A quick note on implied volatility
Applied Stochastic Processes (FIN 514)

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Overview

- The Black-Scholes-Merton (BSM), Bachelier (Normal) models are seldom used to *predict* the option price. The option prices are determined from the supply and demand of market.
- However, the pricing models are still important because they provide a consistent measure to intuitively understand the prices of the options with different strikes, time to maturity, etc.
- Implied volatility (IV) is the value of the volatility in pricing model which returns (or solve) the price of an option given (i.e., from market):

$$C(K, S_0, \sigma, T_e) = \text{Price}$$

- How can you compare the two prices?

$$C(K = 105, S_0 = 100, T_e = 1) = 5.9 \quad P(K = 98, S_0 = 100, T_e = 1) = 8.7$$

The implied volatility is same as 20%.

General rule

- Option prices (both call and put) increase as the volatility increases: $C(K, S_0, \sigma, T_e)$ is monotonically increasing function.
- Option value = Time value + Intrinsic Value
  - Intrinsic value: the value you get by exercising option now. (> 0)
  - Time value: the extra value from the change of the underlying price until the expiry. (> 0)
- The intrinsic value can be understood as the option value wih $\sigma = 0$, $C(K, S_0, \sigma = 0, T_e)$, hence the minimum value.
- The call option value as $\sigma \to \infty$:
  - BSM model ($S_T \geq 0$): S_0. The underlying stock is always worth more than any call option with $K > 0$.
  - Bachelier (Normal) model (S_T can be negative): ∞. Call option can protect the (infinite) loss.

IV computation

- The computation of IV depends on numerical root-solving method. [Demo]
  - Newton method: using vega,

$$\sigma^{(k+1)} = \sigma^{(k)} - \frac{C(\sigma^{(k)}, \cdots) - \text{Price}}{V(\sigma^{(k)})}, \quad V(\sigma) = \frac{\partial C(\sigma, \cdots)}{\partial \sigma}$$

  - Brent's method: [Demo]
- BSM model:
  - Newton's method with good initial guess (`PyFeng` package)
  - Let's Be Rational (Jackel, 2015): Machine epsilon error within two step iterations.
- Normal model:
  - Choi et al. (2007). Numerical Approximation of the Implied Volatility Under Arithmetic Brownian Motion:
    Polynomial approximation with error ($< 10^{-9}$)

IV under the Bachelier Model (Choi et al., 2007)

$$\sigma_{\rm N} = \sqrt{\frac{\pi}{2T}}\,(2C - \theta(F_0 - K))\; h(\eta),$$

where C is the undiscounted price of either call ($\theta = 1$) or for put ($\theta = -1$) option. The function $h(\eta)$ and the argument η are defined in a sequence,

$$h(\eta) = \sqrt{\eta}\,\frac{\sum_{k=0}^{7} a_k \eta^k}{1 + \sum_{k=1}^{9} b_k \eta^k}, \quad \eta = \frac{v}{\tanh^{-1}(v)}, \quad \text{and} \quad v = \frac{|F_0 - K|}{2C - \theta(F_0 - K)},$$

with the coefficients,

a_0	= 3.99496 16873 45134 e-1	b_1	= 4.99053 41535 89422 e+1
a_1	= 2.10096 07950 68497 e+1	b_2	= 3.09357 39367 43112 e+1
a_2	= 4.98034 02178 55084 e+1	b_3	= 1.49510 50083 10999 e+3
a_3	= 5.98876 11026 90991 e+2	b_4	= 1.32361 45378 99738 e+3
a_4	= 1.84848 96954 37094 e+3	b_5	= 1.59891 96976 79745 e+4
a_5	= 6.10632 24078 67059 e+3	b_6	= 2.39200 88917 20782 e+4
a_6	= 2.49341 52853 49361 e+4	b_7	= 3.60881 71083 75034 e+3
a_7	= 1.26645 80513 48246 e+4	b_8	= -2.06771 94864 00926 e+2
		b_9	= 1.17424 05993 06013 e+1.