Mid 1 Review

Qualitative Theory of D.E.

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Overview

Reference: VV256 Lecture Slides by Professor Jing, Professor Olga, VV256 TA Group19FA

Key Ideas

1. non-linear $\Longrightarrow$ linear
2. non-homogeneous $\Longrightarrow$ homogeneous
3. higher-order $\Longrightarrow$ reduced order
4. implicit $\Longrightarrow$ explicit
5. Multi-variable $\Longrightarrow$ Reduced-variable (parametric method)
6. Non-exact equation $\Longrightarrow$ Exact equation

Dynamics behind D.E.

Interval of Existence

Definition. Interval of Existence

The largest open interval J on which an IVP has a unique solution is called the maximal interval of existence for that solution.

Geometric interpretation: Dynamics

$$\begin{cases} \textit{Initial Condition } (\textit{point}, \textit{initialstate}) \\ \textit{Differential Equation } (\textit{trajectory}, \textit{mechanism}) \end{cases} \Rightarrow \textit{Interval Estimation}$$

The interval just interprets how far we can go given those I.C. And it is actually the thought behind the continuation of the solution.

Well-posed Solution

Definition

It is also named as correctly-formulated, and contains mainly three parts of information.

$$\begin{cases} \textit{existence of solution} \\ \textit{uniqueness} \\ \textit{continuously depending on the condition} \end{cases}$$

Approaches to identify the largest interval of existence J

1. Solve directly the equation
   $\Rightarrow$ by definition(geometric interpret) find.
2. Apply existence and uniqueness theorem to identify the existence.
   $\Rightarrow$ Then estimate by applying inequalities.

Existence and Uniqueness Theorem

★★ All of the following are sufficient condition

$$\begin{cases} \textit{Lipschitz Condition} & \textit{nonlinear} \\ \textit{Theorem for First Order, Linear, IVP} & \textit{linear} \\ \textit{Theorem for First Order, explicit, IVP} & \textit{nonlinear} \end{cases}$$

Lipschitz Condition

Consider the Initial Value Problem (IVP)

$$y' = f(x, y), \quad y(x_0) = y_0$$

The condition provided in the next slide provides the sufficient condition for the existence of unique solution of the IVP in the interval

$$|x - x_0| \leq \alpha, \qquad \alpha = min\{a, \frac{b}{K}\}$$

And L here is called the Lipschitz constant.

Lipschitz Condition

Lipschitz, continued

And the condition is

1. $f(x,y), \dfrac{\partial f}{\partial y}(x,y)$ are continuous functions in some open rectangle $R = \{(x,y) : |x - x_0| < a, |y - y_0| < b\}$
2. $\exists K, L,\ s.t.$

$$\begin{cases} |f(x,y)| \leq K \\ |\dfrac{\partial f}{\partial y}| \leq L \end{cases}$$

First Order, Linear, IVP

Theorem

Consider the IVP

$$y' + p(t)y = q(t), \quad y(t_0) = y_0$$

If p, q are both continuous on an open interval J who contains the initial point, i.e.

$$J : a < t < b, \quad t_0 \in J$$

Then the IVP has a unique solution on J for any y_0.

First Order, explicit, IVP

Theorem

Consider the IVP,

$$y' = f(t, y), \quad y(t_0) = y_0$$

If

1. $f, \dfrac{\partial f}{\partial y}$ are both continuous in some open rectangle $R = \{(x, y) : a < t < b, c < y < d\}$
2. Initial point $(t_0, y_0) \in R$

Then the IVP has a unique solution in some open interval J of the form

$$J : t_0 - h < t < t_0 + h, \quad J \subseteq (a < t < b)$$

Continuation of solution

See the Worksheet_ Qualitative_Theory for reference.

Comment

Actually, it can be shown that the Lipschitz condition converges to B.L.'s estimation after applying the Picard iterations if the interval does exist.

Singular Sol: Conception and Theorem

1. Object: All cases for first O.D.E.

$$F(t, y, y') = 0$$

Def. Singular

Singularity occurs when uniqueness property fails.

Def. Singular Solution

A solution $y = \phi(t)$ of the differential equation is called singular if the uniqueness property does not hold at any of its points:

1. There is another solution of the same ODEs passing through each point (t_0, y_0) of the singular solution.
2. Both solutions have the same tangent at the point (t_0, y_0).
3. Another non-singular solution is different form the singular one in any arbitrary small neighbourhood of the point (t_0, y_0).

Singular Sol: Percept the Def.

Behind the def. Special Sol v.s. Singular Sol

If a given curve (equation) satisfies the equation, then it is called the special sol of the equation.
Moreover, if it cannot be expressed as any form of the general sol, then it is the singular sol of the equation.
With the understanding above, we then can by def. identify the singular sol.

However, always it is the case that we can not by eyesight find the form of singular sol. The following two necessary check, however, helps to narrow down the range of our search.

Necessary Check I. p-discriminant

Theorem. Necessary Condition for Singular Sol

If $F(t, y, y')$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial y'}$ are continuous for all t, y, y',
Then any singular solution satisfies the equations

$$\begin{cases} F(t, y, y') = 0 \\ \dfrac{\partial F(t, y, y')}{\partial y'} = 0 \end{cases}$$

Def. p-discriminant

Eliminate the parameter $p = y'$ we then obtain immediately the p-discriminant (equation!)

$$\psi_p(t, y) = 0$$

. Comment : Satisfying p-discriminant may not be singular solution. But singular solution must satisfy p-discriminant.

Exercise: p-discriminant as Singular Solution

Check the p-discriminant for the following equations and verify if they are the singular solution, general solution or not the solution of the D.E.

1. $$(y')^2 + y - x = 0$$
2. $$(y')^2 - y^2 = 0$$

Steps to identify the p-discriminant as singular sol

1. Solve for p-discriminant of the equation.
2. Check if it satisfies the original D.E. If not, then it fails to be sol.
3. Solve the D.E. to obtain the general sol.
4. Check if it can be expressed by the general sol. If not, then it confirms to be the singular sol.

Necessary Check II. C-discriminant

Geometric Equivalence of Singular Sol

Geometrically, the singular sol of the D.E. is defined by the envelope of the family of the D.E.'s general sol, which is denoted as E.

Def. C-discriminant

The C-discriminant $\psi_C(t, y) = 0$ of the given family of one-parametric curves $\Phi(x, y, C)$ is defined by

$$\begin{cases} \Phi(t, y, C) = 0 \\ \dfrac{\partial \Phi(t, y, C)}{\partial C} = 0 \end{cases}$$

Theorem. Necessary Check for envelope

The envelope defined by the given D.E. must satisfy the C-discriminant given by its general sol.

C-discriminant: continued

Comment

Similarly, due to the necessary essence, an envelope must be a part of C-discriminant curve; however, a part of C-discriminant curve may not be the envelope.

Sufficient Condition for identifying C-curve as Envelope

If when plugging in the C-discriminant $\psi_C(t, y) = 0$, the original family of sol $\Phi(t, y, C)$ satisfies

1. $$\left|\frac{\partial \Phi}{\partial t}\right| \leq M, \left|\frac{\partial \Phi}{\partial y}\right| \leq N, \quad M, N = \text{const}$$

2. $$\left|\frac{\partial \Phi}{\partial t}\right|^2 + \left|\frac{\partial \Phi}{\partial y}\right|^2 \neq 0$$

Then the C-discriminant is actually the singular sol.

Exercises: C-discriminant as Singular Sol

Find Singular solutions of the equation

$$t(y')^2 - 2yy' + 4t = 0$$

with the general solution $t^2 = C(y - C)$

Solution

1. *Find the C-discriminant curve. ($y = 2C \Rightarrow y = \pm 2t$)*
2. *Verify if they are solutions of the equation.*
3. *Use the sufficient condition to find if it is a singular sol.*

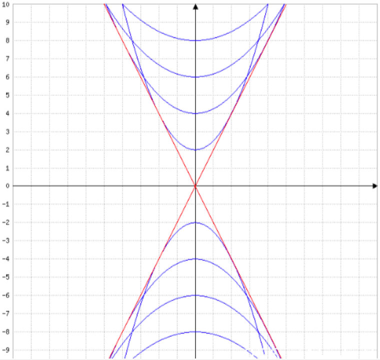
$$\left|\frac{\partial \Phi}{\partial t}\right| = |2t|, \quad \left|\frac{\partial \Phi}{\partial y}\right| = |C| = \left|\frac{y}{2}\right| = |t|$$

when $t = 0$, the second condition doesn't satisfies; since it is the sufficient condition, the test provides no additional information for its singularity seemingly. However, it is rather reasonable for us to doubt its validity to becoming a singularity.

Exercise: continued

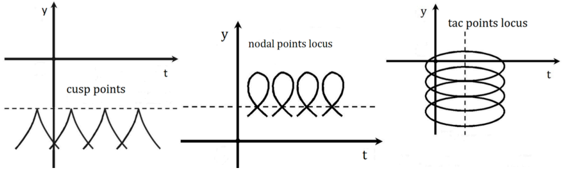
Solution

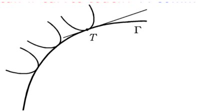
And we shall refer only to the graph in the case to verify our guess for its not being a singular sol,



Geometric Interpretation

Envelope, Cusp, Nodal, Tac





Comment: Envelope is the geometric equivalence of singular sol. While the others are even not the solution of the D.E. (doesn't satisfy the dynamics defined by the D.E.)

Singular solution: Steps to find

1. By inspection, identity the existence (linear case or special form of non-linear).
2. If not sure, first solve the ordinary differential equation and apply discriminant to check.

Comment

Generally, if the family of solution $\Phi(t, y, C) = 0$ can be expressed in exact the form, then we tend to apply C discriminant;
Otherwise, if it can only be expressed in the parametric form (as it is in the case of implicit first order), then we prefer to apply the p-discriminant.