

# Hyperbolic normal stochastic volatility (NSVh) model

## Applied Stochastic Processes (FIN 514)

Jaehyuk CHOI<sup>1</sup>  
in collaboration with Chenru LIU<sup>2</sup> and Byoung Ki SEO<sup>3</sup>

<sup>1</sup>Peking University HSBC Business School, Shenzhen, China

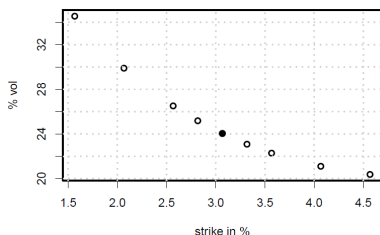
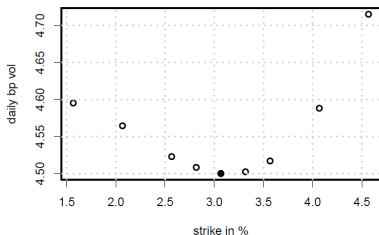
<sup>2</sup>Department of Management Science and Engineering, Stanford University

<sup>3</sup>School of Business Administration, UNIST, Korea

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# Motivation 1: 'Normal' is more normal for some asset classes

- Daily price changes may not be proportional to the price level
- Underlying price can be negative: interest rate, inflation rate, spread, etc.
- Two swaption volatility smiles (in implied normal vs BSM volatility):



- Swaptions are quoted and risk-managed under 'normal' volatility (bp vol). Merrill Lynch Option Volatility Estimate (MOVE) index is a weighted average of the 'normal' implied volatility on 1m Treasury options.
- Therefore, a need for [stochastic volatility model with normal backbone](#) arises. The literature is rare except for the normal case of SABR model.

# Motivation 1: A tale of two backbones

Model	Normal	Lognormal
Reference	Bachelier (1900)	Black and Scholes (1973)
SDE	Arithmetic BM: $dF_t = \sigma dW_t$	Geometric BM: $dF_t/F_t = \sigma dW_t$
Asset class	Interest rate, Inflation, Spread	Equity, FX
Call option price	$(F - K)N(d) + \sigma\sqrt{T}n(d)$ $d = \frac{(F-K)}{\sigma\sqrt{T}}$	$F N(d_1) - K N(d_2)$ $d_{1,2} = \frac{\log(F_0/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$
Volatility conversion	$\sigma_N \approx F_0 \sigma_{BS}$	
Digital, $P(F_T > K)$	$N(d)$	$N(d_2)$
Delta ( $\partial/\partial F_0$ )	$N(d)$	$N(d_1)$
Gamma ( $\partial^2/\partial F_0^2$ )	$n(d)/\sigma\sqrt{T}$	$n(d_1)/F_0\sigma\sqrt{T}$
Vega ( $\partial/\partial\sigma$ )	$\sqrt{T}n(d)$	$F_0\sqrt{T}n(d_1)$
Implied Volatility	Choi et al. (2009)	Jäckel (2015)

# Motivation 1: normal SABR model

- Stochastic-alpha-beta-rho (SABR) model (Hagan et al., 2002)

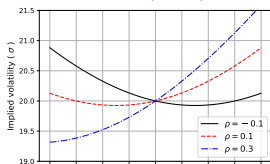
$$\frac{dF_t}{F_t^\beta} = \sigma_t (\rho dZ_t + \rho_* dX_t) \quad \text{and} \quad \frac{d\sigma_t}{\sigma_t} = \alpha dZ_t \quad \text{for} \quad 0 \leq \beta \leq 1, \quad \rho^2 + \rho_*^2 = 1$$

- One of the most popular SV models used in financial engineering.
  - range of backbone choice:  $0 \leq \beta \leq 1$
  - parsimonious and intuitive parameters:  $\rho$  and  $\alpha$  (and  $\beta$ )
  - asymptotic approximation for the implied vol for  $\beta = 0$  (accurate for  $\alpha\sqrt{T} \ll 1$ ) (Hagan et al., 2002)

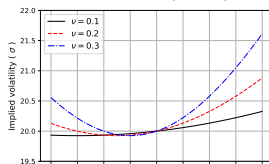
$$\sigma_N = \sigma_0 \left( \frac{\zeta}{\chi(\zeta)} \right) \left( 1 + \frac{2 - 3\rho^2}{24} \alpha^2 T \right)$$

$$\text{where } \zeta = \frac{\alpha}{\sigma_0} (F_0 - K) \quad \text{and} \quad \chi(\zeta) = \log \left( \frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho} \right)$$

$\rho$ : slope(skew)



$\alpha$ : curvature(smile)



## Motivation 2: normal SV for heavy-tailed distribution?

- Most heavy-tailed distributions lack dynamics (SDE).
- Heavy-tailed distribution associated with normal SV model?  
E.g., normal/lognormal distribution vs arithmetic/geometric BM

### Johnson's distribution family

Johnson (1949) proposed a family of transformation from standard normal distribution:

$$\frac{X - \gamma_X}{\delta_X} = f\left(\frac{Z - \gamma_Z}{\delta_Z}\right) \text{ where } f(x) = \begin{cases} \frac{1}{1+e^{-x}} & \text{for } S_B \text{ (bounded) family} \\ x & \text{for } S_N \text{ (normal) family} \\ \sinh x & \text{for } S_U \text{ (unbounded) family} \\ e^x & \text{for } S_L \text{ (log-normal) family} \end{cases}$$

The  $S_U$  (unbounded) Johnson family is an increasingly popular choice for modeling distributions with heavy tail and skewness. (Simonato, 2011; Corlu and Corlu, 2015; Choi and Nam, 2008; Gurrola, 2007; Venkataraman and Rao, 2016)

- PDF/CDF in closed-forms
- wide range of skewness and ex-kurtosis to explain
- trivial random number (RN) draw

Hyperbolic normal stochastic volatility (NSVh) model brings together the normal SABR model ( $\lambda = 0$ ) and Johnson's  $S_U$  distribution ( $\lambda = 1$ ).

Based on the two generalizations (Alili and Gruet, 1997) of Bougerol's identity, we equip NSVh model with unprecedented analytic tools:

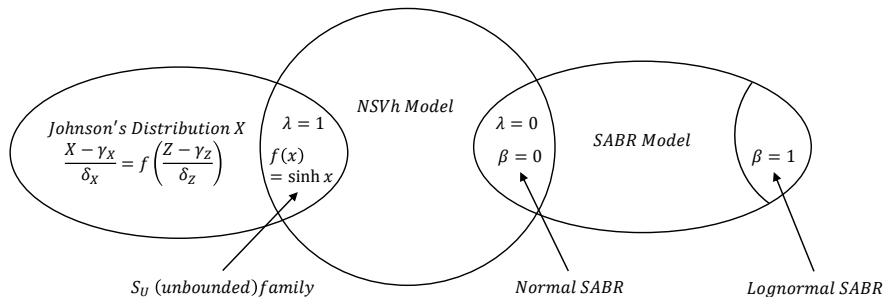
- closed-form (exact and one-step) MC scheme (1 RN draw per 1.5 normal RNs)

For Johnson's  $S_U$  distribution ( $\lambda = 1$ ), we provide

- theoretical foundation as a solution of SDE
- intuitive re-parametrization
- possibility for substituting normal SABR ( $\lambda = 0$ ) with various closed-form solutions (option price, VaR, ES, etc).

# Contribution

The Overview of the NSVh Model in Relation to Other Previously Known Models and Distributions.



# Hyperbolic normal stochastic volatility (NSVh) model

- SDE:

$$dF_t = \sigma_t \left( \rho dZ_t^{[\lambda\alpha/2]} + \rho_* dX_t \right), \quad \frac{d\sigma_t}{\sigma_t} = \alpha dZ_t^{[\lambda\alpha/2]}$$

where  $\rho_* = \sqrt{1 - \rho^2}$ ,  $Z_t^{[\mu]} = Z_t + \mu t$ , and  $X_t$  and  $Z_t$  are independent BMs.

- The vol process is solved to  $\sigma_t = \sigma_0 \exp \left( \alpha Z_t^{[\frac{1}{2}(\lambda-1)\alpha]} \right)$
- The power  $(\sigma_t)^{1-\lambda}$  is a martingale:

$$\frac{d(\sigma_t)^{1-\lambda}}{(\sigma_t)^{1-\lambda}} = (1-\lambda)\alpha dZ_t \quad (\lambda \neq 1), \quad d(\log \sigma_t) = \alpha dZ_t \quad (\lambda = 1).$$

- The price process  $F_t$  is a martingale only when  $\lambda = 0$ , which is normal SABR model. The case  $\lambda \neq 0$  is understood as a perturbation from the baseline case.
- However, using  $\bar{F}_T = F_0 + (\sigma_0 \rho / \alpha) (e^{\frac{1}{2}\lambda\alpha^2 T} - 1)$ ,

$$F_T - \bar{F}_T = \frac{\rho}{\alpha} (\sigma_T - e^{\frac{1}{2}\lambda\alpha^2 T} \sigma_0) + \rho_* \int_0^T \sigma_t dX_t$$



# NSVh: Canonical form with exp. functionals of BM and

- After changes of variables,

$$s = \alpha^2 t \ (S = \alpha^2 T), \quad \tilde{\sigma}_s = \frac{\sigma_t}{\sigma_0}, \quad \text{and} \quad \tilde{F}_s = \frac{\alpha}{\sigma_0} (F_t - \bar{F}_T),$$

the canonical form is given as

$$\begin{aligned} d\tilde{F}_s &= \tilde{\sigma}_s (\rho dZ_s^{[\lambda/2]} + \rho_* dX_s) \quad \text{and} \quad d\tilde{\sigma}_s = \tilde{\sigma}_s dZ_s^{[\lambda/2]} \quad (\tilde{\sigma}_0 = 1), \\ \tilde{\sigma}_S &= \exp \left( Z_S^{[\frac{1}{2}(\lambda-1)]} \right) \quad \text{and} \quad \tilde{F}_S \stackrel{d}{=} \rho \left( e^{Z_S^{[\frac{1}{2}(\lambda-1)]}} - e^{\frac{1}{2}\lambda S} \right) + \rho_* X_{A_S^{[\frac{1}{2}(\lambda-1)]}}. \end{aligned}$$

- Distribution parameterized by  $(\bar{F}_T, \sigma_0/\alpha, S = \alpha^2 T, \rho, \lambda)$
- Motivated from continuously averaged Asian option, the exponential functional of BM has been an important subject of researches. ([Matsumoto and Yor, 2005a,b](#); [Yor, 2012](#)).

$$A_T^{[\mu]} = \int_{t=0}^T e^{2Z_t^{[\mu]}} dt$$

In the context of NSVh (SABR),  $A_S^{[\frac{1}{2}(\lambda-1)]} = \int_0^S \tilde{\sigma}_s^2 ds$  is the time-integrated variance under the stochastic volatility.

# Main results

## Bougerol's identity (Bougerol, 1983)

For the independent BMs,  $X_t$ ,  $Z_t$  and  $W_t$ , the followings are equal in law ( $\stackrel{d}{=}$ )

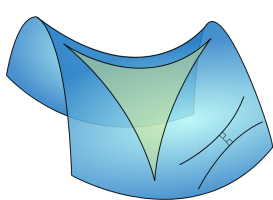
$$\int_0^T e^{Z_t} dX_t \stackrel{d}{=} X_{A_T^{[0]}} \stackrel{d}{=} \sinh(W_T) \quad \text{for any fixed time } T.$$

(See the generalized proof in the 2nd extension.)

Our main results are based on two generalization of Bougerol's identity by Alili and Gruet (1997), *An explanation of a generalized Bougerol's identity in terms of hyperbolic Brownian motion, in Exponential Functionals and Principal Values related to Brownian Motion* and Alili et al. (1997).

- BM in hyperbolic geometry (all  $\lambda$ :  $Z^{[\mu]}$ ,  $A_T^{[\mu]}$ )
- BM with an arbitrary starting point ( $\lambda = 1$ )

# Hyperbolic geometry: 2-d and 3-d Poincaré half-plane



- Geometry with constant negative curvature (e.g., saddle)
- Poincaré disc model [Escher, 1959]:  $r < 1$  with  $\infty$  corresponding to  $r = 1$ .
- Poincaré half-plane model (Dunham, 2012): unit length  $\propto z$

Dimension	$\mathbb{H}_2 = \{(x, z) : z > 0\}$	$\mathbb{H}_3 = \{(x, y, z) : z > 0\}$
Metric $(ds)^2$	$(dx^2 + dz^2)/z^2$	$(dx^2 + dy^2 + dz^2)/z^2$
Volume element $dV$	$dx dz/z^2$	$dx dy dz/z^3$
Geodesic distance $D$	$\text{acosh} \left( \frac{(x'-x)^2 + z^2 + z'^2}{2zz'} \right)$	$\text{acosh} \left( \frac{(x'-x)^2 + (y'-y)^2 + z'^2 + z^2}{2zz'} \right)$
Laplace-Beltrami operator $\Delta_H$	$z^2 (\partial_x^2 + \partial_z^2)$	$z^2 (\partial_x^2 + \partial_y^2 + \partial_z^2) - z \partial_z$
Standard BM	$dx_t = z_t dX_t,$ $dz_t/z_t = dZ_t$	$dx_t = z_t dX_t,$ $dy_t = z_t dY_t,$ $dz_t/z_t = dZ_t - dt/2$
Heat kernel $p_n(t, D)$ for $n = 2$ or $3$ $(\partial_t - \frac{1}{2} \Delta_{\mathbb{H}_n}) p_n = 0$	$\frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}} \int_D^\infty ds \frac{se^{-s^2/2t}}{\sqrt{\cosh s - \cosh D}}$	$\frac{1}{(2\pi t)^{3/2}} \frac{D}{\sinh D} e^{-(t^2 + D^2)/2t}$

# The 1st extension: BM in hyperbolic geometry (all $\lambda$ )

## Proposition 3 in Alili and Gruet (1997)

For the independent BMs  $X_t$  and  $Z_t$ , the followings are equal in law

$$\int_0^T e^{Z_t^{[\mu]}} dX_t \stackrel{d}{=} X_{A_T^{[\mu]}} \stackrel{d}{=} \cos \theta \phi \left( Z_T^{[\mu]}, \sqrt{R_T^2 + (Z_T^{[\mu]})^2} \right)$$

where  $\phi(Z, D) = e^{Z/2} \sqrt{2 \cosh D - 2 \cosh Z}$  for  $Z \leq D$ .

$\theta$  is a uniform random angle,  $R_t$  is a 2-d Bessel process (the radius of a 2-d BM,  $R_t = \sqrt{X_t^2 + Y_t^2}$ ). The equality in law ( $\stackrel{d}{=}$ ) even holds conditional on  $Z_T^{[\mu]}$ .

### Sketch of Proof:

- The proof is related to a drifted 3-d hyperbolic BM  $(x_t, y_t, z_t)$  started at  $(0, 0, 1)$ :

$$dx_t = z_t dX_t, \quad \text{and} \quad dy_t = z_t dY_t \quad (x_0 = y_0 = 0),$$
$$\frac{dz_t}{z_t} = dZ_t + \left( \frac{1}{2} + \mu \right) dt \quad (z_0 = 1).$$

# Sketch of Proof

- Let  $D_t$  be the hyperbolic distance between  $(x_t, y_t, z_t)$  and the starting point  $(0, 0, 1)$ ,

$$D_t = \operatorname{acosh} \left( \frac{1}{2} \left( \frac{x_t^2 + y_t^2}{z_t} + z_t + \frac{1}{z_t} \right) \right) \quad \text{or} \quad r_t = \sqrt{x_t^2 + y_t^2} = \phi(Z_t^{[\mu]}, D_t)$$

$\phi(\cdot)$  is understood as the Euclidean radius of  $(x_t, y_t)$ .

- For a fixed time  $T$ , the **hyperbolic distance**  $D_t$  and **Euclidean distance of**  $(X_T, Y_T, Z_T^{[\mu]})$  from  $(0, 0, 0)$  have same distribution **conditional on**  $Z_T$  (or  $z_T$ ):

$$D_T \stackrel{d}{=} \sqrt{X_T^2 + Y_T^2 + (Z_T^{[\mu]})^2}$$

We have a proof using  $\mathbb{H}_3$  heat kernel, which is more straight-forward than the original proof by **Alili and Gruet (1997)**.

- The result immediately follows

$$\begin{aligned} \sqrt{x_T^2 + y_T^2} &\stackrel{d}{=} \phi \left( Z_T^{[\mu]}, \sqrt{X_T^2 + Y_T^2 + (Z_T^{[\mu]})^2} \right) \\ x_T &\stackrel{d}{=} X_{A_T^{[\mu]}} \stackrel{d}{=} \cos \theta \phi \left( Z_T^{[\mu]}, \sqrt{X_T^2 + Y_T^2 + (Z_T^{[\mu]})^2} \right). \end{aligned}$$

# Closed-form price transition and MC scheme (all $\lambda$ )

The joint transition of price and volatility from time 0 to  $S$

$$\begin{aligned}\tilde{\sigma}_S &= \exp \left( Z_S^{[\frac{1}{2}(\lambda-1)]} \right), \\ \tilde{F}_S &\stackrel{d}{=} \rho \left( e^{Z_S^{[\frac{1}{2}(\lambda-1)]}} - e^{\frac{1}{2}\lambda S} \right) + \rho_* \cos \theta \phi \left( Z_S^{[\frac{1}{2}(\lambda-1)]}, \sqrt{R_S^2 + (Z_S^{[\frac{1}{2}(\lambda-1)]})^2} \right).\end{aligned}$$

## MC simulation

The transition is simulated with three standard normal RNs,  $X_1, Y_1$  and  $Z_1$ ,

$$(Z_S, R_S^2, \cos \theta) \stackrel{d}{=} \left( Z_1 \sqrt{S}, (X_1^2 + Y_1^2)S, \frac{X_1 \text{ (or } Y_1)}{\sqrt{X_1^2 + Y_1^2}} \right),$$

- Exact and one-step jump from  $s = S_1$  to  $s = S_2$ : no Euler method, no bias!
- One RN draw for 1.5 normal RNs.
- For normal SABR, more efficient than the scheme of [Cai et al. \(2017\)](#).

## The 2nd extension: an arbitrary starting point ( $\lambda = 1$ )

### Proposition 4 in Alili and Gruet (1997)

$$e^{Z_T} \sinh(a) + \int_0^T e^{Z_t} dX_t \stackrel{d}{=} \sinh(a + W_T) \quad \text{for any fixed time } T.$$

Remark: The original Bougerol's identity is just a special case with  $a = 0$ .

Following the proof in Alili et al. (1997), the two processes,

$$P_t = \sinh(W_t + a) \quad \text{and} \quad Q_t = e^{Z_t} \left( \sinh(a) + \int_0^t e^{-Z_s} dX_s \right),$$

are equal in law as they have the same SDE with the same initial condition  $P_0 = Q_0 = \sinh(a)$ ,

$$\begin{aligned} dP_t &= \frac{1}{2} P_t dt + \sqrt{1 + P_t^2} dW_t \quad \text{and} \\ dQ_t &= \frac{1}{2} Q_t dt + dX_t + Q_t dZ_t \stackrel{d}{=} \frac{1}{2} Q_t dt + \sqrt{1 + Q_t^2} dW_t. \end{aligned}$$

The invariance of BM under time reversal  $Z_T - Z_{T-s} \leftrightarrow Z_s$  completes the proof.

# Closed-form price transition and MC scheme ( $\lambda = 1$ )

## Price transition given by Johnson's $S_U$ distribution

$$\begin{aligned}\tilde{F}_S &\stackrel{d}{=} \rho_* \sinh(W_S + a \tanh \rho) - \rho e^{S/2} \\ &= \sinh(W_S) + \rho \left( \cosh(W_S) - e^{S/2} \right).\end{aligned}$$

- Johnson's  $S_U$  distribution is the solution of NSVh with  $\lambda = 1$ .
- $S_U$  re-parametrized with intuitive SV parameters:

$$\delta_Z = \frac{1}{\sqrt{S}}, \quad \frac{\gamma_Z}{\delta_Z} = -a \tanh \rho, \quad \delta_X = \frac{\sigma_0 \rho_*}{\alpha}, \quad \text{and} \quad \gamma_X = \bar{F}_T - \delta_X e^{S/2} \left( \frac{\rho}{\rho_*} \right).$$

- Two limiting cases: lognormal ( $S_L$ ) for  $\rho = \pm 1$  and normal ( $S_N$ ) as  $S \rightarrow 0$ .
- Clear split between heavy-tail ( $\sinh$ ) and skewness ( $\cosh$ ) components.
- Closed-form simulation from  $s = 0$  to  $S$ . One RN draw per 1 normal RN.
- Level of analytic tractability same as that of BSM model.
- The distributions are similar for different  $\lambda$  if the parameters are calibrated to the same observations (e.g., moments or volatility smiles)



# Various analytics for NSVh $\lambda = 1$

- PDF,  $f_{\lambda=1}(x)$ , and CDF,  $F_{\lambda=1}(x)$ :

$$f_{\lambda=1}(x) = \frac{n(d)}{\rho_* \sigma_0 \sqrt{T} \sqrt{1 + \xi^2}}, \quad F_{\lambda=1}(x) = N(-d)$$

$$\text{where } \xi = \frac{\alpha}{\rho_* \sigma_0} (\bar{F}_T - x) - \frac{\rho}{\rho_*} e^{S/2}, \quad d = \frac{1}{\sqrt{S}} (\operatorname{asinh} \xi + \operatorname{atanh} \rho)$$

- The forward price of the call/put ( $\pm$ ) option struck at  $K$ :

$$V_{\pm} = \frac{\sigma_0 e^{S/2}}{2\alpha} \left( (1 + \rho)N(d + \sqrt{S}) - (1 - \rho)N(d - \sqrt{S}) - 2\rho N(d) \right) \pm (\bar{F}_T - K) N(\pm d)$$

$$\text{where } d = \frac{1}{\sqrt{S}} \left( \operatorname{asinh} \left( \frac{\alpha}{\rho_* \sigma_0} (\bar{F}_T - K) - \frac{\rho}{\rho_*} e^{S/2} \right) + \operatorname{atanh} \rho \right)$$

- Value-at-Risk (VaR):

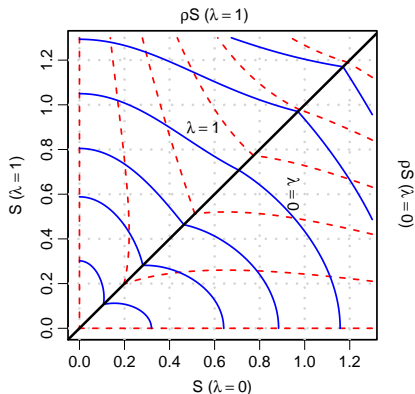
$$\text{VaR}(q) = \bar{F}_T + \frac{\sigma_0}{\alpha} \left( \rho_* \sinh \left( \sqrt{S} Z_q + \operatorname{atanh} \rho \right) - \rho e^{S/2} \right) \quad \text{for } Z_q = N^{-1}(q)$$

- Expected Shortfall (ES):

$$\text{ES}(q) = \bar{F}_T + \frac{\sigma_0 e^{S/2}}{2\alpha q} \left( (1 + \rho)N(Z_q - \sqrt{S}) - (1 - \rho)N(Z_q + \sqrt{S}) - 2\rho q \right) \quad \text{for } Z_q = N^{-1}(q)$$

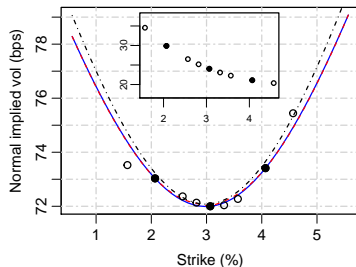
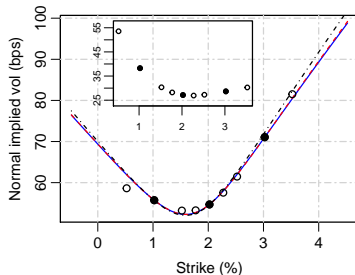
# Moments of NSVh distribution

Contours of skewness and ex-kurtosis for N-SABR ( $\lambda = 0$ ) and  $S_U$  ( $\lambda = 1$ )



- Like  $S_U$ , NSVh generates all possible combinations of skewness and positive ex-kurtosis. In particular, ex-kurtosis is unbounded for a given skewness.
- The moments, skewness and kurtosis have closed-forms for all  $\lambda$ .
- The moment matching can be simplified to 1-d root finding for  $\lambda = 1$  (Tuentner, 2001) and  $\lambda = 0$ . The parameters can be quickly converted between  $\lambda = 0$  and 1 for the same moments.

# Empirical study 1: Swaption smile



- Swaption smiles for 1y1y and 10y10y in normal volatility observed on March 2017. The indistinguishable solid lines (blue and red) are from NSVh with  $\lambda = 0$  and 1 while the dotted lines (black) are from Hagan's approximation.
- Calibrated parameters:

Parameters	1y1y ( $T = 1$ )		10y10y ( $T = 10$ )	
	$\lambda = 0$	$\lambda = 1$	$\lambda = 0$	$\lambda = 1$
$\rho$ (%)	35.280	32.244	1.811	1.580
$\alpha$ (%)	64.634	62.181	23.535	22.196
$\sigma_0$ (%)	0.532	0.477	0.689	0.609
$\bar{F}_T$ (%)	2.0221		3.0673	

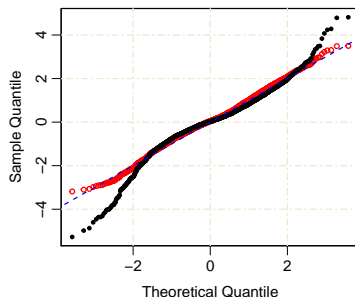
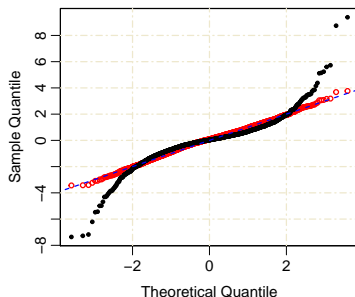
# Empirical study 1 (cont'd): Vanilla option price

- Vanilla Options Pricing Tested Against the Parameters for 10y10y Swaption. Analytic prices ( $P_{ANA}$ ) are computed from Hagan's approximation and exact  $S_U$  analytics and MC price ( $P_{MC}$ ) is shown relative to  $P_{ANA}$  with the standard deviation. The MC simulation with  $10^6$  paths is repeated 100 times. Prices are in the unit of the annuity of the underlying swap.

$K - \bar{F}_T$ (bps)	$\lambda = 0$		$\lambda = 1$	
	$P_{ANA}$	$P_{MC} - P_{ANA}$	$P_{ANA}$	$P_{MC} - P_{ANA}$
-200	2.275E-2	-4.1E-5 $\pm$ 1.8E-5	2.274E-2	6.4E-7 $\pm$ 1.8E-5
-100	1.506E-2	-2.1E-5 $\pm$ 1.6E-5	1.506E-2	6.0E-7 $\pm$ 1.6E-5
0	9.083E-3	-1.2E-5 $\pm$ 1.3E-5	9.083E-3	5.3E-7 $\pm$ 1.3E-5
100	5.108E-3	-2.6E-5 $\pm$ 1.1E-5	5.108E-3	3.3E-7 $\pm$ 1.1E-5
200	2.807E-3	-4.8E-5 $\pm$ 8.8E-6	2.804E-3	3.9E-7 $\pm$ 9.1E-6
300	1.567E-3	-6.0E-5 $\pm$ 7.0E-6	1.559E-3	3.7E-7 $\pm$ 7.3E-6

# Empirical study 2: Distribution of stock index return

- Daily returns of S&P 500 index (left) and China 300 ETF index (right) since 2005.
- Fit  $\lambda = 0$  (N-SABR) and  $\lambda = 1$  ( $S_U$ ) distributions to the first 4 moments.
- Normal probability Q-Q plots of raw (black) and asinh-transformed (red) sample quantile.
- Distributions with  $\lambda = 0$  and  $\lambda = 1$  indistinguishable.



# Conclusion

Hyperbolic normal stochastic volatility (NSVh) model brings together the normal SABR model ( $\lambda = 0$ ) and Johnson's  $S_U$  distribution ( $\lambda = 1$ ).

Based on the two generalizations (Alili and Gruet, 1997) of Bougerol's identity, we equip NSVh model with unprecedented analytic tools:

- closed-form (exact and one-step) MC scheme (1 RN draw per 1.5 normal RNs)

For Johnson's  $S_U$  distribution ( $\lambda = 1$ ), we provide

- theoretical foundation as a solution of SDE
- intuitive re-parametrization
- possibility for substituting normal SABR ( $\lambda = 0$ ) with various closed-form solutions (option price, VaR, ES, etc).

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