Applied Stochastic Processes (FIN 514) Midterm Exam

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BM stands for Brownian motion. Assume that B_t, W_t, and Z_t are standard BMs unless stated otherwise. **RN** and **RV** stand for random number and random variable, respectively. $P(A)$ denotes the probability of the event A.

1. (7 points) [**RN generation**] We want to generate the RNs of X with PDF $f(x)$ and CDF $F(x)$. Suppose that it is <u>not</u> possible to draw X by the inversion, $Y = F^{-1}(U)$, for a uniform RN U (probably because $F^{-1}(u)$ is not analytically available). Instead, we are going to sample X by taking advantage of another RV Y with PDF $g(x)$ and CDF $G(x)$, whose RNs we can easily generate. Suppose that Y is *similar* to X in the sense that the ratio of the two PDFs are bounded everywhere by $C > 0$:

$$\frac{f(x)}{g(x)} \le C \quad \text{for all} \quad x. \tag{1}$$

Now let us consider an RV, Y', obtained as a result of the following algorithm:

Step 1 Independently draw Y and a uniform RN U.

Step 2 If Y and U satisfy the condition,

$$U \le \frac{f(Y)}{Cg(Y)}, \tag{2}$$

accept $Y' = Y$. Otherwise, reject Y and repeat **Step 1** until you get an accepted Y'.

In this question, we are going to prove that the above algorithm actually draws the RNs of X by showing that

$$P(Y' \le x) = F(x) = P(X \le x).$$

For the proof, let us define two events:

$$A_x = \{Y \le x\} \quad \text{(for a given value } x\text{)} \quad \text{and} \quad B = \left\{U \le \frac{f(Y)}{Cg(Y)}\right\}$$

(a) (2 points) What is $P(A_x)$? What is $P\left(U \le \frac{f(x)}{Cg(x)}\right)$? Hint: x is a given number, not an RV.

(b) (3 points) What are $P(A_x \cap B)$ and $P(B)$? Hint: work on $P(A_x \cap B)$ first because $P(B) = \lim_{x\to\infty} P(A_x \cap B)$.

(c) (2 points) The probability $P(Y' \le y)$ can be written as the conditional probability:

$$P(Y' \le x) = P(A_x|B).$$

Using the conditional probability law and the results from (b), verify that $P(A_x|B) = F(x)$ (and complete the proof).

Solution: This algorithm is called rejection sampling (WIKIPEDIA) or acceptance-rejection method. It is a powerful method to sample RVs.

(a)

$$P(A_x) = P(Y \le x) = G(x) \quad \text{and} \quad P\left(U \le \frac{f(x)}{Cg(x)}\right) = \frac{f(x)}{Cg(x)} \ (\le 1).$$

(b)

$$\begin{aligned} P(A_x \cap B) &= P\left(U \le \frac{f(Y)}{Cg(Y)} \cap Y \le x\right) = \int_{-\infty}^{x} P\left(U \le \frac{f(y)}{Cg(y)} \cap Y \in (y, y+dy)\right) \\ &= \int_{-\infty}^{x} P\left(U \le \frac{f(y)}{Cg(y)}\right) g(y)dy = \int_{-\infty}^{x} \frac{f(y)}{Cg(y)} g(y)dy = \frac{F(x)}{C}. \end{aligned}$$

It follows that

$$P(B) = \lim_{x\to\infty} P(A_x \cap B) = \frac{F(\infty)}{C} = \frac{1}{C}.$$

(c)

$$P(Y' \le x) = P(A_x|B) = \frac{P(A_x \cap B)}{P(B)} = \frac{F(x)/C}{1/C} = F(x).$$

2. (5 points) We want to sample standard normal RNs using the algorithm from Problem 1. We will draw $X = |Z|$ for a standard normal RV, Z, and use an exponential RV with $\lambda = 1$ as Y. Reminded that the two PDFs are given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad g(x) = e^{-x} \quad (x \ge 0),$$

and that you can draw $Y = -\log U'$ for a uniform RV U'. (**You can solve this problem even though you did not answer Problem 1.**)

(a) (2 points) Prove that Equation (1) holds between X and Y. What is C?

(b) (2 points) Express the acceptance condition, Equation (2), using the two uniform RVs, U and U'.

(c) (1 point) For the final step, how can you draw Z from X?

Solution:

(a)

$$\frac{f(x)}{g(x)} = \frac{2}{\sqrt{2\pi}} e^{-x^2/2+x} = \frac{2}{\sqrt{2\pi}} e^{-(x-1)^2/2+1/2} = \sqrt{\frac{2e}{\pi}} e^{-(x-1)^2/2} \le \sqrt{\frac{2e}{\pi}}.$$

Therefore, $C = \sqrt{2e/\pi} \approx 1.315$ and the maximum occurs at $x = 1$.

(b) Therefore, the condition becomes

$$U \le e^{-(Y-1)^2/2} = e^{-(-\log U'-1)^2/2}$$

This is further simplified to

$$-2\log U \ge (\log U' + 1)^2.$$

(c) Z is obtained from X by randomly selecting the sign (e.g., + or −). To be specific,

$$Z = \begin{cases} X = -\log U' & \text{if} \quad U'' > 0.5 \\ -X = \log U' & \text{if} \quad U'' \le 0.5. \end{cases}$$

for another independent uniform random variable U''.

3. (12 points) **[Conditional Monte Carlo Simulation]** Suppose that an SV model is given by

$$\frac{dS_t}{S_t} = \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2},$$
$$\frac{dv_t}{v_t} = \kappa dt + \nu\, dZ_t$$

where X_t and Z_t are independent standard BMs. We are going to formulate the conditional Monte Carlo simulation for this SV model. (Notice that this SV model is different from the SABR model because (i) κdt term exists (ii) $v_t = \sigma_t^2$ is used for the SDE. But what you learned from the SABR would be still useful.)

(a) (2 points) Solve v_T (i.e., express v_T as a function of v_0, Z_T, and the model parameters). Hint: v_t follows a geometric BM.

(b) (2 points) From (a), how can you simulate $v_{t+\Delta t}$ from v_t?

(c) (3 points) Derive the SDE for $\sigma_t = \sqrt{v_t}$. Hint: consider $\log \sigma_t = \frac{1}{2}\log v_t$.

(d) (3 points) Using the result of (c), express S_T in terms of v_T, V_T, and U_T, and a standard normal RV X_1, where V_T and U_T are respectively the integrated variance and volatility,

$$V_T = \int_0^T v_t\, dt \quad \text{and} \quad U_T = \int_0^T \sigma_t\, dt.$$

(e) (2 points) What are $E(S_T | v_T, V_T)$ and the equivalent BS volatility of S_T conditional on v_T and V_T?

Solution:

(a) Using Itô's lemma,

$$d\log v_t = \left(\kappa - \frac{\nu^2}{2}\right) dt + \nu dZ_t$$
$$v_T = v_0 \exp\left(\left(\kappa - \frac{\nu^2}{2}\right) T + \nu Z_T\right)$$

(b) $v_{t+\Delta t}$ is obtained from v_t by

$$v_{t+\Delta t} = v_t \exp\left(\left(\kappa - \frac{\nu^2}{2}\right) \Delta t + \nu\sqrt{\Delta t} Z\right),$$

where Z is a standard normal RN.

(c)

$$d\log\sigma_t = \frac{1}{2}d\log v_t = \left(\frac{\kappa}{2} - \frac{\nu^2}{4}\right)dt + \frac{\nu}{2}dZ_t$$

$$\frac{d\sigma_t}{\sigma_t} = \left(\frac{\kappa}{2} - \frac{\nu^2}{4} + \frac{1}{2}\left(\frac{\nu}{2}\right)^2\right)dt + \frac{\nu}{2}dZ_t = \left(\frac{\kappa}{2} - \frac{\nu^2}{8}\right)dt + \frac{\nu}{2}dZ_t$$

(d) Integrating the result of (c),

$$\sigma_T - \sigma_0 = \left(\frac{\kappa}{2} - \frac{\nu^2}{8}\right)\int_0^T \sigma_t dt + \frac{\nu}{2}\int_0^T \sigma_t dZ_t$$

$$\int_0^T \sigma_t dZ_t = \frac{2}{\nu}(\sqrt{v_T} - \sqrt{v_0}) + \left(\frac{\nu}{4} - \frac{\kappa}{\nu}\right)U_T$$

Therefore,

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho\int_0^T \sigma_t dZ_t + \rho_*\int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\ &= \frac{2\rho}{\nu}(\sigma_T - \sigma_0) + \rho\left(\frac{\nu}{4} - \frac{\kappa}{\nu}\right)U_T - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\,X_1.\end{aligned}$$

(e) Accordingly, we obtain the equivalent spot and volatility as

$$\begin{aligned}E(S_T|\sigma_T, V_T) &= S_0\,\exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2}V_T\right) \\ &= S_0\exp\left(\frac{2\rho}{\nu}(\sqrt{v_T} - \sqrt{v_0}) + \rho\left(\frac{\nu}{4} - \frac{\kappa}{\nu}\right)U_T - \frac{\rho^2}{2}V_T\right) \\ \sigma_{\mathrm{BS}} &= \rho_*\sqrt{V_T/T}.\end{aligned}$$