Volt

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Apply Kirchhoff's Second Law (the sum of all voltage drops around a closed circuit is zero) to obtain the differential equation w.r.t the electrical current I and the rate of change of charge Q

$$L\dot{I}(t) + RI(t) + \frac{Q}{C} = U(t)$$

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With $\dot{Q}(t) = I(t)$, initial conditions $Q(0) = 0$ and $I(0) = 0$ and the given values for L, R, C, the IVP becomes

$$\ddot{Q}(t) + 6\dot{Q}(t) + 9Q(t) = \frac{1}{4}\cos t,\ Q(0) = 0,\ \dot{Q}(0) = 0.$$

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Express $\bar{Q}(s)$ and apply partial fraction decomposition:

$$\bar{Q}(s) = \frac{s}{4(s^2+1)(s^2+6s+9)} =$$

$$= \frac{1}{200}\left(\frac{4s}{s^2+1} + \frac{3}{s^2+1} - \frac{4}{s+3} - \underbrace{\frac{15}{(s+3)^2}}_{\text{apply the shift formula}}\right)$$

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Extract the solution

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The solution for large time is dominated by the inhomogeneous term in the differential equation, while the contributions from the complementary function are damped out.

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The long-term behavior of the solution is described by the result

$$\lim_{t\to\infty}\left[Q(t) - \frac{1}{200}(4\cos t + 3\sin t)\right] = 0.$$

The solution for large time is dominated by the inhomogeneous term in the differential equation, while the contributions from the complementary function are damped out.

In this example, the forcing function $U(t) = 250\cos t$ is a continuous function of time. In many applications, such a function is a step function or an impulse function.

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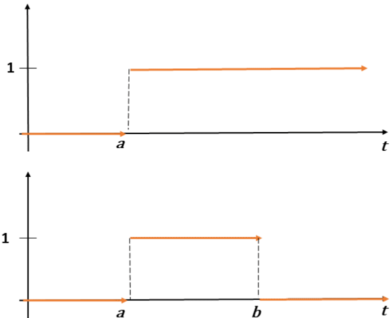

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Take the Laplace transform of the equation to obtain

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The solution is continuous at $t = 1$ even though the driving force $f(t)$ is not.

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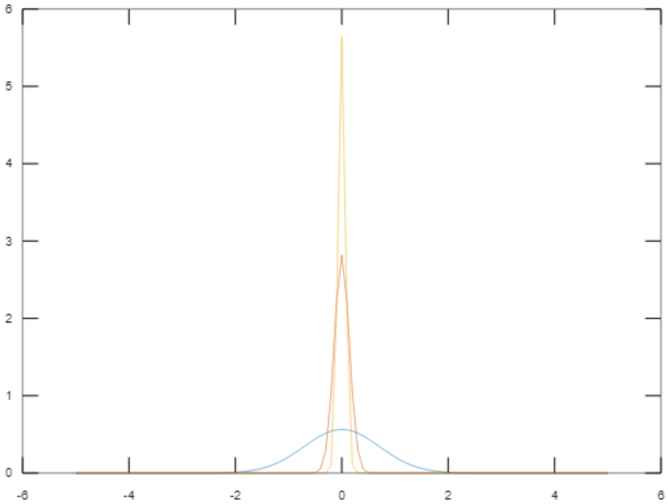
Sequence 2: Another popular description of $\delta(t)$ $(a = 0)$ is the limit of the sequence

$$\delta_n(t) = \lim_{n \to \infty} \frac{n}{\sqrt{\pi}} e^{-n^2 t^2}$$

The Unit Impulse Function



The graph of the function $(n/\sqrt{\pi})e^{-n^2t^2}$ for $n = 1, 5, 10$.

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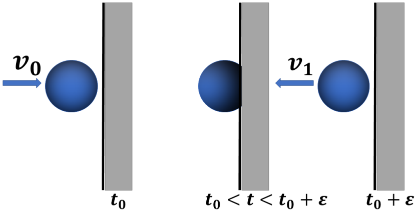

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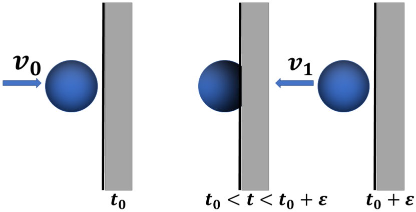

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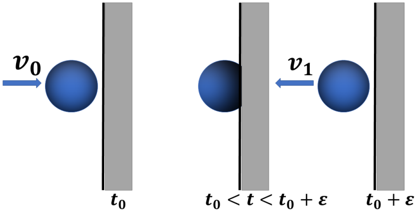

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- During this time, the ball is in contact with the wall and the velocity of the ball reduces from v_0 to 0 and then changes its direction, finally leaving the wall with velocity $-v_1$.

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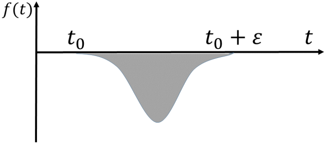

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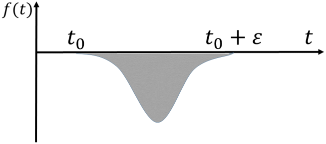

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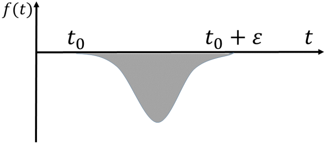

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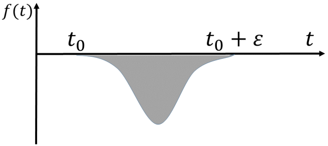

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- The Impulse-Momentum Principle: the change in momentum of mass m is equal to the total impulse on m: $m(-v_1) - mv_0 = I$.

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- For $I = 1$, the limiting function

$$\lim_{\varepsilon \to 0} f(t) = \lim_{\varepsilon \to 0} \frac{H(t - t_0) - H(t - (t_0 + \varepsilon))}{\varepsilon}$$

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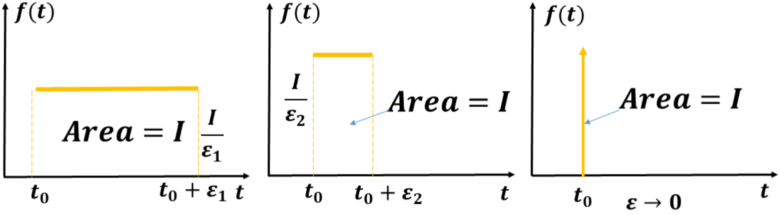

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Solve the IVP

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after taking partial fractions. Apply the first and the second shifting properties to obtain

$$y(t) = 2e^{-t} + 5te^{-t} + (t-1)e^{-(t-1)}H(t-1)$$

Laplace Transforms of Selected Functions

$f(t) = L^{-1}\left[\bar{f}(s) \colon s \to t\right]$	$\bar{f}(s) = L\left[f(t) \colon t \to s\right]$	
$H(t-a)$	$\dfrac{e^{-as}}{s}$	$Re(s) > 0$
$H(t-a)f(t-a)$	$e^{-as}\bar{f}(s)$	
$e^{at}f(t)$	$\bar{f}(s-a)$	
$f(at)$	$\dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$	$a > 0$
$\int_0^t f(t-w)g(w)\,dw$	$\bar{f}(s)\bar{g}(s)$	
$\delta(t-a)$	e^{-as}	
$(-t)^n f(t)$	$\bar{f}^{(n)}(s)$	
$f^{(n)}(t)$	$s^n\bar{f}(s) - \sum_{j=0}^{n-1} s^{n-j-1} f^{(j)}(0)$	

Periodic Functions

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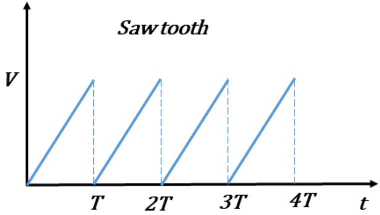
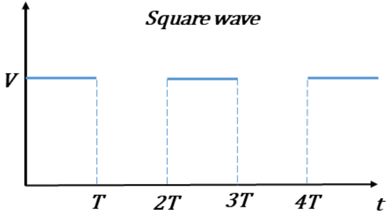
Common periodic waveforms

The periodic waveforms saw tooth and square wave of amplitude V and periods T and $2T$ are shown below





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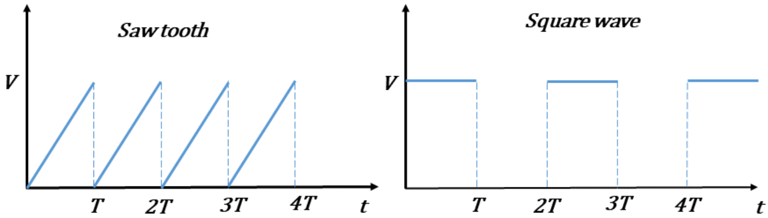

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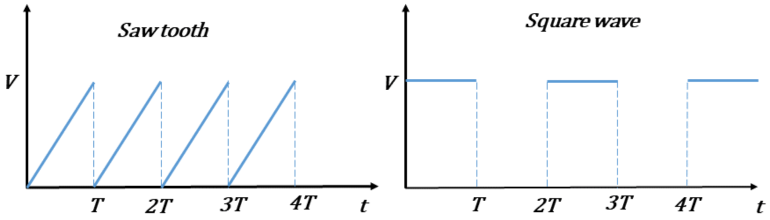

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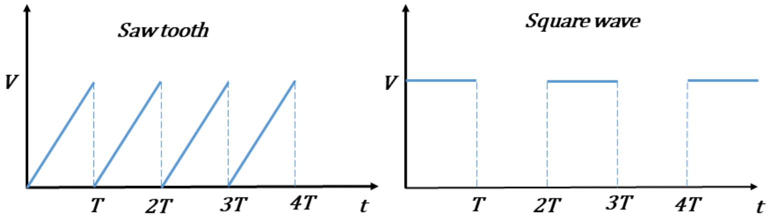

Since

$$\int_0^T f(t)e^{-st}dt = \frac{V}{T}\int_0^T te^{-st}dt = \frac{V}{Ts^2}(1 - e^{-sT} - sTe^{-sT}),$$

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Other periodic pulses.

Meander function

$$f(t) = \begin{cases} V & t \in (0, T) \\ -V & t \in (T, 2T) \end{cases}$$

Triangular wave

$$f(t) = \begin{cases} tV/T & t \in (0, T) \\ \frac{(2T-t)V}{T} & t \in (T, 2T) \end{cases}$$

Full wave rectification

$$f(t) = \begin{cases} V\sin(\frac{\pi t}{T}) & t \in (0, T) \end{cases}$$

Half wave rectification

$$f(t) = \begin{cases} V\sin(\pi t/T) & t \in (0, T) \\ 0 & t \in (T, 2T) \end{cases}$$

Example

Determine the current in a series circuit containing an inductor of L Henry and a resistor of R Ohm when the circuit is driven by a saw tooth voltage of period T and amplitude V. Assume that no current is flowing initially.

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$$\bar{I} = \frac{V}{RT}\frac{a}{s^2(s+a)}\left(1 - \frac{sTe^{-sT}}{1 - e^{-sT}}\right) = \frac{V}{RT}\frac{1}{s^2}\left(1 - \frac{sTe^{-sT}}{1 - e^{-sT}}\right) -$$

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we obtain that

$$I(t) = \frac{U_{saw}(t)}{R} - \frac{VL}{R^2 T}(1 - e^{-at}) + \frac{V}{R}\sum_{k=1}^{\infty} e^{-a(t-kT)}H(t-kT)$$

Exercises

1. Find the Laplace transform of $f(t) = e^{-t-1/2}$ and
$f(t) = \cos(at + b)$.
2. Find the inverse Laplace transform of $\bar{f}(s) = (s-2)/(s^2-2)$,
$\bar{f}(s) = 3/(s^2 + 4s + 9)$.
3. Solve the convolution integral equations

$$A. \quad f(t) = 1 + \int_0^t f(u) \cos(t-u)\, du,$$

$$B. \quad \sin t - t = \int_0^t (t-u)^2 f(u) du.$$

4. Use Laplace transforms to solve the initial value problems

$$A. \, \ddot{y}(t) - 5\dot{y}(t) + 6y(t) = 0, \; y(0) = 2, \; \dot{y}(0) = 1,$$

$$B. \, \ddot{y}(t) - y(t) = te^{2t}, \; y(0) = 0, \; \dot{y}(0) = 1.$$

5. Using the definition of the Laplace transform, prove the second shifting property.

Exercises

6.Write $f(t)$ as a step function and calculate its Laplace transform

$$f(t) = \begin{cases} 1 & t < 0 \\ 3 & 1 \leq t < 7 \\ 5 & t \geq 7. \end{cases}$$

7. Use Laplace transforms to solve the initial value problem

$$4\ddot{y}(t) + 4\dot{y}(t) + 5y(t) = g(t)\, y(0) = \dot{y}(0) = 0,$$

$$g(t) = \begin{cases} 4 & 0 \leq t < \pi \\ 0 & t \geq \pi \end{cases}$$ and sketch the solution $y(t)$.

8. Solve the initial value problem

$$\ddot{y}(t) + 2\dot{y}(t) + 2y(t) = \delta(t - \pi)\, y(0) = \dot{y}(0) = 0,$$

by using Laplace transforms and sketch the solution $y(t)$.
9. Determine the current in a series circuit containing an inductor of L Henry and a resistor of R Ohm when the circuit is driven by a square wave voltage of period $2T$ and amplitude V. No current is flowing initially.

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```
>> syms s t Y
f = exp(-2*t)*3
F=laplace(f) % define the Laplace transform of the RHS part f(t)
y2=Y*s.^2-1*s-(-1) % define the Laplace transform of y''
y1=Y*s-1   % define the Laplace transform of y'
Yimage = solve(y2 + 6*y1 + 5*Y-F,Y) % express the image of y
ilaplace(Yimage,s,t) % find the original, i.e. the solution of the IVP
```

```
Yimage =

(s + 3/(s + 2) + 5)/(s^2 + 6*s + 5)


ans =

(7*exp(-t))/4 - exp(-2*t) + exp(-5*t)/4
```

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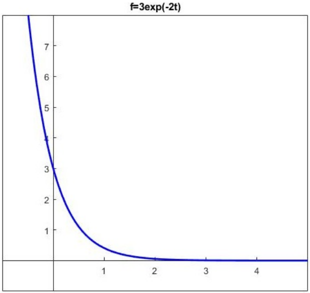

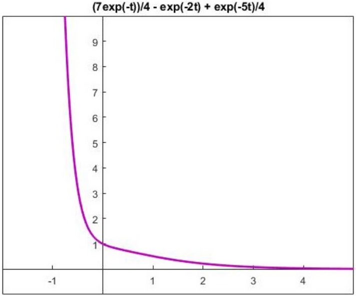

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```
>> f = 1 + 2*dirac(t-1)

f =

2*dirac(t - 1) + 1

>> F=laplace(f)

F =

2*exp(-s) + 1/s
```