Introduction to Variance Derivatives
Applied Stochastic Processes (FIN 514)

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Stochastic Volatility (SV) Models

We assume that the stock price S_t follows an SV model:

$$\frac{dS_t}{S_t^\beta} = \sigma_t dW_t = \sqrt{v_t}(\rho dZ_t + \rho_* dX_t)\text{for} \quad \rho_* = \sqrt{1-\rho^2}$$

$$dv_t = a(t, \sigma_t)dt + b(t, v_t)dZ_t \quad (dW_t\, dZ_t = \rho\; dt)$$

Also define the integrated variance V_T and average variance R_T by

$$V_T = \int_0^T v_t dt \quad \text{and} \quad R_T = \frac{1}{T}\int_0^T v_t dt.$$

Integrated variance as time-clock

- From Itô's isometry, V_t plays the role of time clock of the Brownian motion.

$$\sqrt{v_t}\,W_t \sim W_{V_t} \quad (e.g., \sigma W_t = W_{\sigma^2 t})$$

- How to measure V_T (R_T) in real world?
- We discretize time period $[0,T]$ into N time steps of size h: $t_i = ih/N$ ($Nh = T$). Then, we sum the realized log return variance.

$$R^h_{0,T} = \frac{1}{T}\sum_{i=1}^{N} \log^2(S_i/S_{i-1})$$

- In the limit of very small h:

$$\lim_{h\downarrow 0} R^h_{0,T} = \frac{1}{T}\int_0^T \left(\sqrt{v_t}(\rho dZ_t + \rho_* dX_t)\right)^2 = \frac{1}{T}\int_0^T v_t dt = R_{0,T}.$$

- The variance $R^h_{0,T}$ can be treated like an asset to bet!

Variance Swap

- In general, a swap product is an agreement to exchange the floating and fixed legs of payment for future time T.
- In variance swap, floating leg is $R^h_{0,T}$ and fixed leg is K^h.
- Variance swap is a pure bet on the variance (or volatility) to be realized in the future. Unlike trading European options, you don't have to delta-hedge.
- The fair strike $K_{\text{v-s}}$ is determined to make NPV zero on the trading day:

$$K_{\text{v-s}} = E\left(R^h_{0,T}\right)$$

- It is quoted by the volatility:

$$\sigma_{\text{v-s}} = \sqrt{K_{\text{v-s}}} \quad \text{(e.g., 20\%, 40\%)}.$$

- Variance option is also traded:

$$C_{\text{v-s}} = E\left(\left[R^h_{0,T} - K\right]^+\right)$$

Variance Swap under the Heston Model

- Heston model:

$$dv_t = \kappa(\theta - v_t)dt + \nu\,\sqrt{v_t}dZ_t.$$

- The fair strike of the continuously monitored variance swap is analytically available:

$$K^0_{\text{v-s}} = E(R_{0,T}) = \theta + (v_0 - \theta)\frac{1 - e^{-\kappa T}}{\kappa T}.$$

- Discretely monitored variance swap is also known as a correction term:

$$K^h_{\text{v-s}} = E(R^h_{0,T}) = K_{\text{v-s}} + \Delta^h_{\text{v-s}},$$

$$\begin{aligned}\Delta^h_{\text{swap}} = &\frac{h(\theta + 2q - 2r)}{4}\left[(\theta + 2q - 2r) + 2(v_0 - \theta)\frac{1 - e^{-\kappa T}}{\kappa T}\right] \\ &+ \frac{\theta\nu}{\kappa}\left(\frac{\nu}{4\kappa} - \rho\right)\left(1 - \frac{1 - e^{-\kappa h}}{\kappa h}\right) + (v_0 - \theta)\frac{\nu}{\kappa}\left(\frac{\nu}{2\kappa} - \rho\right)\frac{1 - e^{-\kappa T}}{\kappa T}\left(1 + \frac{\kappa h}{1 - e^{\kappa h}}\right) \\ &+ \left[\frac{\nu^2}{\kappa^2}(\theta - 2v_0) + \frac{2}{\kappa}(v_0 - \theta)^2\right]\frac{1 - e^{-2\kappa T}}{8\kappa T}\,\frac{1 - e^{-\kappa h}}{1 + e^{-\kappa h}}.\end{aligned}$$

Timer Option

- The payout is the same as that of European option:

$$C_{\text{timer}} = E\left(e^{-r\tau}[S_\tau - K]^+ \,|\, \mathcal{F}_0\right)$$

- But the expiry τ is not specified. It is randomly determined when the integrated variance V_t hits $\sigma_B^2 T$ (variance budget):

$$\tau = \min(\tau_B, T) \quad \text{and} \quad \tau_B = \min\{t : V_t = \sigma_B^2 T\},$$

  where σ_B (volatility budget) is agreed between buyer and seller.
- It can reduce the pricing risk; buyer does not over/under-pay the option. Buyer choose the level of volatility (σ_B) to pay.
- Simple case: when $T \gg 1$ and $\rho = 0$, the timer option price is same as that of the Black–Scholes option price with volatility $\sigma_{\text{BS}} = \sigma_B$ even though the expiry is not always T.