Separable equations

Definition. A separable ODE is an equation of the form

$$\frac{dy}{dx} = f(x)g(y),$$

where f and g are given functions.
Remark. If there is any value y_0 such that $g(y_0) = 0$, then $y = y_0$ is a solution (equilibrium solution). To find other (non-constant) solutions, we assume that $g(y) \neq 0$.

- Apply the definition of the differential

$$dy = y'(x)dx = \frac{dy}{dx}dx = f(x)g(y)dx. \tag{1}$$

- Separate variables

$$\frac{1}{g(y)}dy = f(x)dx, \quad \text{and}$$

- Integrate each side with respect to its variable to obtain

$$G(y) = F(x) + C, \tag{2}$$

where F and G are anti-derivatives of f and $1/g$, and C is an arbitrary constant.

Linear equations

A general form of a first-order linear DE is

$$y' + p(t)y = q(t),$$

where p and q are coefficient functions.To solve the equation,

- multiply the equation by an unknown nonzero function $\mu(t)$ (integrating factor)

$$\mu y' + \mu py = \mu q,$$

- choose μ so that the left-hand side is the derivative of the product μy:

$$\mu y' + \mu py = (\mu y)' = \mu y' + \mu' y \Rightarrow \mu' = \mu p$$

- find the solution of the equation $\mu' = \mu p$ by separation of variables

$$\int \frac{d\mu}{\mu} = \int p\, dt \quad \Rightarrow \quad \ln|\mu| = \int p\, dt,$$

$$\text{So,} \quad \mu = Ce^{\int p\, dt}, \quad C = const \neq 0.$$

- We need just one such function, let $C = 1$ and hence, $\mu = e^{\int p\, dt}$.
- Substitute μ into $\mu y' + \mu p y = \mu q \Rightarrow \quad (\mu y)' = \mu q$ and,
- finally find

$$y(t) = \frac{1}{\mu(t)} \left(\int \mu(t) q(t)\, dt + C \right).$$

Can we include an IC $y(t_0) = y_0$ into the solution directly? Yes.
Integrate $(\mu y)' = \mu q$ from t_0 to t, then

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(\tau) q(\tau)\, d\tau$$

and

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(\tau) q(\tau)\, d\tau + \mu(t_0) y(t_0) \right).$$

Homogeneous polar equations

What is a homogeneous polar equation?

$$y'(x) = f\left(\frac{y}{x}\right), \quad x \neq 0,$$

where f is a given one-variable function.

- Make the substitution $y(x) = xv(x)$. What is the derivative of y? $y' = v + xv'$
- The new unknown function v satisfies the DE

$$\frac{dv}{dx} = \frac{f(v) - v}{x}$$

What is the type of this equation? Separable

- For $f(v) - v \neq 0$ $\quad \displaystyle\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$
- What happens when $f(v) - v = 0$? $v' = 0$ The equation has singular solutions of the form $y = cx$, $c = const$.

Bernoulli equations

The general form of a Bernoulli equation is

$$y' + p(t)y = q(t)y^n, \quad n \neq 1.$$

Is it a linear equation? No.

- Make the substitution $y(t) = (w(t))^{1/(1-n)}$.
- Compute the derivative y'.

$$\frac{1}{1-n} w^{n/(1-n)} w' + p w^{1/(1-n)} = q w^{n/(1-n)}.$$

- Divide by $w^{n/(1-n)}$ and multiply by $1-n$ to obtain

$$w' + (1-n)pw = (1-n)q.$$

Is it a linear equation? Yes. $\Rightarrow$ Apply the known methods to solve it.

Riccati eqautions

The general form of a Riccati equation is

$$y' = q_0(t) + q_1(t)y + q_2(t)y^2,$$

where q_0, q_1, and q_2 are given functions, with $q_0, q_2 \neq 0$.

- Assume that we know a particular solution y_1 of the R. equation.
- Make the substitution $y = y_1 + \dfrac{1}{w}$ to obtain

$$y' = y_1' - \frac{w'}{w^2} = q_0 + q_1(y_1 + \frac{1}{w}) + q_2(y_1 + \frac{1}{w})^2.$$ What is y_1'?

- $q_0(t) + q_1(t)y_1 + q_2(t)y_1^2 - \dfrac{w'}{w^2} = q_0(t) + q_1(t)y_1 + $
$+ \dfrac{q_1}{w} + q_2 y_1^2 + 2\dfrac{q_2 y_1}{w} + \dfrac{q_2}{w^2}$

We obtain the linear equation $\quad w' + (q_1 + 2q_2 y_1)w = -q_2.$

Exact equations

The DE

$$P(x,y) + Q(x,y)y' = 0$$

is called an exact equation when $Pdx + Qdy$ is the differential of a function $f(x,y)$.

- If such a function $f(x,y)$ exists then the exact equation has the form $df(x,y) = 0$ with the general solution $f(x,y) = C,\ C = const.$
- If $f(x,y)$ exists then $df = f_x dx + f_y dy$ and hence,
- $f_x = P,\ f_y = Q.$
- $P_y = Q_x \quad \Leftarrow$ this is the necessary and sufficient condition for the equation to be exact.

Consider the IVP

$$y^2 - 4xy^3 + 2 + (2xy - 6x^2y^2)y' = 0, \quad y(1) = 1$$

$P(x,y),\ Q(x,y)$? $P(x,y) = y^2 - 4xy^3 + 2,\ Q(x,y) = 2xy - 6x^2y^2$

Is the equation exact? $P_y = 2y - 12xy^2 = Q_x \Rightarrow$ the equation is exact

Then there is a function $f = f(x,y)$ such that

$$f_x(x,y) = P(x,y) = y^2 - 4xy^3 + 2,\ f_y(x,y) = Q(x,y) = 2xy - 6x^2y^2.$$

We can integrate either A. the first equation with respect to x or B. the second equation with respect to y.

A.

$$f(x,y) = \int f_x(x,y)\,dx = \int (y^2 - 4xy^3 + 2)dx =$$

$$= xy^2 - 2x^2y^3 + 2x + g(y),$$

where $g(y)$ is an arbitrary function of y.

$$f_y(x,y) = 2xy - 6x^2y^2 + g'(y) = 2xy - 6x^2y^2 \quad \Rightarrow g = Const.$$

The general solution has the form $xy^2 - 2x^2y^3 + 2x = C$.
What is the value of C? $C = 1$.

$$xy^2 - 2x^2y^3 + 2x = 1$$

Integrating factors

恰当方程可以通过积分求出它的通解. 因此能否将一个非恰当方程化为恰当方程就有很大的意义. 积分因子就是为了解决这个问题而引进的概念.

如果存在连续可微的函数 $\mu=\mu(x,y)\neq 0$, 使得

$$\mu(x,y)M(x,y)dx+\mu(x,y)N(x,y)dy=0$$

为一恰当方程, 即存在函数 v, 使

$$\mu M dx+\mu N dy\equiv dv \tag{2.57}$$

则称 $\mu(x,y)$ 为方程(2.43)的**积分因子**.

这时 $v(x,y)=c$ 是(2.57) 的通解. 因而也就是(2.43) 的通解.

由此可知, 方程(2.43)有只与 x 有关的积分因子的充要条件是

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\psi(x) \tag{2.60}$$

这里 $\psi(x)$仅为 x 的函数. 假如条件(2.60)成立, 则根据方程(2.59), 可以求得方程(2.43)的一个积分因子

$$\mu=\mathrm{e}^{\int\psi(x)dx} \tag{2.61}$$

同样, (2.43)有只与 y 有关的积分因子的充要条件是

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=\varphi(y)$$

这里 $\varphi(y)$仅为 y 的函数. 从而求得方程(2.43)的一个积分因子

$$\mu=\mathrm{e}^{\int\varphi(y)dy}$$

Implicit first-order ODEs

$$1)\ y=f(x,y'),\quad 2)\ x=f(y,y')$$
$$3)\ F(x,y')=0,\quad 4)\ F(y,y')=0$$

1)

1) 首先讨论形如

$$y=f\left(x,\frac{dy}{dx}\right) \tag{2.63}$$

的方程的解法，这里假设函数 $f\left(x,\frac{dy}{dx}\right)$有连续的偏导数.

引进参数$\frac{dy}{dx}=p$，则(2 63)变为

$$y=f(x,p) \tag{2.64}$$

将(2.64)两边对 x 求导数，并以$\frac{dy}{dx}=p$ 代入，得到

$$p=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{dp}{dx} \tag{2.65}$$

方程(2.65)是关于 x,p 的一阶微分方程，但它的导数已解出．于是我们可按§2.1—§2.3的方法求出它的解.

若已求得(2.65)的通解的形式为

$$p=\varphi(x,c)$$

将它代入(2.64)，得到

$$y=f(x,\varphi(x,c))$$

这就是(2.63)的通解.

若求得(2.65)的通解的形式为

$$x=\psi(p,c)$$

则得到(2.63)的参数形式的通解为

$$\begin{cases}x=\psi(p,c)\\y=f(\psi(p,c),p)\end{cases}$$

其中 p 是参数，c 是任意常数.

若求得(2.65)的通解的形式为

$$\Phi(x,p,c)=0$$

则得到(2.63)的参数形式的通解

$$\begin{cases}\Phi(x,p,c)=0\\y=f(x,p)\end{cases}$$

其中 p 是参数，c 为任意常数.

2)

2) 形如

$$x=f\left(y,\frac{dy}{dx}\right) \tag{2.70}$$

的方程的求解方法与方程(2.63)的求解方法完全类似．这里假定函数 $f\left(y,\frac{dy}{dx}\right)$ 有连续偏导数．

引进参数 $\frac{dy}{dx}=p$，则(2.70)变为

$$x=f(y,p) \tag{2.71}$$

将(2.71)两边对 y 求导数，然后以 $\frac{dx}{dy}=\frac{1}{p}$ 代入，得到

$$\frac{1}{p}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial p}\frac{dp}{dy} \tag{2.72}$$

方程(2.72)是关于 y,p 的一阶微分方程，但它的导数 $\frac{dp}{dy}$ 已解出，于是可按 § 2.1—§ 2.3 的办法去求解．设求得通解为

$$\Phi(y,p,c)=0$$

则得(2.70)的通解为

$$\begin{cases}x=f(y,p)\\ \Phi(y,p,c)=0\end{cases}$$

3)

3) 现在讨论形如

$$F(x,y')=0 \tag{2.74}$$

的方程的解法．

记 $p=y'=\frac{dy}{dx}$. 从几何的观点看，$F(x,p)=0$ 代表 xp 平面上的一条曲线．设把这曲线表为适当的参数形式

$$x=\varphi(t),\quad p=\psi(t) \tag{2.75}$$

这里 t 为参数．再注意到，沿方程(2.74)的任何一条积分曲线上，恒满足基本关系

$$dy=p\,dx$$

以(2.75)代入上式得

$$dy=\psi(t)\varphi'(t)dt$$

两边积分，得到

$$y=\int\psi(t)\varphi'(t)dt+c$$

于是得到方程(2.74)的参数形式的通解为

$$\begin{cases}x=\varphi(t)\\ y=\int\psi(t)\varphi'(t)dt+c\end{cases}$$

c 为任意常数．

4)

4）形如

$$F(y, y')=0 \tag{2.76}$$

的方程，其求解方法同方程(2.74)的求解方法类似.

记 $p=y'$，引入参数 t，将方程表为适当的参数形式:

$$y=\varphi(t),\ p=\psi(t)$$

由关系式 $dy=pdx$ 得 $\varphi'(t)dt=\psi(t)dx$，由此得

$$dx=\frac{\varphi'(t)}{\psi(t)}dt,\ x=\int\frac{\varphi'(t)}{\psi(t)}dt+c$$

于是

$$\begin{cases} x=\int\frac{\varphi'(t)}{\psi(t)}dt+c \\ y=\varphi(t) \end{cases}$$

为方程的参数形式的通解，其中 c 为任意常数.

此外，不难验证，若 $F(y,0)=0$ 有实根 $y=k$，则 $y=k$ 也是方程的解.

Autonomous equations

见课件 Week2-3

Intervals of existence

Consider the Initial Value Problem (IVP)

$$y'=f(x,y), \quad y(x_0)=y.$$

Suppose $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous functions in some open rectangle $R=\{(x,y)\colon |x-x_0|<a, |y-y_0|<b\}$, $a,b>0$, and hence, there exist $K, L>0$ such that

$$(a)\ |f(x,y)|\le K, \quad (b)\ \left|\frac{\partial f}{\partial y}\right|\le L \quad \forall(x,y).$$

Then the IVP has a unique solution in the interval $|x-x_0|\le\alpha$, where $\alpha=\min\{a, \frac{b}{K}\}$.

Theorem: Let J be an open interval of the form $a < t < b$ and t_0 be a point in J. Consider the IVP

$$y' + p(t)y = q(t),\ y(t_0) = y_0,$$

where y_0 is a given initial value.
If p and q are continuous on J, then the IVP has a unique solution on J for any y_0.
Definition: The largest open interval J on which an IVP has a unique solution is called the maximal interval of existence for that solution.
Theorem. Consider the IVP

$$y' = f(t, y),\ y(t_0) = y_0,$$

where f, f_y are continuous in an open rectangle R functions, $R = \{(t, y) : a < t < b,\ c < y < d\}$. If $(t_0, y_0) \in R$ then the IVP has a unique solution in some open interval J of the form $t_0 - h < t < t_0 + h$ contained in the interval $a < t < b$.
The conditions of the theorems are sufficient. Are they nessecary?
Definition: We say that a mathematical model is well-posed (correctly formulated) if there exists its unique solution.

Singular solution

We have already considered ODEs with singular solutions (check examples above). Intuitively, a singular solution is a special solution that is not contained in the general solution for any values of the constant C including $C = \pm\infty$.What do singular solution mean geometrically? How can we plot them?
Definition. A solution $y = \varphi(t)$ of the differential equation

$$F(t, y, y') = 0 \tag{5}$$

is called singular if the uniqueness property does not hold at any of its points, that is,

- there is another solution of the same ODEs passing through each point (t_0, y_0) of the singular solution, and
- both solutions have the same tangent at the point (t_0, y_0) but
- another non-singular solution is different form the singular one in any arbitrary small neighborhood of the point (t_0, y_0).

Does a singular solution satisfies the equation (5)? Yes. Moreover, if $F(t, y, y')$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial y'}$ are continuous with respect to all arguments t, y, y' then any singular solution satisfies the equation

$$\frac{\partial F(t, y, y')}{\partial y'} = 0. \tag{6}$$

How can we find a singular solutions from (5) and (6)? $\Rightarrow$ Eliminate y'. Elimination gives us an equation

$$\psi_p(t, y) = 0$$

which is called p-discriminant of the equation (5), and the integral curve corresponding p-discriminant is called the p-discriminant integral curve.

Is a p-discriminant curve unique? Does it define a singular solution? In general, no. $\Rightarrow$ Double-check.

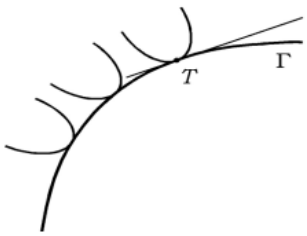
为了求微分方程的奇解，可以先求其通解，再求通解的包络。

Consider the equation

$$\Phi(t, y, C) = 0$$

with a parameter C and continuous Φ_t, Φ_y, Φ_C. It defines a family of curves depending on one parameter.

An **envelope** of the family of curves with a parameter is a smooth curve Γ that touches one curve of the family at any of its points and any its segment is touched by an infinite number of curves from the family. What does it mean if curves touch? A common tangent.



Does the envelope satisfies the definition of a singular integral curve? Yes. $\Rightarrow$ the envelope defines a singular solution.
An envelope is a part of a C-discriminant curve defined a by

$$\begin{cases} \Phi(t,y,C)=0 \\ \frac{\partial\Phi(t,y,C)}{\partial C}=0 \end{cases}$$

To make sure that a branch of a C-discriminant curve is an envelope, we check the following conditions.

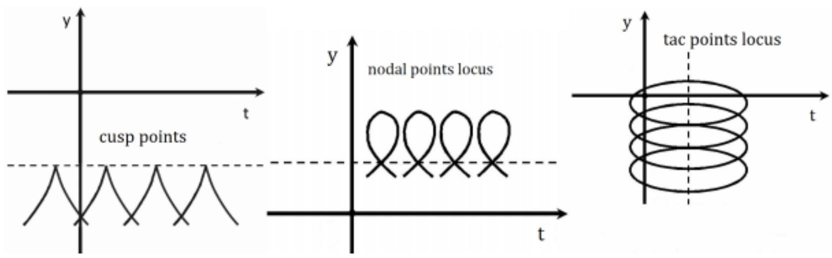
- there exist bounded partial derivatives $\left|\dfrac{\partial\Phi}{\partial t}\right| \le M$, $\left|\dfrac{\partial\Phi}{\partial y}\right| \le N$, $M, N = const$,
- $\dfrac{\partial\Phi}{\partial t} \neq 0$, or, $\dfrac{\partial\Phi}{\partial y} \neq 0$

Are these condition are necessary or sufficient? Sufficient. $\Rightarrow$ if they are not satisfied on a branch of the C-discriminant curve, it can still be an envelope.

The equations of p-discriminant and C-discriminant have a certain structure

$$\psi_p(t,y) = E \cdot C \cdot T^2 = 0,$$
$$\psi_C(t,y) = E \cdot N^2 \cdot C^3 = 0,$$

where $E=0$ is the equation of the envelope, $C=0$ is the equation of the cusp locus, $N=0$ is the equation of nodal locus, $T=0$ is the equation of the tac locus. Attention! Over all locus points only the envelope is a singular solution.



$$\int \frac{dx}{1+x^2} = arctanx + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = arcsinx + c$$

$$\int \frac{dx}{cos^2x} = tanx + c$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\ |x + \sqrt{1+x^2}| + c$$

$$\int secxdx = \ln|secx + tanx| + c$$

$$\int cscxdx = -\ln|cotx + cscx| + c$$

$$e^x = 1 + \frac{x}{1!} + \frac{x}{2!} + \cdots$$

$$sinx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$cosx = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + o\left(x^3\right)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o\left(x^3\right)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + o\left(x^5\right)$$

$$\arcsin x = x + \frac{1}{2}\times\frac{x^3}{3} + \frac{1\times 3}{2\times 4}\times\frac{x^5}{5} + \frac{1\times 3\times 5}{2\times 4\times 6}\times\frac{x^7}{7} + o\left(x^7\right)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^4\right)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + o\left(x^3\right)$$

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + o\left(x^3\right)$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots$$

Vibration
Variation of parameters
Undetermined coefficient

$f(t)e^{\lambda t}$	$y_p(t)$
$f(t)$	$t^k g(t)$
$f(t)e^{\alpha t}$	$t^k g(t)e^{\alpha t}$
$f(t)e^{\alpha t}\sin\beta t$	$t^k e^{\alpha t}[g(t)\sin\beta t + h(t)\cos\beta t]$
$f(t)e^{\alpha t}\cos\beta t$	$t^k e^{\alpha t}[g(t)\sin\beta t + h(t)\cos\beta t]$

Higher-order ODEs: Properties

P5. The Liouville formula for $W(t)$ is

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} a_1(\tau)\,d\tau\right)$$

Remark: Consider two solutions $y_1(t)$ and $y_2(t)$ of the second-order linear homogeneous ODE

$$y'' + a_1(t)y' + a_2(t)y = 0.$$

What is the Wronskian of y_1, y_2?

$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1y_2' - y_1'y_2$

Then $\left(\dfrac{y_2}{y_1}\right)' = \dfrac{y_2'y_1 - y_1'y_2}{y_1^2} = \dfrac{W[y_1, y_2](t)}{y_1^2}$

$$y_2(t) = y_1(t)\left(C_1\int \frac{\exp(-\int a_1(\tau)\,d\tau)}{y_1^2(t)}\,dt + C_2\right)$$

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

A point x_0 is called an ordinary point of the given differential equation if each of the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ and $f(x)$ are analytic at $x = x_0$, that is $p_i(x)$, $i = 1\ldots n-1$ and $f(x)$ can be expressed as power series about x_0 that are convergent for $|x - x_0| < r$, $r > 0$:

$$p_i(x) = \sum_{n=0}^{\infty} p_{i,n}(x - x_0)^n, \quad f(x) = \sum_{n=0}^{\infty} f_n(x - x_0)^n.$$

Definition

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_0(x)y(x) = f(x).$$

A point x_0 is called a singular point of the given differential equation if it is not an ordinary point, that is, not all of the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are analytic at $x = x_0$.
A point x_0 is called a regular singular point of the given differential equation if it is not an ordinary point, BUT all of $(x - x_0)^{n-k}p_k(x)$ are analytic for $k = 0, 1, \ldots, n-1$.
A point x_0 is irregular point of the given differential equation if it is neither an ordinary point nor a regular singular point.

Fuch's theorem: 用来找 two linearly independent solutions

Fuch's Theorem

For the second-order linear homogeneous ordinary differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0,$$

if $x = 0$ is a regular singular point then

$$xP(x) = \sum_{n=0}^{\infty} P_n x^n, \quad x^2Q(x) = \sum_{n=0}^{\infty} Q_n x^n, \ |x| < r.$$

Let the **indicial equation**

$$\alpha(\alpha - 1) + \alpha P_0 + Q_0 = 0$$

has two real roots $\alpha_1 \geq \alpha_2$. Then the DE has at least one Frobenius series solution given by

$$y_1(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n,\ a_0 \neq 0,\ 0 < x < r,$$

where the coefficients a_n can be determined by substituting $y_1(x)$ into the differential equation. A second linearly independent solution is obtained as follows:

1. If $\alpha_1 - \alpha_2$ is not equal to an integer, then a second **Frobenius series solution** is given by

$$y_2(x) = x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n,\ 0 < x < r,$$

in which the coefficients b_n can be determined by substituting $y_2(x)$ into the differential equation.

2. If $\alpha_1 = \alpha_2 = \alpha$, then

$$y_2(x) = y_1(x) \ln x + x^{\alpha} \sum_{n=0}^{\infty} b_n x^n,\ 0 < x < r,$$

where b_n can be determined by substituting $y_2(x)$ into the differential equation, once $y_1(x)$ is known. In this case, the second solution $y_2(x)$ is not a Frobenius series solution.

3. If $\alpha_1 - \alpha_2$ is a positive integer, then

$$y_2(x) = a y_1(x) \ln x + x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n,\ 0 < x < r,$$

where b_n and a can be determined by substituting y_2 into the differential equation. The parameter a may be zero, in which case the second solution $y_2(x)$ is also a Frobenius series solution.

A Fourier series of a periodic function $y = f(x)$ with $T = 2l$ has the following representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n}{l} x + b_n \sin \frac{\pi n}{l} x \right)$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{\pi n}{l} x \, dx, \quad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{\pi n}{l} x \, dx,$$

Boundary value problem of second order

A general boundary value problem of second order consists of the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = h(x), \; a < x < b$$

and two boundary conditions

- of the first kind

$$y(a) = \eta_1, \quad y(b) = \eta_2$$

- of the second kind

$$y'(a) = \eta_1, \quad y'(b) = \eta_2$$

- of the third kind

$$\alpha_1 y(a) + \alpha_2 y'(a) = \eta_1, \quad \beta_1 y(b) + \beta_2 y'(b) = \eta_2$$

- or periodic

$$y(a) = y(b), \quad y'(a) = y'(b)$$