

# Stochastic Volatility Models and Simulation

## Applied Stochastic Processes (FIN 514)

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# Stochastic Volatility (SV) Models

- The price process (martingale):

$$\frac{dS_t}{S_t^\beta} = \sigma_t dW_t = \sigma_t(\rho dZ_t + \rho_* dX_t), \quad \text{for } \rho_* = \sqrt{1 - \rho^2}.$$

BSM-base:  $\beta = 1$ , normal-base:  $\beta = 0$ .

For the models except SABR, the base model is BSM (i.e.,  $\beta = 1$ ).

- The (stochastic) volatility process may vary:

$$d\sigma_t = a(t, \sigma_t)dt + b(t, \sigma_t)dZ_t,$$

- The correlation between the two Brownian motions:

$$dW_t dZ_t = \rho dt.$$

The correlation explains the *leverage effect*: equity volatility increases as price goes down.

# Various SV models

- SABR model ([Hagan et al, 2002](#)):

$$d\sigma_t = \nu \sigma_t dZ_t.$$

- [Heston \(1993\)](#) model ([Cox et al, 1985](#), CIR process):

$$dv_t = \kappa(\theta - v_t)dt + \nu \sqrt{v_t} dZ_t.$$

- 3/2 model ([Heston, 1997](#); [Lewis, 2000](#)):

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu v_t^{3/2} dZ_t.$$

- Ornstein–Uhlenbeck-driven SV model ([Stein and Stein, 1991](#)):

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu dZ_t.$$

- GARCH diffusion model (relatively new):

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t dZ_t.$$

- [Rough volatility](#): use a fractional BM  $Z_t^H$  instead.

# Integrated variance

In all SV models, the integrated variance  $V_T$  plays an important role:

$$V_T = \int_0^T \sigma_t^2 dt = \int_0^T v_t dt$$

Conditional on  $V_T$  (and other variables),  $S_T$  has lognormal distribution. Therefore, we can use BS model. From Itô's isometry,

$$\int_0^T \rho_* \sigma_t dX_t \sim \rho_* \sqrt{V_T} X_1 \sim N(0, \rho_*^2 V_T)$$

So the volatility between 0 and  $T$  is

$$\sigma = \rho_* \sqrt{V_T/T}$$

# Simulation scheme for the SV models

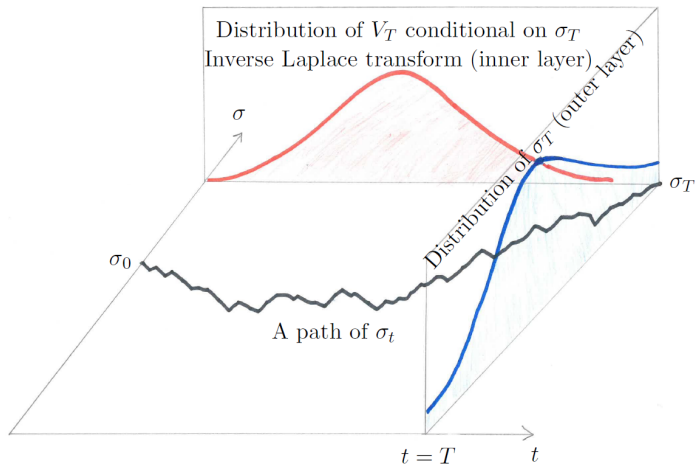
- Time discretization (Euler/Milstein):
  - Easy to implement, but possible bias and computationally expensive.
- Conditional MC:
  - Can skip the simulation of price  $S_t$ . (Simulate volatility  $\sigma_t$  only)
  - The final price  $S_T$  should be expressed by  $\sigma_T$  and  $V_T = \int_0^T \sigma_t^2 dt$ .
- Exact Simulation:
  - No need for time-discretization: jump from  $t = 0$  to  $T$ .
  - $\sigma_T$  follows a well-known distribution.
  - Conditional Laplace transform of  $V_T | v_T$  is analytically available (Heston, 3/2, SABR, etc)

$$E(e^{-sV_T} | v_T = x) = f(s, x)$$

The CDF of  $V_T | v_T$  can be obtained by (numerical) Laplace inverse transform although computationally expensive.

- No Laplace inversion required in some special cases: Normal SABR (Choi et al, 2019, NSVh model), OUSV (working paper)

# Exact MC scheme



# Simulation Procedures

Conditional MC:

- 1) Simulate the path of  $\sigma_t$  for  $(0 \leq t_1 \leq t_2 \leq \dots \leq T)$ . Obtain  $\sigma_T$ .
- 2) Obtain  $V_T$  with time-integral (trapezoidal / Simpson's rule).

Exact MC:

- 1) Sample  $v_T$  from the (well-known) distribution
- 2) Sample  $V_T$  from the (numerical) CDF of  $V_T | v_T$ .

In Common:

- 3) Obtain  $E(S_T | v_T, V_T)$  and effective volatility (usually  $\rho_* \sqrt{V_T/T}$ )
- 4-1) Price sampling: draw normal / log-normal distribution
- 4-2) Option price: Bachelier / BSM option price formula with  $S_0 := E(S_T | v_T)$  and  $\sigma := \rho_* \sqrt{V_T/T}$ .  
Then, average over the simulations.

# Heston model (conditional MC)

Integrating  $v_t$ ,

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t \quad (v_t = \sigma_t^2)$$

$$v_T - v_0 = \kappa(\theta T - V_T) + \nu \int_0^T \sqrt{v_t}dZ_t$$

$$\int_0^T \sqrt{v_t} dZ_t = \frac{1}{\nu} \left( v_T - v_0 + \kappa(V_T - \theta T) \right).$$

You can also express  $S_T$  by  $v_T$  and  $V_T$  (conditional MC possible!).

$$d \log S_t = \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) - \frac{1}{2}v_t dt$$

$$\log(S_T/S_0) = \int_0^T \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) - \int_0^T \frac{1}{2}v_t dt$$

$$\log(S_T/S_0) = \frac{\rho}{\nu}(v_T - v_0 + \kappa(V_T - \theta T)) + \rho_*\sqrt{V_T}X_1 - \frac{1}{2}V_T$$



Therefore, we can sample  $S_T$  as

$$S_T = S_0 \exp \left( \frac{\rho}{\nu} (v_T - v_0 + \kappa(V_T - \theta T)) - \frac{1}{2} V_T + \rho_* \sqrt{V_T} X_1 \right).$$

But, instead of sampling  $S_T$ , we use the BS model. For BS, we need to spot and volatility.

$$\begin{aligned} E(S_T \mid v_T, V_T) &= S_0 \exp \left( \frac{\rho}{\nu} (v_T - v_0 + \kappa(V_T - \theta T)) - \frac{1}{2} V_T + \frac{\rho_*^2}{2} V_T \right) \\ &= S_0 \exp \left( \frac{\rho}{\nu} (v_T - v_0 + \kappa(V_T - \theta T)) - \frac{\rho^2}{2} V_T \right) \\ \sigma_{\text{BS}} &= \rho_* \sqrt{V_T / T} \end{aligned}$$

# Heston model (exact MC)

Broadie and Kaya (2006) pioneered the exact MC scheme:

- $v_T$  is distributed as a noncentral chi-square distribution,  $\chi^2(\delta, \lambda)$  (see **2017ME Bessel process** problem):

$$v_T = \frac{\nu^2(1 - e^{-\kappa T})}{4\kappa} \chi^2(\delta, \lambda) = \frac{e^{-\kappa T/2}}{\phi(\kappa)} \chi^2(\delta, \lambda),$$

where the degrees of freedom  $\delta$  and the noncentrality  $\lambda$  are

$$\delta = \frac{4\kappa\theta}{\nu^2}, \quad \lambda = \frac{4v_0\kappa e^{-\kappa T}}{\nu^2(1 - e^{-\kappa T})} = v_0 e^{-\kappa T/2} \phi(\kappa), \quad \phi(\kappa) = \frac{2\kappa/\nu^2}{\sinh(\kappa T/2)}.$$

Standard library is available for drawing  $\chi^2$  random number.

- The conditional Laplace transform of  $V_T$  (Pitman and Yor, 1982):

$$E\left(e^{-aV_T} \middle| v_T\right) = \frac{\phi(\gamma(a))}{\phi(\kappa)} \frac{\exp\left(-\frac{v_0 + v_T}{2} \cosh\left(\frac{\gamma(a)T}{2}\right) \phi(\gamma(a))\right)}{\exp\left(-\frac{v_0 + v_T}{2} \cosh\left(\frac{\kappa T}{2}\right) \phi(\kappa)\right)} \frac{I_\nu\left(\sqrt{v_0 v_T} \phi(\gamma(a))\right)}{I_\nu\left(\sqrt{v_0 v_T} \phi(\kappa)\right)}$$

- See Glasserman and Kim (2011) for improvement.

# SABR model (conditional MC)

$$\frac{d\sigma_t}{\sigma_t} = \nu dZ_t \quad \Rightarrow \quad \sigma_T = \sigma_0 \exp \left( -\frac{1}{2}\nu^2 T + \nu Z_T \right)$$

Integrating  $\sigma_t$ ,

$$\nu \int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp \left( -\frac{1}{2}\nu^2 T + \nu Z_T \right) - \sigma_0$$

$S_T$  is expressed by  $\sigma_T$  and  $V_T$  (conditional MC possible) !

$$\text{Normal SABR}(\beta = 0) : S_T = S_0 + \frac{\rho}{\nu}(\sigma_T - \sigma_0) + \rho_* \sqrt{V_T} X_1$$

$$\text{BS SABR}(\beta = 1) : \log \left( \frac{S_T}{S_0} \right) = \frac{\rho}{\nu}(\sigma_T - \sigma_0) - \frac{1}{2}V_T + \rho_* \sqrt{V_T} X_1$$

See the **SABR Model** slides for detail.

# SABR Model (exact MC)

- $\sigma_T$  is distributed by a log-normal distribution. Sampling is trivial.
- The conditional Laplace transform of  $1/V_T$  is also known:

$$E\left(e^{-s/V_T} \middle| v_T\right) = \exp\left(-\frac{\phi_x(s)^2 - x^2}{2T}\right)$$

where  $\phi_x(s) = \text{acosh}(se^{-x} + \cosh(x))$  and  $v_T = \exp(\nu x)$

- From above, we can sample  $1/V_T$  and get  $V_T$ .
- Reference: [Cai et al \(2017\)](#)

## 3/2 model (conditional MC)

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu v_t^{3/2} dZ_t.$$

The change of variable,  $x_t = 1/v_t$  yields (a good Itô calculus exercise!)

$$dx_t = -\frac{dv_t}{v_t^2} + \frac{(dv_t)^2}{v_t^3} = (\kappa + \nu^2 - \kappa\theta x_t)dt - \nu\sqrt{x_t} dZ_t.$$

This is same as  $v_t$  in Heston model with new parameters:

$$\begin{aligned} \nu' &= -\nu, & \kappa' &= \kappa\theta, & \text{and} & \theta' = (\kappa + \nu^2)/\kappa\theta \\ (\nu &= -\nu', & \kappa &= \theta'\kappa' - \nu'^2, & \text{and} & \nu' = \kappa'/(\kappa'\theta' - \nu'^2)) \end{aligned}$$

We can express  $S_T$  as a function of  $V_T$  and  $v_T$  (conditional MC possible)!

$$d\log(x_t) = \left( \frac{\kappa + \nu^2/2}{x_t} - \kappa\theta \right) dt - \frac{\nu}{\sqrt{x_t}} dZ_t$$

$$\int_0^T \frac{1}{\sqrt{x_t}} dZ_t = \frac{1}{\nu} \left( \log\left(\frac{x_0}{x_T}\right) + (\kappa + \nu^2/2)V_T - \kappa\theta T \right),$$

$$\log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\nu} \left( \log\left(\frac{v_T}{v_0}\right) + (\kappa + \nu^2/2)V_T - \kappa\theta T \right) - \frac{1}{2}V_T + \rho_*\sqrt{V_T} X_1$$

## 3/2 model (exact MC)

- From the Heston model,  $1/v_T$  is distributed as a noncentral chi-square distribution,  $\chi^2(\delta', \lambda')$  where the degrees of freedom  $\delta'$  and the noncentrality  $\lambda'$  are

$$\delta' = \frac{4\kappa'\theta'}{\nu^2}, \quad \lambda' = \frac{4\kappa' e^{-\kappa'T}}{v_0\nu^2(1 - e^{-\kappa'T})}.$$

Standard library is available for drawing  $\chi^2$  random number.

- The conditional Laplace transform of  $V_T$  is also known.
- Reference: [Baldeaux \(2012\)](#)

# OUSV model (conditional MC). 2019ME Question

Let  $U_T = \int_0^T \sigma_t dt$  and  $V_T = \int_0^T \sigma_t^2 dt$ .

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu dZ_t.$$

$$d\sigma_t^2 = 2\sigma_t d\sigma_t + (d\sigma_t)^2 = (\nu^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\nu\sigma_t dZ_t$$

$$\sigma_T^2 - \sigma_0^2 = \nu^2 T + 2\kappa(\theta U_T - V_T) + 2\nu \int_0^T \sigma_t dZ_t$$

$$\int_0^T \sigma_t dZ_t = \frac{1}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\nu}{2}T - \frac{\kappa\theta}{\nu}U_T + \frac{\kappa}{\nu}V_T$$

$S_T$  is expressed by  $v_T$  and  $V_T$  (conditional MC possible)!

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\ &= \frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{1}{2}\right)V_T + \rho_*\sqrt{V_T} X_1\end{aligned}$$

$$S_0 := E(S_T) = S_0 \exp\left(\frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{\rho^2}{2}\right)V_T\right)$$

$$\sigma_{BS} := \rho_*\sqrt{V_T/T}.$$

# GARCH model (conditional MC): 2020ME Question

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t dZ_t$$

We derive the SDE for  $\sigma_t = \sqrt{v_t}$ ,

$$\begin{aligned} d\sigma_t &= d\sqrt{v_t} = \frac{1}{2} \frac{dv_t}{\sqrt{v_t}} - \frac{1}{8} \frac{(dv_t)^2}{v_t \sqrt{v_t}} = \frac{1}{2} \kappa \left( \frac{\theta}{\sigma_t} - \sigma_t \right) dt + \frac{\nu}{2} \sigma_t dZ_t - \frac{\nu^2}{8} \sigma_t dt \\ &= \frac{1}{2} \left( \frac{\kappa\theta}{\sigma_t} - \left( \kappa + \frac{\nu^2}{4} \right) \sigma_t \right) dt + \frac{\nu}{2} \sigma_t dZ_t. \end{aligned}$$

Integrating above,

$$\begin{aligned} \sigma_T - \sigma_0 &= \frac{1}{2} \left( \kappa\theta Y_T - \left( \kappa + \frac{\nu^2}{4} \right) U_T \right) + \frac{\nu}{2} \int_0^T \sigma_t dZ_t \\ \Rightarrow \int_0^T \sigma_t dZ_t &= \frac{2}{\nu} (\sigma_T - \sigma_0) - \left( \frac{\kappa\theta}{\nu} Y_T - \left( \frac{\kappa}{\nu} + \frac{\nu}{4} \right) U_T \right) \\ \text{where } Y_T &= \int_0^T \frac{1}{\sigma_t} dt, \quad U_T = \int_0^T \sigma_t dt \quad \text{and} \quad V_T = \int_0^T \sigma_t^2 dt. \end{aligned}$$



Therefore,

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2} V_T \\ &= \frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu} Y_T + \rho\left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right) U_T - \frac{1}{2} V_T + \rho_* \sqrt{V_T} X_1.\end{aligned}$$

Finally,

$$\begin{aligned}E(S_T | \sigma_T, Y_T, U_T, V_T) &= S_0 \exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2} V_T\right) \\ &= S_0 \exp\left(\frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu} Y_T + \rho\left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right) U_T - \frac{\rho^2}{2} V_T\right) \\ \sigma_{BS} &= \rho_* \sqrt{V_T/T}.\end{aligned}$$

- For the Euler/Milstein/Log schemes, see **2019ME** problem.
- Analytic approximations are available only for  $\rho = 0$  ([Barone-Adesi et al, 2005](#)).

# Project Suggestion: *Almost* Exact MC (by Choi)

- General scheme:
  - Implementing existing paper is OK:
  - Improving Euler / Milstein scheme? or exact simulation?
  - Or try something new (see below):
- Simulation for **GARCH diffusion**:

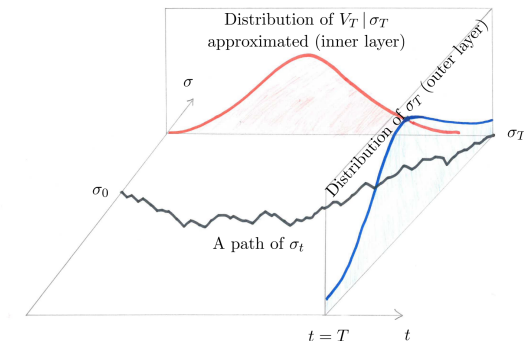
$$dv_t = \kappa(\theta - v_t)dt + \nu v_t dZ_t.$$

- Currently, there is no easy way to solve the SDE.
  - Conditional MC possible? Option pricing with conditional MC?
  - How to express  $S_T$  as a function of  $v_T$  and  $V_T$  (or something else)?
  - Exact simulation possible?
- *Almost* Exact MC (by Choi)

# Project Suggestion: *Almost* Exact MC (by Choi)

The illustration of the proposed double layer approximation method:

- 1 The outer layer distribution of  $\sigma_T$  (in blue) is typically known
- 2 The inner layer distribution of  $V_T | \sigma_T$  (in red) is approximated as well-known distributions such as log-normal or inverse Gaussian.



# Project Suggestion: *Almost* Exact MC (by Choi)

Drawback of the exact simulation methods:

- Inverse of the Laplace transform,  $E(e^{-sV_T}|v_T) = f(s, v_T)$ , is complicated.
- Drawing random number from the numerical CDF is also slow.

How can we simplify this step with some approximation?

- Approximate  $V_T|v_T$  with a well-known distribution by matching the first two moments,  $M_1 = E(V_T|v_T)$  and  $M_2 = E(V_T^2|v_T)$ .
- The RN sampling should be easy from the approximate distribution.

# Almost Exact MC: Candidates for distributions

- Log-normal (LN):

$$Y \sim \mu \exp(\sigma Z - \sigma^2/2) \quad \text{for} \quad Z \sim N(0, 1)$$

The parameters  $(\mu, \sigma)$  can be obtained from the two moments:

$$\mu = M_1 \quad \text{and} \quad \sigma = \sqrt{\log(M_2/M_1^2)}.$$

- Inverse-Gaussian (IG):

$$f_{\text{IG}}(x | \gamma, \delta) = \frac{\delta}{\sqrt{2\pi x^3}} \exp\left(-\frac{(\gamma x - \delta)^2}{2x}\right) \quad \text{for} \quad \gamma \geq 0, \delta > 0.$$

How to determine  $(\gamma, \mu)$  from  $M_1$  and  $M_2$ ?

- The sampling methods for LN and IG are available. See [Michael et al \(1976\)](#) ([WIKIPEDIA](#)) for IG.

# Almost Exact MC: How to obtain $M_1$ and $M_2$ ?

Keep in mind for a random variable  $X \geq 0$ , the MGF and Laplace transform are same:

$$M_X(-s) = E(e^{-sX}) = \int_{x=0}^{\infty} e^{-sx} f_X(x) dx = f(s)$$
$$f(s) = 1 - M_1 s + \frac{1}{2} M_2 s^2 + \dots,$$

where  $M_1 = E(V_T|v_T)$  and  $M_2 = E(V_T^2|v_T)$ .

- Numerical method: [Choudhury and Lucantoni \(1996\)](#)
- Analytic method (from Taylor's expansion or etc):
  - SABR: available in [Kennedy et al \(2012\)](#).
  - Heston, 3/2, OUSV?

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