

Spread and Basket Option Pricing

Applied Stochastic Processes (FIN 514)

Instructor: Jaehyuk Choi

Peking University HSBC Business School, Shenzhen, China

2023-24 Module 3 (Spring 2024)

- Options written on multiple underlying assets
- Extension of Black-Scholes model from uni-variate to multi-variate
- Write a MC pricing routine with control variate
- Analytic approximations (to be used as control variates)
 - Normal model price
 - Kirk's approximation (Margrabe's formula)
 - Geometric basket option

Background: popular derivatives in non-vanilla class

Spread options: $(S_1 - S_2 - K)^+$

- Crack spread option: $(P \text{ of oil products} - P \text{ of oil} - K)^+$
- Spark spread option: $(P \text{ of electricity} - P \text{ of gas} - K)^+$
- Non-inversion note: digital call on rates term-structure, $(30y - 2y)^+$

Basket options: $(\sum w_k S_k - K)^+$ with $w_k > 0$

- Popular as OTC derivatives in FX and commodities market
- Index options

Asian options (path-dependent): $(\sum_1^N S(t_k)/N - K)^+$

- Efficient hedge over average cost, safe from market manipulation.
- Fed fund swaps: daily compounded trade-averaged FF rate
- China interest swaps: 3-month average of 7-day repo rate as a standard floating rate (cf. 3-month LIBOR)

Problem setup

- N asset prices, $S_k(t)$, following the correlated geometric Brownian motions (GBM)

$$\frac{dS_k(t)}{S_k(t)} = (r - q_k) dt + \sigma_k dW_k(t) \quad \text{for } 1 \leq k \leq N$$

for volatilities σ_k , dividend rates q_k , risk-free rate r and BMs $W_k(t)$, the correlation $\mathbb{E}\{dW_k(t)dW_j(t)\} = \rho_{kj}dt$ ($\rho_{kk} = 1$).

- N observation times: t_k ($1 \leq k \leq N$), with expiry at T
- The payout of the option at the maturity T

$$C(T) = \left(\sum_{k=1}^N w_k S_k(t_k) - K \right)^+$$

$$C(T) = \left(\sum_{k=1}^N w_k S_k(t_k) - K \right)^+$$

- **European basket option:**

$w_k > 0$ and $t_k = T$ for all k ; $N < 10$

- **European spread option:**

$w_k < 0$ for some k , $t_k = T$ for all k ; $N = 2$

- **Discretely monitored Asian option** (covered later):

$w_k = 1/N$, $0 \leq t_1 < \dots < t_N = T$ and $S_k(t)$'s are identical; $N \gg 10$

Challenges

Mathematical problem

Multi-dimensional integration over the domain of positive payoff

Difficulties

- The lognormal RV sum is neither lognormal nor has analytic distribution.
- Numerical valuation is cursed by dimensionality: $O(M^N)$. E.g., $h = 0.1$ grid between ± 7 std. dev. for 4 assets: $140^4 \approx 400 \times 10^6$
- Monte-Carlo simulation is used for pricing in industry and academics.

Quote from Broadie and Detemple (2004, MS Survey)

"Many problems are effectively exponential Efficient and convergent methods for pricing high-dimensional and path-dependent American securities depend on the development of new algorithms, not faster computers."

Normal model approximation

- Spread Option: $\sigma_{N1} = \sigma_1 S_1(0)$, $\sigma_{N2} = \sigma_2 S_2(0)$

$$\text{Var}(S_1(T) - S_2(T)) = (\sigma_{N1}^2 + \sigma_{N2}^2 - 2\rho\sigma_{N1}\sigma_{N2}) \cdot T$$

$$\sigma_N = \sqrt{\sigma_{N1}^2 + \sigma_{N2}^2 - 2\rho\sigma_{N1}\sigma_{N2}}$$

and use the normal model formula.

- Basket Option: $\sigma_{Nj} = \sigma_j S_j(0)$ or $= \sigma_j F_j(0)$

$$\text{Var}\left(\sum_k w_k S_k(T)\right) = \left(\sum_j w_j^2 \sigma_{Nj}^2 + 2 \sum_{j \neq k} \rho_{jk} w_j w_k \sigma_{Nj} \sigma_{Nk}\right) \cdot T$$

$$\Sigma_{jk} = \rho_{jk} \sigma_{Nj} \sigma_{Nk}, \quad \sigma_N = \sqrt{\mathbf{w}^T \Sigma \mathbf{w}}$$

and use the normal model formula.

Normal model approximation for Control Variate

- Use the result as a control variate of Spread and Asian option:

$$C_{BS}^{CV}(T, K) = C_{BS}^{MC}(T, K) + \left(C_N^{EXACT}(T, K) - C_N^{MC}(T, K) \right)$$

Use the same sequence of RNs for C^{MC} and C_N^{MC} . [Py Demo]

- **Homework Set 2:**

Implement Monte-Carlo pricer for Spread and Basket options with control variate with normal model price.

- Final project (past years): implement other analytic approximation methods or CV methods (e.g., Kirk's approximation for spread options)

Exchange option: Margrabe's formula

- Option to exchange one asset S_1 for another S_2 : $(S_1(T) - S_2(T))^+$
- Spread option with zero strike $K = 0$: $(S_1(T) - S_2(T) - 0)^+$
- Max (best-of) option in terms of exchange option:

$$\max(S_1(T), S_2(T)) = S_2(T) + (S_1(T) - S_2(T))^+,$$

where $(x)^+ = \max(x, 0)$.

Margrabe's exchange option formula

$$C_{\text{EX}} = S_1(0)N(d_+) - S_2(0)N(d_-),$$

where $d_{\pm} = \frac{\log(S_1(0)/S_2(0))}{\sigma_R \sqrt{T}} \pm \frac{1}{2}\sigma_R \sqrt{T}$ and $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

SDE on S_1/S_2

$$\frac{dS_k(t)}{S_k(t)} = r dt + \sigma_k dW_k(t) \quad (k = 1, 2), \quad dW_1 dW_2 = \rho dt$$

Applying Itô's lemma to S_1/S_2 ,

$$d\left(\frac{S_1}{S_2}\right) = \frac{dS_1}{S_2} - \frac{S_1}{S_2^2} dS_2 + \frac{S_1}{S_2^3} (dS_2)^2 - \boxed{\frac{dS_1 dS_2}{S_2^2}}$$

For $R = S_1/S_2$,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2) dt + (\sigma_1 dW_1 - \sigma_2 dW_2)$$

Alternatively,

$$d \log R = d \log S_2 - d \log S_1 = -\frac{1}{2}(\sigma_1^2 - \sigma_2^2) dt + \sigma_1 dW_1 - \sigma_2 dW_2$$

$$\frac{dR}{R} = d \log R + \frac{1}{2}(\sigma_1 dW_1 - \sigma_2 dW_2)^2 = \text{same result}$$

Equivalent Martingale Measure with Numeraire S_2

Decorrelating the SDE on R , for $dW'_1 dW_2 = 0$,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2) dt + \sigma_1(\rho dW_2 + \sqrt{1-\rho^2} dW'_1) - \sigma_2 dW_2.$$

Now we change the measure from P (risk-less saving numeraire) to Q (S_2);

$$\begin{aligned} C_{\text{EX}} &= 1 \cdot E^P \left(\frac{(S_1(T) - S_2(T))^+}{e^{rT}} \right) \\ &= S_2(0) E^Q \left(\frac{(S_1(T) - S_2(T))^+}{S_2(T)} \right) = S_2(0) E^Q \left((R(T) - 1)^+ \right) \end{aligned}$$

Under the new measure, the standard BM is defined as

$$dW_2^P = dW_2^Q + \sigma_2 dt \quad \text{and} \quad dW_1^P = dW_1^Q.$$

Then the SDE on R becomes drift-less and can be written with a single BM

$$\begin{aligned} \frac{dR}{R} &= (\sigma_2^2 - \rho\sigma_1\sigma_2) dt + \sqrt{1-\rho^2}\sigma_1 dW_1^P - (\sigma_2 - \rho\sigma_1)(dW_2^Q + \sigma_2 dt) \\ &= \sqrt{1-\rho^2}\sigma_1 dW_1^Q + (\sigma_2 - \rho\sigma_1) dW_2^Q = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} dZ^Q \end{aligned}$$

Margrabe's exchange option formula

The exchange option price is obtained from just another Black-Scholes formula on the ratio $R(t) = S_1(t)/S_2(t)$ with

- $K = 1$
- $\sigma_R = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$
- Prefactor (unit of options): $S_2(0)$.

Finally we obtain

$$C_{\text{EX}} = S_2(0) E^Q \left((R(T) - 1)^+ \right) = S_2(0) \left(\frac{S_1(0)}{S_2(0)} N(d_+) - 1 \cdot N(d_-) \right)$$

where $d_{\pm} = \frac{\log(S_1(0)/S_2(0))}{\sigma_R \sqrt{T}} \pm \frac{1}{2} \sigma_R \sqrt{T}$ and $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

Spread Option: Kirk's Approximation

If we assume S_2 follows displaced GBM with $L = K$, $S_2^D = S_2 + K$, then we can apply Margrabe's formula!

$$(S_1 - S_2 - K)^+ = (S_1 - \boxed{(S_2 + K)})^+ = (S_1 - S_2^D)^+$$

The volatility of S^D should be 'calibrated'. We match the local vol at ATM

$$\sigma_2^D(S_2(0) + K) = \sigma_2 S_2(0)$$

Plug in $S_2(0) + K \rightarrow S_2(0)$ and $\sigma_2 S_2(0)/(S_2(0) + K) \rightarrow \sigma_2$ to Margrabe:

Kirk's Approximation Formula

$$C_{\text{KIRK}} = S_1(0)N(d_+) - (S_2(0) + K)N(d_-),$$

$$d_{\pm} = \log\left(\frac{S_1(0)}{S_2(0) + K}\right) / \sigma_R \sqrt{T} \pm \frac{1}{2} \sigma_R \sqrt{T}$$

$$\sigma_R = \sqrt{\sigma_1^2 + \sigma_2'^2 - 2\rho\sigma_1\sigma_2'}, \quad \sigma_2' = \sigma_2 S_2(0)/(S_2(0) + K)$$

Basket Option: Levy's lognormal approximation

- The first two moments of a lognormal distribution with $(\lambda = \sigma\sqrt{T})$, $Y = \mu_1 \exp(\lambda Z - \lambda^2/2)$ for standard normal Z are

$$E(Y) = \mu_1, \quad E(Y^2) = \mu_1^2 \exp(\lambda^2) = \mu_2 \quad \Rightarrow \quad \lambda = \sqrt{\log(\mu_2/\mu_1^2)}$$

- Approximate the final basket price $B(T)$ by a lognormal distribution.

$$B(T) = \sum_{k=1}^N w_k S_k(T) \sim \mu_1 \exp(\lambda Z - \lambda^2/2)$$

- Obtain the first two moments from the original variables:

$$E(B(T)) = \sum_{k=1}^N w_k F_k(T), \quad E(B^2(T)) = \sum_{i,j} w_i w_j F_i F_j e^{\sigma_i \sigma_j \rho_{ij} T}.$$

- Use Black-Scholes formula with $\sigma = \lambda/\sqrt{T}$. (Implemented in PyFENG.)

- Discretely monitored:

$$A(T) = \frac{1}{N} \sum_{k=1}^N S(t_k)$$

- Continuously monitored:

$$\begin{aligned} A(T) &= \frac{1}{T - T_0} \int_{t=T_0}^T S(t) dt \\ &= \frac{\Delta t}{T - T_0} \left(\frac{S(T_0)}{2} + S(T_0 + \Delta t) + \cdots S(T - \Delta t) + \frac{S(T)}{2} \right) \end{aligned}$$

- A special case of basket option:
 $S(t_i)$ and $S(t_j)$ with $t_i < t_j$ are correlated by t_i/t_j .
- Suggested topics for final project.