Heston Model Simulations with Poisson Conditioning
Applied Stochastic Processes (FIN 514)

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Outline

1. Introduction

2. Review of Current Methods
   - Time Discretization Schemes (e.g., Andersen, 2008)
   - Exact MC Simulation (Broadie and Kaya, 2006)
   - Gamma Expansion (Glasserman and Kim, 2011)
   - Inverse Gamma Approximation (Tse and Wan, 2013)

3. Enhanced Simulation Methods with Poisson Conditioning

4. Numerical Results

Introduction: Heston Stochastic Volatility Model

Price (S_t) and variance (V_t) dynamics:

$$\frac{dS_t}{S_t} = r dt + \sqrt{V_t}\, dW_t,$$
$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t} dZ_t,$$

where W_t and Z_t are the standard BMs correlated by ρ, κ is the mean reversion speed, ξ is the volatility of volatility, and θ is the long-term volatility.

- Variance V_t is driven by the Cox et al. (1985, CIR) process.
- One of the most popular stochastic volatility (SV) models.
- European option can be efficiently priced with inverse Fourier transform.
- Path-dependent derivatives require Monte–Carlo (MC) simulation.
- Efficient MC schemes (i.e., simulating S_T) are critical but challenging.

Integrated Variance in the MC Simulation

- Define the (conditional) integrated variance:

$$I_{0,T} = \int_0^T V_t dt \quad \text{or} \quad I_{0,T} = \left(\int_0^T V_t dt \,\middle|\, V_0, V_T \right)$$

- Integration of V_t:

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dZ_t$$

$$\int_0^T \sqrt{V_t}\, dZ_t = \frac{1}{\xi}\Big(V_T - V_0 + \kappa(I_{0,T} - \theta T)\Big).$$

- Integration of S_t:

$$d\,\log S_t = rdt + \sqrt{V_t}(\rho dZ_t + \rho_* dX_t) - \frac{1}{2}V_t dt \quad (\rho_* = \sqrt{1-\rho^2},\ dX_t dZ_t = 0)$$

$$\log\left(\frac{S_T}{S_0}\right) = rT + \int_0^T \sqrt{V_t}(\rho dZ_t + \rho_* dX_t) - \frac{1}{2}\int_0^T V_t dt$$

$$= rT + \frac{\rho}{\xi}\Big(V_T - V_0 + \kappa(I_{0,T} - \theta T)\Big) - \frac{I_{0,T}}{2} + \rho_* \sqrt{I_{0,T}}\, X_1 \quad (X_1 \sim N(0,1)).$$

- Conditional on V_T and $I_{0,T}$, S_T is log-normal with volatility $\rho_*\sqrt{I_{0,T}/T}$.
- Any simulation scheme boils down to the simulation of $(V_T, I_{0,T})$ given V_0.

Outline

Time Discretization: Quadratic Exponential Scheme

- In the CIR model, V_t can reach 0: the Feller condition $(2\kappa\theta > \xi^2)$ is violated in typical Heston parameters.
- Euler/Milstein scheme (V_t floored above 0) introduces large bias.
- Quadratic Exponential (QE) scheme (Andersen, 2008)
  - If V_t is close to 0:

$$V_{t+h} = \begin{cases} 0 & \text{if} \quad 0 \le U \le p \\ \frac{1}{\beta}\log\left(\frac{1-p}{1-U}\right) & \text{if} \quad p < U \le 1 \end{cases} \quad \text{for uniform RV} \quad U$$

  - If V_t is away from 0:

$$V_{t+h} = a(b+Z)^2 \quad \text{for} \quad Z \sim N(0,1)$$

  - The coefficients (a, b, β, and p) are determined to match:

$$E(V_{t+h}) = \theta + (V_t - \theta)e^{-\kappa h}, \quad \text{Var}(V_{t+h}) = \frac{\xi^2}{\kappa}(1-e^{-\kappa h})\left(V_0 e^{-\kappa h} + \frac{\theta}{2}(1-e^{-\kappa h})\right)$$

  - Trapezoidal rule for the integrated variance:

$$I_{t,t+h} = (V_t + V_{t+h})\frac{h}{2} \;\Rightarrow\; I_{0,T} = (V_0 + 2V_h + \cdots 2V_{T-h} + V_T)\frac{h}{2}$$

Some notations and properties of random variables (RV)

- POIS(λ) denotes the Poisson RV ($\#$ of Poisson events in $T = 1$) with rate λ. Poisson RVs are additive.

$$P(\mu = n) = \lambda^n e^{-\lambda}/n! \quad \text{and} \quad \text{POIS}(\lambda_1) + \text{POIS}(\lambda_2) \sim \text{POIS}(\lambda_1 + \lambda_2).$$

- $\Gamma(\alpha)$ denotes the Gamma RV with shape parameter α and unit scale ($\beta = 1$). Waiting time until α Poisson events.

$$f_\Gamma(x) = x^{\alpha-1} e^{-x} \;/\; \Gamma(\alpha)$$

- Gamma RVs are additive;

$$\Gamma(\alpha_1) + \Gamma(\alpha_2) \sim \Gamma(\alpha_1 + \alpha_2).$$

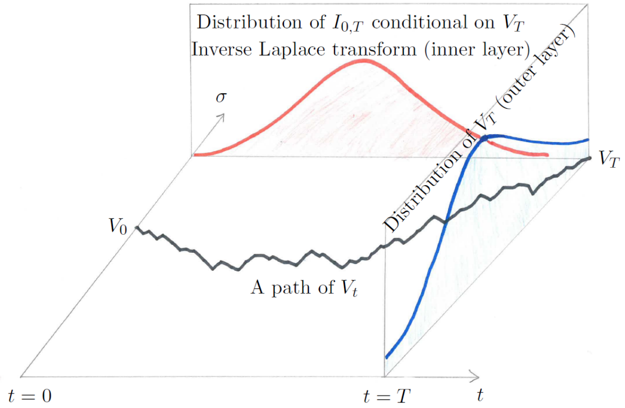
- The exponential RV with rate λ is equivalent to $\Gamma(1)/\lambda$.
- $\chi^2(\delta, \lambda)$ is the noncentral chi-squared RV with degree of freedom δ and noncentrality λ. The ordinary chi-square RV, $\chi^2(\delta, 0)$, is a special case of Gamma RV:

$$\chi^2(\delta, 0) \sim 2\Gamma(\delta/2).$$

- $X^{(j)}$ for $j = 1, 2, \cdots$ denote independent copies of a random variable X.

Exact MC scheme

- Broadie and Kaya (2006) pioneered the so-called *exact MC scheme*, followed by similar schemes for other SV models (Baldeaux, 2012; Cai et al., 2017).
- Possible to sample V_T and $I_{0,T}$ for any large time step T without bias.



Exact MC scheme: Sampling V_T

- The terminal variance V_T given V_0 is distributed by non-central chi-squared RV, $\chi^2(\delta, \lambda)$:

$$V_T \;\sim\; \frac{e^{-\frac{\kappa T}{2}}}{\phi_T(\kappa)} \chi^2\left(\delta,\; V_0\, e^{-\frac{\kappa T}{2}}\, \phi_T(\kappa)\right)$$

  where degree of freedom, δ, and non-centrality, λ, are given by

$$\phi_T(\kappa) = \frac{2\kappa/\xi^2}{\sinh(\frac{\kappa T}{2})} \quad \text{and} \quad \delta = \frac{4\kappa\theta}{\xi^2}.$$

- The Feller condition is equivalent to $\delta > 2$.
- It turns out that the exact sampling of $\chi^2(\delta, \lambda)$ with the standard numerical libraries (e.g., `NumPy` and `Matlab`) is as fast as Andersen (2008)'s QE step.

Exact MC Scheme: Conditional Laplace Transform of $I_{0,T}$

- The conditional Laplace transform of $I_{0,T}$ (Pitman and Yor, 1982):

$$E\left(e^{-uI_{0,T}}\right) = \frac{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa_u T}{2})\phi_T(\kappa_u)\right)}{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa T}{2})\phi_T(\kappa)\right)} \frac{\phi_T(\kappa_u)}{\phi_T(\kappa)} \frac{I_\nu\left(\sqrt{V_0\,V_T}\phi_T(\kappa_u)\right)}{I_\nu\left(\sqrt{V_0\,V_T}\phi_T(\kappa)\right)}$$

  where $\kappa_u = \sqrt{\kappa^2 + 2\xi^2\,u}$, $\nu = \delta/2 - 1$, and $I_\nu(z)$ is the modified Bessel function of the first kind,

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\,\Gamma(k+\nu+1)}.$$

- The CDF is computed by the numerical Laplace inversion (e.g., Abate-Whitt).
- Costly evaluation of $I_\nu(\cdot)$.
- To draw an RV of $X \sim I_{0,T}$, need to find the root of $U = \mathrm{CDF}(X)$ numerically.
- Overall, the algorithm is pioneering in theory, but is not practical (too slow).

Gamma Expansion

Glasserman and Kim (2011) significantly improve the efficiency by avoiding the numerical inverse Laplace transform. Using Pitman and Yor (1982), they decompose the conditional integrated variance to infinite series of Γ RVs:

$$I_{0,T} \sim X + Z_{\delta/2} + \sum_{j=1}^{\eta_{0,T}} Z_2^{(j)},$$

where

$$X \sim \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \Gamma(n_k) \quad \text{for} \quad n_k \sim \text{POIS}((V_0 + V_T)\lambda_k)$$

$$\text{and} \quad Z_\alpha \sim \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \Gamma(\alpha),$$

$$\lambda_k = \frac{16(k\pi)^2}{\xi^2 T((\kappa T)^2 + (2k\pi)^2)} \quad \text{and} \quad \gamma_k = \frac{(\kappa T)^2 + (2k\pi)^2}{2\xi^2 T^2},$$

Gamma Expansion: $\eta_{0,T}$

- $\eta_{0,T} \sim \text{BES}(\nu, z = \sqrt{V_0\,V_T}\,\phi_T(\kappa))$ is a Bessel RV defined by the expansion terms of $I_\nu(z)$:

$$P(\eta_{0,T} = k) = \frac{(z/2)^{2k+\nu}}{I_\nu(z)\,k!\,\Gamma(k+\nu+1)}.$$

- The decomposition of the conditional Laplace transform:

$$E\left(e^{-uI_{0,T}}\right) = \frac{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa_u T}{2})\phi_T(\kappa_u)\right)}{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa T}{2})\phi_T(\kappa)\right)}\,\frac{\phi_T(\kappa_u)}{\phi_T(\kappa)}\,\frac{I_\nu\left(\sqrt{V_0\,V_T}\phi_T(\kappa_u)\right)}{I_\nu\left(\sqrt{V_0\,V_T}\phi_T(\kappa)\right)}$$

$$= \underbrace{\frac{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa_u T}{2})\phi_T(\kappa_u)\right)}{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa T}{2})\phi_T(\kappa)\right)}}_{X}\,\underbrace{\frac{\phi_T(\kappa_u)^{\delta/2}}{\phi_T(\kappa)^{\delta/2}}}_{Z_{\delta/2}}\sum_{k=0}^{\infty}\underbrace{P(\eta_{0,T}=k)\left(\frac{\phi^2(\kappa_u)}{\phi^2(\kappa)}\right)^k}_{\eta_{0,T}\text{ copies of }Z_2}$$

- Sampling $\eta_{0,T}$ requires costly evaluation of $I_\nu(\cdot)$.

Gamma Expansion: Truncation of Infinite Sums

- X depends on V_0 and V_T via n_k.
- Z_δ and Z_2 are independent from V_0 and V_T. η depends on $V_0 V_T$.
- Infinite sums must be truncated, and the truncated terms are approximated with one Gamma RV.

$$X \sim \sum_{k=1}^{K} \frac{1}{\gamma_k}\Gamma(n_k) + X^K \quad \text{and} \quad Z_\alpha \sim \sum_{k=1}^{K} \frac{1}{\gamma_k}\Gamma(\alpha) + Z_\alpha^K,$$
$$X^K \sim a\Gamma(b), \quad Z_\alpha^K \sim a'\Gamma(b')$$

The scale $(a,\, a')$ and shape $(b,\, b')$ of X^K and Z_α^K are determined to match the mean and variance of the truncated Gamma RVs.

Gamma Expansion: Mean and Variance of X and Z_α

- The mean and variance of X and Z_α are analytically available:

$$E(X) = (V_0 + V_T)\sum_{k=1}^{\infty}\frac{\lambda_k}{\gamma_k} = (V_0+V_T)m_X T, \quad E(Z_\alpha) = \sum_{k=1}^{\infty}\frac{\alpha}{\gamma_k} = \alpha m_Z \xi^2 T^2,$$

$$m_X = \frac{c_1 - ac_2}{2a} \quad \text{and} \quad m_Z = \frac{ac_1 - 1}{4a^2},$$

$$\text{Var}(X) = (V_0 + V_T)\sum_{k=1}^{\infty}\frac{2\lambda_k}{\gamma_k^2} = (V_0+V_T)v_X \xi^2 T^3, \quad \text{Var}(Z_\alpha) = \sum_{k=1}^{\infty}\frac{\alpha}{\gamma_k^2} = \alpha v_Z \xi^4 T^4,$$

$$v_X = \frac{c_1 + ac_2 - 2a^2 c_1 c_2}{8a^3} \quad \text{and} \quad v_Z = \frac{ac_1 + a^2 c_2 - 2}{16a^4},$$

$$a = \kappa T/2, \quad c_1 = 1/\tanh a, \qquad \text{and} \quad c_2 = 1/\sinh^2 a$$

- The coefficients, m_X, m_Z, v_X, and v_Z are the functions of κT.
- These results are used to compute the mean and variance of X^K and Z_α^K. For example,

$$E(X^K) = E(X) - (V_0 + V_T)\sum_{k=1}^{K}\frac{\lambda_k}{\gamma_k}, \quad E(Z_\alpha^K) = E(Z_\alpha) - \sum_{k=1}^{K}\frac{\alpha}{\gamma_k}$$

Inverse Gaussian (IG) Approximation

- Tse and Wan (2013) approximates $I_{0,T}$ with one IG RV:

$$f_{\mathsf{IG}}(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\,\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \quad \text{for} \quad \mu,\lambda>0.$$

- Find μ and λ to match $E(I_{0,T})$ and $\mathsf{Var}(I_{0,T})$:

$$\begin{aligned} E(I_{0,T}) &= E(X) + E(Z_{\delta/2}) + E(\eta_{0,T})E(Z_2) \\ \mathsf{Var}(I_{0,T}) &= \mathsf{Var}(X) + \mathsf{Var}(Z_{\delta/2}) + E(\eta_{0,T})\mathsf{Var}(Z_2) + \mathsf{Var}(\eta_{0,T})E(Z_2)^2, \\ \text{where} \quad E(\eta_{0,T}) &= \frac{z\,I_{\nu+1}(z)}{2I_\nu(z)}, \quad \mathsf{Var}(\eta_{0,T}) = \frac{z^2\,I_{\nu+2}(z)}{4I_\nu(z)} + E(\eta_{0,T}) - E(\eta_{0,T})^2. \end{aligned}$$

- Once λ and μ are calibrated, trivial to draw IG RVs (Michael et al., 1976).
- While better than Γ RV in GE, it can not control the error (no K). It needs multiple steps for more accurate results.
- Requires 3 evaluations of $I_\nu(\cdot)$ per path. Otherwise, it needs caching $E(\eta_{0,T})$ and $\mathsf{Var}(\eta_{0,T})$ for the grid of $z = \sqrt{V_0 V_T}\phi_T(\kappa)$.

Relative Strength of the Current Methods

Each simulation method has strength in different monitoring frequencies in path-dependent derivatives.

Gamma series: infrequent monitoring ($>$ year)

- Computation cost for one step is high
- Works well for any time step

IG Approximation: mid-frequency monitoring (quarterly)

- Computation cost for one step is intermediate.
- Accurate when the time step is reasonable.

QE: frequent monitoring (daily/weekly)

- Computation cost for one step is low
- Accurate when the time step is small

Outline

Key Observation: Poisson Conditioning

- $\chi^2(\delta,\lambda)$ is equivalently simulated as Poisson-mixture Gamma:

$$\chi^2(\delta,\lambda) \sim \chi^2(\delta + 2\text{POIS}(\lambda/2), 0) \sim 2\,\Gamma(\delta/2 + \text{POIS}(\lambda/2)).$$

- V_T can be exactly sampled by Broadie and Kaya (2006); Glasserman and Kim (2011)

$$\mu_0 \sim \text{POIS}\left(\frac{V_0\, e^{-\frac{\kappa T}{2}}\,\phi_T(\kappa)}{2}\right), \quad \text{then} \quad V_T \sim \frac{2e^{-\kappa T/2}}{\phi_T(\kappa)}\Gamma\left(\frac{\delta}{2} + \mu_0\right).$$

- $\mu \sim \text{BES}(\nu, z)$ is equivalent to the conditional Poisson RV (Pitman and Yor, 1982):

$$\mu \sim \text{POIS}(\lambda) \mid \Gamma(\nu + 1 + \mu) = z^2/(4\lambda) \quad \text{for any} \quad \lambda > 0.$$

- By choosing $\lambda = V_0\, e^{-\frac{\kappa T}{2}}/2$,

$$\eta_{0,T} \sim \text{POIS}\left(\frac{V_0\, e^{-\frac{\kappa T}{2}}\,\phi_T(\kappa)}{2}\right) \quad \Bigg| \quad \Gamma\left(\frac{\delta}{2} + \eta_{0,T}\right) = \frac{z^2}{2V_0\, e^{-\frac{\kappa T}{2}}} = \frac{V_T\phi_T(\kappa)}{2e^{-\kappa T/2}},$$

- $\eta_{0,T}$ is μ_0 conditional on V_T! We can replace $\eta_{0,T}$ with μ_0.

Poisson Gamma Expansion (POIS–GE)

$$
\begin{aligned}
I_{0,T}|\mu_0 &\sim X + Z_{\delta/2} + \sum_{j=1}^{\eta_0} Z_2^{(j)} \sim \sum_{k=1}^{\infty} \frac{1}{\gamma_k}\Gamma(\delta/2 + 2\mu_0 + n_k)\\
&\approx \sum_{k=1}^{K} \frac{1}{\gamma_k}\Gamma(\delta/2 + 2\mu_0 + n_k) + \mathrm{IG}(\lambda,\mu) \quad \text{for} \quad n_k \sim \mathrm{POIS}((V_0+V_T)\lambda_k).
\end{aligned}
$$

- No longer generate $\eta_{0,T} \sim \mathrm{BES}(\nu, z)$. Just use μ_0 from the V_T simulation.
- Merge the Γ RVs using the additive property.
- Adopt Tse and Wan (2013): use IG for approximating the truncated series.
- $K = 0$ version is similar to Tse and Wan (2013), but no more $I_\nu(\cdot)$ evaluations because μ_0 is fixed.

$$
\begin{aligned}
E(I_{0,T}\,|\,\mu_0) &= E(X) + E(Z_{\delta/2}) + \mu_0 E(Z_2)\\
&= (V_0+V_T) m_X T + \left(\frac{\delta}{2} + 2\mu_0\right) m_Z \xi^2 T^2\\
\mathrm{Var}(I_{0,T}|\mu_0) &= \mathrm{Var}(X) + \mathrm{Var}(Z_{\delta/2}) + \mu_0 \mathrm{Var}(Z_2)\\
&= (V_0+V_T) v_X \xi^2 T^3 + \left(\frac{\delta}{2} + 2\mu_0\right) v_Z \xi^4 T^4
\end{aligned}
$$

Exact MC Scheme under Poisson Conditioning

- The conditional Laplace transform in Broadie and Kaya (2006) can be simplified. Under the joint condition on V_T and μ_0,

$$E\left(e^{-uI_{0,T}}\middle|\mu_0\right) = \frac{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa_u T}{2})\phi_T(\kappa_u)\right)}{\exp\left(-\frac{V_0+V_T}{2}\cosh(\frac{\kappa T}{2})\phi_T(\kappa)\right)}\left(\frac{\phi_T(\kappa_u)}{\phi_T(\kappa)}\right)^{\delta/2+2\mu_0}$$

$$(\kappa_u = \sqrt{\kappa^2 + 2\xi^2\,u},\ \nu = \delta/2-1).$$

- The evaluation of $I_\nu(\cdot)$ can be avoided.
- The original Laplace transform (unconditional on μ_0) can be reconstructed with the probability-weighted sum over μ_0:

$$\sum_{n=0}^{\infty} P(\eta=n)\left(\frac{\phi_T(\kappa_u)}{\phi_T(\kappa)}\right)^{\delta/2+2n} = \sum_{n=0}^{\infty}\frac{(\sqrt{V_0V_T}\phi_T(\kappa)/2)^{\delta/2-1+2n}}{I_\nu(\sqrt{V_0V_T}\phi_T(\kappa))n!\,\Gamma(n+\nu+1)}\left(\frac{\phi_T(\kappa_u)}{\phi_T(\kappa)}\right)^{\delta/2+2n}$$

$$= \sum_{n=0}^{\infty}\frac{\phi_T(\kappa_u)}{\phi_T(\kappa)}\frac{(\sqrt{V_0V_T}\phi_T(\kappa_u)/2)^{\delta/2-1+2n}}{I_\nu(\sqrt{V_0V_T}\phi_T(\kappa))n!\,\Gamma(n+\nu+1)} = \frac{\phi_T(\kappa_u)}{\phi_T(\kappa)}\frac{I_\nu(\sqrt{V_0V_T}\phi_T(\kappa_u))}{I_\nu(\sqrt{V_0V_T}\phi_T(\kappa))}$$

Time Discretization under Poisson Conditioning (POIS–Quad)

- Simulate V_{t+h} from V_t with η_t.
- Instead of the trapezoidal rule, use $E(I_{t,t+h}|V_t, V_{t+h}, \eta_T)$:

$$\hat{I}_{t,t+h} = E(I_{t,t+h}\,|\,\mu_t) = (V_t + V_{t+h}) m_X h + \left(\frac{\delta}{2} + 2\mu_t\right) m_Z \xi^2 h^2,$$

$$\Rightarrow\ \hat{I}_{0,T} = (V_0 + 2V_1 + \cdots + 2V_{T-h} + V_T) m_X h + \left(\frac{N\delta}{2} + 2(\mu_0 + \mu_h + \cdots + \mu_{T-h})\right) m_Z \xi^2 h^2.$$

- Unlike the QE scheme, the mean of $I_{0,T}$ is preserved. It is also possible to estimate the missing variance (i.e., error) in $I_{0,T}$.
- After some algebra, the missing variance ratio is given by

$$\mathcal{E} \approx \frac{-(2v_X + 4v_Z)(\kappa h)^2 \left(\theta + (V_0 - \theta)(1 - e^{-\kappa T})/(\kappa T)\right)}{\theta - 2(V_0 - \theta)e^{-\kappa T} + \left(V_0 - \frac{5\theta}{2} + \left(V_0 - \frac{\theta}{2}\right) e^{-\kappa T}\right)(1 - e^{-\kappa T})/(\kappa T)}$$

Summary of Methods

Exact Simulation Methods (Infrequent Monitoring)

Gamma Expansion (GE)

- V_T: $\chi^2(\delta, \lambda)$, $\eta_{0,T} \sim$ BES
- $I_{0,T}$: 3 Gamma RV series
- Truncated terms: Gamma RV Each

IG–Approximation

- $I_{0,T}$: moment-matched IG

Poison-Gamma Expansion (POIS–GE)

- $\mu_0 \sim$ POIS
- V_T: Poisson-mixture Gamma
- Truncated terms: 1 IG RV

Time-discretization Methods (Frequent Monitoring)

QE–Trapezoid

- V_i: moment-matched ad-hoc rules
- $I_{i,i+1}$: $\frac{1}{2} + \frac{1}{2}$ weights

POIS–Quad

- V_i: Poisson-mixture Gamma.
- $I_{i,i+1}$: Mean-preserving weights

Outline

Spot and European Options (Conditional MC)

- S_T is log-normal given V_T and $I_{0,T}$:

$$S_T \;=\; S_0 \exp\left(rT + \frac{\rho}{\xi}\big(V_T - V_0 + \kappa(I_{0,T} - \theta T)\big) - \frac{1}{2} I_{0,T} + \rho_* \sqrt{I_{0,T}}\, X_1 \right).$$

- Instead of sampling S_T, we use the BS model with the conditional spot and volatility:

$$S_{\text{BS}} = S_0 \exp\left(\frac{\rho}{\xi}\big(V_T - V_0 + \kappa(I_{0,T} - \theta T)\big) - \frac{\rho^2}{2} I_{0,T} \right), \;\; \sigma_{\text{BS}} = \rho_* \sqrt{I_{0,T}/T}$$

- European option price is the MC average of the BS prices:

$$C_{\text{Heston}}(K, S_0, \cdots) = E_{\text{MC}}\left[C_{\text{BS}}(K, S_{\text{BS}}, \sigma_{\text{BS}}) \right]$$

  The so-called conditional MC (Willard, 1997) significantly reduces the MC variance (Broadie and Kaya, 2006; Cai et al., 2017).

- Additionally check the equality of the spot price:

$$S_0 = E_{\text{MC}}(S_{\text{BS}}) = E_{\text{MC}}(E(e^{-rT} S_T \mid V_T, I_{0,T}))$$

Variance Swap: $E(I_{0,T})$ and Var$(I_{0,T})$

- Discretely monitored variance swap:

$$\text{Floating Leg (realized variance)} = \frac{A}{N}\sum_{i=1}^{N} R_i^2 \quad \text{for} \quad R_i = \log\left(\frac{S_i}{S_{i-1}}\right).$$

  Typically daily return is used with the annualization factor $A = 252$.

- Continuously monitored variance swap:

$$\text{Floating Leg} = \bar{V}_T = \frac{I_{0,T}}{T}.$$

- E(Floating Leg) is the fair value of the fixed leg (fair strike).
- We price continuously monitored variance swap with time-discretization methods because $E(\bar{V}_T)$ and Var$(\bar{V}_T)$ are analytically available.

Test Cases

We test four cases:

Case	V_0	θ	ξ	ρ	κ	$\delta = 4\kappa\theta/\xi^2$	T	r	Exact Price
A1	0.04	0.04	1	-0.9	0.5	0.08	10	0	13.08467014
A2	0.04	0.04	0.9	-0.5	0.3	0.06	15	0	16.64922292
B1	0.010201	0.019	0.61	-0.7	6.21	1.27	1	0.0319	6.80611331
B2	0.09	0.09	1	-0.3	2	0.72	5	0.05	34.99975835

$$(S_0 = K = 100)$$

The parameter sets are previously used in the following literature:

- **Case A1**: Andersen (2008), Van Haastrecht and Pelsser (2010), Lord et al. (2010), Tse and Wan (2013)
- **Case A2**: Andersen (2008), Van Haastrecht and Pelsser (2010)
- **Case B1**: Broadie and Kaya (2006), Tse and Wan (2013)
- **Case B2**: Broadie and Kaya (2006), Lord et al. (2010), Tse and Wan (2013)

Case A1: European Option and Spot

Exact simulation methods:

	GE (Truncated Γ)			POIS–GE (Truncated Γ)			POIS–GE (Truncated IG)		
K	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
0				0.07	2.608 (2.4)	2.063 (8.6)	0.07	0.154 (2.0)	0.070 (7.7)
1	0.20	0.985 (2.2)	0.339 (7.8)	0.08	1.097 (2.2)	0.399 (7.9)	0.08	0.151 (2.0)	0.046 (7.4)
2	0.19	0.408 (2.1)	0.073 (7.5)	0.10	0.469 (2.1)	0.091 (7.7)	0.10	0.084 (1.9)	0.016 (7.4)
5	0.24	0.044 (1.9)	0.005 (7.3)	0.14	0.054 (1.9)	0.004 (7.4)	0.14	0.012 (1.9)	-0.001 (7.6)
10	0.31	0.002 (1.9)	0.002 (7.6)	0.21	0.002 (1.9)	0.002 (7.5)	0.21	0.000 (1.9)	0.000 (7.5)

Time discretization schemes:

	QE–Trapezoid			POIS–Trapezoid			POIS–Quad		
h	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
1/2	0.63	0.33 (2.0)	0.09 (8.0)	0.47	0.09 (1.9)	0.06 (7.3)	0.51	-0.15 (1.8)	-0.08 (7.2)
1/4	1.18	0.06 (1.9)	0.00 (7.9)	0.85	0.02 (1.9)	0.02 (7.4)	1.68	-0.04 (1.8)	-0.02 (7.4)
1/8	2.09	0.00 (1.9)	-0.01 (7.7)	1.69	0.01 (2.0)	0.01 (7.8)	2.13	-0.01 (2.0)	0.00 (7.8)

Note: The exact European option price is 16.64922292.
The standard error (SE) is in the unit of 10^{-2}.

Case A2: European Option and Spot

Exact simulation methods:

	GE (Truncated Γ)			POIS–GE (Truncated Γ)			POIS–GE (Truncated IG)		
K	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
0				0.07	-2.054 (1.1)	0.766 (6.1)	0.07	-0.108 (1.1)	-0.004 (5.5)
1	0.19	-0.823 (1.2)	0.092 (5.6)	0.09	-0.914 (1.2)	0.111 (5.6)	0.09	-0.121 (1.1)	0.010 (5.6)
2	0.20	-0.365 (1.2)	0.018 (5.5)	0.10	-0.416 (1.2)	0.022 (5.7)	0.10	-0.074 (1.1)	0.005 (5.7)
5	0.27	-0.047 (1.1)	0.002 (5.5)	0.15	-0.056 (1.1)	0.002 (5.3)	0.15	-0.016 (1.1)	0.000 (5.5)
10	0.34	0.000 (1.0)	-0.001 (5.4)	0.22	0.000 (1.1)	0.002 (5.5)	0.22	-0.002 (1.1)	0.000 (5.6)

Time discretization schemes:

	QE–Trapezoid			POIS–Trapezoid			POIS–Quad		
h	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
1/2	0.77	-0.11 (1.0)	0.00 (5.6)	0.83	-0.02 (1.1)	0.01 (5.7)	0.69	0.07 (1.0)	-0.01 (5.7)
1/4	1.42	-0.01 (1.0)	0.00 (5.6)	1.23	-0.01 (1.0)	0.00 (5.4)	1.33	0.02 (1.0)	0.00 (5.4)
1/8	2.81	0.01 (1.0)	0.00 (5.3)	2.34	0.00 (1.0)	0.00 (5.2)	2.69	0.00 (1.0)	0.00 (5.2)

Note: The exact European option price is 13.08467014.
The standard error (SE) is in the unit of 10^{-2}.

Case B1: European Option and Spot

Exact simulation methods:

	GE (Truncated Γ)			POIS–GE (Truncated Γ)			POIS–GE (Truncated IG)		
K	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
0				0.06	0.084 (1.1)	0.004 (2.3)	0.06	0.005 (1.0)	-0.001 (2.2)
1	0.18	0.007 (1.0)	0.000 (2.2)	0.08	0.009 (1.0)	0.001 (2.2)	0.08	0.002 (1.0)	0.001 (2.2)
2	0.19	0.001 (1.0)	0.000 (2.1)	0.09	0.001 (1.0)	0.000 (2.2)	0.09	0.000 (1.0)	0.000 (2.3)
5	0.25	0.000 (1.0)	0.000 (2.2)	0.14	0.000 (1.0)	0.000 (2.2)	0.15	0.000 (1.0)	0.000 (2.2)
10	0.35	0.000 (1.0)	-0.001 (2.1)	0.23	0.000 (1.0)	-0.001 (2.2)	0.23	0.000 (1.0)	0.000 (2.3)

Time discretization schemes:

	QE–Trapezoid			POIS–Trapezoid			POIS–Quad		
h	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
1/2	0.09	1.11 (0.6)	0.91 (1.6)	0.08	1.02 (1.2)	0.91 (2.7)	0.09	-0.56 (0.8)	-0.15 (1.8)
1/4	0.21	0.38 (1.0)	0.27 (2.4)	0.13	0.35 (1.1)	0.26 (2.4)	0.13	-0.19 (0.9)	-0.05 (2.1)
1/8	0.31	0.10 (1.1)	0.07 (2.4)	0.20	0.10 (1.0)	0.07 (2.3)	0.29	-0.05 (1.0)	-0.01 (2.2)

Note: The exact European option price is 6.80611331.
The standard error (SE) is in the unit of 10^{-2}.

Exact simulation methods:

	GE (Truncated Γ)			POIS–GE (Truncated Γ)			POIS–GE (Truncated IG)		
K	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
0				0.07	0.200 (2.2)	0.067 (4.9)	0.07	0.014 (2.0)	0.004 (4.5)
1	0.17	0.069 (2.0)	0.017 (4.6)	0.08	0.074 (2.1)	0.018 (4.6)	0.10	0.012 (1.9)	0.004 (4.4)
2	0.19	0.023 (2.0)	0.004 (4.5)	0.10	0.027 (2.1)	0.007 (4.6)	0.12	0.007 (1.9)	0.004 (4.4)
5	0.27	0.002 (2.0)	0.001 (4.6)	0.15	0.002 (2.0)	0.001 (4.6)	0.18	0.000 (1.9)	-0.002 (4.4)
10	0.35	0.001 (2.0)	0.002 (4.4)	0.24	0.000 (2.0)	0.000 (4.5)	0.26	0.000 (1.9)	0.000 (4.4)

Time discretization schemes:

	QE–Trapezoid			POIS–Trapezoid			POIS–Quad		
h	Time (sec)	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)	Time	Option Bias (SE)	Spot Bias (SE)
1/2	0.25	0.18 (1.9)	0.14 (4.3)	0.24	0.13 (2.1)	0.13 (4.7)	0.25	-0.10 (2.0)	-0.11 (4.4)
1/4	0.52	0.04 (2.0)	0.04 (4.6)	0.45	0.03 (2.1)	0.03 (4.7)	0.48	-0.03 (2.0)	-0.03 (4.6)
1/8	1.00	0.01 (2.0)	0.01 (4.5)	0.86	0.01 (2.0)	0.01 (4.5)	1.05	-0.01 (2.0)	-0.01 (4.4)

Note: The exact European option price is 34.99975835.
The standard error (SE) is in the unit of 10^{-2}.

Case B1/B2: Variance Swap

B1 ($\kappa = 6.21,\ T = 1$): relative error of mean and variance of $I_{0,T}$.

	QE–Trapezoid		POIS–Trapezoid		POIS–Quad		
h	Mean	Variance	Mean	Variance	Mean	Variance	Analytic Var
1/2	-5.60%	41.51%	-5.61%	41.50%	0.00%	-46.99%	-41.12%
1/4	-1.55%	15.28%	-1.55%	15.32%	0.00%	-15.94%	-15.46%
1/8	-0.40%	4.22%	-0.40%	4.20%	0.00%	-4.38%	-4.34%

B2 ($\kappa = 2,\ T = 5$): relative error of mean and variance of $I_{0,T}$.

	QE–Trapezoid		POIS–Trapezoid		POIS–Quad		
h	Mean	Variance	Mean	Variance	Mean	Variance	Analytic Var
1/2	0.00%	6.63%	0.00%	6.67%	0.00%	-6.20%	-6.12%
1/4	0.00%	1.67%	0.00%	1.71%	0.00%	-1.60%	-1.61%
1/8	0.00%	0.42%	0.00%	0.41%	0.00%	-0.43%	-0.41%

- **POIS–Quad** preserves the mean of integrated variance.
- **POIS–Quad** underestimates the variance, but an error estimate is provided.
- Case **A1** ($\kappa = 0.5$) **A2** ($\kappa = 0.3$) not reported because the errors $< 0.1\%$.

Conclusion

- The MC simulation under the Heston model has been a widely studied topic.
- The methods are broken into two classes:
  - Time-discretization method for frequent monitoring: Andersen (2008)
  - Exact MC method for infrequent monitoring: Glasserman and Kim (2011)

We find that

- The representation for the integrated variance is simplified when conditioned by the Poisson RV used in simulating the variance process.
- Poisson conditioning enhances the Heston simulation methods both classes.
- For the exact MC schemes, Poisson conditioning resolves the bottleneck of costly $I_\nu(\cdot)$ evaluation.
- For the time-discretization schemes, Poisson conditioning formulates a new quadrature rule (replacing the trapezoidal rule) preserving the expected integrated variance.

PyFENG (More tomorrow)

Python implementation available online:

PyFENG

- PyPI package: `PyFENG` (Python Financial ENGineering)
- `https://pypi.org/project/pyfeng/`
- `pip install pyfeng`
- Source: `https://github.com/PyFE/PyFENG/`

PyFengForPapers

- `https://github.com/PyFE/PyfengForPapers/`
- A collection of Jupyter Notebook scripts reproducing quantitative finance papers (mostly in derivative pricing and stochastic volatility so far).
- Uses PyFENG.
- Inspired by PapersWithCode project.

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