vv256: Linear systems of ODEs with constant coefficients.

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Elimination

The elimination method is based on the correspondence between normal systems of linear ODEs and higher-order linear DEs. Consider a normal system of two linear equations with constant coefficients

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + b_1, \\ y_2' = a_{21}y_1 + a_{22}y_2 + b_2 \end{cases}$$

and the initial conditions $y_1(t_0) = y_{10}$ and $y_2(t_0) = y_{20}$.

- Differentiate the first equation and substitute y_2'

$$y_1'' = a_{11}y_1' + a_{12}y_2' = a_{11}y_1' + a_{12}(a_{21}y_1 + a_{22}y_2 + b_2)$$

- Eliminate y_2 from the second term by solviving the first equation explicitly for y_2

$$y_1'' = a_{11}y_1' + a_{12}a_{21}y_1 + a_{12}b_2 + a_{22}(y_1' - a_{11}y_1 - b_1)$$

or

Elimination

$$y_1'' - (a_{11} + a_{22})y_1' + (a_{11}a_{22} - a_{12}a_{21})y_1 = a_{12}b_2 - a_{22}b_1$$

- If λ_1 and λ_2 are two roots of the characteristic equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

  then $y_1(t) = y_{1c}(t) + y_{1p}(t)$, where the complementary solution is

$$y_{1c}(t) = \begin{cases} C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \ \lambda_1 \neq \lambda_2 \\ (C_1 + C_2 t)e^{\lambda_1 t}, \ \lambda_1 = \lambda_2 \end{cases}$$

  and the particular solution y_{1p} can be obtained in the usual way.
- Compute $y_2(t) = 1/a_{12}(y_1' - a_{11}y_1 - b_1)$. What if $a_{12} = 0$?

Elimination

The trajectory $x = x(t)$, $y = y(t)$ of a golf ball of mass m struck with initial speed v_0 and rising initially at angle θ_0 satisfies the differential equations

$$m\ddot{x} = -R_x, \quad m\ddot{y} = -mg - R_y$$

with initial conditions

$$x(0) = y(0) = 0, \, \dot{x}(0) = v_0 \cos\theta_0, \, \dot{y}(0) = v_0 \sin\theta_0.$$

In these equations,

- g is the gravitational acceleration,
- $x(t)$ and $y(t)$ are the horizontal range and vertical height of the ball at time t, and
- R_x, R_y are respectively the horizontal and vertical components of air resistance.

Elimination

(a) Write down the given initial value problem as a fourth order system using the dependent variables

$$y_1(t) = x(t), \quad y_2(t) = y(t), \quad y_3(t) = \dot{x}(t), \quad y_4(t) = \dot{y}(t)$$

$\dot{y}_1 = y_3$, $\dot{y}_2 = y_4$ and $\dot{y}_3 = \ddot{x} = -\dfrac{R_x}{m}$, $\dot{y}_4 = \ddot{y} = -g - \dfrac{R_y}{m}$

$$\begin{cases} \dot{y}_1 = y_3 & y_1(0) = 0 \\ \dot{y}_2 = y_4 & y_2(0) = 0 \\ \dot{y}_3 = -\dfrac{R_x}{m} & y_3(0) = v_0 \cos\theta_0 \\ \dot{y}_4 = -g - \dfrac{R_y}{m} & y_4(0) = v_0 \sin\theta_0 \end{cases}$$

Elimination

(b) What is the trajectory of a golf ball assuming that air resistance is proportional to velocity $R_x = mk\dot{x}$, $R_y = mk\dot{y}$. The model for air resistance gives

$$R_x = mky_3, \quad R_y = mky_4$$

and hence,

$$\begin{cases} \dot{y}_1 = y_3 & y_1(0) = 0 \\ \dot{y}_2 = y_4 & y_2(0) = 0 \\ \dot{y}_3 = -ky_3 & y_3(0) = v_0 \cos\theta_0 \\ \dot{y}_4 = -g - ky_4 & y_4(0) = v_0 \sin\theta_0 \end{cases}$$

Elimination

We can find y_3 and y_4 directly

$$y_3(t) = v_0 \cos\theta_0 e^{-kt},$$

$$y_4(t) = -\frac{g}{k} + \left(\frac{g}{k} + v_0 \sin\theta_0\right) e^{-kt} \quad \text{Verify it}$$

and then,

$$y_1(t) = \frac{v_0 \cos\theta_0}{k}(1 - e^{-kt}),$$

$$y_2(t) = -\frac{gt}{k} + \left(\frac{g}{k^2} + \frac{v_0 \sin\theta_0}{k}\right)(1 - e^{-kt})$$

The Matrix Method

Consider a normal homogeneous nth-dimensional system with constant coefficients

$$y'(t) = Ay(t) \quad \text{How do we find its general solution ?}$$

and look for a solution of the form $y(t) = e^{\lambda t}v$, where v is the constant vector. Substitute $y(t)$ into the equation to obtain

$$e^{\lambda t}(A - \lambda I)v = 0 \Rightarrow det(A - \lambda I) = 0$$

What are λ_i and the corresponding v_i? Eigenvalues and eigenvectors of the matrix A.

Since for distinct $\lambda_1, \ldots, \lambda_n$ the functions $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ and the eigenvectors $v_1, \ldots, v_n$ are linearly independent, so

$$e^{\lambda_1 t}v_1, \ldots, e^{\lambda_n t}v_n$$

are n linearly independent solutions.

The Matrix Method

If all eigenvalues $\lambda_1, \dots, \lambda_n$ are distinct then the complementary solution is

$$y(t) = C_1 e^{\lambda_1 t} v_1 + \ldots + C_n e^{\lambda_n t} v_n.$$

That is, $y(t) = \Phi(t) C$, where $\Phi(t)$ is the fundamental matrix of the system and C is a vector with coordinates $C_1, \dots, C_n$.
For the homogeneous system $y'(t) = Ay(t)$ with the initial condition $y(t_0) = y_0$,

$$y(t_0) = \Phi(t_0) C = y_0 \Rightarrow C = \Phi^{-1}(t_0) y_0$$

Example: Solve

$$\begin{cases} y_1' - y_2' - 6y_2 = 0, \\ y_1' + 2y_2' - 3y_1 = 0. \end{cases}$$

- Represent the system in the normal form

$$\begin{cases} y_1' = y_1 + 4y_2, \\ y_2' = y_1 - 2y_2 \end{cases} \Rightarrow A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$$

The Matrix Method

- The characteristic equation is
$$\det(A-\lambda I) = \left| \begin{array}{cc} -\lambda+1 & 4 \\ 1 & -2-\lambda \end{array} \right| = (\lambda+3)(\lambda-2) = 0$$
and there are two distinct eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$.
- $\lambda_1 = -3$
$$(A-\lambda_1 I)v_1 = \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$
$$\Rightarrow v_1^1 + v_1^2 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
- $\lambda_2 = 2$
$$(A-\lambda_2 I)v_2 = \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix} = 0$$
$$\Rightarrow v_2^1 - 4v_2^2 = 0 \Rightarrow v_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The Matrix Method

- The complimentary solution is

$$y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 = C_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

or

$$\begin{cases} y_1(t) = C_1 e^{-3t} + 4 C_2 e^{2t}, \\ y_2(t) = -C_1 e^{-3t} + C_2 e^{2t} \end{cases}$$

The Matrix Method: Complex Eigenvalues

If the matrix A of the homogeneous system $y'(t) = Ay(t)$ is a real matrix and $\lambda = \alpha + i\beta$ is its eigenvalue with the corresponding eigenvector v then $y_1(t) = \Re(e^{\lambda t} v)$ and $y_2(t) = \Im(e^{\lambda t} v)$ are two linearly independent real-valued solutions and the complementary solution is

$$y(t) = C_1 \Re(e^{\lambda t} v) + C_2 \Im(e^{\lambda t} v).$$

Example: Solve

$$\begin{cases} y_1' + y_1 - 5y_2 = 0, \\ 4y_1 + y_2' + 5y_2 = 0. \end{cases}$$

$$A = \begin{pmatrix} 11 & 5 \\ -4 & -5 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 5 \\ -4 & -5-\lambda \end{vmatrix} = 0$$

The eigenvalues are $\lambda_{1,2} = -3 \pm 4i$.

Example: Complex Eigenvalues

For $\lambda = -3+4i$, the corresponding eigenvector v satisfies

$$(A-\lambda I)v = \begin{pmatrix} 2-4i & 5 \\ -4 & -2-4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and hence,$(2-4i)v_1 + 5v_2 = 0$ or $-4v_1 - (2+4i)v_2 = 0$
Then $v_2 = -\frac{1}{5}(2-4i)v_1$. Taking $v_1 = 5 \Rightarrow v_2 = -2+4i$,

$$v = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + i \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Therefore,

$$e^{\lambda t} v = e^{-3t}(\cos 4t + i \sin 4t) \left(\begin{pmatrix} 5 \\ -2 \end{pmatrix} + i \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right)$$

$$= e^{-3t} \left[\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix} \cos 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \sin 4t \right) + \right.$$

$$\left. + i \left(\begin{pmatrix} 5 \\ -2 \end{pmatrix} \sin 4t + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \cos 4t \right) \right]$$

Example: Complex Eigenvalues

Thus, the complementary solution is

$$y(t) = C_1 e^{-3t}\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\sin 4t\right)$$

$$+C_2 e^{-3t}\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\sin 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\cos 4t\right)$$

and

$$y_1(t) = 5e^{-3t}(C_1\cos 4t + B\sin 4t),$$

$$y_2(t) = 2e^{-3t}[(-C_1 + 2C_2)\cos 4t - (2C_1 + C_2)\sin 4t]$$

The Matrix Method: Multiple Eigenvalues

- Recall, that if a matrix $A_{n\times n}$ has n distinct eigenvalues λ_i, $i = 1..n$, then the corresponding eigenvectors are linearly independent and form a complete basis of eigenvectors.
- What happens if $A_{n\times n}$ has repeated eigenvalues?In general case, the matrix $A_{n\times n}$ may not have n linearly independent eigenvectors!
- To obtain a FSS, we augment the eigenvectors with generalized eigenvectors.Let λ is an eigenvalue of multiplicity m, and there are only $k < m$ linearly independent eigenvectors corresponding to λ. A FSS is obtained by including $(m-k)$ generalized eigenvectors.

$$
\begin{array}{ll}
(A-\lambda I)v_i = 0 & \Rightarrow v_i,\ i = 1..k \text{ are lin. independent} \\
(A-\lambda I)v_{k+1} = v_k & \Rightarrow (A-\lambda I)^2 v_{k+1} = 0 \\
(A-\lambda I)v_{k+2} = v_{k+1} & \Rightarrow (A-\lambda I)^3 v_{k+2} = 0 \\
\dots & \\
(A-\lambda I)v_m = v_{m-1} & \Rightarrow (A-\lambda I)^{m-k+1} v_m = 0
\end{array}
$$

The Matrix Method: Multiple Eigenvalues

If the matrix A of the homogeneous system $y'(t) = Ay(t)$ has an eigenvalue λ of algebraic multiplicity $m > 1$, and a sequence of generalized eigenvectors corresponding to λ is $v_1, v_2, \ldots, v_m$. Then the corresponding m linearly independent solutions of the homogeneous system are

$$y_i(t) = e^{\lambda t} v_i,\ i = 1, \ldots, k,$$

$$y_{k+2} = e^{\lambda t}\left(v_k \frac{t^2}{2!} + v_{k+1} t + v_{k+2}\right),$$

$$\cdots$$

$$y_m = e^{\lambda t}\left(v_k \frac{t^{m-k}}{(m-k)!} + v_{k+1} \frac{t^{m-k-1}}{(m-k-1)!} + \ldots \right.$$
$$\left. + \ldots + v_{m-2} \frac{t^2}{2!} + v_{m-1} t + v_m\right).$$

Example: Multiple Eigenvalues

Example: Solve

$$\begin{cases} y_1' - 4y_1 + y_2 = 0, \\ 3y_1 - y_2' + y_2 - y_3 = 0, \\ y_1 - y_3' + y_3 = 0. \end{cases}$$

$$A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 & 0 \\ 3 & 1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

The eigenvalues are $\lambda_{1,2,3} = 2$.

$$(A - \lambda I)v_1 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{pmatrix} = \begin{pmatrix} 2v_1^1 - v_1^2 \\ 3v_1^1 - v_1^2 - v_1^3 \\ v_1^1 - v_1^3 \end{pmatrix} = \bar{0}$$

Take $v_1^1 = 1 \Rightarrow v_1^2 = 2v_1^1 = 2,\ v_1^3 = v_1^1 = 1$.
It is not possible to find two more linearly independent eigenvectors $\Rightarrow$complete basis of eigenvectors by including two generalized eigenvectors

Example: Multiple Eigenvalues

$$(A - \lambda I)v_2 = v_1 \Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{pmatrix}$$

$$= \begin{pmatrix} 2v_2^1 - v_2^2 \\ 3v_2^1 - v_2^2 - v_2^3 \\ v_2^1 - v_2^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Taking $v_2^1 = 2$, then $v_2^2 = 2v_2^1 - 1 = 3, v_2^3 = v_2^1 - 1 = 1.$

Example: Multiple Eigenvalues

$$(A - \lambda I)v_3 = v_2 \Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_3^1 \\ v_3^2 \\ v_3^3 \end{pmatrix}$$

$$= \begin{pmatrix} 2v_3^1 - v_3^2 \\ 3v_3^1 - v_3^2 - v_3^3 \\ v_3^1 - v_3^3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Taking $v_3^1 = 1$, then $v_3^2 = 2v_3^1 - 2 = 0, v_3^3 = v_3^1 - 1 = 0$.

Example: Multiple Eigenvalues

Three linearly independent solutions are

$$y_1(t) = e^{\lambda t} v_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

$$y_2(t) = e^{\lambda t}(v_1 t + v_2) = e^{2t} \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right],$$

$$y_3(t) = e^{\lambda t}(v_1 t^2/2 + v_2 t + v_3) = e^{2t} \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

Example: Multiple Eigenvalues

The complementary solution is

$$y(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{2t} \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right] +$$

$$+2C_3 e^{2t} \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

or

$$y_1(t) = e^{2t}[C_3 t^2 + (C_4 + 4C_3)t + (C_1 + 2C_2 + 2C_3)],$$

$$y_2(t) = e^{2t}[2C_3 t^2 + 2(C_2 + 3C_3)t + (2C_1 + 3C_2)],$$

$$y_3(t) = e^{2t}[C_3 t^2 + (C_2 + 2C_3)t + (C_1 + C_2)].$$

Example: Multiple Eigenvalues

Example: Solve $y'(t) = Ay(t), A = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & 2 \\ 3 & -3 & 5 \end{pmatrix}$.

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & -2 \\ 1 & -2-\lambda & 2 \\ 3 & -3 & 5-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-3) = 0$$

$\lambda_{1,2} = -1$:

$$(A - \lambda I)v = \begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 3 & -3 & 6 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} =$$

$$= \begin{pmatrix} -v^1 + v^2 - 2v^3 \\ -(-v^1 + v^2 - 2v^3) \\ -3(-v^1 + v^2 - 2v^3) \end{pmatrix} = \bar{0} \Rightarrow v^1 = v^2 - 2v^3$$

Example: Multiple Eigenvalues

Taking

$$v^2 = 1, v^3 = 0 \Rightarrow v^1 = 1$$

and taking

$$v^2 = 0, v^3 = 1 \Rightarrow v^1 = -2.$$

Therefore, although $\lambda = -1$ is an eigenvalue of multiplicity 2, two linearly independent eigenvectors do exist.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_3 = 3$:

$$(A - \lambda I)v_3 = \begin{pmatrix} -5 & 1 & -2 \\ 1 & -5 & 2 \\ 3 & -3 & 2 \end{pmatrix} \begin{pmatrix} v_3^1 \\ v_3^2 \\ v_3^3 \end{pmatrix} = \bar{0}$$

If $v_3^3 = 3 \Rightarrow v_3^1 = -1, v_3^2 = 1$ and

Example: Multiple Eigenvalues

$$v_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

The complementary solution is

$$y(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

and

$$y_1(t) = (C_1 - 2C_2)e^{-t} - C_3 e^{3t},$$

$$y_2(t) = C_1 e^{-t} + C_3 e^{3t},$$

$$y_3(t) = C_2 e^{-t} + 3C_3 e^{3t}.$$

Non-homogeneous Systems

Consider a non-homogeneous system of linear ODEs with constant coefficients

$$y'(t) = Ay(t) + b(t).$$

The complementary solution of the homogeneous system $y'(t) = Ay(t)$ has been obtained in the form

$$y(t) = \Phi(t)C,$$

where $\Phi(t)$ is a fundamental matrix with linearly independent columns-solutions of the homogeneous equation.
Therefore, $\Phi'(t) = A\Phi(t)$, and C is an n dimensional constant vector.
Apply variation of parameters, that is assume that $C = C(t)$ to obtain $y(t) = \Phi(t)C(t)$ and hence,

$$\Phi'(t)C(t) + \Phi(t)C'(t) = Ay(t) + b(t)$$

$$A\Phi(t)C(t) + \Phi(t)C'(t) = A\Phi(t)C(t) + b(t)$$

$$\Rightarrow \Phi(t)C'(t) = b(t) \Rightarrow C'(t) = \Phi^{-1}(t)b(t)$$

Non-homogeneous Systems

Integrate with respect to t to obtain

$$C(t) = C + \int \Phi^{-1}(t) b(t)\, dt.$$

Therefore, the general solution of the non-homogeneous system is

$$y(t) = \Phi(t) \left(C + \int \Phi^{-1}(t) b(t)\, dt \right).$$

Example: Solve

$$\begin{cases} y_1' + 3y_1 + 4y_2 = 2e^{-t}, \\ y_1 - y_2' + y_2 = 0. \end{cases}$$

Here

$$A = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix},\; b(t) = \begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix}$$

The roots of the characteristic equation $\det(A - \lambda I) = 0$ are $\lambda_{1,2} = -1$

Non-homogeneous Systems

$$(A-\lambda I)v_1 = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1^1 + 2v_1^2 = 0 \Rightarrow v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

But a second linearly independent eigenvector does not exist. We need to find a generalized eigenvector:

$$(A-\lambda I)v_2 = v_1 \Rightarrow \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$v_2^1 = -2v_2^2 - 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Two linearly independent solutions are

Non-homogeneous Systems

$$y_1(t) = e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix},\ y_2(t) = e^{-t}\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

and the fundamental matrix is

$$\Phi(t) = [y_1(t)\, y_2(t)] = \begin{pmatrix} 2e^{-t} & (2t+1)e^{-t} \\ -e^{-t} & -(t+1)e^{-t} \end{pmatrix}, \quad \det\Phi = -e^{-2t}$$

The inverse $\Phi^{-1}(t)$ of the fundamental matrix is

$$\Phi^{-1}(t) = \begin{pmatrix} (t+1)e^{t} & (2t+1)e^{t} \\ -e^{t} & -2e^{t} \end{pmatrix}.$$

Evaluate

$$\int \Phi^{-1}(t)b(t)\,dt = \int \begin{pmatrix} (t+1)e^{t} & (2t+1)e^{t} \\ -e^{t} & -2e^{t} \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix} dt$$

$$= \int \begin{pmatrix} 2(t+1) \\ -2 \end{pmatrix} dt = \begin{pmatrix} t^2 + 2t \\ -2t \end{pmatrix}$$

Non-homogeneous Systems

The general solution of the non-homogeneous system is

$$y(t) = \Phi(t)\left(C + \int \Phi^{-1}(t)b(t)\,dt\right)$$

$$= \begin{pmatrix} 2e^{-t} & (2t+1)e^{-t} \\ -e^{-t} & -(t+1)e^{-t} \end{pmatrix} \begin{pmatrix} C_1 + t^2 + 2t \\ C_2 - 2t \end{pmatrix}$$

Therefore,

$$y_1(t) = e^{-t}(-2t^2 + 2(C_2+1)t + (2C_1 + C_2),$$

$$y_2(t) = e^{-t}(t^2 - C_2 t - (C_1 + C_2)).$$

The Matrix Exponent Method

Consider the system

$$y'(t) = A(t)y(t) + b(t), \quad y(t_0) = y_0.$$

Can we find its solution $y(t)$ by integrating the equation?
Yes, but we need to understand how to deal with $\exp A$.
Use the linearity of the system and integrate it using a matrix integrating factor.

The Matrix Exponent Method

Let A be a $n \times n$ matrix. We define the exponent of A by

$$\exp A = I + A + \frac{A^2}{2} + \ldots + \frac{A^k}{k!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

where $A^2 = A \cdot A$, ect.

Is $\exp A$ well-defined? The convergence of the power series for e^t for all values of t guarantees that the series for $\exp A$ converges for all matrices A.

Properties of $\exp A$**:**

P1. $\exp 0 = I$

P2. For a constant matrix A,

$$\frac{d \exp(At)}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A \exp(At).$$

The Matrix Exponent Method

P3. $\exp A$ commutes with any power of A.
P4. If B commutes with A, that is $AB = BA$ then B commutes with $\exp A$.
P5. If A, B are $n \times n$ matricies then

$$\exp A \exp B = \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{j=0}^{\infty} \frac{A^j}{j!} = \sum_{m=0}^{\infty} \frac{1}{n!} \underbrace{\left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} A^k B^{n-k} \right)}_{\substack{(A+B)^n \text{ only when} \\ A, B \text{ are commuting matrices}}}$$

Therefore,

$$\exp A \exp B = \exp(A + B) = \exp B \exp A$$

only for commuting matrices A and B.
P6. Since A and $-A$ commute then

$$\exp A \exp(-A) = \exp 0 = I = \exp(-A) \exp A$$

Thus $\exp A$ has inverse $\exp(-A)$ for any matrix A.

The Matrix Exponent Method

The concept of the exponent of a matrix can now be employed to solve

$$y'(t) = A(t)y(t) + b(t), \quad y(t_0) = y_0.$$

Let $M(t)$ be the solution of the matrix equation

$$\frac{d(M(t))}{dt} = -M(t)A(t), \quad M(t_0) = I.$$

Then

$$\frac{d(My)}{dt} = M\frac{dy}{dt} + \frac{dM}{dt}y = M(Ay + b) - (MA)y = Mb.$$

By formal integration of this equation

$$M(t)y(t) = \int_{t_0}^{t} M(u)b(u)\,du + M(t_0)y(t_0) = \int_{t_0}^{t} M(u)b(u)\,du + y(t_0)$$

The Matrix Exponent Method

If $M(t)$ is non-singular for all t, then

$$y(t) = M^{-1}(t) \int_{t_0}^{t} M(u)b(u)\, du + M^{-1}(t)y(t_0)$$

Does such a matrix $M(t)$ exist for all A? We can prove existence of M based on the iterative construction

$$M_0(t) = I, \quad M_{k+1}(t) = \int_{t_0}^{t} M_k(s)A(s)\, ds,\ k = 0, 1, \ldots$$

and it gives us the definition

$$M(t) = \sum_{k=0}^{\infty} (-1)^{k+1} M_k(t)$$

that can be used that $M(t)$ satisfies the corresponding matrix equation.

The Matrix Exponent Method

If A is a constant matrix then

$$M(t) = \exp(-A(t - t_0))$$

and hence,

$$y(t) = \int_{t_0}^{t} [\exp(A(t-u))] b(u)\, du + \exp(A(t - t_0)) y(t_0)$$

Example: Consider a system $y'(t) = Ay(t)$ with

$$A = \begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$$

We shall use the matrix exponent method to determine the general solution of the system.
The general solution of the system is $y(t) = \exp(At)C$ where C is a vector of two arbitrary constants.

The Matrix Exponent Method

We need to calculate $\exp(At)$ in order to find the solution.
Notice that

$$y(t) = \exp[(A-I)t + It]C = \exp[(A-I)t]\exp(It)C$$

(You need to check that I and $A-I$ are commuting matrices)

$$\exp(It) = e^t I, \ \exp[(A-I)t] = \sum_{k=0}^{\infty} \frac{t^k (A-I)^k}{k!} = I + t(A-I)$$

(You need to verify that $(A-I)^k = 0$ for $k \geq 2$)
Therefore, the general solution is

$$y = e^t C + te^t (A-I)C$$

with the component form

$$y_1(t) = C_1 e^t + 2(2C_2 - 3C_1)te^t, \ y_2(t) = 3(2C_2 - 3C_1)te^t + C_2 e^t$$

The Matrix Exponent Method

- Let D be a diagonal matrix, $D = diag(d_1, d_2, \ldots, d_n)$. What is D^2, D^3, $\ldots, D^n, \ldots$? A direct verification shows that $\exp(Dt) = diag(\exp(d_1 t), \exp(d_2 t), \ldots, \exp(d_n t))$, where
$$\exp(d_i t) = 1 + \sum_{k=0}^{\infty} \frac{d_i^k t^k}{k!}, \quad i = 1 \ldots n$$
- Recall, that if a matrix A is diagonalizable then A is similar to a diagonal matrix D: $D = T^{-1}AT$, T is the transformation matrix (its columns are eigenvectors of A!)
- A and D have the same eigenvalues.Moreover, the elements of D are eigenvalues of A!!!
- Introduce a new function $y = Tx \Rightarrow Tx' = ATx$ and
$$x' = T^{-1}ATx = Dx$$

The Matrix Exponent Method

- The solution of this equation is

$$x = \exp(Dt)C = \begin{pmatrix} e^{d_1 t} & 0 & \dots & 0 \\ 0 & e^{d_2 t} & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & e^{d_n t} \end{pmatrix} C$$

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$$y = Tx = \begin{pmatrix} \varphi_1^1 e^{d_1 t} & \dots & \varphi_n^1 e^{d_n t} \\ \dots & & \\ \varphi_1^n e^{d_1 t} & \dots & \varphi_n^n e^{d_n t} \end{pmatrix} C$$

- For example, the solution of the DE $y' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} y$ is $y = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ with eigenvalues $\lambda_1 = 2\, \lambda_2 = -1$ of A and the corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The Matrix Exponent Method

Exercise: Find $\exp(A)$ for the following matrices

$$1.\, A = (3, 0; 0, -2); \quad 2.\, A = (0, 1; -1, 0); \quad 3.\, A = (2, 1; 0, 2);$$

$$4.\, A = (3, -1; 2, 0); \quad 5.\, A = (-2, -4; 1, 2);$$

$$6.\, A = (0, 1, 0; 0, 0, 0; 0, 0, 2).$$

See more worked examples in the class.