# Numerical Integration Methods
# Applied Stochastic Processes (FIN 514)

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# Numerical Integration

- In stochastic finance, numerical integral is often required to calculate the expectation (over the probability density):

$$E(f(X)) = \int_{-\infty}^{\infty} f(x)w(x)dx,$$

where $w(x)$ is the PDF of $X$.

# Trapezoid Rule

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$$

Chained rule:

$$\int_a^b f(x)dx = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{N-1}) + f(x_N)\right) + E,$$

where $x_0 = a$, $x_N = b$, and $\Delta x = x_k - x_{k-1} = (b-a)/N$. The error is given by

$$E = -\frac{(b-a)(\Delta x)^2}{12} f''(\xi) \quad \text{for} \quad \xi \in (a,b)$$

# Simpson's Rule

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}$$

(How can you derive?)
Chained rule (for even $N$):

$$\int_a^b f(x)dx = \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)\cdots$$
$$+2f(x_{N-2})+4f(x_{N-1})+f(x_N))+E,$$

where $x_0=a$, $x_N=b$, and $\Delta x = x_k - x_{k-1} = (b-a)/N$. The error is given by

$$E = -\frac{(b-a)(\Delta x)^4}{180}f^{(4)}(\xi) \quad \text{for} \quad \xi \in (a,b)$$

# Gaussian Quadrature

- A set of point and weight $\{(x_k, w_k) : k = 1, \dots N\}$ associated with a weight function $w(x)$.
- When $w(x)$ is a PDF of $X$,

$$E(f(X)) = \int_a^b f(x)w(x)dx \approx \sum_{k=1}^{N} f(x_k)w_k$$

- $\{(x_k, w_k)\}$ are the most optimal in that they exactly evaluate the integral when $g(x)$ is a polynomial up to degree $2N-1$.
- When $w(x)$ is a PDF, the moments up to $2n-1$ order is accurate!

$$E(X^n) = \sum_{k=1}^{N} x_k^n w_k \quad \text{for} \quad n = 0, 1, \dots, 2N-1$$

# Various Gauss Quadratures

Quadratures are associated with various distributions, $w(x)$:

- Gauss–Hermite: normal distribution $w(x) = e^{-x^2/2}$
- Gauss–Legendre: uniform distribution $w(x) = 1,\ x \in (0,1)$.
- Gauss–Laguerre: exponential distribution $w(x) = e^{-x},\ x \geq 0$.
- Generalized Gauss–Laguerre: gamma distribution $w(x) = x^{a-1}e^{-x},\ x \geq 0$.
- Gauss–Jacobi: beta distribution $w(x) = (x-1)^{\alpha}(x+1)^{\beta},\ x \in (-1,1)$.

See Gaussian quadrature (WIKIPEDIA) and an online calculator.

# Lagrange Interpolation (WIKIPEDIA)

- We want to interpolate $\{(x_k, y_k) : k = 1, \ldots, N \text{ and } a \le x_k \le b\}$.
- Define $L(x) = (x - x_1)(x - x_2) \cdots (x - x_N)$ (degree $N$) and

$$l_k(x) = \frac{L(x)}{L'(x_k)(x - x_k)}$$

- Then, $l_k(x)$ has nice properties:

$$l_k(x_j) = 0 \text{ if } j \ne k, \quad l_k(x_k) = 1 \text{ (L'hopital rule)}.$$

  So $\{l_k(x)\}$ serve as a basis of the interpolation.
- The following polynomial satisfy $y_k = p(x_k)$ for $k = 1, \ldots, N$:

$$p(x) = y_1 l_1(x) + \cdots + y_N l_N(x).$$

- If we pre-calculate the integral of the $l_k(x)$, we can them for integrating $f(x)$.

$$w_k = \int_a^b l_k(x)dx. \quad \int_a^b p(x)dx = w_1 y_1 + \cdots w_N y_N.$$

# Integral using Lagrange Interpolation

- How to approximate $\int_a^b f(x)dx$ ?
- Assume we know $\{(x_k, f(x_k)) : k = 1, \ldots, N \text{ and } a \le x_k \le b\}$.
- The Lagrange polynomial $p(x)$ is a close approximation of $f(x)$:

$$f(x) = f(x_1)l_1(x) + \cdots + f(x_N)l_N(x).$$

  So $\{l_k(x)\}$ serve as a basis for the integration.
- If we pre-calculate the integral of the $l_k(x)$, we can use them for integrating $f(x)$.

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = \sum_{k=1}^N w_k f(x_k) \quad \text{for} \quad w_k = \int_a^b l_k(x)dx.$$

# Quadratures and Special Polynomials

- How to find the points and weights with respect to $w(x)$?
- Orthogonal polynomials $p_n(x)$ (of degree $n$) with respect to $w(x)$ is well-known.

$$\int_a^b p_i(x)p_j(x)w(x)dx = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$

- The points of the Gaussian quadrature is chosen as the roots of $p_n(x)$:

$$p_n(x_k) = 0.$$

- The weights are (pre)calculated as

$$w_k = \frac{1}{p_n'(x_k)} \int_a^b \frac{p_n(x)}{(x - x_k)} w(x)dx.$$