① $F(t, y', y'', \dots, y^{(n)}) = 0$
$y' = p(t)$

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③ if $F(t, y, y', \dots, y^{(n)}) = 0$ is homogeneous, then
$y' = y \cdot z$ ← new function to reduce the order

(1) $y''' = \underbrace{\sin t + \cos t}$ does not depend on y, y', y''

$\Downarrow$

$$y'' = \int (\sin t + \cos t)\,dt = -\cos t + \sin t + C_1$$

$$y' = \int (-\cos t + \sin t + C_1)\,dt = -\sin t - \cos t + C_1 t + C_2$$

$$y = \cos t - \sin t + C_1 \frac{t^2}{2} + C_2 t + C_3$$

(2) $t(y'' + 1) + y' = 0$

The equation does not contain $y \Rightarrow$ define $y' = p(t)$

$\Downarrow$

$y'' = p'(t)$

$$\Rightarrow t(p' + 1) + p = 0 \Rightarrow p' + \frac{p}{t} = -1$$

↑ it is a linear equation, solve it using a substitution $p = u \cdot v$

$\Downarrow$

$p' = u'v + uv'$

$$\Rightarrow u'v + u\left(v' + \frac{v}{t}\right) = -1; \quad \frac{dv}{v} = -\frac{dt}{t} \Rightarrow v = \frac{1}{t}$$

$\Downarrow$

$$\frac{u'}{t} = -1 \Rightarrow u = -\frac{t^2}{2} + C_1$$

$\Downarrow$

$$p = -\frac{t}{2} + \frac{C_1}{t}$$

$\Downarrow$

$$y = -\frac{t^2}{4} + C_1 \ln|t| + C_2$$

(3) $y'y''' - 3(y'')^2 = 0$

The equation does not contain t and y $\Rightarrow$

$\Rightarrow$ define $y' = p(y) \Rightarrow y'' = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dt}}_{p} = p\frac{dp}{dy}$

$$\Rightarrow y''' = p\left(p\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right)$$

$$\Rightarrow p\left(p\left(\frac{dp}{dy}\right)^2 + p^2\frac{d^2p}{dy^2}\right) - 3p^2\left(\frac{dp}{dy}\right)^2 = 0$$

$\Downarrow$ $\Leftarrow$ we divided by p, so we need to keep the solution $p=0$, i.e. $y = \text{Const}$

$$p\frac{d^2p}{dy^2} - 2\left(\frac{dp}{dy}\right)^2 = 0$$

Define $\frac{dp}{dy} = z \Rightarrow \frac{d^2p}{dy^2} = z\frac{dz}{dp} \Rightarrow pz\frac{dz}{dp} - 2z^2 = 0 \quad | : z \neq 0$

$\Downarrow$ keep the solution $z=0 \Rightarrow p = C_1$ $\Downarrow$ $y = C_1t + C_2$

$$p\frac{dz}{dp} - 2z = 0 \Rightarrow \ln|z| - \ln p^2 = \ln|C_1|$$

$$\Downarrow$$

$$z = \frac{dp}{dy} = C_1p^2$$

$$\Downarrow$$

$$-\frac{1}{p} = C_1y + C_2 \Rightarrow \frac{dt}{dy} = -(C_1y + C_2)$$

$$\Downarrow$$

$$\boxed{t = \bar{C}_1y^2 + \bar{C}_2y + C_3}$$

$$\bar{C}_1 = -\frac{C_1}{2}, \quad \bar{C}_2 = -C_2$$

(4) $y'' = \sqrt{1-(y')^2}$

The equation does not contain t, y

Let $y'=p \Rightarrow y''=p' \Rightarrow p' = \sqrt{1-p^2}$

$$\Downarrow$$

$$\frac{dp}{\sqrt{1-p^2}} = dt$$

$$\Downarrow$$

$$\sin^{-1} p = t + C_1 \Rightarrow p = \sin(t+C_1)$$

$$\Rightarrow \frac{dy}{dt} = p = \sin(t+C_1) \Rightarrow y = -\cos(t+C_1) + C_2$$

(5) $(1+t^2)y'' + 2ty' = t^3$

$$\Downarrow$$

$$\left((1+t^2)y'\right)' = \left(\frac{t^4}{4}\right)' \Rightarrow (1+t^2)y' = \frac{t^4}{4} + \frac{C_1}{4}$$

$$\Downarrow$$

$$y = \int \frac{t^4 + C_1}{4(1+t^2)}\, dt$$

$$\Downarrow$$

$$y = \frac{1}{12}t^3 + \frac{t}{4} + \bar{C}_1 \tan^{-1} t + C_2,$$

$$\bar{C}_1 = \frac{C_1+1}{4}$$

(6) $tyy'' - t(y')^2 - yy' = 0 \Leftarrow$ homogeneous $\Rightarrow y' = y \cdot z$

$$\Rightarrow y'' = y(z^2+z') \Rightarrow ty^2(z^2+z') - ty^2z^2 - y^2 z = 0 \quad \Big| \div y^2 \neq 0$$

keep the solution $y=0$

$$\Rightarrow tz' - z = 0 \Rightarrow \frac{dz}{z} - \frac{dt}{t} = 0 \Rightarrow z = C_1 t$$

$$\Downarrow$$

$$y' = C_1 t y \Rightarrow y = C_2 e^{\bar{C}_1 t^2}, \quad \bar{C}_1 = \frac{C_1}{2}$$