vv256: Fourier Series. Sturm-Liouville Eigenvalue Problems.

Dr.Olga Danilkina

UM-SJTU Joint Institute

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Outline

Inner product and orthigonality

Fourier series: real form

Fourier series: complex exponential form

Boundary-value problems

Sturm boundary value problem

Sturm-Liouville eigenvalue problem

Inner product

Let X be a linear space.
A complex-valued function $(\cdot,\cdot)\colon X \times X \to \mathbb{C}$ satisfying

1. $(x,x) \geq 0 \quad \forall x \in X$,
2. $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z), \quad \forall x,\ y,\ z \in X, \forall \alpha, \beta \in \mathbb{K}$,
3. $\overline{(x,y)} = (y,x) \quad \forall x, y \in X$

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Examples: $\mathbb{R}^n:\ (x,y)=\sum_{i=1}^{n}x_iy_i$, $\mathbb{C}^n:\ (x,y)=\sum_{i=1}^{n}x_i\bar{y}_i$,

$$l_2:\ (x,y)=\sum_{i=1}^{\infty}x_i\bar{y}_i, \quad \underbrace{C[a,b]}_{\text{incomplete}}:\ (x,y)=\int_a^b x(t)y(t)\,dt$$

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4. Parallelogram identity

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$$

Hint: Use it to verify if a NLS is an inner product space.

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A system $\{e_i\}$ is said to be complete if the equality $(e_i, x) = 0$ for all $i = 1..\infty$, implies that $x = 0$.

Outline

Fourier series

A functional series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where the coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx,$$

is called a Fourier series of the function $f(x)$.

It is a real form of the Fourier series.

Theorem 1

A Fourier series of a periodic ($\omega = 2\pi$), piecewise continuous bounded function $f(x)$ converges at all points $x \in \mathbb{R}$ and its sum equals

$$S(x) = \frac{f(x-0) + f(x+0)}{2}.$$

Remark: $S(x) = f(x)$ at the points where $f(x)$ is continuous, and $S(x)$ equals to the average of left-hand side and right-hand side limits at the points where $f(x)$ has jump discontinuities.

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Find a Fourier series expansion of the periodic ($T = 2\pi$) function

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 \leq x \leq \pi. \end{cases}$$

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Substitute the obtained coefficients into the Fourier series:

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi(2n-1)^2}\cos(2n-1)x + \frac{(-1)^{n-1}}{n}\sin nx\right)$$

The series converges to $f(x)$ at all $x \neq (2n-1)\pi$.

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The sum of the Fourier series equals $(\pi + 0)/2 = \frac{\pi}{2}$ at the points $x = (2n-1)\pi$.

A Fourier series of a periodic function $y = f(x)$ with $T = 2l$ has the following representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n}{l} x + b_n \sin \frac{\pi n}{l} x \right)$$

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Substitute the obtained coefficients:

$$f(x) = \frac{1}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{2}$$

Even and odd functions

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- If a periodic function $y = f(x)$ is even then its Fourier series is a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n}{l}x$$

with coefficients

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- $a_n = \begin{cases} -8/(\pi n)^2 & n = 2k-1 \\ 0 & n = 2k \end{cases}$

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- The function $y = |x|$ is even $\Rightarrow b_n = 0 \Rightarrow$ Fourier cosine series
- Determine the coefficients

$$a_0 = \frac{2}{2}\int_0^2 |x|\, dx = 2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{\pi n}{l}x\, dx = \int_0^l x\cos\frac{\pi n}{2}x\, dx =$$

$$= \frac{2x}{\pi n}\sin\frac{\pi n}{2}x\Big|_0^2 + \frac{4}{\pi^2 n^2}\cos\frac{\pi n}{2}x\Big|_0^2 = \frac{4}{\pi^2 n^2}((-1)^n - 1)$$

- $a_n = \begin{cases} -8/(\pi n)^2 & n = 2k-1 \\ 0 & n = 2k \end{cases}$

The Fourier series

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi}{2}x$$

equals to the function $f(x)$ on $[-2, 2]$.

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$$f(x) = \begin{cases} 0 & -l < x < -b \\ -f(x) & -b \leq x \leq -a \\ 0 & -a < x < a \\ f(x) & a \leq x \leq b \\ 0 & b < x < l \end{cases}$$

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- The sum $S(x)$ is $f(x)$ in (a, b), $S(a) = f(a)/2$, $S(b) = f(b)/2$

Example 4

Find the Fourier sine series of the function $f(x) = 2 - x$ on $[0, 2]$.

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The Fourier series

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{\pi n}{2} x$$

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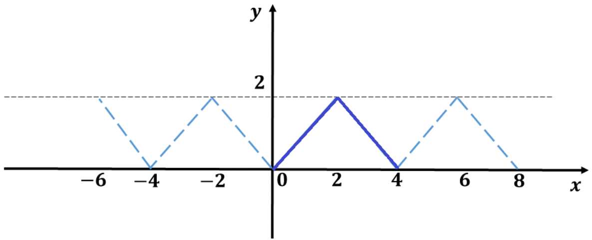
Find the Fourier cosine series of the function shown below



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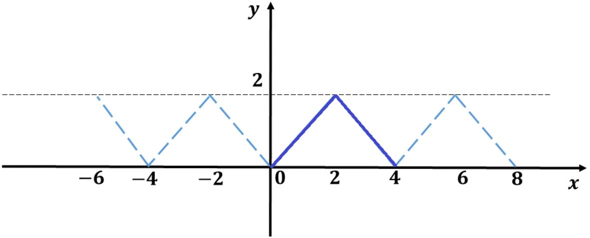

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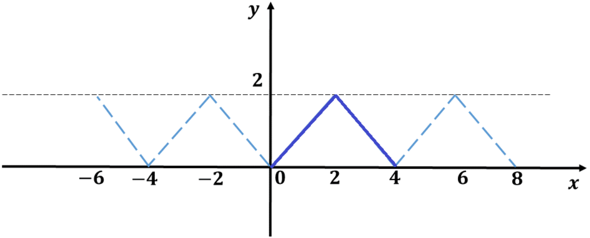

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- The cosine Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n}{2} x$$

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$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi}{2}x$$

Remark

- A Fourier series converges to the value of the corresponding function at the points of continuity
  $\Rightarrow$ we may use Fourier series to find sums of series.
- For example, let $x = 2$ in the Fourier series (Example 5):

$$2 = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example 6

Expand the function $y = x^2$ in cosine Fourier series on the interval $[0, \pi]$, and use thus obtained series to find sums of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$$

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$$= -\frac{4}{\pi n} \left(-\frac{x}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right) = \frac{4(-1)^n}{n^2}$$

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- The function is continuous $\Rightarrow$ it converges to its Fourier series

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$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2} = -\frac{\pi^2}{12}$$

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- Take $x = \pi$:

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Gibbs Phenomenon

- Explore the behavior of the Fourier series at the points of discontinuity of the square wave function

$$f(x) = \begin{cases} 1 & 0 < x < 1/2, \\ -1 & 1/2 < x < 1. \end{cases}$$

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- The Fourier series (over/under) shoots the actual value of $x(t)$ at points of discontinuity.

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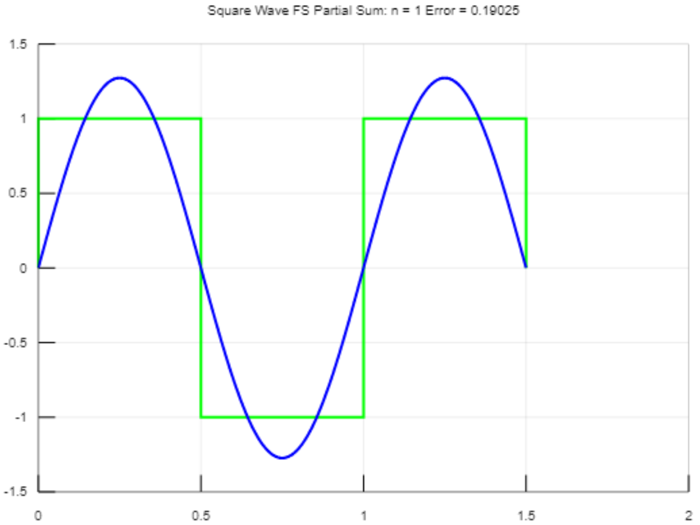

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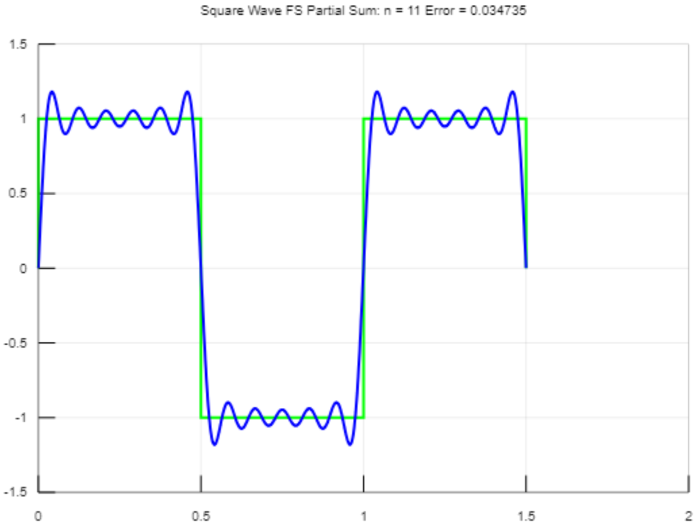

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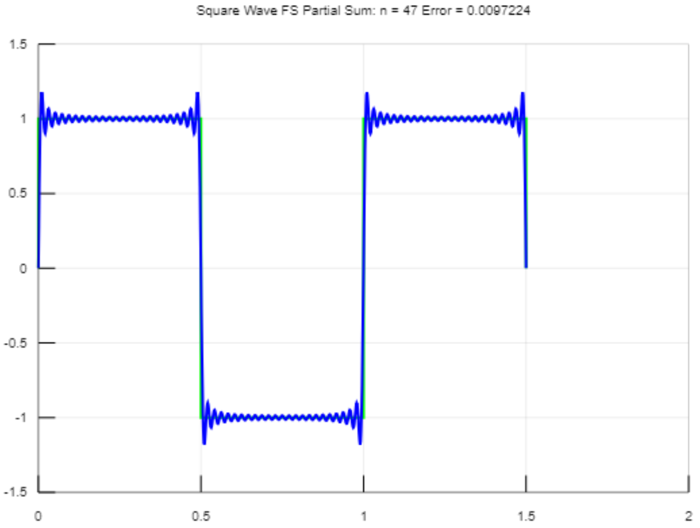

Outline

Inner product and orthigonality

Fourier series: real form

Fourier series: complex exponential form

Boundary-value problems

Sturm boundary value problem

Sturm-Liouville eigenvalue problem

The Complex Exponential Form

- Consider a piecewise continuous periodic ($T = 2\pi$) real-valued function $f(x)$. It's Fourier series expansion is

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- The real form of the Fourier series becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{inx} + \frac{a_n + ib_n}{2} e^{-inx}$$

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- The system $\{\frac{e^{inx}}{\sqrt{2\pi}}\}$ is orthonormal check it! $\Rightarrow$ the obtained series is also a partial case of the general Fourier series
$$f(x) = \sum_{i=1}^{\infty} (f(x), e_i)\, e_i, \quad \{e_i\} \text{ is orhonormal and complete}$$

The Complex Exponential Form

- Express the coefficients of the real form of the Fourier series from the coefficients of the complex form

$$\begin{aligned} a_n - ib_n &= 2c_n, \\ a_n + ib_n &= 2c_{-n} \end{aligned} \quad \Rightarrow$$

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$$f(x) = c_0 + \sum_{n=1}^{\infty} 2|c_n| cos(nx + \varphi_n)$$

The Complex Exponential Form

- Express the coefficients of the real form of the Fourier series from the coefficients of the complex form

$$\begin{array}{c} a_n - ib_n = 2c_n, \\ a_n + ib_n = 2c_{-n} \end{array} \Rightarrow \begin{array}{c} a_n = c_n + c_{-n} \\ ib_n = c_{-n} - c_n \end{array}$$

- If $f(x)$ is real then $c_{-n} = \overline{c_n} \Rightarrow$

$$c_n = |c_n|e^{i\varphi_n},\ c_{-n} = |c_n|e^{-i\varphi_n}$$

- The real form becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} |c_n| \left(e^{i(nx+\varphi_n)} + e^{-i(nx+\varphi_n)} \right)$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} 2|c_n| cos(nx + \varphi_n)$$

- Derive that

$$2|c_n| = \sqrt{a_n^2 + b_n^2},\ \varphi_n = -\tan^{-1}(b_n/a_n)$$

Multiplication

- Can we multiply Fourier series? What happens then? Let

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n e^{inx}, \; g(x) = \sum_{n=-\infty}^{+\infty} g_n e^{inx}$$

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- The Fourier coefficients
$$h_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} f_k e^{ikx} g(x) e^{-inx}\,dx$$
$$= \sum_{k=-\infty}^{+\infty} f_k \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-k)x} g(x)\,dx}_{g_{n-k}} = \sum_{k=-\infty}^{+\infty} f_k g_{n-k}$$

Parceval's Identity

- Notice that
$$\overline{f_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)e^{-inx}}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-i(-n)x}\,dx = g_{-n}$$

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Exercise: Derive Parseval's identity in the form
$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Mean-Square Error Approximation

- Consider a problem of approximation a periodic function with the Fourier series expansion

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by a finite sum, say

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- The mean-square error ε is

$$\varepsilon_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - f_N(x)|^2 \, dx = \underbrace{\sum_{n=-N}^{N} |f_n - \alpha_n|^2 + \sum_{|n|>N} |f_n|^2}_{\text{we applied Parseval's identity}}$$

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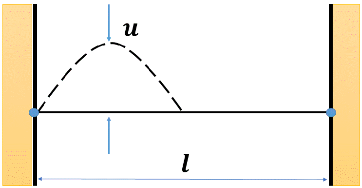
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx - \sum_{n=-N}^{N} |f_n|^2 \to 0 \text{ as } N \to \infty$$

Outline

- Inner product and orthigonality
- Fourier series: real form
- Fourier series: complex exponential form
- Boundary-value problems
- Sturm boundary value problem
- Sturm-Liouville eigenvalue problem

Motivation

- Consider the vibration of an elastic string stretched to tension T between two fixed points:



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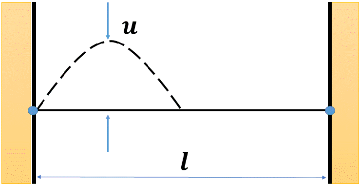

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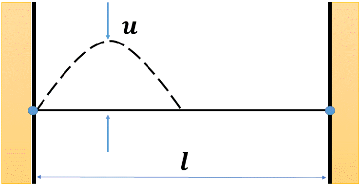

- Let $u(x,t)$ be the vertical displacement of the string at point x and at time t
- Then the equation of motion of the string is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where $c^2 = T/\rho$, ρ is the string mass per unit length.

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- Specify initial conditions

$$u(x, t_0) = r(x), \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=t_0} = s(x)$$

(the shape of the string and its velocity at time $t = t_0$)

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- and substitute it into the wave equation to derive

$$\frac{d^2F}{dx^2} + \alpha F = 0, \quad \frac{d^2G}{dt^2} + c^2\alpha G = 0$$

for some arbitrary constant α.

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with arbitrary constants a_n, $n = 1, \ldots, \infty$.

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Non-trivial solution $\Rightarrow \sqrt{\lambda} = n\pi, \quad n = 1, 2, 3, \ldots$ The solutions of the boundary value problem are therefore the functions

$$y_n(x) = k_n e^x \sin\pi n x,\; n = 1, 2, \ldots$$

where k_n are arbitrary constants.

Boundary value problem of second order

Example 2. Find the values of the parameter λ for which the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0,\, y(\pi) = 0$$

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- The choice $k_n = \sqrt{2/\pi}$ gives normalized eigenfunctions

$$y_n(x) = \sqrt{\frac{2}{\pi}}\cos(n-1/2)x$$

Outline

Sturm boundary value problems

Let $\hat{L}$ be a linear differential operator defined by

$$\hat{L}y(x) = (p(x)y'(x))' + q(x)y(x)$$

The boundary value problem defined by

$$\hat{L}y(x) = h(x), a < x < b,$$

$$\hat{R}_1 y = \alpha_1 y(a) + \alpha_2 y'(a) = \eta_1,$$

$$\hat{R}_2 y = \beta_1 y(b) + \beta_2 y'(b) = \eta_2,$$

where $\alpha_1^2 + \alpha_2^2 > 0$, $\beta_1^2 + \beta_2^2 > 0$, $p(x)$ is a non-negative continuously differentiable function, $q(x)$ is a continuous function for $a < x < b$, is called the Sturm boundary value problem.

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Remark: Any differential equation

$$y'' + a_1(x)y' + a_2(x)y = g(x)$$

can be represented in this form. Show it

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- Find the integration constants from

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- For the uniqueness of the solution,

$$\begin{vmatrix} \hat{R}_1 y_1 & \hat{R}_1 y_2 \\ \hat{R}_2 y_1 & \hat{R}_2 y_2 \end{vmatrix} \neq 0$$

  Attention: the non-homogeneous problem has a unique solution and the corresponding homogeneous problem has only the trivial solution.

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The non-homogeneous problem has no solution, the homogeneous problem has the solution $y = C\sin x$

Outline

- Inner product and orthigonality
- Fourier series: real form
- Fourier series: complex exponential form
- Boundary-value problems
- Sturm boundary value problem
- Sturm-Liouville eigenvalue problem

The Sturm-Liouville eigenvalue problem

The differential equation

$$\hat{L}y(x) + \lambda r(x)y(x) = 0, \quad a < x < b \tag{2}$$

where $\hat{L}y(x) = (p(x)y'(x))' + q(x)y(x)$, and the boundary conditions

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form a Sturm-Liouville eigenvalue problem.

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- Assume that p, p', q and r are continuous, $p > 0$ and $r > 0$ for $a < x < b$.
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$$\int_a^b \left[u_2(\hat{L}u_1) - u_1(\hat{L}u_2)\right] dx = \int_a^b [(pu_1')'u_2 - u_1(pu_2')']\, dx =$$

$$= \left(p(u_1'u_2 - u_1u_2')\right)_a^b$$

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  and apply the Lagrange identity in combination with the boundary conditions $\Rightarrow \lambda = \lambda^*$
- All the eigenvalues are simple: each eigenvalue has an eigenfunction that is unique up to a constant scaling factor. The proof is established by a contradiction argument.
- The eigenvalues may be ordered into an unbounded sequence

  $$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

  Given any $M > 0$, there is an integer n such that $\lambda_k > M$ for all $k > n$

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- The eigenfunctions $\varphi_n(x)$ can be normalized by the requirement

$$\int_a^b r(x)\varphi_n^2(x) = 1, \quad n = 1, 2, \ldots \tag{5}$$

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- The eigenfunctions $\varphi_n(x)$, $n = 1, 2, \ldots$ satisfying (5), (4) are said to form an orthonormal set.
- Attention! the most important property
  Let $y(x)$ be a such function that $\hat{R}_1 y = \hat{R}_2 = 0$ and y, y' be piecewise continuous. Then $y(x)$ can be expanded in terms of the normalized eigenfunctions $\varphi_n(x)$ in the form

$$y(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x), \quad a_n = \int_a^b r(x)\varphi_n(x)y(x)\,dx$$

The Sturm-Liouville eigenvalue problem

Return to the motivational problem with the elastic string.

- After separation of variables,

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- In terms of a Sturm-Liouville eigenvalue problem, the string problem corresponds to the special case $r(x) = 1$.
- Therefore, the normalizing coefficient is obtained from the condition

$$\int_0^l r(x)\varphi_n(x)\varphi(x)\,dx = A_n^2 \int_0^l \sin^2 \frac{n\pi x}{l}\,dx = 1 \Rightarrow A_n = \sqrt{\frac{2}{l}}$$

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- The series above converges everywhere if $y(x)$ is a continuous function. Such expansions are closely related to the concept of Fourier series.

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This is a transcendental equation and can only be solved approximately using numerical or graphical means.

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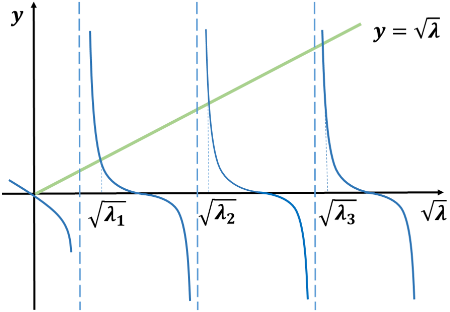

We can see that

$$
\begin{array}{cc}
\sqrt{\lambda_1} \approx 2.0 & \lambda_1 \approx 4 \\
\sqrt{\lambda_2} \approx 4.5 & \lambda_1 \approx 20.3 \\
\dots & \\
\sqrt{\lambda_n} \approx \pi(n-1/2) & \lambda_n \approx \pi^2(n-1/2)^2,\ n = 3, 4, \dots
\end{array}
$$

The unnormalized eigenfunctions: $y_n(x) = k_n \sin\sqrt{\lambda_n}x, n = 1, 2, ...$

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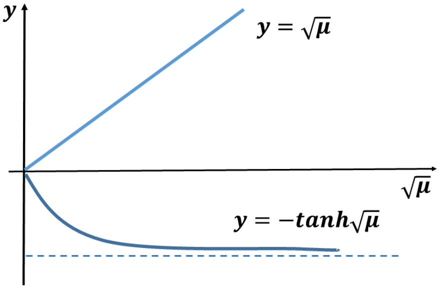

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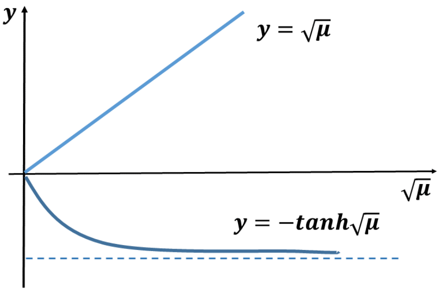

4. If λ is complex-valued, then the trivial solution $y = 0$ is the only possible solution. Explain why.

Non-homogeneous eigenvalue problems

- Non-homogeneous Sturm-Liouville eigenvalue problems are of the form
$$\hat{L}y(x) + \mu r(x)y(x) = f(x), \quad a < x < b \tag{6}$$
with the boundary conditions
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$$\hat{L}\varphi_n(x) + \lambda_n r(x)\varphi_n(x) = 0, \quad \hat{R}_1\varphi_n(x) = \hat{R}_2\varphi_n(x) = 0$$

with

$$\int_a^b r(x)\varphi_n(x)\varphi_m(x)\, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Non-homogeneous eigenvalue problems

- Second, multiply both sides of (6) by $\varphi_n(x)$ and integrate from a to b
$$\int_a^b \Big((\hat{L}y(x))\varphi_n(x) + \mu r(x)y(x)\varphi_n(x) \Big)\, dx = \int_a^b f(x)\varphi_n(x)\, dx$$

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- Use the definition of $\hat{L}$ and integration by parts

$$\int_a^b f(x)\varphi_n\, dx = \int_a^b \left((py')'\varphi_n + qy\varphi_n + \mu ry\varphi_n \right) dx =$$

$$= \left(py'\varphi_n \right)_a^b + \int_a^b \left(-py'\varphi_n' + qy\varphi_n + \mu ry\varphi_n \right) dx =$$

$$= \left(p(y'\varphi_n - y\varphi_n' \right)_a^b + \int_a^b y \left(\underbrace{(p\varphi_n')' + q\varphi_n}_{-\lambda_n r\varphi_n} + \mu r\varphi_n \right) dx$$

-

$$(\mu - \lambda_n) \int_a^b r(x)y(x)\varphi_n(x) dx = \int_a^b f(x)\varphi_n(x)\, dx \qquad (8)$$

Non-homogeneous eigenvalue problems

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$$y(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x) \tag{9}$$

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- The solution (10) is defined if μ is not an eigenvalue of the homogeneous Sturm-Liouville eigenvalue problem.

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The non-homogeneous Sturm-Liouville eigenvalue problem (6), (7) has a unique solution for every continuous function $f(x)$ provided μ is not an eigenvalue of the corresponding homogeneous Sturm-Liouville eigenvalue problem.
The solution (10) converges uniformly for every continuous function $f(x)$.

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Example. Solve the non-homogeneous boundary value problem

$$y'' + 2y = -x + \frac{2x^2}{3}, \quad y(0) = 0,\ y(1) + y'(1) = 0$$

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- Find the eigenfunctions of the corresponding homogeneous SL eigenvalue problem

$$\varphi_n''(x) + \lambda_n \varphi_n(x) = 0, \quad \varphi_n(0) = 0,\ \varphi_n(1) + \varphi_n'(1) = 0$$

$$\Rightarrow \varphi_n(x) = A_n \sin\sqrt{\lambda_n}x \Rightarrow \sqrt{\lambda_n} = -\tan\sqrt{\lambda_n}$$

with the normalization constant $A_n = \sqrt{2/(1+\cos^2\sqrt{\lambda_n})}$.

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- The solution is

$$y(x) = \frac{8}{3} \sum_{n=1}^{\infty} \frac{(1 - \cos \sqrt{\lambda_n})}{\lambda_n\sqrt{\lambda_n}(\lambda_n - 2)(1 + \cos^2 \lambda_n)} \sin \sqrt{\lambda_n} x$$

Non-homogeneous eigenvalue problems

Next, we shall solve this problem using the usual approach.

- The complementary solution is

$$y_C(x) = A\cos\sqrt{2}x + B\sin\sqrt{2}x$$

- Calculate

$$\begin{vmatrix} \hat{R}_1 y_1 & \hat{R}_1 y_2 \\ \hat{R}_2 y_1 & \hat{R}_2 y_2 \end{vmatrix} = \sin\sqrt{2} + \sqrt{2}\cos\sqrt{2} \neq 0$$

- The general solution is

$$y(x) = A\cos\sqrt{2}x + B\sin\sqrt{2}x + y_P(x), \quad y_P(x) = a + bx + cx^2$$

- $a = -1/3$, $b = -1/2$, $c = 1/3$

$$y(x) = A\cos\sqrt{2}x + B\sin\sqrt{2}x - \frac{1}{3} - \frac{x}{2} + \frac{x^2}{3}$$

- Determine A and B using boundary conditions

$$y(x) = \frac{1}{3}\cos\sqrt{2}x + \frac{1+\sqrt{2}\sin\sqrt{2} - \cos\sqrt{2}}{3(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2})}\sin\sqrt{2}x - \frac{1}{3} - \frac{x}{2} + \frac{x^2}{3}$$

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  Formally,

$$y(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x), \quad b_n = \int_a^b r(x) y(x) \varphi_n(x)\, dx$$

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At the boundary point $x = 1$ the expansion of y in terms of φ_n forces an incorrect boundary value on y. Near $x = 1$ the partial sums provide a very poor representation of y.

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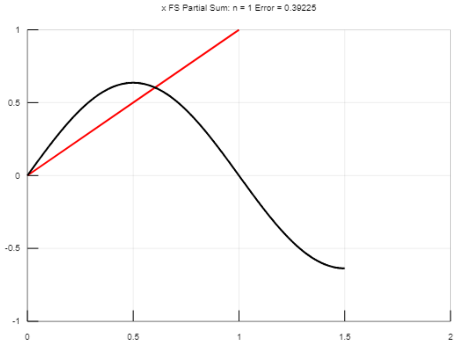
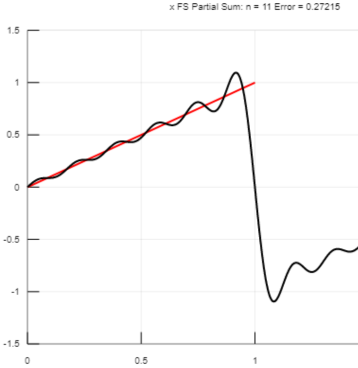

x FS Partial Sum: n = 1 Error = 0.39225
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2


x FS Partial Sum: n = 11 Error = 0.27215
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1

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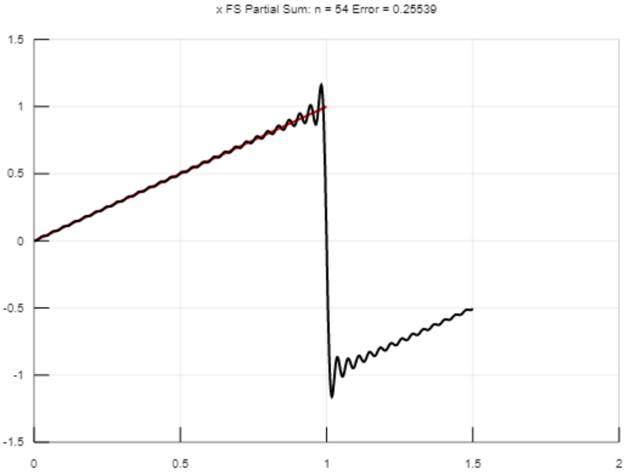

Exercises

1. Find the eigenvalues and eigenfunctions of the boundary value problem

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Establish if $\lambda = 0$ is an eigenvalue. Find an approximate value for the eigenvalue of smallest value. Estimate λ_n for large values of n.

Exercises

4. Determine real-valued eigenvalues of the boundary value problem

$$y''(x) + (\lambda + 1)y'(x) + \lambda y(x) = 0, \quad y'(0) = 0, \, y(1) = 0,$$

if any exist, and the form of the corresponding eigenfunction(s).

5. Solve the eigenvalue problem

$$x(xy')' + \lambda y = 0, \quad y'(1) = 0, \, y'(e^{2\pi}) = 0$$

for $y(x)$ and obtain the eigenfunctions.

6. Solve the non-homogeneous Sturm-Liouville boundary value problem

$$y''(x) + 9y(x) = \cos x, \quad 0 < x < \pi/4, \, y'(0) = 0, \, y(\pi/4) + y'(\pi/4) = 0.$$

Does it have a unique solution? If yes, obtain the solution in two ways.