# Exact simulation scheme for the Ornstein–Uhlenbeck driven stochastic volatility (OUSV) model with the Karhunen–Loève (KL) expansions
# Applied Stochastic Processes (FIN 514)

Jaehyuk CHOI

Peking University HSBC Business School, Shenzhen, China

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# Introduction: OUSV model

## Ornstein–Uhlenbeck driven stochastic volatiliy (OUSV) model

$$\frac{dS_t}{S_t} = r dt + \sigma_t dW_t \quad \text{and} \quad d\sigma_t = \kappa(\theta - \sigma_t)dt + \xi dZ_t,$$

where $W_t$ and $Z_t$ are the standard BMs correlated by $\rho$, $\kappa$ is the mean reversion speed, $\xi$ is the volatility of volatility (vol-of-vol), and $\theta$ is the long-term volatility.

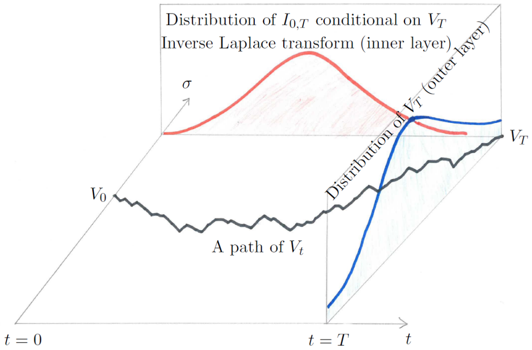
- Volatility $\sigma_t$ is driven by the Ornstein–Uhlenbeck (OU) process.
- One of the earliest stochastic volatility (SV) models:
  Uncorrelated OUSV (Stein and Stein, 1991)
  v.s. Hull and White (1987) and Heston (1993) models.
- Vanilla option can be efficiently priced with inverse Fourier transform (Schöbel and Zhu, 1999).
- Efficient and accurate Monte–Carlo (MC) simulation (i.e., sampling $S_T$) under the OUSV model is recently studied.

# Introduction: Exact simulation

## Exact simulation for various SV models

- Heston (1993) model: Broadie and Kaya (2006) and Glasserman and Kim (2011, Gamma RV series)
- 3/2 volatility model: Baldeaux (2012)
- Stochastic-alpha-beta-rho (SABR): Cai et al. (2017), Choi et al. (2019, $\beta = 0$ SABR), and Cui et al. (2021, CTMC)
- **OUSV model**: Li and Wu (2019)

- Directly sample $S_T$ from $S_0$ for any time step $T$ without time-discretization.
- Avoid possible discretization bias from the Euler/Milstein scheme.
- Typical procedures:
  1. Sample terminal volatility $\sigma_T$ from a known distribution (SDE)
  2. Sample integrated variance $V_{0,T}$ by numerically inverting its Laplace transform.
  3. Sample price $S_T$ from a log-normal distribution.
- Heavy computation cost from the numerical inversion, step (2).
  Not necessarily more efficient than the simple Euler/Milstein scheme.


Distribution of $I_{0,T}$ conditional on $V_T$
Inverse Laplace transform (inner layer)
$\sigma$
Distribution of $V_T$ (outer layer)
$V_T$
$V_0$
A path of $V_t$
$t=0$
$t=T$
$t$

# Contribution summary

## Existing studies

- Li and Wu (2019, EJOR) propose the first exact simulation scheme for the OUSV model.
- As in other exact simulation schemes, Li and Wu (2019) suffer from costly numerical inversion.

## Contribution of this study

- Using the Karhunen–Loève (KL) expansions, the volatility path ($\sigma_t$, OU process) is expressed as the infinite sine series.
- The time integrals of $\sigma_t$ and $\sigma_t^2$ are analytically derived as the sum of independent normal variates.
- Only $N \leq 10$ sine terms required after handling the truncation error.
- The new simulation is **several hundred times faster than** Li and Wu (2019) by avoiding the transform inversion approach.

# The average volatility ($U_{0,T}$) and variance ($V_{0,T}$)

- Define the time averages of volatility and variance, respectively:

$$U_{0,T} = \frac{1}{T}\int_0^T \sigma_t \, dt \quad \text{and} \quad V_{0,T} = \frac{1}{T}\int_0^T \sigma_t^2 \, dt.$$

- The final price, $S_T$, conditional on $(\sigma_T,\, U_{0,T},\, V_{0,T})$, follows a log-normal distribution (Li and Wu, 2019, Proposition 1):

$$\log(S_T/S_0) = \mu_{0,T} + \Sigma_{0,T} Z \quad (Z : \text{standard normal}),$$

$$\mu_{0,T} = rT + \frac{\rho}{2\xi}\left[\left(-\xi^2 - 2\kappa\theta U_{0,T} + (2\kappa - \xi/\rho)\, V_{0,T}\right) T + (\sigma_T^2 - \sigma_0^2)\right]$$

$$\Sigma_{0,T}^2 = \rho_*^2\, T V_{0,T} \quad (\rho_* = \sqrt{1-\rho^2}).$$

## Li and Wu (2019)'s sampling

1. $(\sigma_T, U_{0,T})$ from a bi-variate normal distribution.
2. $V_{0,T}$ from the inverse transform of $V_{0,T}$ conditional on $(\sigma_T, U_{0,T})$.
3. $S_T$ from a lognormal distribution.

# Derivation

- Let $U_{0,T} = \frac{1}{T}\int_0^T \sigma_t\, dt$ and $V_{0,T} = \frac{1}{T}\int_0^T \sigma_t^2\, dt$.
- Derive SDE for $\sigma_t^2$:

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \xi\, dZ_t.$$

$$d\sigma_t^2 = 2\sigma_t d\sigma_t + (d\sigma_t)^2 = (\xi^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\xi\sigma_t dZ_t$$

$$\sigma_T^2 - \sigma_0^2 = \xi^2 T + 2\kappa T(\theta U_{0,T} - V_{0,T}) + 2\xi\int_0^T \sigma_t dZ_t$$

- Stochastic integral of $\sigma_t$:

$$\int_0^T \sigma_t dZ_t = \frac{1}{2\xi}(\sigma_T^2 - \sigma_0^2) - T\left(\frac{\xi}{2} + \frac{\kappa\theta}{\xi}U_{0,T} - \frac{\kappa}{\xi}V_{0,T}\right)$$

- $S_T$ is expressed by $\sigma_T$ and $V_{0,T}$!

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho\int_0^T \sigma_t dZ_t + \rho_*\int_0^T \sigma_t dX_t - \frac{1}{2}TV_{0,T} \\ &= \rho\Big[\cdots\Big] - \frac{1}{2}TV_{0,T} + \rho_*\sqrt{TV_{0,T}}\, Z\end{aligned}$$

# Sampling the terminal volatility

- Define the centered (de-meaned) OU process $\hat{\sigma}_T$:

$$\hat{\sigma}_T = \sigma_T - \underbrace{\theta - (\sigma_0 - \theta)e^{-\kappa T}}_{E(\sigma_T)} = \xi e^{-\kappa T} \int_0^T e^{\kappa t} dZ_t \quad (\hat{\sigma}_0 = 0),$$

- $\hat{\sigma}_T$ follows a Gaussian process with zero mean and the covariance given by

$$\text{Cov}(\hat{\sigma}_t, \hat{\sigma}_T) = \frac{\xi^2}{2\kappa}\left(e^{-\kappa(T-t)} - e^{-\kappa(T+t)}\right) = \xi^2 \frac{\sinh(\kappa t)}{\kappa e^{\kappa T}} \quad \text{for} \quad 0 \le t \le T.$$

- $\hat{\sigma}_T$ can be sampled by

$$\hat{\sigma}_T \overset{d}{=} \xi \sqrt{\frac{\sinh(\kappa T)}{\kappa e^{\kappa T}}}\, Z_0 \quad \text{for} \quad Z_0 \sim N(0,1).$$

# Auxiliary processes (1/2)

- Remove the long-term volatility $\theta$:

$$\bar\sigma_t = \sigma_t - \theta, \quad \bar U_{0,T} = \frac{1}{T}\int_0^T \bar\sigma_t\,dt \quad \text{and} \quad \bar V_{0,T} = \frac{1}{T}\int_0^T \bar\sigma_t^2\,dt.$$

- The process $\bar\sigma_t$ satisfies a simpler form of the OU process:

$$d\bar\sigma_t = -\kappa\bar\sigma_t\,dt + \xi\,dZ_t \quad (\bar\sigma_0 = \sigma_0 - \theta).$$

- Its solution is well-known as

$$\bar\sigma_T = \bar\sigma_0 e^{-\kappa T} + \xi e^{-\kappa T}\int_0^T e^{\kappa t}\,dZ_t.$$

- The introduction of $(\bar\sigma_T, \bar U_{0,T}, \bar V_{0,T})$ will simplify algebra. The original triplets $(\sigma_T, U_{0,T}, V_{0,T})$ and the new triplet $(\bar\sigma_T, \bar U_{0,T}, \bar V_{0,T})$ are interchangeable by

$$\sigma_T = \theta + \bar\sigma_T, \quad U_{0,T} = \theta + \bar U_{0,T}, \quad \text{and} \quad V_{0,T} = \theta^2 + 2\theta\bar U_{0,T} + \bar V_{0,T}.$$

# Auxiliary processes (2/2)

- Define the centered OU process $\hat\sigma_T$ by removing its mean from $\bar\sigma_T$,

$$\hat\sigma_T = \bar\sigma_T - \bar\sigma_0 e^{-\kappa T} = \xi e^{-\kappa T}\int_0^T e^{\kappa t}\,dZ_t \quad (\hat\sigma_0 = 0).$$

- $\hat\sigma_T$ is a Gaussian process with zero mean and the covariance given by

$$\mathrm{Cov}(\hat\sigma_t, \hat\sigma_T) = \frac{\xi^2}{2\kappa}\left(e^{-\kappa(T-t)} - e^{-\kappa(T+t)}\right) \quad \text{for} \quad 0 \le t \le T.$$

- The terminal value $\hat\sigma_T$ can be sampled by

$$\hat\sigma_T \sim \xi\sqrt{\frac{1-e^{-2\kappa T}}{2\kappa}}\,Z_0 = \xi\sqrt{T\phi(2\kappa T)}\,Z_0,$$

$$\text{where} \quad Z_0 \sim N(0,1) \quad \text{and} \quad \phi(x) = \frac{1-e^{-x}}{x} \quad (\phi(0)=1).$$

- $\hat\sigma_T$ is interchangeably used with $\sigma_T$ or $\bar\sigma_T$:

$$\sigma_T - \theta = \bar\sigma_T = \hat\sigma_T + \bar\sigma_0 e^{-\kappa T}.$$

# The KL expansion of the OU bridge

- Given $\hat{\sigma}_T$, construct the OU bridge process, $B_t$, of $\hat{\sigma}_t$ for $0 \le t \le T$:

$$B_t = \hat{\sigma}_t - \frac{\mathrm{Cov}(\hat{\sigma}_t, \hat{\sigma}_T)}{\mathrm{Cov}(\hat{\sigma}_T, \hat{\sigma}_T)} \hat{\sigma}_T = \hat{\sigma}_t - \frac{\sinh(\kappa t)}{\sinh(\kappa T)} \hat{\sigma}_T \quad (B_0 = B_T = 0).$$

- The OU bridge can be decomposed into the KL expansions (Daniluk and Muchorski, 2016, Theorem 2.3)

$$B_t = \xi \sum_{n=1}^{\infty} a_n \sqrt{T} \sin\left(\frac{n\pi t}{T}\right) Z_n \quad \text{for} \quad a_n = \sqrt{\frac{2}{(\kappa T)^2 + (n\pi)^2}},$$

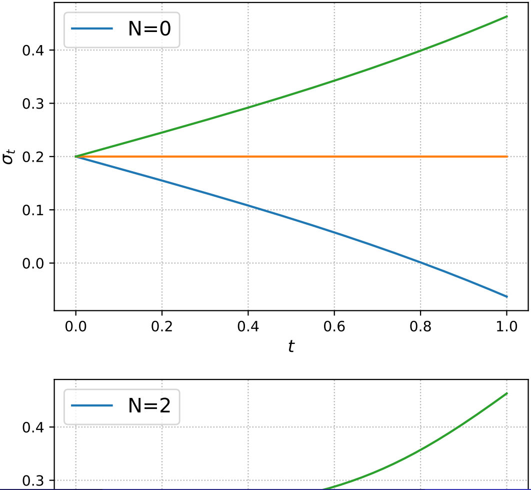
  where $Z_n \sim N(0,1)\ i.i.d.$
- Effectively, **KL expansions are the PCA on** $B_t$ ($\infty$-dimensional data).
- When $\kappa = 0$, the expansions are nested to the KL expansions of the Brownian bridge, $a_n = \sqrt{2}/(n\pi)$.
- Finally, the volatility (i.e., OU process) path is represented by

$$\bar{\sigma}_t = \bar{\sigma}_0\, e^{-\kappa t} + \hat{\sigma}_T \frac{\sinh(\kappa t)}{\sinh(\kappa T)} + \xi\sqrt{T} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right) Z_n \quad (0 \le t \le T).$$

# Illustration of $\sigma_t$ paths from the KL expansions

Three sample volatility (OU process) paths generated by KL expansions with increasing $N = 0$, 2, $\cdots$, 64 sine terms for

$$\sigma_0 = \theta = \xi = 0.2, \quad \text{and} \quad \kappa = 1.$$



# Exact simulation scheme

$\bar{U}_{0,T}$ and $\bar{V}_{0,T}$ are analytically integrated to,

$$\bar{U}_{0,T} = \underbrace{\left[\bar{\sigma}_0 + \frac{\hat{\sigma}_T}{1+e^{-\kappa T}}\right]\phi(\kappa T)}_{:=E(\bar{U}_{0,T}\,|\,\hat{\sigma}_T)} + 2\xi\sqrt{T}\sum_{\substack{n=1\\ n:\text{ odd}}}^{\infty}\frac{a_n}{n\pi}Z_n,$$

$$\bar{V}_{0,T} = \underbrace{\bar{\sigma}_0^2\,\phi(2\kappa T) + \hat{\sigma}_T^2\frac{\sinh(2\kappa T)-2\kappa T}{4\kappa T\sinh^2(\kappa T)} + \frac{\xi^2}{2\kappa}\left[\coth(\kappa T)-\frac{1}{\kappa T}\right] + \bar{\sigma}_0\hat{\sigma}_T\frac{e^{-\kappa T}}{\kappa T}\left[\frac{1}{\phi(2\kappa T)}-1\right]}_{:=E(\bar{V}_{0,T}\,|\,\hat{\sigma}_T)}$$

$$+\,\xi\sqrt{T}\left[\bar{\sigma}_0\sum_{n=1}^{\infty}n\pi\,a_n^3 Z_n + \bar{\sigma}_T\sum_{n=1}^{\infty}(-1)^{n-1}n\pi\,a_n^3 Z_n\right] + \frac{\xi^2 T}{2}\sum_{n=1}^{\infty}a_n^2(Z_n^2-1).$$

$$\bar{U}_{0,T} = \underbrace{\left[\bar{\sigma}_0 + \frac{\hat{\sigma}_T}{1+e^{-\kappa T}}\right]\phi(\kappa T)}_{:=E(\bar{U}_{0,T}\,|\,\hat{\sigma}_T)} + 2\xi\sqrt{T}\sum_{\substack{n=1\\ n:\text{ odd}}}^{L}\frac{a_n}{n\pi}Z_n,$$

$$\bar{V}_{0,T} = \underbrace{\bar{\sigma}_0^2\,\phi(2\kappa T) + \hat{\sigma}_T^2\frac{\sinh(2\kappa T)-2\kappa T}{4\kappa T\sinh^2(\kappa T)} + \frac{\xi^2}{2\kappa}\left[\coth(\kappa T)-\frac{1}{\kappa T}\right] + \bar{\sigma}_0\hat{\sigma}_T\frac{e^{-\kappa T}}{\kappa T}\left[\frac{1}{\phi(2\kappa T)}-1\right]}_{:=E(\bar{V}_{0,T}\,|\,\hat{\sigma}_T)}$$

$$+\,\xi\sqrt{T}\left[\bar{\sigma}_0\sum^{L}n\pi\,a_n^3 Z_n + \bar{\sigma}_T\sum^{L}(-1)^{n-1}n\pi\,a_n^3 Z_n\right] + \frac{\xi^2 T}{2}\sum^{L}a_n^2(Z_n^2-1).$$

# Truncated terms (1/3)

We also need to simulate the truncated terms in $\bar{U}_{0,T}$ and $\bar{V}_{0,T}$:

$$G_L = \sum_{\substack{n=L+1 \\ n \text{ odd}}}^{\infty} \frac{a_n}{n\pi} Z_n, \quad P_L = \sum_{\substack{n=L+1 \\ n \text{ odd}}}^{\infty} n\pi\, a_n^3 Z_n, \quad Q_L = \sum_{\substack{n=L+1 \\ n \text{ even}}}^{\infty} n\pi\, a_n^3 Z_n.$$

- $G_L$, $P_L$, and $Q_L$ follow a multivariate normal distribution with

$$\mu = 0, \quad \Sigma = \begin{pmatrix} f_L^{\text{odd}} & c_L^{\text{odd}} & 0 \\ c_L^{\text{odd}} & g_L^{\text{odd}} & 0 \\ 0 & 0 & g_L^{\text{even}} \end{pmatrix},$$

where

$$c_L = \sum_{n=L+1}^{\infty} a_n^4,\ f_L = \sum_{n=L+1}^{\infty} \frac{a_n^2}{(n\pi)^2},\ g_L = \sum_{n=L+1}^{\infty} (n\pi)^2 a_n^6.$$

and **odd** and **even** superscripts are defined by

$$c_L^{\text{odd}} = \sum_{\substack{n=L+1 \\ n \text{ odd}}}^{\infty} a_n^4 \quad \text{and} \quad c_L^{\text{even}} = \sum_{\substack{n=L+1 \\ n \text{ even}}}^{\infty} a_n^4.$$

# Truncated terms (2/3)

- We can exactly sample $G_L$, $P_L$, and $Q_L$:

$$G_L = \sqrt{f_L^{\text{odd}}} \left( \sqrt{1-\rho_{GP}^2}\, W_1 + \rho_{GP}\, W_2 \right) \quad \text{with} \quad \rho_{GP} = c_L^{\text{odd}} \Big/ \sqrt{f_L^{\text{odd}}\, g_L^{\text{odd}}}\,,$$

$$P_L = \sqrt{g_L^{\text{odd}}}\, W_2 \quad \text{and} \quad Q_L = \sqrt{g_L^{\text{even}}}\, W_3,$$

  where $W_1$, $W_2$, and $W_3$ are independent normal random variates.
- Even more, the infinite sums have analytic expressions: for $\lambda = \kappa T$,

$$c_0 = \frac{1}{\lambda^4}\left(\frac{\lambda}{\tanh(\lambda)} + \frac{\lambda^2}{\sinh(\lambda)^2} - 2\right),$$

$$f_0 = \frac{b_0(0) - b_0(\lambda)}{\lambda^2}, \quad \text{and} \quad g_0(\lambda) = 2c_0 - \lambda^2 d_0.$$

- No approximation so far.

# Truncated terms (3/3)

$$R_L = \sum_{n=L+1}^{\infty} a_n^2 (Z_n^2 - 1) \approx \sqrt{c_L}(W_4^2 - 1)$$

- Mean and variance of $R_L$ are

$$E(R_L) = 0 \quad \text{and} \quad \mathrm{Var}(R_L) = 2c_L = 2\textstyle\sum_{n=L+1}^{\infty} a_n^4.$$

- Not independent from, but zero-correlation with $G_L$, $P_L$, and $Q_L$.
- $W_4$ is a normal variate independent from $W_1$, $W_2$, and $W_3$.
- Approximation of $R_L$ is the only source of error.

# Exact MC procedure

## The procedure for simulating $\bar\sigma_T$, $\bar U_{0,T}$, and $\bar V_{0,T}$

$$
\begin{aligned}
\bar\sigma_T =& \bar\sigma_0 e^{-\kappa T} + \hat\sigma_T \quad \text{where} \quad \hat\sigma_T = \xi\sqrt{T\phi(2\kappa T)}\, Z_0,\\
\bar U_{0,T} =& E(\bar U_{0,T}\,|\,\hat\sigma_T) + 2\xi\sqrt{T}\left(\sum_{\substack{n=1\\ n:\ \text{odd}}}^{L} \frac{a_n}{n\pi} Z_n + G_L\right),\\
\bar V_{0,T} \approx& E(\bar V_{0,T}|\hat\sigma_T) + \xi\sqrt{T}\left[\bar\sigma_0\left(\sum_{n=1}^{L} n\pi\, a_n^3 Z_n + P_L + Q_L\right)\right.\\
&\left. + \bar\sigma_T\left(\sum_{n=1}^{L}(-1)^{n-1} n\pi\, a_n^3 Z_n + P_L - Q_L\right)\right] + \frac{\xi^2 T}{2}\left(\sum_{n=1}^{L} a_n^2(Z_n^2-1) + R_L\right),
\end{aligned}
$$

- We use $L+5$ ($Z_0, \cdots, Z_L$, $W_1$, $W_2$, $W_3$, $W_4$) independent normal variates for simulating from $t=0$ to $T$.
- $R_T$ is the only source of approximation error.

# Numerical test

- We test the new MC scheme against the spot and vanilla option prices as non-trivial examples.
- Exact option values are available from the inverse Fourier transform (Schöbel and Zhu, 1999).

## Parameter set to test

- For direct comparison, we test the parameter set from Li and Wu (2019, Table 2):

$$S_0 = K = 100,\ \sigma_0 = \theta = 0.2,\ \kappa = 4,\ \xi = 0.1,\ \rho = -0.7,\ \text{and } r = 0.09531,$$
$$T = 1, \quad 5, \quad \text{and} \quad 10$$

- As in Fourier series, we expect more sine terms ($L$) is required for bigger $T$.

# MC variance reduction

## Conditional MC (Willard, 1997)

Instead of sampling $S_T$, MC-average conditional BS price over $(\sigma_T, U_{0,T}, V_{0,T})$:

$$C_{\text{OUSV}} = E_{\text{MC}}\left\{ C_{\text{BS}}\left(K,\, F_T,\, \rho_* \sqrt{V_{0,T}}\right) \right\}, \quad \text{where}$$

$$F_T = E\{S_T \,|\, \sigma_T, U_{0,T}, V_{0,T}\} = S_0 \exp\left(\mu_{0,T} + \Sigma^2_{0,T}/2\right)$$

$$= S_0 \exp\left( rT + \frac{\rho}{2\xi}\left[\left(-\xi^2 - 2\kappa\theta U_{0,T} + (2\kappa - \rho\xi)\, V_{0,T}\right) T + ({\sigma_T}^2 - \sigma_0^2)\right] \right).$$

- Cai et al. (2017) reports 99% variance reduction (in the SABR model).
- With the reduced variance, we aim to measure the bias accurately.

## Martingale-correcting control variate

Because $S_0 = e^{-rT}\, E_{\text{MC}}\{F_T\}$, correct $F_T$ by

$$F_T^{\text{cv}} = \mu F_T \quad \text{for} \quad \mu = S_0\, e^{rT} \big/ E_{\text{MC}}\{F_T\}.$$

# Numerical results: $T = 1$

Table: The simulation result for $T = 1$. "Spot Price" and "Option Price" evaluate $\bar{S}_T$ and $C_{\text{OUSV}}$, respectively. "Option Price with CV" uses the control variate $\bar{S}_T^{\text{cv}}$. The true option price is 13.21492.

| $N$ | $n_{\text{path}}$ (number of paths) | Spot Price | | Option Price | | Option Price with CV | | |
|---|---|---|---|---|---|---|---|---|
| | | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | CPU Time (Seconds) |
| 2 | 10,000 | 0.3 | 1.74 | -0.4 | 4.04 | -0.7 | 3.48 | 0.006 |
| | 40,000 | 0.3 | 0.87 | -0.4 | 2.33 | -0.6 | 1.74 | 0.024 |
| | 160,000 | 0.3 | 0.44 | -0.4 | 1.17 | -0.6 | 0.88 | 0.096 |
| 4 | 10,000 | -0.2 | 1.73 | -0.9 | 4.03 | -0.7 | 3.47 | 0.008 |
| | 40,000 | -0.2 | 0.87 | -0.9 | 2.33 | -0.7 | 1.74 | 0.028 |
| | 160,000 | -0.2 | 0.44 | -0.9 | 1.17 | -0.7 | 0.87 | 0.109 |
| 6 | 10,000 | -0.2 | 1.74 | -0.4 | 4.04 | -0.3 | 3.48 | 0.008 |
| | 40,000 | -0.2 | 0.87 | -0.4 | 2.33 | -0.2 | 1.74 | 0.030 |
| | 160,000 | -0.2 | 0.44 | -0.4 | 1.17 | -0.2 | 0.87 | 0.122 |

# Numerical results: $T = 5$

Table: The simulation result for $T = 5$. "Spot Price" and "Option Price" evaluate $\bar{S}_T$ and $C_{\text{OUSV}}$, respectively. "Option Price with CV" uses the control variate $\bar{S}_T^{\text{cv}}$. The true option price is 40.797689.

| $N$ | $n_{\text{path}}$ (number of paths) | Spot Price | | Option Price | | Option Price with CV | | |
|---|---|---|---|---|---|---|---|---|
| | | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | CPU Time (Seconds) |
| 4 | 10,000 | 19.7 | 9.56 | 10.7 | 12.13 | -6.8 | 4.11 | 0.006 |
| | 40,000 | 19.7 | 4.78 | 10.7 | 6.35 | -6.9 | 2.06 | 0.026 |
| | 160,000 | 19.7 | 2.41 | 10.7 | 3.19 | -7.0 | 1.04 | 0.103 |
| 6 | 10,000 | 5.4 | 9.57 | 3.4 | 12.13 | -1.2 | 4.12 | 0.007 |
| | 40,000 | 5.4 | 4.79 | 3.4 | 6.37 | -1.4 | 2.06 | 0.028 |
| | 160,000 | 5.4 | 2.39 | 3.4 | 3.18 | -1.4 | 1.03 | 0.110 |
| 8 | 10,000 | -0.2 | 9.54 | -1.2 | 12.11 | -0.7 | 4.11 | 0.007 |
| | 40,000 | -0.2 | 4.76 | -1.2 | 6.33 | -0.9 | 2.05 | 0.028 |
| | 160,000 | -0.2 | 2.37 | -1.2 | 3.15 | -0.9 | 1.02 | 0.111 |

# Numerical results: $T = 10$

Table: The simulation result for $T = 10$. "Spot Price" and "Option Price" evaluate $\bar{S}_T$ and $C_{\text{OUSV}}$, respectively. "Option Price with CV" uses the control variate $\bar{S}_T^{\text{cv}}$. The true option price is 62.76312.

| $N$ | $n_{\text{path}}$ (number of paths) | Spot Price | | Option Price | | Option Price with CV | | |
|---|---|---|---|---|---|---|---|---|
| | | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | Bias $\times 10^{-4}$ | RMSE $\times 10^{-2}$ | CPU Time (Seconds) |
| 6 | 10,000 | 19.3 | 19.14 | 16.8 | 20.68 | -1.3 | 2.65 | 0.007 |
| | 40,000 | 19.3 | 9.59 | 16.8 | 10.56 | -1.6 | 1.33 | 0.028 |
| | 160,000 | 19.3 | 4.78 | 16.8 | 5.27 | -1.7 | 0.66 | 0.111 |
| 8 | 10,000 | -1.5 | 19.06 | -1.9 | 20.63 | 0.0 | 2.64 | 0.007 |
| | 40,000 | -1.5 | 9.50 | -1.9 | 10.47 | -0.3 | 1.32 | 0.029 |
| | 160,000 | -1.5 | 4.73 | -1.9 | 5.20 | -0.4 | 0.65 | 0.117 |
| 10 | 10,000 | -4.6 | 19.10 | -4.5 | 20.65 | 0.5 | 2.65 | 0.008 |
| | 40,000 | -4.6 | 9.57 | -4.5 | 10.54 | 0.2 | 1.33 | 0.033 |
| | 160,000 | -4.6 | 4.82 | -4.5 | 5.31 | 0.1 | 0.67 | 0.132 |

# Conclusion

## Contribution of this study

- Using the Karhunen–Loève (KL) expansions, the volatility path (OU process) is expressed as the infinite sine series.
- The time integrals of $\sigma_t$ and $\sigma_t^2$ are analytically derived as the sum of independent normal variates.
- Only $L \leq 10$ sine terms required after handling the truncation error.
- The new simulation is **several hundred times faster than** Li and Wu (2019) by avoiding the transform inversion approach.

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