Applied Stochastic Processes (FIN 514) Midterm Exam

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BM stands for Brownian motion. Assume that B_t, W_t, X_t and Z_t are standard BMs unless stated otherwise. **RN** and **RV** stand for random number and random variable, respectively. $P(A)$ denotes the probability of the event A.

1. (12 points) From the 2020 exam problem, we learn how to sample Gamma(k, β) RV when k is a positive integer. In this problem, we are going to generate Gamma(α, β) when $\alpha > 0$ is any number using the acceptance-rejection sampling from the 2021 exam problem. To make problem simple, we can assume $\beta = 1$ because Gamma$(\alpha, \beta) \sim$ Gamma$(\alpha, 1)/\beta$.

   Let $X \sim$ Gamma$(\alpha, 1)$ for $\alpha > 0$. Reminded that its PDF is given by

   $$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x},$$

   where $\Gamma(\alpha)$ is the gamma function.

   (a) (2 points) Consider the following RV, Y (≥ 0), whose PDF $g(x)$ is given by

   $$g(x) = \begin{cases} A\, x^{\alpha-1} & \text{if} \quad 0 \leq x \leq 1 \\ A\, e^{-x} & \text{if} \quad 1 < x. \end{cases}$$

   Determine the constant A so that $g(x)$ is a proper PDF.

   (b) (2 points) Find the CDF of Y, $G(x) = \int_0^x g(s)ds$.

   (c) (3 points) How can we sample Y using $G(x)$? (Give detail.)

   (d) (2 points) **First assume that** $0 < \alpha < 1$. We are going to use Y with $g(x)$ for sampling $X \sim$ Gamma$(\alpha, 1)$ in the acceptance-rejection method. For this, $f(x)$ and $g(x)$ must satisfy

   $$\frac{f(x)}{g(x)} \leq C \quad \text{for all} \quad x \geq 0.$$

   Show that the condition is satisfied. (Why do you need $0 < \alpha < 1$?) What is C?

   (e) (3 points) From (d), we can now sample $X \sim$ Gamma$(\alpha, 1)$ for $0 < \alpha < 1$. Then, how can we sample X when $\alpha \geq 1$?

Solution:

(a) We need to ensure that $\int_0^\infty g(x)dx = 1$.

$$\begin{aligned}\int_0^\infty g(x)dx &= A \int_0^1 x^{\alpha-1} dx + A \int_1^\infty e^{-x} dx \\ &= A\left(\frac{1}{\alpha} + \frac{1}{e}\right) = 1.\end{aligned}$$

Therefore,

$$A = \frac{\alpha\, e}{\alpha + e}.$$

(b) When $0 \le x \le 1$,

$$G(x) = \int_0^x g(s)ds = \frac{e}{\alpha + e} x^\alpha.$$

When $1 < x$,

$$G(x) = \int_0^1 g(s)ds + \int_1^x g(s)ds = \frac{e}{\alpha+e} + \frac{\alpha\, e}{\alpha+e}\left(e^{-1} - e^{-x}\right) = 1 - \frac{\alpha\, e}{\alpha+e} e^{-x}.$$

Therefore,

$$G(x) = \begin{cases} \frac{e}{\alpha+e} x^\alpha & \text{if} \quad 0 \le x \le 1 \\ 1 - \frac{\alpha\, e}{\alpha+e} e^{-x} & \text{if} \quad 1 \le x. \end{cases}$$

(c) We can sample Y using the inverse CDF, $Y = G^{-1}(U)$, for a uniform RV, U.

$$Y = \begin{cases} \left[\frac{\alpha+e}{e} U\right]^{1/\alpha} & \text{if } 0 \le U \le \frac{e}{\alpha+e} \\ -\log\left(\frac{\alpha+e}{\alpha e}(1-U)\right) = \log\frac{\alpha e}{\alpha+e} - \log(1-U) & \text{if } \frac{e}{\alpha+e} \le U. \end{cases}$$

(d) When $0 \le x \le 1$,

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \le \frac{1}{\Gamma(\alpha)} x^{\alpha-1} = \frac{\alpha+e}{\alpha e \Gamma(\alpha)} g(x).$$

When $1 < x$, we know $0 < x^{\alpha-1} \le 1$ because $0 < \alpha < 1$. So,

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \le \frac{1}{\Gamma(\alpha)} e^{-x} = \frac{\alpha+e}{\alpha e \Gamma(\alpha)} g(x).$$

From the two cases, $f(x) \le Cg(x)$ is satisfied with $C = \dfrac{\alpha+e}{\alpha e \Gamma(\alpha)} \quad \left(= \dfrac{\alpha+e}{e\Gamma(\alpha+1)}\right)$.

(e) If $\alpha > 1$, we separate α into the largest integer and remainder parts:

$$\alpha = k + \alpha_0 \quad \text{where } k \text{ is the integer part of } \alpha \text{ and } 0 \le \alpha_0 < 1.$$

Then, from the 2020 exam, we can separate Gamma$(\alpha, 1)$ by

$$\text{Gamma}(\alpha, 1) \sim \text{Gamma}(k, 1) + \text{Gamma}(\alpha_0, 1).$$

We can sample Gamma$(k, 1)$ from the 2020 exam problem and sample Gamma$(\alpha_0, 1)$ from above.

2. (9 points) [λ–**SABR Model Simulation**] The λ–SABR model is given by

$$\frac{dS_t}{S_t^\beta} = \sigma_t(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2},$$
$$d\sigma_t = \lambda(\theta - \sigma_t)dt + \nu\sigma_t\, dZ_t,$$

where X_t and Z_t are independent standard BMs. This model is an extension of the SABR model because of the additional $\lambda(\theta - \sigma_t)dt$ term. We are going to formulate the conditional Monte Carlo simulation for this model.

(a) (3 points) How can you simulate $\sigma_{t+\Delta t}$ from σ_t? Write the Euler and Milstein schemes for σ_t. (Hint: the SDE for σ_t is same as the SDE for v_t in the GARCH diffusion model.)

(b) (3 points) When $\beta = 0$, express S_T in terms of σ_T, V_T, U_T, and a standard normal RV X_1, where V_T and U_T are respectively the integrated variance and volatility,

$$V_T = \int_0^T \sigma_t^2\, dt \quad \text{and} \quad U_T = \int_0^T \sigma_t\, dt.$$

(c) (3 points) When $\beta = 1$, express S_T in terms of σ_T, V_T, U_T, and X_1. What are $E(S_T|\sigma_T, V_T, U_T)$ and the equivalent BS volatility σ_{BS} conditional on σ_T, V_T, and U_T?

Solution:

(a) The Euler/Milstein schemes are given by

$$\sigma_{t+\Delta t} = \sigma_t + \lambda(\theta - \sigma_t)\Delta t + \nu\sigma_t\sqrt{\Delta t}\, Z + \frac{\nu^2}{2} v_t \Delta t\, (Z^2 - 1),$$

where Z is standard normal RN. The term in red is for Milstein scheme.

(b) By integration,

$$\sigma_T - \sigma_0 = \lambda(\theta T - U_T) + \nu \int_0^T \sigma_t dZ_t$$

$$\int_0^T \sigma_t dZ_t = \frac{1}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right).$$

Therefore,

$$\begin{aligned} S_T - S_0 &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t \\ &= \frac{\rho}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right) + \rho_* \sqrt{V_T} X_1. \end{aligned}$$

(c) When $\beta = 1$, we use

$$d\log S_t = \sigma_t(\rho dZ_t + \rho_* dX_t) - \frac{\sigma_t^2}{2} dt.$$

Integrating the equation,

$$\begin{aligned} \log(S_T/S_0) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \int_0^T \frac{\sigma_t^2}{2} dt \\ &= \frac{1}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right) - \frac{V_T}{2} + \sqrt{V_T} X_1. \end{aligned}$$

Accordingly, we obtain the conditional spot and volatility as

$$\begin{aligned} E(S_T|\sigma_T, V_T, U_T) &= S_0 \exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2} V_T\right) \\ &= S_0 \exp\left(\frac{1}{\nu}\left(\sigma_T - \sigma_0 - \lambda(\theta T - U_T)\right) - \frac{\rho^2}{2} V_T\right) \\ \sigma_{\text{BS}} &= \rho_* \sqrt{V_T/T}. \end{aligned}$$

3. (9 points) [**Andersen's Heston model simulation**] The variance under the Heston model is given by the CIR process:

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}\, dZ_t.$$

It is well known that the Euler/Milstein scheme for v_t is not accurate because v_t becomes negative easily. To handle this problem, Andersen (2008) proposed an advanced scheme. First, let m and s^2 be the mean and variance of $v_{t+\Delta t}$ given v_t, and their ratio be $\psi = s^2/m^2$. We know the analytic solution for m and s^2:

$$\begin{aligned} m &= E(v_{t+\Delta t}|v_t) = \theta + (v_t - \theta)e^{-\kappa\Delta t} \\ s^2 &= \mathrm{Var}(v_{t+\Delta t}|v_t) = \frac{\nu^2}{\kappa}(1 - e^{-\kappa\Delta})\left[v_t e^{-\kappa\Delta t} + \frac{\theta}{2}(1 - e^{-\kappa\Delta t})\right]. \end{aligned} \tag{1}$$

High ψ implies high probability of v_t hitting zero. Therefore, when $\psi \geq 1$, Andersen proposed to approximate $v_{t+\Delta t}$ by

$$v_{t+\Delta t} \approx \begin{cases} 0 & \text{with probability } p \\ \mathrm{Exp}(\lambda) & \text{with probability } 1-p, \end{cases}$$

where $\mathrm{Exp}(\lambda)$ is the exponential RV with rate λ. We are going to determine the two parameters, p and λ.

(a) (3 points) How can you simulate $v_{t+\Delta t}$ using only one uniform RV U? Give detail.

(b) (3 points) What are $E(v_{t+\Delta t})$ and $\mathrm{Var}(v_{t+\Delta t})$ of Andersen's approximation? (Hint: the mean and variance of $\mathrm{Exp}(\lambda)$ are $1/\lambda$ and $1/\lambda^2$, respectively.)

(c) (3 points) Determine p and λ by matching the result of (b) to m and s^2. Why do you need to assume $\psi \geq 1$. (Express your answers with m, s^2, and $\psi = s^2/m^2$. No need to use Eq. (1).)

Solution:

(a) We use $U \leq p$ to determine the probability p. If $U > p$, the re-scaled value $\frac{U-p}{1-p}$ (or $\frac{1-U}{1-p}$) is also a RV, so we use this to sample $\mathrm{Exp}(\lambda)$.

$$v_{t+\Delta t} \approx \begin{cases} 0 & \text{if} \quad 0 \leq U \leq p \\ \frac{1}{\lambda}\log\left(\frac{1-p}{U-p}\right), & \text{if} \quad p < U \leq 1 \end{cases}$$

(b) The two moments are

$$\begin{aligned} E(v_{t+\Delta t}) &= 0 \cdot p + \frac{1}{\lambda}(1-p) = \frac{1-p}{\lambda} \\ E(v_{t+\Delta t}^2) &= 0 \cdot p + E\left(\mathrm{Exp}(\lambda)^2\right)(1-p) = \left(E(\mathrm{Exp}(\lambda))^2 + \mathrm{Var}(\lambda)\right)(1-p) = \frac{2(1-p)}{\lambda^2}. \end{aligned}$$

Therefore,

$$\mathrm{Var}(v_{t+\Delta t}) = E(v_{t+\Delta t}^2) - E(v_{t+\Delta t})^2 = \frac{2(1-p)}{\lambda^2} - \frac{(1-p)^2}{\lambda^2} = \frac{1-p^2}{\lambda^2}$$

(c) Matching,

$$\frac{1-p}{\lambda} = m \quad \text{and} \quad \frac{1-p^2}{\lambda^2} = s^2,$$

leads to

$$\frac{1-p}{\lambda} = m \quad \text{and} \quad \frac{1+p}{\lambda} = \frac{s^2}{m}.$$

We solve

$$p = \frac{\psi-1}{\psi+1} \quad \text{and} \quad \lambda = \frac{1-p}{m} = \frac{2}{m(\psi+1)}.$$

We need $\psi \geq 1$ in order for the probability p to be positive.